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Stabilité de profils de chocs totalement discrets pour les systèmes de lois de conservation

Thèse présentée et soutenue, le 12 juillet 2024 par

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Stabilité de profils de choc totalement discrets pour les systèmes de lois de conservation

Résumé

Cette thèse aborde l'analyse de la stabilité des profils de choc totalement discrets pour les systèmes de lois de conservation. Ces profils correspondent à l'approximation d'ondes progressives discontinues par des schémas aux différences finies conservatifs. De telles solutions discontinues apparaissent naturellement dans l'étude des systèmes de lois de conservation qui peuvent modéliser de nombreuses situations physiques comme par exemple la dynamique des gaz.
L'étude des profils de choc totalement discrets se divise essentiellement en deux axes, le premier étant de construire de tels profils discrets et donc de prouver leur existence, et le second étant d'étudier leur stabilité. L'objectif principal de cette thèse est d'approfondir cette seconde direction. De nombreux résultats existants sur la stabilité des profils de chocs totalement discrets introduisent des hypothèses contraignantes, telles que la restriction aux lois de conservation scalaires ou encore le fait d'imposer que les discontinuités approchées soient de faible amplitude. Les résultats de cette thèse visent à ouvrir la voie vers des résultats de stabilité non linéaire qui traiterait de systèmes de lois de conservation et non pas seulement de lois scalaires, et qui remplacerait l'hypothèse de faible amplitude des discontinuités par une hypothèse spectrale sur le linéarisé du schéma autour du profil de choc discret considéré.
Au niveau des résultats obtenues, dans un premier temps, la thèse se focalise sur l'obtention d'estimées de décroissance fines sur le linéarisé du schéma aux niveaux de solutions particulières. On se concentrera d'abord sur le linéarisé au niveau des solutions constantes avant de passer au cas plus compliqué du linéarisé au niveau des profils de choc totalement discrets. D'un point de vue spectral, l'analyse du problème des chocs fait apparaitre une valeur propre plongée dans le spectre essentiel. Il en résulte de nouveaux termes dans l'analyse de la fonction de Green du schéma linéarisé et on détaille les propriétés de décroissance de chacun de ces termes. Dans une dernière partie, on utilise les estimations obtenues sur l'opérateur linéarisé pour établir un argument de stabilité non linéaire.

Mots clés : schémas aux différences finies, stabilité, systèmes de lois de conservation, profils de choc

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Stability of discrete shock profiles for systems of conservation laws

Abstract

This thesis deals with the stability analysis of discrete shock profiles for systems of conservation laws. These profiles correspond to approximations of discontinuous traveling waves by conservative finite difference schemes. Such discontinuous solutions appear naturally in the study of conservation law systems, which can model many physical situations, such as gas dynamics.
The study of discrete shock profiles is essentially divided into two directions, the first one focusing on the construction of such discrete profiles and thus on the proof of their existence, and the second one studying their stability. The main objective of this thesis is to investigate this second direction. Many existing results on the stability of discrete shock profiles introduce constraining hypotheses, such as the restriction to scalar conservation laws or the requirement that the approximated discontinuities should be of small amplitude. The results of this thesis aim to pave the way towards nonlinear stability results that would deal with systems of conservation laws and not just scalar laws, and that would replace the smallness assumption on the amplitude of the discontinuities by a spectral assumption on the linearization of the numerical scheme about the discrete shock profile under consideration.
In terms of the results obtained, the thesis initially focuses on obtaining sharp decay estimates for the linearization of the numerical scheme about particular solutions. We will first focus on the linearization about constant solutions before moving on to the more complicated case of the linearization about discrete shock profiles. From a spectral point of view, the analysis of the shock problem implies the existence of an eigenvalue located within the essential spectrum. This results in new terms in the analysis of the Green's function of the linearized scheme and decay properties of each of these terms will be presented. In a final section, we use the estimates obtained on the linearized operator to establish a nonlinear stability argument.

Keywords: finite difference schemes, stability, systems of conservation laws, shock profiles

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Chapitre 1

Introduction

Sommaire du présent chapitre

Lorsque l'on considère une approximation d'un système de lois de conservation par un schéma aux différences finies conservatif, les profils de choc totalement discrets correspondent aux solutions qui sont des ondes progressives reliant deux états. Dans cette thèse, nous allons présenter des résultats sur la stabilité de ces fameux profils de choc totalement discrets. Le premier objectif de cette introduction est de rappeler l'intérêt et l'état actuel de la recherche autour de ces objets afin de clarifier les apports des résultats de cette thèse. On commencera par redéfinir succinctement et rappeler des éléments centraux autour des systèmes de lois de conservation hyperboliques monodimensionnels et des approximations de ceux-ci par des schémas aux différences finies conservatifs (Section 1.1). Cela nous mènera à définir les profils de choc totalement discrets. Nous ferons une présentation relativement générale de cette notion et des propriétés connues concernant les profils des chocs totalement discrets (Section 1.2). La troisième partie de l'introduction (Section 1.3) portera sur les différentes contributions de la thèse concernant la stabilité de profils de choc totalement discrets. Cette section présentera en particulier les résultats des Chapitres 4 et 5 ainsi que la méthodologie pour arriver à ces résultats. Le Chapitre 4 portera sur le contenu de l'article [Coe23].

Les Chapitres 2 et 3 présentent des contributions de la thèse qui ne portent pas immédiatement sur la stabilité des profils de choc totalement discrets. La Section 1.4 présente la contribution du Chapitre 2 sur une généralisation du théorème de la limite locale, un résultat connu de probabilités, et, plus précisément, sur le contenu de l'article [Coe22]. Enfin, la Section 1.5 quant à elle porte sur le résultat du Chapitre 3 qui concerne la stabilité de schémas aux différences finies pour une équation de transport à vitesse négative sur la demi-droite positive. Plus précisément, le Chapitre 3 présentera le contenu de l'article [Coe24].

1.1 Contexte général

1.1.1 Systèmes de lois de conservation, solutions faibles et chocs

Les systèmes de lois de conservation monodimensionnels sont les équations aux dérivées partielles de la forme :

$$\begin{aligned} \partial_t u + \partial_x f(u) = 0, \quad t \in]0,+\infty[, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{U} \subset \mathbb{R}^d, \end{aligned} \tag{1.1.1}$$

où l'entier d correspond au nombre de quantités conservées, la variable $u := (u_1, \dots, u_d)$ correspond aux quantités conservées, l'ensemble $\mathcal{U}$ appelé espace d'états est un ouvert (parfois supposé convexe) de $\mathbb{R}^d$ et la fonction $f : \mathcal{U} \to \mathbb{R}^d$ est appelée le flux. On considèrera toujours que les systèmes de lois de conservation étudiés dans cette thèse sont hyperboliques, c'est-à-dire que pour tout état $u \in \mathcal{U}$, la jacobienne $df(u)$ est diagonalisable dans $\mathbb{R}$. L'étude des solutions des systèmes de lois de conservation (1.1.1) et du problème de Cauchy qui lui est associé est un domaine vaste et dont la théorie est encore incomplète (en particulier dans le cadre multidimensionnel). On redirige les lecteurs intéressés vers [Ser99 ; Ser00 ; Bre00 ; Daf16] pour une présentation générale des systèmes de lois de conservation.

Ces systèmes de lois de conservation (plus généralement leurs versions multidimensionnelles) apparaissent dans de multiples domaines : mécanique des fluides (avec les équations d'Euler ou encore les équations de Saint-Venant), électromagnétisme (équations de Maxwell), magnétohydrodynamique, etc...

Précisons deux exemples de lois de conservation scalaires (i.e. $d = 1$ et $\mathcal{U} = \mathbb{R}$ dans (1.1.1)) qui apparaissent dans la thèse.

— En considérant le flux linéaire $f(u) = cu$ avec une constante $c \in \mathbb{R}$, on retrouve la fameuse équation de transport

$$\begin{aligned} \partial_t u + c\partial_x u = 0, \quad t \in]0,+\infty[, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{U} \subset \mathbb{R}^d. \end{aligned}$$

Précisons d'emblée que les chapitres 2 et 3 présenteront des résultats associés à des schémas aux différences finies appliqués à cette fameuse équation de transport dans divers cadres.

— Si l'on prend le flux $f(u) = u^2/2$, on retrouve l'équation dite de Burgers. C'est l'exemple le plus simple de loi de conservation nonlinéaire et il nous servira de modèle jouet dans cette thèse. Pour faire des simulations associées aux profils de choc totalement discrets, on considérera généralement ce choix.

Perte de régularité des solutions

Une particularité des systèmes de lois de conservation est que, même en imposant une condition initiale régulière, des discontinuités peuvent apparaître en temps fini. En effet, si l'on se place dans le cas scalaire (i.e. $d = 1$ dans (1.1.1)) et que l'on considère le problème de Cauchy associé à (1.1.1) où la condition initiale u_0 est régulière, la méthode des caractéristiques permet de construire "explicitement" une solution régulière sur un ensemble $[0, T[\times\mathbb{R}$ où $T > 0$. Cependant, la valeur maximale de T est généralement bornée et correspond au temps où les caractéristiques se croisent. Cela conduit à introduire la notion de solutions faibles du problème de Cauchy associé à (1.1.1).

Définition (Solution faible du problème de Cauchy)**.** Si l'on considère une condition initiale $u_0 \in L^1_{loc}(\mathbb{R},\mathcal{U})$. On dit que la fonction $u \in L^1_{loc}([0,+\infty[\times\mathbb{R},\mathbb{R}^d)$ est une solution faible du problème de Cauchy associé à (1.1.1) avec comme condition initiale u_0 lorsque $u(t, x) \in \mathcal{U}$ pour presque tout $(t, x) \in [0,+\infty[\times\mathbb{R}$, $f(u)$ appartient aussi à $L^1_{loc}([0,+\infty[\times\mathbb{R},\mathbb{R}^d)$ et que pour toute fonction test $\phi \in \mathcal{C}^\infty_c([0,+\infty[\times\mathbb{R})$:

$$\iint_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial\phi}{\partial t}(t,x)u(t,x) + \frac{\partial\phi}{\partial x}(t,x)f(u(t,x))dtdx + \int_{\mathbb{R}} \phi(x,0)u_0(x)dx = 0.$$

Condition de Rankine-Hugoniot et chocs

Une conséquence de cette définition des solutions faibles est la condition de Rankine-Hugoniot qui permet de caractériser les discontinuités admissibles pour les systèmes de lois de conservation.

On considère une fonction $X : [0,+\infty[\to \mathbb{R}$ de classe $\mathcal{C}^1$ paramétrisant la localisation d'une discontinuité. Soit une fonction $u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{U}$ telle que :

— La fonction u est $\mathcal{C}^1$ sur les ensembles

$$\{(t,x)\in\mathbb{R}_+\times\mathbb{R},\quad x<X(t)\}\quad \text{et}\quad \{(t,x)\in\mathbb{R}_+\times\mathbb{R},\quad x>X(t)\}$$

et est une solution classique de (1.1.1) sur ces ensembles.

— Pour tout $t\in\mathbb{R}_+$, les états

$$u^+(t):=\lim_{x\to X(t)^+}u(t,x)\in\mathcal{U}\quad \text{et}\quad u^-(t):=\lim_{x\to X(t)^-}u(t,x)\in\mathcal{U}$$

sont bien définis.

Alors, la fonction u est une solution faible de (1.1.1) si et seulement si la condition dite de Rankine-Hugoniot est vérifiée :

$$\forall t\in\mathbb{R}_+,\quad f(u^+(t))-f(u^-(t))=\frac{dX}{dt}(t)(u^+(t)-u^-(t)). \tag{1.1.2}$$

Un exemple de solutions faibles sont les chocs.

Définition. On appellera choc les fonctions de la forme :

$$\begin{array}{rccc} u: & \mathbb{R}_+\times\mathbb{R} & \to & \mathcal{U} \\ & (t,x) & \mapsto & \begin{cases} u^- & \text{si } x\leq st, \\ u^+ & \text{sinon,}\end{cases}\end{array} \tag{1.1.3}$$

où les deux états $u^-,u^+\in\mathcal{U}$ et la vélocité $s\in\mathbb{R}$ vérifient :

$$f(u^+)-f(u^-)=s(u^+-u^-). \tag{1.1.4}$$

La condition (1.1.4) correspond à la condition de Rankine-Hugoniot (1.1.2) et implique que les chocs sont des solutions faibles de (1.1.1). Précisons cependant que l'on ajoute la plupart du temps des conditions supplémentaires, dont on parlera plus bas, que l'on appelle condition d'admissibilité afin de définir quels chocs sont "physiquement admissibles".

Problématique générale

Essayons d'ores et déjà de présenter la problématique générale qui justifiera l'introduction des profils de choc totalement discrets.

Les observations de la Section 1.1.1 nous poussent à envisager que la solution du problème de Cauchy (1.1.1) pour une donnée initiale régulière par morceaux sera elle aussi régulière par morceaux avec des discontinuités qui peuvent apparaître ou disparaître, mais qui sont toujours caractérisées par la condition de Rankine-Hugoniot (1.1.2). Si l'on cherche à approcher ces solutions à l'aide d'un schéma aux différences finies, une problématique qui apparaît alors clairement est de déterminer si la méthode numérique sera capable de capturer les discontinuités de ces solutions. En effet, l'étude de la consistance des schémas aux différences finies repose généralement sur des hypothèses de régularité des solutions sous-jacentes (par exemple, le développement de Taylor pour déterminer l'ordre de consistance du schéma).

Pour étudier ce vaste problème qu'est la capacité des schémas numériques à capturer les discontinuités des solutions, on se ramène essentiellement à se poser la question plus "élémentaire" de l'approximation des chocs par ces méthodes numériques. Les profils de choc totalement discrets qui seront définis en Section 1.2 sont des objets centraux pour répondre à cette problématique.

Condition d'admissibilité et chocs de Lax

Avant de passer à une partie liée aux schémas aux différences finies, finissons par discuter de conditions supplémentaires que l'on va imposer aux chocs qui seront étudiés au sein de cette thèse. En effet, l'introduction de la notion de solutions faibles pose un problème fondamental dans la théorie des systèmes de lois de conservation : on perd l'unicité des solutions du problème de Cauchy associé au système de lois de conservation (1.1.1) Afin de résoudre ce problème, il est nécessaire d'ajouter des conditions supplémentaires que les solutions doivent vérifier pour essayer d'isoler une solution qui serait "physiquement admissible". Plusieurs conditions d'admissibilité existent et les systèmes étudiés approchant généralement des phénomènes physiques, ces conditions ont généralement pour but d'identifier une solution qui ait un sens physique.

Comme nous l'avons intuité dans le paragraphe précédent, nous allons nous concentrer sur l'approximation des chocs par les schémas aux différences finies. Plus précisément, nous allons nous focaliser sur une famille particulière de chocs dans cette thèse vérifiant une condition d'admissibilité nommée *condition de Lax* que l'on va présenter ci-dessous.

On considère deux états $u^-, u^+ \in \mathcal{U}$ et une vélocité $s \in \mathbb{R}$ telle que la condition de Rankine-Hugoniot (1.1.4) est vérifiée. Le choc (1.1.3) est alors une solution faible de (1.1.1). Le système de lois de conservation que l'on étudie est supposé hyperbolique au niveau des états u^- et u^+. On peut alors noter les valeurs propres $\lambda_1^\pm, \dots, \lambda_d^\pm$ des jacobiennes $df(u^+)$ et $df(u^-)$ en les ordonnant telles que

$$\lambda_1^\pm \leq \ldots \leq \lambda_d^\pm$$

où certaines des valeurs propres peuvent être égales.

On considèrera par la suite des chocs dits non-caractéristiques, c'est-à-dire que la vélocité s du choc n'appartiendra pas au spectre des jacobiennes $df(u^+)$ et $df(u^-)$. Il existe alors deux entiers $p^+ \in \{0, \dots, d-1\}$ et $p^- \in \{1, \dots, d\}$ tels que :

$$\lambda_{p^+}^+ < s < \lambda_{p^++1}^+ \quad \text{and} \quad \lambda_{p^--1}^- < s < \lambda_{p^-}^- \tag{1.1.5}$$

où l'on prend la convention que $\lambda_0^\pm = -\infty$ et $\lambda_{d+1}^\pm = +\infty$. Un choc vérifiant la condition $p^+ = p^- =: p$ (*condition de Lax*) est appelé p-choc de Lax.

Cette condition a été introduite dans [Lax57]. Elle est particulièrement prisée pour la simplicité de sa vérification comparée à d'autres conditions d'admisibilité telle que la condition d'entropie qui nécessite d'identifier une entropie associée au système de lois de conservation. Dans certains cas, cette condition partage des liens forts avec d'autres types de condition d'admissibilité. Typiquement, pour les lois de conservation scalaires avec un flux convexe, les chocs entropiques sont les chocs de Lax, i.e. ceux qui vérifient

$$f'(u^+) < s < f'(u^-).$$

Dans le cas plus général d'un système de lois de conservation, si celui-ci admet une entropie, il est aussi connu que les p-chocs de Lax de faible amplitude associés à un champ caractéristique vraiment non linéaire sont aussi des solutions entropiques. Cependant, dès lors que l'on considère des chocs sans hypothèse de faible amplitude, les liens entre les conditions d'admissibilité tiennent moins. Précisons d'ores et déjà que cela va nous pousser à nous poser des questions dans les perspectives sur la généralisation de certains résultats de la thèse pour des chocs plus généraux, typiquement les chocs sur- et sous-compressifs (où les entiers p^+ et p^- définis par (1.1.5) ne sont pas égaux).

1.1.2 Schémas aux différences finies conservatifs

Dans cette thèse, on s'intéresse à l'approximation des solutions du système de lois de conservation (1.1.1) par des schémas aux différences finies conservatifs que l'on va définir ci-dessous. Concernant l'étude des schémas aux différences finies conservatifs, on peut citer [GR21 ; LeV92].

On considère une constante $\boldsymbol{\nu} > 0$ et des pas d'espace $\Delta x > 0$ et de temps $\Delta t := \boldsymbol{\nu}\Delta x > 0$. La constante $\boldsymbol{\nu}$ correspond alors au ratio entre les pas de temps et d'espace. On introduit l'opérateur d'évolution discret $\mathcal{N} : \mathcal{U}^\mathbb{Z} \to (\mathbb{R}^d)^\mathbb{Z}$ défini pour une donnée $u = (u_j)_{j\in\mathbb{Z}} \in \mathcal{U}^\mathbb{Z}$ par

$$\forall j \in \mathbb{Z}, \quad (\mathcal{N}(u))_j := u_j - \boldsymbol{\nu}\left(F\left(\boldsymbol{\nu}; u_{j-p+1}, \dots, u_{j+q}\right) - F\left(\boldsymbol{\nu}; u_{j-p}, \dots, u_{j+q-1}\right)\right), \tag{1.1.6}$$

où les entiers $p, q \in \mathbb{N}\backslash\{0\}$ correspondent au stencil du schéma et la fonction

$$F : (\nu; u_{-p}, \dots, u_{q-1}) \in]0, +\infty[\times \mathcal{U}^{p+q} \to \mathbb{R}^d$$

de classe $\mathcal{C}^1$ est appelé le flux numérique du schéma. Le schéma aux différences finies conservatif correspond alors aux familles $(u^n)_{n\in\mathbb{N}}$ telles que $u^n \in \mathcal{U}^\mathbb{Z}$ pour tout $n \in \mathbb{N}$ et qui soient solutions de la récurrence :

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n). \tag{1.1.7}$$

Généralement, on impose de plus que la constante $\boldsymbol{\nu}$ doit vérifier (ou au moins la vérifier sur un compact raisonnable de $\mathcal{U}$) la condition dite de Courant-Friedrichs-Lewy (CFL) définie par :

$$\forall u \in \mathcal{U}, \quad \boldsymbol{\nu} \min \sigma(df(u)) > -q \quad \text{and} \quad \boldsymbol{\nu} \max \sigma(df(u)) < p. \tag{1.1.8}$$

Consistance, Théorème de Lax-Wendroff et comportement diffusif

Une hypothèse importante à introduire est la fameuse condition de consistance qui se traduit par l'égalité suivante :

$$\forall \nu \in]0, +\infty[, \forall u \in \mathcal{U}, \quad F(\nu; u, \dots, u) = f(u). \tag{1.1.9}$$

Cette hypothèse est essentielle car elle est centrale pour le Théorème de Lax-Wendroff (originellement introduit dans [LW60]) qui permet de conclure que si les solutions du schéma aux différences finies (1.1.7) convergent vers une fonction u, alors u est une solution faible de (1.1.1). L'énoncé détaillé est le suivant.

Théorème (Théorème de Lax-Wendroff)**.** *On fixe dans cet énoncé $\boldsymbol{\nu} > 0$ le ratio entre les pas de temps et d'espace. On considère un schéma numérique (1.1.7) consistant avec le système de lois de conservation (1.1.1), i.e. telle que le flux numérique F apparaissant dans l'opérateur d'évolution nonlinéaire $\mathcal{N}$ défini par (1.1.6) vérifie la condition (1.1.9). On considère une suite de pas d'espace $(\Delta x_k)_{k\in\mathbb{N}}$ convergeant vers 0 et on définit les pas de temps $\Delta t_k := \boldsymbol{\nu}\Delta x_k$ pour $k \in \mathbb{N}$. Pour $k \in \mathbb{N}$, on considère la fonction par morceaux $v_k : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{U}$ définie par :*

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad v_k(t,x) := u_j^n \quad \text{si } t \in [n\Delta t_k, (n+1)\Delta t_k[\text{ et } x \in \left[\left(j - \frac{1}{2}\right)\Delta x_k, \left(j + \frac{1}{2}\right)\Delta x_k\right[$$

où les suites u^n sont construites par le schéma aux différences finies (1.1.7) avec la condition initiale

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \frac{1}{\Delta x_k} \int_{(j-\frac{1}{2})\Delta x_k}^{(j+\frac{1}{2})\Delta x_k} u_0(x)dx.$$

Si la suite de fonctions $(v_k)_{k\in\mathbb{N}}$ est bornée dans $L^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}^d)$ et converge vers une fonction u appartenant à $L^1_{loc}(]0,+\infty[\times\mathbb{R}, \mathbb{R}^d)$, alors la fonction u est une solution faible du problème de Cauchy associé à (1.1.1) pour la condition initiale u_0.

Comme on l'a présenté dans le paragraphe "Problématique générale" de la Section 1.1.1, ce qui nous intéresse, c'est de discuter de la capacité des schémas aux différences finies à capturer les discontinuités. Sur cette question, le Théorème de Lax-Wendroff nous permet de conclure que les discontinuités capturées par les schémas aux différences finies conservatifs et consistants vont vérifier la condition de Rankine-Hugoniot. Essentiellement, l'objectif des profils de choc totalement discrets est de répondre à la question réciproque : Si l'on considère un choc vérifiant la condition de Rankine-Hugoniot ainsi que des conditions d'admissibilité convenables, est-il bien approché par un tel schéma aux différences finies ?

Finissons cette section en introduisant la notion d'ordre de consistance.

Définition. On dit que le schéma aux différences finies conservatif (1.1.7) est d'ordre de consistance $r \in \mathbb{N}$ si r est l'entier le plus grand tel que, en fixant le ratio $\boldsymbol{\nu} := \frac{\Delta t}{\Delta x} > 0$ et en considérant une solution régulière u du système de lois de conservation (1.1.1), on ait que :

$$\begin{aligned} & u(t+\Delta t, x) - u(t,x) \\ & \quad + \boldsymbol{\nu}\left(F\left(\boldsymbol{\nu}; u(t, x+(-p+1)\Delta x), \dots, u(t, x+q\Delta x)\right) - F\left(\boldsymbol{\nu}; u(t, x-p\Delta x), \dots, u(t, x+(q-1)\Delta x)\right)\right) \\ & \qquad = O(\Delta x^{r+1}). \end{aligned} \tag{1.1.10}$$

Le terme de gauche dans (1.1.10) est appelé erreur de troncature locale du schéma et il correspond à l'erreur effectuée en remplaçant dans l'expression du schéma (1.1.7) les valeurs approchées u_j^n, calculées à l'aide du schéma numérique, par les valeurs exactes de la solution $u(t,x)$ du système de lois de conservation (1.1.1). Pour déterminer l'ordre de consistance d'un schéma, on utilise la régularité de la solution u que l'on considère pour effectuer des développements limités et obtenir une estimation de la forme (1.1.10). Précisons que les schémas consistants, c'est-à-dire les schémas satisfaisant la condition (1.1.9), sont au moins d'ordre 1.

Les contributions de cette thèse se concentreront sur le cas des schémas aux différences finies conservatifs d'ordre impair. En effet, les schémas de ce type impliquent des propriétés spectrales particulières pour les opérateurs linéarisés du schéma numérique au niveau des états constants (voir le paragraphe suivant). Précisons de plus que si l'on considère un schéma d'ordre impair, alors on peut s'attendre à voir apparaître des comportements diffusifs sur les solutions du schéma numérique (1.1.7). En effet, il est par exemple possible de prouver que, sous des hypothèses de régularité suffisantes sur le flux numérique F, en appliquant des développements limités sur l'erreur de troncature locale (i.e. le membre de gauche de (1.1.10)) et en utilisant la condition de consistance (1.1.9), on a que :

$$\begin{aligned} & u(t+\Delta t, x) - u(t,x) \\ & \quad + \boldsymbol{\nu}\left(F\left(\boldsymbol{\nu}; u(t, x+(-p+1)\Delta x), \dots, u(t, x+q\Delta x)\right) - F\left(\boldsymbol{\nu}; u(t, x-p\Delta x), \dots, u(t, x+(q-1)\Delta x)\right)\right) \\ & \qquad = -\boldsymbol{\nu}\Delta x^2 \partial_x\left(B(\boldsymbol{\nu}; u(t,x))\partial_x u(t,x)\right) + O(\Delta x^3) \end{aligned} \tag{1.1.11}$$

où la matrice $B(\boldsymbol{\nu}; u(t,x))$, parfois appelée matrice de viscosité, est définie par :

$$B(\boldsymbol{\nu}; u(t,x)) = -\frac{\boldsymbol{\nu}}{2} df(u(t,x))^2 - \sum_{k=-p}^{q-1} \frac{2k-1}{2} \partial_{u_k} F(\boldsymbol{\nu}, u(t,x), \dots, u(t,x)).$$

Le cas des schémas consistants d'ordre 1 correspond au cas où la matrice de viscosité B ne s'annule pas. On a alors que le schéma approche bien mieux les solutions de l'EDP :

$$\partial_t u + \partial_x f(u) = -\Delta x \partial_x \left(B(u) \partial_x u \right), \quad t > 0, x \in \mathbb{R} \tag{1.1.12}$$

où l'on voit un phénomène diffusif apparaître, que le système de lois de conservation (1.1.1). Précisons que l'équation (1.1.12) correspond à ce qui est connu comme étant l'*équation modifiée* du schéma. Pour plus de détails, on invite les lecteurs intéressés par la notion d'équation modifiée à s'intéresser à l'article [Hed75] qui présente cette notion plus en détail dans le cadre plus élémentaire des schémas aux différences finies pour l'équation de transport.

Les schémas d'ordres impairs plus élevés font apparaître des phénomènes visqueux eux-aussi d'ordre plus élevé. Au contraire, un schéma d'ordre 2 par exemple, comme le schéma de Lax-Wendroff (voir (1.1.21) ci-dessous), fait apparaître un comportement dispersif (voir Figure 1.1).

Stabilité du linéarisé au niveau des constantes

On remarque que pour un état $\overline{u} \in \mathcal{U}$, la condition de consistance (1.1.9) implique que la suite constante $(\overline{u})_{j\in\mathbb{Z}}$ est un point fixe de l'opérateur d'évolution discret $\mathcal{N}$ associé au schéma. De plus, dans le cas où l'on étudie l'approximation des solutions régulières du système de lois de conservation (1.1.1) par les schémas aux différences finies (1.1.7), on remarque que, pour des choix de pas de temps et d'espace suffisamment petits, les valeurs $u_{j-p}^n, \dots, u_{j+q}^n$ qui apparaissent sont proches d'une valeur constante. Il est donc utile d'étudier la linéarisation de l'opérateur $\mathcal{N}$ au niveau des états constants.

Fixons un état constant $\overline{u} \in \mathcal{U}$. La linéarisation de $\mathcal{N}$ au niveau de l'état constant $\overline{u}$ est donnée par l'opérateur $\mathscr{L}_{\overline{u}}$ défini lorsqu'il agit sur l'espace $\ell^r(\mathbb{Z}, \mathbb{C}^d)$ avec $r \in [1, +\infty]$ par :

$$\forall h \in \ell^r(\mathbb{Z}, \mathbb{C}^d), \forall j \in \mathbb{Z}, \quad (\mathscr{L}_{\overline{u}} h)_j := \sum_{k=-p}^{q} A_k^{\overline{u}} h_{j+k}, \tag{1.1.13}$$

où pour $k \in \{-p, \dots, q\}$, les matrices $A_k^{\overline{u}} \in \mathcal{M}_d(\mathbb{C})$ sont définies par :

$$A_k^{\overline{u}} := \begin{cases} \delta_{k,0} Id + \boldsymbol{\nu} \partial_{u_k} F(\boldsymbol{\nu}; \overline{u}, \dots, \overline{u}) - \boldsymbol{\nu} \partial_{u_{k-1}} F(\boldsymbol{\nu}; \overline{u}, \dots, \overline{u}) & \text{si } k \in \{-p+1, \dots, q-1\} \\ -\boldsymbol{\nu} \partial_{u_{q-1}} F(\boldsymbol{\nu}; \overline{u}, \dots, \overline{u}) & \text{si } k = q, \\ \boldsymbol{\nu} \partial_{u_{-p}} F(\boldsymbol{\nu}; \overline{u}, \dots, \overline{u}) & \text{si } k = -p. \end{cases} \tag{1.1.14}$$

Définition. Pour $r \in [1, +\infty]$, on dit que le schéma aux différences finies conservatif (1.1.7) est linéairement ℓ^r-stable au niveau de l'état constant $\overline{u}$ lorsque le semi-groupe $(\mathscr{L}_{\overline{u}}^n)_{n\in\mathbb{N}}$ est borné quand il agît dans $\ell^r(\mathbb{Z}, \mathbb{C}^d)$, i.e. il existe une constante $C > 0$ telle que :

$$\forall n \in \mathbb{N}, \forall h \in \ell^r(\mathbb{Z}, \mathbb{C}^d), \quad \|\mathscr{L}_{\overline{u}}^n h\|_{\ell^r} \leq C \|h\|_{\ell^r}.$$

⋆ *Objectif de ce paragraphe*

Par souci de clarté, on se permet de présenter d'ores et déjà les objectifs de ce paragraphe pour guider le lecteur. L'objectif est de présenter un lien entre la stabilité linéaire du schéma numérique au niveau de l'état $\overline{u}$, l'étude du spectre de l'opérateur $\mathscr{L}_{\overline{u}}$ et enfin l'ordre de consistance du schéma. En effet, comme dit précédemment, dans cette thèse, l'ensemble des résultats obtenus vont se concentrer exclusivement sur le cas particulier des schémas d'ordre de consistance impair, c'est-à-dire les schémas aux différences finies conservatifs faisant apparaître des comportements diffusifs. Nous allons voir que les opérateurs $\mathscr{L}_{\overline{u}}$ satisfont des propriétés spectrales intéressantes quand on considère des schémas diffusifs.

⋆ *Matrices d'amplification et spectre de l'opérateur* $\mathscr{L}_{\overline{u}}$

Les opérateurs $\mathscr{L}_{\overline{u}}$ étant à coefficients constants, ils peuvent être étudiés par analyse de Fourier. Cela nous pousse à introduire les **matrices d'amplification** $\mathcal{F}_{\overline{u}}(\kappa)$:

$$\forall \kappa \in \mathbb{C} \backslash \{0\}, \quad \mathcal{F}_{\overline{u}}(\kappa) := \sum_{k=-p}^{q} \kappa^k A_k^{\overline{u}} \in \mathcal{M}_d(\mathbb{C}). \tag{1.1.15}$$

Précisons que l'expression de la matrice d'amplification $\mathcal{F}_{\overline{u}}(\kappa)$ implique que celle-ci possède d valeurs propres qui dépendent continûment de κ. Nous allons maintenant nous permettre une hypothèse simplificatrice pour cette introduction en supposant qu'il existe d fonctions holomorphes $\lambda_1, \dots, \lambda_d$ définies sur $\mathbb{C}\backslash\{0\}$, à valeurs complexes, telles que pour tout $\kappa \in \mathbb{C}\backslash\{0\}$, les valeurs propres de la matrice d'amplification $\mathcal{F}_{\overline{u}}(\kappa)$ soient $\lambda_1(\kappa), \dots, \lambda_d(\kappa)$. Ceci est bien évidemment nontrivial à imposer et nous introduirons dans le Chapitre 4 des hypothèses qui nous permettront de nous retrouver dans ce genre de situation.

L'expression (1.1.14) des matrices $A_k^{\overline{u}}$ et la condition de consistance (1.1.9) impliquent que :

$$\mathcal{F}_{\overline{u}}(1) := Id \quad \text{et} \quad \mathcal{F}_{\overline{u}}'(1) := -\boldsymbol{\nu} df(\overline{u}),$$

et donc

$$\forall l \in \{1, \dots, d\}, \quad \lambda_l(1) := 1 \quad \text{et} \quad \lambda_l'(1) := -\boldsymbol{\nu}\gamma_l \tag{1.1.16}$$

où γ_l est une des valeurs propres de $df(\overline{u})$, qui est bien défine par hyperbolicité du système de lois de conservation (1.1.1).

À l'aide du Théorème de Wiener, dont une preuve succincte dans le cadre scalaire peut être trouvée dans [New75], on prouve que, lorsque l'on considère que l'opérateur $\mathscr{L}_{\overline{u}}$ agît sur $\ell^r(\mathbb{Z}, \mathbb{C}^d)$, on a :

$$\sigma(\mathscr{L}_{\overline{u}}) = \bigcup_{\kappa \in \mathbb{S}^1} \sigma(\mathcal{F}_{\overline{u}}(\kappa)) = \bigcup_{l=1}^{d} \lambda_l(\mathbb{S}^1). \tag{1.1.17}$$

où la notation σ est utilisée pour indiquer le spectre d'un opérateur ou d'une matrice en fonction de son application. Ainsi, le spectre de $\mathscr{L}_{\overline{u}}$ est l'union de d courbes continues paramétrisées par les valeurs propres $\lambda_1(\kappa), \dots, \lambda_d(\kappa)$ des matrices d'amplification $\mathcal{F}_{\overline{u}}(\kappa)$ quand κ parcourt le cercle unité $\mathbb{S}^1$ dans le plan complexe.

$\star$ *ℓ^2-stabilité linéaire et condition de dissipativité*

Si l'on s'intéresse au cas particulier de la ℓ^2-stabilité linéaire, l'analyse de Fourier implique que la famille d'opérateurs $(\mathscr{L}_{\overline{u}}^n)_{n\in\mathbb{N}}$ est bornée quand elle agît sur $\ell^2(\mathbb{Z}, \mathbb{C}^d)$ si et seulement si la fameuse condition de Von Neumann est vérfiée :

$$\sigma(\mathscr{L}_{\overline{u}}) = \bigcup_{l=1}^{d} \lambda_l(\mathbb{S}^1) \subset \{z \in \mathbb{C}, |z| \leq 1\},$$

i.e. les courbes de valeurs propres des matrices d'amplification $\mathcal{F}_{\overline{u}}(\kappa)$ quand $\kappa \in \mathbb{S}^1$ doivent se trouver dans le disque unité fermé. Cette condition est généralement vérifiée en imposant une condition CFL sur le choix de $\boldsymbol{\nu}$.

Afin d'étudier la ℓ^r-stabilité linéaire pour $r \in [1, +\infty]$, on impose généralement une condition plus stricte que la condition de Von Neumann connue sous le nom de *condition de dissipativité*. Elle suppose l'existence d'un entier $\mu \in \mathbb{N}\backslash\{0\}$ et d'une constante $c > 0$ tels que, pour tout $\xi \in [-\pi, \pi]\backslash\{0\}$, on ait que les valeurs propres $\lambda_1(e^{i\xi}), \dots, \lambda_d(e^{i\xi}) \in \mathbb{C}$ de la matrice d'amplification $\mathcal{F}_{\overline{u}}(e^{i\xi})$ vérifient :

$$\forall l \in \{1, \dots, d\}, \quad |\lambda_l(e^{i\xi})| \leq 1 - c\xi^{2\mu}. \tag{1.1.18}$$

En $\xi = 0$, on remarque que (1.1.16) implique que $\lambda_l(1) = 1$. La condition de dissipativité implique donc que les courbes de valeurs propres des matrices d'amplification $\mathcal{F}_{\overline{u}}(\kappa)$ quand $\kappa \in \mathbb{S}^1$ vont toucher le disque unité en 1 puis vont rentrer à l'intérieur du disque unité ouvert. Grâce à cette hypothèse de dissipativité, le problème de la ℓ^r-stabilité linéaire pour tout $r \in [1, +\infty]$ se ramène à regarder plus précisément le spectre de $\mathscr{L}_{\overline{u}}$ au niveau de 1. C'est cette partie de l'étude qui va nous pousser à nous concentrer sur les schémas aux différences finies d'ordre impair.

$\star$ *Article [Tho65] de Thomée et ℓ^r-stabilité linéaire pour tout $r \in [1, +\infty]$*

Dans la contribution fondamentale [Tho65] de Thomée, celui-ci caractérise, pour l'équation de transport scalaire, l'ensemble des schémas aux différences finies consistants qui sont ℓ^r-stables pour tout $r \in [1, +\infty]$ (plus précisément, il se concentre sur le cas $r = +\infty$ mais un schéma ℓ^∞-stable est ℓ^r-stable pour tout $r \in [1, +\infty]$). Adaptée dans notre cadre, la condition de Thomée se traduit comme suit :

Hypothèse 1.1. *Pour tout $l \in \{1, \dots, d\}$, on a :*

$$\forall \kappa \in \mathbb{S}^1\backslash\{1\}, \quad |\lambda_l(\kappa)| < 1. \tag{1.1.19}$$

De plus, on suppose qu'il existe un entier $\mu \in \mathbb{N}\backslash\{0\}$ et pour tout entier $l \in \{1, \dots, d\}$, il existe un nombre complexe β_l à partie réelle strictement positive tels que :

$$\lambda_l(e^{i\xi}) \underset{\xi\to 0}{=} \exp(-i\boldsymbol{\nu}\gamma_l\xi - \beta_l\xi^{2\mu} + O(|\xi|^{2\mu+1})), \quad \textit{(Condition de diffusivité)} \tag{1.1.20}$$

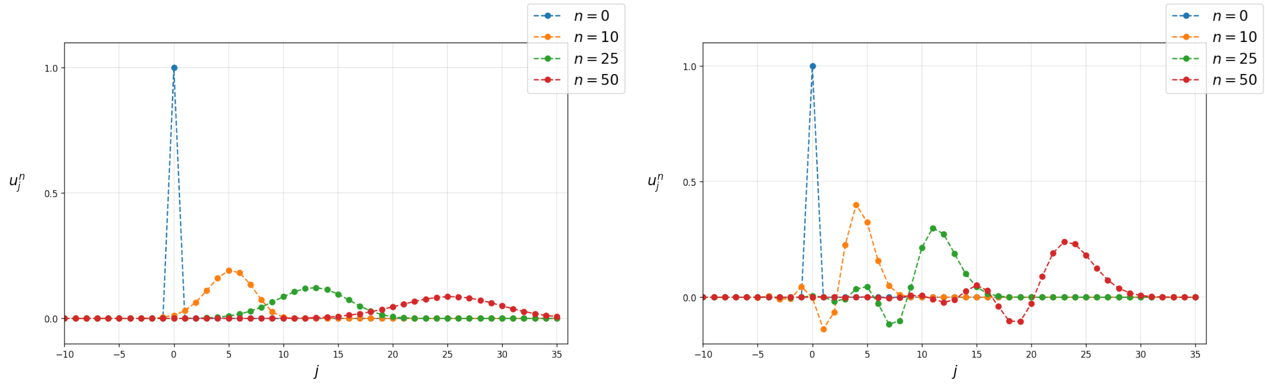

FIGURE 1.1 – On représente le comportement en temps long des solutions u^n des schémas de Lax-Friedrichs modifié (à gauche) et de Lax-Wendroff (à droite) approchant une équation de transport à vitesse positive (i.e. $f(u) = cu$ avec $c > 0$). La condition initiale est un Dirac $u_j^0 = \delta_{0,j}$ centré en 0. Puisque l'on étudie le cas d'une EDP linéaire sans dépendance spatiale, le schéma est linéaire et comprendre le comportement en temps long d'une condition initiale qui soit un Dirac fournit la solution fondamentale du schéma aux différences finies qui permet de comprendre le comportement en temps long de la solution du schéma pour n'importe quelle condition initiale. On voit apparaître les caractères diffusifs et dispersifs respectifs des schémas.

où γ_l est défini par (1.1.16).

En combinant (1.1.19) et (1.1.20), on peut retrouver la condition de dissipativité (1.1.18) présentée précédemment. Sous cette hypothèse, on peut prouver que le semi-groupe associé à l'opérateur $\mathscr{L}_{\overline{u}}$ est bornée lorsqu'il agît sur $\ell^r(\mathbb{Z}, \mathbb{C}^d)$ pour n'importe quel choix de $r \in [1, +\infty]$.

On remarque que la condition de diffusivité (1.1.20) correspond à avoir une forme particulière au développement limité du logarithme de la fonction λ_l en 1, où tous les termes d'ordre 2 jusqu'à $2\mu - 1$ doivent être nuls et où le coefficient à l'ordre pair 2μ doit avoir une partie réelle strictement positive. On prétend que cela impose au schéma d'être d'ordre $2\mu - 1$ et donc en particulier d'être d'ordre impair. Les schémas d'ordre pair, tels que le schéma de Lax-Wendroff, font apparaître dans le développement limité (1.1.20) des termes de la forme $ic\xi^k$ où c est un réel non nul et k est un entier dans $\{2, \dots, 2\mu - 1\}$. L'article [Tho65] de Thomée prouve qu'en particulier, ces schémas ne sont pas ℓ^∞-stables.

Exemples de schémas aux différences finies conservatifs

Précisons deux exemples de schémas aux différences finies conservatifs :

— Le schéma de Lax-Friedrichs modifié correspond au choix de flux numérique :

$$\forall \nu \in]0, +\infty[, \forall u_{-1}, u_0 \in \mathbb{R}, \quad F(\nu; u_{-1}, u_0) := \frac{f(u_{-1}) + f(u_0)}{2} + D(u_{-1} - u_0)$$

où $D \in \mathbb{R}$. Pour avoir la stabilité de ce schéma au niveau des états constants, il nous faut choisir un ratio fixe $\boldsymbol{\nu} := \frac{\Delta t}{\Delta x}$ et un coefficient D tels que :

$$\forall u \in \mathcal{U}, \quad |f'(u)| < 2D < \frac{1}{\boldsymbol{\nu}}.$$

Ce schéma est d'ordre 1 et fait apparaître une viscosité numérique (voir Figure 1.1). Ce sera le schéma exemple que l'on utilisera régulièrement à travers la thèse.

— Le schéma de Lax-Wendroff correspond au choix de flux numérique :

$$\forall \nu \in]0, +\infty[, \forall u_{-1}, u_0 \in \mathbb{R}, \quad F(\nu; u_{-1}, u_0) := \frac{f(u_{-1}) + f(u_0)}{2} + \frac{\boldsymbol{\nu}}{2} df \left(\frac{u_{-1} + u_0}{2}\right) (f(u_{-1}) - f(u_0)). \tag{1.1.21}$$

Au contraire du schéma de Lax-Friedrichs modifié, ce schéma ne fait pas apparaître un comportement diffusif mais plutôt un comportement qualifié de dispersif (voir Figure 1.1). En effet, c'est un schéma d'ordre 2. Les résultats prouvés dans cette thèse ne s'appliqueront pas à ce type de schémas, mais notons que certains résultats sur les profils de choc totalement discrets se concentrent sur ces schémas.

1.2 Profils de choc totalement discrets

1.2.1 Définition et objectifs

Nous allons introduire la notion de profil de choc totalement discret. Ils sont dénommés "discrete shock profiles" en anglais. Nous nous permettrons généralement l'utilisation de l'abréviation DSP pour remplacer "discrete shock profile" tout au long de la thèse. Ce seront les objets centraux de cette thèse.

Définition. Pour deux états $u^+, u^- \in \mathcal{U}$ et une vélocité $s \in \mathbb{R}$, un profil de choc totalement discret associé au triplet $(u^+, u^-; s)$ est une solution $(u^n)_{n\in\mathbb{N}} \in (\mathcal{U}^{\mathbb{Z}})^{\mathbb{N}}$ du schéma numérique (1.1.6) de la forme :

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^n = U(j - s\boldsymbol{\nu} n) \tag{1.2.1}$$

où $U : \mathbb{Z} + s\boldsymbol{\nu}\mathbb{Z} \to \mathcal{U}$ vérifie

$$\lim_{x\to\pm\infty} U(x) = u^{\pm}. \tag{1.2.2}$$

Les profils de choc totalement discrets sont donc les solutions du schéma numérique (1.1.6) qui sont des ondes progressives connectant deux états u^- et u^+. La vitesse $s\boldsymbol{\nu}$ du profil de choc totalement discret correspond essentiellement au nombre de cellules parcourues en un pas de temps. Pour une introduction relativement exhaustive sur la notion des profils de choc totalement discrets, on renvoie à [Ser07].

Précisons quelques propriétés et remarques fondamentales liées à ces profils de choc totalement discrets :

— Il est important de remarquer que l'existence et les propriétés d'un profil de choc totalement discret associé à un triplet $(u^-, u^+; s)$ sont très dépendantes du choix du ratio $\boldsymbol{\nu}$ entre les pas de temps et d'espace choisis pour le schéma numérique. Un profil de choc totalement discret pour un choix de $\boldsymbol{\nu}$ ne sera pas nécessairement aussi un profil de choc totalement discret pour un autre choix de $\boldsymbol{\nu}$. De plus, il est relativement évident que la définition des profils de choc totalement discrets diffère en fonction de la rationalité du paramètre $s\boldsymbol{\nu}$. On parlera de profil de choc totalement discret à vitesse rationnelle quand $s\boldsymbol{\nu} \in \mathbb{Q}$ et à vitesse irrationnelle quand $s\boldsymbol{\nu} \notin \mathbb{Q}$. Dans le cas des profils de choc totalement discrets à vitesse irrationnelle, l'ensemble $\mathbb{Z} + s\boldsymbol{\nu}\mathbb{Z}$ est dense dans $\mathbb{R}$. Généralement, dans le cas des vitesses irrationnelles, on demandera alors à la fonction U associée au profil de choc totalement discret d'être définie sur $\mathbb{R}$ entier et d'être (au moins) continue.

— Par l'expression (1.1.6) de l'opérateur d'évolution $\mathcal{N}$, la définition des profils de choc totalement discrets se ramène essentiellement à la recherche des fonctions $U : \mathbb{Z} + s\boldsymbol{\nu}\mathbb{Z} \to \mathcal{U}$ vérifiant (1.2.2) et qui soient solutions de :

$$\begin{aligned}&\forall y \in \mathbb{Z} + s\boldsymbol{\nu}\mathbb{Z},\\ &U(y - s\boldsymbol{\nu}) = U(y) - \boldsymbol{\nu}\left(F\left(\boldsymbol{\nu}; U(y-p+1), \dots, U(y+q)\right) - F\left(\boldsymbol{\nu}; U(y-p), \dots, U(y+q-1)\right)\right).\end{aligned} \tag{1.2.3}$$

Dans son habilitation à diriger la recherche [Ben98], Benzoni-Gavage prouve que, pour un schéma consistant, dans le cas des profils de choc totalement discrets à vitesse rationnelle, en sommant intelligemment (1.2.3), on retrouve que les états u^-, u^+ et la vélocité s associés à un profil de choc totalement discret doivent vérifier la condition de Rankine Hugoniot (1.1.4). Dans [Ser07], cet argument est présenté à nouveau et est étendu au cas des profils de choc à vitesse irrationnelle en intégrant plutôt qu'en sommant. L'argument essentiel de la preuve dans les deux cas est l'utilisation de la condition de consistance (1.1.9). La conclusion de cette observation est que les profils de choc totalement discrets ne peuvent exister que pour des chocs qui sont des solutions faibles du système de lois de conservation avec lequel le schéma est consistant. C'est en réalité relativement logique lorsque l'on met cette observation en parallèle avec le théorème de Lax-Wendroff qui énonçait sensiblement la même chose : les chocs qui peuvent être approchés par un schéma aux différences finies conservatifs consistants avec (1.1.1) sont ceux qui vérifient la condition de Rankine-Hugoniot (1.1.4).

— La thèse va se concentrer en particulier sur les profils de choc totalement discrets *stationnaires*, qui seront donc associés aux chocs *stationnaires* du système de lois de conservation d'après la remarque ci-dessus. Ceux-ci ont une forme particulière car ils correspondent aux points fixes de l'opérateur (1.1.7) d'évolution $\mathcal{N}$ reliant deux états, i.e. ce sont les suites $(u_j)_{j\in\mathbb{Z}}$ telles que

$$\lim_{j\to\pm\infty} u_j = u^{\pm} \quad \text{et} \quad \mathcal{N}((u_j)_{j\in\mathbb{Z}}) = (u_j)_{j\in\mathbb{Z}}.$$

On remarque que l'équation (1.2.3) devient dans le cas des profils de chocs totalement discrets stationnaires :

$$\forall j \in \mathbb{Z}, \quad F\left(\boldsymbol{\nu}; u_{j-p+1}, \dots, u_{j+q}\right) = F\left(\boldsymbol{\nu}; u_{j-p}, \dots, u_{j+q-1}\right),$$

i.e. la suite $\left(F\left(\boldsymbol{\nu}; u_{j-p}, \dots, u_{j+q-1}\right)\right)_{y\in\mathbb{Z}}$ est constante.

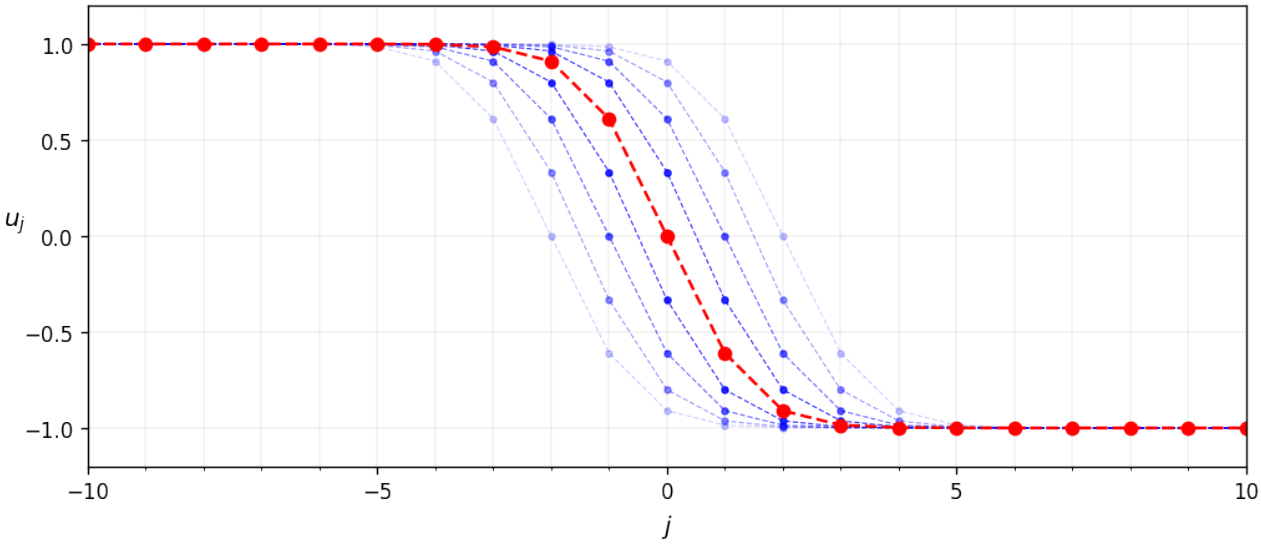

FIGURE 1.2 – Un exemple de profils de choc totalement discrets stationnaires. La loi de conservation est l'équation de Burgers, le schéma aux différences finies est le schéma de Lax-Friedrichs modifié avec $\boldsymbol{\nu} = 0.5$ et $D = 1/6$ et le choc stationaire a pour états $u^- = 1$ et $u^+ = -1$. La solution en rouge est un profil de choc totalement discret ainsi que tous les profils en bleu. Il existe un continuum de profils de choc totalement discrets. De plus, en se concentrant sur le profil en rouge, on remarque que la transition entre les deux états limites u^- et u^+ est essentiellement contenue dans un nombre fini de cellules.

Précisons que des observations similaires peuvent être faites pour les profils de choc totalement discrets à vitesses rationnelles en itérant le schéma numérique, mais au prix d'une augmentation notable du nombre de paramètres à prendre en compte.

Dans la Figure 1.2, on représente un exemple de profils de choc totalement discrets stationaires. Faisons deux remarques à ce stade :

— On remarque que, pour un profil de choc totalement discret, la transition entre les deux états du choc est généralement contenue dans un nombre "raisonnable" de cellules. Il est en effet cohérent de s'attendre à une convergence exponentielle de la fonction U vers les états limites pour les chocs non-caractéristiques. Ainsi, en termes de paramètres temporels et spatiaux t et x, la transition entre les deux états u^- et u^+ est confinée dans une bande de taille $O(\Delta x)$. La discontinuité est donc bien capturée.

— Pour les *chocs de Lax*, qui seront les chocs étudiés dans cette thèse, on prouve généralement non pas l'existence d'un unique profil de choc totalement discret associé à ce choc mais d'une famille continue de tels profils. Nous parlerons plus en détail de la théorie de l'existence des profils de choc totalement discrets ci-dessous. Précisons cependant que dans le cadre particulier des *chocs sous-compressifs* (qui ne sera pas considéré dans cette thèse), on prouve au contraire l'existence de profils de chocs totalement discrets isolés.

Revenons à la problématique qui nous occupe depuis le début de cette introduction. Notre objectif est de démontrer que, pour tout choc solution faible de (1.1.1) répondant à des conditions d'admissibilité appropriées (essentiellement liées à la stabilité du choc pour le système de lois de conservation), il existe une famille de profils de choc totalement discrets qui présentent également des propriétés de stabilité satisfaisantes. Ainsi, les thématiques de recherche sur les profils de choc totalement discrets se scindent en deux directions distinctes : l'existence de ces profils et leur stabilité. Bien que cette thèse se concentre principalement sur la seconde direction, nous aborderons tout d'abord brièvement les résultats connus en matière d'existence de profils de choc totalement discrets.

1.2.2 Existence de profils de choc totalement discrets

Dans cette section, nous allons présenter certains des résultats connus sur l'existence des profils de choc totalement discrets. On notera dorénavant u^- et u^+ les états caractérisant un choc et s sa vélocité. On supposera d'ailleurs toujours que la condition de Rankine-Hugoniot (1.1.4) est vérifiée.

Le résultat de Jennings [Jen74] pour les schémas monotones

Dans son article fondamental [Jen74] sur les profils de choc totalement discrets, Jennings prouve un résultat d'existence dans le cas des lois de conservation *scalaires* (i.e. $d = 1$) et pour les schémas aux différences finies monotones. Ce sont les schémas numériques (1.1.7) dont le flux numérique F vérifie que la fonction :

$$(u_{-p}, \dots, u_q) \in \mathbb{R}^{p+q+1} \mapsto u_0 - \boldsymbol{\nu} \left(F\left(\boldsymbol{\nu}; u_{-p+1}, \dots, u_q\right) - F\left(\boldsymbol{\nu}; u_{-p}, \dots, u_{q-1}\right)\right) \in \mathbb{R}$$

est croissante en fonction de chacune de ses variables. Plus précisément, pour les chocs vérifiant la condition de Rankine-Hugoniot et la fameuse condition d'entropie d'Oleinik définie par :

$$\forall u \in]u^-, u^+[, \quad \frac{f(u) - f(u^+)}{u - u^+} < \frac{f(u^-) - f(u^+)}{u^- - u^+},$$

si l'on considère une valeur $\overline{u}$ dans l'intervalle $]u^-, u^+[$, alors il existe un unique profil de choc totalement discret qui vaut $\overline{u}$ en $n = 0$ et $j = 0$. De plus, le profil de choc totalement discret dépend continûment de $\overline{u}$. Ce résultat a essentiellement deux avantages :

— Il s'applique pour des chocs de n'importe quelle amplitude. Comme on va le voir par la suite, une des limites récurrentes liées aux résultats sur les profils de choc totalement discrets sera le fait qu'ils s'appliquent généralement dans le cadre des chocs de faible amplitude.

— Le résultat tient pour des profils de choc totalement discrets à vitesse rationnelle ou irrationnelle. Les résultats d'existence et de stabilité dans ce second cadre sont rares. Précisons cependant que la preuve de Jennings était fausse pour les profils de choc totalement discrets à vitesse irrationnelle. Une preuve correcte a été donnée plus tard par Serre dans [Ser04].

Cependant, il est impossible de passer outre les fortes limitations du résultat de Jennings. En effet, le cadre qu'il considère est particulièrement favorable pour faire son étude, mais aussi extrêmement limité. La preuve repose entièrement sur la monotonie du schéma considéré. Précisons que les schémas monotones consistants sont d'ordre 1 mais que la réciproque est fausse (voir [LeV92]). Il est inenvisageable de l'étendre pour des schémas aux différences conservatifs plus généraux (même les schémas d'ordre 1) et encore moins pour étudier le cas des systèmes de lois de conservation (i.e. $d > 1$).

