Stability of discrete shock profiles for systems of conservation laws

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Conservation laws and shocks

We consider a one-dimensional scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R},$$
$$u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R},$$

where the flux $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

Some of the results that will be presented also hold for systems of conservation laws.

Observation: This type of PDE tends to have solutions with discontinuities.

Overarching goal: When considering a conservative finite difference scheme, understand if it will be able to handle/capture discontinuities of solutions.

We consider two distinct states $u^-, u^+ \in \mathbb{R}^2$ and a velocity $s \in \mathbb{R}$ such that:

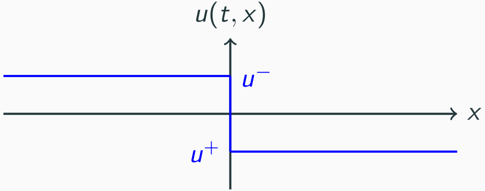
$$f(u^-) - f(u^+) = s(u^- - u^+), \quad \text{(Rankine-Hugoniot condition)}$$
$$f'(u^+) < s < f'(u^-). \quad \text{(Lax shock inequalities)}$$

The function u defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t,x) := \begin{cases} u^- & \text{if } x < st, \\ u^+ & \text{else,} \end{cases}$$

which is a weak solution of the scalar conservation law, is known as a Lax shock.

We focus on steady Lax shock, i.e. $s = 0$.



Conservative finite difference schemes

We introduce a cartesian grid with a space step $\Delta x > 0$ and a time step $\Delta t > 0$.

<u>**Goal:**</u> Compute sequences $u^n := (u_j^n)_{j\in\mathbb{Z}}$ such that u_j^n is close to the solution u on $[n\Delta t, (n+1)\Delta t[\times \left[\left(j-\frac{1}{2}\right)\Delta x, \left(j+\frac{1}{2}\right)\Delta x\right[$.

We consider a conservative explicit finite difference scheme:

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z},$$
$$u_j^{n+1} = u_j^n - \nu\left(F\left(\nu; u_{j-p+1}^n, \dots, u_{j+q}^n\right) - F\left(\nu; u_{j-p}^n, \dots, u_{j+q-1}^n\right)\right)$$

- $u^0 \in \mathbb{R}^{\mathbb{Z}}$: initial condition
- $F :]0, +\infty[\times \mathbb{R}^{p+q} \to \mathbb{R}$: numerical flux
- $p, q \in \mathbb{N}\backslash\{0\}$: integers defining the size of the stencil of the scheme.
- $\nu = \frac{\Delta t}{\Delta x} > 0$: ratio between the time and space steps.

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We consider a conservative explicit finite difference scheme:

$$\boxed{\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n)}$$

where the nonlinear discrete evolution operator $\mathcal{N} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ is defined as:

$$\mathcal{N}(u)_j := u_j - \nu \left(F\left(\nu; u_{j-p+1}, \ldots, u_{j+q}\right) - F\left(\nu; u_{j-p}, \ldots, u_{j+q-1}\right)\right).$$

"Usual" assumptions on the numerical scheme:

- $\forall u \in \mathbb{R}, \quad F(\nu; u, \ldots, u) = f(u)$ (consistency condition)
- For some neighborhood $\mathcal{U}$ of the states $u^{\pm}$

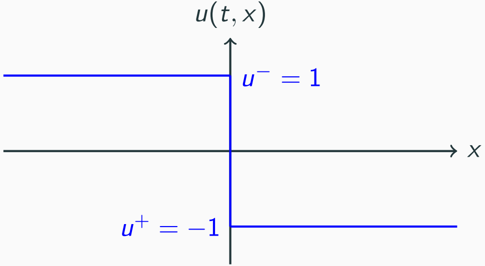
$$\forall u \in \mathcal{U}, \quad -q \leq \nu f'(u) \leq p \quad \text{(CFL condition on } \nu\text{)}$$

- Linear-ℓ^2 stability for constant states $u \in \mathcal{U}$

Diffusion assumption on the numerical scheme:

- The scheme introduces numerical viscosity. This excludes dispersive schemes like for instance the Lax-Wendroff scheme.
  In the present presentation, we consider a first order scheme.

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$.