Le résultat de Smyrlis [Smy90] et discussion sur la paramétisation des profils de choc totalement discrets dans le cas scalaire $d = 1$

Précisons qu'il est parfois possible de construire explicitement des profils de choc totalement discrets pour certains schémas aux différences finies. C'est typiquement le cas du schéma de Lax-Wendroff. En effet, si l'on considère un choc stationnaire (i.e. $s = 0$) vérifiant la condition de Rankine-Hugoniot (1.1.4), alors la suite $(u_j)_{j\in\mathbb{Z}}$ définie ci-dessous est un profil de choc totalement discret stationnaire (i.e. un point fixe de l'opérateur $\mathcal{N}$ définie par (1.1.6)) :

$$\forall j \in \mathbb{Z}, \quad u_j := \begin{cases} u^- & \text{si } j \leq -1, \\ u^+ & \text{si } j \geq 0. \end{cases}$$

Dans [Smy90], Smyrlis se concentre sur le schéma de Lax-Wendroff pour les lois de conservation scalaires. Pour les chocs de Lax stationnaires (i.e. $s = 0$) d'amplitude quelconque, il prouve en utilisant la remarque présentée ci-dessus l'existence d'une famille continue de profils de choc totalement discrets associée au choc (u^-, u^+). De plus, il paramétrise cette famille de façon élégante avec une notion "d'excès de masse". Nous qualifions cette paramétrisation d'élégante car elle repose sur le caractère conservatif des schémas et surtout car cette paramétrisation jouera un rôle important dans la stabilité des profils de choc totalement discrets. Cela sera discuté dans la section suivante.

Le résultat de Smyrlis a, comme pour le résultat de Jennings, plusieurs limites. En particulier, il ne s'applique que dans le cas scalaire et seulement pour le schéma de Lax-Wendroff. Cependant, la fameuse paramétrisation par excès de masse est très intéressante. En effet, dans le Chapitre 5 de cette thèse, dans la Section 5.1.4, nous retrouvons cette paramétrisation dans le cas scalaire pour des schémas aux différences finies conservatifs qui exhibent cette fois un caractère diffusif. Ce n'est pas un résultat central de cette thèse mais potentiellement une remarque intéressante.

Les résultats de Majda-Ralston [MR79] et de Michelson [Mic84] dans le cas sytème pour les vitesses rationnelles

Parlons maintenant des résultats d'existence des profils de choc totalement discrets dans le cas des systèmes de lois de conservation. Le résultat de Majda-Ralston [MR79], qui a été ensuite généralisé par Michelson dans [Mic84], se concentre sur les profils de choc totalement discrets à vitesse rationnelle et associés à des chocs de Lax. Pour des schémas à comportement visqueux (plus précisément d'ordre de consistance 1 dans [MR79] et plus généralement d'ordre 1 et 3 dans [Mic84]), ces articles énoncent des résultats d'existence de familles continues de profils de choc totalement discrets. Cependant, ils imposent que les chocs soient de faible amplitude. Plus précisément si la vitesse $s\boldsymbol{\nu}$ s'écrit comme $\frac{p}{q}$ avec $p \in \mathbb{Z}$ et $q \in \mathbb{N}\backslash\{0\}$, alors l'amplitude du choc doit être petite vis-à-vis de $\frac{1}{q}$. Cette condition de faible amplitude est une conséquence de la preuve utilisée qui repose sur l'utilisation du théorème de la variété centrale (voir [Bre07]). Précisons qu'il n'est pas envisageable d'écrire un résultat aussi général sur l'existence des profils de choc totalement discrets pour des chocs d'amplitude quelconque.

Le résultat de Liu et Yu [LY99a] dans le cas système pour les vitesses irrationnelles

Comme dit plus haut, le cas des profils de choc totalement discrets à vitesse irrationnelle est particulièrement compliqué car la preuve utilisant le théorème de la variété centrale ne pourra pas fonctionner dans ce cas. Si ce n'est le résultat de Jennings [Jen74] qui s'applique dans le cadre des lois de conservation scalaires et pour les schémas aux différences finies conservatifs monotones, le résultat principal connu pour des vitesses irrationnelles est celui de [LY99a] par Liu et Yu qui s'applique dans le cas système et pour un choix particulier de schéma de Lax-Friedrichs modifié. Leur résultat d'existence porte sur les chocs de Lax de faible amplitude et lorsque la vitesse $s\boldsymbol{\nu}$ vérifie une condition dite Diophantienne qui signifie qu'il existe des constantes C et v positives telles que :

$$\forall p \in \mathbb{Z}, \forall q \in \mathbb{N}\backslash \{0\}, \quad \left| s\boldsymbol{\nu} - \frac{p}{q}\right| > \frac{C}{q^v}.$$

Cette condition implique que la vitesse $s\boldsymbol{\nu}$ est "mal" approchée par les rationnels.

1.2.3 Résultats connus sur la stabilité des profils de choc totalement discrets

Maintenant que l'on a présenté des résultats d'existence liés aux profils de choc totalement discrets, venons en au coeur de la thèse qui concerne les résultats de stabilité. Pour présenter les choses plus clairement, on considèrera dans cette section un choc de Lax stationnaire et on notera les états de part et d'autre de la discontinuités u^+ et u^-. De par les résultats d'existence de la section précédente, il semble raisonnable de supposer qu'il existe une famille à un paramètre $(\overline{u}^\delta)_{\delta\in]-\Delta,\Delta[}$ de profils de choc totalement discrets stationnaires associés à ce choc. En particulier, puisque l'on s'est placé dans le cas stationnaire, les suites $\overline{u}^\delta$ sont des points fixes de $\mathcal{N}$. Essentiellement, l'existence d'une famille continue de profils de choc totalement discrets pouvant approcher le même choc implique que la notion raisonnable de stabilité à introduire sera la notion de *stabilité orbitale*. Les résultats que l'on peut espérer prouver auront l'allure suivante, où X et Y désignent des espaces vectoriels normés :

Théorème. *Il existe une constante ε positive telles que si l'on considère une perturbation initiale $h^0 \in X$ telle que*

$$\left\|h^0\right\|_X < \varepsilon,$$

alors la solution $(u^n)_{n\in\mathbb{N}}$ du schéma numérique (1.1.7) pour la condition initiale $u^0 := \overline{u}^0 + h^0$ est bien définie pour tout $n \in \mathbb{N}$ et de plus, on a :

$$\min_{\delta\in]-\Delta,\Delta[} \left\|u^n - \overline{u}^\delta\right\|_Y \underset{n\to+\infty}{\to} 0. \tag{1.2.4}$$

Les résultats de ce type sont appelés *résultat de stabilité orbitale nonlinéaire*. Il nécessite d'identifier des espaces normés X et Y convenables tels que si l'on perturbe un profil de choc totalement discret avec une perturbation h^0 assez petite dans X, alors la solution du schéma numérique (1.1.7) qui en découle est bien définie en tout temps $n \in \mathbb{N}$ et tend à rester proche de la famille des profils de choc totalement discrets ou même à converger vers l'un d'entre eux. La Figure 1.3 représente exactement ce comportement. Précisons qu'il n'est pas trivial que la solution $(u^n)_{n\in\mathbb{N}}$ du schéma numérique (1.1.7) initialisé par $u^0 := \overline{u}^0 + h^0$ soit bien définie pour tout $n \in \mathbb{N}$. En effet, il est tout à fait possible qu'un mauvais choix de perturbation initiale h^0 implique que la solution du schéma numérique sorte du domaine de définition $\mathcal{U}^\mathbb{Z}$ de l'opérateur $\mathcal{N}$ d'évolution du schéma (1.1.7), i.e. il peut exister un temps $n \in \mathbb{N}$ et un entier $j \in \mathbb{Z}$ tel que $u_j^n \notin \mathcal{U}$.

Une partie non négligeable des articles présentés dans la section précédente prouvent des résultats de stabilité nonlinéaire orbitale en plus des résultats d'existence des profils de choc totalement discrets que l'on a présentés.

• Dans la continuité de leur résultat d'existence, Liu et Yu ont prouvé dans [LY99b] un résultat de stabilité nonlinéaire sur les profils dans le cas système de lois de conservation pour profils de choc totalement discrets à vitesse irrationnelle. Ce résultat constitue d'ailleurs le seul résultat connu de stabilité sur ce type de famille de profils de choc totalement discrets.

• On rappelle que les résultats de Jennings [Jen74] et de Smyrlis [Smy90] se concentraient sur les profils de choc totalement discrets dans le cas des lois de conservations scalaires, bien qu'ils considéraient des schémas numériques très différents, l'un se focalisant sur les schémas monotones et l'autre sur le schéma de Lax-Wendroff. Chacun dans leurs articles respectifs prouve de plus un résultat de stabilité orbitale nonlinéaire. Le résultat de Jennings [Jen74, Théorème 2] s'applique pour les chocs de Lax et pour les profils de choc totalement discrets à vitesse rationnelle (ce qui est moins général que pour son résultat d'existence). Les perturbations qui sont

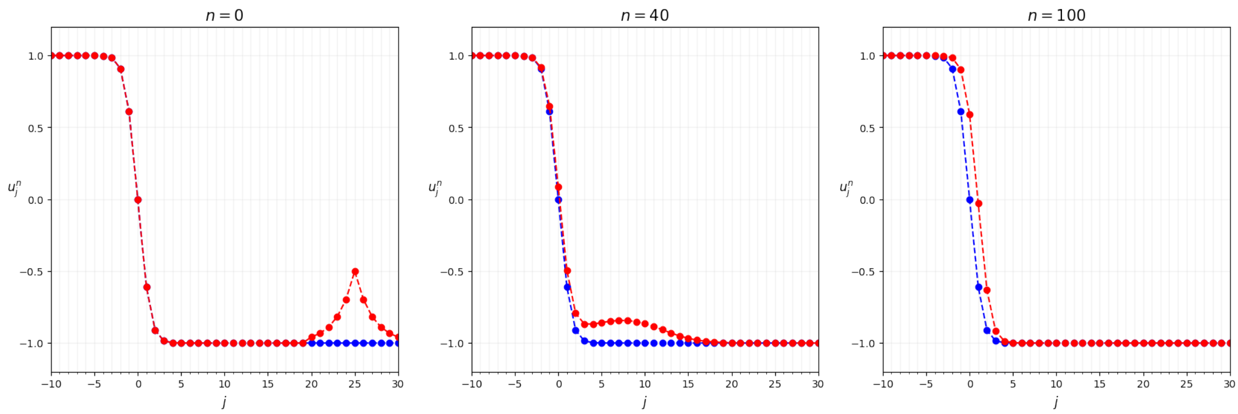

FIGURE 1.3 – Un exemple numérique de stabilité orbitale d'un profil de choc totalement discret. Sur l'image de gauche, on représente en bleu un profil de choc totalement discret $\overline{u}^0$ et en rouge une perturbation du profil discret $u^0 := \overline{u}^0 + h^0$. Les images suivantes représente l'évolution dans le temps de la solution u^n du schéma numérique (1.1.7). On voit qu'en temps long, la suite $(u^n)_{n\in\mathbb{N}}$ converge vers un autre profil de choc totalement discret associé au même choc, i.e. la distance entre u^n et la famille des $(\overline{u}^\delta)_{\delta\in]-\Delta,\Delta[}$ tend vers 0.

permises sont mesurées dans la norme ℓ^1 ce qui est une condition relativement faible. Rappelons cependant que le cadre dans lequel Jennings travaille est particulièrement favorable et lui permet de prouver un résultat de stabilité aussi fort. Cela n'est pas le cas du résultat de Smyrlis [Smy90, Théorème 4.1] qui impose des perturbations qui doivent appartenir à des espaces ℓ^1 à poids exponentiel. Il s'agît donc d'un cadre fonctionnel beaucoup plus restreint.

Résultats sur les profils discrets à vitesse rationnelle pour les systèmes de lois de conservation

Les résultats présentés ci-dessus traitent de cas très particuliers. En effet, le résultat [LY99b] ne s'applique que pour les profils de choc totalement discrets associés à une vitesse irrationnelle. C'est un cas particulier extrêmement complexe à traiter et cette thèse ne se préoccupera pas d'essayer d'étendre des résultats dans cette voie. D'un autre côté, les résultats de Jennings [Jen74] et Smyrlis [Smy90] ne s'appliquant que pour les profils de choc totalement discrets de lois de conservation scalaires, on comprend aisément que de fortes limitations en terme d'applications concrètes de ces résultats sont à anticiper. En effet, la plupart des phénomènes physiques approchés par des EDP de la forme (1.1.1) font intervenir plusieurs variables conservées qui interagissent entre elles, ce qui nécessite de trouver des résultats qui ne s'appliquent pas seulement au cas des lois de conservation scalaires, mais bien dans le cas des systèmes de lois de conservation.

Nous allons donc maintenant mettre en avant quelques résultats supplémentaires qui s'appliquent particulièrement aux systèmes de lois de conservation et pour des profils de choc totalement discrets à vitesse rationnelle associés à des chocs de Lax.

• Dans [LX93b ; LX93a], Liu et Xin prouvent des résultats de stabilité nonlinéaire sur les profils de choc totalement discrets dans le cas des systèmes de lois de conservation pour le choix particulier du schéma de Lax-Friedrichs modifié et pour des chocs de *faible amplitude*. L'article [LX93a] se concentre initialement sur les profils de choc totalement discrets stationnaires avant d'être généralisé pour les profils discrets à vitesse rationnelle dans [LX93b]. Les deux articles imposent aux perturbations initiales d'être petites dans un espace ℓ^2 à poids polynomial. Précisons qu'une telle condition est relativement raisonnable comparée à d'autres résultats tels que celui de Smyrlis [Smy90] qui peuvent imposer des décroissances exponentielles sur les conditions initiales. Cependant, il est aussi imposé dans [LX93a ; LX93b] aux perturbations initiales h^0 d'être de masse nulle, i.e.

$$\sum_{j\in\mathbb{Z}} h_j^0 = 0. \tag{1.2.5}$$

Le résultat obtenu est alors la convergence de la solution du schéma numérique vers le profil discret initial dans toutes les normes ℓ^p. La condition de masse nulle (1.2.5) impose la convergence vers le profil discret initial et non pas vers l'un des autres profils discrets de la famille continue qui est associée aux états limites du choc. Un résultat similaire est prouvé par Ying dans [Yin97] concernant les profils de choc totalement discrets à vitesse rationnelle pour le schéma de Lax-Friedrichs en retirant l'hypothèse (1.2.5) de masse nulle sur les perturbations initiales.

• Enfin, on cite le résultat [Mic02] par Michelson. Il prouve que les chocs de Lax stationnaires de *faible am-*

plitude admettent des profils totalement discrets vérifiant une propriété de stabilité nonlinéaire. La convergence de la solution perturbée u^n obtenue est exponentielle en norme ℓ^p. La preuve repose sur la stabilité de certaines ondes visqueuses pour des problèmes scalaires qui restreignent le résultat aux schémas d'ordre de consistance 1 ou 3.

1.3 Contribution de la thèse sur la stabilité des profils de choc totalement discrets (Chapitres 4 et 5)

1.3.1 Objectif initial de la thèse

Limitations des résultats connus de stabilité nonlinéaire et question ouverte de [Ser07]

Dans cette thèse, nous allons nous concentrer sur les profils de choc totalement discrets à vitesse rationnelle (et même plus particulièrement au cas des profils discrets stationaires) associés à des chocs de Lax. Comme nous l'avons dit plus tôt, le cas des profils discrets à vitesse irrationnelle est particulièrement ardu et ne sera pas considéré ici.

Comme présenté dans la Section 1.2.3, la plupart des résultats de stabilité nonlinéaire qui ont été prouvés font intervenir de multiples limitations. Si l'on devait les classifier, les limitations majeures seraient les suivantes :

— Certains résultats tels que celui de Jennings [Jen74] ou encore de Smyrlis [Smy90] ne s'appliquent que dans le cadre des lois de conservation *scalaires*, et les preuves ne peuvent être étendues au cas des systèmes de lois de conservation. Comme précisé précedemment, il serait intéressant de prouver des résultats dans le cas des systèmes de lois de conservation car ce sont généralement ceux qui apparaissent pour modéliser des situations physiques.

— Une partie des résultats est réalisée pour des choix spécifiques de schémas aux différences finies. On peut par exemple citer le résultat de Smyrlis [Smy90] qui se concentre sur le schéma de Lax-Wendroff, les résultats [LX93a ; LX93b] de Liu et Xin qui se concentrent sur le schéma de Lax-Friedrichs modifié ou encore le résultat [Yin97] qui considère le schéma de Lax-Friedrichs. Il serait intéressant de prouver des résultats de stabilité des profils de choc totalement discrets qui fonctionnent pour de larges familles de schémas aux différences finies conservatifs. Plus précisément, il serait préférable de prouver un résultat valable pour des schémas d'ordre de consistance (potentiellement) élevé.

— Pour les résultats qui s'appliquent dans le cas des systèmes de lois de conservation, les résultats ne s'appliquent généralement que pour les profils de choc totalement discrets approchant des discontinuités de *faible amplitude* (voir [LX93a ; LX93b ; Yin97 ; Mic02]). C'est un cadre qui est à première vue relativement logique car les chocs de Lax vraiment non-linéaires sont des solutions entropiques de (1.1.1) lorsqu'ils sont de faible amplitude. De plus, les résultats d'existence de profils de choc totalement discrets [MR79 ; Mic84] reposent sur cette condition de faible amplitude des discontinuités approchées. Il serait cependant préférable d'étendre l'étude aux profils de choc totalement discrets approchant des discontinuités d'amplitude quelconque. Précisons que cela remet en question le fait de se concentrer sur les chocs de Lax. Une problématique non traitée dans la thèse viserait à généraliser cette étude au cas des chocs sur- et sous-compressifs (i.e. où la différence entre p^+ et p^- prend des valeurs différentes dans (1.1.5)).

L'objectif initial de la thèse était donc de prouver un résultat de stabilité non-linéaire qui éviterait le plus possible les limitations décrites ci-dessus. Plus précisément, le but est d'obtenir une réponse la plus convaincante possible à la question ouverte suivante :

Question ouverte [Ser07, Open Question 5.3] : *On se place dans le cadre des systèmes de lois de conservation et l'on considère un schéma conservatif aux différences finies qui introduit de la viscosité numérique. Pour un choc de Lax* ***stationnaire****, on suppose l'existence d'une famille continue de profils de choc totalement discrets* $(\overline{u}^\delta)_{\delta\in]-\Delta,\Delta[}$. *On suppose que chacun des profils discrets* $\overline{u}^\delta$ *est* ***spectralement stable****. Prouver la stabilité nonlinéaire de la famille de profils discrets* $(\overline{u}^\delta)_{\delta\in]-\Delta,\Delta[}$.

Remarque. La question ouverte [Ser07, Open Question 5.3] impose plus spécifiquement que la famille de profils discrets soit issue d'un profil continu $U:\mathbb{R}\to\mathcal{U}$, c'est-à-dire qu'elle soit de la forme :

$$\forall\delta\in\mathbb{R},\forall j\in\mathbb{Z},\quad \overline{u}_j^\delta = U(j+\delta).$$

Nous n'imposerons pas cette restriction sur les profils considérés dans la thèse.

Revenons sur quelques points intéressants autour de cette question. Tout d'abord, on remarque qu'une réponse à cette question éviterait les limitations citées plus haut. L'hypothèse de *faible amplitude* qui était introduite régulièrement dans les résultats de stabilité nonlinéaire cités précédemment a cette fois été remplacée par une hypothèse de *stabilité spectrale*. Cette hypothèse impose essentiellement que le spectre du linéarisé de l'opérateur d'évolution $\mathcal{N}$ au niveau du profil discret $\overline{u}^\delta$ (qui est un point fixe de $\mathcal{N}$ dans le cas stationnaire considéré ici) ne contienne pas de valeurs propres de module supérieur ou égal à 1 autre que 1. Cette définition de la stabilité spectrale n'est que partielle et cette hypothèse sera détaillée plus en détails un peu plus tard.

Méthodologie pour répondre à la question ouverte

La méthodologie pour aboutir au résultat de stabilité nonlinéaire attendu est relativement claire. Elle peut être séparée en deux étapes :

1. **Stabilité spectrale ⇒ Stabilité linéaire :**
   Les différentes hypothèses introduites sur le profil discret et le schéma (choc de Lax, schéma aux différences finies introduisant de la viscocité,...) ainsi que l'hypothèse de *stabilité spectrale* implique que le linéarisé de l'opérateur $\mathcal{N}$ au niveau du profil de choc totalement discret satisfait certaines propriétés spectrales. L'objectif est de traduire ces propriétés spectrales en estimées de décroissance fines pour le semi-groupe associé au linéarisé au niveau du profil de choc totalement discret.
2. **Stabilité linéaire ⇒ Stabilité nonlinéaire :**
   Sous réserve que les estimées obtenues sur le semi-groupe associé au linéarisé au niveau du profil discret soient assez fines, on espère qu'il soit ensuite possible d'utiliser ces estimées pour boucler un argument de stabilité nonlinéaire.

Ce schéma de preuve n'est bien entendu pas nouveau et a été utilisé à de multiples reprises dans d'autres contextes. On peut par exemple citer [ZH98 ; MZ03] qui traitent de la stabilité nonlinéaire des chocs visqueux et [MZ02] du cas des chocs de relaxation, ou encore [BHR03 ; Bec+10] qui traitent d'un problème très proche de celui que l'on étudie dans cette thèse : la stabilité nonlinéaire des approximations *semi-discrètes* des chocs de systèmes de lois de conservation. Essentiellement, les chapitres de la thèse se séparent eux aussi selon les deux étapes présentées ci-dessus.

1. Les Chapitres 2, 3 et 4 traitent du passage de propriétés spectrales d'opérateurs linéaires à des estimées fines sur les fonctions de Green associées à ces opérateurs. En restant sur le thème de la stabilité des profils de choc totalement discrets, le point d'orgue est l'obtention du Théorème 4.1 du Chapitre 4 qui donne essentiellement, dans le cadre des systèmes de lois de conservation et pour une famille de schémas aux différences finies conservatifs assez large, des estimées fines sur la fonction de Green du linéarisé au niveau d'un profil de choc totalement discret associé à un choc de Lax stationnaire, sous réserve que celui-ci vérifie la fameuse condition de *stabilité spectrale*. Précisons que les Chapitres 2 et 3 ne traitent pas à proprement parler du problème de la stabilité des profils de choc totalement discrets mais ils constituent des étapes préliminaires importantes dans l'aboutissement de ce programme. Ils présentent par ailleurs des liens avec d'autres problèmes sur lesquels nous reviendrons. Ils partagent de plus les mêmes outils mathématiques que l'on détaillera plus tard.
2. Le Chapitre 5 est consacré au passage des estimations sur la fonction de Green obtenues dans le Théorème 4.1 du Chapitre 4 à un résultat de stabilité nonlinéaire. Cela correspond au Théorème 5.1 qui répond partiellement à la question ouverte [Ser07, Open Question 5.3]. Le résultat de stabilité nonlinéaire du Théorème 5.1 a cependant deux limitations importantes : Il ne s'applique pour l'instant que pour les lois de conservation *scalaires* et pour des perturbations de *masse nulle*. Cependant, la Section 5.1.4 du Chapitre 5 présente une discussion détaillée de ces limitations pour justifier qu'un résultat plus général que le Théorème 5.1 peut être prouvé, où les deux limitations présentées ci-dessus n'interviendraient pas.

L'article [God03] et la thèse [Laf01] de Lafitte-Godillon

Avant de passer à une description plus précise des résultats de la présente thèse, il nous faut parler de l'article [God03] et de la thèse [Laf01] de Lafitte-Godillon qui, au sein de la présente thèse, joueront un rôle central. En effet, au cours de sa thèse, Lafitte-Godillon s'était d'ores et déjà attelée à la tâche de prouver le passage de l'hypothèse de stabilité spectrale sur un profil de choc totalement discret associé à un choc de Lax statonnaire vers des estimées sur la fonction de Green du linéarisé du schéma au niveau du profil discret. Le résultat obtenu sur la fonction de Green dans [God03] ne semblait malheureusement pas assez précis pour conclure un argument de stabilité nonlinéaire et était limité au cas du schéma de Lax-Friedrichs. Essentiellement, le résultat du Chapitre 4 sera une amélioration du résultat de [God03] qui permettra enfin de conclure un argument nonlinéaire. Cependant, il est important de remarquer que beaucoup des outils qui seront utilisés dans cette thèse sont directement inspirés ou même issus de [God03] et [Laf01].

Dans la suite de cette introduction, nous proposons donc de rentrer plus dans le détail sur les résultats des chapitres que l'on a cités ci-dessus en respectant la séparation entre les deux parties de preuve que nous avons identifiées.

1.3.2 Linéarisé du schéma au niveau du profil discret et étude spectrale

Fixons les principaux objets et les principales hypothèses qui interviendront par la suite.

— On considèrera dorénavant un choc de Lax stationnaire dont les états à gauche et à droite de la discontinuité sont respectivement notés u^- et u^+.

— Le schéma aux différences finies conservatif étudié est d'ordre impair et introduira de la viscosité au niveau des états constants u^- et u^+ dans le sens défini dans l'Hypothèse 1.1 du paragraphe "Stabilité du linéarisé au niveau des constantes" de la Section 1.1.2. On peut donc observer des comportements numériques paraboliques d'ordre élevé, et non pas seulement des viscosités numériques que l'on trouve pour les schémas d'ordre 1.

— On suppose qu'il existe un profil de choc totalement discret associé à la discontinuité que l'on note $\overline{u}^s$ qui converge exponentiellement vite vers ses états limites u^+ et u^-. On utilisera par moment une famille dérivable selon δ de profils discrets $(\overline{u}^\delta)_{\delta\in]-\Delta,\Delta[}$ qui lui sera associée. Les théorèmes ne nécessiteront pas cette famille mais nous l'utiliserons afin d'illustrer certaines observations importantes.

Linéarisé du schéma au niveau du profil $\overline{u}^s$

On linéarise l'opérateur d'évolution discret $\mathcal{N}$ au niveau du profil de choc totalement discret $\overline{u}^s$ et l'on définit alors l'opérateur bornée $\mathscr{L}$ agissant sur $\ell^r(\mathbb{Z},\mathbb{C}^d)$ avec $r\in[1,+\infty]$ par :

$$\forall h\in\ell^r(\mathbb{Z},\mathbb{C}^d),\forall j\in\mathbb{Z},\quad (\mathscr{L}h)_j:=\sum_{k=-p}^{q}A_{j,k}h_{j+k}, \tag{1.3.1}$$

où pour $j\in\mathbb{Z}$ et $k\in\{-p,\dots,q-1\}$, on commence par définir la matrice :

$$B_{j,k}:=\boldsymbol{\nu}\partial_{u_k}F\left(\boldsymbol{\nu};\overline{u}^s_{j-p},\dots,\overline{u}^s_{j+q-1}\right)\in\mathcal{M}_d(\mathbb{R})$$

puis pour $j\in\mathbb{Z}$ and $k\in\{-p,\dots,q\}$:

$$A_{j,k}:=\begin{cases}-B_{j+1,q-1} & \text{si } k=q,\\ B_{j,-p} & \text{si } k=-p,\\ \delta_{k,0}Id+B_{j,k}-B_{j+1,k-1} & \text{sinon.}\end{cases}$$

On remarque que la convergence du profil de choc totalement discret à l'infini vers les états limites :

$$\overline{u}^s_j\underset{j\to\pm\infty}{\to}u^\pm$$

implique que les matrices $A_{j,k}$ vérifient :

$$A_{j,k}\underset{j\to\pm\infty}{\to}A_k^{u^\pm} \tag{1.3.2}$$

où les matrices $A_k^{u^\pm}$ sont définies par (1.1.14). Cette remarque nous permet essentiellement de conclure que le "comportement à l'infini" de l'opérateur linéaire $\mathscr{L}$ est le même que celui des opérateurs $\mathscr{L}_{u^\pm}$ définis par (1.1.13). Le linéarisé de $\mathcal{N}$ autour d'un état constant apparaît donc de façon naturelle dans l'étude des profils de choc totalement discrets, et c'est ceci qui motivera l'étude du Chapitre 2 ci-après.

Étude du problème aux valeurs propres de l'opérateur $\mathscr{L}$ et réécriture en système dynamique

Comme nous l'avons dit plus tôt, le point central de l'analyse produite dans cette thèse repose sur une compréhension précise des propriétés spectrales de l'opérateur linéarisé $\mathscr{L}$ et sur la traduction de ces propriétés spectrales en estimations sur la fonction de Green associée à l'opérateur $\mathscr{L}$. Pour ce faire, il faut étudier le problème aux valeurs propres associé à l'opérateur $\mathscr{L}$, c'est-à-dire chercher les éléments $u\in\ell^2(\mathbb{Z},\mathbb{C}^d)$ tels que :

$$(zId_{\ell^2}-\mathscr{L})u=0 \tag{1.3.3}$$

pour $z\in\mathbb{C}$. L'étude de ce type de problème est en réalité bien connue de par la forme de l'opérateur $\mathscr{L}$ défini par (1.3.1). En s'inspirant de ce qui se fait dans le cadre de l'étude des linéarisations d'EDPs nonlinéaires au niveau d'ondes progressives (voir [KP13, Section 3] pour plus d'informations), l'idée est de se rendre compte que le problème aux valeurs propres (1.3.3) se ramène essentiellement à chercher les solutions d'un système dynamique discret homogène de la forme :

$$\forall j\in\mathbb{Z},\quad W_{j+1}=M_j(z)W_j \tag{1.3.4}$$

où

$$\forall j\in\mathbb{Z},\quad W_j:=\begin{pmatrix}u_{j+q-1}\\ \vdots\\ u_{j-p}\end{pmatrix}\in\mathbb{C}^{d(p+q)}. \tag{1.3.5}$$

Les matrices $M_j(z)\in\mathcal{M}_{d(p+q)}(\mathbb{C})$ sont des matrices par blocs explicites faisant intervenir les matrices $A_{j,k}$ de la définition de l'opérateur linéarisé $\mathscr{L}$. Faisons quelques remarques importantes :

— Si $z\in\mathbb{C}$ est une valeur propre de $\mathscr{L}$, alors il existe une suite $u\in\ell^2(\mathbb{Z},\mathbb{C}^d)$ qui soit solution non nulle de (1.3.3). Cela implique alors qu'il existe une solution $(W_j)_{j\in\mathbb{Z}}$ non nulle du système dynamique (1.3.4) qui converge 0 en $+\infty$ et en $-\infty$. Il se trouve que le sens réciproque est aussi vrai pour beaucoup de $z\in\mathbb{C}$. Ceci nous pousse donc à introduire les ensembles $E_0^\pm(z)$ des solutions du système dynamique (1.3.4) qui tendent respectivement vers 0 en $+\infty$ et en $-\infty$:

$$E_0^\pm(z):=\left\{W_0\in\mathbb{C}^{(p+q)d},\quad (W_j)_{j\in\mathbb{Z}}\text{ solution de (1.3.4) telle que } W_j\underset{j\to\pm\infty}{\to}0\right\}. \tag{1.3.6}$$

De plus, pour la plupart des $z\in\mathbb{C}$, si $W_0\in E_0^\pm(z)$, on prouve que la solution $(W_j)_{j\in\mathbb{Z}}$ (1.3.4) converge exponentiellement vite vers 0. On a alors que z est une valeur propre de $\mathscr{L}$ si et seulement si $E_0^+(z)\cap E_0^-(z)$ n'est pas réduit à 0. Précisons que pour une grande partie des $z\in\mathbb{C}$, les solutions $(W_j)_{j\in\mathbb{Z}}$ de (1.3.4) associées à $W_0\in E_0^\pm(z)$ vont converger exponentiellement vite vers 0 quand j tend vers $\pm\infty$.

— On remarque que la limite (1.3.2) implique que les matrices $M_j(z)$ convergent vers des matrices $M^\pm(z)$ quand j tend vers l'infini. Cela nous permet de dire que le système dynamique (1.3.4) peut être vu comme une perturbation des systèmes dynamiques

$$\forall j\in\mathbb{N},\quad W_{j+1}=M^+(z)W_j, \tag{1.3.7a}$$
$$\forall j\in-\mathbb{N},\quad W_{j+1}=M^-(z)W_j. \tag{1.3.7b}$$

De plus, les solutions du système dynamique (1.3.7a) tendant vers 0 en $+\infty$ sont données par l'espace des vecteurs propres "généralisés" stables $E_s(M^+(z))$ de $M^+(z)$. Il en est de même pour les solutions tendant vers 0 de (1.3.7b) et l'espace des vecteurs propres "généralisés" instables $E_u(M^-(z))$ de $M^-(z)$. Or, on connaît des résultats (en particulier [Kre68]) qui nous permettent de déterminer le spectre des matrices $M^\pm(z)$. On connaît donc explicitement les solutions tendant vers 0 des systèmes dynamiques (1.3.7a) et (1.3.7b).

Essentiellement, à partir de ce moment, l'étude du problème aux valeurs propres (1.3.3) se ramène à trouver comment lier les deux observations ci-dessus. Typiquement, puisque le système dynamique (1.3.4) est une perturbation du système dynamique (1.3.7a), on peut s'attendre à ce que les dimensions des espaces $E_0^+(z)$ et $E_s(M^+(z))$ soient les mêmes et que l'on puisse construire une base de $E_0^+(z)$ à partir des éléments de $E_s(M^+(z))$. Le même genre d'idées est vrai pour $E_0^-(z)$.

Une partie importante de l'article [God03] et des deux dernières chapitres de la thèse [Laf01] de Lafitte-Godillon visait à formaliser ce lien. Les deux choses importantes qui peuvent en être tirées (que l'on ne va pas détailler dans cette introduction) sont :

— La notion de *dichotomie géométrique* qui vise à exhiber des projecteurs projetant sur les espaces $E_0^\pm(z)$.

— La construction de bases des espaces $E_0^\pm(z)$ qui permettent ultimement la construction d'une fonction d'Evans. Cette fonction d'Evans jouera le même rôle que le polynôme caractéristique pour une matrice. C'est une fonction holomorphe s'annulant au niveau des valeurs propres de l'opérateur $\mathscr{L}$.

Dans le Chapitre 4 de la présente thèse, nous allons réintroduire ces différents résultats prouvés par Lafitte-Godillon dans [Laf01 ; God03] et parfois les améliorer.

Résultat sur le spectre de $\mathscr{L}$ et hypothèse de stabilité spectrale

À l'aide des résultats obtenus sur le problème aux valeurs propres (1.3.3) que l'on a présentées dans la section précédente, il est possible d'obtenir des informations importantes sur le spectre de l'opérateur $\mathscr{L}$. En particulier, à l'aide de la fameuse dichotomie géométrique formalisée par Lafitte-Godillon dans sa thèse [Laf01], Serre prouve dans [Ser07, Theorem 4.1] que le spectre essentiel de l'opérateur $\mathscr{L}$ ne peut pas se localiser dans la composante connexe non bornée de $\mathbb{C}\backslash(\sigma(\mathscr{L}_{u^+})\cup\sigma(\mathscr{L}_{u^-}))$ (zone grise sur la Figure 1.4) où les opérateurs $\mathscr{L}_{u^+}$ et $\mathscr{L}_{u^-}$ sont définis par (1.1.13) et correspondent aux linéarisés du schéma au niveau des états constants u^+ et u^-. La conséquence immédiate de cette observation est que les éléments se trouvant dans la composante connexe non bornée de $\mathbb{C}\backslash(\sigma(\mathscr{L}_{u^+})\cup\sigma(\mathscr{L}_{u^-}))$ font soit partie de l'ensemble résolvant de l'opérateur $\mathscr{L}$, soit sont des valeurs propres de $\mathscr{L}$. De plus, on rappelle que, d'après les explications du paragraphe "Stabilité du linéarisé au niveau des constantes" de la Section 1.1.2, le spectre des opérateurs $\mathscr{L}_{u^+}$ et $\mathscr{L}_{u^-}$ correspond à d courbes continues, contenues strictement dans le disque unité ouvert sauf au niveau du point 1 où elles sont

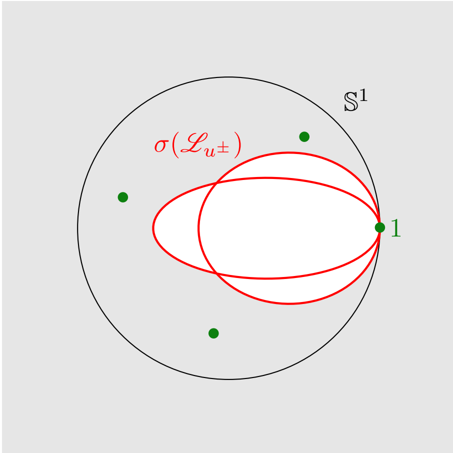

FIGURE 1.4 – Une représentation du spectre de $\mathscr{L}$. En rouge, on trouve le spectre des opérateurs $\mathscr{L}_{u^\pm}$. Ces courbes délimitent une zone au sein de laquelle doit se trouver le spectre essentiel de l'opérateur $\mathscr{L}$. À l'extérieur de ces courbes rouges, c'est-à-dire dans la zone grise, les éléments ne peuvent que soit se trouver dans l'ensemble résolvant de $\mathscr{L}$, soit être des valeurs propres de l'opérateur $\mathscr{L}$ (représentées en vert). L'hypothèse de stabilité spectrale impose aux valeurs propres de se situer dans le disque unité ouvert. 1 est la seule valeur propre qui fait exception.

tangentes au cercle unité (voir Figure 1.4).

On rappelle que l'on va considérer que le profil de choc totalement discret que l'on considère est spectralement stable. Cela implique que toutes les valeurs propres ne peuvent pas être de module supérieur ou égal à 1, sauf 1 qui est un cas particulier que l'on va traiter ci-dessous. On obtient alors la représentation faite sur la Figure 1.4 d'un spectre typique pour l'opérateur $\mathscr{L}$.

Le cas particulier de la valeur propre 1

La valeur propre 1 joue un rôle central dans l'étude de l'opérateur $\mathscr{L}$. En effet, on peut prouver que 1 est toujours une valeur propre pour le linéarisé d'un profil de choc totalement discret associé à un choc de Lax. Cette propriété est intimement liée au fameux résultat d'existence d'une famille $(\overline{u}^\delta)_{\delta\in]-\Delta,\Delta[}$ de profils de choc totalement discrets associée à $\overline{u}^s$. En effet, on a formellement :

$$\text{"}\left(\forall \delta \in]-\Delta,\Delta[, \quad \mathcal{N}(\overline{u}^\delta) = \overline{u}^\delta\right) \quad \Rightarrow \quad \mathscr{L}\frac{\partial \overline{u}^\delta}{\partial \delta}|_{\delta=0} = \frac{\partial \overline{u}^\delta}{\partial \delta}|_{\delta=0}.\text{"}$$

De plus, l'hypothèse de convergence exponentielle des profils de choc totalement discrets vers leurs états limites que l'on fait impose que la suite $\frac{\partial \overline{u}^\delta}{\partial \delta}|_{\delta=0}$ décroît exponentiellement vite vers 0 en $+\infty$ et $-\infty$. On a donc en particulier que la suite $\frac{\partial \overline{u}^\delta}{\partial \delta}|_{\delta=0}$ appartient à $\ell^2(\mathbb{Z},\mathbb{C}^d)$. L'hypothèse de stabilité spectrale implique en particulier que 1 doit être une valeur propre **simple** de $\mathscr{L}$ (quand il agît sur $\ell^2(\mathbb{Z},\mathbb{C}^d)$). Cela implique donc formellement que

$$\text{"}\ker(Id_{\ell^2} - \mathscr{L}) = \operatorname{Span}\left(\frac{\partial \overline{u}^\delta}{\partial \delta}\right).\text{"}$$

Remarque. La véritable expression de la condition de stabilité spectrale qui sera utilisée dans le Chapitre 4 est légèrement plus précise que ce qui est présenté ci-dessus et demande à 1 d'être un zéro simple de la fonction d'Evans dont on a parlé plus tôt (que l'on peut construire dans un voisinage de 1).

1.3.3 Passage de la stabilité spectrale à des estimations sur la fonction de Green

Le but de cette section est de présenter le résultat du Chapitre 4 qui correspond au passage de la stabilité spectrale à la stabilité linéaire. L'analyse qui va être présentée ici a été réalisée au moins en partie dans l'article [God03].

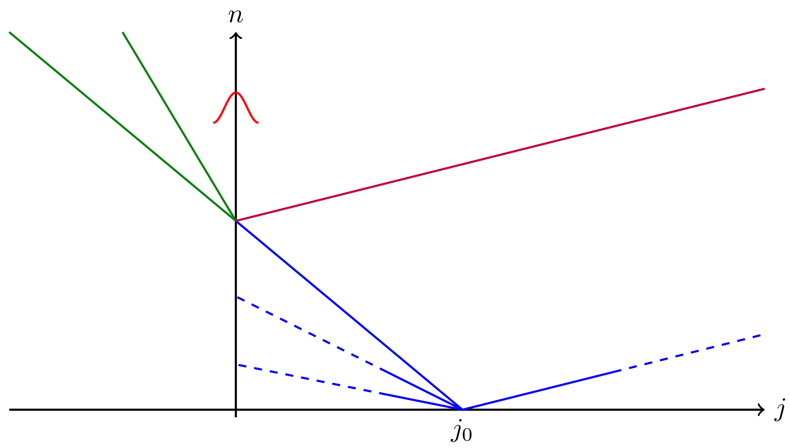

FIGURE 1.5 – Une représentation schématique du résultat du Théorème 4.1 sur la fonction de Green $\mathscr{G}(n, j_0, j)$. On représente le cas $j_0 \in \mathbb{N}$ et $d = 4$.

Définition de la fonction de Green

Pour un entier $j_0 \in \mathbb{Z}$, on définit la fonction de Green de l'opérateur $\mathscr{L}$ récursivement par :

$$\begin{aligned} \mathscr{G}(0, j_0, \cdot) &:= \delta_{j_0} \\ \forall n \in \mathbb{N}, \quad \mathscr{G}(n+1, j_0, \cdot) &:= \mathscr{L}\mathscr{G}(n, j_0, \cdot), \end{aligned} \tag{1.3.8}$$

où la suite δ_{j_0} est définie par la suite de Dirac en j_0 constamment égale à 0 sauf en j_0 où elle vaut l'identité :

$$\delta_{j_0} := (\delta_{j_0,j} Id)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathcal{M}_d(\mathbb{C})).$$

L'intérêt de la fonction de Green est qu'elle fournit la solution fondamentale pour la récurrence :

$$h^0 \in \ell^r(\mathbb{Z}, \mathbb{C}^d) \quad \text{et} \quad \forall n \in \mathbb{N}, \quad h^{n+1} = \mathscr{L} h^n.$$

En effet, pour toute suite $h \in \ell^r(\mathbb{Z}, \mathbb{C}^d)$ avec $r \in [1, +\infty]$, on a

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad (\mathscr{L}^n h)_j = \sum_{j_0 \in \mathbb{Z}} \mathscr{G}(n, j_0, j) h_{j_0}. \tag{1.3.9}$$

Ainsi, il est clair que l'égalité (1.3.9) implique qu'une compréhension assez précise du comportement en temps long de la fonction de Green permet de déterminer le comportement en temps du semi-groupe $(\mathscr{L}^n)_{n \in \mathbb{N}}$ associé à l'opérateur $\mathscr{L}$.

Estimées sur la fonction de Green

Le résultat essentiel du Chapitre 4 est le Théorème 4.1. L'expression de ce théorème est légèrement chargée, en particulier dans le cas des systèmes de lois de conservation. On se propose donc dans cette introduction de faire une présentation informelle du résultat à l'aide d'un schéma et de conserver son expression concrète dans le Chapitre 4. On attire l'attention du lecteur sur la Figure 1.5 qui représente une version schématique du résultat du Théorème 4.1.

Nous allons faire une description du comportement de la fonction de Green $\mathscr{G}(n, j_0, j)$ de l'opérateur $\mathscr{L}$ décrit par le Théorème 4.1 pour un point de départ $j_0 \geq 0$, c'est-à-dire quand la masse de Dirac se situe initialement à droite du choc. En regardant la figure ci-dessus, nous espérons qu'il est clair que le comportement quand la masse de Dirac est à gauche du choc sera similaire.

Dans un premier temps, la masse de Dirac est décomposée selon d ondes correspondant aux caractéristiques associées à l'état u^+. Pour chaque caractéristique de l'état de droite u^+ du choc, on verra une gaussienne (ou une gaussienne généralisée pour des schémas d'ordre élevé) être transportée et diffusée le long de celle-ci (en bleu sur la Figure 1.5). Certaines caractéristiques ne sont pas orientées vers le choc et, dans ce cas, l'onde gaussienne parcourra la caractéristique en se diffusant sans encombre. Ce comportement correspond à une propagation "libre" pour l'opérateur de convolution $\mathscr{L}_{u^+}$ définie par (1.1.13). Ces opérateurs sont étudiés en détail dans le Chapitre 2

Concernant les caractéristiques orientées vers le choc, une nouvelle décomposition de l'onde a lieu dès lors que celle-ci atteint le choc. On observe l'apparition d'ondes réfléchies (en violet) et transmises (en vert), ainsi que l'activation d'un profil constant (en rouge). Les ondes réfléchies sont transportées le long des caractéristiques de l'état u^+ qui sortent du choc. De même, les ondes transmises sont transportées le long des caractéristiques de l'état u^- qui sortent du choc. Enfin, en rouge, on voit l'activation d'un profil constant. Ce profil est lié à la valeur propre 1 et correspond donc à la dérivée de la famille de profils discrets d'après les remarques faites plus haut.

Cette description de la fonction de Green est similaire à celle observée dans d'autres résultats traitant d'approximations différentes des chocs (chocs visqueux [ZH98; MZ03] ou encore approximations semi-discrètes des ondes de chocs [BHR03; Bec+10]). La seule différence est la nature totalement discrète du cas qui nous intéresse ici. Comparé au résultat [God03, Theorem 1.1], la différence est que la description donnée dans le Théorème 4.1 est uniforme en $j_0 \in \mathbb{Z}$ alors que les bornes du [God03, Theorem 1.1] sont locales en j_0 (i.e. valables pour un nombre fini de valeurs de j_0). Cette différence est fondamentale car l'utilisation de la formule (1.3.9) nécessite la plupart du temps de faire des sommes sur l'ensemble des $j_0 \in \mathbb{Z}$.

De plus, on isole le terme principal dans la fonction de Green et on montre une meilleure estimation sur le reste. On donne donc un premier développement asymptotique en temps de la fonction de Green.

Idée de la preuve : Formule de Laplace inverse et fonction de Green spatiale

Les techniques qui vont être présentées ici sont initialement dues à Zumbrun et Howard dans [ZH98]. Elles ont été utilisées dans une quantité importante de problèmes depuis leur introduction comme, pour citer les exemples qui apparaissent dans la thèse, [ZH98; MZ02; MZ03; BHR03; Bec+10; God03; CF22; CF23]. Elles sont au centre des résultats des Chapitres 2, 3 et 4.

Le point de départ de l'analyse est la transformée de Laplace inverse qui nous fournit l'égalité suivante sur la fonction de Green :

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad \mathscr{G}(n, j_0, j) = \frac{1}{2i\pi} \int_\Gamma z^n G(z, j_0, j) dz \tag{1.3.10}$$

où Γ est un chemin qui entoure le spectre de l'opérateur $\mathscr{L}$ et la suite $G(z, j_0, \cdot) \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$ aussi connue sous le nom de fonction de Green *spatiale* est définie comme l'unique solution pour z dans l'ensemble résolvant de l'opérateur de $\mathscr{L}$ de :

$$(zId_{\ell^2} - \mathscr{L})G(z, j_0, \cdot) = \delta_{j_0}. \tag{1.3.11}$$

Précisons le chemin Γ intervenant dans (1.3.10) entoure le spectre de $\mathscr{L}$ sans le croiser d'où la définition (1.3.11) de la fonction de Green spatiale a du sens pour $z \in \Gamma$.

Si l'on regarde l'allure du spectre de $\mathscr{L}$ que l'on a déterminé plus tôt (voir Figure 1.4), on comprend que la difficulté se concentre dans l'absence de trou spectral car le spectre de l'opérateur $\mathscr{L}$ touche le cercle unité en 1. Cependant, la situation est plus délicate encore. En effet, 1 fait à la fois partie du spectre essentiel de $\mathscr{L}$ mais est aussi une valeur propre de $\mathscr{L}$ comme on l'a vu plus tôt.

L'idée de Zumbrun et Howard dans [ZH98] se sépare en deux temps :

1. Tout d'abord, la définition de la fonction de Green spatiale (1.3.11) permet d'exprimer celle-ci à l'aide de solutions d'un problème inhomogène associé à (1.3.4). Les résultats prouvés sur le système dynamique (1.3.4) permettent alors d'exprimer la fonction de Green spatiale en fonction des éléments des ensembles $E_0^\pm(z)$ définis par (1.3.6). Cela va permettre de tout d'abord prouver des bornes sur la fonction de Green spatiale mais aussi, et c'est certainement le plus important, de prouver un prolongement *méromorphe* de la fonction $z \mapsto G(z, j_0, j)$ dans un voisinage de 1 avec 1 qui est un pôle d'ordre 1.
2. La seconde étape est donc de revenir à la formule (1.3.10) reliant la fonction de Green $\mathscr{G}(n, j_0, j)$ et la fonction de Green spatiale $G(z, j_0, j)$. L'idée est maintenant d'utiliser les bornes obtenues sur la fonction de Green spatiale et un bon choix de chemin Γ pour obtenir les estimées les plus fines possibles sur la fonction de Green. La déformation du chemin Γ se fait au moyen de la formule de Cauchy ou du théorème des résidus. La dissipativité du problème que l'on considère fait que la difficulté dans le choix du chemin Γ est uniquement concentrée au niveau du point 1. Cependant, le prolongement méromorphe de la fonction de Green spatiale que l'on a prouvé permet d'avoir une marge de manoeuvre plus conséquente. L'article [ZH98] offre essentiellement un choix optimal de chemin quand $\mu = 1$ dans l'Hypothèse 1.1, ce qui permet de répondre à de multiples situations faisant apparaître des comportements visqueux. Le choix optimal de chemin quand $\mu > 1$ dans l'Hypothèse 1.1 est quant à lui obtenu dans les articles [CF22; CF23].

1.3.4 Passage de la stabilité linéaire à la stabilité nonlinéaire

Comme expliqué à la fin de la Section 1.3.1, nous avons réussi à utiliser les estimées sur la fonction de Green données par le Théorème 4.1 pour conclure un argument nonlinéaire. Le résultat est le Théorème 5.1 obtenu dans le Chapitre 5 qui est un résultat de stabilité nonlinéaire ne fonctionnant pour l'instant que dans le cadre des lois de conservations *scalaires*.

Théorème 1.1 (Stabilité spectrale implique stabilité nonlinéaire)**.** *On se place dans le contexte des lois de conservation **scalaires**. On considère un paramètre $\boldsymbol{p} \in [0,+\infty[$. Si le profil de choc totalement discret $\overline{u}^s$ est **spectralement stable** et sous quelques hypothèses techniques supplémentaires sur le profil discret $\overline{u}^s$ considéré, il existe deux constantes $\varepsilon, C \in [0,+\infty[$ telles que si l'on considère une perturbation initiale $h^0 \in \mathbb{R}^{\mathbb{Z}}$ vérifiant :*

$$\sum_{j\in\mathbb{Z}} (1+|j|)^{\max(1,\boldsymbol{p})+\boldsymbol{p}} |h_j^0| \leq \varepsilon \quad \text{et} \quad \sum_{j\in\mathbb{Z}} h_j^0 = 0 \tag{1.3.12}$$

alors la solution $(u_n)_{n\in\mathbb{N}}$ du schéma numérique (1.1.7) initialisé avec la condition initiale $\overline{u}^s + h^0$ est définie pour tout $n \in \mathbb{N}$ et l'on a pour tout $r \in [1,+\infty]$:

$$\forall n \in \mathbb{N}\backslash\{0\}, \quad \left\| \left((1+|j|)^{\max(1,\boldsymbol{p})} \left| u_j^n - \overline{u}_j^s \right| \right)_{j\in\mathbb{Z}} \right\|_{\ell^r} \leq \frac{C}{n^{\boldsymbol{p}+\frac{1}{2\mu}\left(1-\frac{1}{r}\right)}} \left\| \left((1+|j|)^{\max(1,\boldsymbol{p})+\boldsymbol{p}} |h_j^0| \right)_{j\in\mathbb{Z}} \right\|_{\ell^1} \tag{1.3.13}$$

où l'entier μ est donné par l'Hypothèse 1.1.

Ce résultat constitue donc une première réponse à la question ouverte [Ser07, Open Question 5.3]. Dans la section 5.1.4 du Chapitre 5, on revient sur la possibilité d'étendre ce résultat au cas des systèmes lois de conservation. En effet, puisque le Théorème 4.1 sur la fonction de Green prouvé dans le chapitre 4 fonctionne dans le cas des systèmes de lois de conservation, il est raisonnable de penser qu'une généralisation est envisageable.

Finissons cette section en décortiquant un petit peu l'énoncé du théorème 1.1. On observe que l'on impose aux perturbations initiales h^0 la condition (1.3.12) qui se sépare en deux parties :

- La première partie de (1.3.12) impose que les perturbations h^0 soient petites dans un espace ℓ^1 à poids polynomial. Le poids que l'on impose est défini à l'aide de la constante $\boldsymbol{p} \in [0,+\infty[$ qui intervient également dans la limite (1.3.13). Essentiellement, plus l'on impose un poids fort sur les perturbations initiales h^0, plus on peut espérer un taux de convergence rapide sur $u^n - \overline{u}^s$.
- La seconde partie est une condition de masse nulle sur les perturbations initiales, comme dans les articles [LX93a ; LX93b] cités plus tôt. Cela offre une justification au fait de trouver des bornes seulement sur la différence entre u^n et le profil initial $\overline{u}^s$ et non pas sur la distance entre la solution et la courbe des profils discrets stationnaires :
$$\min_{\delta\in]-\Delta,\Delta[} \left\| u^n - \overline{u}^\delta \right\|_{\ell^r}.$$
En effet, la condition de masse nulle des perturbations implique que l'on va converger vers le profil initial et non pas vers l'un de ses translatés. Dans le cas où l'on aurait considéré des perturbations de masse non nulle, il aurait fallu essayer de traquer à tout temps $n \in \mathbb{N}$ quel élément de la famille des profils discrets est le plus proche de u^n et l'on prétend que ceci est particulièrement difficile dans le contexte totalement discret dans lequel on se trouve (voir section 5.1.4 du Chapitre 5 pour une discussion détaillée à ce sujet).

1.4 Généralisation du théorème de la limite locale (Chapitre 2)

1.4.1 Présentation des opérateurs de Laurent et du théorème de la limite locale

Le contenu du Chapitre 2 a fait l'objet d'un article [Coe22] et porte sur l'étude de la fonction de Green des opérateurs de convolution sur l'ensemble $\mathbb{Z}$, aussi connu sous le nom d'opérateurs de Laurent. Ce problème se trouve, comme on le précisera ci-dessous, dans la périphérie de l'étude de la fonction de Green du linéarisé des profils de choc totalement discrets. Commençons par redéfinir les opérateurs de Laurent ainsi que certaines propriétés importantes les concernant.

On commence par rappeler que l'inégalité de Young pour les convolutions permet en particulier de déduire que, pour $b \in \ell^1(\mathbb{Z},\mathbb{C})$ et $u \in \ell^r(\mathbb{Z},\mathbb{C})$ avec $r \in [1,+\infty]$, la suite $b * u$ définie par :

$$\forall j \in \mathbb{Z}, \quad (b*u)_j := \sum_{j\in\mathbb{Z}} b_k u_{j-k}$$

est bien définie et appartient à $\ell^r(\mathbb{Z},\mathbb{C})$. Plus précisément, on a que :

$$\|b*u\|_{\ell^r} \leq \|b\|_{\ell^1} \|u\|_{\ell^r}.$$

Ainsi, pour une suite $a \in \ell^1(\mathbb{Z},\mathbb{C})$ et pour $r \in [1,+\infty]$, si l'on définit de plus la suite $b := (a_{-k})_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z},\mathbb{C})$, l'opérateur de Laurent L associé à la suite a défini par :

$$\forall u \in \ell^r(\mathbb{Z},\mathbb{C}), \forall j \in \mathbb{Z}, \quad (Lu)_j := \sum_{k\in\mathbb{Z}} a_k u_{j+k} = (b*u)_j \tag{1.4.1}$$

est bien défini et borné lorsqu'il agît sur $\ell^r(\mathbb{Z},\mathbb{C})$. Précisons que l'expression (1.4.1) de l'opérateur de Laurent à l'aide de la suite a ou de la suite b sont deux faces d'une même pièce :

— L'écriture de l'opérateur L à l'aide de la suite b pointe clairement le fait que les opérateurs de Laurent sont les opérateurs de convolution sur l'ensemble $\mathbb{Z}$. Cela implique immédiatement un lien fort avec l'utilisation de la théorie de Fourier pour étudier ces opérateurs. De plus, cette écriture des opérateurs de Laurent sera particulièrement adaptée dans le paragraphe plus bas qui concernera les marches aléatoires sur $\mathbb{Z}$. Enfin, on verra plus bas qu'introduire la suite b permet d'exprimer simplement la fonction de Green de l'opérateur L comme les itérés de produits de convolution de la suite b.

— L'écriture de l'opérateur L à l'aide de la suite a est particulièrement récurrent dans le cadre de l'approximation de problèmes d'évolution linéaires par des méthodes aux différences finies, comme cela sera présenté plus en détail dans un paragraphe ci-dessous.

Cette thèse se plaçant de manière évidente dans la seconde direction, on utilisera plus souvent la notation des opérateurs de Laurent à l'aide de la suite a dans le reste de la thèse. Nous sauterons cependant entre les deux notations relativement librement à certains moments dans le reste de l'introduction.

Le théorème de Wiener, que l'on a déjà cité plus tôt dans le paragraphe "Stabilité du linéarisé au niveau des constantes" de la Section 1.1.2 et dont on peut trouver une preuve dans [New75], nous permet de décrire le spectre de l'opérateur L en fonction de la série de Fourier F associée à la suite a :

$$\begin{gathered}\forall \kappa \in \mathbb{S}^1, \quad F(\kappa) := \sum_{k\in\mathbb{Z}} a_k \kappa^k, \\ \sigma(L) = \left\{ F(e^{i\xi}), \quad \xi \in \mathbb{R} \right\}.\end{gathered} \tag{1.4.2}$$

On remarquera que le spectre de L est indépendant de l'espace $\ell^r(\mathbb{Z},\mathbb{C})$ sur lequel il agît et que la série de Fourier F est bien définie et continue car la suite a appartient à $\ell^1(\mathbb{Z},\mathbb{C})$.

On s'intéresse aux solutions du problème d'évolution discret dans $\ell^r(\mathbb{Z},\mathbb{C})$ pour $r \in [1,+\infty]$:

$$\begin{cases} \forall n \in \mathbb{N}, \quad u^{n+1} = Lu^n = b * u^n, \\ \qquad u^0 \in \ell^r(\mathbb{Z},\mathbb{C}). \end{cases} \tag{1.4.3}$$

Les solutions $(u^n)_{n\in\mathbb{N}}$ de (1.4.3) sont immédiatement données par le semi-groupe $(L^n)_{n\in\mathbb{N}}$ associé à l'opérateur L à l'aide de l'expression suivante :

$$\forall n \in \mathbb{N}, \quad u^n = L^n u^0 = b^n * u^0 \tag{1.4.4}$$

où l'on note $b^n := b * \ldots * b \in \ell^1(\mathbb{Z},\mathbb{C})$ les produits de convolution itérés de b. D'après l'expression (1.4.4), on souhaite utiliser l'appelation fonction de Green de l'opérateur L pour parler des suites b^n. En effet, si l'on introduit la suite δ, qui appartient à l'espace $\ell^r(\mathbb{Z},\mathbb{C})$ pour tout $r \in [1,+\infty]$, definie par :

$$\forall j \in \mathbb{Z}, \quad \delta_j := \begin{cases} 1 & \text{si } j = 0, \\ 0 & \text{sinon,} \end{cases}$$

alors la suite $(b^n)_{n\in\mathbb{N}}$ est la solution fondamentale du problème d'évolution discret (1.4.3) pour $u^0 = \delta$ dans $\ell^r(\mathbb{Z},\mathbb{C})$ pour $r \in [1,+\infty]$.

Dorénavant et dans tout le reste de cette section, on se permettra de noter $(b^n)_{n\in\mathbb{N}}$ la fonction de Green l'opérateur L associé la suite a (en se rappelant du lien avec les produits de convolution itérés de la suite $b = (a_{-k)})_{k\in\mathbb{Z}}$).

Le point de départ de notre étude repose sur l'observation suivante : Pour étudier le comportement en temps long des solutions du problème d'évolution (1.4.3), il suffit d'étudier le comportement en temps long de la fonction de Green $(b^n)_{n\in\mathbb{N}}$ associée à l'opérateur L.

Renormalisation de la suite a

Avant de continuer, précisons que l'on va dorénavant considérer que l'on a :

$$\max_{\mathbb{S}^1} |F| = 1 \tag{1.4.5}$$

quitte à renormaliser la suite a, en ayant au préalable évacué le cas trivial $a = 0$. En effet, un résultat de Beurling [RS55, p.428] implique que :

$$\lim_{n\to+\infty} \|b^n\|_{\ell^1}^{\frac{1}{n}} = \max_{\mathbb{S}^1} |F|.$$

Ainsi, lorsque $\max_{\mathbb{S}^1} |F| < 1$, la suite $(b^n)_{n\in\mathbb{N}}$ tend exponentiellement vite vers 0 en norme ℓ^1. De plus, la suite $(\|b^n\|_{\ell^1})_{n\in\mathbb{N}}$ croît exponentiellement vite lorsque $\max_{\mathbb{S}^1} |F| > 1$. Ainsi, lorsque la condition (1.4.5) est vérifiée, on se trouve dans le cas limite où le comportement de b^n quand n tend vers $+\infty$ n'est pas évident.

Liens avec les profils de choc totalement discrets et les schémas aux différences finies pour l'équation de transport

On commence par observer que l'opérateur de Laurent L correspond à une version scalaire de l'opérateur $\mathscr{L}_{\overline{u}}$, défini par (1.1.13), correspondant au linéarisé du schéma numérique (1.1.7) au niveau de l'état constant $\overline{u}$. Comprendre le comportement en temps long de la fonction de Green associée à l'opérateur L dans ce cas scalaire permet, sous des hypothèses raisonnables de diagonalisabilité, de passer à l'étude de la fonction de Green des opérateurs $\mathscr{L}_{\overline{u}}$.

Cette observation est importante dans le cadre des résultats de stabilité des profils de choc totalement discrets que l'on présente dans les Chapitres 4 et 5. En effet, une partie du comportement principal observé sur la fonction de Green du linéarisé au niveau des profils de choc totalement discrets dans le résultat principal du Chapitre 4 correspond à une propagation "libre" pour les opérateurs de convolution $\mathscr{L}_{u^+}$ ou $\mathscr{L}_{u^-}$ en fonction de la position initiale du Dirac par rapport au choc, où u^+ et u^- sont les états limites du profil de choc totalement discret (ondes tracées en bleu sur la Figure 1.5). Ainsi, comprendre le comportement asymptotique ponctuel de la fonction de Green b^n de l'opérateur de Laurent L correspond à une première étape "élémentaire" de l'étude de la fonction de Green pour les profils de choc totalement discrets.

Les opérateurs de Laurent apparaissent aussi en étant les opérateurs d'évolution associés aux schémas aux différences finies approchant une équation de transport scalaire, ou même tout autre problème d'évolution linéaire scalaire. En effet, si l'on fixe une vélocité $c > 0$, alors pour l'équation de transport

$$\partial_t u + c\partial_x u = 0, \quad t > 0, x \in \mathbb{R}, \tag{1.4.6}$$

les schémas aux différences finies (à un pas de temps) peuvent être écrit sous la forme du problème d'évolution discret (1.4.3) pour un certain choix de suite a. Par exemple, le schéma upwind défini par :

$$\begin{aligned} &\forall n \in \mathbb{N}, \forall j \in \mathbb{Z} \quad u_j^{n+1} = c\boldsymbol{\nu} u_{j-1}^n + (1 - c\boldsymbol{\nu})\, u_j^n, \\ &u^0 \in \mathbb{C}^{\mathbb{Z}}, \end{aligned} \tag{1.4.7}$$

où $\boldsymbol{\nu}$ correspond au ratio entre les pas de temps Δt et d'espace Δx du schéma, se réécrit comme (1.4.3) pour la suite a définie par :

$$a_0 = 1 - c\boldsymbol{\nu}, \quad a_{-1} = c\boldsymbol{\nu} \quad \text{et} \quad \forall j \in \mathbb{Z}\backslash \{-1, 0\}, \quad a_j = 0.$$

Si l'on revient à un choix arbitraire de schéma, remarquons que les suites a associées aux schémas *explicites* seront à support fini. De plus, l'hypothèse de consistance avec l'équation de transport (1.4.6) pour le schéma aux différences finies que l'on considère se traduit par les égalités :

$$F(1) = \sum_{k\in\mathbb{Z}} a_k = 1 \quad \text{et} \quad F'(1) = \sum_{k\in\mathbb{Z}} k a_k = -c\boldsymbol{\nu}. \tag{1.4.8}$$

Cette condition ne contredit pas (1.4.5). On remarque enfin que la condition (1.4.5) correspond en terme de schémas aux différences finies à la condition de Von Neumann qui implique la ℓ^2-stabilité du schéma numérique, i.e. la famille d'opérateurs $(L^n)_{n\in\mathbb{N}}$ est bornée lorsqu'ils agissent sur $\ell^2(\mathbb{Z}, \mathbb{C})$.

Marche aléatoire sur $\mathbb{Z}$ et Théorème de la limite locale

Nous allons présenter un premier résultat sur le comportement en temps long de la fonction de Green b^n de l'opérateur de Laurent L : le théorème de la limite locale. Précisons d'ores et déjà que le domaine d'étude dans lequel se place ce résultat, c'est-à-dire les marches aléatoires sur l'ensemble $\mathbb{Z}$, limitera le cadre d'application de ce qui suit à la famille très restreinte des opérateurs de Laurent associés à des suites b à valeurs réelles positives.

On considère deux variables aléatoires X, Y à valeurs dans $\mathbb{Z}$. La première variable aléatoire X correspondra à la distribution de probabilité de la position initiale de notre marche aléatoire. La seconde variable aléatoire Y, quant à elle, sera la distribution de probabilité pour chacun des pas de la marche aléatoire. On considère donc des copies $(Y_n)_{n\in\mathbb{N}\backslash\{0\}}$ indépendantes et identiquement distribuées de Y qui soient aussi indépendantes avec X. On considère la marche aléatoire définie par :

$$\forall n \in \mathbb{N}, \quad S_n := X + \sum_{k=1}^{n} Y_k.$$

On introduit alors des suites b et u^n correspondant respectivement aux distributions de probabilité des variables aléatoires Y et S_n :

$$\begin{cases} \forall j \in \mathbb{Z}, & b_j := \mathbb{P}(Y = j), \\ \forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, & u_j^n := \mathbb{P}(S_n = j). \end{cases}$$

On remarque que u^0 est aussi la distribution de probabilité de la position initiale X :

$$\forall j \in \mathbb{Z}, \quad u_j^0 = \mathbb{P}(S_0 = j) = \mathbb{P}(X = j).$$

On observe que la suite b appartient à $\ell^1(\mathbb{Z}, \mathbb{C})$ et est à valeurs positives. De plus, par indépendance des variables aléatoires, on a que :

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^{n+1} = \mathbb{P}(S_{n+1} = j) = \sum_{k \in \mathbb{Z}} \mathbb{P}(Y_{n+1} = k)\mathbb{P}(S_n = j - k) = \sum_{k \in \mathbb{Z}} b_k u_{j-k}^n = (b * u^n)_j.$$

et donc par récurence :

$$\forall n \in \mathbb{N}, \quad u^n = b^n * u^0 = L^n u^0.$$

Ainsi, la distribution de probabilité u^n de la marche aléatoire S_n est déterminée par l'application du semi-groupe $(L^n)_{n \in \mathbb{N}}$ à la distribution de probabilité u^0 associée à la position initiale de la marche aléatoire. Plus précisément, si l'on considère que la variable aléatoire X indiquant la position initiale de la marche aléatoire est nulle $\mathbb{P}$-presque sûrement (i.e. le point de départ de notre marche aléatoire est 0), alors la distribution de probabilité de la marche aléatoire S_n est la fonction de Green b^n de l'opérateur L. On se place dans ce cas dorénavant.

Il existe de multiples résultats abordant le comportement en temps long des marches aléatoires sur $\mathbb{Z}$, tels que la loi des grands nombres ou encore le théorème central limite qui énonce la convergence en loi de la variable aléatoire $\sqrt{n}\left(\frac{S_n}{n} - \mathbb{E}(Y)\right)$ vers la loi normale $\mathcal{N}(0, V(Y))$, où $\mathbb{E}(Y) := \sum_{k \in \mathbb{Z}} k a_k$ et $V(Y) := \sum_{k \in \mathbb{Z}} k^2 a_k - \mathbb{E}(Y)^2$ correspondent à l'espérance et à la variance de la variable aléatoire Y.

Le théorème qui nous intéresse particulièrement est le théorème de la limite locale car il détermine le comportement ponctuel en temps long de la distribution de probabilité de la variable aléatoire S_n, ce qui, d'après ce que l'on a dit précédemment, revient à déterminer le comportement ponctuel en temps long de la fonction de Green b^n associée à l'opérateur L. Sous certaines hypothèses sur la distribution de probabilité de la variable aléatoire Y qui détermine les pas de la marche aléatoire, et donc sur la suite b, le théorème de la limite locale énonce qu'il existe une famille *explicite* de fonctions $(q_\sigma : \mathbb{R} \to \mathbb{R})_{\sigma \in \mathbb{N} \setminus \{0,1\}}$ telle que pour tout $s \in \mathbb{N} \setminus \{0\}$, la suite b^n correspondant à la distribution de probabilité de la variable aléatoire S_n satisfait le développement asymptotique suivant :

$$b_j^n - \frac{1}{\sqrt{2\pi V(Y) n}} \exp\left(-\frac{X_{n,j}^2}{2}\right) - \sum_{\sigma=2}^{s} \frac{q_\sigma(X_{n,j})}{n^{\frac{\sigma}{2}}} \underset{n \to +\infty}{=} o\left(\frac{1}{n^{\frac{s}{2}}}\right) \tag{1.4.9}$$

avec $X_{n,j} = \frac{j - n\mathbb{E}(Y)}{\sqrt{V(Y) n}}$ et où le terme en o est uniforme en $j \in \mathbb{Z}$. Ainsi, dans le cadre où b est une distribution de probabilité, le théorème de la limite locale offre une description précise de la fonction de Green b^n associée à l'opérateur L. On obtient un développement asymptotique à *n'importe quel ordre* uniforme en espace et où l'on retrouve dans le premier terme le comportement gaussien qui est attendu d'après le théorème central limite. Les fonctions q_σ sont des combinaisons linéaires de fonctions de la forme :

$$x \in \mathbb{R} \mapsto x^k \exp\left(-\frac{x^2}{2}\right) \in \mathbb{R} \quad \text{où } k \in \mathbb{N}$$

ce qui peut aussi être exprimé en terme de dérivées de la gaussienne. On redirige les lecteurs intéressés vers [Pet75, Chapter VII, Theorem 13] pour plus de détails.

1.4.2 Généralisations connues du théorème de la limite locale et résultat du Chapitre 2

Généralisations attendues

Pour l'étude de la fonction de Green b^n de l'opérateur de Laurent L, le théorème de la limite locale que l'on a présenté ci-dessus ne peut être appliqué que lorsque la suite b, et donc aussi la suite a, correspond à la distribution de probabilité d'une variable aléatoire à valeurs dans $\mathbb{Z}$. Une première perspective qui serait souhaitable est de trouver une version du théorème de la limite locale applicable pour des suites a vérifiant (1.4.5) à valeurs réelles ou même complexes. Cela permettrait d'appliquer ce résultat dans d'autres situations, comme par exemple pour discuter du comportement en temps long des solutions de schémas aux différences finies pour l'équation de transport, comme discuté plus haut. En effet, le comportement gaussien décrit par le théorème de la limite locale peut par exemple expliquer la diffusion numérique observée dans les solutions de schémas numériques dont on dit qu'ils introduisent de la viscosité numérique.

De plus, on souhaiterait prouver des estimées précises et ponctuelles sur le reste de l'égalité (1.4.9). En effet, il peut être souhaitable de pouvoir sommer les erreurs que l'on fait en approchant le comportement de la suite b^n par la gaussienne qui est le terme principal du développement asymptotique (1.4.9). Cependant, le fait de n'avoir qu'une estimation de l'erreur en $o\left(\frac{1}{n^{\frac{s}{2}}}\right)$ uniforme en j ne permet pas de sommer sur l'ensemble des entiers $\mathbb{Z}$ ou même sur une partie infinie de celui-ci.

Nous allons dorénavant discuter de généralisations du théorème de la limite locale dans les directions citées ci-dessus. Nous allons commencer par fixer certaines limitations importantes à préciser sur le choix de suite a que l'on va étudier en discutant de l'article fondamental de Thomée [Tho65]. La seconde partie de cette section portera sur la présentation des résultats récents [DS14 ; RS15 ; CF22] qui généralisent le théorème de la limite locale. Enfin, nous parlerons de la contribution du Chapitre 2 et de son rapport avec les articles cités précédemment.

Limitations sur le choix des suites a et l'article de Thomée [Tho65]

Par la suite, nous imposerons deux limitations sur les suites a étudiées. Nous préciserons les moments où l'une de ces hypothèses n'est pas nécessaire pour un résultat. Tout d'abord, on considèrera que la suite a est à *support fini*. Dans le cadre de l'analyse numérique, cela revient à ne regarder que les schémas aux différences finies explicites.

Concernant la seconde limitation, on rappelle que l'on a d'ores et déjà imposé que la série de Fourier F associée à la suite a doit vérifier la condition (1.4.5). Nous allons imposer une condition plus forte insipirée par l'article de Thomée [Tho65].

Hypothèse 1.2. *On suppose que :*

$$\forall \kappa \in \mathbb{S}^1 \backslash \{1\}, \quad |F(\kappa)| < 1. \quad (\textbf{\textit{Condition de dissipativité}}). \tag{1.4.10}$$

De plus, on supposera qu'il existe une constante réelle $\alpha \in \mathbb{R}$, une constante complexe $\beta \in \mathbb{C}$ à partie réelle strictement positive et un entier $\mu \in \mathbb{N}\backslash\{0\}$ tels que :

$$F(e^{i\xi}) \underset{\xi \to 0}{=} \exp(-i\alpha\xi - \beta\xi^{2\mu} + o(\xi^{2\mu})). \quad (\textbf{\textit{Condition de diffusivité}}) \tag{1.4.11}$$

En particulier, on a $F(1) = 1$.

Par l'égalité (1.4.2) sur le spectre de l'opérateur L, la condition de dissipativité (1.4.10) impose que le spectre $\sigma(L)$ de l'opérateur L n'a qu'un unique point de tangence en 1 avec le cercle unité $\mathbb{S}^1$ et que le reste du spectre doit être contenu dans le disque unité ouvert. L'information spectrale pour comprendre le semi-groupe $(L^n)_{n\in\mathbb{N}}$ et la fonction de Green $(b^n)_{n\in\mathbb{N}}$ se trouve donc au niveau du point 1. La condition de diffusivité (1.4.11) impose quand à elle un comportement particulier du spectre de l'opérateur L au niveau du point 1. Elle impose en particulier au logarithme de F d'avoir un développement limité en 1 dont tous les termes d'ordre 2 jusqu'à $2\mu - 1$ doivent être nuls et où le coefficient à l'ordre pair 2μ doit avoir une partie réelle strictement positive.

Faisons quelques remarques qui nous semblent importantes :

— Nous allons présenter ci-dessous des généralisations du théorème de la limite locale pour des suites a dont la série de Fourier F vérifie l'Hypothèse 1.2. Ces résultats peuvent être résumés *très grossièrement* de la façon suivante :

"La fonction de Green $(b^n)_{n\in\mathbb{N}}$ de l'opérateur L se comporte en temps long comme une *gaussienne généralisée* dont les propriétés (vitesse de déplacement, coefficient de diffusion, etc...) sont données par (1.4.11)."

où le terme *gaussienne généralisée* sera défini plus bas. Essentiellement, l'information du comportement en temps long de la fonction de Green b^n se trouve au niveau du point 1 du spectre de l'opérateur L. Bien évidemment, le reste de cette section de l'introduction sera amené à préciser ce que l'on vient de dire le plus clairement possible.

Cependant, il nous semble important de garder à l'esprit que, dans les résultats que l'on va présenter ci-dessous, c'est une version plus faible de l'Hypothèse 1.2 qui est souvent utilisée. Cette hypothèse "affaiblie" impose à l'ensemble $F(\mathbb{S}^1) = \sigma(L)$ d'avoir un nombre fini de points de tangence avec le cercle unité $\mathbb{S}^1$, et non pas nécessairement qu'un seul comme dans l'Hypothèse 1.2. De plus, au niveau de chacun de ces points de tangence, une condition de diffusivité doit être vérifiée. Pour une suite a dont la série de Fourier vérifie cette hypothèse affaiblie, la fonction de Green b^n fera apparaître autant de *gaussiennes généralisées* qu'il y aura de points de tangence entre $F(\mathbb{S}^1)$ et le cercle unité $\mathbb{S}^1$. Les résultats dont nous parlerons sont alors adaptés pour soutenir cette description de la fonction de Green b^n.

Dans le reste de cette introduction, nous allons présenter tous les résultats dans le cadre restreint de l'Hypothèse 1.2 mais soulignerons quand ceux-ci sont généralisables pour des suites a dont le spectre $\sigma(L)$ de l'opérateur de Laurent associé à la suite a admet plusieurs points de tangence avec le cercle unité $\mathbb{S}^1$ sans rentrer beaucoup plus dans le détail de l'impact sur l'énoncé des résultats. Nous invitons les lecteurs en recherche de détails à se tourner vers le Chapitre 2 où tout ceci est clairement présenté.

— Dans l'étude de la stabilité des profils de choc totalement discrets que l'on a présentée précédemment et qui est le centre thématique de cette thèse, on retrouve ces conditions de dissipativité et de diffusivité dans l'Hypothèse 1.1 que l'on avait imposée aux opérateurs $\mathscr{L}_{\overline{u}}$, définis par (1.1.13), correspondant aux linéarisés de l'opérateur $\mathcal{N}$ au niveau des états constants $\overline{u} \in \mathcal{U}$.

— Dans le cadre présenté plus haut des marches aléatoires, sous certaines hypothèses sur la suite a comme par exemple le fait d'avoir deux termes successifs non nuls, on a que la série de Fourier F vérifie l'Hypothèse 1.2 avec $\mu = 1$, $\alpha = \mathbb{E}(Y)$ et $\beta = \frac{V(Y)}{2}$. Ainsi, le cas probabiliste présenté précédemment se trouve dans le giron de l'Hypothèse 1.2.

— Dans le contexte présenté plus haut des schémas aux différences finies pour l'équation de transport, on remarque que la condition de consistance (1.4.8) implique que la constante α de (1.4.11) est égale à $\boldsymbol{\nu} c$. De plus, si le schéma que l'on considère vérifie effectivement la condition de consistance (1.4.8), alors la condition de diffusivité (1.4.11) implique que le schéma aux différences finies associé a est d'ordre de consistance $2\mu - 1$. Par exemple, le schéma Upwind (1.4.7) d'ordre 1 vérifie la condition de dissipativité (1.4.10) sous une condition CFL et la condition de diffusivité (1.4.11) pour $\mu = 1$. Il en est de même pour le schéma O3 [Des08] d'ordre 3 mais cette fois avec $\mu = 2$. Cependant, les schémas de Lax-Wendroff et de Beam-Warming qui sont d'ordre 2 ne vérifient pas la condition de diffusivité (1.4.11) bien qu'ils vérifient la condition de dissipation (1.4.10) sous condition CFL (voir [Cou22] pour plus de détail). On appelle "condition de diffusivité" la condition (1.4.11) car les schémas qui vérifient cette condition introduisent de la viscosité numérique.

Comme dit précédemment, l'Hypothèse 1.2 est inspirée de l'article [Tho65] de Thomée. Dans cet article, Thomée prouve que pour une suite $a \in \ell^1(\mathbb{Z})$ à support fini, dont la série de Fourier vérifie l'Hypothèse 1.2, il existe une constante $C > 0$ telle que la fonction de Green b^n de l'opérateur L vérifie :

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \quad \begin{cases} \left|b_j^n\right| \leq \dfrac{C}{n^{\frac{1}{2\mu}}}, \\ \left|b_j^n\right| \leq \dfrac{Cn^{\frac{1}{\mu}}}{|j - n\alpha|^2}, \end{cases} \tag{1.4.12}$$

où les constantes α et μ sont celles de l'Hypothèse 1.2.

À l'aide des estimées (1.4.12) sur la fonction de Green b^n de l'opérateur L, Thomée prouve que, pour n'importe quel $r \in [1, +\infty]$, le semi-groupe $(L^n)_{n\in\mathbb{N}}$ est borné lorsque l'opérateur L agit sur $\ell^r(\mathbb{Z})$. Dans le contexte des schémas aux différences finies approchant les problèmes d'évolution linéaires, cela correspond à avoir la ℓ^r-stabilité du schéma pour tout $r \in [1, +\infty]$. De plus, Thomée prouve que, si la suite a vérifie la condition de dissipativité (1.4.10) mais pas la condition de diffusivité (1.4.11), alors on peut trouver des bornes inférieures sur les normes $|b_j^n|$ des coefficients de la fonction de Green b^n qui permettent de prouver que le semi-groupe $(L^n)_{n\in\mathbb{N}}$ n'est pas borné lorsque l'opérateur L agit sur $\ell^\infty(\mathbb{Z})$.

Le résultat de Thomée a un impact fort sur l'étude de la stabilité des schémas aux différences finies pour les équations d'évolution linéaire car il permet de caractériser les schémas qui sont ℓ^r-stables pour tout $r \in [1, +\infty]$. Précisons que Thomée considère dans son article le cas plus général où l'ensemble $F(\mathbb{S}^1)$ peut avoir plusieurs points de tangence avec le cercle unité et prouve qu'il suffit de ne pas avoir la condition de diffusivité (1.4.11) au niveau de l'un de ces points de tangence pour que le semi-groupe $(L^n)_{n\in\mathbb{N}}$ ne soit pas borné en agissant sur $\ell^\infty(\mathbb{Z})$.

Les résultats [DS14] de Diaconis et Saloff-Coste et [CF22] de Coulombel et Faye

Le résultat de Thomée permet de prouver des bornes (1.4.12) ponctuelles algébriques élémentaires pour la fonction de Green b^n. Le résultat [DS14] de Diaconis et Saloff-Coste améliore les estimées (1.4.12) en prouvant des bornes gaussiennes généralisées sur les coefficients de la fonction de Green b^n. Plus précisément, si une suite $a \in \ell^1(\mathbb{Z})$ est à support fini et vérifie l'Hypothèse 1.2, alors il existe deux constantes $C, c > 0$ telles que :

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \quad \left|b_j^n\right| \leq \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{|j - n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{1.4.13}$$

On retrouve pour $\mu = 1$ des bornes gaussiennes usuelles. Ces bornes sont attendues car le terme principal du théorème de la limite locale (1.4.9) pour les marches aléatoires sur $\mathbb{Z}$ est une gaussienne. Dans le cas $\mu \neq 1$, on parle de bornes gaussiennes généralisées. On observe que la fonction de Green b^n est transportée à vitesse α.

Ce résultat est d'ores et déjà à la fois une généralisation du théorème de la limite locale pour des suites a à valeurs complexes, mais aussi une amélioration du théorème de la limite locale dans le cadre des marches aléatoires, permettant de prouver des estimées de décroissance plus précises que celle de (1.4.9).

Dans [CF22], Coulombel et Faye généralisent le résultat de [DS14] dans deux directions :

- Ils considèrent le cas où l'ensemble $F(\mathbb{S}^1)$ a plusieurs points de tangence avec le cercle unité. Cela fait apparaître une somme de bornes gaussiennes généralisées.
- Ils généralisent le résultat pour certaines suites $a \in \ell^1(\mathbb{Z})$ qui ne sont pas à support fini. Plus précisément, ils considèrent le cas où la suite $b = (a_{-k})_{k\in\mathbb{Z}}$ est de la forme $b_1{}^{-1} * b_0$ où les suites $b_0, b_1 \in \ell^1(\mathbb{Z})$ sont à support fini et où la suite b_1 est inversible pour le produit de convolution, i.e., par [New75], la série de Fourier de b_1 ne s'annule pas sur $\mathbb{S}^1$. Cela permet d'étudier le cas des suites a associées à des schémas aux différences finies *implicites*.

Le résultat de [RS15] de Randles et Saloff-Coste

L'article [RS15] de Randles et Saloff-Coste généralise le théorème de la limite locale dans une direction différente de [Tho65 ; DS14 ; CF22] en réussissant à extraire le comportement principal de la fonction de Green b^n.

On considère une suite $a \in \ell^1(\mathbb{Z})$ à support fini qui vérifie l'Hypothèse 1.2. On définit la fonction $H^\beta_{2\mu} : \mathbb{R} \to \mathbb{R}$ par :

$$\forall x \in \mathbb{R}, \quad H^\beta_{2\mu}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixu} e^{-\beta u^{2\mu}} du$$

où les constantes μ et β sont déterminées par la condition de diffusivité (1.4.11). On appelle ces fonctions $H^\beta_{2\mu}$ des *gaussiennes généralisées* car, pour $\mu = 1$, on retrouve la fonction gaussienne :

$$\forall x \in \mathbb{R}, \quad H^\beta_2(x) = \frac{1}{\sqrt{4\pi\beta}} e^{-\frac{x^2}{4\beta}}.$$

Le résultat principal de [RS15] est que la fonction de Green b^n associée à l'opérateur L vérifie :

$$b^n_j - \frac{1}{n^{\frac{1}{2\mu}}} H^\beta_{2\mu}\left(\frac{j - n\alpha}{n^{\frac{1}{2\mu}}}\right) = o\left(\frac{1}{n^{\frac{1}{2\mu}}}\right) \tag{1.4.14}$$

où la notation de Landau o est uniforme en $j \in \mathbb{Z}$. Essentiellement, Randles et Saloff-Coste ont réussi à déterminer le comportement principal de la fonction de Green b^n. Leur résultat correspond à une expression dans le cadre général des suites à valeurs complexes du théorème de la limite locale (1.4.9) pour $s = 1$, i.e. un développement à l'ordre 1 de la fonction de Green b^n.

On précise que le résultat de [RS15] peut être généralisé au cas où l'ensemble $F(\mathbb{S}^1)$ a plusieurs points de tangence avec le cercle unité. Chaque point de tangence correspond alors à une gaussienne généralisée qui peut être exprimée à l'aide des fonctions $H^\beta_{2\mu}$ où les coefficients μ et β sont associés au comportement de F au niveau du point de tangence. De plus, le résultat [RS15] peut aussi être exprimé dans le cas où la condition de diffusivité (1.4.11) n'est pas vérifiée, permettant ainsi par exemple d'étudier la fonction de Green b^n associée à des schémas aux différences finies non diffusifs tels que les schémas de Lax-Wendroff ou Beam-Warming.

Contribution du Chapitre 2

Le résultat principal du Chapitre 2 est le Théorème 2.1. On considère une suite $a \in \ell^1(\mathbb{Z})$ à support fini vérifiant l'Hypothèse 1.2. Le Théorème 2.1 énonce alors que pour tout $s \in \mathbb{N}$, il existe une famille de fonctions $q_1, \dots, q_s : \mathbb{R} \to \mathbb{R}$ et deux constantes positives C, c telles que :

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \quad \left| b^n_j - \sum_{\sigma=1}^{s} \frac{1}{n^{\frac{\sigma}{2}}} q_\sigma\left(\frac{j - n\alpha}{n^{\frac{1}{2\mu}}}\right)\right| \leq \frac{C}{n^{\frac{s+1}{2\mu}}} \exp\left(-c\left(\frac{|j - n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \tag{1.4.15}$$

où les constantes α, μ et β sont déterminées par la condition de diffusivité (1.4.11). De plus, la fonction q_1 est égale à $H^\beta_{2\mu}$ et les fonctions q_σ peuvent *explicitement calculées* à l'aide des fonctions :

$$x \in \mathbb{R} \mapsto x^{k_1} {H^\beta_{2\mu}}^{(k_2)}(x) \in \mathbb{R} \quad \text{où } k_1, k_2 \in \mathbb{N}.$$

Précisons que l'expression des fonctions q_σ est facilement implémentable pour les calculer sur machine. Les inégalités (1.4.15) nous donnent donc le comportement asymptotique *à tout ordre* des éléments b_j^n de la fonction de Green avec des *estimées précises sur le reste*. Replaçons ce résultat au sein de l'état de l'art dressé précédemment :

— Si l'on prend $s = 0$ dans (1.4.15), on retrouve le résultat (1.4.13) de [DS14; CF22]. On a donc étendu le résultat [DS14; CF22] en prouvant des estimées du même type mais sur le reste du développement asymptotique des coefficients b_j^n la fonction de Green à n'importe quel ordre.

— Comme dit précédemment, la fonction q_1 est égale à la fonction $H_{2\mu}^\beta$. Ainsi, en prenant $s = 1$ dans (1.4.15), on retrouve le résultat (1.4.14) de [RS15] mais avec une estimation précise et surtout sommable du reste. On peut estimer l'erreur effectuée en remplaçant la fonction de Green b^n par son comportement principal plus efficacement.

— Le résultat (1.4.15) est aussi une amélioration dans le cadre probabiliste du théorème de la limite locale (1.4.9) en prouvant des estimées précises sur le reste du développement asymptotique à tout ordre.

Tout comme les résultats de [Tho65; CF22; RS15], le Théorème 2.1 est généralisé au cas où l'ensemble $F(\mathbb{S}^1)$ a plusieurs points de tangence avec le cercle unité.

Contrairement aux articles [Tho65; DS14; RS15] qui utilise l'analyse de Fourier pour prouver leur résultat, la preuve du Théorème 2.1 est inspiré de [CF22] et repose sur les mêmes idées que celles développées dans le Paragraphe "Idée de la preuve : Formule de Laplace inverse et fonction de Green spatiale" de la Section 1.3.3, i.e. l'adaptation dans un cadre totalement discret des techniques de [ZH98] d'étude de fonctions de Green pour des problèmes paraboliques. Précisons que, contrairement aux études des fonctions de Green dans les Chapitres 3 et 4, on note une absence de valeurs propres plongées dans le spectre essentiel de l'opérateur L sur le disque unité. L'utilisation de la transformée de Laplace dans ce contexte pourrait paraître artificielle mais elle sert d'échauffement en vue de l'étude de problèmes aux limites comme on le détaille maintenant.

1.5 Stabilité des schémas aux différences finies pour l'équation de transport sur la demi-droite (Chapitre 3)

1.5.1 Contexte général

Cette dernière section de l'introduction porte sur le contenu de l'article [Coe24] qui est présenté dans le Chapitre 3. On s'intéresse aux approximations par des schémas aux différences finies de l'équation de transport à vélocité négative $c < 0$ sur la demi-droite $\mathbb{R}_+$:

$$\begin{aligned} \forall t \geq 0, \forall x \geq 0, \quad & \partial_t u + c\partial_x u = 0, \\ \forall x \geq 0, \quad & u(0,x) = u_0(x) \in \mathbb{R}. \end{aligned} \tag{1.5.1}$$

On remarque que la vélocité c de l'équation de transport étant negative, il n'est pas nécessaire d'imposer de condition de bord en $x = 0$ sur l'EDP (1.5.1) pour que le problème soit bien posé.

On fixe $\boldsymbol{\nu} > 0$ le ratio entre les pas de temps Δt et d'espace Δx que l'on va considérer. Pour les cellules à l'intérieur du domaine de définition $[(j-1)\Delta x, j\Delta x[$ avec $j \in \mathbb{N}\backslash\{0\}$, le schéma aux différences finies s'écrit :

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{N}\backslash\{0\}, \quad u_j^{n+1} = \sum_{k=-p}^{q} a_k u_{j+k}^n \tag{1.5.2a}$$

où les entiers $p, q \in \mathbb{N}$ définissent la taille du stencil du schéma et les coefficients a_k sont des réels pouvant dépendre de la velocité c et du ratio $\boldsymbol{\nu}$ et sont choisis pour satisfaire la condition de consistance (1.4.8) où F est la série de Fourier définie par (1.4.2) associée à la suite a. On voit cependant apparaître des valeurs $u_{1-p}^n, \ldots, u_0^n$ qui ne sont pas définies. On considère que ces valeurs sont associées à des "cellules fantômes" et on les détermine à l'aide de combinaisons linéaires des valeurs u_j^n proches du bord :

$$\forall n \in \mathbb{N}, \forall j \in \{1-p, \ldots, 0\}, \quad u_j^n = \sum_{k=1}^{q_b} b_{k,j} u_k^n, \tag{1.5.2b}$$

où q_b est un entier, les coefficients $b_{k,j}$ sont aussi des réels pouvant dépendre de la velocité c et du ratio $\boldsymbol{\nu}$. On fera attention au fait que les coefficients $b_{k,j}$ n'ont aucun lien avec la suite b que l'on avait introduite dans la section précédente. Les équations (1.5.2b) correspondent à des conditions de bord purement numériques. Le choix de ces conditions de bord (1.5.2b) est central dans les propriétés de stabilité du schéma numérique (1.5.2). En effet, les conditions de bord ont un impact direct sur la localisation des valeurs propres de l'opérateur d'évolution $\mathcal{T}$, qui sera défini plus bas dans (1.5.3), associé au schéma aux différences (1.5.2).

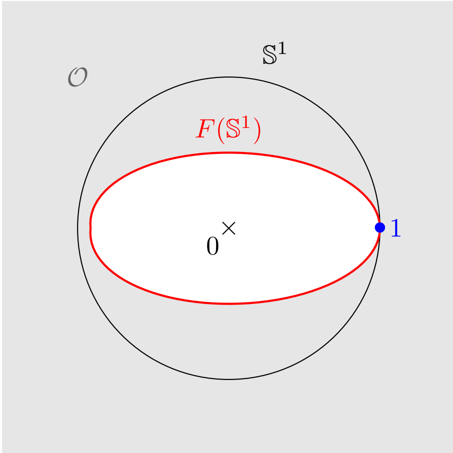

FIGURE 1.6 – Un exemple de courbe $F(\mathbb{S}^1)$ où F satisfait l'Hypothèse 1.2. La condition de dissipativité (1.4.10) implique que la courbe $F(\mathbb{S}^1)$ (en rouge) reste dans le disque unité ouvert sauf en son unique point de tangence 1 avec le cercle unité $\mathbb{S}^1$. En gris, on trouve la composante connexe non bornée $\mathcal{O}$ de $\mathbb{C}\backslash F(\mathbb{S}^1)$.

Les valeurs initiales u_j^0 pour $j \in \mathbb{N}\backslash\{0\}$ sont calculées à l'aide de la condition initiale u_0 de l'équation de transport (1.5.1), par exemple comme :

$$\forall j \in \mathbb{N}\backslash\{0\}, \quad u_j^0 := \frac{1}{\Delta x}\int_{(j-1)\Delta x}^{j\Delta x} u_0(x)dx.$$

On s'intéresse à la stabilité du schéma aux différences finies (1.5.2). Pour ce faire, nous allons commencer par définir l'opérateur d'évolution $\mathcal{T}$ du schéma ainsi que l'espace vectoriel normé sur lequel il agit.

Pour $r \in [1, +\infty]$, on introduit l'espace de Banach $\mathcal{H}_r$ défini par :

$$\mathcal{H}_r := \left\{ (w_j)_{j\geq 1-p} \in \ell^r\left(\{j \in \mathbb{Z}, j \geq 1-p\}, \mathbb{R}\right), \quad \forall j \in \{1-p, \dots, 0\}, w_j = \sum_{k=1}^{q} b_{k,j} w_k \right\}$$

avec la norme

$$\forall w \in \mathcal{H}_r, \quad \|w\|_{\mathcal{H}_r} := \left\|(w_j)_{j\in\mathbb{N}\backslash\{0\}}\right\|_{\ell^r(\mathbb{N}\backslash\{0\})}.$$

Les espace de Banach $\mathcal{H}_r$ sont des adaptations de l'espace $\ell^r(\mathbb{N}\backslash\{0\})$ qui prennent en compte les conditions de bord (1.5.2b). On définit ensuite l'opérateur linéaire borné $\mathcal{T}$ agissant sur $\mathcal{H}^r$:

$$\forall w \in \mathcal{H}_r, \forall j \in \mathbb{N}\backslash\{0\}, \quad (\mathcal{T}w)_j := \sum_{k=-p}^{q} a_k w_{j+k}. \tag{1.5.3}$$

Les valeurs $(\mathcal{T}w)_j$ pour $j \in \{1-p, \dots, 0\}$ sont déterminées par la condition $\mathcal{T}w \in \mathcal{H}_r$. On remarque alors que pour une condition initiale $u^0 \in \mathcal{H}_r$, le schéma aux différences finies (1.5.2) se réécrit sous la forme :

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{T}u^n.$$

Pour $r \in [1, +\infty]$, on dit alors que le schéma aux différences finies (1.5.2) est ℓ^r-stable lorsque le semi-groupe $(\mathcal{T}^n)_{n\in\mathbb{N}}$ est borné en agissant sur $\mathcal{H}_r$, i.e. il existe une constante $C > 0$ telle que

$$\forall u^0 \in \mathcal{H}_r, \forall n \in \mathbb{N}, \quad \left\|\mathcal{T}^n u^0\right\|_{\mathcal{H}_r} \leq C \left\|u^0\right\|_{\mathcal{H}_r}.$$

1.5.2 Hypothèses spectrales et déterminant de Lopatinskii

Tout comme pour l'étude de la fonction de Green des opérateurs de Laurent présentée dans la Section 1.4, pour obtenir des propriétés de stabilité pour l'opérateur $\mathcal{T}$, il est nécessaire de déterminer précisément des informations sur le spectre de l'opérateur $\mathcal{T}$. Cette section va s'atteler à présenter très succinctement les outils nécessaires pour cette étude spectrale.

On commence par imposer que la série de Fourier F associée à la suite a vérifie l'Hypothèse 1.2. En effet, comme on l'a vu lors de la discussion sur le résultat [Tho65] de Thomée, cette hypothèse implique que, si l'on considère l'opérateur de Laurent L, défini par (1.4.1), associé à la suite a intervenant dans la discrétisation de l'équation de transport (1.5.2a), alors le semi-groupe $(L^n)_{n\in\mathbb{N}}$ est borné quand il agit sur $\ell^r(\mathbb{Z})$ pour tout $r \in [1, +\infty]$. C'est un point de départ évident pour construire des schémas aux différences finies (1.5.2) stable pour le plus grand nombre de ℓ^r.

On note $\mathcal{O}$ la composante connexe non bornée de $\mathbb{C}\backslash F(\mathbb{S}^1)$ (voir Figure 1.6). On prouve dans [CF23] ainsi que dans le Chapitre 3 au Lemme 3.4.1, que tout élément $z \in \mathcal{O}$ est soit une valeur propre de l'opérateur $\mathcal{T}$, soit que z appartient à l'ensemble résolvant de l'opérateur $\mathcal{T}$. Ce qui se cache ici est le fait que $F(\mathbb{S}^1)$ correspond au spectre essentiel de l'opérateur $\mathcal{T}$. Ainsi, le comportement du semi-groupe $(\mathcal{T}^n)_{n\in\mathbb{N}}$ dépend de la l'existence et de la localisation de valeurs propres de $\mathcal{T}$ de module supérieur ou égal à 1.

Afin de discuter de la localisation de ces valeurs propres, on construit alors une fonction Δ appelée *déterminant de Lopatinskii* au voisinage de tout point z de module supérieur ou égal à 1 qui jouera le rôle de polynôme caractéristique pour l'opérateur $\mathcal{T}$. La construction de ce déterminant de Lopatinskii, qui est relativement technique, est présentée dans son intégralité au sein du Chapitre 3. Concernant les éléments principaux sur le déterminant de Lopatinskii, il faut savoir que c'est une fonction définie localement et holomorphe au voisinage de tout point de l'ensemble $\mathcal{O} \cup \{1\}$ et donc en particulier de tout point de module plus grand ou égal à 1 (voir Figure 1.6). Précisons que la définition de Δ localement au niveau d'un point est faite à la multiplication par une fonction holomorphe non nulle près. Cela ne pose cependant pas de problème car ce sont les zéros de la fonction Δ qui nous intéressent. En effet, le déterminant de Lopatinskii s'annule en un point si et seulement si celui-ci est une valeur propre de l'opérateur $\mathcal{T}$. Le déterminant de Lopatinskii partage ainsi le même rôle que la fonction d'Evans, dont on a parlé bien plus tôt dans l'introduction, qui s'annule au niveau des valeurs propres de l'opérateur $\mathscr{L}$ défini par (1.3.1), correspondant au linéarisé au niveau d'un profil de choc totalement discret.

1.5.3 Résultats connus et contribution du Chapitre 3

Condition de Godunov-Ryabenkii et condition uniforme de Kreiss-Lopatinskii

Nous allons maintenant décrire les résultats connus de ℓ^r-stabilité des schémas aux différences finies (1.5.2) en commençant par deux cas centraux.

— Un premier cas évident, connu sous le nom de *Condition de Godunov-Ryabenkii* et introduit dans [GR63], porte sur le cas où le déterminant de Lopatinskii Δ s'annule en un point z de module strictement plus grand que 1. Il est alors évident que le schéma (1.5.2) n'est pas ℓ^r-stable pour tout $r \in [1, +\infty]$ car si l'on considère un vecteur propre $u^0 \in \mathcal{H}_r$ non nul de l'opérateur $\mathcal{T}$ associé à la valeur propre z, on a immédiatement :
$$\forall n \in \mathbb{N}, \quad \mathcal{T}^n u^0 = z^n u^0.$$
Ainsi, la *Condition de Godunov-Ryabenkii* est une condition nécessaire de stabilité qui énonce que le déterminant de Lopatinskii Δ ne peut pas s'annuler hors du disque unité *fermé*, i.e. l'opérateur $\mathcal{T}$ ne peut pas avoir de valeur propre de module strictement plus grand que 1.

— Introduite dans [GKS72], on dit que le schéma (1.5.2) vérifie la *condition uniforme de Kreiss-Lopatinskii* dès lors que le déterminant de Lopatinskii Δ ne s'annule pas hors du disque unité *ouvert*, i.e. l'opérateur $\mathcal{T}$ n'admet pas de valeur propre de module plus grand ou égal 1. Il est connu que la vérification par un schéma (1.5.2) de cette *condition uniforme de Kreiss-Lopatinskii* est équivalent à prouver la *stabilité forte* du schéma. Cette notion de *stabilité forte* est différente de la notion de ℓ^r-stabilité qui nous intéressé et on redirige les lecteurs cherchant plus de détail sur cette notion vers [GKO13]. Cependant, en utilisant des estimées d'énergie, il est possible de prouver que la stabilité forte, et donc la vérification de la *condition uniforme de Kreiss-Lopatinskii*, implique la ℓ^2-stabilité du schéma (1.5.2) (voir [Wu95 ; CG11 ; Cou13] qui traitent respectivement le cas scalaire monodimensionnel avec un pas de temps, le cas système multdimensionnel à un pas de temps et enfin le cas scalaire multidimensionnel à plusieurs pas de temps). On prétend qu'il est même tout à fait possible de prouver que la vérification de la *condition uniforme de Kreiss-Lopatinskii* implique la ℓ^r-stabilité pour tout $r \in [1, +\infty]$ en adaptant les idées de l'article [CF23], dont on va parler juste après, ou même du Chapitre 3.

L'article [CF23] de Coulombel et Faye

Ces deux premières conditions laissent un trou dans la théorie : Que peut-on dire de la ℓ^r-stabilité du schéma (1.5.2) dès lors que la condition de Godunov-Ryabenkii est satisfaite mais pas la condition uniforme de Kreiss-Lopatinskii, i.e. quand le déterminant de Lopatinskii Δ a des zéros de module 1 ? Cette question a été soulevée dans [Tre84 ; KW93] et un premier élément de réponse se trouve dans [CF23]. On se place dans le cas où la série de Fourier F associée à la suite a vérifie l'Hypothèse 1.2. Alors, $1 \in F(\mathbb{S}^1)$ appartient au spectre essentiel de l'opérateur $\mathcal{T}$ mais pas les autres points du disque unité. Alors, Coulombel et Faye prouvent dans [CF23] que si le déterminant de Lopatinskii Δ s'annule en un des points du disque unité autre que 1 et que ce zéro est simple, alors le schéma (1.5.2) est ℓ^2-stable. On prétend cependant que la preuve peut être généralisée pour prouver que le schéma (1.5.2) est même ℓ^r-stable pour tout $r \in [1, +\infty]$.

La preuve repose sur l'introduction et une description précise de la fonction de Green de l'opérateur $\mathcal{T}$. Tout comme pour l'étude de la fonction de Green des opérateurs de Laurent, la méthodologie suivie est basée sur celle présentée dans le Paragraphe "Idée de la preuve : Formule de Laplace inverse et fonction de Green spatiale" de la Section 1.3.3, i.e. l'adaptation dans un cadre totalement discret des techniques de [ZH98] d'étude de fonctions de Green pour des problèmes paraboliques. La difficulté qui est rencontrée dans [CF23], i.e. l'existence d'un zéro du déterminant de Lopatinskii Δ sur le cercle unité $\mathbb{S}^1$ non plongé dans le spectre essentiel de l'opérateur $\mathcal{T}$, implique que la fonction de Green spatiale ne sera pas définie au niveau de ce point. Cela crée une difficulté supplémentaire en comparaison avec l'étude de la fonction de Green des opérateurs de Laurent, qui eux ne possèdent pas de valeurs propres de module supérieur ou égal à 1. Il faudra prouver que la fonction de Green spatiale est en réalité définie méromorphiquement avec un pôle d'ordre 1 au niveau du zéro du déterminant de Lopatinskii. L'utilisation du théorème des résidus dans l'égalité (1.3.10) reliant fonction de Green temporelle et fonction de Green spatiale permettra alors de résoudre le problème.

Contribution du Chapitre 3

Le résultat principal du Chapitre 3 comble le dernier trou laissé par les articles cités plus haut, en tout cas pour les schémas (1.5.2) approchant l'équation de transport à *vitesse négative* (1.5.1).

On se place dans le cas où la série de Fourier F associée à la suite a vérifie l'Hypothèse 1.2 et l'on suppose que le déterminant de Lopatinskii n'a aucun zéro de module supérieur ou égal à 1 sauf en 1 où il admet un zéro simple. Dans ce cas, le Théorème 3.1 énonce que le schéma (1.5.2) est bien ℓ^1-stable mais qu'il est, en fonction d'une condition algébrique *explicite*, soit ℓ^r-stable pour tout $r \in]1, +\infty]$, soit ℓ^r-instable pour tout $r \in]1, +\infty]$. On précise de plus que cette fameuse condition algébrique semble indiquer que, des deux cas cités ci-dessus, c'est le second qui a le plus tendance à se produire.

Ainsi, la contribution vient compléter le résultat [CF23] et répondre à la question de [Tre84 ; KW93] présentée plus tôt. En combinant et généralisant légèrement les résultats des articles [CF23 ; Coe24], on a que si la condition de Godunov-Ryabenkii est vérifiée mais pas la condition uniforme de Kreiss-Lopatinskii, il suffit de vérifier si les zéros du déterminant de Lopatinskii sont plongés ou pas dans le spectre essentiel de l'opérateur $\mathcal{T}$ pour en déduire la stabilité du schéma numérique (1.5.2).

En remarquant que les propriétés spectrales de l'opérateur $\mathcal{T}$ considérées dans le Théorème 3.1 (i.e. existence d'un zéro simple du déterminant de Lopatinskii plongé dans le spectre essentiel de l'opérateur $\mathcal{T}$) correspondent exactement aux propriétés spectrales imposées par l'hypothèse de stabilité spectrale à l'opérateur $\mathscr{L}$ correspondant linéarisé au niveau d'un profil de choc totalement discret dans le Chapitre 4 (i.e. existence d'un zéro simple de la fonction d'Evans plongé dans le spectre essentiel de l'opérateur $\mathscr{L}$), on espère transmettre aux lecteurs le fait que le cadre du Chapitre 3 s'intègre au sein de cette thèse comme une préparation en vue de l'obtention du résultat central du Chapitre 4 sur l'étude de la fonction de Green de l'opérateur $\mathscr{L}$.

En effet, la contribution du Chapitre 3 est non seulement intéressante pour la réponse qu'elle apporte à la question de la ℓ^r-stabilité du schéma (1.5.2), mais aussi pour ce qui relève de l'étude de la fonction de Green de l'opérateur $\mathcal{T}$. En effet, la preuve du Théorème 3.1 du Chapitre 3 est similaire à celle de [CF23] à la différence près que le zéro de module 1 du déterminant de Lopatinskii se trouve cette fois au sein du spectre essentiel. On prouve avec des éléments de preuves similaires à ceux de [CF23] que la fonction de Green spatiale peut être prolongée méromorphiquement dans un voisinage de 1 avec un pôle d'ordre 1 en 1. La difficulté est cependant que, là où dans [CF23] l'utilisation du théorème des résidus permettait de s'occuper du pôle dans l'égalité (1.3.10) reliant la fonction de Green temporelle et la fonction de Green spatiale, le cas traité dans le Chapitre 3 nécessite une analyse plus méticuleuse afin d'obtenir des estimées pour la fonction de Green de l'opérateur $\mathcal{T}$ qui soient assez précises. Cette analyse plus méticuleuse est connue et a été traitée dans [MZ02] pour l'étude de la fonction de Green d'un opérateur linéaire associé à un problème parabolique continu en temps et en espace. Pour résumer, une contribution centrale et nontriviale du Chapitre 3 repose sur la généralisation d'une partie de l'analyse présentée dans [MZ02] de la fonction de Green, en rapport avec l'existence de valeurs propres de module 1 plongées dans le spectre essentiel de l'opérateur, dans un cadre cette fois *discret en temps et en espace*.

Chapter 2

Local limit theorem for complex valued sequences

This chapter presents the content of the preprint [Coe22].

Abstract of the current chapter

In this chapter, we study the pointwise asymptotic behavior of iterated convolutions on the one dimensional lattice $\mathbb{Z}$. We generalize the so-called local limit theorem in probability theory to complex valued sequences. A sharp rate of convergence towards an explicitly computable attractor is proved together with a generalized Gaussian bound for the asymptotic expansion up to any order of the iterated convolution.

Outline of the current chapter

Notations

For $1 \leq q < +\infty$, we let $\ell^q(\mathbb{Z})$ denote the Banach space of complex valued sequences indexed by $\mathbb{Z}$ and such that the norm:

$$\|u\|_{\ell^q} := \left(\sum_{j\in\mathbb{Z}} |u_j|^q \right)^{\frac{1}{q}}$$

is finite. We also let $\ell^\infty(\mathbb{Z})$ denote the Banach space of bounded complex valued sequences indexed by $\mathbb{Z}$ equipped with the norm

$$\|u\|_{\ell^\infty} := \sup_{j\in\mathbb{Z}} |u_j|.$$

Throughout this article, we define the following sets:

$$\mathcal{U} := \{z \in \mathbb{C}, |z| > 1\}, \quad \mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}, \quad \mathbb{S}^1 := \{z \in \mathbb{C}, |z| = 1\},$$

$$\overline{\mathcal{U}} := \mathbb{S}^1 \cup \mathcal{U}, \quad \overline{\mathbb{D}} := \mathbb{S}^1 \cup \mathbb{D}.$$

For $z \in \mathbb{C}$ and $r > 0$, we let $B_r(z)$ denote the open ball in $\mathbb{C}$ centered at z with radius r.

For E a Banach space, we denote $\mathcal{L}(E)$ the space of bounded operators acting on E and $\|\cdot\|_{\mathcal{L}(E)}$ the operator norm. For T in $\mathcal{L}(E)$, the notation $\sigma(T)$ stands for the spectrum of the operator T.

Lastly, we let $\mathcal{M}_n(\mathbb{C})$ denote the space of complex valued square matrices of size n and for an element M of $\mathcal{M}_n(\mathbb{C})$, the notation M^T stands for the transpose of M.

We use the notation $\lesssim$ to express an inequality up to a multiplicative constant. Eventually, we let C (resp. c) denote some large (resp. small) positive constants that may vary throughout the text (sometimes within the same line).

2.1 Introduction and main result

2.1.1 Context of the problem

We define the convolution $a * b$ of two elements a and b of $\ell^1(\mathbb{Z})$ by

$$\forall j \in \mathbb{Z}, \quad (a * b)_j := \sum_{l\in\mathbb{Z}} a_l b_{j-l}.$$

When equipped with this product, $\ell^1(\mathbb{Z})$ is a Banach algebra. For $a \in \ell^1(\mathbb{Z})$, we define the Laurent operator L_a associated with a which acts on $\ell^q(\mathbb{Z})$ for $q \in [1, +\infty]$ as

$$\forall u \in \ell^q(\mathbb{Z}), \quad L_a u := a * u \in \ell^q(\mathbb{Z}).$$

Young's inequality implies that those operators are well defined and are bounded for all $q \in [1, +\infty]$. Furthermore, we have that $L_{a*b} = L_a \circ L_b$ for $a, b \in \ell^1(\mathbb{Z})$. Finally, Wiener's theorem [New75] characterizes the invertible elements of $\ell^1(\mathbb{Z})$ and thus allows us to describe the spectrum of L_a via the Fourier series F associated with a:

$$\sigma(L_a) = \left\{ F(t) := \sum_{k\in\mathbb{Z}} a_k e^{itk}, t \in \mathbb{R} \right\}.$$

We observe that the spectrum is independent of the index q and that F is continuous since a belongs to $\ell^1(\mathbb{Z})$.