For the numerical scheme, we consider the modified Lax Friedrichs scheme

$$\forall u \in \mathbb{R}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \quad \mathcal{N}(u)_j := \frac{u_{j+1} + u_j + u_{j-1}}{3} - \nu \frac{f(u_{j+1}) - f(u_{j-1})}{2}.$$

Discrete shock profiles

- Discrete shock profile (DSP): A discrete shock profile is a solution of the numerical scheme of the form

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^n = \overline{u}(j - s\nu n)$$

where the function $\overline{u} : \mathbb{Z} + s\nu\mathbb{Z} \to \mathbb{R}$ verifies that

$$\overline{u}(x) \underset{x\to\pm\infty}{\to} u^{\pm}.$$

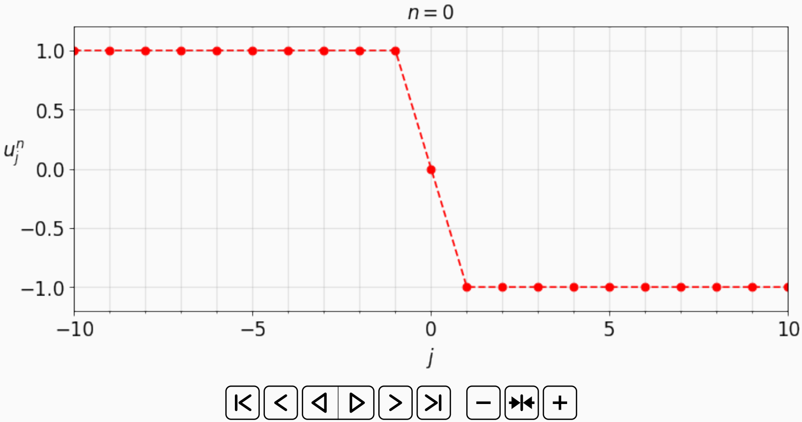
- The particular case of steady Lax shocks, i.e. $s = 0$:

Stationary discrete shock profiles (SDSP) are sequences $\overline{u} = (\overline{u}_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfy

$$\mathcal{N}(\overline{u}) = \overline{u} \quad \text{and} \quad \overline{u}_j \underset{j\to\pm\infty}{\to} u^{\pm}.$$

Example : We consider the following initial condition (mean of the standing shock on each cell):

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$



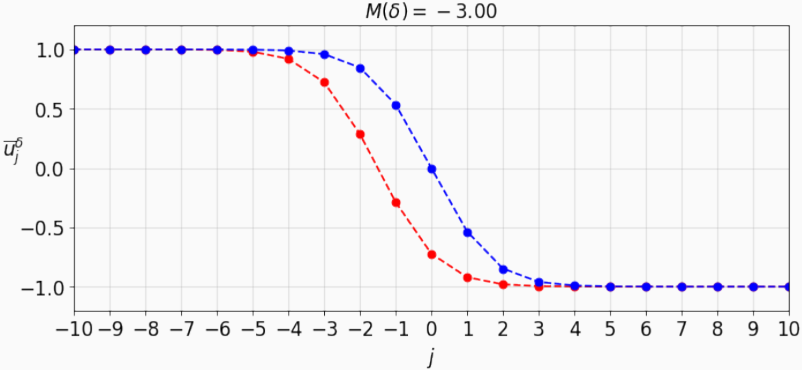
Main goal: Finding conditions on the numerical schemes so that:

stable shock waves for the conservation law	$\Longrightarrow$	stable DSPs for the numerical scheme

This separates the theory surrounding DSPs in two parts:

- Existence of DSPs
- Stability of DSPs

Existence results on SDSPs

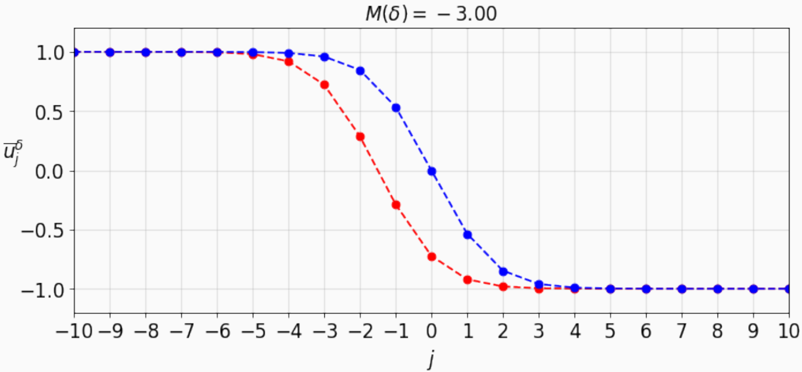


Existence of a continuous one-parameter family $(\overline{u}^\delta)_{\delta \in]-\Delta,\Delta[}$ of SDSPs around [Jennings '74, Majda-Ralston '79, Michelson '84, ...]

From now on, we denote a reference discrete shock profile:

$$\overline{u} := \overline{u}^0.$$

Existence results on SDSPs



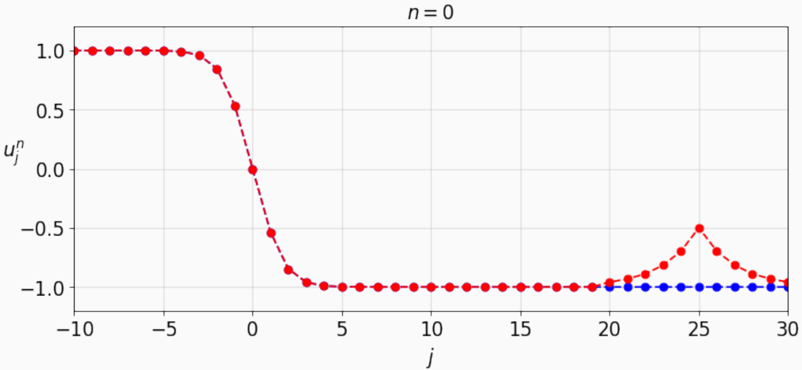
Assumption: The *mass* function M is injective, where:

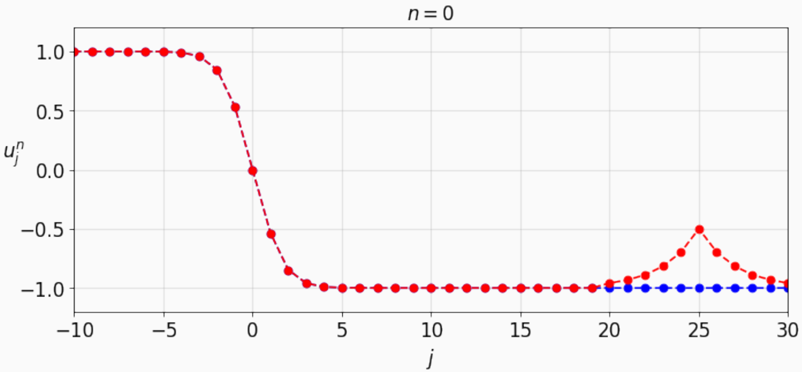
$$\forall \delta \in]-\Delta, \Delta[, \quad M(\delta) := \sum_{j \in \mathbb{Z}} \overline{u}_j^\delta - \overline{u}_j.$$

Nonlinear orbital stability of discrete shock profiles

The end goal would be the nonlinear orbital stability of those DSPs:

For small admissible perturbations $\mathbf{h}$, prove that the solution u^n of the numerical scheme for the initial condition $u^0 = \overline{u} + \mathbf{h}$ converges towards the set of translations of the DSP $\{\overline{u}^\delta, \delta \in]-\Delta, \Delta[\}$.



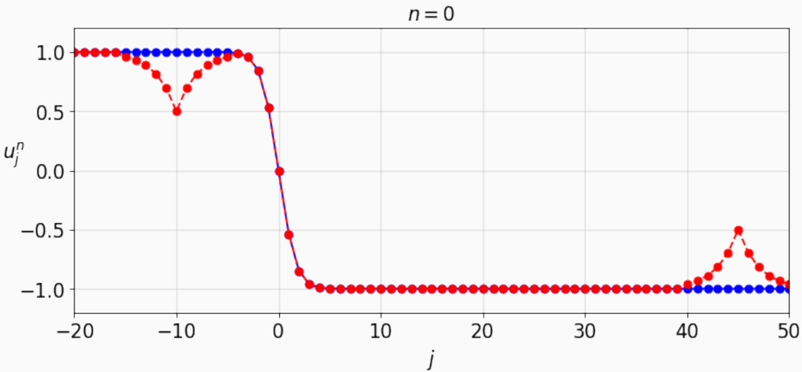


We have conservation of mass:

$$\forall n \in \mathbb{N}, \quad \sum_{j \in \mathbb{Z}} u_j^n - \overline{u}_j = \sum_{j \in \mathbb{Z}} u_j^0 - \overline{u}_j = \sum_{j \in \mathbb{Z}} \mathbf{h}_j.$$

Thus, if $u^n \underset{n \to +\infty}{\to} \overline{u}^\delta$ in ℓ^1, then:

$$\boxed{M(\delta) = \sum_{j \in \mathbb{Z}} \mathbf{h}_j.}$$



The case of zero-mass pertubations $\mathbf{h}$.

[Jennings, '74]

- scalar case
- conservative monotone scheme
- nonlinear orbital stability for ℓ^1 perturbations

[Smyrlis, '90]

- scalar case
- stationnary Lax shocks
- Lax-Wendroff scheme
- exponential weight on the initial perturbation

[Liu-Xin, '93]

- system case
- Lax-Friedrichs scheme
- weak Lax shocks
- zero mass assumption (dropped in [Ying, 97']) and polynomial weight on the initial perturbation

[Michelson, '02]

- system case
- weak stationary Lax shocks
- First and third order schemes

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Goal: We want to prove a result of nonlinear orbital stability that works:

- for systems of conservation laws,
- for a fairly large class of numerical schemes,
- with the fewest restrictions possible on the initial perturbations $\mathbf{h}$,
- replacing the smallness assumption on the amplitude of the shock with a spectral stability assumption on the discrete shock profile.

The spectral stability assumption corresponds to additional information on the point spectrum of the linearization about the SDSP $\overline{u}$.

Technique: Adaptation of the ideas "*à la Zumbrun*" which study the stability of traveling waves for parabolic PDEs.

Spectral stability $\Longrightarrow$ Linear stability $\Longrightarrow$ Nonlinear stability

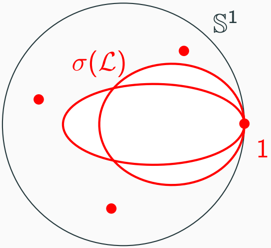
Linearization of the numerical scheme about the SDSP

We define the bounded operator $\mathcal{L} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ obtained by linearizing $\mathcal{N}$ about $\overline{u}$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}h)_j := \sum_{k=-p}^{q} a_{j,k} h_{j+k},$$

with $a_{j,k} \to a_k^\pm$ as $j \to \pm\infty$.

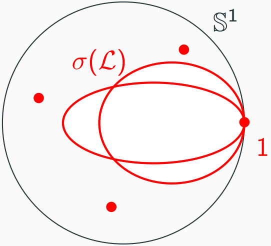
The coefficient $a_{j,k}$ are expressed using the partial derivatives $\partial_k F(\nu; \overline{u}_{j-p}, \dots, \overline{u}_{j+q-1})$.



Observation on the spectrum of $\mathcal{L}$

- There are curves of essential spectrum corresponding to the spectras of the linearized operators about the constant states u^+ and u^-.
- Outside of those essential spectrum curves, the spectrum is only composed of eigenvalues.
- 1 is an eigenvalue of the operator $\mathcal{L}$:

$$\text{"}\ \left(\forall\delta \in]-\Delta,\Delta[,\quad \mathcal{N}(\overline{u}^\delta) = \overline{u}^\delta\right) \Rightarrow \left.\frac{\partial \overline{u}^\delta}{\partial \delta}\right|_{\delta=0} \in \ker(Id - \mathcal{L}).\text{"}$$



Spectral stability assumption

- We construct a so-called Evans function. We assume that 1 is a simple zero of the Evans function. As a consequence, 1 is a simple eigenvalue of the operator $\mathcal{L}$:

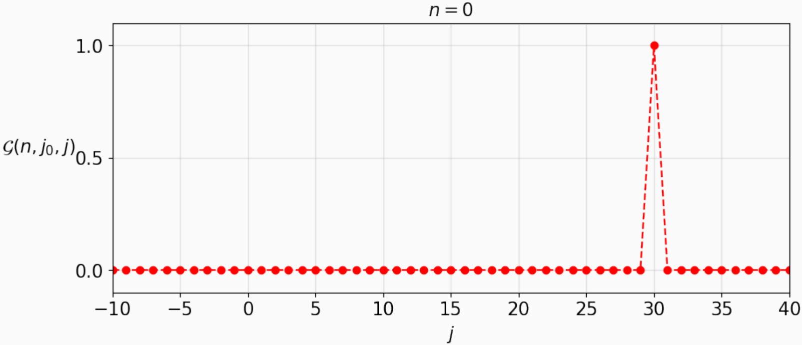
$$\text{"}\ \ker(Id - \mathcal{L}) = \mathrm{Span}\left(\left.\frac{\partial \overline{u}^\delta}{\partial \delta}\right|_{\delta=0} \right).\text{"}$$

- The operator $\mathcal{L}$ has no other eigenvalue of modulus equal or larger than 1.

We define the Green's function associated to $\mathcal{L}$:

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j \in \mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Green's function associated to the operator $\mathcal{L}$ for $j_0 = 30$



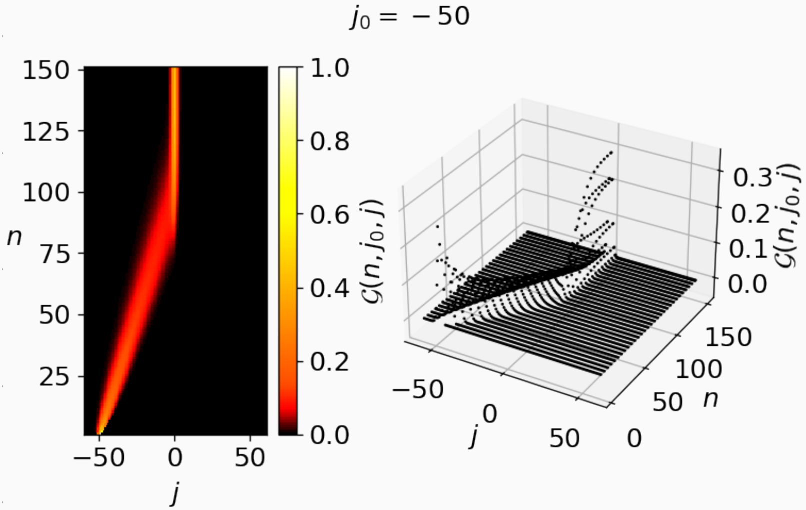
Theorem [C. 23']

Under some more precise assumptions, there exist a positive constant c, an element V of $\ker(Id - \mathcal{L})$ and an (explicit) function $E : \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$\begin{aligned}
&\mathcal{G}(n, j_0, j) \\
=&E\left(\frac{nf'(u^+)\nu + j_0}{\sqrt{n}}\right) V(j) \quad \text{(Excited eigenvector)} \\
+&\mathbb{1}_{j\in\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\frac{\left|nf'(u^+)\nu - (j-j_0)\right|^2}{n}\right)\right) \quad \text{(Gaussian wave)} \\
+&\mathbb{1}_{j\in-\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\frac{\left|nf'(u^+)\nu + j_0\right|^2}{n}\right) e^{-c|j|}\right) \quad \text{(Exponential residual)} \\
+&O(e^{-cn - c|j-j_0|})
\end{aligned}$$