If we suppose that the sequence a has real nonnegative coefficients and $\sum_{k\in\mathbb{Z}} a_k = 1$, then the sequence $a^n := a * \ldots * a$ is the probability distribution [1] of the sum of n independent random variables supported on $\mathbb{Z}$ each with the probability distribution a. A lot is known on the pointwise asymptotic behavior of the sequence a^n in this case. In particular, the local limit theorem states, under suitable hypotheses on the sequence a, that there exists a family of functions $(q_\sigma : \mathbb{R} \to \mathbb{R})_{\sigma\in\mathbb{N}\setminus\{0,1\}}$ such that for all $s \in \mathbb{N}\setminus\{0\}$ we have the following asymptotic expansion for the elements a_j^n

$$a_j^n - \frac{1}{\sqrt{2\pi V n}} \exp\left(-\frac{X_{n,j}^2}{2} \right) - \sum_{\sigma=2}^{s} \frac{q_\sigma(X_{n,j})}{n^{\frac{\sigma}{2}}} \underset{n\to+\infty}{=} o\left(\frac{1}{n^{\frac{s}{2}}} \right) \tag{2.1.1}$$

1. We say that a sequence a is the probability distribution of a random variable Y with values in $\mathbb{Z}$ when $\mathbb{P}(Y = j) = a_j$ for all $j \in \mathbb{Z}$.

with $X_{n,j} = \frac{j-n\alpha}{\sqrt{Vn}}$ where $\alpha = \sum_{k\in\mathbb{Z}} k a_k$ and $V = \sum_{k\in\mathbb{Z}} k^2 a_k - \alpha^2$ are respectively the mean and the variance of a random variable with probability distribution a and where the error term is uniform with respect to $j \in \mathbb{Z}$ (see [Pet75, Chapter VII, Theorem 13] for more details). Furthermore, the terms in the asymptotic expansion (2.1.1) can be explicitely computed using Hermite polynomials since the functions q_σ are explicit linear combinations of derivatives of the Gaussian function $x \mapsto \exp\left(-\frac{x^2}{2}\right)$. The asymptotic expansion (2.1.1) gives a precise description of the asymptotic behavior of a_j^n in the range $|j - n\alpha| \lesssim \sqrt{n}$ and implies that the convolution powers of a are attracted towards the heat kernel.

Following, among other works, [DS14; RS15; CF22], we are interested in generalizing the local limit theorem to the case where a is complex valued. This problem is relevant for instance when one studies the large time behavior of finite difference approximation of evolution equations such as the transport equation or the heat equation (see for instance [GKO13]). Extending the works of Schoenberg [Sch53], Greville [Gre66] and Diaconis and Saloff-Coste [DS14, Theorem 2.6], the article [RS15] of Randles and Saloff-Coste already provides a generalization of the local limit theorem for a large class of complex valued finitely supported sequences. By doing so, the authors of [RS15] describe an asymptotic expansion similar to (2.1.1) for $s = 1$ and identify the leading asymptotic term (the so-called "attractors" in [RS15]). Our goal in this paper is to generalize the result of [RS15] by obtaining an asymptotic expansion similar to (2.1.1) for any $s \in \mathbb{N}$ with explicitly computable terms. We also prove a sharp rate of convergence together with a generalized Gaussian bound for the remainder of our new-found asymptotic expansion (see Theorem 2.1). In the case where a is the probability distribution of a random variable, as above, the main theorem of this paper would translate in saying that, under suitable assumptions on a (for instance that a is finitely supported with at least two nonzero elements), for all $s \in \mathbb{N}\backslash\{0\}$, there exist two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \qquad \left| a_j^n - \frac{1}{\sqrt{2\pi V n}} \exp\left(-\frac{X_{n,j}^2}{2}\right) - \sum_{\sigma=2}^{s} \frac{q_\sigma(X_{n,j})}{n^{\frac{\sigma}{2}}} \right| \leq \frac{C}{n^{\frac{s+1}{2}}} \exp\left(-c{X_{n,j}}^2\right)$$

with $X_{n,j} = \frac{j-n\alpha}{\sqrt{Vn}}$. As an example of application, these improvements on the local limit theorem allow us in the probabilistic case to prove the well-known Berry-Esseen inequality (see [Ber41; Ess42]) which states that there exists a constant $C > 0$ such that

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall J \in \mathbb{Z}, \qquad \left| \sum_{j\leq J} a_j^n - \sum_{j\leq J} \frac{1}{\sqrt{2\pi V n}} \exp\left(-\frac{|j-n\alpha|^2}{2nV}\right) \right| \leq \frac{C}{\sqrt{n}}.$$

However, we will need stronger hypotheses on the elements of $\ell^1(\mathbb{Z})$ than the conditions imposed in [RS15]. We will consider here elements a of $\ell^1(\mathbb{Z})$ which are finitely supported and such that the sequence $(a^n)_{n\in\mathbb{N}}$ is bounded in $\ell^1(\mathbb{Z})$. The fundamental contribution [Tho65] by Thomée completely characterizes such elements and is an important starting point for our work.

In the articles [DS14] and [RS15], the proofs mainly rely on the use of Fourier analysis to express the elements a_j^n via the Fourier series associated with a. In this paper, we will rather follow an approach usually referred to in partial differential equations as "spatial dynamics". It aims at using the functional calculus (see [Con90, Chapter VII]) to express the temporal Green's function (here the coefficients a_j^n) with the resolvent of the operator L_a via the spatial Green's function which is the unique solution of

$$(zId - L_a)u = \delta, \quad z \in \mathbb{C}\backslash\sigma(L_a),$$

where δ is the discrete Dirac mass $\delta := (\delta_{j,0})_{j\in\mathbb{Z}}$. This approach has already been used in [CF22] to extend the result of [DS14, Theorem 1.1] and obtain a uniform generalized Gaussian bound for the elements a_j^n. It has also been used in [CF23] to prove similar results on finite rank perturbations of Toeplitz operators (convolution operators on $\ell^q(\mathbb{N})$ rather than on $\ell^q(\mathbb{Z})$). The present paper is very much inspired by [CF22; CF23] and we will use notations and methods similar to those articles. We will now present in more details the hypotheses we need on the elements $a \in \ell^1(\mathbb{Z})$ that we shall consider and we shall then present our main theorem.

2.1.2 Hypotheses

We consider a given sequence $a \in \ell^1(\mathbb{Z})$. We let $\mathscr{L}_a$ be the bounded operator acting on $\ell^q(\mathbb{Z})$ defined as

$$\forall u \in \ell^q(\mathbb{Z}), \quad \mathscr{L}_a u := \left(\sum_{l\in\mathbb{Z}} a_l u_{j+l}\right)_{j\in\mathbb{Z}}.$$

This operator is obviously linked to Laurent operators and could be written as one of them ($\mathscr{L}_a = L_b$ for $b := (a_{-j})_{j\in\mathbb{Z}}$). Our goal will be to study the powers $\mathscr{L}_a^n$ for n large. This problem arises for instance as the large time behavior of finite difference approximations of partial differential equations and is equivalent to studying the asymptotic behavior of the coefficients of $b^n := b * \ldots * b$ as n tends to infinity. We define the symbol F associated with a as

$$\forall \kappa \in \mathbb{S}^1, \quad F(\kappa) := \sum_{j\in\mathbb{Z}} a_j \kappa^j. \tag{2.1.2}$$

The Wiener theorem [New75] allows us to conclude that the spectrum of $\mathscr{L}_a$ is given, for any $q \in [1, +\infty]$, by:

$$\sigma(\mathscr{L}_a) = F(\mathbb{S}^1).$$

We are now going to introduce some hypotheses that are necessary for the rest of the paper.

Hypothesis 2.1. *The sequence a is finitely supported and has at least two nonzero coefficients.*

Looking at the definition of the operator $\mathscr{L}_a$, in terms of applications for numerical analysis, this hypothesis translates the fact that we are only considering the case of explicit finite difference schemes. Hypothesis 2.1 implies that we can extend the definition (2.1.2) of F to the pointed plane $\mathbb{C}\backslash\{0\}$ and F becomes a holomorphic function on this domain. We introduce the two following elements

$$k_m := \min\{k \in \mathbb{Z}, \quad a_k \neq 0\}, \quad k_M := \max\{k \in \mathbb{Z}, \quad a_k \neq 0\}.$$

Observing that Hypothesis 2.1 implies $k_m < k_M$, we then distinguish three different possibilities:

— Case 1: $k_M \leq -1$. We then define $r := -k_m$ and $p := 0$.

— Case 2: $k_m \leq 0 \leq k_M$. We then define $r := -k_m$ and $p := k_M$.

— Case 3: $1 \leq k_m$. We then define $r := 0$ and $p := k_M$.

In every case, we have $r, p \in \mathbb{N}$ and $-r < p$. Also, we have that

$$\forall u \in \ell^q(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathscr{L}_a u)_j = \sum_{l=-r}^{p} a_l u_{j+l}. \tag{2.1.3}$$

The natural integers r and p we just introduced define the common stencil of the operators $\mathscr{L}_a$ and the identity operator and they will be useful to study the so-called resolvent equation (2.2.3) below. We now introduce an assumption on the Laurent series F which is based on [Tho65]. Just like in [DS14; RS15; CF22], we normalize the sequence a so that the maximum of F on $\mathbb{S}^1$ is 1.

Hypothesis 2.2. *There exists a finite set of distinct points $\{\underline{\kappa}_1, \ldots, \underline{\kappa}_K\}$, $K \geq 1$, in $\mathbb{S}^1$ such that for all $k \in \{1, \ldots, K\}$, $\underline{z}_k := F(\underline{\kappa}_k)$ belongs to $\mathbb{S}^1$ and*

$$\forall \kappa \in \mathbb{S}^1 \backslash \{\underline{\kappa}_1, \ldots, \underline{\kappa}_K\}, \quad |F(\kappa)| < 1.$$

Moreover, we suppose that for each $k \in \{1, \ldots, K\}$, there exist a nonzero real number α_k, an integer $\mu_k \geq 1$ and a complex number β_k with positive real part such that

$$F(\underline{\kappa}_k e^{i\xi}) \underset{\xi\to 0}{=} \underline{z}_k \exp(-i\alpha_k \xi - \beta_k \xi^{2\mu_k} + O(|\xi|^{2\mu_k+1})). \tag{2.1.4}$$

Geometrically, this means that the spectrum $\sigma(\mathscr{L}_a)$ is contained in the disk $\overline{\mathbb{D}}$ and it intersects $\mathbb{S}^1$ at finitely many points (see Figure 2.1 for an example with $K = 2$, $\underline{z}_1 = 1$, $\underline{z}_2 = -1$) and that the logarithm of F has a specific asymptotic expansion at those intersection points. From a general point of view, it is proved in [Tho65, Theorem 1] that Hypothesis 2.2 is one of two conditions that characterize the elements a of $\ell^1(\mathbb{Z})$ such that the geometric sequence $(a^n)_{n\in\mathbb{N}}$ is bounded in $\ell^1(\mathbb{Z})$. In the more specific field of numerical analysis, the condition (2.1.4) has been studied closely because of its link with the stability of finite difference approximations in the maximum norm (see [Tho65]). We can observe that, under Hypotheses 2.1 and 2.2, there holds

$$\forall n \in \mathbb{N}\backslash\{0\}, \quad \|\mathscr{L}_a^n\|_{\mathcal{L}(\ell^2(\mathbb{Z}))} = \|F^n\|_{L^\infty(\mathbb{S}^1)} = 1.$$

It assures the ℓ^2-stability, or strong stability (see [Str68], [Tad86]), of the numerical scheme defined as

$$\begin{cases} u^{n+1} = \mathscr{L}_a u^n, \quad n \geq 0, \\ u^0 \in \ell^2(\mathbb{Z}). \end{cases} \tag{2.1.5}$$

However, it has further consequences, as the asymptotic expansion (2.1.4) assures the ℓ^q-stability of the scheme (2.1.5) for every q in $[1, +\infty]$ (see [Tho65, Theorem 1] which focuses on the ℓ^∞-stability but also studies the

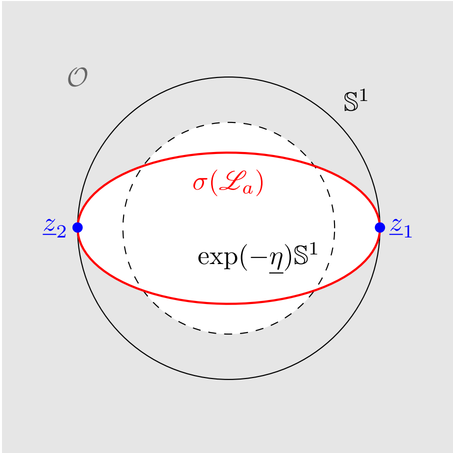

Figure 2.1 – An example of spectrum $\sigma(\mathscr{L}_a)$. The spectrum $\sigma(\mathscr{L}_a)$ (in red) is inside the closed disk $\bar{\mathbb{D}}$ and touches the boundary $\mathbb{S}^1$ in finitely many points. In gray, we have the set $\mathcal{O}$ defined below in Section 2.2 which is the intersection of the unbounded connected component of $\mathbb{C}\backslash\sigma(\mathscr{L}_a)$ and $\left\{z \in \mathbb{C}, |z| > \exp(-\underline{\eta})\right\}$.

ℓ^q-stability as a consequence). In terms of numerical scheme, the meaning of (2.1.4) is that the numerical scheme introduces an artificial numerical diffusion (like the Lax-Friedrichs scheme for example).

We now introduce yet another hypothesis.

Hypothesis 2.3. *For all $k \in \{1, \dots, K\}$, the set*

$$\mathcal{I}_k := \{\nu \in \{1, \dots, K\}, \quad \underline{z}_\nu = \underline{z}_k\}$$

has either one or two elements, where we recall that $\underline{z}_\nu := F(\underline{\kappa}_\nu)$. Moreover, if there are two distinct elements $\nu_{k,1}$ and $\nu_{k,2}$ in $\mathcal{I}_k$, then $\alpha_{\nu_{k,1}}\alpha_{\nu_{k,2}} < 0$.

Hypothesis 2.3 will simplify part of the analysis when we will study the spatial Green's function defined in (2.2.3). It will allow us to study precisely the spectrum of the matrix $\mathbb{M}(z)$ defined below as (2.2.2) near the tangency points $\underline{z}_k$. Combining Hypothesis 2.3 with the fact that the α_k's are nonzero real numbers (see Hypothesis 2.2) implies that, for $k \in \{1, \dots, K\}$, we have three different possibilities:

— **Case I:** $\mathcal{I}_k$ is the singleton $\{k\}$ and $\alpha_k > 0$,

— **Case II:** $\mathcal{I}_k$ is the singleton $\{k\}$ and $\alpha_k < 0$,

— **Case III:** $\mathcal{I}_k$ has two distinct elements $\nu_{k,1}$ and $\nu_{k,2}$ such that $\alpha_{\nu_{k,1}} > 0$ and $\alpha_{\nu_{k,2}} < 0$.

Distinguishing between those three cases will be useful later on. The three hypotheses we presented above will be crucial in the rest of the paper. Some hypotheses might be relaxable, but this would be considerations for future works.

Finally, by defining the discrete Dirac mass $\delta := (\delta_{j,0})_{j\in\mathbb{Z}}$, we introduce the so-called temporal Green's function defined by

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad \mathscr{G}_j^n := (\mathscr{L}_a^n \delta)_j. \tag{2.1.6}$$

It is interesting to observe that the equality between the operator $\mathscr{L}_a$ and the Laurent operator L_b with $b = (a_{-j})_{j\in\mathbb{Z}}$ implies that

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad \mathscr{G}_j^n = b_j^n$$

where $b^n = b * \dots * b$.

2.1.3 Main results and comparison to previous results

Our main goal is to determine the asymptotic behavior of $\mathscr{G}_j^n$ when n becomes large. The identification of the leading asymptotic term was achieved in [RS15, Theorem 1.2]. We aim here at extending the result of [RS15, Theorem 1.2] into a complete asymptotic expansion up to any order and at proving sharp bounds for the remainder. To express the asymptotic expansion of $\mathscr{G}_j^n$, we introduce the functions $H_{2\mu}^\beta : \mathbb{R} \to \mathbb{C}$, where $\mu \in \mathbb{N}\backslash\{0\}$ and $\beta \in \mathbb{C}$ has positive real part, which are defined as

$$\forall x \in \mathbb{R}, \quad H_{2\mu}^\beta(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixu}e^{-\beta u^{2\mu}}du.$$

We call those functions generalized Gaussians since for $\mu = 1$, we have

$$\forall x \in \mathbb{R}, \quad H_2^\beta(x) = \frac{1}{\sqrt{4\pi\beta}} e^{-\frac{x^2}{4\beta}}.$$

Let us state the main result of this paper.

Theorem 2.1. *Let $a \in \ell^1(\mathbb{Z})$ which verifies Hypotheses 2.1, 2.2 and 2.3. Then, for all integers $s_1, \dots, s_K \in \mathbb{N}$ there exist a family of polynomials $(\mathscr{P}_\sigma^k)_{\sigma \in \{1,\dots,s_k\}}$ in $\mathbb{C}[X,Y]$ for each $k \in \{1, \dots, K\}$ and two positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$ and $j \in \mathbb{Z}$, there holds:*

$$\left| \mathscr{G}_j^n - \sum_{k=1}^K \sum_{\sigma=1}^{s_k} \frac{z_k^n \kappa_k^j}{n^{\frac{\sigma}{2\mu_k}}} \left(\mathscr{P}_\sigma^k \left(X_{n,j,k}, \frac{d}{dx} \right) H_{2\mu_k}^{\beta_k} \right) (X_{n,j,k}) \right| \leq \sum_{k=1}^K \frac{C}{n^{\frac{s_k+1}{2\mu_k}}} \exp\left(-c |X_{n,j,k}|^{\frac{2\mu_k}{2\mu_k - 1}} \right) \tag{2.1.7}$$

where $X_{n,j,k} = \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$.

Theorem 2.1 gives the asymptotic behavior of the elements $\mathscr{G}_j^n$ up to any order with a sharp generalized Gaussian estimate of the remainder. We would also like to point out that the proof of Theorem 2.1 (mainly Propositions 2.2, 2.3 and equality (2.3.7)) gives us an explicit expression of the polynomials $\mathscr{P}_\sigma^k$ of Theorem 2.1. Examples are provided in Section 2.5 where we compute these polynomials for $\sigma = 1, 2$. In the same section, we will also numerically verify the claim of Theorem 2.1 for some sequences a.

The following lemma, which is proved using integration by parts, implies that we cannot prove the uniqueness of the polynomials $\mathscr{P}_\sigma^k$ of Theorem 2.1.

Lemma 2.1.1. *For $\mu \in \mathbb{N}\backslash\{0\}$, $\beta \in \mathbb{C}$ with positive real part and $m \in \mathbb{N}\backslash\{0\}$, we have*

$$\forall x \in \mathbb{R}, \quad x {H_{2\mu}^\beta}^{(m)}(x) = (-1)^\mu 2\mu\beta {H_{2\mu}^\beta}^{(m+2\mu-1)}(x) - m {H_{2\mu}^\beta}^{(m-1)}(x),$$

and

$$\forall x \in \mathbb{R}, \quad x H_{2\mu}^\beta(x) = (-1)^\mu 2\mu\beta {H_{2\mu}^\beta}^{(2\mu-1)}(x).$$

In other words, one can either choose to multiply $H_{2\mu}^\beta$ by a polynomial or to differentiate it sufficiently many times. Hence, there may hold

$$P(\cdot, \frac{d}{dx}) H_{2\mu}^\beta = 0$$

for a nonzero $P \in \mathbb{C}[X,Y]$.

In our proof of Theorem 2.1, the polynomials $\mathscr{P}_\sigma^k$ depend on the chosen integers $s_1, \dots, s_k$. It might be possible to prove the existence of a family of polynomials $\left(\mathscr{P}_\sigma^k\right)_{(k,\sigma)\in\{1,\dots,K\}\times\mathbb{N}\backslash\{0\}}$ in $\mathbb{C}[X,Y]$ for which the estimates (2.1.7) are verified for all $s_1, \dots, s_K \in \mathbb{N}$. However, we do not yet have a proof of this fact in full generality. We now compare Theorem 2.1 with prior results:

• In the probabilistic case presented in the introduction, Theorem 2.1 allows us to prove sharp bounds with Gaussian estimates on the remainder of the asymptotic expansion of $\mathscr{G}_j^n$ that were not proved via the asymptotic expansion (2.1.1) of the local limit theorem.

• [DS14, Theorem 3.1] gives sharp generalized Gaussian estimates for the elements $\mathscr{G}_j^n$ when the sequence a satisfies Hypotheses 2.1, 2.2 and 2.3 with a single tangency point (i.e. $K = 1$), which in comparison to Theorem 2.1 would match the case $s_k = 0$. [CF22, Theorem 1.6] generalizes those generalized Gaussian estimates for sequences a with any number $K \in \mathbb{N}\backslash\{0\}$ of tangency points and a relaxed Hypothesis 2.1. Theorem 2.1 thus improves those results by proving similar sharp generalized Gaussian estimates for the remainder of the asymptotic expansion of the elements $\mathscr{G}_j^n$ up to any order $s_1, \dots, s_K \in \mathbb{N}$.

• For a sequence $a \in \ell^1(\mathbb{Z})$ which satisfies Hypotheses 2.1 and 2.2, we introduce the the so-called "attractors":

$$\forall k \in \{1, \dots, K\}, \forall n \in \mathbb{N}\backslash\{0\}, \quad \mathscr{H}_k^n = \left(\mathscr{H}_{k,j}^n\right)_{j\in\mathbb{Z}} := \left(\frac{{z_k}^n {\kappa_k}^j}{n^{\frac{1}{2\mu_k}}} H_{2\mu_k}^{\beta_k} \left(\frac{j - n\alpha_k}{n^{\frac{1}{2\mu_k}}} \right) \right)_{j\in\mathbb{Z}}.$$

In [RS15, Theorem 1.2], it is proved that if we introduce

$$\mu = \max_{k\in\{1,\dots,K\}} \mu_k \quad \text{and} \quad \mathcal{K}_\mu = \{k \in \{1, \dots, K\}, \mu_k = \mu\},$$

then

$$\mathscr{G}_j^n - \sum_{k\in\mathcal{K}_\mu} \mathscr{H}_{k,j}^n \underset{n\to+\infty}{=} o\left(\frac{1}{n^{\frac{1}{2\mu}}} \right) \tag{2.1.8}$$

where the error term in (2.1.8) is uniform on $\mathbb{Z}$. Compared to Theorem 2.1, this is equivalent to finding the asymptotic expansion up to order $s_1, \dots, s_K = 1$. The result of Randles and Saloff-Coste gives a precise description of the behavior of $\mathscr{G}_j^n$ for j such that

$$|j - n\alpha_k| \lesssim n^{\frac{1}{2\mu}}, \tag{2.1.9}$$

where $k \in \mathcal{K}_\mu$. Theorem 2.1 allows us to extend the result of [RS15] by going even farther in the asymptotic expansion of the elements $\mathscr{G}_j^n$, and proving sharp generalized Gaussian bounds on the remainder with a more precise speed of convergence. As a consequence, one can prove using Theorem 2.1 that there exists a positive constant C such that for all initial condition $u^0 \in \ell^2(\mathbb{Z})$, the solution of the numerical scheme (2.1.5) verifies

$$\forall n \in \mathbb{N}\backslash\{0\}, \quad \left\| u^n - \sum_{k=1}^K \mathscr{H}_k^n * u^0 \right\|_{\ell^2(\mathbb{Z})} \leq \frac{C}{n^{\frac{1}{2\mu}}} \left\| u^0 \right\|_{\ell^2(\mathbb{Z})}.$$

We have thus identified the main behavior of the solutions of the numerical scheme (2.1.5) as the time n becomes large. Such a result was not attainable using only [RS15, Theorem 1.2] (or even [CF22, Theorem 1.6]).

However, [RS15, Theorem 1.2] also treats the case where the asymptotic expansion (2.1.4) has the form

$$F(\underline{\kappa}_k e^{i\xi}) \underset{\xi \to 0}{=} \underline{z}_k \exp(-i\alpha_k \xi + i\gamma_k \xi^{\nu_k} + O(|\xi|^{\nu_k + 1})),$$

where γ_k is a real number and the integer $\nu_k \in \mathbb{N}\backslash\{0,1\}$ can be even or odd. A generalization of Theorem 2.1 in this difficult case has not yet been found, even though the result of [Cou22] indicates that such a result might be attainable.

2.1.4 Extending the result when the drift vanishes

As we have seen, Theorem 2.1 allows us to generalize the local limit theorem for complex valued sequences but it still has some limits. Relaxing some of the hypotheses we introduced could be interesting and theoretically doable in some cases. For instance, Theorem 2.1 is constrained by Hypothesis 2.2 which imposes that α_k is nonzero even though the result [RS15, Theorem 1.2] does not have this kind of restriction. The hypothesis $\alpha_k \neq 0$ is essential in the proof of Theorem 2.1 below but it seems to be a technical hypothesis that we would want to avoid. The following corollary will allow us to extend Theorem 2.1 to some sequences a for which we allow α_k to be equal to 0. First, we introduce a relaxed version of Hypothesis 2.2.

Hypothesis 2.4 (Hypothesis 2 bis)**.** *The sequence a verifies Hypothesis 2.2 but with the possibility that some α_k are equal to* 0.

We now consider a finitely supported sequence $a \in \ell^1(\mathbb{Z})$ which verifies Hypothesis 2.4 and let $J \in \mathbb{Z}$. Then, if we define the sequence $b = (a_{j+J})_{j\in\mathbb{Z}}$ and $\widetilde{F}$ the symbol associated with b, we have that b satisfies Hypothesis 2.4 since

$$\forall \kappa \in \mathbb{S}^1, \quad \widetilde{F}(\kappa) = \kappa^{-J} F(\kappa),$$

and therefore

$$\forall \kappa \in \mathbb{S}^1, \quad \left| \widetilde{F}(\kappa) \right| = |F(\kappa)|.$$

Also, we have for $k \in \{1, \dots, K\}$

$$\widetilde{F}(\underline{\kappa}_k e^{i\xi}) \underset{\xi \to 0}{=} \underline{\kappa}_k^{-J} \underline{z}_k \exp(-i(\alpha_k + J)\xi - \beta_k \xi^{2\mu_k} + o(|\xi|^{2\mu_k})).$$

Considering this new sequence b allows us to "shift" the elements α_k. In particular, if we choose J large enough, then b satisfies Hypothesis 2.2. However, it is not clear that the sequence b would satisfy Hypothesis 2.3. We can then prove the following corollary which generalizes Theorem 2.1 in the case where α_k can be equal to 0.

Corollary 1. *Let $a \in \ell^1(\mathbb{Z})$ which verifies Hypotheses 2.1 and 2.4. If there exists some integer $J \in \mathbb{Z}$ such that the sequence $(a_{j+J})_{j\in\mathbb{Z}}$ verifies Hypotheses 2.2 and 2.3, then for all $s_1, \dots, s_K \in \mathbb{N}$ there exist a family of polynomials $(\mathscr{P}_\sigma^k)_{\sigma\in\{1,\dots,s_k\}}$ in $\mathbb{C}[X,Y]$ for each $k \in \{1, \dots, K\}$ and two positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$ and $j \in \mathbb{Z}$*

$$\left| \mathscr{G}_j^n - \sum_{k=1}^K \sum_{\sigma=1}^{s_k} \frac{\underline{z}_k^n \underline{\kappa}_k^j}{n^{\frac{\sigma}{2\mu_k}}} \left(\mathscr{P}_\sigma^k \left(X_{n,j,k}, \frac{d}{dx} \right) H_{2\mu_k}^{\beta_k} \right) (X_{n,j,k}) \right| \leq \sum_{k=1}^K \frac{C}{n^{\frac{s_k+1}{2\mu_k}}} \exp\left(-c |X_{n,j,k}|^{\frac{2\mu_k}{2\mu_k - 1}} \right)$$

with $X_{n,j,k} = \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$.

We prove Corollary 1 in Section 2.4.2.

2.1.5 Plan of the paper

The main goal of the paper is the proof of Theorem 2.1. As explained in the introduction, the proof of Theorem 2.1 will rely on an approach referred to as spatial dynamics. In Section 2.2, we will introduce the spatial Green's function on which Coulombel and Faye proved holomorphic extension properties and sharp bounds in [CF22, Section 2]. Our goal in Section 2.2 is to improve the analysis of [CF22] and to obtain the precise behavior of the spatial Green's function for z close to $\underline{z}_k$ and to prove sharp bounds on the remainder. More precisely, the main novelty of this section is the introduction of the explicit function f_k in Lemmas 2.2.4 and 2.2.5 which allows us to properly describe the spatial Green's function for z close to $\underline{z}_k$.

In Section 2.3, we prove Theorem 2.1 while assuming that the elements α_k are distinct. This assumption will allow us to separate the different Gaussian waves in the estimate (2.1.7). Section 2.3.1 will be dedicated to the easier part of the proof which is proving estimate (2.1.7) when j is far from the axes $j = n\alpha_k$. The bulk of the proof resides in Sections 2.3.2-2.3.5 which will be dedicated to proving estimate (2.1.7) when j is close to the axes $j = n\alpha_k$. In Section 2.3.3, we will express the elements $\mathscr{G}_j^n$ with the spatial Green's function using functional calculus. We will then use the results of Section 2.2 on the spatial Green's function to prove generalized Gaussian estimates on the difference of the elements $\mathscr{G}_j^n$ and a linear combination of terms of the form

$$\frac{1}{\left(\frac{j}{\alpha_k}\right)^{\frac{l}{2\mu_k}}} {H_{2\mu_k}^{\beta_k}}^{(m)}(Y_{n,j,k}) \quad \text{where } l \in \mathbb{N}\backslash\{0\}, m \in \mathbb{N}, Y_{n,j,k} := \frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}. \tag{2.1.10}$$

Keeping in mind that we are considering the case where j is close to $n\alpha_k$, Section 2.3.4 will deal with approaching the terms (2.1.10) with linear combinations of the following terms appearing in Theorem 2.1:

$$\frac{1}{n^{\frac{l}{2\mu_k}}} (X_{n,j,k})^{m_2} {H_{2\mu_k}^{\beta_k}}^{(m_1)}(X_{n,j,k}) \quad \text{where } l \in \mathbb{N}\backslash\{0\}, m_1, m_2 \in \mathbb{N}, X_{n,j,k} := \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}.$$

Section 2.3.5 will combine the results of the previous sections to conclude the proof of Theorem 2.1 by constructing the polynomials $\mathscr{P}_\sigma^k$.

In Section 2.4, we prove Theorem 2.1 when the elements α_k can be equal. We also prove Corollary 1.

Finally, in Section 2.5, we will explicitly compute the polynomials $\mathscr{P}_\sigma^k$ of Theorem 2.1 for $\sigma = 1, 2$ for any $s_k \in \mathbb{N}\backslash\{0,1\}$ and numerically verify the estimate (2.1.7) of Theorem 2.1 in two cases. The first one is the probabilistic case, i.e. a sequence a with non negative coefficients. We will compare the result of Theorem 2.1 with the local limit theorem. The second example will be the sequence a associated with the so-called O3 scheme for the transport equation (see [Des08]). This is an example of sequence a where $\mu = 2$ in the asymptotic expansion (2.1.4).

2.2 Spatial Green's function

From now on, we consider a sequence a that satisfies Hypotheses 2.1, 2.2 and 2.3. In this section, we are going to introduce the spatial Green's function and prove some estimates for it. We will start by defining the necessary objects for our study. First, we can observe the following lemma for which the proof can be found in the Appendix (Section 2.A).

Lemma 2.2.1. *For $a \in \ell^1(\mathbb{Z})$ which verifies Hypotheses 2.1 and 2.2, we have that a_{-r} and a_p belong to $\mathbb{D}$.*

We define for $z \in \mathbb{C}$ and $j \in \{-r, \dots, p\}$

$$\mathbb{A}_j(z) := z\delta_{j,0} - a_j. \tag{2.2.1}$$

The definition of r and p implies that the functions $\mathbb{A}_{-r}$ and $\mathbb{A}_p$ can vanish at most on one point which are respectively a_{-r} and a_p. Lemma 2.2.1 allows us to find $\underline{\eta} > 0$ such that $\mathbb{A}_{-r}$ and $\mathbb{A}_p$ do not vanish on $\{z \in \mathbb{C}, |z| > \exp(-\underline{\eta})\}$. We can therefore define for all $z \in \mathbb{C}$ such that $|z| > \exp(-\underline{\eta})$ the matrix

$$\mathbb{M}(z) := \begin{pmatrix} -\frac{\mathbb{A}_{p-1}(z)}{\mathbb{A}_p(z)} & \dots & \dots & -\frac{\mathbb{A}_{-r}(z)}{\mathbb{A}_p(z)} \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{M}_{p+r}(\mathbb{C}). \tag{2.2.2}$$

The application which associates z with $\mathbb{M}(z)$ is holomorphic on the annulus $\{z \in \mathbb{C}, |z| > \exp(-\underline{\eta})\}$. Moreover, since $\mathbb{A}_{-r}(z) \neq 0$, the upper right coefficient of $\mathbb{M}(z)$ is always nonzero and $\mathbb{M}(z)$ is invertible. We define the open set $\mathcal{O}$ which corresponds to the intersection of the unbounded connected component of $\mathbb{C}\backslash F(\mathbb{S}^1)$ and $\{z \in \mathbb{C}, |z| > \exp(-\underline{\eta})\}$ (see Figure 2.1). Hypothesis 2.2 implies that $\overline{\mathcal{U}}\backslash \{\underline{z}_1, \dots, \underline{z}_K\}$ is contained within $\mathcal{O}$. By recalling that $\sigma(\mathscr{L}_a) = F(\mathbb{S}^1)$, when we consider that the operator $\mathscr{L}_a$ acts on $\ell^2(\mathbb{Z})$, we have the existence for every $z \in \mathcal{O}$ of a unique sequence $G(z) := (G_j(z))_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$(zI - \mathscr{L}_a)G(z) = \delta, \tag{2.2.3}$$

where δ still denotes the discrete Dirac mass. The sequence $G(z)$ is the so-called spatial Green's function which has already been studied in [CF22]. In [CF22, Lemma 2.2], we can find a proof of local sharp exponential bounds on $G_j(z)$ when $z \in \mathcal{O}$ is far from the tangency points $\underline{z}_k$. This bound will be sufficient for our purpose. Furthermore, in [CF22, Lemmas 2.3 and 2.4], the authors proved that the spatial Green's function evaluated at some index $G_j(z)$ could be holomorphically extended near the points $\underline{z}_k$ through the spectrum of the operator $\mathscr{L}_a$ which is not immediate based on the definition (2.2.3) of the spatial Green's function and they proved sharp bounds on $G_j(z)$ in this case. To prove Theorem 2.1, we will need to get a more precise description of the behavior of the function $G_j(z)$ close to any tangency point $\underline{z}_k$. This section will therefore follow [CF22, Section 2] and make it more precise by specifying where our study of the sequence $G(z)$ differs from [CF22, Section 2].

Using the functions $\mathbb{A}_l$ which are defined by (2.2.1), the equation (2.2.3) can be rewritten as

$$\forall z \in \mathcal{O}, \forall j \in \mathbb{Z}, \quad \sum_{l=-r}^{p} \mathbb{A}_l(z) G_{j+l}(z) = \delta_{j,0}.$$

We introduce the vectors

$$\forall z \in \mathcal{O}, \forall j \in \mathbb{Z}, \quad W_j(z) := \begin{pmatrix} G_{j+p-1}(z) \\ \vdots \\ G_{j-r}(z) \end{pmatrix} \in \mathbb{C}^{p+r}, \quad e := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{p+r}.$$

We then end up with the following dynamical system

$$\forall z \in \mathcal{O}, \forall j \in \mathbb{Z}, \quad W_{j+1}(z) - \mathbb{M}(z) W_j(z) = \frac{\delta_{j,0}}{\mathbb{A}_p(z)} e. \tag{2.2.4}$$

The study of the recurrence relation (2.2.4) relies on the following lemma introduced in [Kre68] that studies the eigenvalues of $\mathbb{M}(z)$ for $z \in \mathcal{O}$ and $z \in \{\underline{z}_k, 1 \leq k \leq K\}$. We recall that we defined cases **I**, **II** and **III** according to the cardinality of $\mathcal{I}_k$ and the sign of α_k right after Hypothesis 2.3. We also recall that we consider that the sequence a verifies Hypotheses 2.1, 2.2 and 2.3.

Lemma 2.2.2 (Spectral Splitting). *For $z \in \mathbb{C}$ such that $|z| > \exp(-\underline{\eta})$, the eigenvalues $\kappa \in \mathbb{C}$ of the matrix $\mathbb{M}(z)$ are nonzero and satisfy the equality*

$$F(\kappa) = z.$$

Let $z \in \mathcal{O}$. Then the matrix $\mathbb{M}(z)$ has
- *no eigenvalue on $\mathbb{S}^1$,*
- *r eigenvalues in $\mathbb{D}\backslash\{0\}$ (that we call stable eigenvalues),*
- *p eigenvalues in $\mathcal{U}$ (that we call unstable eigenvalues).*

We now consider $k \in \{1, \dots, K\}$. The eigenvalues of the matrix $\mathbb{M}(\underline{z}_k)$ are described by the following possibilities depending on k.
- *In case **I**, $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_k \in \mathbb{S}^1$ as a simple eigenvalue, $r-1$ eigenvalues in $\mathbb{D}$ and p eigenvalues in $\mathcal{U}$.*
- *In case **II**, $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_k \in \mathbb{S}^1$ as a simple eigenvalue, r eigenvalues in $\mathbb{D}$ and $p-1$ eigenvalues in $\mathcal{U}$.*
- *In case **III**, if we denote $\nu_{k,1}$ and $\nu_{k,2}$ the two distinct elements of $\mathcal{I}_k$, then $\mathbb{M}(\underline{z}_k)$ has $\underline{\kappa}_{\nu_{k,1}} \in \mathbb{S}^1$ and $\underline{\kappa}_{\nu_{k,2}} \in \mathbb{S}^1$ as simple eigenvalues, $r-1$ eigenvalues in $\mathbb{D}$ and $p-1$ eigenvalues in $\mathcal{U}$.*

Lemma 2.2.2 is proved in [CF22, Lemma 2.1] and is the key to study the recurrence relation (2.2.4). We now want to prove some estimates on the spatial Green's function $G(z)$. We recall that the set $\mathcal{O}$ is the intersection of the set $\{z \in \mathbb{C}, |z| \geq \exp(-\underline{\eta})\}$, where the matrix $\mathbb{M}(z)$ is defined, and the set $\sigma(\mathscr{L}) = \mathbb{C}\backslash F(\mathbb{S}^1)$, where the spatial Green's function $G(z)$ is defined. We begin with the following lemma.

Lemma 2.2.3 (Bounds far from the tangency points [CF22]). *For all $\underline{z} \in \mathcal{O}$, there exist a radius $\delta > 0$ and* *constants $C, c > 0$ such that for all $j \in \mathbb{Z}$, $z \mapsto G_j(z)$ is holomorphic on $B_\delta(\underline{z})$ and satisfies*

$$\forall z \in B_\delta(\underline{z}), \forall j \in \mathbb{Z}, \quad |G_j(z)| \leq C \exp(-c|j|).$$

Lemma 2.2.3 is proved in [CF22, Lemma 2.2] and allows us to study the spatial Green's function far from the points $\underline{z}_k$, where the spectrum of $\mathscr{L}_a$ intersects the unit circle $\mathbb{S}^1$. We will now have to study the spatial Green's function $G(z)$ near those points $\underline{z}_k$ while still remembering that $G_j(z)$ and the vector $W_j(z)$ are only defined on $\mathcal{O}$ in the neighborhood of $\underline{z}_k$. We are going to extend holomorphically $G_j(z)$ in a whole neighborhood of $\underline{z}_k$, and thus pass through the spectrum $\sigma(\mathscr{L}_a)$.

Lemma 2.2.4 (Bounds close to the tangency points : cases **I** and **II**)**.** *Let $k \in \{1, \dots, K\}$ so that we are either in case **I** or **II**. Then, there exist a radius $\varepsilon > 0$, some constants $C, c > 0$ and some holomorphic functions $\kappa_k, f_k : B_\varepsilon(\underline{z}_k) \to \mathbb{C}$ such that for all $z \in B_\varepsilon(\underline{z}_k)$, $\kappa_k(z)$ is a simple eigenvalue of $\mathbb{M}(z)$ with $\kappa_k(\underline{z}_k) = \underline{\kappa}_k$, for all $j \in \mathbb{Z}$, the function $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{O} \mapsto G_j(z)$ can be holomorphically extended on $B_\varepsilon(\underline{z}_k)$ and*

Case I: *($\alpha_k > 0$)*

$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \geq 1, \quad |G_j(z) - f_k(z)\kappa_k(z)^j| \leq C \exp(-cj). \tag{2.2.5}$$
$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \leq 0, \quad |G_j(z)| \leq C \exp(-c|j|). \tag{2.2.6}$$

Case II: *($\alpha_k < 0$)*

$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \geq 1, \quad |G_j(z)| \leq C \exp(-cj). \tag{2.2.7}$$
$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \leq 0, \quad |G_j(z) - f_k(z)\kappa_k(z)^j| \leq C \exp(-c|j|). \tag{2.2.8}$$

Furthermore, we have

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad f_k(z) = -\mathrm{sgn}(\alpha_k)\frac{\kappa_k'(z)}{\kappa_k(z)}. \tag{2.2.9}$$

Lemma 2.2.5 (Bounds close to the tangency points : case **III**)**.** *Let $k \in \{1, \dots, K\}$ so that we are in case **III**. The set $\mathcal{I}_k$ has two elements $\nu_{k,1}$ and $\nu_{k,2}$ so that $\alpha_{\nu_{k,1}} > 0$ and $\alpha_{\nu_{k,2}} < 0$. Then, there exist a radius $\varepsilon > 0$, some constants $C, c > 0$ and some holomorphic functions $\kappa_{\nu_{k,1}}, \kappa_{\nu_{k,2}}, f_{\nu_{k,1}}, f_{\nu_{k,2}} : B_\varepsilon(\underline{z}_k) \to \mathbb{C}$ such that for all $z \in B_\varepsilon(\underline{z}_k)$, $\kappa_{\nu_{k,1}}(z)$ and $\kappa_{\nu_{k,2}}(z)$ are simple eigenvalues of $\mathbb{M}(z)$ with $\kappa_{\nu_{k,1}}(\underline{z}_k) = \underline{\kappa}_{\nu_{k,1}}$ and $\kappa_{\nu_{k,2}}(\underline{z}_k) = \underline{\kappa}_{\nu_{k,1}}$, for all $j \in \mathbb{Z}$, the function $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{O} \mapsto G_j(z)$ can be holomorphically extended on $B_\varepsilon(\underline{z}_k)$ and*

$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \geq 1, \quad |G_j(z) - f_{\nu_{k,1}}(z)\kappa_{\nu_{k,1}}(z)^j| \leq C \exp(-cj). \tag{2.2.10}$$
$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \leq 0, \quad |G_j(z) - f_{\nu_{k,2}}(z)\kappa_{\nu_{k,2}}(z)^j| \leq C \exp(-c|j|). \tag{2.2.11}$$

Furthermore, knowing that $\underline{z}_k = \underline{z}_{\nu_{k,1}} = \underline{z}_{\nu_{k,2}}$, we have that

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad f_{\nu_{k,1}}(z) = -\frac{\kappa_{\nu_{k,1}}'(z)}{\kappa_{\nu_{k,1}}(z)}, \quad f_{\nu_{k,2}}(z) = \frac{\kappa_{\nu_{k,2}}'(z)}{\kappa_{\nu_{k,2}}(z)}. \tag{2.2.12}$$

Lemmas 2.2.4 and 2.2.5 are similar to [CF22, Lemmas 2.3 and 2.4] but instead of proving sharp bounds on the spatial Green's function, we express its precise behavior near the points $\underline{z}_k$. This is the crucial improvement with respect to [CF22] that will allow us to find their asymptotic behavior and prove a sharp bound for the remainder.

Proof of Lemma 2.2.4 Our proof will follow that of [CF22, Lemmas 2.3, 2.4]. First, we observe that case **II** would be dealt similarly as case **I** and that case **III** is a mixture of both cases **I** and **II**. Therefore, we will only detail the proof of Lemma 2.2.4 in case **I** and leave the proof of Lemma 2.2.5 to the interested reader. We therefore consider $k \in \{1, \dots, K\}$ so that we are in case **I**. Lemma 2.2.2 implies that $\underline{\kappa}_k$ is a simple eigenvalue of $\mathbb{M}(\underline{z}_k)$. Thus, we can find a holomorphic function κ_k defined on a neighborhood $B_\varepsilon(\underline{z}_k)$ of $\underline{z}_k$ such that for all $z \in B_\varepsilon(\underline{z}_k)$, $\kappa_k(z)$ is an algebraically simple eigenvalue of $\mathbb{M}(z)$ and $\kappa_k(\underline{z}_k) = \underline{\kappa}_k$. We also know that for all $z \in B_\varepsilon(\underline{z}_k)$, the vector

$$R_k(z) := \begin{pmatrix} \kappa_k(z)^{p+r-1} \\ \vdots \\ \kappa_k(z) \\ 1 \end{pmatrix} \in \mathbb{C}^{p+r}$$

is an eigenvector of $\mathbb{M}(z)$ associated with $\kappa_k(z)$. Because of Lemma 2.2.2, even if we have to take a smaller radius ε, we can assume that for all $z \in B_\varepsilon(\underline{z}_k)$, $\mathbb{M}(z)$ has $\kappa_k(z)$ as a simple eigenvalue, $r-1$ eigenvalues different from $\kappa_k(z)$ in $\mathbb{D}$ and p eigenvalues different from $\kappa_k(z)$ in $\mathcal{U}$. We define $E^s(z)$ (resp. $E^u(z)$) the strictly stable (resp. strictly unstable) subspace of $\mathbb{M}(z)$ which corresponds to the subspace spanned by the generalized eigenvectors of $\mathbb{M}(z)$ associated with eigenvalues different from $\kappa_k(z)$ in $\mathbb{D}$ (resp. $\mathcal{U}$). We therefore know that $E^s(z)$ (resp. $E^u(z)$) has dimension $r-1$ (resp. p) thanks to Lemma 2.2.2 and we have the decomposition

$$\mathbb{C}^{p+r} = E^s(z) \oplus E^u(z) \oplus \text{ Span } R_k(z).$$

The associated projectors are denoted $\pi^s(z)$, $\pi^u(z)$ and $\pi^k(z)$. Those linear maps commute with $\mathbb{M}(z)$ and depend holomorphically on $z \in B_\varepsilon(\underline{z}_k)$ (see [Kat95, I. Problem 5.9]).

For all $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{O}$ and $j \in \mathbb{Z}$, $G_j(z)$ and the vector $W_j(z)$ are well defined. Also, by Lemma 2.2.2, we have that $|\kappa_k(z)| < 1$ for all $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{O}$. By reasoning in the same manner as in the proof of [CF22, Lemma 2.3], we have for all $z \in B_\varepsilon(\underline{z}_k) \cap \mathcal{O}$ and $j \in \mathbb{Z}$

$$\pi^u(z)W_j(z) = -\frac{\mathbb{1}_{j\in]-\infty,0]}}{\mathbb{A}_p(z)}\mathbb{M}(z)^{j-1}\pi^u(z)e, \tag{2.2.13}$$

$$\pi^s(z)W_j(z) = \frac{\mathbb{1}_{j\in[1,+\infty[}}{\mathbb{A}_p(z)}\mathbb{M}(z)^{j-1}\pi^s(z)e, \tag{2.2.14}$$

$$\pi^k(z)W_j(z) = \frac{\mathbb{1}_{j\in[1,+\infty[}}{\mathbb{A}_p(z)}\mathbb{M}(z)^{j-1}\pi^k(z)e = \frac{\mathbb{1}_{j\in[1,+\infty[}}{\mathbb{A}_p(z)}\kappa_k(z)^{j-1}\pi^k(z)e. \tag{2.2.15}$$

We observe that the right hand side in the equations (2.2.13), (2.2.14) and (2.2.15) can be holomorphically extended on $B_\varepsilon(\underline{z}_k)$. Therefore, we can extend holomorphically the applications which associates z to $\pi^s(z)W_j(z)$, $\pi^u(z)W_j(z)$ and $\pi^k(z)W_j(z)$ on the whole open ball $B_\varepsilon(\underline{z}_k)$ and this allows us to extend $W_j(z)$ on $B_\varepsilon(\underline{z}_k)$. Since $G_j(z)$ is a coordinate of the vector $W_j(z)$, the holomorphic extension property is proved.

By reasoning in the same manner as in the proof of the inequality [CF22, (2.12)], we prove that there exist two constants $C, c > 0$ such that

$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \in \mathbb{Z}, \quad \|\pi^s(z)W_j(z) + \pi^u(z)W_j(z)\| \leq C\exp(-c|j|).$$

This implies that

$$\forall z \in B_\varepsilon(\underline{z}_k), \forall j \in \mathbb{Z}, \quad \|W_j(z) - \pi^k(z)W_j(z)\| \leq C\exp(-c|j|).$$

This is now where our proof differs from the proof of [CF22, Lemmas 2.3, 2.4]. In [CF22], the authors find bounds on $\pi^k(z)W_j(z)$ and thus obtain estimates on $G_j(z)$. In our case, we have a stronger hypothesis (Hypothesis 2.1) that allows us to have a much simpler expression (2.2.15) of $\pi^k(z)W_j(z)$ and this will enable us to find the precise behavior of $G_j(z)$.

For $j \leq 0$, we observe from (2.2.15) that $\pi^k(z)W_j(z) = 0$ and that $G_j(z)$ is a component of $W_j(z)$. We therefore get the inequality (2.2.6).

We now consider the case $j \geq 1$. We have that $G_j(z) = (W_j(z))_p$ for all $z \in B_\varepsilon(\underline{z}_k)$ where $(X)_p$ refers to the p-th coordinate of a vector $X \in \mathbb{C}^{p+r}$. Then,

$$\forall z \in B_\varepsilon(\underline{z}_k), \quad |G_j(z) - (\pi^k(z)W_j(z))_p| \leq C\exp(-c|j|).$$

We then define the holomorphic function

$$f_k: \begin{array}{ccc} B_\varepsilon(\underline{z}_k) & \to & \mathbb{C} \\ z & \mapsto & \frac{1}{\mathbb{A}_p(z)\kappa_k(z)}(\pi^k(z)e)_p \end{array}.$$

By observing that $(\pi^k(z)W_j(z))_p = f_k(z)\kappa_k(z)^j$, we get the inequality (2.2.5) and it now remains to obtain the expression (2.2.9). We first need to determine the spectral projector $\pi^k(z)$. We recall that $\kappa_k(z) \in \mathbb{S}^1$ is a simple eigenvalue of $\mathbb{M}(z)$ and the vector

$$R_k(z) = \begin{pmatrix} \kappa_k(z)^{p+r-1} \\ \vdots \\ \kappa_k(z) \\ 1 \end{pmatrix} \in \mathbb{C}^{p+r}$$

is an eigenvector of $\mathbb{M}(z)$ associated with $\kappa_k(z)$. We also know that there exists a unique eigenvector $L_k(z) = (l_j(z))_{j\in\{1,\dots,p+r\}} \in \mathbb{C}^{p+r}$ of $\mathbb{M}(z)^T$ associated with the eigenvalue $\kappa_k(z)$ such that

$$L_k(z) \cdot R_k(z) = 1$$

where the symmetric bilinear form $\cdot$ on $\mathbb{C}^{p+r}$ is defined as [2]

$$\forall X, Y \in \mathbb{C}^{p+r}, \quad X \cdot Y := \sum_{l=1}^{p+r} X_i Y_i.$$

Then, we have that

$$\forall Y \in \mathbb{C}^{p+r}, \quad \pi^k(z) Y = (L_k(z) \cdot Y) R_k(z).$$

Thus, applying to the vector e implies that

$$f_k(z) = \frac{l_1(z)\kappa_k(z)^{r-1}}{\mathbb{A}_p(z)}. \tag{2.2.16}$$

We thus need to find the value of the coefficient $l_1(z)$. Since $L_k(z)$ is an eigenvector of $\mathbb{M}(z)^T$ for the eigenvalue $\kappa_k(z)$, we get

$$\forall j \in \{1, \dots, p+r\}, \quad l_j(z) = -\left(\sum_{l=-r}^{p-j} \frac{\mathbb{A}_l(z)}{\kappa_k(z)^{p-j+1-l}}\right) \frac{l_1(z)}{\mathbb{A}_p(z)}.$$

We now have an expression of each $l_j(z)$ depending on $l_1(z)$. To determine the value of $l_1(z)$, we have to use the normalization condition that we have made between $L_k(z)$ and $R_k(z)$. We have

$$1 = L_k(z) \cdot R_k(z) = \sum_{j=1}^{p+r} \kappa_k(z)^{p+r-j} l_j(z) = -\left(\sum_{j=1}^{p+r} \sum_{l=-r}^{p-j} \mathbb{A}_l(z)\kappa_k(z)^{l+r-1}\right) \frac{l_1(z)}{\mathbb{A}_p(z)}.$$

By the expression of $\mathbb{A}_l(z)$, this implies that

$$\begin{aligned} 1 = -\left(\sum_{l=-r}^{p} (p-l)\mathbb{A}_l(z)\kappa_k(z)^{l+r-1}\right) \frac{l_1(z)}{\mathbb{A}_p(z)} &= -\left(p\kappa_k(z)^{r-1} z - \sum_{l=-r}^{p} (p-l) a_l \kappa_k(z)^{l+r-1}\right) \frac{l_1(z)}{\mathbb{A}_p(z)} \\ &= -\left(p\kappa_k(z)^{r-1}(z - F(\kappa_k(z))) + \kappa_k(z)^r F'(\kappa_k(z))\right) \frac{l_1(z)}{\mathbb{A}_p(z)}. \end{aligned}$$

Since $\kappa_k(z)$ is an eigenvalue of $\mathbb{M}(z)$, Lemma 2.2.2 implies that

$$F(\kappa_k(z)) = z \quad \text{and} \quad \kappa_k'(z) F'(\kappa_k(z)) = 1.$$

Thus,

$$1 = -\frac{\kappa_k(z)^r l_1(z)}{\kappa_k'(z)\mathbb{A}_p(z)}.$$

Combining this equality with (2.2.16) implies the equality (2.2.9). □

2.3 Temporal Green's function

We are now ready to start proving Theorem 2.1. In Section 2.3.1, we will prove the result of the theorem far from the axes $j = n\alpha_k$. In this regime, the estimates proved in [CF22, Theorem 1.6] on $\mathscr{G}_j^n$ and estimates on the derivatives of the function $H_{2\mu}^{\beta}$ will allow us to prove bounds that are even stronger than those claimed in Theorem 2.1. The bulk of the proof will happen in the case where $j - n\alpha_k$ is close to 0 as the limiting estimates of Theorem 2.1 occur in this case. Section 2.3.2 will summarize the idea of the proof in the case where j is close to $n\alpha_k$ and Sections 2.3.3-2.3.5 give the details. The main tools are the use of functional calculus (see [Con90, Chapter VII]) to express the elements $\mathscr{G}_j^n$ with the spatial Green's function $G_j(z)$ and the estimates on the spatial Green's function proved in Section 2.2.

Before we start, we are going to make two hypotheses to simplify the proof. The first one is that $-1 \notin \{\underline{z}_1, \dots, \underline{z}_K\}$. This hypothesis is actually not restrictive. If it were not verified, we would just have to multiply the sequence a by some well chosen element of $\mathbb{S}^1$ to find a new sequence b that will verify this hypothesis and prove the theorem for this new sequence. The theorem for our previous sequence a would directly follow.

2. Observe that this symetric bilinear form is not the Hermitian product on $\mathbb{C}^{p+r}$.

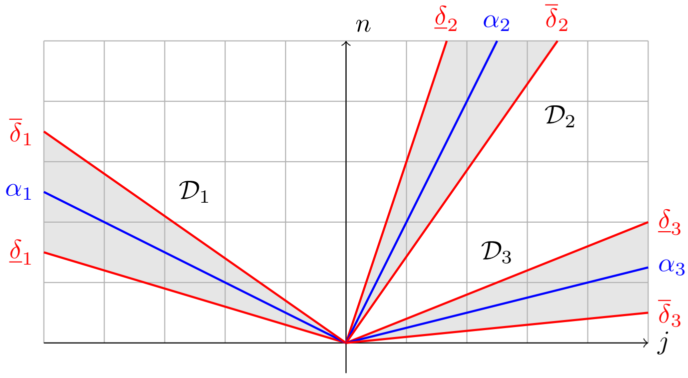

Figure 2.2 – An illustration of the sectors $\mathcal{D}_k$. Here, we have $\alpha_1 = -2$, $\alpha_2 = 0.5$ and $\alpha_3 = 4$. The rays labeled α_k (resp. $\underline{\delta}_k$, $\overline{\delta}_k$) correspond to the ray $j = n\alpha_k$ (resp. $j = n\underline{\delta}_k$, $j = n\overline{\delta}_k$). We observe that, because $\underline{\delta}_k$, α_k and $\overline{\delta}_k$ have the same sign, j and α_k have the same sign for $(n,j) \in \mathcal{D}_k$. Also, the sectors $\mathcal{D}_k$ do not intersect each other.

The second hypothesis we make is that all α_k are distinct from one another. This hypothesis has a real impact on the proof, symplifying greatly some parts of the calculations. We will come back in Section 2.4.1 to the case where the elements α_k can be equal and explain which elements of the proof should be modified.