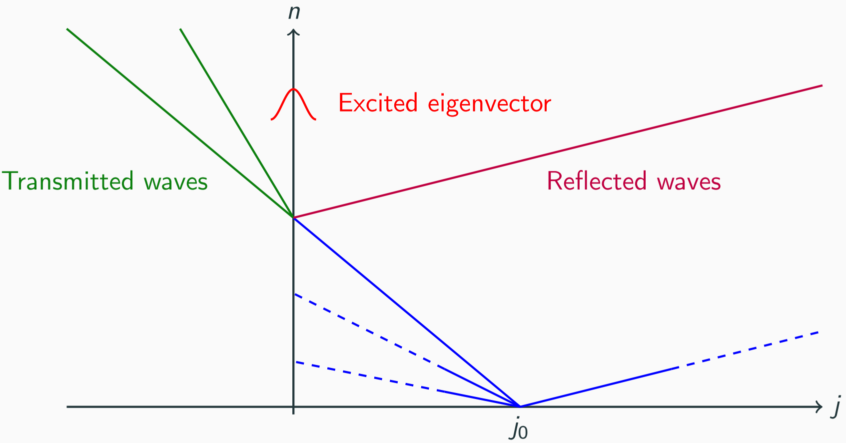
where $E(x) \underset{x\to-\infty}{\to} 1$ and $E(x) \underset{x\to+\infty}{\to} 0$.

There is a similar result for $j_0 \in -\mathbb{N}$.



j0 = −50
150
125
100
75
50
25
n
−50
0
50
j
1.0
0.8
0.6
0.4
0.2
0.0
G(n, j0, j)
0.3
0.2
0.1
0.0
150
100
50
0
n

Result on the Green's function in the case of systems of conservation laws



For $h \in \ell^\infty(\mathbb{Z})$ and $n \in \mathbb{N}$, we have:

$$\forall j \in \mathbb{Z}, \quad (\mathcal{L}^n h)_j = \sum_{j_0 \in \mathbb{Z}} \mathcal{G}(n, j_0, j) h_{j_0}.$$

Also, for $j_0 \in \mathbb{N}$ (a similar description exists for $j_0 \in -\mathbb{N}$):

$$\begin{aligned}
&\mathcal{G}(n, j_0, j) \\
=& E\left(\frac{nf'(u^+)\nu + j_0}{\sqrt{n}}\right) V(j) \quad \text{(Excited eigenvector)} \\
+& \mathbb{1}_{j\in\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu - (j-j_0)|^2}{n}\right)\right)\right) \quad \text{(Gaussian wave)} \\
+& \mathbb{1}_{j\in-\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu + j_0|^2}{n}\right)\right) e^{-c|j|}\right) \quad \text{(Exponential residual)} \\
+& O(e^{-cn - c|j-j_0|})
\end{aligned}$$

The description of the Green's function allows to prove decay estimates on the semigroup $(\mathcal{L}^n)_{n\in\mathbb{N}}$.

Linear stability implies nonlinear orbital stability

We consider an initial perturbation $\mathbf{h} \in \ell^1$ such that there exists $\delta \in]-\Delta, \Delta[$ satisfying:

$$M(\delta) = \sum_{j \in \mathbb{Z}} \mathbf{h}_j.$$

For the solution of the numerical scheme $(u^n)_{n\in\mathbb{N}}$ with $u^0 := \overline{u} + \mathbf{h}$, then the sequence u^n should converge towards $\overline{u}^\delta$.

We thus want to estimate $h^n := u^n - \overline{u}^\delta$ in some suitable norm.

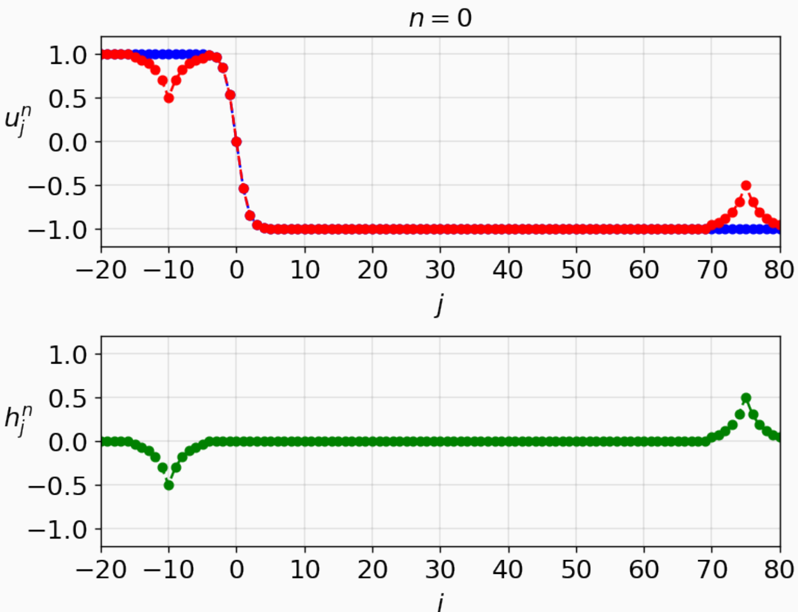
The polynomially weighted norms

For $r \in [1, +\infty]$ and $\gamma \in [0, +\infty[$, we define the polynomial-weighted spaces ℓ^r_γ:

$$\ell^r_\gamma := \{(h_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad \left((1+|j|)^\gamma h_j\right)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z})\}$$

with the norm:

$$\forall h \in \ell^r_\gamma, \quad \|h\|_{\ell^r_\gamma} = \left\|\left((1+|j|)^\gamma h_j\right)_{j\in\mathbb{Z}}\right\|_{\ell^r}.$$



$n = 0$
1.0
0.5
0.0
−0.5
−1.0
u_j^n
−20
−10
0
10
20
30
40
50
60
70
80
j
h_j^n

Theorem [C. '24]

Let us assume that the same assumptions are verified (and thus especially the spectral stability assumption). We consider a constant $\mathbf{p} > 0$. There exist two constants $\varepsilon, C \in [0, +\infty[$ such that, if we consider a initial perturbation $\mathbf{h} \in \ell^1_{\mathbf{p}+\max\left(\frac{1}{2},\mathbf{p}\right)}$ such that:

$$\|\mathbf{h}\|_{\ell^1_{\mathbf{p}+\max\left(\frac{1}{2},\mathbf{p}\right)}} < \varepsilon \quad \text{(polynomial weight condition)},$$

then there exists $\delta \in]-\Delta, \Delta[$ such that:

$$M(\delta) = \sum_{j\in\mathbb{Z}} \mathbf{h}_j$$

Furthermore, for the solution $(u^n)_n$ of the numerical scheme initialized with $u^0 = \overline{u}$, we have that for all $n \in \mathbb{N}\backslash\{0\}$, the sequence $h^n = u^n - \overline{u}^\delta$ satisfies:

$$\|h^n\|_{\ell^1_{\mathbf{p}}} \leq \frac{C}{n^{\mathbf{p}}} \|\mathbf{h}\|_{\ell^1_{\mathbf{p}+\max\left(\frac{1}{2},\mathbf{p}\right)}} \qquad \text{and} \qquad \|h^n\|_{\ell^\infty_{\mathbf{p}}} \leq \frac{C}{n^{\mathbf{p}+\frac{1}{2}}} \|\mathbf{h}\|_{\ell^1_{\mathbf{p}+\max\left(\frac{1}{2},\mathbf{p}\right)}}.$$

Conclusion/ Perspective / Open questions

About the Green's function theorem:

- Bounds on Green's function uniform in j_0
- Proved for a large family of schemes
- The result is proved for systems

About the nonlinear stabilty theorem:

- The result is not **yet** proved for systems

Other Perspectives:

- What can we say for moving shocks (with rational speed) and/or for under/over-compressive shocks?
- What can we say for dispersive schemes?
  - $\rightarrow$ *Stability of SDSPs for the Lax-Wendroff scheme* [Coulombel and Faye '24]
- Spectral stability of the SDSPs?
- Study of the stability for multi-dimensional conservation laws

Thank you for your attention!

Brief idea of the proof

Using the inverse Laplace tranform, we have:

$$\forall n \in \mathbb{N}\setminus\{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi}\int_{\Gamma} z^n G(z, j_0, j)dz$$

where the spatial Green's function $G(z, j_0, j)$ is defined by:

$$\forall z \notin \sigma(\mathcal{L}), \forall j_0, j \in \mathbb{Z}, \quad G(z, j_0, j) := \left((zId - \mathcal{L})^{-1}\delta_{j_0}\right)_j$$

and the integration path Γ surrounds the spectrum $\sigma(\mathcal{L})$.

Goal:

- Meromorphic extension of the spatial Green's function $G(\cdot, j_0, j)$ near 1
- Find a suitable choice of path Γ