2.3.1 Estimates far from the axes $j = n\alpha_k$

As explained at the beginning of the section, we suppose that all α_k are distinct from one another. Without loss of generality, we suppose that we arranged them so that there holds:

$$\alpha_1 < \ldots < \alpha_k < \ldots < \alpha_K.$$

For all $k \in \{1, \ldots, K\}$, we define two elements $\underline{\delta}_k, \overline{\delta}_k \in \mathbb{R}\backslash\{0\}$ such that $\underline{\delta}_k, \overline{\delta}_k$ and α_k have the same sign and

$$\underline{\delta}_1 < \alpha_1 < \overline{\delta}_1 < \ldots < \underline{\delta}_k < \alpha_k < \overline{\delta}_k < \ldots < \underline{\delta}_K < \alpha_K < \overline{\delta}_K.$$

We now define for every $k \in \{1, \ldots, K\}$ the sector

$$\mathcal{D}_k := \left\{(n,j) \in \mathbb{N}\backslash\{0\} \times \mathbb{Z}, \quad n\underline{\delta}_k \leq j \leq n\overline{\delta}_k\right\}$$

that do not intersect each other. We also introduce

$$\mathcal{D} := \bigcup_{k=1}^{K} \mathcal{D}_k.$$

We represent the sectors $\mathcal{D}_k$ on the Figure 2.2. In this section, we are going to prove the following two lemmas, which give estimates on the Green's function $\mathscr{G}_j^n$ and on the elements in its asymptotic expansion (2.1.7) outside of the sectors $\mathcal{D}_k$.

Lemma 2.3.1. *We have that*

$$\forall (n,j) \in \mathbb{N} \times \mathbb{Z}, \quad j < -np \text{ or } j > nr \Rightarrow \mathscr{G}_j^n = 0.$$

Furthermore, there exist two constants $C, c > 0$ such that

$$\forall (n,j) \in (\mathbb{N}\backslash\{0\} \times \mathbb{Z})\backslash\mathcal{D}, \quad -np \leq j \leq nr \Rightarrow \left|\mathscr{G}_j^n\right| \leq C\exp(-c(n+|j|)). \tag{2.3.1}$$

Lemma 2.3.2. *We consider $k \in \{1, \ldots, K\}$ and $\mathscr{P} \in \mathbb{C}[X,Y]$. For all $s \in \mathbb{N}$, there exist two constants $C, c > 0$ such that*

$$\forall (n,j) \in (\mathbb{N}\backslash\{0\} \times \mathbb{Z})\backslash\mathcal{D}_k, \quad \left|\left(\mathscr{P}\left(X_{n,j,k}, \frac{d}{dx}\right) H_{2\mu_k}^{\beta_k}\right)(X_{n,j,k})\right| \leq \frac{C}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-c\left|X_{n,j,k}\right|^{\frac{2\mu_k}{2\mu_k-1}}\right) \tag{2.3.2}$$

where $X_{n,j,k} := \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$.

Both lemmas are proved in a similar way.

Proof of Lemma 2.3.1 The first part of Lemma 2.3.1 is directly proved recursively using the definition (2.1.6) of the elements $\mathscr{G}_j^n$ and the equality (2.1.3) on the operator $\mathscr{L}_a$. We now focus our attention on the inequality (2.3.1) of Lemma 2.3.1. The result [CF22, Theorem 1.6] gives us the existence of two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \quad |\mathscr{G}_j^n| \leq \sum_{k=1}^{K} \frac{C}{n^{\frac{1}{2\mu_k}}} \exp\left(-c\left(\frac{|j-n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

For a sufficiently small $\tilde{c} > 0$, we have that

$$\forall k \in \{1, \ldots, K\}, \forall (n,j) \in (\mathbb{N}\backslash\{0\} \times \mathbb{Z})\backslash\mathcal{D}, \quad -np \leq j \leq nr \Rightarrow c\left(\frac{|j-n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}} \geq \tilde{c}(n+|j|). \qquad (2.3.3)$$

Therefore, we prove that there exist two positive constants C, c such that

$$\forall (n,j) \in (\mathbb{N}\backslash\{0\} \times \mathbb{Z})\backslash\mathcal{D}, \quad -np \leq j \leq nr \Rightarrow |\mathscr{G}_j^n| \leq C\exp(-c(n+|j|)).$$

□

To prove Lemma 2.3.2, we use the following lemma which gives sharp estimates on the derivatives of the function $H_{2\mu}^{\beta}$.

Lemma 2.3.3. *For $\mu \in \mathbb{N}\backslash\{0\}$, $\beta \in \mathbb{C}$ with positive real part and $m \in \mathbb{N}$, there exist two constants $C, c > 0$ such that*

$$\forall x \in \mathbb{R}, \quad \left|{H_{2\mu}^{\beta}}^{(m)}(x)\right| \leq C\exp\left(-c|x|^{\frac{2\mu}{2\mu-1}}\right).$$

This lemma is proved in [Rob91, Proposition 5.3]. For the sake of completeness, we give a complete proof in the appendix (Section 2.A).

Proof of Lemma 2.3.2 We fix a $k \in \{1, \ldots, K\}$ and we verify the estimate of Lemma 2.3.2 for the monomial $\mathscr{P} = X^{l_X}Y^{l_Y}$ where $l_X, l_Y \in \mathbb{N}$. We use Lemma 2.3.3 which implies the existence of two constants $C, c > 0$ such that

$$\forall (n,j) \in \mathbb{N}\backslash\{0\} \times \mathbb{Z}, \quad \left|\left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}\right)^{l_X}\left(H_{2\mu_k}^{\beta_k}\right)^{(l_Y)}\left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}\right)\right|$$
$$\leq C\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{l_X}\exp\left(-c\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

This implies that there exists $\tilde{C} > 0$ such that

$$\forall (n,j) \in \mathbb{N}\backslash\{0\} \times \mathbb{Z}, \quad \left|\left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}\right)^{l_X}\left(H_{2\mu_k}^{\beta_k}\right)^{(l_Y)}\left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}\right)\right| \leq \tilde{C}\exp\left(-\frac{c}{2}\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

Using the definition of the set $\mathcal{D}_k$, we prove the existence of a constant $\tilde{c} > 0$ such that

$$\forall (n,j) \in (\mathbb{N}\backslash\{0\} \times \mathbb{Z})\backslash\mathcal{D}_k, \quad \frac{c}{4}\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}} \geq \tilde{c}n.$$

Therefore, we easily conclude that there exist two positive constants C, c such that the inequality (2.3.2) of Lemma 2.3.2 is verified for $\mathscr{P} = X^{l_X}Y^{l_Y}$. □

Now that the two Lemmas 2.3.1 and 2.3.2 are proved, we observe that for any family of polynomials $(\mathscr{P}_\sigma^k)_{k\in\{1,\ldots,K\},\sigma\in\mathbb{N}\backslash\{0\}}$ which belong to $\mathbb{C}[X,Y]$, for all $s_1, \ldots, s_K \in \mathbb{N}$, there exist two positive constants C, c such that for all $(n,j) \in \mathbb{N}\backslash\{0\} \times \mathbb{Z}\backslash\mathcal{D}$

$$\left|\mathscr{G}_j^n - \sum_{k=1}^{K}\sum_{\sigma=1}^{s_k}\frac{z_k^n\kappa_k^j}{n^{\frac{\sigma}{2\mu_k}}}\left(\mathscr{P}_\sigma^k\left(X_{n,j,k}, \frac{d}{dx}\right)H_{2\mu_k}^{\beta_k}\right)(X_{n,j,k})\right| \leq \sum_{k=1}^{K}\frac{C}{n^{\frac{s_k+1}{2\mu_k}}}\exp\left(-c\left|X_{n,j,k}\right|^{\frac{2\mu_k}{2\mu_k-1}}\right), \qquad (2.3.4)$$

and for any $k_0 \in \{1, \dots, K\}$ and for all $(n,j) \in \mathcal{D}_{k_0}$, since the sets $\mathcal{D}_k$ do not intersect each other

$$\left| \mathscr{G}_j^n - \sum_{k=1}^{K} \sum_{\sigma=1}^{s_k} \frac{z_k^n \kappa_k^j}{n^{\frac{\sigma}{2\mu_k}}} \left(\mathscr{P}_\sigma^k \left(X_{n,j,k}, \frac{d}{dx} \right) H_{2\mu_k}^{\beta_k} \right)(X_{n,j,k}) \right| \leq \sum_{\substack{k=1\\k\neq k_0}}^{K} \frac{C}{n^{\frac{s_k+1}{2\mu_k}}} \exp\left(-c \left| X_{n,j,k} \right|^{\frac{2\mu_k}{2\mu_k-1}} \right)$$
$$+ \left| \mathscr{G}_j^n - \sum_{\sigma=1}^{s_{k_0}} \frac{z_{k_0}^n \kappa_{k_0}^j}{n^{\frac{\sigma}{2\mu_{k_0}}}} \left(\mathscr{P}_\sigma^{k_0} \left(X_{n,j,k_0}, \frac{d}{dx} \right) H_{2\mu_{k_0}}^{\beta_{k_0}} \right)(X_{n,j,k_0}) \right| \qquad (2.3.5)$$

with $X_{n,j,k} = \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$. There just remains to find a family of polynomials $(\mathscr{P}_\sigma^k)_{k,\sigma}$ to bound the last term in (2.3.5) when $(n,j) \in \mathcal{D}_{k_0}$.

2.3.2 Plan of the proof of Theorem 2.1 close to the axes $j = n\alpha_k$

We claim that to conclude the proof of Theorem 2.1, there only remains to prove the following proposition:

Proposition 2.1. *For all $k \in \{1, \dots, K\}$ and $s_k \in \mathbb{N}$, there exist a family of polynomials $(\mathscr{P}_\sigma^k)_{\sigma \in \{1,\dots,s_k\}}$ in $\mathbb{C}[X,Y]$ and two positive constants C, c such that*

$$\forall (n,j) \in \mathcal{D}_k, \qquad \left| \mathscr{G}_j^n - \sum_{\sigma=1}^{s_k} \frac{z_k^n \kappa_k^j}{n^{\frac{\sigma}{2\mu_k}}} \left(\mathscr{P}_\sigma^k \left(X_{n,j,k}, \frac{d}{dx} \right) H_{2\mu_k}^{\beta_k} \right)(X_{n,j,k}) \right| \leq \frac{C}{n^{\frac{s_k+1}{2\mu_k}}} \exp\left(-c |X_{n,j,k}|^{\frac{2\mu_k}{2\mu_k-1}} \right) \qquad (2.3.6)$$

with $X_{n,j,k} = \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$.

Once the existence of families of polynomials $(\mathscr{P}_\sigma^k)_{k,\sigma}$ satisfying Proposition 2.1 is proved, the inequalities (2.3.4) and (2.3.5) we deduced from Lemmas 2.3.1 and 2.3.2 imply that Theorem 2.1 is also verified for the same family of polynomials. It is important to observe that we use intensively the fact that the sectors $\mathcal{D}_k$ do not intersect each other. In Section 2.4.1, we will see that when the elements α_k are not supposed to be different, we will need to adapt Proposition 2.1 to take into account that for each sector there could be multiple generalized Gaussian waves that are superposed in the estimate (2.1.7).

We now focus our attention on proving Proposition 2.1. We fix $k \in \{1, \dots, K\}$ and $s \in \mathbb{N}$. For $s = 0$, the result has been proved in [CF22, Theorem 1.6]. Therefore, we will focus on the case where $s \geq 1$. The proof of Proposition 2.1 in this case will be separated in three steps:

• **Step 1:** In Section 2.3.3, we will express the elements $\mathscr{G}_j^n$ using the spatial Green's function $G_j(z)$ via the inverse Laplace transform and use the results of Section 2.2 to prove the following proposition:

Proposition 2.2. *For all $k \in \{1, \dots, K\}$ and for all $s \in \mathbb{N} \backslash \{0\}$, there exist two positive constants C, c such that for all $(n,j) \in \mathcal{D}_k$*

$$\left| \mathscr{G}_j^n - \frac{z_k^n \kappa_k^j}{2\pi} \int_{-\infty}^{+\infty} P_{s,k}(it + \underline{\tau}_k) \left(\sum_{l=0}^{s-1} \frac{(jR_{s,k}(it + \underline{\tau}_k))^l}{l!} \right) \exp\left(it \left(n - \frac{j}{\alpha_k} \right) - \frac{j}{\alpha_k} \frac{\beta_k}{\alpha_k^{2\mu_k}} t^{2\mu_k} \right) dt \right|$$
$$\leq \frac{C}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-c \left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k-1}} \right)$$

where $\underline{\tau}_k := i\theta_k$ is the only element of $i]-\pi, \pi[$ such that

$$\underline{z}_k = \exp(\underline{\tau}_k) = \exp(i\theta_k)$$

and the polynomial functions $P_{s,k}$ and $R_{s,k}$ have explicit expressions defined in Lemma 2.3.4.

• **Step 2:** We observe that in Proposition 2.2, we approach the elements $\mathscr{G}_j^n$ for $(n,j) \in \mathcal{D}_k$ by an explicit linear combination of the following terms where $l, m \in \mathbb{N}$ and $m \geq (2\mu_k + 1)l$

$$\frac{j^l}{2\pi} \int_{-\infty}^{+\infty} (it)^m \exp\left(it \left(n - \frac{j}{\alpha_k} \right) - \frac{j}{\alpha_k} \frac{\beta_k}{\alpha_k^{2\mu_k}} t^{2\mu_k} \right) dt = \frac{\alpha_k^{m+l} |\alpha_k|}{\left(\frac{j}{\alpha_k} \right)^{\frac{m - 2\mu_k l + 1}{2\mu_k}}} {H_{2\mu_k}^{\beta_k}}^{(m)} \left(\frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k} \right)^{\frac{1}{2\mu_k}}} \right). \qquad (2.3.7)$$

If we compare the terms in (2.3.7) with the terms appearing in the estimate (2.1.7) of Theorem 2.1, since we are considering $(n,j) \in \mathcal{D}_k$, we see that $\frac{j}{\alpha_k}$ is close to n. Therefore, once Proposition 2.2 is proved, we will only need some standard analysis in Section 2.3.4 to prove the following proposition.

Proposition 2.3. *For all $s \in \mathbb{N}$, $m \in \mathbb{N}$, $l \in \mathbb{N}\backslash\{0\}$ and $k \in \{1, \dots, K\}$, if we consider $d \in \mathbb{N}$ such that*

$$d \geq \frac{s+1}{2\mu_k - 1}$$

then there exist two constants $C, c > 0$ such that for all $(n, j) \in \mathcal{D}_k$,

$$\left| \frac{H_{2\mu_k}^{\beta_k\ (m)}(Y_{n,j,k})}{\left(\frac{j}{\alpha_k}\right)^{\frac{l}{2\mu_k}}} - \sum_{k_1=0}^{d-1}\sum_{k_3=0}^{d-1} \frac{\mathscr{B}^k_{l,k_1,k_3}}{n^{\frac{l+(2\mu_k-1)k_3}{2\mu_k}}} (X_{n,j,k})^{k_1+k_3} H_{2\mu_k}^{\beta_k\ (m+k_1)}(X_{n,j,k}) \right| \leq \frac{C}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-c\left|X_{n,j,k}\right|^{\frac{2\mu_k}{2\mu_k-1}}\right)$$

where $Y_{n,j,k} := \frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}$, $X_{n,j,k} := \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$ and

$$\mathscr{B}^k_{l,k_1,k_3} := \sum_{k_2=0}^{k_1} \frac{\binom{k_1}{k_2}(-1)^{k_1-k_2}}{k_1!k_3!\alpha_k^{k_3}} \left(\prod_{k_4=0}^{k_3-1} \frac{l+k_2}{2\mu_k} + k_4\right).$$

- **Step 3:** In Section 2.3.5, we will explicitly construct the polynomials $\mathscr{P}^k_\sigma$ satisfying Proposition 2.1 using Propositions 2.2 and 2.3. This will conclude the proof of Proposition 2.1 and Theorem 2.1 in the case where the elements α_k are distinct.

2.3.3 Step 1: Link between the spatial and temporal Green's functions and proof of Proposition 2.2

As explained at the end of the previous section, we start by proving Proposition 2.2. The first step will be to express the elements $\mathscr{G}^n_j$ via the spatial Green's function $G_j(z)$. The equation (2.2.3) implies by using the inverse Laplace transform that if we define a path which surrounds $\sigma(\mathscr{L}_a) = F(\mathbb{S}^1)$, like for example $\widetilde{\Gamma}_\rho = \exp(\rho)\mathbb{S}^1$ for $0 < \rho \leq \pi$, then

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \quad \mathscr{G}^n_j = \frac{1}{2i\pi}\int_{\widetilde{\Gamma}_\rho} z^n G_j(z)dz.$$

We fix this choice of path for now but we are going to modify it in what follows. The idea will be to deform the path on which we integrate so that we can best use the estimates on $G_j(z)$ proved in Section 2.2. We start with a change of variable $z = \exp(\tau)$ in the previous equality. Therefore, if we define $\Gamma_\rho := \{\rho + il, l \in [-\pi, \pi]\}$ and $\mathbf{G}_j(\tau) = e^\tau G_j(e^\tau)$, then

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \quad \mathscr{G}^n_j = \frac{1}{2i\pi}\int_{\Gamma_\rho} e^{n\tau}\mathbf{G}_j(\tau)d\tau. \tag{2.3.8}$$

We will therefore need a lemma that allows us to get from estimates on $G_j(z)$ to estimates on $\mathbf{G}_j(\tau)$. First, recalling that $\underline{z}_k \neq -1$, we define for all $k \in \{1, \dots, K\}$ the unique element $\underline{\tau}_k := i\theta_k$ of $i]-\pi, \pi[$ such that

$$\underline{z}_k = \exp(\underline{\tau}_k) = \exp(i\theta_k).$$

We also introduce for all $k \in \{1, \dots, K\}$ the unique $\tilde{\theta}_k \in]-\pi, \pi]$ such that

$$\underline{\kappa}_k = e^{i\tilde{\theta}_k}.$$

We now introduce a lemma to pass from estimates on $G_j(z)$ to estimates on $\mathbf{G}_j(\tau)$.

Lemma 2.3.4. *There exist a radius $\varepsilon_\star > 0$ and for all $k \in \{1, \dots, K\}$ two holomorphic functions $\varpi_k : B_{\varepsilon_\star}(\underline{\tau}_k) \to \mathbb{C}$ and $g_k : B_{\varepsilon_\star}(\underline{\tau}_k) \to \mathbb{C}$ such that for all $\varepsilon \in]0, \varepsilon_\star[$, there exist a width $\eta_\varepsilon \in]0, \varepsilon[$ and two constants $C, c > 0$ such that if we define*

$$U_\varepsilon := \{\tau \in \mathbb{C}, \Re(\tau) \in]-\eta_\varepsilon, \pi], \operatorname{Im}(\tau) \in [-\pi, \pi]\} \quad \text{and} \quad \Omega_\varepsilon := U_\varepsilon \backslash \bigcup_{k=1}^K B_\varepsilon(\underline{\tau}_k),$$

then for all $j \in \mathbb{Z}$, the application $\tau \mapsto \mathbf{G}_j(\tau)$ can be holomorphically extended on $U_\varepsilon \cup \bigcup_{k=1}^K B_\varepsilon(\underline{\tau}_k)$ and we have that

$$\forall \tau \in \Omega_\varepsilon, \forall j \in \mathbb{Z}, \quad |\mathbf{G}_j(\tau)| \leq Ce^{-c|j|}. \tag{2.3.9}$$

Also, for all $k \in \{1, \dots, K\}$, depending on the case, we have that

Case I:

$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \geq 1, \quad |\boldsymbol{G}_j(\tau) - e^\tau g_k(\tau) e^{j\varpi_k(\tau)}| \leq Ce^{-c|j|}, \tag{2.3.10}$$
$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \leq 0, \quad |\boldsymbol{G}_j(\tau)| \leq Ce^{-c|j|}, \tag{2.3.11}$$

Case II:

$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \geq 1, \quad |\boldsymbol{G}_j(\tau)| \leq Ce^{-c|j|}, \tag{2.3.12}$$
$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \leq 0, \quad |\boldsymbol{G}_j(\tau) - e^\tau g_k(\tau) e^{j\varpi_k(\tau)}| \leq Ce^{-c|j|}, \tag{2.3.13}$$

Case III:

$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \geq 1, \quad |\boldsymbol{G}_j(\tau) - e^\tau g_{\nu_{k,1}}(\tau) e^{j\varpi_{\nu_{k,1}}(\tau)}| \leq Ce^{-c|j|}, \tag{2.3.14}$$
$$\forall \tau \in B_\varepsilon(\underline{\tau}_k), \forall j \leq 0, \quad |\boldsymbol{G}_j(\tau) - e^\tau g_{\nu_{k,2}}(\tau) e^{j\varpi_{\nu_{k,2}}(\tau)}| \leq Ce^{-c|j|}, \tag{2.3.15}$$

where we have $\mathcal{I}_k = \{\nu_{k,1}, \nu_{k,2}\}$, $\alpha_{\nu_{k,1}} > 0$ *and* $\alpha_{\nu_{k,2}} < 0$.

For all $k \in \{1, \dots, K\}$, *we have*

$$\varpi_k(\tau) \underset{\tau \to \underline{\tau}_k}{=} i\tilde{\theta}_k - \frac{(\tau - \underline{\tau}_k)}{\alpha_k} + (-1)^{\mu_k+1} \frac{\beta_k}{\alpha_k^{2\mu_k+1}} (\tau - \underline{\tau}_k)^{2\mu_k} + o(|\tau - \underline{\tau}_k|^{2\mu_k}). \tag{2.3.16}$$

and

$$\forall \tau \in B_{\varepsilon_\star}(\underline{\tau}_k), \quad e^\tau g_k(\tau) = -\mathrm{sgn}(\alpha_k) \varpi_k'(\tau) \tag{2.3.17}$$

For $s \in \mathbb{N} \backslash \{0\}$, *we define the functions*

$$\begin{array}{rcl}
P_{s,k}: \quad \tau \in \mathbb{C} & \mapsto & -\mathrm{sgn}(\alpha_k) \displaystyle\sum_{l=0}^{s-1} \frac{\varpi_k^{(l+1)}(\underline{\tau}_k)}{l!} (\tau - \underline{\tau}_k)^l, \\
\varphi_k: \quad \tau \in \mathbb{C} & \mapsto & i\tilde{\theta}_k - \dfrac{(\tau - \underline{\tau}_k)}{\alpha_k} + (-1)^{\mu_k+1} \dfrac{\beta_k}{\alpha_k^{2\mu_k+1}} (\tau - \underline{\tau}_k)^{2\mu_k}, \\
Q_{s,k}: \quad \tau \in \mathbb{C} & \mapsto & \displaystyle\sum_{l=0}^{2\mu_k+s-1} \frac{\varpi_k^{(l)}(\underline{\tau}_k)}{l!} (\tau - \underline{\tau}_k)^l, \\
R_{s,k}: \quad \tau \in \mathbb{C} & \mapsto & Q_{s,k}(\tau) - \varphi_k(\tau).
\end{array}$$

The functions $P_{s,k}$, $Q_{s,k}$ *and* φ_k *are asymptotic expansions of the function* $e^\tau g_k$ *and* ϖ_k *at* $\underline{\tau}_k$ *up to different orders. We can then define a bounded holomorphic function* $\xi_{s,k} : B_{\varepsilon_\star}(\underline{\tau}_k) \mapsto \mathbb{C}$ *such that*

$$\forall \tau \in B_{\varepsilon_\star}(\underline{\tau}_k), \quad \varpi_k(\tau) = Q_{s,k}(\tau) + \xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}.$$

We then can prove that there exist two positive constants A_R, A_I *such that for all* $\tau \in B_{\varepsilon_\star}(\underline{\tau}_k)$

$$\alpha_k \Re(\varphi_k(\tau)) \leq -\Re(\tau - \underline{\tau}_k) + A_R \Re(\tau - \underline{\tau}_k)^{2\mu_k} - A_I \mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k}, \tag{2.3.18}$$
$$\alpha_k \Re(\varpi_k(\tau)) + |\alpha_k||\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}| \leq -\Re(\tau - \underline{\tau}_k) + A_R \Re(\tau - \underline{\tau}_k)^{2\mu_k} - A_I \mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k}, \tag{2.3.19}$$
$$\alpha_k \Re(\varphi_k(\tau)) + |\alpha_k||R_{s,k}(\tau)| \leq -\Re(\tau - \underline{\tau}_k) + A_R \Re(\tau - \underline{\tau}_k)^{2\mu_k} - A_I \mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k}. \tag{2.3.20}$$

Proof Using the Lemmas 2.2.4 and 2.2.5 and writing $\kappa_k(z) = \exp(\omega_k(z))$ for z near $\underline{z}_k$ with $\omega_k(\underline{z}_k) = i\tilde{\theta}_k$, we can define for a choice of $\varepsilon_\star$ small enough two holomorphic functions ϖ_k and g_k such that

$$\forall \tau \in B_{\varepsilon_\star}(\underline{\tau}_k), \quad \varpi_k(\tau) = \omega_k(e^\tau), g_k(\tau) = f_k(e^\tau).$$

Lemmas 2.2.4 and 2.2.5 directly imply the inequalities (2.3.10), (2.3.11), (2.3.12), (2.3.13), (2.3.14) and (2.3.15) on the open balls $B_{\varepsilon_\star}(\underline{\tau}_k)$ and the fact that the functions $\tau \mapsto \mathbf{G}_j(\tau)$ are holomorphic on $B_{\varepsilon_\star}(\underline{\tau}_k)$. We now consider $\varepsilon \in]0, \varepsilon_\star[$. The inequalities we just proved remain true on $B_\varepsilon(\underline{\tau}_k)$. Using a compactness argument and Lemma 2.2.3, we also get the existence of η_ε and the inequality (2.3.9).

We observe that the asymptotic expansion (2.1.4) implies that

$$\tau - \underline{\tau}_k \underset{\tau \to \underline{\tau}_k}{=} -\alpha_k(\varpi_k(\tau) - i\tilde{\theta}_k) + (-1)^{\mu_k+1}\beta_k(\varpi_k(\tau) - i\tilde{\theta}_k)^{2\mu_k} + O\left(\left|\varpi_k(\tau) - i\tilde{\theta}_k\right|^{2\mu_k+1}\right).$$

We then deduce the equation (2.3.16).

For $\tau \in B_{\varepsilon_\star}(\underline{\tau}_k)$, the equations (2.2.9) and (2.2.12) imply the equality (2.3.17).

There only remains to prove the existence of A_R and A_I to verify the inequalities (2.3.18) - (2.3.20).

We are going to prove (2.3.18) first. Because of Young's inequality, we have that for $l \in \{1, \dots, 2\mu_k - 1\}$, for all $\delta > 0$, there exists $C_\delta > 0$ such that for all $\tau \in \mathbb{C}$

$$|\Re(\tau)|^l |\mathrm{Im}\,(\tau)|^{2\mu_k - l} \leq \delta \mathrm{Im}\,(\tau)^{2\mu_k} + C_\delta \Re(\tau)^{2\mu_k}.$$

Furthermore, we have that

$$\alpha_k \Re(\varphi_k(\tau)) = -\Re(\tau - \underline{\tau}_k) + (-1)^{\mu_k+1}\left(\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}}\Re((\tau - \underline{\tau}_k)^{2\mu_k}) - \frac{\mathrm{Im}\,(\beta_k)}{\alpha_k^{2\mu_k}}\mathrm{Im}\,((\tau - \underline{\tau}_k)^{2\mu_k})\right).$$

Then, for $\delta > 0$, there exists $C_\delta > 0$ such that

$$\alpha_k \Re(\varphi_k(\tau)) \leq -\Re(\tau - \underline{\tau}_k) + \Re(\tau - \underline{\tau}_k)^{2\mu_k}\left(\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}} + C_\delta\right) + \mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k}\left(-\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}} + \delta\right).$$

Therefore, by taking δ small enough, we can end the proof of inequality (2.3.18). The proof of inequality (2.3.19) is similar. We have for $\tau \in B_{\varepsilon_\star}(\underline{\tau}_k)$

$$\begin{aligned}\alpha_k \Re(\varpi_k(\tau)) + |\alpha_k||\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}| \leq &-\Re(\tau - \underline{\tau}_k) + |\alpha_k|\left(2|\xi_{s,k}(\tau)||\tau - \underline{\tau}_k|^{2\mu_k+s} + |R_{s,k}(\tau)|\right)\\ &+ (-1)^{\mu_k+1}\left(\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}}\Re((\tau - \underline{\tau}_k)^{2\mu_k}) - \frac{\mathrm{Im}\,(\beta_k)}{\alpha_k^{2\mu_k}}\mathrm{Im}\,((\tau - \underline{\tau}_k)^{2\mu_k})\right).\end{aligned}$$

We know there exists $c_1, c_2 > 0$ such that

$$\forall k \in \{1, \dots, K\}, \forall \tau \in \mathbb{C}, \quad |\tau|^{2\mu_k} \leq c_1 \Re(\tau)^{2\mu_k} + c_2 \mathrm{Im}\,(\tau)^{2\mu_k}.$$

Since $\xi_{s,k}$ and $\frac{R_{s,k}}{X^{2\mu_k+1}}$ can be bounded by some constant $\tilde{C} > 0$ on $B_{\varepsilon_\star}(\underline{\tau}_k)$, using the same reasoning as previously gives us

$$\begin{aligned}\alpha_k \Re(\varpi_k(\tau)) + |\alpha_k||\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}| \leq &-\Re(\tau - \underline{\tau}_k) + |\alpha_k|\tilde{C}\left(2\varepsilon_\star^s + \varepsilon_\star\right)\left(c_1\Re(\tau - \underline{\tau}_k)^{2\mu_k} + c_2 \mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k}\right)\\ &+ \Re(\tau - \underline{\tau}_k)^{2\mu_k}\left(\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}} + C_\delta\right) + \mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k}\left(-\frac{\Re(\beta_k)}{\alpha_k^{2\mu_k}} + \delta\right).\end{aligned}$$

Taking δ and $\varepsilon_\star$ small enough allows us to prove (2.3.19). We prove the inequality (2.3.20) the same way. □

Remark 1. We observe that the constants in the inequalities (2.3.10), (2.3.11), (2.3.12), (2.3.13), (2.3.14) and (2.3.15) can (and will) be chosen uniformly with respect to $\varepsilon \in]0, \varepsilon_\star[$. However, it is not the case for the constants in inequality (2.3.9).

Choice of integration paths for the proof of Proposition 2.2

From now on, we fix a $k \in \{1, \dots, K\}$ and an integer $s \in \mathbb{N}\backslash\{0\}$ and our goal is to prove the claim of Proposition 2.2 for this k and s, i.e. we want to prove the existence of two positive constants C, c such that for all $(n, j) \in \mathcal{D}_k$ we have

$$\begin{aligned}&\left|\mathscr{G}_j^n - \frac{z_k^n \kappa_k^j}{2\pi}\int_{-\infty}^{+\infty} P_{s,k}(it + \underline{\tau}_k)\left(\sum_{l=0}^{s-1}\frac{(jR_{s,k}(it + \underline{\tau}_k))^l}{l!}\right)\exp\left(it\left(n - \frac{j}{\alpha_k}\right) - \frac{j}{\alpha_k}\frac{\beta_k}{\alpha_k^{2\mu_k}}t^{2\mu_k}\right)dt\right|\\ &\qquad\qquad \leq \frac{C}{n^{\frac{s+1}{2\mu_k}}}\exp\left(-c\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).\end{aligned} \qquad (2.3.21)$$

We will suppose that $\alpha_k > 0$. The major consequence is that for $(n, j) \in \mathcal{D}_k$, we have $j \geq 1$. This implies that we will use the inequalities (2.3.10), (2.3.12) and (2.3.14). The case where $\alpha_k < 0$ would need some little modifications, in particular we will have that $j \leq 0$ for $(n, j) \in \mathcal{D}_k$ and we would rather use the inequalities (2.3.11), (2.3.13) and (2.3.15).

Before we begin with the proof, we will need to introduce some lemmas and define some elements. First, we can easily prove the following lemma which allows us to pass from bounds that are exponentially decaying in n to the generalized Gaussian bounds expected in (2.3.21).

Lemma 2.3.5. *We consider $C, c > 0$. Then, for all $s \in \mathbb{N}\backslash\{0\}$, there exist $\tilde{C}, \tilde{c} > 0$ such that*

$$\forall (n,j) \in \mathcal{D}_k, \quad C\exp(-cn) \leq \frac{\tilde{C}}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-\tilde{c}\,|X_{n,j,k}|^{\frac{2\mu_k}{2\mu_k-1}}\right)$$

with $X_{n,j,k} := \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$.

We now apply Lemma 2.3.4 and consider $\varepsilon \in]0, \varepsilon_\star[$ small enough so that

$$\forall i,j \in \{1,\dots,K\}, \quad \underline{z}_i \neq \underline{z}_j \Rightarrow B_\varepsilon(\underline{\tau}_i) \cap B_\varepsilon(\underline{\tau}_j) = \emptyset$$

and

$$\forall l \in \{1,\dots,K\}, \quad B_\varepsilon(\underline{\tau}_l) \subset \{\tau \in \mathbb{C}, \quad \mathrm{Im}\,(\tau) \in [-\pi,\pi]\}.$$

This can be done because we supposed that $\underline{z}_l \neq -1$ which implies $\underline{\tau}_l \notin \{-i\pi, i\pi\}$ for all l. We also introduce some conditions on the values η_ε we defined in Lemma 2.3.4 which will be useful later on in the proof, especially for Lemma 2.3.10. We define the function

$$\begin{array}{rccc} r_\varepsilon: &]0,\varepsilon[& \to & \mathbb{R} \\ & \eta & \mapsto & \sqrt{\varepsilon^2-\eta^2} \end{array} \tag{2.3.22}$$

which serves to define the extremities of $-\eta + i\mathbb{R} \cap B_\varepsilon(\underline{\tau}_k)$ for any $k \in \{1,\dots,K\}$. We impose that η_ε is small enough so that

$$\eta_\varepsilon < \sqrt{\frac{3}{4}}\varepsilon. \tag{2.3.23}$$

This condition implies that

$$r_\varepsilon(\eta_\varepsilon) > \frac{\varepsilon}{2}.$$

Finally, we also impose that

$$\forall k \in \{1,\dots,K\}, \quad \eta_\varepsilon + A_R\eta_\varepsilon^{2\mu_k} - A_I\left(\frac{\varepsilon}{2}\right)^{2\mu_k} < 0. \tag{2.3.24}$$

We now fix a constant $\eta \in]0, \eta_\varepsilon[$ which we will use to express the modified path on which we will integrate the right-hand term of equality (2.3.8).

We will now follow a strategy developed in [ZH98], which has also been used in [God03], [CF22] and [CF23], and introduce a family of parameterized curves. For $\tau_p \in \mathbb{R}$, we introduce

$$\Psi_k(\tau_p) = \tau_p - A_R{\tau_p}^{2\mu_k}.$$

The function Ψ_k is continuous and strictly increasing on $\left]-\infty, \left(\frac{1}{2\mu_k A_R}\right)^{\frac{1}{2\mu_k-1}}\right[$. We choose ε small enough so that it is strictly increasing on $]-\infty,\varepsilon]$. We can therefore introduce for $\tau_p \in [-\eta,\varepsilon]$

$$\Gamma_{k,p} = \left\{\tau \in \mathbb{C}, -\eta \leq \Re(\tau) \leq \tau_p, \quad \Re(\tau-\underline{\tau}_k) - A_R\Re(\tau-\underline{\tau}_k)^{2\mu_k} + A_I\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k} = \Psi_k(\tau_p)\right\}.$$

It is a symmetric curve with respect to the axis $\mathbb{R}+\underline{\tau}_k = \mathbb{R}+i\theta_k$ which intersects this axis on the point $\tau_p+\underline{\tau}_k$. If we introduce $\ell_{k,p} = \left(\frac{\Psi_k(\tau_p)-\Psi_k(-\eta)}{A_I}\right)^{\frac{1}{2\mu_k}}$, then $-\eta+i(\theta_k+\ell_{k,p})$ and $-\eta+i(\theta_k-\ell_{k,p})$ are the end points of $\Gamma_{k,p}$. We can also introduce a parametrization of this curve by defining $\gamma_{k,p}: [-\ell_{k,p},\ell_{k,p}] \to \mathbb{C}$ such that

$$\forall \tau_p \in [-\eta,\varepsilon], \forall t \in [-\ell_{k,p},\ell_{k,p}], \quad \mathrm{Im}\,(\gamma_{k,p}(t)) = t+\theta_k, \quad \Re(\gamma_{k,p}(t)) = h_{k,p}(t) := \Psi_k^{-1}\left(\Psi_k(\tau_p) - A_It^{2\mu_k}\right). \tag{2.3.25}$$

The above parametrization immediately yields that there exists a constant $M>0$ such that

$$\forall \tau_p \in [-\eta,\varepsilon], \forall t \in [-\ell_{k,p},\ell_{k,p}], \quad |h'_{k,p}(t)| \leq M. \tag{2.3.26}$$

Also, there exists a constant $c_\star > 0$ such that

$$\forall \tau_p \in [-\eta,\varepsilon], \forall \tau \in \Gamma_{k,p}, \quad \Re(\tau-\underline{\tau}_k) - \tau_p \leq -c_\star\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k}. \tag{2.3.27}$$

We introduce those integration paths $\Gamma_{k,p}$ because they allow us to use optimally the inequalities (2.3.18)-(2.3.20). For example, if we seek to bound $e^{n\tau+j\varpi_k(\tau)}$ when $(n,j)\in\mathcal{D}_k$ and $\tau\in\Gamma_{k,p}$, it follows from the equality $\mathrm{sgn}(j)=\mathrm{sgn}(\alpha_k)$ and the inequalities (2.3.19) and (2.3.27) that

$$\begin{aligned} n\Re(\tau-\underline{\tau}_k)+j\Re(\varpi_k(\tau)) &\leq n\Re(\tau-\underline{\tau}_k)-\frac{j}{\alpha_k}\left(\Re(\tau-\underline{\tau}_k)-A_R\Re(\tau-\underline{\tau}_k)^{2\mu_k}+A_I\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k}\right)\\ &\leq -nc_\star\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k}-\left(\frac{j}{\alpha_k}-n\right)\tau_p+\frac{j}{\alpha_k}A_R\tau_p^{2\mu_k}. \end{aligned} \tag{2.3.28}$$

Such calculations will happen regularly in the following proof (see Lemmas 2.3.8 and 2.3.9). There remains to make an appropriate choice of τ_p depending on n and j that minimizes the right-hand side of the inequality (2.3.28) whilst the paths $\Gamma_{k,p}$ remain within the ball $B_\varepsilon(\underline{\tau}_k)$. Even if we have to consider a smaller η, we can define a real number $0<\varepsilon_{k,0}<\varepsilon$ such that the curve $\Gamma_{k,p}$ associated to $\tau_p=\varepsilon_{k,0}$ intersects the axis $-\eta+i\mathbb{R}$ within $B_\varepsilon(\underline{\tau}_k)$. Then, recalling that we consider $\alpha_k>0$ and thus that $j\geq 0$, we let

$$\zeta_k=\frac{j-n\alpha_k}{2\mu_k n},\quad \gamma_k=\frac{A_Rj}{n},\quad \rho_k\left(\frac{\zeta_k}{\gamma_k}\right)=\mathrm{sgn}(\zeta_k)\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{1}{2\mu_k-1}}.$$

The inequality (2.3.28) thus becomes

$$n\Re(\tau-\underline{\tau}_k)+j\Re(\varpi_k(\tau))\leq -nc_\star\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k}+\frac{n}{\alpha_k}(\gamma_k\tau_p^{2\mu_k}-2\mu_k\zeta_k\tau_p). \tag{2.3.29}$$

Our limiting estimates will come from the case where ζ_k is close to 0. We observe that the condition $(n,j)\in\mathcal{D}_k$ implies

$$A_R\underline{\delta}_k\leq\gamma_k\leq A_R\overline{\delta}_k. \tag{2.3.30}$$

Moreover, we have that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right)$ is the unique real root of the polynomial

$$\gamma_k x^{2\mu_k-1}=\zeta_k.$$

Then, we take

$$\tau_p:=\begin{cases}\rho_k\left(\frac{\zeta_k}{\gamma_k}\right), & \text{if } \rho_k\left(\frac{\zeta_k}{\gamma_k}\right)\in[-\frac{\eta}{2},\varepsilon_{k,0}], \quad (\text{Case } \mathbf{A})\\ \varepsilon_{k,0}, & \text{if } \rho_k\left(\frac{\zeta_k}{\gamma_k}\right)>\varepsilon_{k,0}, \quad (\text{Case } \mathbf{B})\\ -\frac{\eta}{2}, & \text{if } \rho_k\left(\frac{\zeta_k}{\gamma_k}\right)<-\frac{\eta}{2}. \quad (\text{Case } \mathbf{C})\end{cases}$$

The case **A** corresponds to the choice to minimize the right-hand side of (2.3.29). The cases **B** and **C** allow the path $\Gamma_{k,p}$ to stay within $B_\varepsilon(\underline{\tau}_k)$.

There just remains to define the path Γ_k defined on the Figure 2.3. As we can see, it follows the ray $-\eta+i[-\pi,\pi]$ and is deformed inside $B_\varepsilon(\underline{\tau}_k)$ into the path $\Gamma_{k,p}$. We define

$$\begin{aligned}\Gamma_{k,res}&:=\{-\eta+it,t\in[-\pi,\pi]\backslash[\theta_k-\ell_{k,p},\theta_k+\ell_{k,p}]\}\cap B_\varepsilon(\underline{\tau}_k),\\ \Gamma_{k,out}&:=\{-\eta+it,t\in[-\pi,\pi]\}\cap B_\varepsilon(\underline{\tau}_k)^c,\\ \Gamma_{k,in}&:=\Gamma_{k,p}\cup\Gamma_{k,res},\\ \Gamma_k&:=\Gamma_{k,in}\cup\Gamma_{k,out}.\end{aligned}$$

Using Cauchy's formula and taking into account the "$2i\pi$-periodicity" of $\mathbf{G}_j(\tau)$, we have that for all $n\in\mathbb{N}\backslash\{0\}$ and $j\in\mathbb{Z}$

$$\mathscr{G}_j^n=\frac{1}{2i\pi}\int_{\Gamma_\rho}e^{n\tau}\mathbf{G}_j(\tau)d\tau=\frac{1}{2i\pi}\int_{\Gamma_k}e^{n\tau}\mathbf{G}_j(\tau)d\tau. \tag{2.3.31}$$

In order to prove Proposition 2.2, we will start by proving the following lemma.

Lemma 2.3.6. *For all $k\in\{1,\dots,K\}$ and for all $s\in\mathbb{N}\backslash\{0\}$, there exist two positive constants C,c such that for all $(n,j)\in\mathcal{D}_k$*

$$\left|\mathscr{G}_j^n-\frac{1}{2i\pi}\int_{\Gamma_{k,in}}P_{s,k}(\tau)\left(\sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right)e^{n\tau}e^{j\varphi_k(\tau)}d\tau\right|\leq\frac{C}{n^{\frac{s+1}{2\mu_k}}}\exp\left(-c\left(\frac{|n\alpha_k-j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

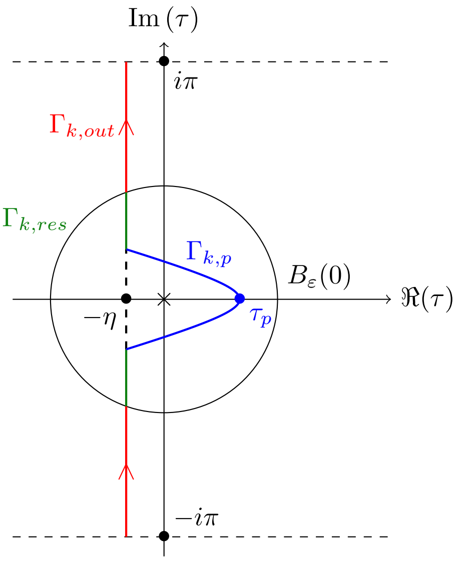

Figure 2.3 – A representation of the path Γ_k for $\underline{\tau}_k = 0$. It is composed of $\Gamma_{k,out}$ (in red), $\Gamma_{k,res}$ (in green) and $\Gamma_{k,p}$ (in blue). The section of Γ_k which lies inside the ball $B_\varepsilon(\underline{\tau}_k)$ (i.e. the reunion of $\Gamma_{k,res}$ and $\Gamma_{k,p}$) is notated $\Gamma_{k,in}$.

Our main focus now will be to prove Lemma 2.3.6. We observe that the triangular inequality implies

$$\left| \mathscr{G}_j^n - \frac{1}{2i\pi} \int_{\Gamma_{k,in}} P_{s,k}(\tau) \left(\sum_{l=0}^{s-1} \frac{(jR_{s,k}(\tau))^l}{l!} \right) e^{n\tau} e^{j\varphi_k(\tau)} d\tau \right| \leq \frac{1}{2\pi} \sum_{l=1}^{8} E_l \tag{2.3.32}$$

where

$$E_1 = \left| \int_{\Gamma_{k,out}} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right|, \qquad E_2 = \left| \int_{\Gamma_{k,in}} e^{n\tau} \left(\mathbf{G}_j(\tau) - e^{\tau} g_k(\tau) \exp(j\varpi_k(\tau)) \right) d\tau \right|,$$

$$E_3 = \left| \int_{\Gamma_{k,p}} e^{n\tau + j\varpi_k(\tau)} \left(e^{\tau} g_k(\tau) - P_{s,k}(\tau) \right) d\tau \right|, \qquad E_4 = \left| \int_{\Gamma_{k,res}} e^{n\tau + j\varpi_k(\tau)} \left(e^{\tau} g_k(\tau) - P_{s,k}(\tau) \right) d\tau \right|,$$

$$E_5 = \left| \int_{\Gamma_{k,p}} P_{s,k}(\tau) e^{n\tau} \left(e^{j\varpi_k(\tau)} - e^{jQ_{s,k}(\tau)} \right) d\tau \right|, \qquad E_6 = \left| \int_{\Gamma_{k,res}} P_{s,k}(\tau) e^{n\tau} \left(e^{j\varpi_k(\tau)} - e^{jQ_{s,k}(\tau)} \right) d\tau \right|,$$

$$E_7 = \left| \int_{\Gamma_{k,p}} P_{s,k}(\tau) e^{n\tau + j\varphi_k(\tau)} \left(e^{jR_{s,k}(\tau)} - \sum_{l=0}^{s-1} \frac{(jR_{s,k}(\tau))^l}{l!} \right) d\tau \right|,$$

$$E_8 = \left| \int_{\Gamma_{k,res}} P_{s,k}(\tau) e^{n\tau + j\varphi_k(\tau)} \left(e^{jR_{s,k}(\tau)} - \sum_{l=0}^{s-1} \frac{(jR_{s,k}(\tau))^l}{l!} \right) d\tau \right|.$$

We will now have to determine estimates on all these terms depending on k (case **I**, **II** and **III**) and also on τ_p and $\Gamma_{k,p}$:

— **Case A:** $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) \in \left[-\frac{\eta}{2}, \varepsilon_{k,0}\right]$,

— **Case B:** $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) > \varepsilon_{k,0}$,

— **Case C:** $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) < -\frac{\eta}{2}$.

The main contribution will come from the terms E_3, E_5 and E_7 in the case **A**. We will prove much sharper estimates for the other terms.

Preliminary lemmas

Before we start to determine the estimates on the different terms, we are going to introduce some lemmas to simplify the redaction. Those lemmas assemble inequalities in the different cases (**A**, **B** and **C**) for which the proofs are similar with variations depending on the case we are in. They mainly rely on the inequalities (2.3.18), (2.3.19) and (2.3.20). The proofs of those lemmas can be found in the appendix. Lemmas here will be proved for $\alpha_k > 0$. When $\alpha_k < 0$, they would sometimes require some slight modifications.

We start with a lemma which will be useful to study the terms E_5, E_6, E_7 and E_8.

Lemma 2.3.7 (Inequalities in $B_{\varepsilon_\star}(\underline{\tau}_k)$)**.** *There exists $C > 0$ such that for all $\tau \in B_{\varepsilon_\star}(\underline{\tau}_k)$ and $(n,j) \in \mathcal{D}_k$, we have*

$$\left|e^{n\tau}\left(e^{j\varpi_k(\tau)} - e^{jQ_{s,k}(\tau)}\right)\right| \leq Cn|\tau - \underline{\tau}_k|^{2\mu_k+s}\exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s+1}|))$$

and

$$\left|e^{n\tau + j\varphi_k(\tau)}\left(e^{jR_{s,k}(\tau)} - \sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right)\right| \\ \leq C\left(n|\tau - \underline{\tau}_k|^{2\mu_k+1}\right)^s\exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varphi_k(\tau)) + \operatorname{sgn}(\alpha_k)|R_{s,k}(\tau)|)).$$

This next lemma will be useful for terms where the integral is defined along the path $\Gamma_{k,p}$ (terms E_3, E_5 and E_7).

Lemma 2.3.8 (Inequalities on $\Gamma_{k,p}$)**.** *For $(n,j) \in \mathbb{N}\backslash\{0\} \times \mathbb{N}$ and $\tau \in \Gamma_{k,p}$, we have*

- *Case **A**:* $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) \in \left[-\frac{\eta}{2}, \varepsilon_{k,0}\right]$

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|) \\ \leq -nc_\star\mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k} - \frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}, \tag{2.3.33}$$

$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) \leq -nc_\star\mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k} - \frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}, \tag{2.3.34}$$

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varphi_k(\tau)) + \operatorname{sgn}(\alpha_k)|R_{s,k}(\tau)|) \leq -nc_\star\mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k} - \frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}. \tag{2.3.35}$$

- *Case **B**:* $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) > \varepsilon_{k,0}$

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\varepsilon_{k,0}^{2\mu_k}, \tag{2.3.36}$$

$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\varepsilon_{k,0}^{2\mu_k}, \tag{2.3.37}$$

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varphi_k(\tau)) + \operatorname{sgn}(\alpha_k)|R_{s,k}(\tau)|) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\varepsilon_{k,0}^{2\mu_k}. \tag{2.3.38}$$

- *Case **C**:* $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) < -\frac{\eta}{2}$

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\left(\frac{\eta}{2}\right)^{2\mu_k}, \tag{2.3.39}$$

$$n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\left(\frac{\eta}{2}\right)^{2\mu_k}, \tag{2.3.40}$$

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varphi_k(\tau)) + \operatorname{sgn}(\alpha_k)|R_{s,k}(\tau)|) \leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\left(\frac{\eta}{2}\right)^{2\mu_k}. \tag{2.3.41}$$

Finally, we introduce in the next lemma some inequalities that will help us for the terms with integrals defined on $\Gamma_{k,res}$ (terms E_4, E_6 and E_8).

Lemma 2.3.9 (Inequalities on $\Gamma_{k,res}$)**.** *For $(n,j) \in \mathbb{N}\backslash\{0\} \times \mathbb{N}$ and $\tau \in \Gamma_{k,res}$, we have in all cases*

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)\left|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}\right|) \leq -n\frac{\eta}{2}, \tag{2.3.42}$$

$$n\Re(\tau-\underline{\tau}_k)+j(\Re(\varpi_k(\tau))\leq -n\frac{\eta}{2}, \tag{2.3.43}$$

$$n\Re(\tau-\underline{\tau}_k)+j(\Re(\varphi_k(\tau))+\mathrm{sgn}(\alpha_k)|R_{s,k}(\tau)|)\leq -n\frac{\eta}{2}. \tag{2.3.44}$$

Estimates of part of the terms

We are going to first prove estimates for the terms where the proof will not depend on the case **A**, **B** or **C** in which we are.

• Estimate for E_2:

We introduce the path $\Gamma_{\eta,k}$ defined as

$$\Gamma_{\eta,k}:=\{-\eta+it,t\in[-\pi,\pi]\}\cap B_\varepsilon(\underline{\tau}_k).$$

Using Cauchy's formula, we have that

$$\int_{\Gamma_{k,in}} e^{n\tau}\left(\mathbf{G}_j(\tau)-e^\tau g_k(\tau)\exp(j\varpi_k(\tau))\right)d\tau=\int_{\Gamma_{\eta,k}} e^{n\tau}\left(\mathbf{G}_j(\tau)-e^\tau g_k(\tau)\exp(j\varpi_k(\tau))\right)d\tau.$$

Because we supposed that $\alpha_k>0$, depending on whether we are in case **I** or **III**, the previous equality and the inequalities (2.3.10) and (2.3.14) imply

$$\left|\int_{\Gamma_{k,in}} e^{n\tau}\left(\mathbf{G}_j(\tau)-e^\tau g_k(\tau)\exp(j\varpi_k(\tau))\right)d\tau\right|\lesssim e^{-n\eta-cj}.$$

• Estimate for E_4:

The inequality (2.3.43) implies

$$\left|\int_{\Gamma_{k,res}} e^{n\tau+j\varpi_k(\tau)}\left(e^\tau g_k(\tau)-P_{s,k}(\tau)\right)d\tau\right|\lesssim\int_{\Gamma_{k,res}}\exp\left(n\Re(\tau)+j\Re(\varpi_k(\tau))\right)|d\tau|\lesssim e^{-n\frac{\eta}{2}}.$$

• Estimate for E_6:

If we use Lemma 2.3.7, we have

$$\left|\int_{\Gamma_{k,res}} P_{s,k}(\tau)e^{n\tau}\left(e^{j\varpi_k(\tau)}-e^{jQ_{s,k}(\tau)}\right)d\tau\right|$$
$$\lesssim\int_{\Gamma_{k,res}}\exp(n\Re(\tau-\underline{\tau}_k)+j(\Re(\varpi_k(\tau))+\mathrm{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau-\underline{\tau}_k)^{2\mu_k+s}|))n|\tau-\underline{\tau}_k|^{2\mu_k+s}|d\tau|.$$

Therefore, the inequality (2.3.42) implies

$$\left|\int_{\Gamma_{k,res}} P_{s,k}(\tau)e^{n\tau}\left(e^{j\varpi_k(\tau)}-e^{jQ_{s,k}(\tau)}\right)d\tau\right|\lesssim ne^{-n\frac{\eta}{2}}\lesssim e^{-n\frac{\eta}{4}}.$$

• Estimate for E_8:

If we use Lemma 2.3.7, we have

$$\left|\int_{\Gamma_{k,res}} P_{s,k}(\tau)e^{n\tau+j\varphi_k(\tau)}\left(e^{jR_{s,k}(\tau)}-\sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right)d\tau\right|$$
$$\lesssim\int_{\Gamma_{k,res}}\exp(n\Re(\tau-\underline{\tau}_k)+j(\Re(\varpi_k(\tau))+\mathrm{sgn}(\alpha_k)|R_{s,k}(\tau)|))(n|\tau-\underline{\tau}_k|^{2\mu_k+1})^s|d\tau|.$$

Therefore, the inequality (2.3.44) implies

$$\left|\int_{\Gamma_{k,res}} P_{s,k}(\tau)e^{n\tau+j\varphi_k(\tau)}\left(e^{jR_{s,k}(\tau)}-\sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right)d\tau\right|\lesssim n^se^{-n\frac{\eta}{2}}\lesssim e^{-n\frac{\eta}{4}}.$$

It remains to study the terms E_1, E_3, E_5 and E_7.

The terms E_3, E_5 and E_7, Case A : $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) \in \left[-\frac{\eta}{2}, \varepsilon_{k,0}\right]$

This part of the proof is the most important because those terms will create the limiting estimates.

• Estimate for E_3:

Because of Taylor's theorem, we have

$$E_3 = \left|\int_{\Gamma_{k,p}} \left(e^\tau g_k(\tau) - P_{s,k}(\tau)\right) e^{n\tau + j\varpi_k(\tau)} d\tau\right| \lesssim \int_{\Gamma_{k,p}} |\tau - \underline{\tau}_k|^s \exp\left(n\Re(\tau) + j\Re(\varpi_k(\tau))\right) |d\tau|.$$

The inequality (2.3.34) implies

$$E_3 \lesssim \int_{\Gamma_{k,p}} |\tau - \underline{\tau}_k|^s e^{-nc_\star \mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k}} |d\tau| \exp\left(-\frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

But, the inequality (2.3.30) and the fact that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) = \tau_p$ imply

$$\frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}} \geq \frac{2\mu_k - 1}{\alpha_k} A_R \underline{\delta}_k n |\tau_p|^{2\mu_k}.$$

If we introduce $c > 0$ small enough, then

$$E_3 \lesssim \int_{\Gamma_{k,p}} |\tau - \underline{\tau}_k|^s e^{-nc_\star \mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k}} |d\tau| \exp\left(-cn|\tau_p|^{2\mu_k}\right).$$

Using the parametrization (2.3.25) and the inequality (2.3.26), we have

$$\int_{\Gamma_{k,p}} |\tau - \underline{\tau}_k|^s e^{-nc_\star \mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k}} |d\tau| \lesssim \int_{-\ell_{k,p}}^{\ell_{k,p}} (|\tau_p|^s + |t|^s) e^{-nc_* t^{2\mu_k}} dt.$$

The change of variables $u = n^{\frac{1}{2\mu_k}} t$ and the fact that the function $x \geq 0 \mapsto x^s \exp\left(-\frac{c}{2}x^{2\mu_k}\right)$ is bounded imply

$$\begin{cases} \displaystyle\int_{-\ell_{k,p}}^{\ell_{k,p}} |t|^s e^{-nc_* t^{2\mu_k}} dt \lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}}, \\ \displaystyle\int_{-\ell_{k,p}}^{\ell_{k,p}} |\tau_p|^s e^{-nc_* t^{2\mu_k}} dt \lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}} \exp\left(\frac{c}{2} n|\tau_p|^{2\mu_k}\right). \end{cases}$$

Thus,

$$E_3 \lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-\frac{c}{2} n|\tau_p|^{2\mu_k}\right).$$

Lastly, the inequality (2.3.30) implies that we have a constant $\tilde{c} > 0$ independent from j and n such that

$$\frac{c}{2} n|\tau_p|^{2\mu_k} \geq \tilde{c}\left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}$$

so,

$$E_3 \lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-\tilde{c}\left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

• Estimate for E_5:

Using Lemma 2.3.7 and the inequality (2.3.33), we have

$$\begin{aligned} E_5 &= \left|\int_{\Gamma_{k,p}} P_{s,k}(\tau) e^{n\tau}\left(e^{j\varpi_k(\tau)} - e^{jQ_{s,k}(\tau)}\right) d\tau\right| \\ &\lesssim \int_{\Gamma_{k,p}} n|\tau - \underline{\tau}_k|^{2\mu_k+s} \exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \mathrm{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|))|d\tau| \end{aligned}$$

$$\lesssim \exp\left(-\frac{n}{\alpha_k}(2\mu_k-1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right) n\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|^{2\mu_k+s}\exp(-nc_\star\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k})|d\tau|.$$

Just like in the estimate for the previous term, because of the inequality (2.3.30), if we introduce $c>0$ small enough, we have

$$E_5 \lesssim n\exp\left(-cn|\tau_p|^{2\mu_k}\right)\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|^{2\mu_k+s}\exp(-nc_\star\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k})|d\tau|.$$

The same reasoning as for the estimate of E_3 implies that

$$n\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|^{2\mu_k+s}\exp(-nc_\star\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k})|d\tau| \lesssim n\int_{-\ell_{k,p}}^{\ell_{k,p}}|t|^{2\mu_k+s}e^{-nc_* t^{2\mu_k}}dt + n\int_{-\ell_{k,p}}^{\ell_{k,p}}|\tau_p|^{2\mu_k+s}e^{-nc_* t^{2\mu_k}}dt.$$

The change of variables $u=n^{\frac{1}{2\mu_k}}t$ and the fact that the function $x\geq 0\mapsto x^{2\mu_k+s}\exp\left(-\frac{c}{2}x^{2\mu_k}\right)$ is bounded imply

$$\left\{\begin{aligned} n\int_{-\ell_{k,p}}^{\ell_{k,p}}|t|^{2\mu_k+s}e^{-nc_* t^{2\mu_k}}dt &\lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}},\\ n\int_{-\ell_{k,p}}^{\ell_{k,p}}|\tau_p|^{2\mu_k+s}e^{-nc_* t^{2\mu_k}}dt &\lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}}\exp\left(\frac{c}{2}n|\tau_p|^{2\mu_k}\right).\end{aligned}\right.$$

Thus,

$$E_5 \lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}}\exp\left(-\frac{c}{2}n|\tau_p|^{2\mu_k}\right).$$

Lastly, the inequality on γ_k (2.3.30) implies that we have a constant $\tilde{c}>0$ independent from j and n such that

$$\frac{c}{2}n|\tau_p|^{2\mu_k}\geq \tilde{c}\left(\frac{|j-n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}$$

so,

$$E_5 \lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}}\exp\left(-\tilde{c}\left(\frac{|j-n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

• Estimate for E_7:
Using Lemma 2.3.7 and the inequality (2.3.35), we have

$$\begin{aligned}E_7 &= \left|\int_{\Gamma_{k,p}}P_{s,k}(\tau)e^{n\tau+j\varphi_k(\tau)}\left(e^{jR_{s,k}(\tau)}-\sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right)d\tau\right|\\ &\lesssim \int_{\Gamma_{k,p}}(n|\tau-\underline{\tau}_k|^{2\mu_k+1})^s\exp(n\Re(\tau-\underline{\tau}_k)+j(\Re(\varphi_k(\tau))+\mathrm{sgn}(\alpha_k)|R_{s,k}(\tau)|))|d\tau|\\ &\lesssim \exp\left(-\frac{n}{\alpha_k}(2\mu_k-1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right)n^s\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|^{s(2\mu_k+1)}\exp(-nc_\star\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k})|d\tau|.\end{aligned}$$

Just like in the estimate for the previous term, because of the inequality (2.3.30), if we introduce $c>0$ small enough, we have

$$E_7 \lesssim n^s\exp\left(-cn|\tau_p|^{2\mu_k}\right)\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|^{s(2\mu_k+1)}\exp(-nc_\star\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k})|d\tau|.$$

The same reasoning as for the estimate of E_3 implies that

$$\begin{aligned}&n^s\int_{\Gamma_{k,p}}|\tau-\underline{\tau}_k|^{s(2\mu_k+1)}\exp(-nc_\star\mathrm{Im}\,(\tau-\underline{\tau}_k)^{2\mu_k})|d\tau|\\ &\qquad\lesssim n^s\int_{-\ell_{k,p}}^{\ell_{k,p}}|t|^{s(2\mu_k+1)}e^{-nc_* t^{2\mu_k}}dt + n^s\int_{-\ell_{k,p}}^{\ell_{k,p}}|\tau_p|^{s(2\mu_k+1)}e^{-nc_* t^{2\mu_k}}dt.\end{aligned}$$

The change of variables $u=n^{\frac{1}{2\mu_k}}t$ and the fact that the function $x\geq 0\mapsto x^{s(2\mu_k+1)}\exp\left(-\frac{c}{2}x^{2\mu_k}\right)$ is bounded imply

$$\begin{cases} n^s \displaystyle\int_{-\ell_{k,p}}^{\ell_{k,p}} |t|^{s(2\mu_k+1)} e^{-nc_* t^{2\mu_k}} dt \lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}}, \\ n^s \displaystyle\int_{-\ell_{k,p}}^{\ell_{k,p}} |\tau_p|^{s(2\mu_k+1)} e^{-nc_* t^{2\mu_k}} dt \lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}} \exp\left(\frac{c}{2} n |\tau_p|^{2\mu_k}\right). \end{cases}$$

Thus,

$$E_7 \lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-\frac{c}{2} n |\tau_p|^{2\mu_k}\right).$$

Lastly, the inequality on γ_k (2.3.30) implies that we have a constant $\tilde{c} > 0$ independent from j and n such that

$$\frac{c}{2} n |\tau_p|^{2\mu_k} \geq \tilde{c} \left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k - 1}}$$

so,

$$E_7 \lesssim \frac{1}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-\tilde{c}\left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k - 1}}\right).$$

The terms E_3, E_5 and E_7, Case B and C:

We now consider that we are either in case **B** or case **C** (i.e. $\rho_k(\frac{\zeta_k}{\gamma_k}) \notin \left[-\frac{\eta}{2}, \varepsilon_{k,0}\right]$).

• Estimate for E_3:

Because of Taylor's theorem, we have

$$E_3 = \left| \int_{\Gamma_{k,p}} e^{n\tau + j\varpi_k(\tau)} \left(e^\tau g_k(\tau) - P_{s,k}(\tau)\right) d\tau \right| \lesssim \int_{\Gamma_{k,p}} |\tau - \underline{\tau}_k|^s \exp\left(n\Re(\tau) + j\Re(\varpi_k(\tau))\right) |d\tau|.$$

Using the inequality (2.3.37) or (2.3.40) whether we are in case **B** or **C**, they imply that there exists $c > 0$ independent from j and n such that

$$E_3 \lesssim e^{-cn}.$$

• Estimate for E_5:

Using Lemma 2.3.7, we have

$$\begin{aligned} E_5 &= \left| \int_{\Gamma_{k,p}} e^{n\tau} P_{s,k}(\tau) \left(e^{j\varpi_k(\tau)} - e^{jQ_{s,k}(\tau)}\right) d\tau \right| \\ &\lesssim \int_{\Gamma_{k,p}} n|\tau - \underline{\tau}_k|^{2\mu_k + s} \exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k + s}|))|d\tau|. \end{aligned}$$

Using the inequality (2.3.36) or (2.3.39) whether we are in case **B** or **C**, they imply that there exists $c > 0$ independent from j and n such that

$$E_5 \lesssim n e^{-cn} \lesssim e^{-\frac{c}{2} n}.$$

• Estimate for E_7:

Using Lemma 2.3.7, we have

$$\begin{aligned} E_7 &= \left| \int_{\Gamma_{k,p}} P_{s,k}(\tau) e^{n\tau + j\varphi_k(\tau)} \left(e^{jR_{s,k}(\tau)} - \sum_{l=0}^{s-1} \frac{(jR_{s,k}(\tau))^l}{l!}\right) d\tau \right| \\ &\lesssim \int_{\Gamma_{k,p}} (n|\tau - \underline{\tau}_k|^{2\mu_k + 1})^s \exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varphi_k(\tau)) + \operatorname{sgn}(\alpha_k)|R_{s,k}(\tau)|))|d\tau|. \end{aligned}$$

Using the inequality (2.3.38) or (2.3.41) whether we are in case **B** or **C**, they imply that there exists $c > 0$ independent from j and n such that

$$E_7 \lesssim n^s e^{-cn} \lesssim e^{-\frac{c}{2} n}.$$

Estimate for the term E_1

• Estimate for E_1:

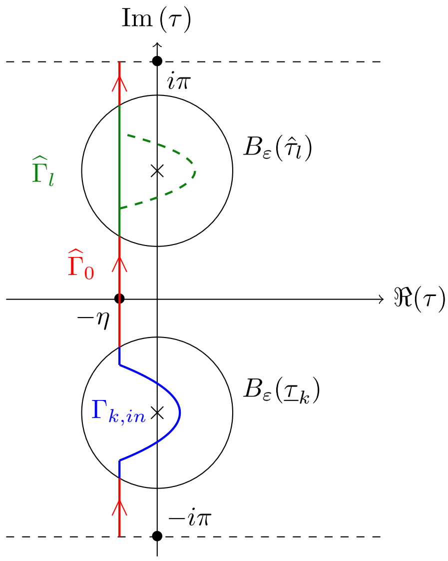

Figure 2.4 – This is a representation of Γ_k where we decompose $\Gamma_{k,out}$. The red path corresponds to $\widehat{\Gamma}_0$ the part of $\Gamma_{k,out}$ which lies outside the balls $B_\varepsilon(\hat{\tau}_l)$. The green path corresponds to $\widehat{\Gamma}_l$ the part of $\Gamma_{k,out}$ which lies inside the ball $B_\varepsilon(\hat{\tau}_l)$. The dashed green path corresponds to the deformation we use in the proof of the estimate for E_1.

We recall that

$$E_1 = \left| \int_{\Gamma_{k,out}} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right|.$$

For $\tau \in \Gamma_{k,out}$, we have different estimates depending on whether we are inside a ball $B_\varepsilon(\underline{\tau}_l)$ or not. Therefore, we introduce the set of distinct points

$$\{\hat{\tau}_1, \dots, \hat{\tau}_R\} = \{\underline{\tau}_l, \quad l \in \{1, \dots, K\}\} \setminus \{\underline{\tau}_k\}.$$

It allows us to decompose the path $\Gamma_{k,out}$ as

$$\Gamma_{k,out} := \bigcup_{l=0}^{R} \widehat{\Gamma}_l,$$

where for all $l \in \{1, \dots, R\}$

$$\widehat{\Gamma}_l := \Gamma_{k,out} \cap B_\varepsilon(\hat{\tau}_l)$$

and

$$\widehat{\Gamma}_0 := \Gamma_{k,out} \setminus \bigcup_{l=1}^{R} \widehat{\Gamma}_l.$$

This decomposition of $\Gamma_{k,out}$ is represented on Figure 2.4. The inequality (2.3.9) gives us that

$$\left| \int_{\widehat{\Gamma}_0} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right| \lesssim e^{-n\eta - c|j|}.$$

We now consider $l \in \{1, \dots, R\}$. There are two possibilities because of Hypothesis 2.3:

• The set $\{i \in \{1, \dots, R\}, \quad \underline{\tau}_i = \hat{\tau}_l\}$ is the singleton $\{i\}$ with $\alpha_i < 0$ (i.e. we are in case **II**). Then, knowing that for $(n, j) \in \mathcal{D}_k$ we have $j \geq 1$, because of the inequality (2.3.12), we have

$$\left| \int_{\widehat{\Gamma}_l} e^{n\tau} \mathbf{G}_j(\tau) d\tau \right| \lesssim e^{-n\eta - c|j|}.$$

• The set $\{i \in \{1, \dots, R\}, \quad \underline{\tau}_i = \hat{\tau}_l\}$ is the singleton $\{i\}$ with $\alpha_i > 0$ (i.e. we are in case **I**) or it has two distinct elements $\{i, j\}$ with $\alpha_i > 0$ and $\alpha_j < 0$ (i.e. we are in case **III**). Either way, the inequalities (2.3.10) and (2.3.14) imply that

$$\left|\int_{\widehat{\Gamma}_l} e^{n\tau}\mathbf{G}_j(\tau)d\tau\right| \leq 2\pi C e^{-n\eta-c|j|} + \left|\int_{\widehat{\Gamma}_l} \exp(n\tau + j\varpi_i(\tau))e^{\tau}g_i(\tau)d\tau\right|.$$

Just like we defined the path $\Gamma_{k,p}$, $\Gamma_{k,res}$ and $\Gamma_{k,in} := \Gamma_{k,p} \sqcup \Gamma_{k,res}$, we can define a path $\Gamma_{i,p}$, $\Gamma_{i,res}$ and $\Gamma_{i,in} := \Gamma_{i,p} \sqcup \Gamma_{i,res}$. The path $\Gamma_{i,in}$ is represented with a dashed green line on the Figure 2.4. Using Cauchy's formula, we then have

$$\int_{\widehat{\Gamma}_l} \exp(n\tau + j\varpi_i(\tau))e^{\tau}g_i(\tau)d\tau = \int_{\Gamma_{i,in}} \exp(n\tau + j\varpi_i(\tau))e^{\tau}g_i(\tau)d\tau$$

The function $\tau \mapsto e^{\tau}g_i(\tau)$ can be bounded so we just have to bound $\displaystyle\int_{\Gamma_{i,in}} \exp(n\Re(\tau - \underline{\tau}_i) + j\Re(\varpi_i(\tau)))d|\tau|$. We observe that the proofs of the Lemmas 2.3.8 and 2.3.9 are also true for $\Gamma_{i,p}$ and $\Gamma_{i,res}$. Using the inequality (2.3.43) for the integral along the path $\Gamma_{i,res}$, we prove that there exists a constant $c > 0$ independent from n and j so that

$$\int_{\Gamma_{i,res}} \exp(n\Re(\tau - \underline{\tau}_i) + j\Re(\varpi_i(\tau)))d|\tau| \lesssim e^{-cn}.$$

It remains to bound the integral along the path $\Gamma_{i,p}$. In the case **A** (i.e. $\rho_i(\frac{\zeta_i}{\gamma_i}) \in [-\frac{\eta}{2}, \varepsilon_{i,0}]$), we observe that for $(n,j) \in \mathcal{D}_k$, γ_i is bounded between two positive constants and

$$|\zeta_i| \geq \frac{1}{2\mu_i}\min(|\alpha_i - \underline{\delta}_k|, |\alpha_i - \overline{\delta}_k|).$$

Therefore, using the inequality (2.3.34) and the previous observation in case **A** and using the inequalities (2.3.37) and (2.3.40) in cases **B** and **C**, we prove that there exists a constant $c > 0$ independent from n and j so that

$$\int_{\Gamma_{i,p}} \exp(n\Re(\tau - \underline{\tau}_i) + j\Re(\varpi_i(\tau)))d|\tau| \lesssim e^{-cn}.$$

Therefore, there exists a constant $c > 0$ such that

$$\forall (n,j) \in \mathcal{D}_k, \quad \left|\int_{\widehat{\Gamma}_l} e^{n\tau}\mathbf{G}_j(\tau)d\tau\right| \lesssim e^{-cn}.$$

This gives a sharp estimate of E_1.

If we recapitulate the estimates we found, we can define two constants $C, c > 0$ such that

$$\forall (n,j) \in \mathcal{D}_k, \forall l \in \{1,2,4,6,8\}, \quad E_l \leq Ce^{-cn},$$

and

$$\forall (n,j) \in \mathcal{D}_k, \forall l \in \{3,5,7\}, \quad E_l \leq \frac{C}{n^{\frac{s+1}{2\mu_k}}}\exp\left(-c\left(\frac{|j - n\alpha_k|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k - 1}}\right).$$

The estimates we proved on all the terms and Lemma 2.3.5 allow us to conclude the proof of Lemma 2.3.6.

From Lemma 2.3.6 to Proposition 2.2

Now that Lemma 2.3.6 is proved, we know that there exist two positive constants C, c such that for all $(n,j) \in \mathcal{D}_k$,

$$\left|\mathscr{G}_j^n - \frac{1}{2i\pi}\int_{\Gamma_{k,in}} P_{s,k}(\tau)\left(\sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right)e^{n\tau}e^{j\varphi_k(\tau)}d\tau\right| \leq \frac{C}{n^{\frac{s+1}{2\mu_k}}}\exp\left(-c\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k - 1}}\right). \quad (2.3.45)$$

Proving Proposition 2.2 amounts to proving a similar estimate as (2.3.45) where the integration path would be $\{it + \underline{\tau}_k, t \in \mathbb{R}\}$. This is the goal of this subsection. We prove the following lemma, which will use the conditions (2.3.23) and (2.3.24) we introduced on η_ε.

Lemma 2.3.10. *We define the path*

$$\Gamma_{k,in}^0 := \{it, \quad t \in [\theta_k - r_\varepsilon(\eta), \theta_k + r_\varepsilon(\eta)]\}$$

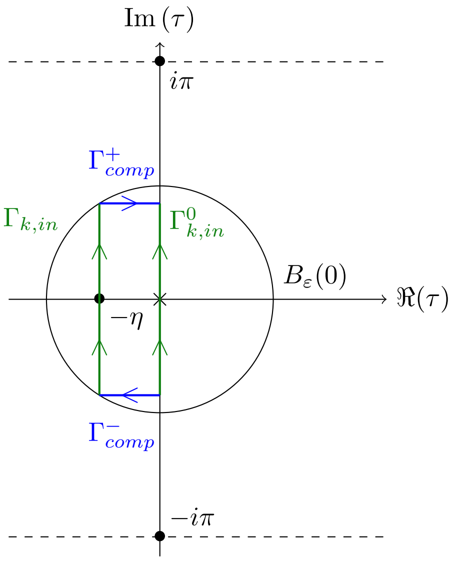

Figure 2.5 – A representation of the path $\Gamma_{k,in}$, $\Gamma^0_{k,in}$ and $\Gamma^\pm_{comp}$ for $\underline{\tau}_k = 0$ used in Lemma 2.3.10.

where the function r_ε is defined in (2.3.22). Then, for all $m \in \mathbb{N}\setminus\{0\}$, there exist two positive constants C, c such that

$$\forall (n,j) \in \mathcal{D}_k, \quad \left| \int_{\Gamma^0_{k,in}} (\tau - \underline{\tau}_k)^m e^{n\tau} e^{j\varphi_k(\tau)} d\tau - \int_{\Gamma_{k,in}} (\tau - \underline{\tau}_k)^m e^{n\tau} e^{j\varphi_k(\tau)} d\tau \right| \leq Ce^{-cn}.$$

Proof As in Figure 2.5, we define the paths

$$\Gamma^+_{comp} := \{t + i(\theta_k + r_\varepsilon(\eta)), \quad t \in [-\eta, 0]\}, \quad \Gamma^-_{comp} := \{t + i(\theta_k - r_\varepsilon(\eta)), \quad t \in [-\eta, 0]\}.$$

Cauchy's formula then implies that

$$\left| \int_{\Gamma^0_{k,in}} (\tau - \underline{\tau}_k)^m e^{n\tau} e^{j\varphi_k(\tau)} d\tau - \int_{\Gamma_{k,in}} (\tau - \underline{\tau}_k)^m e^{n\tau} e^{j\varphi_k(\tau)} d\tau \right|$$
$$\leq \left| \int_{\Gamma^+_{comp}} (\tau - \underline{\tau}_k)^m e^{n\tau} e^{j\varphi_k(\tau)} d\tau \right| + \left| \int_{\Gamma^-_{comp}} (\tau - \underline{\tau}_k)^m e^{n\tau} e^{j\varphi_k(\tau)} d\tau \right|.$$

We need to find estimates for the two terms on the right-hand side. Both terms will be bounded similarly so we will focus on the first one. Since $\Gamma^+_{comp} \subset B_\varepsilon(\underline{\tau}_k)$, we have

$$\left| \int_{\Gamma^+_{comp}} (\tau - \underline{\tau}_k)^m e^{n\tau} e^{j\varphi_k(\tau)} d\tau \right| \lesssim \int_{-\eta}^0 \exp\left(nt + j\Re\left(\varphi_k\left(t + i(\theta_k + r_\varepsilon(\eta))\right)\right)\right) dt.$$

For $t \in]-\eta, 0[$, since $t + i(\theta_k + r_\varepsilon(\eta)) \in B_{\varepsilon_\star}(\underline{\tau}_k)$ and $\frac{j}{\alpha_k} > 0$, using the inequality (2.3.18), we prove

$$nt + j\Re(\varphi_k(t + i(\theta_k + r_\varepsilon(\eta)))) \leq \frac{j}{\alpha_k}\left(\eta + A_R\eta^{2\mu_k} - A_I r_\varepsilon(\eta)^{2\mu_k}\right).$$

Using the inequality (2.3.23), we have that $r_\varepsilon(\eta) \geq r_\varepsilon(\eta_\varepsilon) > \frac{\varepsilon}{2}$. Inequality (2.3.24) then implies that

$$\eta + A_R\eta^{2\mu_k} - A_I r_\varepsilon(\eta)^{2\mu_k} \leq \eta_\varepsilon + A_R\eta_\varepsilon^{2\mu_k} - A_I\left(\frac{\varepsilon}{2}\right)^{2\mu_k} < 0.$$

Since $(n,j) \in \mathcal{D}_k$, we have that $\frac{j}{\alpha_k} \geq \frac{\tilde{\delta}_k}{\alpha_k} n$ so there must exist $c > 0$ such that

$$\forall (n,j) \in \mathcal{D}_k, \forall t \in]-\eta, 0[, \quad nt + j\Re(\varphi_k(t + i(\theta_k + r_\varepsilon(\eta)))) \leq -cn.$$

This concludes the proof of Lemma 2.3.10. □

Using Lemma 2.3.6 and the estimate (2.3.45), we have thus proved that for all $s \in \mathbb{N}\backslash\{0\}$, there exist two positive constants C, c such that for all $(n,j) \in \mathcal{D}_k$

$$\left| \mathscr{G}_j^n - \frac{\underline{z}_k^n \underline{\kappa}_k^j}{2\pi} \int_{-r_\varepsilon(\eta)}^{r_\varepsilon(\eta)} P_{s,k}(it+\underline{\tau}_k) \left(\sum_{l=0}^{s-1} \frac{(jR_{s,k}(it+\underline{\tau}_k))^l}{l!} \right) \exp\left(it\left(n - \frac{j}{\alpha_k}\right) - \frac{j}{\alpha_k} \frac{\beta_k}{\alpha_k^{2\mu_k}} t^{2\mu_k} \right) dt \right| \\ \leq \frac{C}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-c \left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right). \quad (2.3.46)$$

There just remains to prove the following lemma to conclude the proof of Proposition 2.2.

Lemma 2.3.11. *For all $m \in \mathbb{N}$ and $c_0 > 0$, there exist two positive constants $C, c > 0$ such that*

$$\forall (n,j) \in \mathcal{D}_k, \qquad \int_{r_\varepsilon(\eta)}^{+\infty} t^m \exp\left(-\frac{j}{\alpha_k} c_0 t^{2\mu_k} \right) dt \leq Ce^{-cn}.$$

Proof The proof is done recursively and using the following equality proved by integrating by parts

$$\int_{r_\varepsilon(\eta)}^{+\infty} t^m \exp\left(-\frac{j}{\alpha_k} c_0 t^{2\mu_k} \right) dt = \frac{r_\varepsilon(\eta)^{m+1-2\mu_k}}{2\mu_k c_0 \frac{j}{\alpha_k}} \exp\left(-c_0 r_\varepsilon(\eta)^{2\mu_k} \frac{j}{\alpha_k} \right) \\ + \frac{m+1-2\mu_k}{2\mu_k c_0 \frac{j}{\alpha_k}} \int_{r_\varepsilon(\eta)}^{+\infty} t^{m-2\mu_k} \exp\left(-\frac{j}{\alpha_k} c_0 t^{2\mu_k} \right) dt. \quad (2.3.47)$$

— For $m \in \{0, \dots, 2\mu_k - 1\}$, since the second term of the sum on the right hand side of (2.3.47) is non-positive, using the fact that $(n,j) \in \mathcal{D}_k$, we directly prove the result.
— If we consider $\tilde{m} \geq 2\mu_k$ such that the result of lemma has been proved for all $m \in \{0, \dots, \tilde{m} - 1\}$, then the equality (2.3.47) implies the result for $m = \tilde{m}$.

□

Combining Lemmas 2.3.11, 2.3.5 and the inequality (2.3.46), we easily conclude the proof of Proposition 2.2.

2.3.4 Step 2 : Proof of Proposition 2.3

As we explained in Section 2.3.2, Proposition 2.2 and the equality (2.3.7) imply that we proved generalized Gaussian estimates on the difference between the elements $\mathscr{G}_j^n$ and a linear combination of

$$\frac{1}{\left(\frac{j}{\alpha_k}\right)^{\frac{l}{2\mu_k}}} H_{2\mu_k}^{\beta_k \,(m)} \left(\frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}} \right) \quad \text{where } l \in \mathbb{N}\backslash\{0\}, m \in \mathbb{N}.$$

We now need to approach the above terms by the elements appearing in Theorem 2.1, i.e. a linear combination of

$$\frac{1}{n^{\frac{l}{2\mu_k}}} \left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}} \right)^{m_2} H_{2\mu_k}^{\beta_k \,(m_1)} \left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}} \right) \quad \text{where } l \in \mathbb{N}\backslash\{0\}, m_1, m_2 \in \mathbb{N}.$$

This is the goal of Proposition 2.3 that we recall here:

Proposition (Proposition 2.3)**.** *For all $s \in \mathbb{N}$, $m \in \mathbb{N}$, $l \in \mathbb{N}\backslash\{0\}$ and $k \in \{1, \dots, K\}$, if we consider $d \in \mathbb{N}$ such that*

$$d \geq \frac{s+1}{2\mu_k - 1}$$

then there exist two constants $C, c > 0$ such that for all $(n,j) \in \mathcal{D}_k$,

$$\left| \frac{H_{2\mu_k}^{\beta_k \,(m)}(Y_{n,j,k})}{\left(\frac{j}{\alpha_k}\right)^{\frac{l}{2\mu_k}}} - \sum_{k_1=0}^{d-1} \sum_{k_3=0}^{d-1} \frac{\mathscr{B}_{l,k_1,k_3}^k}{n^{\frac{l+(2\mu_k-1)k_3}{2\mu_k}}} (X_{n,j,k})^{k_1+k_3} H_{2\mu_k}^{\beta_k \,(m+k_1)}(X_{n,j,k}) \right| \leq \frac{C}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-c |X_{n,j,k}|^{\frac{2\mu_k}{2\mu_k - 1}} \right)$$

where $Y_{n,j,k} := \frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}$, $X_{n,j,k} := \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$ *and*

$$\mathscr{B}^k_{l,k_1,k_3} := \sum_{k_2=0}^{k_1} \frac{\binom{k_1}{k_2}(-1)^{k_1-k_2}}{k_1! k_3! \alpha_k^{k_3}} \left(\prod_{k_4=0}^{k_3-1} \frac{l+k_2}{2\mu_k} + k_4 \right).$$

First, we prove the following lemma.

Lemma 2.3.12. *For all $s \in \mathbb{N}$, $m \in \mathbb{N}$ and $k \in \{1, \dots, K\}$, if we consider $d \in \mathbb{N}$ such that*

$$d \geq \frac{s+1}{2\mu_k - 1}$$

then there exist two constants $C, c > 0$ such that for all $(n,j) \in \mathcal{D}_k$,

$$\left| {H^{\beta_k}_{2\mu_k}}^{(m)}(Y_{n,j,k}) - \sum_{k_1=0}^{d-1} \frac{{H^{\beta_k}_{2\mu_k}}^{(m+k_1)}(X_{n,j,k})}{k_1!} (n\alpha_k - j)^{k_1} \left(\left(\frac{\alpha_k}{j}\right)^{\frac{1}{2\mu_k}} - \left(\frac{1}{n}\right)^{\frac{1}{2\mu_k}} \right)^{k_1} \right| \\ \leq \frac{C}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-c \left| X_{n,j,k} \right|^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

Proof We will apply Taylor's Theorem to bound the term on the left hand side of the inequality. We observe using the bounds of Lemma 2.3.3 on the derivatives of $H^{\beta_k}_{2\mu_k}$ that there exist two positive constants C, c such that

$$\forall (n,j) \in \mathcal{D}_k, \forall x \in [X_{n,j,k}, Y_{n,j,k}], \quad \left| {H^{\beta_k}_{2\mu_k}}^{(m+d)}(x) \right| \leq C \exp\left(-c \left| X_{n,j,k} \right|^{\frac{2\mu_k}{2\mu_k - 1}} \right). \tag{2.3.48}$$

We also observe that the mean value inequality implies that there exists a constant $C > 0$ such that

$$\forall (n,j) \in \mathcal{D}_k, \quad \left| \left(\frac{\alpha_k}{j}\right)^{\frac{1}{2\mu_k}} - \left(\frac{1}{n}\right)^{\frac{1}{2\mu_k}} \right| \leq \frac{C}{n^{1+\frac{1}{2\mu_k}}} |n\alpha_k - j|. \tag{2.3.49}$$

Combining Taylor's Theorem and both inequalities (2.3.48) and (2.3.49), we can prove the existence of two positive constants C, c such that for all $(n,j) \in \mathcal{D}_k$

$$\left| {H^{\beta_k}_{2\mu_k}}^{(m)}(Y_{n,j,k}) - \sum_{k_1=0}^{d-1} \frac{{H^{\beta_k}_{2\mu_k}}^{(m+k_1)}(X_{n,j,k})}{k_1!} (n\alpha_k - j)^{k_1} \left(\left(\frac{\alpha_k}{j}\right)^{\frac{1}{2\mu_k}} - \left(\frac{1}{n}\right)^{\frac{1}{2\mu_k}} \right)^{k_1} \right| \\ \leq \frac{C}{n^{d\left(1-\frac{1}{2\mu_k}\right)}} \left| X_{n,j,k} \right|^{2d} \exp\left(-c \left| X_{n,j,k} \right|^{\frac{2\mu_k}{2\mu_k - 1}} \right).$$

Since the function $x \mapsto x^{2d} \exp\left(-\frac{c}{2} x^{\frac{2\mu_k}{2\mu_k - 1}} \right)$ is bounded, our choice for d allows us to conclude. $\square$

Using Lemma 2.3.12, we have now approached the elements $\mathscr{G}^n_j$ via a linear combination of

$$\frac{(n\alpha_k - j)^{k_1}}{n^{\frac{k_1-k_2}{2\mu_k}} \left(\frac{j}{\alpha_k}\right)^{\frac{l+k_2}{2\mu_k}}} {H^{\beta_k}_{2\mu_k}}^{(m+k_1)} \left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}} \right) \quad \text{where } l \in \mathbb{N} \backslash \{0\}, m \in \mathbb{N}, k_1 \in \mathbb{N}, k_2 \in \{0, \dots, k_1\}. \tag{2.3.50}$$

We approach the terms in (2.3.50) using the following lemma.

Lemma 2.3.13. *We consider $s \in \mathbb{N}$, $m \in \mathbb{N}$, $l \in \mathbb{N} \backslash \{0\}$, $k_1 \in \mathbb{N}$, $k_2 \in \{0, \dots, k_1\}$ and $k \in \{1, \dots, K\}$. We define the function*

$$\Psi_q : x \in]0, +\infty[\to \frac{1}{x^q}.$$

If we consider $d \in \mathbb{N}$ such that

$$d \geq \frac{s+1}{2\mu_k - 1}$$

then there exist two constants $C, c > 0$ such that for all $(n,j) \in \mathcal{D}_k$,

$$\left| H_{2\mu_k}^{\beta_k\ (m+k_1)}(X_{n,j,k}) \frac{(n\alpha_k - j)^{k_1}}{n^{\frac{k_1-k_2}{2\mu_k}}} \left(\Psi_{\frac{l+k_2}{2\mu_k}}\left(\frac{j}{\alpha_k}\right) - \sum_{k_3=0}^{d-1} \frac{\Psi^{(k_3)}_{\frac{l+k_2}{2\mu_k}}(n)}{k_3!} \left(\frac{j}{\alpha_k} - n\right)^{k_3}\right)\right| \leq \frac{C}{n^{\frac{s+1}{2\mu_k}}} \exp\left(-c\left|X_{n,j,k}\right|^{\frac{2\mu_k}{2\mu_k-1}}\right)$$

with $X_{n,j,k} := \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$.

Proof We will apply Taylor's theorem to bound the term on the left hand side of the inequality. We observe that there exist two positive constants C, c such that

$$\forall (n,j) \in \mathcal{D}_k, \forall x \in \left[n, \frac{j}{\alpha_k}\right], \qquad \left|\Psi^{(d)}_{\frac{l+k_2}{2\mu_k}}(x)\right| \leq \frac{C}{n^{\frac{l+k_2}{2\mu_k}+d}}. \tag{2.3.51}$$

Thus, the inequality (2.3.51) and Taylor's theorem imply the existence of two positive constants C, c such that for all $(n,j) \in \mathcal{D}_k$

$$\left| H_{2\mu_k}^{\beta_k\ (m+k_1)}(X_{n,j,k}) \frac{(n\alpha_k - j)^{k_1}}{n^{\frac{k_1-k_2}{2\mu_k}}} \left(\Psi_{\frac{l+k_2}{2\mu_k}}\left(\frac{j}{\alpha_k}\right) - \sum_{k_3=0}^{d-1} \frac{\Psi^{(k_3)}_{\frac{l+k_2}{2\mu_k}}(n)}{k_3!} \left(\frac{j}{\alpha_k} - n\right)^{k_3}\right)\right| \leq \frac{C}{n^{\frac{l}{2\mu_k}+d\left(1-\frac{1}{2\mu_k}\right)}} \left|X_{n,j,k}\right|^{k_1+d} \exp\left(-c\left|X_{n,j,k}\right|^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

Since the function $x \mapsto x^{k_1+d}\exp\left(-\frac{c}{2}x^{\frac{2\mu_k}{2\mu_k-1}}\right)$ is bounded, our choice for d allows us to conclude. □

Lemmas 2.3.12 and 2.3.13 allow us to conclude the proof of Proposition 2.3.

2.3.5 Step 3: Construction of the polynomials $\mathscr{P}_\sigma^k$ satisfying Proposition 2.1 and Theorem 2.1

Now that Propositions 2.2 and 2.3 are proved, we will construct the polynomials $\mathscr{P}_\sigma^k$ in $\mathbb{C}[X,Y]$ which will verify Proposition 2.1 and Theorem 2.1. We start by introducing some notations.

We fix $k \in \{1,\dots,K\}$ and $s_k \in \mathbb{N}$. For $l \in \{0,\dots,s_k-1\}$, we define the coefficients $\mathscr{A}^k_{s_k,l,m} \in \mathbb{C}$ for $m \in \{(2\mu_k+1)l,\dots,(2\mu_k+s_k-1)l+s_k-1\}$ such that:

$$\forall \tau \in B_{\varepsilon_\star}(\underline{\tau}_k), \quad P_{s_k,k}(\tau)\frac{R_{s_k,k}(\tau)^l}{l!} = \sum_{m=(2\mu_k+1)l}^{(2\mu_k+s_k-1)l+s_k-1} \mathscr{A}^k_{s_k,l,m}(\tau-\underline{\tau}_k)^m. \tag{2.3.52}$$

where the polynomial functions $P_{s_k,k}$ and $R_{s_k,k}$ are defined in Lemma 2.3.4. Using Proposition 2.2 and equality (2.3.7), we prove that there exist two positive constants C, c such that for all $(n,j) \in \mathcal{D}_k$

$$\left| \mathscr{G}_j^n - \underline{z}_k^n \underline{\kappa}_k^j \sum_{l=0}^{s_k-1} \sum_{m=(2\mu_k+1)l}^{(2\mu_k+s_k-1)l+s_k-1} \frac{\mathscr{A}^k_{s_k,l,m}\alpha_k^{m+l}|\alpha_k|}{\left(\frac{j}{\alpha_k}\right)^{\frac{m-2\mu_k l+1}{2\mu_k}}} H_{2\mu_k}^{\beta_k\ (m)}(Y_{n,j,k}) \right| \leq \frac{C}{n^{\frac{s_k+1}{2\mu_k}}} \exp\left(-c\left(\frac{|n\alpha_k - j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right) \tag{2.3.53}$$

where $Y_{n,j,k} := \frac{n\alpha_k - j}{\left(\frac{j}{\alpha_k}\right)^{\frac{1}{2\mu_k}}}$.

We now want to apply Proposition 2.3, so we need to define an integer $d \in \mathbb{N}$ such that

$$d \geq \frac{s_k+1}{2\mu_k-1}.$$

We will consider that $d = s_k+1$ so that when we will do computations of the polynomials $\mathscr{P}_\sigma^k$ in Section 2.5, we will not have to distinguish the value of d depending on the value of μ_k. Then, for $l \in \{0,\dots,s_k-1\}$,

$m \in \{(2\mu_k+1)l, \dots, (2\mu_k+s_k-1)l+s_k-1\}$ and $k_1, k_3 \in \{0, \dots, s_k\}$, we define the coefficients

$$\begin{aligned}\mathscr{C}^k_{s_k,l,m,k_1,k_3} &:= \mathscr{A}^k_{s_k,l,m}\alpha_k^{m+l}|\alpha_k|\mathscr{B}^k_{m-2\mu_k l+1,k_1,k_3} \\ &= \frac{\mathscr{A}^k_{s_k,l,m}\alpha_k^{m+l-k_3}|\alpha_k|}{k_1!k_3!}\sum_{k_2=0}^{k_1}\binom{k_1}{k_2}(-1)^{k_1-k_2}\left(\prod_{k_4=0}^{k_3-1}\frac{m-2\mu_k l+1+k_2}{2\mu_k}+k_4\right)\end{aligned} \tag{2.3.54}$$

where the coefficients $\mathscr{B}^k_{m-2\mu_k l+1,k_1,k_3}$ are defined in Proposition 2.3. Combining the result of Proposition 2.3 with the estimates (2.3.53), we prove the existence of two positive constants C, c such that for all $(n,j) \in \mathcal{D}_k$

$$\left|\mathscr{G}^n_j - \underline{z}^n_k\underline{\kappa}^j_k\sum_{l=0}^{s_k-1}\sum_{m=(2\mu_k+1)l}^{(2\mu_k+s_k-1)l+s_k-1}\sum_{k_1=0}^{s_k}\sum_{k_3=0}^{s_k}\frac{\mathscr{C}^k_{s_k,l,m,k_1,k_3}}{n^{\frac{m-2\mu_k l+k_3(2\mu_k-1)+1}{2\mu_k}}}{X_{n,j,k}}^{k_1+k_3}{H^{\beta_k}_{2\mu_k}}^{(m+k_1)}(X_{n,j,k})\right| \leq \frac{C}{n^{\frac{s_k+1}{2\mu_k}}}\exp\left(-c\left|X_{n,j,k}\right|^{\frac{2\mu_k}{2\mu_k-1}}\right) \tag{2.3.55}$$

with $X_{n,j,k} := \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$. For $\sigma \in \{1, \dots, s_k\}$, we define the polynomial

$$\mathscr{P}^k_\sigma(X,Y) := \sum_{l=0}^{s_k-1}\sum_{m=(2\mu_k+1)l}^{(2\mu_k+s_k-1)l+s_k-1}\sum_{k_1=0}^{s_k}\sum_{k_3=0}^{s_k}\mathbb{1}_{m-2\mu_k l+k_3(2\mu_k-1)+1=\sigma}\mathscr{C}^k_{s_k,l,m,k_1,k_3}X^{k_1+k_3}Y^{m+k_1} \in \mathbb{C}[X,Y]. \tag{2.3.56}$$

Using the estimates on the derivatives of $H^\beta_{2\mu}$ (Lemma 2.3.3) to take care of the terms where $m - 2\mu_k l + k_3(2\mu_k-1)+1 \geq s_k+1$, the inequality (2.3.55) implies that the polynomials $\mathscr{P}^k_\sigma$ verify the estimates (2.3.6) of Proposition 2.1. Proposition 2.1 is proved and Theorem 2.1 in the case where the elements α_k are supposed to be distinct ensues from Proposition 2.1 and inequalities (2.3.4) and (2.3.5).

2.4 Closing arguments on Theorem 2.1 and proof of Corollary 1

2.4.1 Proof of Theorem 2.1 when the elements α_k can be equal

As we said in the beginning on Section 2.3, we supposed in the proof that the elements α_k were distinct from one another. In the case where the α_k can be equal, there are some changes that need to be done but the calculations remain similar. Most modifications will happen on the part of the proof contained in Section 2.3.3.

First, just as in Section 2.3.1, we would define $\overline{\delta}_k$, $\underline{\delta}_k$ and $\mathcal{D}_k$ in the same manner but with the added condition that if $\alpha_k = \alpha_l$, then $\overline{\delta}_k = \overline{\delta}_l$ and $\underline{\delta}_k = \underline{\delta}_l$.

If we consider $k_0 \in \{1, \dots, K\}$, we define

$$\mathcal{J}_{k_0} := \{k \in \{1, \dots, K\}, \quad \alpha_k = \alpha_{k_0}\}.$$

We observe that for $k \in \mathcal{J}_{k_0}$, we have $\mathcal{D}_k = \mathcal{D}_{k_0}$ because of our new condition.

Lemmas 2.3.1 and 2.3.2 remain true. The inequality (2.3.4) thus remains true, however inequality (2.3.5) now becomes that for $k_0 \in \{1, \dots, K\}$, there exist two constants $C, c > 0$ such that for all $(n,j) \in \mathcal{D}_{k_0}$

$$\begin{aligned}\left|\mathscr{G}^n_j - \sum_{k=1}^{K}\sum_{\sigma=1}^{s_k}\frac{\underline{z}^n_k\underline{\kappa}^j_k}{n^{\frac{\sigma}{2\mu_k}}}\left(\mathscr{P}^k_\sigma\left(X_{n,j,k}, \frac{d}{dx}\right)H^{\beta_k}_{2\mu_k}\right)(X_{n,j,k})\right| &\leq \sum_{\substack{k=1\\k\notin\mathcal{J}_{k_0}}}^{K}\frac{C}{n^{\frac{s_k+1}{2\mu_k}}}\exp\left(-c\left|X_{n,j,k}\right|^{\frac{2\mu_k}{2\mu_k-1}}\right) \\ &+ \left|\mathscr{G}^n_j - \sum_{k\in\mathcal{J}_{k_0}}\sum_{\sigma=1}^{s_k}\frac{\underline{z}^n_k\underline{\kappa}^j_k}{n^{\frac{\sigma}{2\mu_k}}}\left(\mathscr{P}^k_\sigma\left(X_{n,j,k}, \frac{d}{dx}\right)H^{\beta_k}_{2\mu_k}\right)(X_{n,j,k})\right|.\end{aligned}$$

Therefore, to prove Theorem 2.1, we now have to prove the following proposition which is a modification of Proposition 2.1.

Proposition 2.4 (Modified Proposition 2.1)**.** *For all $k_0 \in \{1, \dots, K\}$ and $(s_k)_{k\in\mathcal{J}_{k_0}} \in \mathbb{N}^{\mathcal{J}_{k_0}}$, there exist a family of polynomials $(\mathscr{P}^k_\sigma)_{\sigma\in\{1,\dots,s_k\}}$ in $\mathbb{C}[X,Y]$ for each $k \in \mathcal{J}_{k_0}$ and two positive constants C, c such that for*

$(n,j)\in\mathcal{D}_{k_0}$

$$\left|\mathscr{G}_j^n-\sum_{k\in\mathcal{J}_{k_0}}\sum_{\sigma=1}^{s_k}\frac{z_k^n\kappa_k^j}{n^{\frac{\sigma}{2\mu_k}}}\left(\mathscr{P}_\sigma^k\left(X_{n,j,k},\frac{d}{dx}\right)H_{2\mu_k}^{\beta_k}\right)(X_{n,j,k})\right|\leq\sum_{k\in\mathcal{J}_{k_0}}\frac{C}{n^{\frac{s_k+1}{2\mu_k}}}\exp\left(-c\left|X_{n,j,k}\right|^{\frac{2\mu_k}{2\mu_k-1}}\right)$$

with $X_{n,j,k}:=\frac{n\alpha_k-j}{n^{\frac{1}{2\mu_k}}}$.

Just as in the case where the elements α_k were supposed distinct, if Proposition 2.4 is verified, then the families of polynomials $(\mathscr{P}_\sigma^k)_{k,\sigma}$ constructed in Proposition 2.4 will also verify the estimates (2.1.7) of Theorem 2.1. Since the equality (2.3.7) and Proposition 2.3 remain true, to prove Proposition 2.4, we only have to prove the following proposition which is a modification of Proposition 2.2.

Proposition 2.5 (Modified Proposition 2.2)**.** *For all* $k_0\in\{1,\dots,K\}$ *and for all* $(s_k)_{k\in\mathcal{J}_{k_0}}\in\mathbb{N}\backslash\{0\}^{\mathcal{J}_{k_0}}$*, there exist two positive constants* C,c *such that for all* $(n,j)\in\mathcal{D}_{k_0}$

$$\left|\mathscr{G}_j^n-\sum_{k\in\mathcal{J}_{k_0}}\frac{z_k^n\kappa_k^j}{2\pi}\int_{-\infty}^{+\infty}P_{s_k,k}(it+\underline{\tau}_k)\left(\sum_{l=0}^{s_k-1}\frac{(jR_{s_k,k}(it+\underline{\tau}_k))^l}{l!}\right)\exp\left(it\left(n-\frac{j}{\alpha_k}\right)-\frac{j}{\alpha_k}\frac{\beta_k}{\alpha_k^{2\mu_k}}t^{2\mu_k}\right)dt\right|$$
$$\leq\sum_{k\in\mathcal{J}_{k_0}}\frac{C}{n^{\frac{s_k+1}{2\mu_k}}}\exp\left(-c\left(\frac{|n\alpha_k-j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right)$$

where the polynomial functions $P_{s_k,k}$ *and* $R_{s_k,k}$ *have explicit expression defined in Lemma 2.3.4.*

Therefore, there just remains to prove Proposition 2.5 and Theorem 2.1 will ensue. We recall that, to prove Proposition 2.2 in the case where the elements α_k were distinct from one another, we found an expression of the elements $\mathscr{G}_j^n$ as an integral along the path Γ_k

$$\forall n\in\mathbb{N}\backslash\{0\},\forall j\in\mathbb{Z},\quad\mathscr{G}_j^n=\frac{1}{2i\pi}\int_{\Gamma_k}e^{n\tau}\mathbf{G}_j(\tau)d\tau$$

and used the triangular inequality to find the inequality (2.3.32) that we recall here

$$\left|\mathscr{G}_j^n-\frac{1}{2i\pi}\int_{\Gamma_{k,in}}P_{s,k}(\tau)\left(\sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right)e^{n\tau}e^{j\varphi_k(\tau)}d\tau\right|\leq\frac{1}{2\pi}\sum_{l=1}^{8}E_l.$$

We then bounded all the terms E_i to find an estimate on

$$\left|\mathscr{G}_j^n-\frac{1}{2i\pi}\int_{\Gamma_{k,in}}P_{s,k}(\tau)\left(\sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right)e^{n\tau}e^{j\varphi_k(\tau)}d\tau\right|.$$

In the case where the elements α_k are no longer supposed to be distinct, the reasoning is the same but with a better suited choice of path to express the elements $\mathscr{G}_j^n$. We fix $k_0\in\{1,\dots,K\}$ and introduce the path $\widetilde{\Gamma}_{k_0}$ which is the ray $\{-\eta+it,t\in[-\pi,\pi]\}$ deformed into the path $\Gamma_{k,in}$ inside the balls $B_\varepsilon(\underline{\tau}_k)$ for $k\in\mathcal{J}_{k_0}$ (see Figure 2.6). Using Cauchy's formula and taking into account the "$2i\pi$-periodicity" of $\mathbf{G}_j(\tau)$, we have that

$$\forall n\in\mathbb{N}\backslash\{0\},\forall j\in\mathbb{Z},\quad\mathscr{G}_j^n=\frac{1}{2i\pi}\int_{\widetilde{\Gamma}_{k_0}}e^{n\tau}\mathbf{G}_j(\tau)d\tau.$$

We end up with an inequality similar to (2.3.32).

$$\left|\mathscr{G}_j^n-\frac{1}{2i\pi}\sum_{k\in\mathcal{J}_{k_0}}\int_{\Gamma_{k,in}}P_{s_k,k}(\tau)\left(\sum_{l=0}^{s_k-1}\frac{(jR_{s_k,k}(\tau))^l}{l!}\right)e^{n\tau}e^{j\varphi_k(\tau)}d\tau\right|\leq\frac{1}{2\pi}\left(E_{out}+\sum_{k\in\mathcal{J}_{k_0}}\sum_{l=2}^{8}E_{l,k}\right)\tag{2.4.1}$$

where $E_{l,k}$ has the same definition as E_l in (2.3.32) but depends on the $k\in\mathcal{J}_{k_0}$ we consider. The term E_{out} is similar to E_1 in (2.3.32) and is equal to

$$E_{out}=\left|\int_{\widetilde{\Gamma}_{k_0,out}}e^{n\tau}\mathbf{G}_j(\tau)d\tau\right|,$$

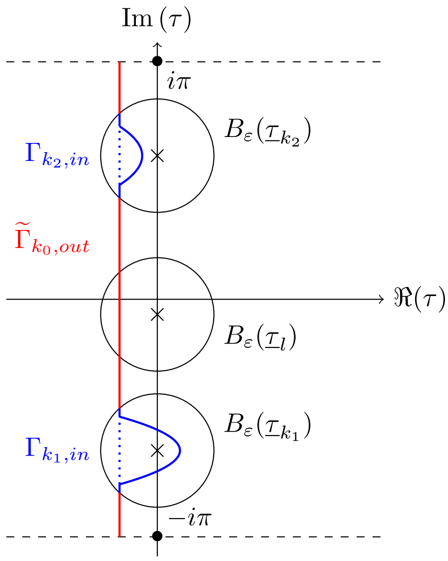

Figure 2.6 – A representation of the path $\widetilde{\Gamma}_{k_0}$. Inside the balls $B_\varepsilon(\underline{\tau}_k)$ where k belongs to $\mathcal{J}_{k_0}$, it follows the path $\Gamma_{k,in}$ composed of $\Gamma_{k,res}$ and $\Gamma_{k,p}$. For $l \in \{1,\dots,K\}$, if there is no $k \in \mathcal{J}_{k_0}$ such that $\underline{\tau}_k = \underline{\tau}_l$, then the path $\widetilde{\Gamma}_{k_0}$ inside $B_\varepsilon(\underline{\tau}_l)$ just corresponds to the ray $\{-\eta + it, t \in [-\pi,\pi]\}$.

where $\widetilde{\Gamma}_{k_0,out}$ corresponds to the part of $\widetilde{\Gamma}_{k_0}$ outside the balls $B_\varepsilon(\underline{\tau}_k)$ for $k \in \mathcal{J}_{k_0}$ (see the red path on Figure 2.6). Reasoning in the same manner as in the case where the elements α_k are different from one another, we get estimates on the different terms. The minor modifications are left to the reader. Notice that Lemmas 2.3.10 and 2.3.11 are still verified, Proposition 2.5 ensues. Therefore, Theorem 2.1 in the case where the elements α_k can be equal is proved for the same polynomials $\mathscr{P}_\sigma^k$ given in Section 2.3.5.

2.4.2 Proof of Corollary 1

We are now going to prove Corollary 1 that we recall here:

Corollary (Corollary 1)**.** *Let $a \in \ell^1(\mathbb{Z})$ which verifies Hypotheses 2.1 and 2.4. If there exists some integer $J \in \mathbb{Z}$ such that the sequence $b := (a_{j+J})_{j\in\mathbb{Z}}$ verifies Hypotheses 2.2 and 2.3, then for all $s_1,\dots,s_K \in \mathbb{N}$ there exist a family of polynomials $(\mathscr{P}_\sigma^k)_{\sigma\in\{1,\dots,s_k\}}$ in $\mathbb{C}[X,Y]$ for each $k \in \{1,\dots,K\}$ and two positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$ and $j \in \mathbb{Z}$*

$$\left| \mathscr{G}_j^n - \sum_{k=1}^K \sum_{\sigma=1}^{s_k} \frac{\underline{z}_k^n \underline{\kappa}_k^j}{n^{\frac{\sigma}{2\mu_k}}} \left(\mathscr{P}_\sigma^k\left(X_{n,j,k}, \frac{d}{dx}\right) H_{2\mu_k}^{\beta_k}\right)(X_{n,j,k}) \right| \leq \sum_{k=1}^K \frac{C}{n^{\frac{s_k+1}{2\mu_k}}} \exp\left(-c|X_{n,j,k}|^{\frac{2\mu_k}{2\mu_k-1}}\right)$$

with $X_{n,j,k} = \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$.

We consider that a satisfies the hypotheses of Corollary 1. As we said just before we introduced the corollary, we observe that if we define $\widetilde{F}$ the symbol associated with b, then we have that

$$\forall \kappa \in \mathbb{S}^1, \quad \widetilde{F}(\kappa) = \kappa^{-J} F(\kappa).$$

and we have for $k \in \{1,\dots,K\}$

$$\widetilde{F}(\underline{\kappa}_k e^{i\xi}) \underset{\xi\to 0}{=} \underline{\kappa}_k^{-J} \underline{z}_k \exp(-i(\alpha_k + J)\xi - \beta_k \xi^{2\mu_k} + o(|\xi|^{2\mu_k})). \tag{2.4.2}$$

We fix $s_1,\dots,s_K \in \mathbb{N}$. Applying Theorem 2.1 for the sequence b, there exist two positive constants C, c and a family of polynomials $(\mathscr{P}_\sigma^k)_{\sigma\in\{1,\dots,s_k\}}$ in $\mathbb{C}[X,Y]$ for each $k \in \{1,\dots,K\}$ such that for all $n \in \mathbb{N}\backslash\{0\}$ and $j \in \mathbb{Z}$:

$$\left| (\mathscr{L}_b^n \delta)_j - \sum_{k=1}^K \sum_{\sigma=1}^{s_k} \frac{\underline{z}_k^n \underline{\kappa}_k^j}{n^{\frac{\sigma}{2\mu_k}}} \left(\mathscr{P}_\sigma^k\left(\frac{n(\alpha_k+J)-j}{n^{\frac{1}{2\mu_k}}}, \frac{d}{dx}\right) H_{2\mu_k}^{\beta_k}\right)\left(\frac{n(\alpha_k+J)-j}{n^{\frac{1}{2\mu_k}}}\right) \right|$$

$$\leq \sum_{k=1}^{K} \frac{C}{n^{\frac{s_k+1}{2\mu_k}}} \exp\left(-c\left(\frac{|n(\alpha_k+J)-j|}{n^{\frac{1}{2\mu_k}}}\right)^{\frac{2\mu_k}{2\mu_k-1}}\right).$$

By observing that

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \quad (\mathscr{L}_b^n \delta)_j = (\mathscr{L}_a^n \delta)_{j-nJ} = \mathscr{G}_{j-nJ}^n,$$

we conclude the proof of Corollary 1.

2.5 Computations of the polynomials $\mathscr{P}_\sigma^k$

Now that Theorem 2.1 is proved, we want to compute the polynomials $\mathscr{P}_\sigma^k$ defined with (2.3.56) in the proof of Theorem 2.1. We separate this section in three parts:

— The coefficients of the polynomials $\mathscr{P}_\sigma^k$ depend on the elements $\mathscr{A}_{s_k,l,m}^k$ defined as (2.3.52). Based on the definition of the polynomials $P_{s_k,k}$ and $Q_{s_k,k}$ defined in Lemma 2.3.4, the elements $\mathscr{A}_{s_k,l,m}^k$ are expressed using derivatives of ϖ_k at $\underline{\tau}_k$. In Section 2.5.1, we present a reliable way to compute the value $\varpi_k^{(n)}(\underline{\tau}_k)$.

— In Section 2.5.2, we compute the polynomials $\mathscr{P}_\sigma^k$ for $\sigma = 1, 2$. We compare those results with the asymptotic expansion determined in [RS15, Theorem 1.2].

— In Section 2.5.3, we compute numerically the polynomials $\mathscr{P}_\sigma^k$ and verify the sharpness of the estimates (2.1.7) in Theorem 2.1 for two specific examples of sequences a:

  ⋆ A case where the sequence a has real non negative coefficients.

  ⋆ The sequence a associated to the O3 scheme for the transport equation.

2.5.1 Computing the derivatives of ϖ_k at $\underline{\tau}_k$

The coefficients $\mathscr{A}_{s_k,l,m}^k$ defined in (2.3.52) are expressed using the derivatives of ϖ_k at $\underline{\tau}_k$. We now present a reliable way to compute $\varpi_k^{(n)}(\underline{\tau}_k)$. For $\tau \in B_{\varepsilon_\star}(\underline{\tau}_k)$, $e^{\varpi_k(\tau)} = \kappa_k(e^\tau)$ is an eigenvalue of $\mathbb{M}(e^\tau)$. Lemma 2.2.2 implies that

$$\forall \tau \in B_{\varepsilon_\star}(\underline{\tau}_k), \quad F(e^{\varpi_k(\tau)}) = e^\tau. \tag{2.5.1}$$

For all $n \in \mathbb{N}$, we define the moment function

$$M_n : \begin{array}{ccc} \mathbb{C}\backslash\{0\} & \to & \mathbb{C} \\ \kappa & \mapsto & \sum_{j\in\mathbb{Z}} j^n a_j \kappa^j \end{array}. \tag{2.5.2}$$

We observe that we have the equality $M_0 = F$ and

$$\forall n \in \mathbb{N}, \forall \kappa \in \mathbb{C}\backslash\{0\}, \quad M_{n+1}(\kappa) = \kappa \frac{dM_n}{d\kappa}(\kappa),$$

thus

$$\forall n \in \mathbb{N}, \forall \tau \in B_{\varepsilon_\star}(\underline{\tau}_k), \quad \frac{d}{d\tau}\left(M_n\left(e^{\varpi_k(\tau)}\right)\right) = \varpi_k'(\tau) M_{n+1}\left(e^{\varpi_k(\tau)}\right). \tag{2.5.3}$$

We will differentiate the equality (2.5.1) and use the equality (2.5.3) to find an expression of $\varpi_k^{(n)}(\underline{\tau}_k)$. To do so, we introduce the Bell polynomials (see [Com74], Chapter 3.3) defined for $n \in \mathbb{N}$ and $j \in \{1, \dots, n\}$ as

$$B_{n,j}(X_1, \dots, X_{n+1-j}) := \sum \frac{n!}{l_1! \dots l_{n+1-j}!}\left(\frac{X_1}{1!}\right)^{l_1} \dots \left(\frac{X_{n+1-j}}{(n+1-j)!}\right)^{l_{n+1-j}}$$

where the sum is taken over the integers $l_1, \dots, l_{n+1-j} \in \mathbb{N}$ such that

$$\begin{aligned} j &= l_1 + l_2 + \dots + l_{n+1-j}, \\ n &= l_1 + 2l_2 + \dots + (n+1-j)l_{n+1-j}. \end{aligned}$$

The Bell polynomials $B_{n,j}$ verify the following equalities:

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \{1, \dots, n\}, \quad B_{n,j} = \sum_{i=1}^{n+1-j} \binom{n-1}{i-1} X_i B_{n-i,j-1}, \tag{2.5.4}$$

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \{1, \dots, n\}, \forall i \in \{1, \dots, n+1-j\}, \quad \frac{\partial B_{n,j}}{\partial X_i} = \binom{n}{i} B_{n-i,j-1}. \tag{2.5.5}$$

We can now prove the following lemma which allows us to express recursively the derivatives of ϖ_k at $\underline{\tau}_k$ with the moments $M_n(\underline{\kappa}_k)$.

Lemma 2.5.1. *For all $k \in \{1, \dots, K\}$, we have*

$$\varpi_k'(\underline{\tau}_k) = \frac{\underline{z}_k}{M_1(\underline{\kappa}_k)},$$
$$\forall n \geq 2, \quad \varpi_k^{(n)}(\underline{\tau}_k) = \frac{1}{M_1(\underline{\kappa}_k)} \left(\underline{z}_k - \sum_{j=2}^{n} M_j(\underline{\kappa}_k) B_{n,j} \left(\varpi_k'(\underline{\tau}_k), \dots, \varpi_k^{(n+1-j)}(\underline{\tau}_k) \right) \right).$$

Proof Using the equalities (2.5.3), (2.5.4) and (2.5.5), we can prove recursively the following equality for all $n \in \mathbb{N} \setminus \{0\}$ and $\tau \in B_{\varepsilon_\star}(\underline{\tau}_k)$ which looks like Faà di Bruno's formula :

$$\frac{d^n}{d\tau^n} \left(M_0(e^{\varpi_k(\tau)}) \right) = \sum_{j=1}^{n} M_j(e^{\varpi_k(\tau)}) B_{n,j} \left(\varpi_k'(\tau), \dots, \varpi_k^{(n+1-j)}(\tau) \right). \tag{2.5.6}$$

Using the equalities (2.5.6), (2.5.1) and $M_0 = F$, we conclude the proof of Lemma 2.5.1. □

2.5.2 Computation of $\mathscr{P}_\sigma^k$ for $\sigma = 1, 2$

In this section, we will compute the polynomials $\mathscr{P}_\sigma^k$ for $\sigma = 1, 2$. The goal is to compare the asymptotic expansion (2.1.7) with the result of [RS15, Theorem 1.2] and with the local limit theorem (see [Pet75, Chapter VII, Theorem 13]). We consider $k \in \{1, \dots, K\}$ and $s_k \in \mathbb{N} \setminus \{0\}$.

- We start to compute the polynomials $\mathscr{P}_1^k$. We have using (2.3.56)

$$\begin{aligned}
\mathscr{P}_1^k &= \sum_{l=0}^{s_k-1} \sum_{m=(2\mu_k+1)l}^{(2\mu_k+s_k-1)l+s_k-1} \sum_{k_1=0}^{s_k} \sum_{k_3=0}^{s_k} \mathbb{1}_{m-2\mu_k l + k_3(2\mu_k-1)+1=1} \mathscr{C}_{s_k,l,m,k_1,k_3}^k X^{k_1+k_3} Y^{m+k_1} \\
&= \sum_{k_1=0}^{s_k} \mathscr{C}_{s_k,0,0,k_1,0}^k X^{k_1} Y^{k_1}.
\end{aligned}$$

Furthermore, for $k_1 \in \{0, \dots, s_k\}$, we have using the definition (2.3.54) of $\mathscr{C}_{s_k,l,m,k_1,k_3}^k$ that

$$\mathscr{C}_{s_k,0,0,k_1,0}^k = \begin{cases} 0 & \text{if } k_1 \geq 1, \\ \mathscr{A}_{s_k,0,0}^k |\alpha_k| & \text{if } k_1 = 0. \end{cases}$$

Furthermore, using the equality (2.3.52) and the asymptotic expansion (2.3.16), we have

$$\mathscr{A}_{s_k,0,0}^k = -\mathrm{sgn}(\alpha_k) \varpi_k'(\underline{\tau}_k) = \frac{1}{|\alpha_k|}.$$

We then have

$$\mathscr{C}_{s_k,0,0,k_1,0}^k = \mathscr{A}_{s_k,0,0}^k |\alpha_k| = 1.$$

Therefore, we have proved that

$$\forall s_k \in \mathbb{N} \setminus \{0\}, \quad \mathscr{P}_1^k = 1. \tag{2.5.7}$$

Theorem 2.1 implies that there exist two positive constants C, c such that

$$\forall (n,j) \in \mathbb{N} \setminus \{0\} \times \mathbb{Z}, \quad \left| \mathscr{G}_j^n - \sum_{k=1}^{K} \frac{\underline{z}_k^n \underline{\kappa}_k^j}{n^{\frac{1}{2\mu_k}}} H_{2\mu_k}^{\beta_k} \left(\frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}} \right) \right| \leq \sum_{k=1}^{K} \frac{C}{n^{\frac{1}{\mu_k}}} \exp\left(-c \left(\frac{|n\alpha - j|}{n^{\frac{1}{2\mu_k}}} \right)^{\frac{2\mu_k}{2\mu_k - 1}} \right). \tag{2.5.8}$$

The estimate (2.5.8) deduced from Theorem 2.1 gives us the same leading term for the asymptotic behavior of $\mathscr{G}_j^n$ as expected from [RS15, Theorem 1.2].

- We now compute the polynomials $\mathscr{P}_2^k$. We have using (2.3.56)

$$\mathscr{P}_2^k = \sum_{l=0}^{s_k-1} \sum_{m=(2\mu_k+1)l}^{(2\mu_k+s_k-1)l+s_k-1} \sum_{k_1=0}^{s_k} \sum_{k_3=0}^{s_k} \mathbb{1}_{m-2\mu_k l + k_3(2\mu_k-1)+1=2} \mathscr{C}_{s_k,l,m,k_1,k_3}^k X^{k_1+k_3} Y^{m+k_1}$$

$$= \sum_{k_1=0}^{s_k} \mathbb{1}_{\mu_k=1}\mathscr{C}^k_{s_k,0,0,k_1,1}X^{1+k_1}Y^{k_1} + \mathscr{C}^k_{s_k,0,1,k_1,0}X^{k_1}Y^{1+k_1} + \mathscr{C}^k_{s_k,1,2\mu_k+1,k_1,0}X^{k_1}Y^{2\mu_k+1+k_1}.$$

Furthermore, for $k_1 \in \{0,\dots,s_k\}$, we have using the definition (2.3.54) of $\mathscr{C}^k_{s_k,l,m,k_1,k_3}$ that

$$\mathscr{C}^k_{s_k,0,1,k_1,0} = \begin{cases} 0 & \text{if } k_1 \geq 1, \\ \mathscr{A}^k_{s_k,0,1}\alpha_k|\alpha_k| & \text{if } k_1 = 0, \end{cases}$$

$$\mathscr{C}^k_{s_k,1,2\mu_k+1,k_1,0} = \begin{cases} 0 & \text{if } k_1 \geq 1, \\ \mathscr{A}^k_{s_k,1,2\mu_k+1}\alpha_k^{2\mu_k+2}|\alpha_k| & \text{if } k_1 = 0, \end{cases}$$

$$\begin{aligned}\mathscr{C}^k_{s_k,0,0,k_1,1} &= \mathscr{A}^k_{s_k,0,0}\alpha_k^{-1}|\alpha_k|\sum_{k_2=0}^{k_1}\binom{k_1}{k_2}(-1)^{k_1-k_2}\frac{k_2+1}{2\mu_k} \\ &= \begin{cases} 0 & \text{if } k_1 \geq 2, \\ \frac{\mathscr{A}^k_{s_k,0,0}\operatorname{sgn}(\alpha_k)}{2\mu_k} & \text{if } k_1 = 0,1. \end{cases}\end{aligned}$$

Also, using the equality (2.3.52) and Lemma 2.5.1, we have

$$\mathscr{A}^k_{s_k,0,0} = -\operatorname{sgn}(\alpha_k)\varpi_k'(\underline{\tau}_k),$$

$$\mathscr{A}^k_{s_k,0,1} = -\operatorname{sgn}(\alpha_k)\varpi_k^{(2)}(\underline{\tau}_k),$$

$$\mathscr{A}^k_{s_k,1,2\mu_k+1} = -\operatorname{sgn}(\alpha_k)\varpi_k'(\underline{\tau}_k)\frac{\varpi_k^{(2\mu_k+1)}(\underline{\tau}_k)}{(2\mu_k+1)!}.$$

Thus, for all $s_k \in \mathbb{N}\setminus\{0\}$,

$$\mathscr{P}^k_2 = \mathbb{1}_{\mu_k=1}\left(-\frac{\varpi_k'(\underline{\tau}_k)}{2\mu_k}\right)\left(X + X^2Y\right) - \alpha_k^2\varpi_k^{(2)}(\underline{\tau}_k)Y - \alpha_k^{2\mu_k+3}\varpi_k'(\underline{\tau}_k)\frac{\varpi_k^{(2\mu_k+1)}(\underline{\tau}_k)}{(2\mu_k+1)!}Y^{2\mu_k+1}. \tag{2.5.9}$$

Theorem 2.1 thus implies that there exist two positive constants C, c such that for all $(n,j) \in \mathbb{N}\setminus\{0\}\times\mathbb{Z}$

$$\left|\mathscr{G}^n_j - \underline{z}_k^n\underline{\kappa}_k^j\sum_{k=1}^{K}\frac{1}{n^{\frac{1}{2\mu_k}}}H^{\beta_k}_{2\mu_k}(X_{n,j,k}) + \frac{1}{n^{\frac{1}{\mu_k}}}\mathscr{P}^k_2\left(X_{n,j,k},\frac{d}{dx}\right)H^{\beta_k}_{2\mu_k}(X_{n,j,k})\right| \leq \sum_{k=1}^{K}\frac{C}{n^{\frac{3}{2\mu_k}}}\exp\left(-c\left|X_{n,j,k}\right|^{\frac{2\mu_k}{2\mu_k-1}}\right) \tag{2.5.10}$$

with $X_{n,j,k} := \frac{n\alpha_k - j}{n^{\frac{1}{2\mu_k}}}$.

⋆ When $\mu_k \geq 2$, the asymptotic expansion (2.3.16) implies that $\varpi_k^{(2)}(\underline{\tau}_k) = 0$. Thus, the equality (2.5.9) becomes

$$\mathscr{P}^k_2 = -\alpha_k^{2\mu_k+3}\varpi_k'(\underline{\tau}_k)\frac{\varpi_k^{(2\mu_k+1)}(\underline{\tau}_k)}{(2\mu_k+1)!}Y^{2\mu_k+1}. \tag{2.5.11}$$

⋆ We now look at the case $\mu_k = 1$. The equality (2.5.9) becomes

$$\mathscr{P}^k_2 = \left(-\frac{\varpi_k'(\underline{\tau}_k)}{2}\right)\left(X + X^2Y\right) - \alpha_k^2\varpi_k^{(2)}(\underline{\tau}_k)Y - \alpha_k^5\varpi_k'(\underline{\tau}_k)\frac{\varpi_k^{(3)}(\underline{\tau}_k)}{6}Y^3.$$

As we said in the introduction of the paper, the polynomials satisfying Theorem 2.1 are not unique. We will now propose a more convenient choice of polynomials to replace $\mathscr{P}_{k,s_k,2}$. Using Lemma 2.1.1, if we define the polynomial

$$\mathcal{Q}^k_2(X,Y) = \left(-2\beta_k\varpi_k'(\underline{\tau}_k) - \alpha_k^2\varpi_k^{(2)}(\underline{\tau}_k)\right)Y + \left(-2\beta_k^2\varpi_k'(\underline{\tau}_k) - \alpha_k^5\varpi_k'(\underline{\tau}_k)\frac{\varpi_k^{(3)}(\underline{\tau}_k)}{6}\right)Y^3 \in \mathbb{C}[X,Y]$$

we have

$$\mathscr{P}^k_2\left(.,\frac{d}{dx}\right)H^{\beta_k}_2 = \mathcal{Q}^k_2\left(.,\frac{d}{dx}\right)H^{\beta_k}_2. \tag{2.5.12}$$

We can then replace $\mathscr{P}^k_2$ with $\mathcal{Q}^k_2$ in the estimate (2.5.10) when $\mu_k = 1$. This allows us to express the second term of the asymptotic expansion using a linear combination of derivatives of $H^{\beta_k}_2$ since $\mathcal{Q}^k_2(X,Y)$ does not have any terms where X intervenes. We notice that the asymptotic expansion (2.3.16) implies that

$$\frac{\varpi_k^{(2)}(\underline{\tau}_k)}{2} = \frac{\beta_k}{\alpha_k^3}. \tag{2.5.13}$$

Using Lemma 2.5.1 and equality (2.5.13), we can prove that actually

$$\mathcal{Q}_2^k(X,Y) = -\frac{1}{6\underline{z}_k^2}\left(\underline{z}_k^2 M_3(\underline{\kappa}_k) - 3\underline{z}_k M_2(\underline{\kappa}_k)M_1(\underline{\kappa}_k) + 2M_1(\underline{\kappa}_k)^3\right)Y^3. \tag{2.5.14}$$

We will see in Section 2.5.3 that, in the probabilistic case we presented in the introduction of the paper that motivated our result, this expression of $\mathcal{Q}_2^k$ gives exactly the second term of the asymptotic expansion (2.1.1) when we apply the local limit theorem (which is fortunate).

2.5.3 Numerical examples

In this section, we consider some examples of elements $a \in \ell^1(\mathbb{Z})$ which satisfy the conditions of Theorem 2.1 and see how sharp the estimations we found are.

Probability distribution : real non negative sequences

First, we consider the case where a has real non negative coefficients. If we introduce the sequence $b = (a_{-j})_{j\in\mathbb{Z}}$, then b is the probability distribution of some random variable X supported on $\mathbb{Z}$. We observe that $L_b = \mathscr{L}_a$, so, recalling that $b^n = b * \ldots * b$, we have

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \quad b_j^n = \mathscr{G}_j^n.$$

We will settle on $a \in \ell^1(\mathbb{Z})$ such that $a_j = 0$ for $j \neq -1, 0, 1$ and

$$a_{-1} = 2/3, a_0 = 1/6, a_1 = 1/6.$$

This sequence verifies Hypothesis 2.1. In this case, we have $r = p = 1$. Also, $F(1) = 1$ and

$$\forall \kappa \in \mathbb{S}^1\backslash\{1\}, \quad |F(\kappa)| < 1.$$

The function F satisfies that

$$F(e^{i\xi}) \underset{\xi\to 0}{=} \exp(-i\alpha\xi - \beta\xi^2 + o(\xi^2))$$

where $\alpha = \mathbb{E}(X) = \frac{1}{2}$ and $\beta = \frac{V(X)}{2} = \frac{7}{24}$. We have $\mu = 1$ in this case and Hypothesis 2.2 is satisfied with $K = 1$, $\underline{\kappa}_1 = 1$ and $\underline{z}_1 = 1$. It also directly satisfies Hypothesis 2.3 since $K = 1$. Since $K = 1$, we lose the subscript k in most the notations that follow. The sequence a verifies Hypotheses 2.1, 2.2 and 2.3, so we can apply Theorem 2.1. As an example, we will apply Theorem 2.1 for $s = 2$ and use the calculations of Section 2.5.2 to determine the terms of the asymptotic expansion:

- Using the equality (2.5.7) on $\mathscr{P}_1^k$, the leading order term of the asymptotic expansion given by Theorem 2.1 is

$$\begin{aligned}\frac{1}{\sqrt{n}}H_2^\beta\left(\frac{n\alpha - j}{\sqrt{n}}\right) &= \frac{1}{\sqrt{4\pi\beta n}}\exp\left(-\frac{|j-n\alpha|^2}{4\beta n}\right)\\ &= \frac{1}{\sqrt{2\pi V(X)n}}\exp\left(-\frac{|j-n\mathbb{E}(X)|^2}{2V(X)n}\right).\end{aligned}$$

- We notice that using the moments function M_n defined with (2.5.2), we have

$$\forall n \in \mathbb{N}, \quad M_n(1) = (-1)^n\mathbb{E}(X^n).$$

Using the equalities (2.5.12) and (2.5.14) that respectively links the polynomials $\mathscr{P}_2^k$ and $\mathcal{Q}_2^k$ and allows us to compute the polynomial $\mathcal{Q}_2^k$, the second order term of the asymptotic expansion given by Theorem 2.1 is

$$\begin{aligned}&\frac{1}{n}\left(-\frac{1}{6}(M_3(1) - 3M_2(1)M_1(1) + 2M_1(1)^3)\right)\left(H_2^\beta\right)^{(3)}(X_{n,j})\\ &= \frac{\mathbb{E}((X-\mathbb{E}(X))^3)}{6(2\beta)^2 n}\left(H_2^{\frac{1}{2}}\right)^{(3)}\left(\frac{X_{n,j}}{\sqrt{2\beta}}\right)\end{aligned}$$

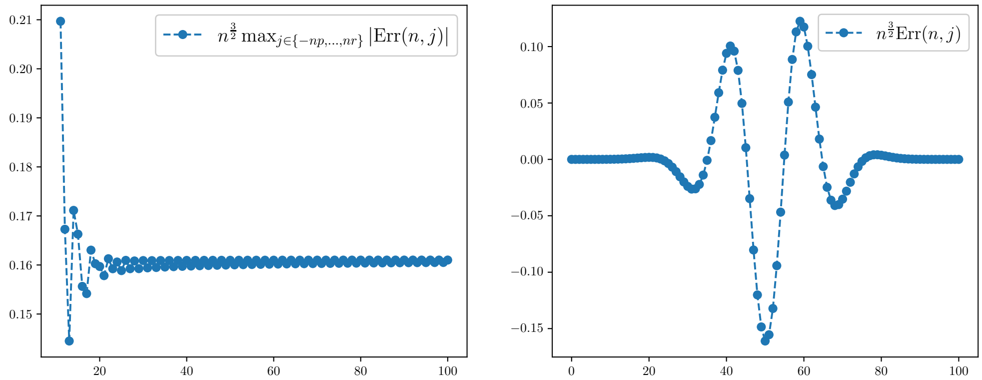

Figure 2.7 – On the left : A representation of $n^{\frac{3}{2}} \max_{j\in\{-nr,\dots,np\}} |\mathrm{Err}(n,j)|$ depending on n. As expected knowing that $-r < \alpha < p$, we see that the function is bounded and even seems to converge. On the right : We fixed $n = 100$ and represented $j \in \mathbb{Z} \mapsto n^{\frac{3}{2}}\mathrm{Err}(n,j)$.

$$= \frac{q_1\left(\frac{X_{n,j}}{\sqrt{V(X)}}\right)}{n}$$

where $X_{n,j} := \frac{n\mathbb{E}(X)-j}{\sqrt{n}}$ and the function $q_1 : \mathbb{R} \to \mathbb{R}$ is defined as

$$\forall x \in \mathbb{R}, \quad q_1(x) := -\frac{\mathbb{E}((X-\mathbb{E}(X))^3)}{6\sqrt{2\pi}V(X)^2}(x^3-3x)e^{-\frac{x^2}{2}}.$$

Theorem 2.1 then states that there exist two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}\setminus\{0\}, \forall j \in \mathbb{Z}, \quad |\mathrm{Err}(n,j)| \leq \frac{C}{n^{\frac{3}{2}}} \exp\left(-c\,|X_{n,j}|^2\right), \tag{2.5.15}$$

with $X_{n,j} = \frac{n\mathbb{E}(X)-j}{\sqrt{n}}$ and

$$\mathrm{Err}(n,j) := \mathscr{G}_j^n - \frac{1}{\sqrt{2\pi V(X) n}} \exp\left(-\frac{|X_{n,j}|^2}{2V(X)}\right) - \frac{q_1\left(\frac{X_{n,j}}{\sqrt{V(X)}}\right)}{n}$$

.

The estimate (2.5.15) is exactly the asymptotic expansion of the elements $b_j^n = \mathscr{G}_j^n$ we expected via the local limit theorem (see [Pet75, Chapter VII, Theorem 13] for more details). This behavior is represented on Figure 2.7 where we even see that the remainder $n^{\frac{3}{2}} Err(n,j)$ seems to scale like $f\left(\frac{n\alpha-j}{\sqrt{n}}\right)$. This would correspond to the next term in the asymptotic expansion of $\mathscr{G}_j^n$.

The O3 scheme for the transport equation

We will now consider an example linked to finite difference schemes. We consider the transport equation

$$\partial_t u + a\partial_x u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}$$

with Cauchy data at $t = 0$. The O3 scheme is an explicit third order accurate finite difference approximation of the previous transport equation. We refer to [Des08] for a detailed analysis of this scheme. It corresponds to the numerical scheme (2.1.5) for $a \in \ell^1(\mathbb{Z})$ such that $a_j = 0$ for $j \notin \{-2,-1,0,1\}$ and

$$a_{-2} = -\frac{\lambda a(1-(\lambda a)^2)}{6}, \quad a_{-1} = \frac{\lambda a(1+\lambda a)(2-\lambda a)}{2},$$
$$a_0 = \frac{(1-(\lambda a)^2)(2-\lambda a)}{2}, \quad a_1 = -\frac{\lambda a(1-\lambda a)(2-\lambda a)}{6},$$

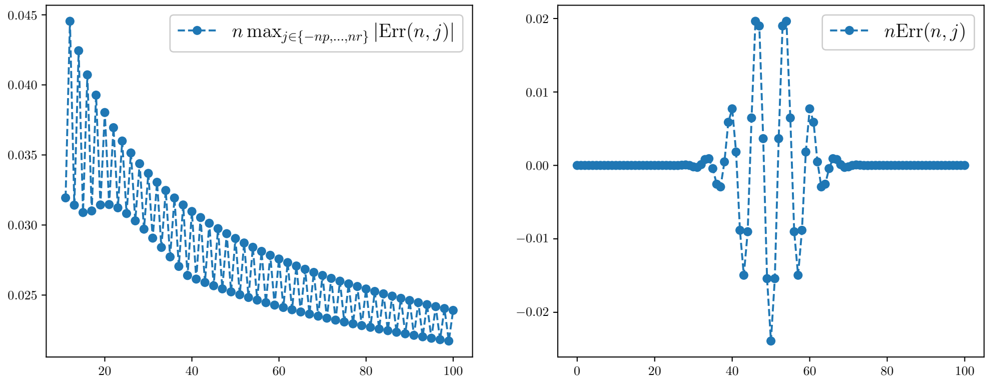

Figure 2.8 – For these figures, we chose $\lambda a = 1/2$. On the left : A representation of $n \max_{j\in\{-nr,\dots,np\}} |\mathrm{Err}(n,j)|$ depending on n. As expected, the function seems to be bounded. On the right : We fixed $n = 100$ and represented $j \in \mathbb{Z} \mapsto n\mathrm{Err}(n,j)$. We observe the exponential decay in j. Also, we can see a particular shape of curve that arises that would correspond to the next term in the asymptotic expansion of $\mathscr{G}_j^n$.

with $\lambda = \frac{\Delta t}{\Delta x} > 0$. The parameter λa is the Courant number. We have in this case that $r = 2$ and $p = 1$. For $\lambda a \in]-1,1[\backslash\{0\}$, we have that $F(1) = 1$ and

$$\forall \kappa \in \mathbb{S}^1 \backslash \{1\}, \quad |F(\kappa)| < 1.$$

Also, there exists $\beta \in \mathbb{R}_+ \backslash \{0\}$ such that

$$F(e^{i\xi}) \underset{\xi\to 0}{=} \exp(-i\lambda a\xi - \beta\xi^4 + o(\xi^4)).$$

We have $\mu = 2$ in this case and Hypothesis 2.2 is satisfied with $K = 1$, $\underline{\kappa}_1 = 1$ and $\underline{z}_1 = 1$. Since $K = 1$, we lose the subscript k in most the notations that follow. The sequence a verifies hypotheses 2.1, 2.2 and 2.3, so we can apply Theorem 2.1. As an example, we will apply Theorem 2.1 for $s = 3$ and $\lambda a = \frac{1}{2}$.

- Using the equality (2.5.7), we have

$$\mathscr{P}_1 = 1.$$

- Using the equality (2.5.11) and Lemma 2.5.1 to compute $\varpi^{(3)}(1)$, we have

$$\mathscr{P}_2 = 0.$$

- Using the equality (2.3.56) to express the polynomial $\mathscr{P}_3$ and Lemma 2.5.1 to compute the coefficients $\mathscr{A}_{l,m}$, we numerically compute the polynomials $\mathscr{P}_3$:

$$\mathscr{P}_3 = p_3 Y^6$$

where $p_3 \approx -1,953125.10^{-3}$.

Theorem 2.1 then states that there exist two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \quad |\mathrm{Err}(n,j)| \leq \frac{C}{n} \exp\left(-c\,|X_{n,j}|^{\frac{4}{3}}\right), \tag{2.5.16}$$

with $X_{n,j} = \frac{n\alpha - j}{n^{\frac{1}{4}}}$ and

$$\mathrm{Err}(n,j) := \mathscr{G}_j^n - \sum_{\sigma=1}^{3} \frac{1}{n^{\frac{\sigma}{4}}} \mathscr{P}_\sigma\left(X_{n,j}, \frac{d}{dx}\right) H_4^\beta(X_{n,j}).$$

This behavior is represented on Figure 2.8 where we even see that the remainder $nErr(n,j)$ seems to scale like $f\left(\frac{n\alpha - j}{\sqrt{n}}\right)$. Hence, the estimate (2.5.16) seems to be sharp.

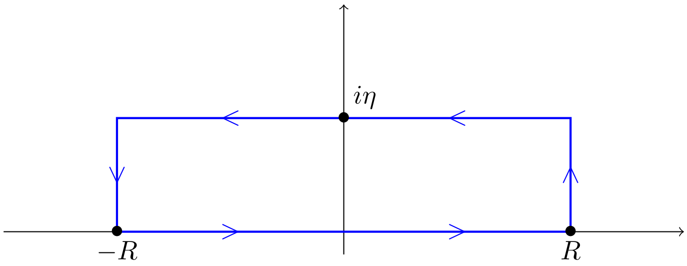

Figure 2.9 – Integrating path for the proof of Lemma 2.3.3.

2.A Appendix of the chapter

Proof of the Lemma 2.2.1

We recall here the statement of Lemma 2.2.1.

Lemma (Lemma 2.2.1). *For $a \in \ell^1(\mathbb{Z})$ which verifies Hypotheses 2.1 and 2.2, we have that a_{-r} and a_p belong to $\mathbb{D}$.*

Proof We introduce the polynomial function g defined by

$$\forall \kappa \in \mathbb{C}, \quad g(\kappa) := \sum_{l=-r}^{p} a_l \kappa^{l+r}.$$

For all $\kappa \in \mathbb{S}^1$, Hypothesis 2.2 implies that

$$|g(\kappa)| = |\kappa^r F(\kappa)| = |F(\kappa)| \leq 1.$$

Observing that g is not a constant function, the maximum principle for holomorphic functions [Rud87] allows us to conclude that

$$|a_{-r}| = |g(0)| < 1.$$

The same kind of argument allows us to conclude for the coefficient a_p. □

Proof of the Lemma 2.3.3

We recall here the statement of Lemma 2.3.3.

Lemma (Lemma 2.3.3). *For $\mu \in \mathbb{N}\backslash\{0\}$, $\beta \in \mathbb{C}$ with positive real part and $m \in \mathbb{N}$, there exist two constants $C, c > 0$ such that*

$$\forall x \in \mathbb{R}, \quad \left|H_{2\mu}^{\beta}{}^{(m)}(x)\right| \leq C \exp\left(-c|x|^{\frac{2\mu}{2\mu-1}}\right).$$

Proof We fix $\eta \in \mathbb{R}$ that we will choose more precisely later. Integrating the function $z \mapsto (iz)^m \exp(izx - \beta z^{2\mu})$ on the rectangle depicted in the Figure 2.9 using the Cauchy formula and passing to the limit $R \to +\infty$, we obtain

$$\forall \eta \in \mathbb{R}, \quad H_{2\mu}^{\beta}{}^{(m)}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} (i(t+i\eta))^m e^{i(t+i\eta)x} e^{-\beta(t+i\eta)^{2\mu}} dt.$$

Thus,

$$\left|H_{2\mu}^{\beta}{}^{(m)}(x)\right| \leq \frac{e^{-\eta x}}{2\pi}\int_{\mathbb{R}} (t^2+\eta^2)^{\frac{m}{2}} \exp\left(-\Re\left(\beta(t+i\eta)^{2\mu}\right)\right) dt.$$

Using Young's inequality, we can show that there exists a constant $c > 0$ such that

$$\forall t \in \mathbb{R}, \quad \Re\left(\beta(t+i\eta)^{2\mu}\right) \geq \frac{\Re(\beta)}{2} t^{2\mu} - c\eta^{2\mu}.$$

and thus there exists $C > 0$ independent from x and η such that

$$\left|H_{2\mu}^{\beta}{}^{(m)}(x)\right| \leq C(1+|\eta|^m) e^{-\eta x + c\eta^{2\mu}}.$$

Optimizing $e^{-\eta x + c\eta^{2\mu}}$ with respect to η yields the desired result. □

Proof of the Lemma 2.3.7

We recall here the statement of Lemma 2.3.7.

Lemma (Lemma 2.3.7, Inequalities in $B_{\varepsilon_\star}(\underline{\tau}_k)$)**.** *There exists $C > 0$ such that for all $\tau \in B_{\varepsilon_\star}(\underline{\tau}_k)$ and $(n, j) \in \mathcal{D}_k$, we have*

$$\left|e^{n\tau}\left(e^{j\varpi_k(\tau)} - e^{jQ_{s,k}(\tau)}\right)\right| \leq Cn|\tau - \underline{\tau}_k|^{2\mu_k+s} \exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|))$$

and

$$\left|e^{n\tau + j\varphi_k(\tau)}\left(e^{jR_{s,k}(\tau)} - \sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right)\right| \\ \leq C\left(n|\tau - \underline{\tau}_k|^{2\mu_k+1}\right)^s \exp(n\Re(\tau - \underline{\tau}_k) + j(\Re(\varphi_k(\tau)) + \operatorname{sgn}(\alpha_k)|R_{s,k}(\tau)|)).$$

Proof We begin with the first inequality. We define the holomorphic function $\mathcal{S}$ such that

$$\forall z \in \mathbb{C}, \quad \mathcal{S}(z) = \begin{cases} 1 & \text{if } z = 0, \\ \frac{\sinh(z)}{z} & \text{else.} \end{cases}$$

We consider $(n, j) \in \mathcal{D}_k$ and $\tau \in B_{\varepsilon_\star}(\underline{\tau}_k)$. We have

$$\left|e^{n\tau}\left(e^{j\varpi_k(\tau)} - e^{jQ_{s,k}(\tau)}\right)\right| = |j||\xi_{s,k}(\tau)||\tau - \underline{\tau}_k|^{2\mu_k+s}\left|\mathcal{S}\left(j\frac{\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}}{2}\right)\right| \\ \exp\left(n\Re(\tau) + j\left(\Re(\varpi_k(\tau)) - \Re\left(\frac{\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}}{2}\right)\right)\right).$$

We observe that the function $z \in \mathbb{C} \mapsto |\mathcal{S}(z)|\exp(-|z|)$ is bounded. Therefore, because the function ξ_k can be bounded on $B_{\varepsilon_\star}(\underline{\tau}_k)$ and $\Re(\tau) = \Re(\tau - \underline{\tau}_k)$,

$$\left|e^{n\tau}\left(e^{j\varpi_k(\tau)} - \underline{\kappa}_k^j e^{j\varphi_k(\tau)}\right)\right| \\ \lesssim n|\tau - \underline{\tau}_k|^{2\mu_k+s} \\ \exp\left(n\Re(\tau - \underline{\tau}_k) + j\left(\Re(\varpi_k(\tau)) - \Re\left(\frac{\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}}{2}\right)\right) + |j|\frac{|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|}{2}\right)$$

Since we have

$$j\left(\Re(\varpi_k(\tau)) - \Re\left(\frac{\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}}{2}\right)\right) + |j|\frac{|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|}{2} \\ \leq j\left(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|\right).$$

The proof of the second inequality is similar. We define the holomorphic function Ψ_s such that

$$\forall z \in \mathbb{C}, \quad \Psi_s(z) = \frac{1}{z^s}\left(e^z - \sum_{l=0}^{s-1}\frac{z^l}{l!}\right).$$

We then have

$$e^{n\tau + j\varphi_k(\tau)}\left(e^{jR_{s,k}(\tau)} - \sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right) = (jR_{s,k}(\tau))^s\Psi_s(jR_{s,k}(\tau))e^{n\tau + j\varphi_k(\tau)}.$$

We observe that the function $z \in \mathbb{C} \mapsto |\Psi_s(z)|\exp(-|z|)$ is bounded. Therefore,

$$\left|e^{n\tau + j\varphi_k(\tau)}\left(e^{jR_{s,k}(\tau)} - \sum_{l=0}^{s-1}\frac{(jR_{s,k}(\tau))^l}{l!}\right)\right| \lesssim (n|\tau - \underline{\tau}_k|^{2\mu_k+1})^s \exp(n\Re(\tau - \underline{\tau}_k) + j\Re(\varphi_k(\tau)) + |jR_{s,k}(\tau)|).$$

We can then conclude the proof of the second inequality. ☐

Proof of the Lemma 2.3.8

We recall the statement of Lemma 2.3.8 and that we consider $\alpha_k > 0$.

Lemma (Lemma 2.3.8, Inequalities on $\Gamma_{k,p}$)**.** *For $(n,j) \in \mathbb{N}\backslash\{0\} \times \mathbb{N}$ and $\tau \in \Gamma_{k,p}$, we have*

- *Case A:* $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) \in \left[-\frac{\eta}{2}, \varepsilon_{k,0}\right]$

$$\begin{aligned} n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|) \\ \leq -nc_\star \mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k} - \frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k \left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}, \\ n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) \leq -nc_\star \mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k} - \frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k \left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}, \\ n\Re(\tau - \underline{\tau}_k) + j(\Re(\varphi_k(\tau)) + \operatorname{sgn}(\alpha_k)|R_{s,k}(\tau)|) \leq -nc_\star \mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k} - \frac{n}{\alpha_k}(2\mu_k - 1)\gamma_k \left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}}. \end{aligned}$$

- *Case B:* $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) > \varepsilon_{k,0}$

$$\begin{aligned} n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|) &\leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\varepsilon_{k,0}^{2\mu_k}, \\ n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) &\leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\varepsilon_{k,0}^{2\mu_k}, \\ n\Re(\tau - \underline{\tau}_k) + j(\Re(\varphi_k(\tau)) + \operatorname{sgn}(\alpha_k)|R_{s,k}(\tau)|) &\leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\varepsilon_{k,0}^{2\mu_k}. \end{aligned}$$

- *Case C:* $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) < -\frac{\eta}{2}$

$$\begin{aligned} n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|) &\leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\left(\frac{\eta}{2}\right)^{2\mu_k}, \\ n\Re(\tau - \underline{\tau}_k) + j\Re(\varpi_k(\tau)) &\leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\left(\frac{\eta}{2}\right)^{2\mu_k}, \\ n\Re(\tau - \underline{\tau}_k) + j(\Re(\varphi_k(\tau)) + \operatorname{sgn}(\alpha_k)|R_{s,k}(\tau)|) &\leq -\frac{n}{\alpha_k}(2\mu_k - 1)A_R\underline{\delta}_k\left(\frac{\eta}{2}\right)^{2\mu_k}. \end{aligned}$$

Proof In every case, the second inequality is a direct consequence of the first one. Furthermore, the proof of the first and third inequalities are very similar. For the first one, we will use inequality (2.3.19) and the third one will rely on inequality (2.3.20). Thus, we will focus in each case on the first inequality.

We consider $(n,j) \in \mathbb{N}\backslash\{0\} \times \mathbb{N}$ and $\tau \in \Gamma_{k,p}$. Using first the inequality (2.3.19), the fact that $\tau \in \Gamma_{k,p}$ and finally the inequality (2.3.27), we have

$$\begin{aligned} n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|) &\leq n\Re(\tau - \underline{\tau}_k) - \frac{j}{\alpha_k}\Psi_k(\tau_p) \\ &\leq -nc_\star \mathrm{Im}\,(\tau - \underline{\tau}_k)^{2\mu_k} + \frac{n}{\alpha_k}\left(\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p\right). \end{aligned}$$

- First, we consider the case A. Then, we have $\tau_p = \rho_k\left(\frac{\zeta_k}{\gamma_k}\right)$. Therefore,

$$\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p = -(2\mu_k - 1)\gamma_k\left(\frac{|\zeta_k|}{\gamma_k}\right)^{\frac{2\mu_k}{2\mu_k-1}} \leq 0. \tag{2.A.1}$$

- We consider the case B. Because $\tau_p = \varepsilon_{k,0}$, we have

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|) \leq \frac{n}{\alpha_k}\left(\gamma_k\varepsilon_{k,0}^{2\mu_k} - 2\mu_k\zeta_k\varepsilon_{k,0}\right).$$

We recall that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) > \varepsilon_{k,0}$ and that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right)$ is the only real root of $-\zeta_k + \gamma_k x^{2\mu_k-1} = 0$. Therefore, $-\zeta_k \leq -\gamma_k\varepsilon_{k,0}^{2\mu_k-1}$ and

$$\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p \leq -(2\mu_k - 1)\gamma_k\varepsilon_{k,0}^{2\mu_k} \leq 0. \tag{2.A.2}$$

Using (2.3.30) to bound γ_k, we deduce the inequality (2.3.36).

- Finally, we place ourselves in case C. We have that $\tau_p = -\frac{\eta}{2}$, so

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}|) \le \frac{n}{\alpha_k}\left(\gamma_k\left(\frac{\eta}{2}\right)^{2\mu_k} + 2\mu_k\zeta_k\frac{\eta}{2}\right).$$

We recall that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right) < -\frac{\eta}{2}$ and that $\rho_k\left(\frac{\zeta_k}{\gamma_k}\right)$ is the only real root of $-\zeta_k + \gamma_k x^{2\mu_k-1} = 0$. Then, $\zeta_k \le -\gamma_k\left(\frac{\eta}{2}\right)^{2\mu_k-1}$ and

$$\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p \le -(2\mu_k - 1)\gamma_k\left(\frac{\eta}{2}\right)^{2\mu_k} \le 0. \tag{2.A.3}$$

Using (2.3.30) to bound γ_k, we deduce the inequality (2.3.39). □

Proof of the Lemma 2.3.9

We recall here the statement of Lemma 2.3.9.

Lemma (Lemma 2.3.9)**.** *For $(n,j) \in \mathbb{N}\setminus\{0\} \times \mathbb{N}$ and $\tau \in \Gamma_{k,res}$, we have in all cases*

$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)\left|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}\right|) \le -n\frac{\eta}{2},$$
$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) \le -n\frac{\eta}{2},$$
$$n\Re(\tau - \underline{\tau}_k) + j(\Re(\varphi_k(\tau)) + \operatorname{sgn}(\alpha_k)|R_{s,k}(\tau)|) \le -n\frac{\eta}{2}.$$

Proof For the same reasons as for the proof of Lemma 2.3.8, we will only focus on the first inequality. We consider $(n,j) \in \mathbb{N}\setminus\{0\} \times \mathbb{Z}$ such that $\operatorname{sgn}(j) = \operatorname{sgn}(\alpha_k)$ and $\tau \in \Gamma_{k,res}$. Using the inequality (2.3.19) and the facts that $\operatorname{Im}(\tau - \underline{\tau}_k)^{2\mu_k} \ge \ell_{k,p}^{2\mu_k}$ and $-\eta + i\ell_{k,p} + \underline{\tau}_k \in \Gamma_{k,p}$, we have

$$\begin{aligned}n\Re(\tau - \underline{\tau}_k) + j(\Re(\varpi_k(\tau)) + \operatorname{sgn}(\alpha_k)\left|\xi_{s,k}(\tau)(\tau - \underline{\tau}_k)^{2\mu_k+s}\right|) &\le -n\eta - \frac{j}{\alpha_k}\left(-\eta - A_R\eta^{2\mu_k} + A_I\operatorname{Im}(\tau - \underline{\tau}_k)^{2\mu_k}\right)\\ &\le -n\eta - \frac{j}{\alpha_k}\Psi_k(\tau_p).\end{aligned}$$

We know that $\eta + \tau_p \ge \frac{\eta}{2}$, so

$$-n\eta - \frac{j}{\alpha_k}(\tau_p - A_R\tau_p^{2\mu_k}) = -n(\eta + \tau_p) + \frac{n}{\alpha_k}\left(\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p\right) \le -n\frac{\eta}{2} + \frac{n}{\alpha_k}\left(\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p\right).$$

We proved at the end of the proof of Lemma 2.3.8 that, in the three cases A, B and C, $\gamma_k\tau_p^{2\mu_k} - 2\mu_k\zeta_k\tau_p$ are non positive (see (2.A.1), (2.A.2) and (2.A.3)). This concludes the proof. □

Chapter 3

Tamed stability of finite difference schemes for the transport equation on the half-line

This chapter presents the content of the article [Coe24] published in *Mathematics of Computation*.

Abstract of the current chapter

In this chapter, we prove that, under precise spectral assumptions, some finite difference approximations of scalar leftgoing transport equations on the positive half-line with numerical boundary conditions are ℓ^1-stable but ℓ^q-unstable for any $q > 1$. The proof relies on the accurate description of the Green's function for a particular family of finite rank perturbations of Toeplitz operators whose essential spectrum belongs to the closed unit disk and with a simple eigenvalue of modulus 1 embedded into the essential spectrum.

Outline of the current chapter

Notations

Throughout this article, we define the following sets:

$$\mathcal{U} := \{z \in \mathbb{C}, |z| > 1\}, \quad \mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}, \quad \mathbb{S}^1 := \{z \in \mathbb{C}, |z| = 1\},$$

$$\overline{\mathcal{U}} := \mathbb{S}^1 \cup \mathcal{U}, \quad \overline{\mathbb{D}} := \mathbb{S}^1 \cup \mathbb{D}.$$

For $z \in \mathbb{C}$ and $r > 0$, we let $B_r(z)$ denote the open ball in $\mathbb{C}$ centered at z with radius r.

For E a Banach space, we denote $\mathcal{L}(E)$ the space of bounded operators acting on E and $\|\cdot\|_{\mathcal{L}(E)}$ the operator norm. For T in $\mathcal{L}(E)$, the notations $\sigma(T)$ and $\rho(T)$ stand respectively for the spectrum and the resolvent set of the operator T.

We let $\mathcal{M}_{n,k}(\mathbb{C})$ denote the space of complex valued $n \times k$ matrices and we use the notation $\mathcal{M}_n(\mathbb{C})$ when $n = k$.

We use the notation $\lesssim$ to express an inequality up to a multiplicative constant. Eventually, we let C (resp. c) denote some large (resp. small) positive constants that may vary throughout the text (sometimes within the same line).

3.1 Introduction

3.1.1 Context

The purpose of this article is to study the so-called semigroup stability for discretizations of hyperbolic initial boundary value problems. More precisely, we focus our attention on explicit finite difference schemes that are consistent with the scalar leftgoing ($v < 0$) transport equation on the positive half-line with a Cauchy initial datum

$$\begin{aligned} \forall t \geq 0, \forall x \geq 0, \quad & \partial_t u + v\partial_x u = 0, \\ \forall x \geq 0, \quad & u(0,x) = u_0(x) \in \mathbb{R}. \end{aligned} \tag{3.1.1}$$

We observe that for an initial datum $u_0 \in L^q(\mathbb{R}_+)$ with $q \in [1,+\infty]$, the solution u of (3.1.1) satisfies

$$\sup_{t\geq 0} \|u(t,\cdot)\|_{L^q(\mathbb{R}_+)} \leq \|u_0\|_{L^q(\mathbb{R}_+)}. \tag{3.1.2}$$

No boundary condition is required here at $x = 0$ for (3.1.1) since the transport operator is outgoing with respect to the boundary. However, the numerical schemes we consider will require the introduction of nonphysical numerical boundary conditions which can generate instabilities.

When considering discretizations of initial boundary value problems, there are several possible definitions for stability that have been introduced. The one interesting us in the present paper is the semigroup stability in the ℓ^q-topology (see Definition 1 of ℓ^q-stability for more details) introduced for instance in [Str64; Kre68; Osh69b; Osh69a]. It corresponds to a discretized equivalent of (3.1.2) and can be rewritten as the following power boundedness property

$$\sup_{n\in\mathbb{N}} \|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_q)} < +\infty \tag{3.1.3}$$

where $\mathcal{T}$ is the discrete evolution operator that allows to compute the solution of the numerical scheme from one time step to the next and the vector space $\mathcal{H}_q$ is a modification of the vector space $\ell^q(\mathbb{N})$ which takes into account the numerical boundary conditions of the numerical scheme. A direct conclusion is that the ℓ^q-stability prevents the existence of eigenvalues of the operator $\mathcal{T}$ in the set $\mathcal{U}$ (unstable eigenvalues). This corresponds to the so-called Godunov-Ryabenkii condition introduced in [GR63]. Let us point out that the existence and position of eigenvalues for the operator $\mathcal{T}$ highly depends on the choice of numerical boundary condition that is done, as will be explained in the article.

One of the other most notable definition of stability is the notion of *strong stability* (also known as GKS-stability) introduced in the fundamental contribution [GKS72]. It can be considered to be one of the most robust definitions of stability in the context of finite difference schemes for initial boundary value problems as it is stable with respect to perturbations. We refer the interested reader to [GKO13] for a complete overview of GKS-theory as it will not be the main focus of this paper. However, we need to point out that the strong stability of a finite difference scheme is fully characterized by the fulfillment of the so-called uniform Kreiss-Lopatinskii condition which in our case corresponds to the operator $\mathcal{T}$ not having eigenvalues or generalized eigenvalues in the set $\overline{\mathcal{U}}$. Even though the two notions of stability we introduced are quite different, it can be shown using energy estimates that the strong stability, i.e. the verification of the uniform Kreiss-Lopatinskii condition, implies the ℓ^2-stability (see [Wu95; CG11; Cou13] which respectively tackle the cases of the scalar one-dimensional problem and one time step scheme, of the multidimensional system and one time step scheme and of the scalar multidimensional problem and multistep scheme) and even the ℓ^q-stability for all $q \in [1,+\infty]$ (using the semigroup estimates in [CF23]).

However, it remains uncertain to conclude on the ℓ^q-stability of a numerical scheme whenever the Godunov-Ryabenkii condition is satisfied but not the uniform Kreiss-Lopatinskii condition. Up to our knowledge, this question was first formalized and tackled in [Tre84; KW93]. In the recent paper [CF23], which highly influenced the study carried in the present article, the authors proved that when the operator $\mathcal{T}$ admits simple eigenvalues on the unit circle that do not belong to the essential spectrum of $\mathcal{T}$, the numerical scheme remains ℓ^q-stable for all $q \in [1, +\infty]$. The goal of the present paper is to carry the same kind of analysis when the operator $\mathcal{T}$ admits a simple eigenvalue of modulus 1 that lies within the essential spectrum of $\mathcal{T}$, which is up to our knowledge a spectral configuration that was not handled before. Let us point out that studying this spectral assumption has two further incentives:

- First, the analysis carried under this type of spectral assumption will have to be carefully dealt with since the spatial Green's function, a tool introduced in Section 3.4, will only be meromorphically extended near some interest point and not holomorphically as in [CF23]. This will motivate the introduction of careful computations that could be reused in other studies with similar spectral configuration. Similar computations have already been presented in the continuous setting when studying relaxation shocks for instance (see [MZ02]) but this is an occasion to extend them to the fully discrete setting.

- Second, this type of spectral configuration also occurs in the context of the study of stability of discrete shock profiles (see [Ser07]). The author hopes that the analysis carried in this paper could be used to extend results on this subject, for instance the result of [God03] (see Remark 3 for more details).

In direct opposition with the main result of [CF23], the main result of the present paper (Theorem 3.1) states that if the operator $\mathcal{T}$ has a simple eigenvalue of modulus 1 that is not isolated from the essential spectrum, then the numerical scheme remains ℓ^1-stable, but, under some explicit algebraic condition, the numerical scheme is also either ℓ^q-stable for all $q \in]1, +\infty]$ or ℓ^q-unstable for all $q \in]1, +\infty]$. In the later case, we even prove a sharp growth rate (3.1.17) of the norm $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_q)}$ depending on n.

The proof of Theorem 3.1 relies on a precise description of the asymptotic behavior of the semigroup associated with the numerical scheme (see Theorem 3.2) using an approach referred to as "spatial dynamics". It follows a series of papers initiated by [ZH98] and aims at using functional calculus (see [Con90, Chapter VII]) to express the temporal Green's function (the fundamental solution of the numerical scheme, defined below by (3.1.19)) using the so-called spatial Green's function (defined in Section 3.4). The general structure of the article is kept quite similar with the one of [CF23]. On the one hand, it allows the interested reader to observe the similarities between both papers that could lead to a more general result combining both the result of the present paper and that of [CF23]. On the other hand, the author thinks that the fundamental differences on the choice of spectral setup will be clearer this way as every step of the proof can be compared with its equivalent in [CF23].

3.1.2 Setup

We seek to approach the real valued solution u of the Cauchy problem (3.1.1). We introduce a time step $\Delta t > 0$ as well as a space step $\Delta x > 0$ and from now on we consider that the ratio $\lambda := \frac{\Delta t}{\Delta x}$ is always kept fixed. To approach the value of the solution u of the Cauchy problem (3.1.1) in the cell $[n\Delta t, (n+1)\Delta t[\times[(j-1)\Delta x, j\Delta x[$, we define an explicit one-step in time finite difference scheme applied to (3.1.1). In the interior cells $[(j-1)\Delta x, j\Delta x[$ for $j \in \mathbb{N}\backslash\{0\}$, the finite difference scheme is defined by

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{N}\backslash\{0\}, \quad u_j^{n+1} = \sum_{k=-r}^{p} a_k u_{j+k}^n. \tag{3.1.4a}$$

where r, p are non negative integers, the coefficients a_k are given real numbers which can depend on the velocity v and on the ratio λ. There remains to define how the values $u_{1-r}^n, \dots, u_0^n$ in the so-called "ghost cells" are dealt with. As is usual, the numerical boundary conditions to compute the values of $u_{1-r}^n, \dots, u_0^n$ are given by a linear combination of the first values close to the boundary:

$$\forall n \in \mathbb{N}, \forall j \in \{1-r, \dots, 0\}, \quad u_j^n = \sum_{k=1}^{p_b} b_{k,j} u_k^n, \tag{3.1.4b}$$

where p_b is a non negative integer, the coefficients $b_{k,j}$ are also given real numbers which can depend on the velocity v and on the ratio λ. The initial values u_j^0 for $j \in \mathbb{N}\backslash\{0\}$ are computed using the Cauchy initial datum u_0 of the PDE (3.1.1), for instance as

$$\forall j \in \mathbb{N}\backslash\{0\}, \quad u_j^0 := \frac{1}{\Delta x}\int_{(j-1)\Delta x}^{j\Delta x} u_0(x)dx.$$

We will assume that the integers r and p are fixed and that $a_{-r}, a_p \neq 0$. We claim that the case $r = 0$ is special as there would be no numerical boundary condition to implement in the numerical scheme (3.1.4) since the discretization of the transport equation would be of the "upwind" type. From now on, we will assume that $r \geq 1$ while keeping the normalization condition $a_{-r} \neq 0$. We will also assume that the integer p_b in (3.1.4b) satisfies $p_b \leq p$. This condition on p_b, which was also made in [CF23], will ease the computations in the paper as it allows us to translate the boundary condition of the numerical scheme (3.1.4b) as

$$\forall n \in \mathbb{N}, \quad \mathcal{B}\begin{pmatrix} u_p^n \\ \vdots \\ u_{1-r}^n \end{pmatrix} = 0$$

where

$$\mathcal{B} := \begin{pmatrix} 0 & \dots & 0 & -b_{p_b,0} & \dots & -b_{1,0} & 1 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & 1 & 0 \\ 0 & \dots & 0 & -b_{p_b,1-r} & \dots & -b_{1,1-r} & 0 & \dots & \dots & 0 & 1 \end{pmatrix} \in \mathcal{M}_{r,p+r}(\mathbb{R}). \tag{3.1.5}$$

This matrix form of the numerical boundary conditions (3.1.4b) will often appear in this article and the matrix $\mathcal{B}$ will play a major role in our stability analysis.

We introduce for $q \in [1, +\infty]$ the Banach space

$$\mathcal{H}_q := \left\{ (w_j)_{j\geq 1-r} \in \ell^q \left(\{ j \in \mathbb{Z}, j \geq 1-r \}, \mathbb{R} \right), \quad \forall j \in \{1-r, \dots, 0\}, w_j = \sum_{k=1}^{p_b} b_{k,j} w_k \right\}$$

with the norm

$$\forall w \in \mathcal{H}_q, \quad \|w\|_{\mathcal{H}_q} := \left\| (w_j)_{j\in\mathbb{N}\setminus\{0\}} \right\|_{\ell^q(\mathbb{N}\setminus\{0\})}.$$

We define the bounded operator $\mathcal{T} \in \mathcal{L}(\mathcal{H}_q)$ defined by

$$\forall w \in \mathcal{H}_q, \forall j \in \mathbb{N}\setminus\{0\}, \quad (\mathcal{T}w)_j := \sum_{k=-r}^{p} a_k w_{j+k}. \tag{3.1.6}$$

The values $(\mathcal{T}w)_j$ for $j \in \{1-r, \dots, 0\}$ are determined by the condition $\mathcal{T}w \in \mathcal{H}_q$. Using the same reasoning, we also allow ourselves occasionally to say that some sequence $(u_j)_{j\geq 1}$ belongs to $\mathcal{H}_q$ without making the values for $j \in \{1-r, \dots, 0\}$ precise in order to alleviate the redaction.

The definition of the operator $\mathcal{T}$ does not depend on q but the Banach space $\mathcal{H}_q$ on which it acts does. We observe that for an initial condition $u^0 \in \mathcal{H}_q$, the numerical scheme (3.1.4) can be rewritten as the following discrete evolution problem

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{T}u^n. \tag{3.1.7}$$

We thus introduce the standard terminology:

Definition 1 (ℓ^q-stability)**.** The numerical scheme (3.1.4) is said to be ℓ^q-stable if there exists some positive constant $C > 0$ such that for all $u^0 \in \mathcal{H}_q$, the solution $(u^n)_{n\in\mathbb{N}}$ of the scheme (3.1.4) computed using the initial condition u^0 satisfies

$$\sup_{n\in\mathbb{N}} \|u^n\|_{\mathcal{H}_q} \leq C \left\|u^0\right\|_{\mathcal{H}_q}.$$

This is equivalent to proving that the family of operators $(\mathcal{T}^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}(\mathcal{H}_q)$.

The purpose of this article is to demonstrate that under a specific type of spectral condition which corresponds to the operator $\mathcal{T}$ having a simple eigenvalue of modulus 1 that is located in its essential spectrum, the numerical scheme (3.1.4) is ℓ^1-stable but ℓ^q-unstable for every $q > 1$. This is in sharp contrast with the result of [CF23] where it is proved that the existence of simple eigenvalues of the operator $\mathcal{T}$ on the unit circle outside the essential spectrum maintains the ℓ^q-stability of the numerical scheme (3.1.4).

3.1.3 Hypotheses and main result

We will introduce a few objects that will allow us to present the main hypotheses we will make in this article.

Consistency, dissipativity and diffusivity conditions

Since the stencil of the numerical scheme (3.1.4) is finite, if we consider an initial condition u^0 with a support that is located far from the boundary, then the numerical boundary condition (3.1.4b) will not have any effect on the computation of u^n for small times n. We then deduce that the solutions of the numerical scheme (3.1.4) are closely linked to the solutions $(u_j^n)_{n\in\mathbb{N},j\in\mathbb{Z}}$ of the following system on the whole one dimensional lattice $\mathbb{Z}$:

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^{n+1} = \sum_{k=-r}^{p} a_k u_{j+k}^n. \tag{3.1.8}$$

The system (3.1.8) corresponds to a numerical scheme for the transport equation on the whole line $\mathbb{R}$

$$\forall t \geq 0, \forall x \in \mathbb{R}, \quad \partial_t u + v\partial_x u = 0,$$
$$\forall x \in \mathbb{R}, \quad u(0,x) = u_0(x).$$

The stability of schemes of the form (3.1.8) has been studied thoroughly in [Tho65; DS14; CF22; Coe22]. We introduce the Laurent operator $\mathscr{L} \in \mathcal{L}(\ell^q(\mathbb{Z}))$ defined by

$$\forall w \in \ell^q(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathscr{L}w)_j := \sum_{k=-r}^{p} a_k w_{j+k} \tag{3.1.9}$$

which allows us to rewrite the numerical scheme (3.1.8) as a discrete evolution problem

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathscr{L}u^n$$

just as the operator $\mathcal{T}$ allowed us to rewrite the numerical scheme (3.1.4) as (3.1.7).

We introduce the symbol F associated with the scheme:

$$\forall \kappa \in \mathbb{C}\backslash\{0\}, \quad F(\kappa) := \sum_{j=-r}^{p} a_j \kappa^j. \tag{3.1.10}$$

We now make the following assumption on the numerical scheme (3.1.4) that we consider to discretize the transport equation.

Hypothesis 3.1. *We assume that*

$$F(1) = 1, \quad \alpha := -F'(1) = \lambda v. \quad \textit{(Consistency condition)}$$

This implies that $\alpha < 0$. Moreover, we suppose that

$$\forall \kappa \in \mathbb{S}^1\backslash\{1\}, \quad |F(\kappa)| < 1 \quad \textit{(Dissipativity condition)}$$

and that there exist an integer $\mu \in \mathbb{N}\backslash\{0\}$ and a complex number $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$ such that

$$F(e^{it}) \underset{t\to 0}{=} \exp(-i\alpha t - \beta t^{2\mu} + o(t^{2\mu})). \quad \textit{(Diffusivity condition)} \tag{3.1.11}$$

In [Tho65], Hypothesis 3.1 and especially the asymptotic expansion (3.1.11) are crucial for the stability analysis of the numerical scheme (3.1.8) in the ℓ^q-topology on the whole line $\mathbb{Z}$. For instance, the dissipativity condition of Hypothesis 3.1 implies the Von Neumann condition and thus the ℓ^2-stability of the numerical scheme (3.1.8) since Fourier analysis implies

$$\forall n \in \mathbb{N}\backslash\{0\}, \quad \|\mathscr{L}^n\|_{\mathcal{L}(\ell^2(\mathbb{Z}))} = \|F^n\|_{L^\infty(\mathbb{S}^1)} = 1.$$

To be more precise, the addition of the diffusivity condition given by (3.1.11) implies that the numerical scheme (3.1.8) on the whole lattice $\mathbb{Z}$ is actually ℓ^q-stable for all $q \in [1, +\infty]$. Thus, Hypothesis 3.1 being verified provides a starting point to study the stability of the numerical scheme (3.1.4) on the half-line in the ℓ^q-topology.

In the rest of the paper, the set $\mathcal{O}$ represented on Figure 3.1 corresponds to the exterior of the curve $F(\mathbb{S}^1)$, i.e. the unbounded connected component of $\mathbb{C}\backslash F(\mathbb{S}^1)$. The following lemma is a consequence of Hypothesis 3.1.

Lemma 3.1.1. *If Hypothesis 3.1 is verified, then $\alpha \in]-p, 0[$.*

The proof of Lemma 3.1.1 is entirely similar to the proof of [CF23, Lemma 6] so we will omit it. The result of Lemma 3.1.1 is comparable to the well-known Courant-Friedrichs-Lewy condition (see [CFL28]) and implies that $p \geq 1$. Consequently, the numerical scheme (3.1.4) which satisfies Hypothesis 3.1 must take information on the right of j to compute u_j^{n+1}.

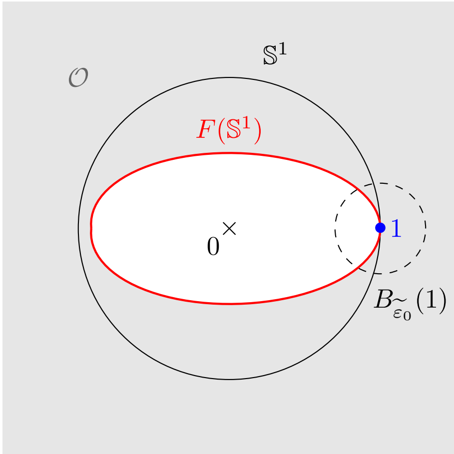

Figure 3.1 – An example of curve $F(\mathbb{S}^1)$. Hypothesis 3.1 implies that the curve $F(\mathbb{S}^1)$ (in red) is inside the closed disk $\overline{\mathbb{D}}$ and touches the boundary $\mathbb{S}^1$ only at 1. In gray, we have $\mathcal{O}$ the unbounded connected component of $\mathbb{C}\backslash F(\mathbb{S}^1)$. In dashed, we find the ball $B_{\widetilde{\varepsilon}_0}(1)$ where we have a more precise spectral decomposition (3.1.14) associated with the matrix $\mathbb{M}(z)$ defined by (3.1.12).

Lopatinskii determinant, spectral hypothesis and main result

We introduce for $z \in \mathbb{C}$ the companion matrix

$$\mathbb{M}(z) := \begin{pmatrix} \frac{z\delta_{p-1,0}-a_{p-1}}{a_p} & \frac{z\delta_{p-2,0}-a_{p-2}}{a_p} & \dots & \dots & \frac{z\delta_{-r,0}-a_{-r}}{a_p} \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \mathcal{M}_{p+r}(\mathbb{C}). \tag{3.1.12}$$

Since $r, p \geq 1$ and $a_{-r}a_p \neq 0$, we observe that the matrix $\mathbb{M}(z)$ is well-defined and invertible for all $z \in \mathbb{C}$ and it depends holomorphically on z. The matrix $\mathbb{M}(z)$ appears when we study the eigenvalue problem for $\mathcal{H}_q$ (see Lemma 3.4.1):

$$(zId_{\mathcal{H}_q} - \mathcal{T})u = 0.$$

It will also serve us later on to describe the so-called spatial Green's function in Section 3.4. The following lemma is due to Kreiss (see [Kre68]) and describes precisely the spectrum of the matrix $\mathbb{M}(z)$ as z belongs to $\mathcal{O} \cup \{1\}$.

Lemma 3.1.2 (Spectral Splitting)**.**

— *For $z \in \mathbb{C}$, $\kappa \in \mathbb{C}$ is an eigenvalue of $\mathbb{M}(z)$ if and only if $\kappa \neq 0$ and*

$$F(\kappa) = z.$$

— *For $z \in \mathcal{O}$, the matrix $\mathbb{M}(z)$ has*
  — *no eigenvalue on $\mathbb{S}^1$,*
  — *r eigenvalues in $\mathbb{D}\backslash\{0\}$ (that we call stable eigenvalues),*
  — *p eigenvalues in $\mathcal{U}$ (that we call unstable eigenvalues).*

— *The matrix $\mathbb{M}(1)$ has 1 as a simple eigenvalue, r eigenvalues in $\mathbb{D}\backslash\{0\}$ and $p-1$ eigenvalues in $\mathcal{U}$.*

A complete proof of Lemma 3.1.2 can be found in [CF22, Lemma 3].

For $z \in \mathcal{O}$, we define $E^s(z)$ (resp. $E^u(z)$) the stable (resp. unstable) subspace of $\mathbb{M}(z)$ which corresponds to the subspace spanned by the generalized eigenvectors of $\mathbb{M}(z)$ associated with eigenvalues in $\mathbb{D}$ (resp. $\mathcal{U}$). We therefore know that the subspace $E^s(z)$ (resp. $E^u(z)$) has dimension r (resp. p) thanks to Lemma 3.1.2 and we have the decomposition

$$\mathbb{C}^{p+r} = E^s(z) \oplus E^u(z). \tag{3.1.13}$$

The associated projectors are denoted $\pi^s(z)$ and $\pi^u(z)$. Those linear maps commute with $\mathbb{M}(z)$ and depend holomorphically on $z \in \mathcal{O}$ (see [Kat95]).

We now need to clarify the situation near $z = 1$. Using Lemma 3.1.2, we know that 1 is a simple eigenvalue of $\mathbb{M}(1)$ and that the matrix $\mathbb{M}(1)$ has r eigenvalues in $\mathbb{D}\backslash\{0\}$ and $p-1$ eigenvalues in $\mathcal{U}$. Therefore, there exist a radius $\widetilde{\varepsilon}_0>0$ and a holomorphic function $\kappa:B_{\widetilde{\varepsilon}_0}(1)\to\mathbb{C}$ such that $\kappa(1)=1$ and for all $z\in B_{\widetilde{\varepsilon}_0}(1)$, $\kappa(z)$ is a simple eigenvalue of $\mathbb{M}(z)$, $\mathbb{M}(z)$ has r eigenvalues distinct from $\kappa(z)$ in $\mathbb{D}\backslash\{0\}$ and $p-1$ eigenvalues distinct from $\kappa(z)$ in $\mathcal{U}$. We then have that for $z\in B_{\widetilde{\varepsilon}_0}(1)$, the vector

$$R_c(z):=\begin{pmatrix}\kappa(z)^{p+r-1}\\ \vdots\\ 1\end{pmatrix}\in\mathbb{C}^{p+r}$$

is a nonzero eigenvector of $\mathbb{M}(z)$ associated with $\kappa(z)$. For $z\in B_{\widetilde{\varepsilon}_0}(1)$, we define $E^c(z):=\mathrm{Span}(R_c(z))$ and $E^{ss}(z)$ (resp. $E^{su}(z)$) the strictly stable (resp. strictly unstable) subspace of $\mathbb{M}(z)$ which corresponds to the subspace spanned by the generalized eigenvectors of $\mathbb{M}(z)$ associated with eigenvalues distinct from $\kappa(z)$ in $\mathbb{D}$ (resp. $\mathcal{U}$). We therefore know that $E^{ss}(z)$ (resp. $E^{su}(z)$) has dimension r (resp. $p-1$) and we have the decomposition

$$\mathbb{C}^{p+r}=E^{ss}(z)\oplus E^c(z)\oplus E^{su}(z). \tag{3.1.14}$$

The associated projectors are denoted $\pi^{ss}(z)$, $\pi^c(z)$ and $\pi^{su}(z)$. Again, those linear maps commute with $\mathbb{M}(z)$ and depend holomorphically on $z\in\mathcal{O}$ (see [Kat95]).

For $z\in B_{\widetilde{\varepsilon}_0}(1)\cap\mathcal{O}$, Lemma 3.1.2 implies that $\kappa(z)\in\mathcal{U}$. In other words, the "central" eigenvalue $\kappa(z)$ that is close to 1 comes from $\mathcal{U}$ as $z\in\mathcal{O}$ approaches 1. Therefore, for $z\in B_{\widetilde{\varepsilon}_0}(1)\cap\mathcal{O}$, we can link the two decompositions (3.1.13) and (3.1.14) of the spectrum of $\mathbb{M}(z)$ and observe that

$$E^s(z)=E^{ss}(z)\quad\text{and}\quad E^u(z)=E^c(z)\oplus E^{su}(z), \tag{3.1.15}$$

so

$$\pi^s(z)=\pi^{ss}(z)\quad\text{and}\quad\pi^u(z)=\pi^c(z)+\pi^{su}(z).$$

This allows us to extend holomorphically the vector spaces E^s and E^u and the projectors π^s and π^u in a neighborhood of $B_{\widetilde{\varepsilon}_0}(1)$. Even if we have to take $\widetilde{\varepsilon}_0>0$ smaller, we can introduce a family of holomorphic functions

$$e_1,\dots,e_r:B_{\widetilde{\varepsilon}_0}(1)\to\mathbb{C}^{p+r}$$

such that for all $z\in B_{\widetilde{\varepsilon}_0}(1)$, the family $(e_1(z),\dots,e_r(z))$ is a basis of $E^s(z)=E^{ss}(z)$ (see [Kat95, Section II.4.2.]). We then define the so-called Lopatinskii determinant Δ near 1 by the following formula:

$$\forall z\in B_{\widetilde{\varepsilon}_0}(1),\quad\Delta(z):=\det(\mathcal{B}e_1(z),\dots,\mathcal{B}e_r(z)), \tag{3.1.16}$$

with $\mathcal{B}$ the matrix defined as (3.1.5). In this way, the function Δ depends holomorphically on z near 1. The Lopatinskii determinant Δ does depend on the choice of the basis $(e_1,\dots,e_r)$ but we will only be interested in its zeroes, and these are independent of the choice of the basis.

We need to point out that for all $\underline{z}\in\mathcal{O}$, we can also define a holomorphic basis of $E^s(z)$ and thus a Lopatinskii determinant $\Delta(z)$ for z in a neighborhood of $\underline{z}$. The Lopatinskii determinant plays in our situation a similar role as the characteristic polynomial for a matrix as it allows to detect the eigenvalues of the operator $\mathcal{T}$. The stability of the numerical scheme (3.1.4) depends on the vanishing points of the Lopatinskii determinant Δ. Let us outline some terminology and results (see [GKO13]):

- The so-called Godunov-Ryabenkii condition introduced in [GR63] states that the Lopatinskii determinant does not vanish on $\mathcal{U}$, i.e. the operator $\mathcal{T}$ acting on $\mathcal{H}_q$ does not have any eigenvalue outside of the closed unit disk. It is a necessary stability condition for the numerical scheme (3.1.4).
- If the Lopatinskii determinant does not vanish on the whole set $\overline{\mathcal{U}}$, this means that the so-called uniform Kreiss-Lopatinskii condition is satisfied (see [GKS72; Cou11]). In that case, the main result in [Wu95] shows that the operator $\mathcal{T}$ on $\mathcal{H}_2$ is power bounded.
- In [CF22], the authors study the stability of the explicit numerical schemes for the scalar rightgoing $(v>0)$ transport equation on the positive half-line (3.1.1). The authors make the assumption that the Godunov-Ryabenkii condition is satisfied and that the Lopatinskii determinant has a finite number of simple zeroes on $\mathbb{S}^1$ that are not in $F(\mathbb{S}^1)$ (i.e. that are different from 1). Thus, the uniform Kreiss-Lopatinskii condition is not satisfied, and yet the authors prove semigroup estimates that lead to the ℓ^q-stability of the numerical scheme (3.1.4) for all $q\in[1,+\infty]$.

In this paper, we make the following assumption.

Hypothesis 3.2.

— *We suppose that for all $z \in \overline{\mathcal{U}} \backslash \{1\}$, we have*

$$E^s(z) \oplus \ker \mathcal{B} = \mathbb{C}^{p+r}.$$

In particular, this implies that the Lopatinskii determinant does not vanish anywhere on the set $\overline{\mathcal{U}} \backslash \{1\}$.

— *We assume that 1 is a simple zero of the Lopatinskii determinant, i.e.*

$$\Delta(1) = 0, \quad \Delta'(1) \neq 0.$$

We will also consider $\widetilde{\varepsilon}_0$ small enough so that the function Δ only vanishes in 1, which implies that

$$\forall z \in B_{\widetilde{\varepsilon}_0}(1) \backslash \{1\}, \quad E^s(z) \oplus \ker \mathcal{B} = \mathbb{C}^{p+r}.$$

Remark 2. We would like to make two observations on Hypothesis 3.2. First, noticing that for $z \in \mathcal{O} \cup B_{\widetilde{\varepsilon}_0}(1)$, we have $\dim E^s(z) = r$ and $\dim \ker \mathcal{B} = p$, it is interesting to observe for future purposes (see Lemma 3.4.1) that

$$E^s(z) \oplus \ker \mathcal{B} = \mathbb{C}^{p+r} \Leftrightarrow E^s(z) \cap \ker \mathcal{B} = \{0\}.$$

This also means that $\mathcal{B}_{|E^s(z)}$ is an isomorphism from $E^s(z)$ to $\mathbb{C}^r$.

Second, we observe that proving for some concrete choice of numerical scheme that Hypothesis 3.2 is verified can be challenging (see Section 3.2 for simple examples). We would like to point out that in the recent papers [BLS23; BPS23], though the study is done for numerical schemes applied to the rightgoing ($v > 0$) transport equation (3.1.1) which does not coincide with the study of the present paper, the authors present a reliable way to study the verification of the Uniform Kreiss-Lopatinskii condition by counting the number of zeroes of a modified version of the Lopatinskii determinant Δ.

Let us settle on the position of Hypothesis 3.2 compared to the previously stated results. If Hypothesis 3.2 is verified, then the Godunov-Ryabenkii condition is verified but not the uniform Kreiss-Lopatinskii condition, i.e. the Lopatinskii determinant Δ does not vanish on the set $\mathcal{U}$ but it vanishes on the unit circle and more precisely at 1, in the essential spectrum of $\mathcal{T}$. A similar situation is tackled in [CF22] but with two differences:

— In [CF22], the transport equation (3.1.1) which is approached using the numerical scheme (3.1.4) is rightgoing (i.e. $v > 0$) which is in direct opposition with the case handled in this paper. The main effect of this change is that the "central" eigenvalue $\kappa(z)$ of the matrix $\mathbb{M}(z)$ that is close to 1 as $z \in \mathcal{O}$ approaches 1 no longer comes from $\mathcal{U}$, but rather from the open unit disk $\mathbb{D}$. This in turn changes the equality (3.1.15) on the stable and unstable subspaces of the matrix $\mathbb{M}(z)$.

— The second and main difference however is on the position of the zeroes of the Lopatinskii determinant Δ which has a direct impact on the analysis of the so-called spatial Green's function, a useful tool that will be defined in Section 3.4. In [CF22], the function Δ can vanish at a finite number of points on the unit circle but not at the point 1. This allows the authors to holomorphically extend the spatial Green's function in a neighborhood of the interest point 1, through the essential spectrum of $\mathcal{T}$. However, in the present paper, we assume that 1 is a simple zero of the Lopatinskii determinant Δ which restricts us to only meromorphically extend the spatial Green's function in a neighborhood of 1 (see Lemma 3.4.3), which will toughen the computations. It is also interesting to point out that the type of spectral configuration we consider here in Hypothesis 3.2 with a simple eigenvalue at 1 which lies in the essential spectral $\mathcal{T}$ also occurs in the study of the linear stability of discrete shock profiles for conservation law approximations (see [Ser07; God03]). We detail a little more on the later point in Remark 3 after introducing the temporal Green's function.

The goal is now to study the stability of the scheme (3.1.4) under Hypotheses 3.1 and 3.2. The main result of this paper is the following theorem.

Theorem 3.1. *We assume that Hypotheses 3.1 and 3.2 are verified.*

— *If $\mathcal{B}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T \in \mathcal{B}E^s(1)$, then the numerical scheme (3.1.4) is ℓ^q-stable for all $q \in [1, +\infty]$ (see Definition 1 of ℓ^q-stability).*

— *If $\mathcal{B}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T \notin \mathcal{B}E^s(1)$, then the numerical scheme (3.1.4) is ℓ^1-stable but ℓ^q-unstable for all $q \in]1, +\infty]$. Furthermore, for all $q \in]1, +\infty]$, there exists a positive constant C such that*

$$\forall n \in \mathbb{N} \backslash \{0\}, \quad \|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_q)} \geq C n^{1-\frac{1}{q}}. \tag{3.1.17}$$

This is in direct opposition with the result of [CF22] which proved that the existence of simple zeroes of the Lopatinskii determinant Δ on the unit circle that are different from 1 does not prevent the ℓ^q-stability of the numerical scheme (3.1.4) for all $q \in [1, +\infty]$. Let us observe that Hypothesis 3.2 implies that $\mathcal{B}E^s(1)$ is a hyperplane of $\mathbb{C}^r$. Thus, the condition $\mathcal{B}(1 \dots 1)^T \in \mathcal{B}E^s(1)$ is "rarely" verified. In Section 3.2, we will present a concrete example for each possibility.

Temporal Green's function

The proof of Theorem 3.1 relies on a precise description of the temporal Green's function associated with $\mathcal{T}$ given in Theorem 3.2 below. We will start by defining the temporal Green's function associated with $\mathcal{T}$.

For all $j_0 \in \mathbb{N}\backslash\{0\}$, we introduce the Dirac mass $\delta_{j_0} \in \bigcap_{q\in[1,+\infty]}\mathcal{H}_q$ such that

$$\forall j \in \mathbb{N}\backslash\{0\}, \quad (\delta_{j_0})_j := \begin{cases} 1 & \text{if } j = j_0, \\ 0 & \text{else.} \end{cases} \tag{3.1.18}$$

We then define the temporal Green's function $\mathcal{G}(n, j_0, j)$ of the operator $\mathcal{T}$ as

$$\forall n \in \mathbb{N}, \quad \mathcal{G}(n, j_0, \cdot) := \mathcal{T}^n\delta_{j_0} \in \bigcap_{q\in[1,+\infty]}\mathcal{H}_q. \tag{3.1.19}$$

For any initial condition $u^0 \in \mathcal{H}_q$, the solution $(u^n)_{n\in\mathbb{N}}$ of (3.1.4) can be written as

$$\forall n \in \mathbb{N}, \forall j \geq 1-r, \quad u_j^n = (\mathcal{T}^n u^0)_j = \sum_{j_0\geq 1} u_{j_0}^0 \mathcal{G}(n, j_0, j). \tag{3.1.20}$$

Thus, the temporal Green's function $\mathcal{G}(n, j_0, j)$ allows us to analyze the behavior of solutions of the numerical scheme (3.1.4) over time. Let us observe that the sequences δ_{j_0} and $\mathcal{G}(n, j_0, \cdot)$ are finitely supported. More precisely, we recursively prove that:

$$\forall n \in \mathbb{N}, \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad j - j_0 \notin \{-np, \dots, nr\} \Rightarrow \mathcal{G}(n, j_0, j) = 0.$$

Just as we introduced the Dirac mass and the temporal Green's function $\mathcal{G}(n, j_0, j)$ of the operator $\mathcal{T}$, we introduce the Dirac mass $\widetilde{\delta} \in \bigcap_{q\in[1,+\infty]}\ell^q(\mathbb{Z})$ and the temporal Green's function $\widetilde{\mathcal{G}}(n, j)$ of the convolution operator $\mathscr{L}$ defined by:

$$\forall j \in \mathbb{Z}, \quad \widetilde{\delta}_j := \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{else,} \end{cases} \tag{3.1.21}$$

and

$$\forall n \in \mathbb{N}, \quad \widetilde{\mathcal{G}}(n, \cdot) := \mathscr{L}^n\widetilde{\delta} \in \bigcap_{q\in[1,+\infty]}\ell^q(\mathbb{Z}). \tag{3.1.22}$$

We also introduce the functions $H_{2\mu}^\beta, E_{2\mu}^\beta : \mathbb{R} \to \mathbb{C}$, where $\mu \in \mathbb{N}\backslash\{0\}$ and $\beta \in \mathbb{C}$ has positive real part, which are defined as

$$\begin{aligned} \forall x \in \mathbb{R}, \quad H_{2\mu}^\beta(x) &:= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixu}e^{-\beta u^{2\mu}}du, \\ \forall x \in \mathbb{R}, \quad E_{2\mu}^\beta(x) &:= \int_x^{+\infty} H_{2\mu}^\beta(y)dy. \end{aligned} \tag{3.1.23}$$

We will recall below why $H_{2\mu}^\beta \in L^1(\mathbb{R}, \mathbb{R})$. We call the functions $H_{2\mu}^\beta$ generalized Gaussians and the functions $E_{2\mu}^\beta$ generalized Gaussian error functions since for $\mu = 1$, we have

$$\forall x \in \mathbb{R}, \quad H_2^\beta(x) = \frac{1}{\sqrt{4\pi\beta}}e^{-\frac{x^2}{4\beta}}.$$

Noticing that the function $H_{2\mu}^\beta$ is the inverse Fourier transform of $u \mapsto e^{-\beta u^{2\mu}}$, we observe that

$$\lim_{x\to-\infty} E_{2\mu}^\beta(x) = \int_{-\infty}^{+\infty} H_{2\mu}^\beta(y)dy = 1. \tag{3.1.24}$$

The temporal Green's function $\widetilde{\mathcal{G}}(n, j)$ of the operator $\mathscr{L}$ has been studied thoroughly in [Tho65; DS14; RS15; CF23; Coe22]. For instance, under Hypothesis 3.1, it is known that the family $\left(\widetilde{\mathcal{G}}(n, \cdot)\right)_{n\in\mathbb{N}}$ is bounded in $\ell^1(\mathbb{Z})$. Furthermore, in [RS15; Coe22], it is proved that the leading order of the asymptotic behavior of $\widetilde{\mathcal{G}}(n, j)$ when n becomes large is the generalized Gaussian wave which travels at speed α. For instance, the main result in [RS15] gives:

$$\widetilde{\mathcal{G}}(n, j) = \frac{1}{n^{\frac{1}{2\mu}}}H_{2\mu}^\beta\left(\frac{j - n\alpha}{n^{\frac{1}{2\mu}}}\right) + o\left(\frac{1}{n^{\frac{1}{2\mu}}}\right) \tag{3.1.25}$$

where the remainder is uniform with respect to $j \in \mathbb{Z}$.

In this paper, we aim to prove the following theorem which describes the long time behavior of the temporal Green's function $\mathcal{G}(n, j_0, j)$:

Theorem 3.2. *Under Hypotheses 3.1 and 3.2, there exist two sequences* $(\mathcal{R}^c(j))_{j\in\mathbb{N}\setminus\{0\}}$ *and* $(\mathcal{R}^u(j_0,j))_{j_0,j\in\mathbb{N}\setminus\{0\}}$ *and two constants* $C, c > 0$ *such that if we define for all* $n, j_0, j \in \mathbb{N}\setminus\{0\}$,

$$\mathrm{Err}(n, j_0, j) := \mathcal{G}(n, j_0, j) - \widetilde{\mathcal{G}}(n, j - j_0) - \mathbb{1}_{np\geq j_0}\mathcal{R}^u(j_0, j) - E^{\beta}_{2\mu}\left(\frac{j_0 + n\alpha}{n^{\frac{1}{2\mu}}}\right)\mathcal{R}^c(j), \tag{3.1.26}$$

then, we have that:

$$\forall n, j_0, j \in \mathbb{N}\setminus\{0\}, \quad |\mathrm{Err}(n, j_0, j)| \leq \frac{Ce^{-cj}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|n\alpha + j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{3.1.27}$$

Furthermore, there exist two positive constants C, c *such that*

$$\forall j_0, j \in \mathbb{N}\setminus\{0\}, \quad |\mathcal{R}^u(j_0, j)| \leq Ce^{-c(j+j_0)} \quad \textit{and} \quad |\mathcal{R}^c(j)| \leq Ce^{-cj}. \tag{3.1.28}$$

Finally, the sequence $\mathcal{R}^c$ *satisfies that*

$$\mathcal{R}^c = 0 \Leftrightarrow \mathcal{B}(1\dots1)^T \in \mathcal{B}E^s(1). \tag{3.1.29}$$

The sequences $\mathcal{R}^u(j_0, j)$ and $\mathcal{R}^c(j)$ correspond to boundary layers which are linked respectively to the vector spaces $E^{su}(1)$ and $E^c(1)$ defined in (3.1.14). The coefficients $\mathbb{1}_{np\geq j_0}$ and $E^{\beta}_{2\mu}\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)$ which are in front of the values $\mathcal{R}^u(j_0, j)$ and $\mathcal{R}^c(j)$ in the definition (3.1.26) of $\mathrm{Err}(n, j_0, j)$ can be described as "activation" coefficients. They are close to 0 for small times n and get closer to 1 as n becomes larger.

Let us portray the description of the temporal Green's function $\mathcal{G}(n, j_0, j)$ that Theorem 3.2 conveys. For an initial condition $u^0 = \delta_{j_0}$, the solution $\mathcal{G}(n, j_0, j)$ of the numerical scheme (3.1.4) does not see the boundary condition for sufficiently small times n since the stencil of the numerical scheme is finite. Therefore, it coincides with $(\widetilde{\mathcal{G}}(n, j-j_0))_{j\geq1}$ the solution of the numerical scheme (3.1.8) with a similar initial condition. Thus, (3.1.25) tells us that the temporal Green's function $\mathcal{G}(n, j_0, j)$ is close to a generalized Gaussian wave for small times n. The boundary layer $\mathcal{R}^c$ and $\mathcal{R}^u$ are not activated yet. However, when the time n gets close to $-\frac{j_0}{\alpha}$, the generalized Gaussian wave associated with $\widetilde{\mathcal{G}}(n, j - j_0)$ reaches the boundary and the boundary layers $\mathcal{R}^u(j_0, j)$ and $\mathcal{R}^c(j)$ get activated. As n becomes large compared to $-\frac{j_0}{\alpha}$, most of the generalized Gaussian wave will have passed through the boundary and the boundary layers $\mathcal{R}^u(j_0, j)$ and $\mathcal{R}^c(j)$ are fully activated.

We can already intuitively deduce Theorem 3.1 from Theorem 3.2 when the boundary layer $\mathcal{R}^c$ is different from zero. The defining element that implies the ℓ^1-stability and the ℓ^q-instability for $q \in]1, +\infty]$ of the numerical scheme (3.1.4) is the independence with respect to j_0 of the boundary layer $\mathcal{R}^c$. If we consider an initial condition u^0, we expect to see a boundary layer

$$\left(\sum_{j_0\geq1} u^0_{j_0}\right)\mathcal{R}^c$$

appear for large times n because of the equality (3.1.20) and Theorem 3.2. We will clarify the proof in Section 3.3.

Remark 3. We make two remarks here:

• It should certainly be possible to prove a generalization of Theorem 3.1 and Theorem 3.2 with a relaxed Hypothesis 3.2 that allows additional simple zeroes of modulus 1 for the Lopatinskii determinant Δ either or not embedded into the essential spectrum. This would be achieved by combining the techniques from this paper and from [CF22].

• As explained earlier, there are connections between the study of approximations of hyperbolic PDEs with boundary conditions and the study of discrete shock profiles for conservation law approximations. The spectral configuration presented in Hypothesis 3.2 with a simple eigenvalue at 1 which lies in the essential spectrum $\mathcal{T}$ also occurs in the study of the linear stability of discrete shock profiles for conservation law approximations for Lax shocks (see [God03]). Using a similar analysis as in Theorem 3.2 and, more precisely, calculations similar as those done in Section 3.5 may improve the description of the temporal Green's function of stationary discrete shock profiles done in [God03]. This could potentially result in an argument for linear (and possibly non-linear) stability for the stationary discrete shock profiles.

3.1.4 Plan of the paper

We now present the outline of the paper.

Firstly, in Section 3.2, we numerically verify Theorems 3.1 and 3.2 on the example of the Lax-Friedrichs scheme with a boundary condition so that

$$\mathcal{B}(1\dots 1)^T \notin \mathcal{B}E^s(1)$$

which implies that the numerical scheme will be ℓ^1-stable but ℓ^q-unstable for all $q \in]1,+\infty]$. We compute the temporal Green's function $\mathcal{G}(n, j_0, j)$ and observe the formation of the boundary layer $\mathcal{R}^c(j)$. We then assess the accuracy of the estimates (3.1.27) on the error term $\mathrm{Err}(n, j_0, j)$ and of the rate of growth (3.1.17) for the family $(\mathcal{T}^n)_{n\in\mathbb{N}}$ acting on $\mathcal{H}_q$. We then consider the example of the O3 scheme with a boundary condition so that

$$\mathcal{B}(1\dots 1)^T \in \mathcal{B}E^s(1)$$

which implies that the scheme is ℓ^q-stable for all $q \in [1,+\infty]$. We will numerically verify this statement.

In Section 3.3, we present the proof of Theorem 3.1 whilst assuming that Theorem 3.2 has been proved.

The main part of this article will be dedicated to the proof of Theorem 3.2, which will rely on an approach referred to as spatial dynamics, also used in [ZH98; God03; CF23; CF22; Coe22]. In Section 3.4, we study the spectrum of the operators $\mathcal{T}$ and $\mathscr{L}$ and define the spatial Green's functions for those operators. We then demonstrate precise estimates for the difference between the two spatial Green's functions and extend them meromorphically in a neighborhood of 1. In Lemma 3.4.3, we also define the boundary layers $\mathcal{R}^u$ and $\mathcal{R}^c$ that occur in Theorem 3.2 and prove the assertion (3.1.29).

The proof of Theorem 3.2 will be presented in Section 3.5. The approach involves expressing the difference of the temporal Green's functions $\mathcal{G}(n, j_0, j) - \widetilde{\mathcal{G}}(n, j - j_0)$ through the spatial Green's functions. Using the results obtained in Section 3.4, we will then prove bounds on $\mathrm{Err}(n, j_0, j)$. We expect three different behaviors depending on the ratio j_0/n: the case where j_0 is large compared to n (i.e. $j_0 > np$), the case where j_0 is small compared to n (i.e. $j_0 < -\frac{n\alpha}{2}$) and the case where j_0 is close to $-n\alpha$ (i.e. $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$). The latter case will be the bulk of the proof.

3.2 Numerical result

3.2.1 Example of unstable boundary condition for the modified Lax-Friedrichs scheme

We consider the modified Lax-Friedrichs scheme for the transport equation (3.1.1)

$$\begin{aligned} \forall n\in\mathbb{N}, \forall j \in \mathbb{N}\backslash\{0\}, \quad & u_j^{n+1} = a_{-1}u_{j-1}^n + a_0 u_j^n + a_1 u_{j+1}^n, \\ \forall n \in \mathbb{N}, \quad & u_0^n = b u_1^n, \end{aligned} \tag{3.2.1}$$

where $p = r = 1$, $\alpha := \lambda v$, $D > 0$, the coefficient $b \in \mathbb{R}$ determines the boundary condition and

$$a_{-1} = \frac{D+\alpha}{2}, \quad a_0 = 1 - D, \quad a_1 = \frac{D-\alpha}{2}.$$

We assume that $D \neq -\alpha$ so that all three coefficients a_{-1}, a_0 and a_1 are nonzero. The symbol F defined by (3.1.10) verifies

$$\forall t \in \mathbb{R}, \quad F(e^{it}) = 1 - D + D\cos(t) - i\alpha\sin(t).$$

If we consider that $\alpha^2 < D < 1$, then there holds:

$$\forall t \in [-\pi,\pi]\backslash\{0\}, \quad |F(e^{it})| < 1.$$

Furthermore, we have that

$$F(e^{it}) \underset{t\to 0}{=} \exp\left(-i\alpha t - \beta t^2 + o(t^2)\right)$$

with $\beta := \frac{D-\alpha^2}{2} > 0$ and $\mu := 1$ in our notation of Hypothesis 3.1. Therefore, Hypothesis 3.1 is verified.

We will now make a choice for the coefficient b so that Hypothesis 3.2 is also verified. The matrix $\mathcal{B}$ and its kernel for the numerical scheme (3.2.1) are equal to

$$\mathcal{B} = \begin{pmatrix} -b & 1 \end{pmatrix} \quad \text{and} \quad \ker \mathcal{B} = \mathrm{Span}\begin{pmatrix} 1 \\ b \end{pmatrix}.$$

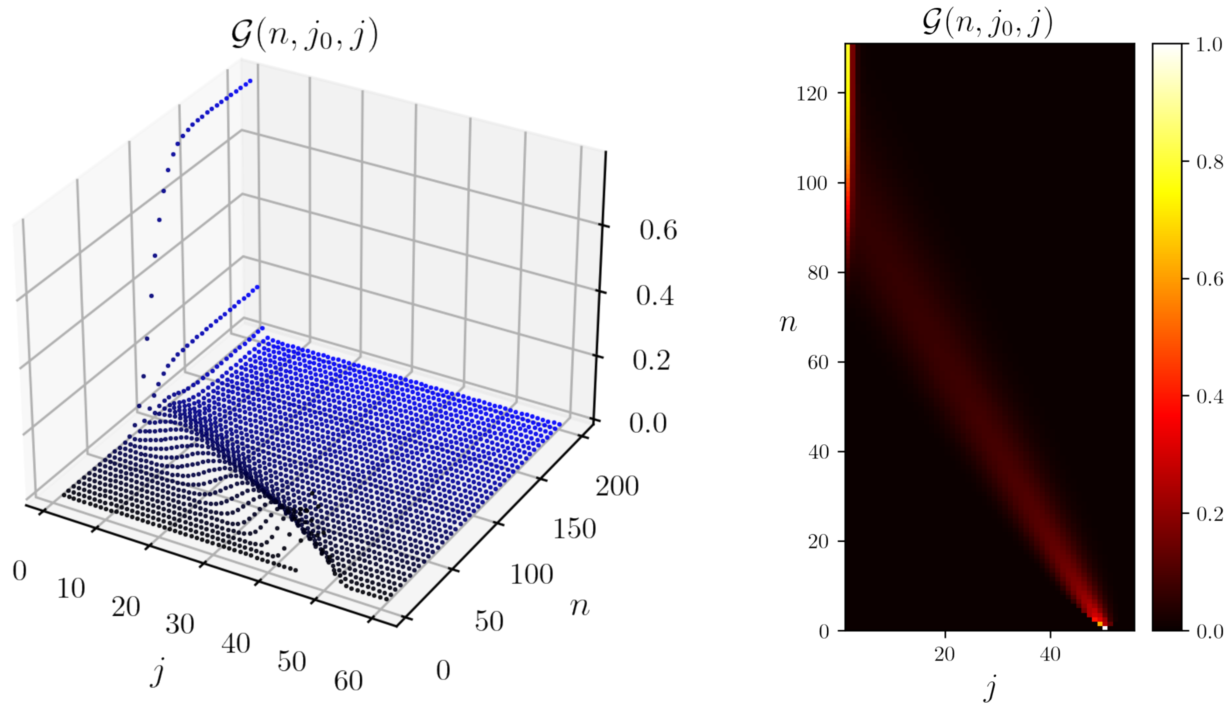

Figure 3.2 – We consider the modified Lax-Friedrichs scheme (3.2.1) with the parameters $\alpha = -\frac{1}{2}$, $D = \frac{3}{4}$ and $b = 5$. Both figures represent the temporal Green's function $\mathcal{G}(n, j_0, j)$ for $j_0 = 50$ which is the solution of the numerical scheme (3.2.1) for the initial condition $u^0 = \delta_{j_0}$.

We now need to determine $E^s(z)$ for $z \in \mathcal{O}$ and $z = 1$. Lemma 3.1.2 implies that for $z \in \mathcal{O}$, the matrix $\mathbb{M}(z)$ has an eigenvalue $\kappa_s(z) \in \mathbb{D}$ and an eigenvalue $\kappa_u(z) \in \mathcal{U}$ and that for $z = 1$, the matrix $\mathbb{M}(1)$ has an eigenvalue $\kappa_s(1) \in \mathbb{D}$ and $\kappa_u(1) := 1$ is a simple eigenvalue of $\mathbb{M}(1)$. We thus have that

$$\forall z \in \mathcal{O} \cup \{1\}, \quad E^s(z) = \mathrm{Span}\begin{pmatrix} \kappa_s(z) \\ 1 \end{pmatrix}.$$

We also observe that the determinant of the matrix $\mathbb{M}(z)$ is constantly equal to $\frac{a_{-1}}{a_1}$. Thus, for $z \in \mathcal{O} \cup \{1\}$, $\kappa_s(z) = \kappa_s(1)$ if and only if 1 is an eigenvalue of $\mathbb{M}(z)$, which is only verified when $z = 1$. Furthermore, the Lopatinskii determinant Δ in a neighborhood of 1 is equal to $1 - b\kappa_s(z)$. To satisfy Hypothesis 3.2, we obviously choose $b = \frac{1}{\kappa_s(1)} = \frac{a_1}{a_{-1}}$ so that 1 is a simple zero of Δ and

$$\forall z \in \mathcal{O}, \quad \ker \mathcal{B} \cap E^s(z) = \{0\}.$$

Thus, Hypothesis 3.2 is satisfied. Furthermore, we observe that

$$\mathcal{B}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 - b \neq 0 \quad \text{and} \quad \mathcal{B}E^s(1) = \{0\}. \tag{3.2.2}$$

It ensues from Theorem 3.1 that the numerical scheme (3.2.1) is ℓ^1-stable but ℓ^q-unstable for all $q \in]1, +\infty]$.

We now consider the numerical scheme (3.2.1) with the parameters $\alpha = -\frac{1}{2}$, $D = \frac{3}{4}$ and $b = 5$. Hypotheses 3.1 and 3.2 are thus fulfilled. We want to observe the decomposition of the temporal Green's function $\mathcal{G}(n, j_0, j)$ presented in Theorem 3.2. In Figure 3.2, we consider $j_0 = 50$ and we apply the numerical scheme (3.2.1) for the initial datum $u^0 = \delta_{j_0}$ in order to compute the temporal Green's function $\mathcal{G}(n, j_0, j)$ defined by (3.1.19). Let us observe that, since the strictly unstable subspace $E^{su}(1)$ is equal to $\{0\}$, following the proof of Theorem 3.2, we can prove that $\mathcal{R}^u = 0$ (see proof of Lemma 3.4.3). Furthermore, because of (3.2.2), Theorem 3.2 states that $\mathcal{R}^c \neq 0$. On Figure 3.2, we observe that:

- For n small compared to $-\frac{j_0}{\alpha}$, we observe that the temporal Green's function $\mathcal{G}(n, j_0, j)$ resembles a Gaussian wave which travels at a speed α, as expected for the temporal Green's function $\widetilde{\mathcal{G}}(n, j - j_0)$ computed using (3.1.22).
- When n is close to $-\frac{j_0}{\alpha}$, the Gaussian wave reaches the boundary and activates the boundary layer $\mathcal{R}^c$.

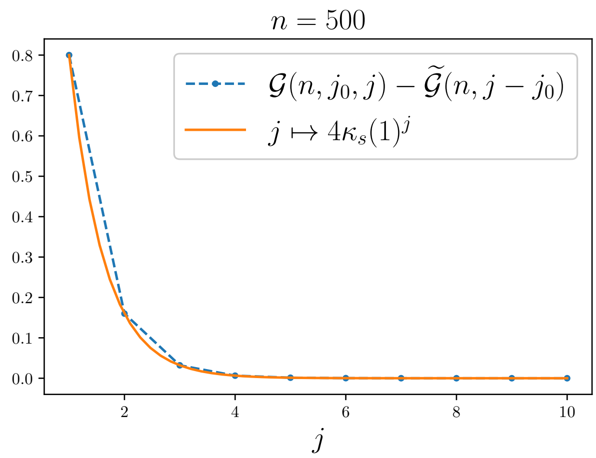

Figure 3.3 – We consider the modified Lax-Friedrichs scheme (3.2.1) with the parameters $\alpha = -\frac{1}{2}$, $D = \frac{3}{4}$ and $b = 5$. We represent the difference of the temporal Green's functions $\mathcal{G}(n, j_0, j) - \widetilde{\mathcal{G}}(n, j - j_0)$ for $j_0 = 50$ and $n = 500$. Since n is large compared to $-\frac{j}{\alpha}$, there only remains the boundary layer $\mathcal{R}^c$ which we can identify.

•When n is large compared to $-\frac{j_0}{\alpha}$, the Gaussian wave has passed the boundary. Furthermore, $E^{\beta}_{2\mu}\left(\frac{j_0 + n\alpha}{n^{\frac{1}{2\mu}}}\right)$ is close to 1 so the boundary layer $\mathcal{R}^c$ is fully activated.

On Figure 3.3, we consider a large time iteration $n = 500$. Theorem 3.2 states that the difference of the temporal Green's functions $\mathcal{G}(n, j_0, j) - \widetilde{\mathcal{G}}(n, j - j_0)$ should be close to the boundary layer $\mathcal{R}^c$. This allows us to identify the expression of the boundary layer $\mathcal{R}^c$ in our case.

Now that we have determined the boundary layer $\mathcal{R}^c$ which does not depend on the value of j_0, we can compute the error term $\mathrm{Err}(n, j_0, j)$ defined by (3.1.26). Let us observe that for the numerical scheme (3.2.1), since Hypotheses 3.1 and 3.2 are verified and $\mu = 1$, Theorem 3.2 states that there exist two positive constants $C, c > 0$ such that the term $\mathrm{Err}(n, j_0, j)$ satisfies the following estimate

$$\forall n, j_0, j \in \mathbb{N}\backslash\{0\}, \quad \sqrt{n}|\mathrm{Err}(n, j_0, j)| \leq Ce^{-cj} \exp\left(-c\frac{|n\alpha + j_0|^2}{n}\right). \tag{3.2.3}$$

In Figure 3.4, we fix $j = 1$ and plot $\sqrt{n}\,\mathrm{Err}(n, j_0, 1)$ against n and j_0. The results support the sharpness of the estimates of Theorem 3.2, showing Gaussian behavior as predicted by (3.2.3).

Finally, because of (3.2.2), Theorem 3.1 states that for $q \in]1, +\infty]$, the numerical scheme (3.2.1) is ℓ^q-unstable and that there exists a constant $C > 0$ such that the family of operators $(\mathcal{T}^n)_{n\geq 0}$ in $\mathcal{L}(\mathcal{H}_q)$ satisfies the inequality (3.1.17) that we recall here

$$\forall n \in \mathbb{N}, \quad \|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_q)} \geq Cn^{1-\frac{1}{q}}. \tag{3.2.4}$$

In order to numerically verify the sharpness of the inequality (3.2.4), we will consider for $J \in \mathbb{N}\backslash\{0\}$ the initial condition $u_J \in \mathcal{H}_q$ defined by

$$u_J = \sum_{j_0=1}^{J} \delta_{j_0} \tag{3.2.5}$$

and compute the solution $(\mathcal{T}^n u_J)_{n\in\mathbb{N}}$ of the numerical scheme (3.2.1).

- On the left-side of Figure 3.5, we choose $q = +\infty$ and we represent for several choices of J the ratio of $\|\mathcal{T}^n u_J\|_{\mathcal{H}_\infty}$ and $\|u_J\|_{\mathcal{H}_\infty}$ which is lower than $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_\infty)}$. We observe a linear increase of $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_\infty)}$ depending on n that supports the inequality (3.2.4).
- On the right-side of Figure 3.5, we choose $q = 2$ and we represent in the logarithmic scale, for several choices of J, the ratio of $\|\mathcal{T}^n u_J\|_{\mathcal{H}_2}$ and $\|u_J\|_{\mathcal{H}_2}$ which is lower than $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_2)}$. We observe a growth of $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_2)}$ at a rate of $\sqrt{n}$ that also supports the inequality (3.2.4).

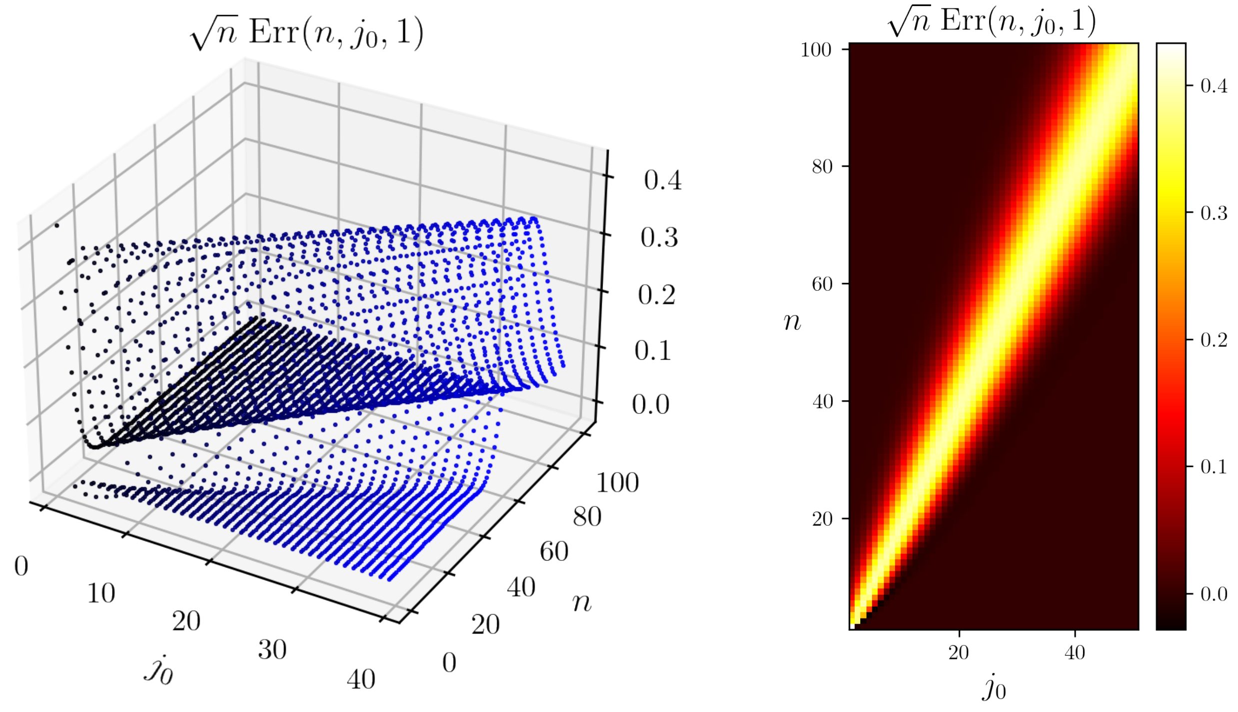

Figure 3.4 – For the modified Lax-Friedrichs scheme (3.2.1), we plot the error term $\sqrt{n}\mathrm{Err}(n, j_0, j)$ defined by (3.1.26). As predicted by Theorem 3.2, we observe that $\sqrt{n}\mathrm{Err}(n, j_0, j)$ satisfies Gaussian estimates of the form (3.2.3).

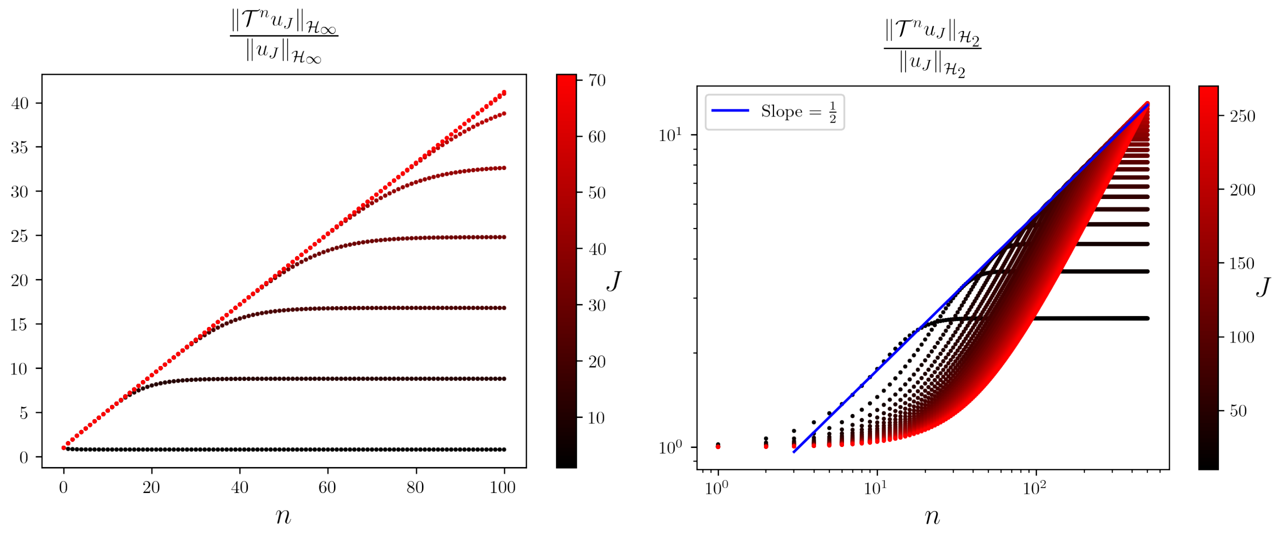

Figure 3.5 – We consider the modified Lax-Friedrichs scheme (3.2.1).
<u>On the left side:</u> For several choices of integers J, we compute the ratio between $\|\mathcal{T}^n u_J\|_{\mathcal{H}_\infty}$ and $\|u_J\|_{\mathcal{H}_\infty}$ depending on n, where the sequence u_J is defined by (3.2.5). For n fixed, this gives a lower bound for $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_\infty)}$. The figure supports the fact that $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_\infty)}$ grows at a linear rate with regards to n.
<u>On the right side:</u> For several choices of integers J, we compute the ratio between $\|\mathcal{T}^n u_J\|_{\mathcal{H}_2}$ and $\|u_J\|_{\mathcal{H}_2}$ depending on n, where the sequence u_J is defined by (3.2.5). For n fixed, this gives a lower bound for $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_2)}$. The representation in the logarithmic scale supports the fact that $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_2)}$ grows at a rate $\sqrt{n}$.

3.2.2 Example of stable boundary condition for the O3 scheme

We consider the O3 scheme for the transport equation (3.1.1)

$$\begin{aligned} \forall n \in \mathbb{N}, \forall j \in \mathbb{N}\backslash\{0\}, \quad & u_j^{n+1} = a_{-1}u_{j-1}^n + a_0 u_j^n + a_1 u_{j+1}^n + a_2 u_{j+2}^n, \\ \forall n \in \mathbb{N}, \quad & u_0^n = b_1 u_1^n + b_2 u_2^n, \end{aligned} \tag{3.2.6}$$

where $r = 1$ and $p = 2$, $\alpha := \lambda v$, the coefficients $b_1, b_2 \in \mathbb{R}$ determine the boundary condition and

$$\begin{aligned} a_{-1} &:= \frac{\alpha(1+\alpha)(2+\alpha)}{6}, \quad a_0 := \frac{(1-\alpha^2)(2+\alpha)}{2}, \\ a_1 &:= -\frac{\alpha(1-\alpha)(2+\alpha)}{2}, \quad a_2 := \frac{\alpha(1-\alpha^2)}{6}. \end{aligned}$$

We refer to [Des08] for a detailed analysis of this scheme for the rightgoing transport equation on the whole line $\mathbb{R}$. For $\alpha \in]-1, 0[$, the symbol F defined by (3.1.10) satisfies $F(1) = 1$ and

$$\forall \kappa \in \mathbb{S}^1\backslash\{1\}, \quad |F(\kappa)| < 1.$$

Furthermore, there exists a constant $\beta > 0$ such that

$$F(e^{it}) \underset{t\to 0}{=} \exp\left(-i\alpha t - \beta t^4 + o(t^4)\right).$$

We have that $\mu = 2$ in our notation of Hypothesis 3.1. Thus, Hypothesis 3.1 is satisfied.

We now make a choice for the coefficients b_1 and b_2 that define the numerical boundary condition for the numerical scheme (3.2.6). Lemma 3.1.2 implies that for $z \in \mathcal{O}$, the matrix $\mathbb{M}(z)$ has an eigenvalue $\kappa_s(z) \in \mathbb{D}\backslash\{0\}$ and two (not necessarily distinct) eigenvalues $\kappa_u^1(z), \kappa_u^2(z) \in \mathcal{U}$ and that for $z = 1$ the matrix $\mathbb{M}(1)$ has an eigenvalue $\kappa_s(1) \in \mathbb{D}\backslash\{0\}$, one eigenvalue $\kappa_u^1(1) \in \mathcal{U}$ and $\kappa_u^2(1) = 1$ is a simple eigenvalue of $\mathbb{M}(1)$. We thus have that

$$\forall z \in \mathcal{O}\cup\{1\}, \quad E^s(z) = \operatorname{Span}\begin{pmatrix} \kappa_s(z)^2 \\ \kappa_s(z) \\ 1 \end{pmatrix}.$$

We observe that the trace and the determinant of $\mathbb{M}(z)$ are respectively equal to $-\frac{a_1}{a_2}$ and $-\frac{a_{-1}}{a_2}$. Let us consider $z \in \mathcal{O}\cup\{1\}$ such that $\kappa_s(z) = \kappa_s(1)$. Since the traces (resp. determinants) of the matrices $\mathbb{M}(z)$ and $\mathbb{M}(1)$ are equal and $\kappa_s(z) = \kappa_s(1) \neq 0$, we have

$$\begin{cases} \kappa_u^1(z) + \kappa_u^2(z) = \kappa_u^1(1) + 1, \\ \kappa_u^1(z)\kappa_u^2(z) = \kappa_u^1(1). \end{cases}$$

This implies that either $\kappa_u^1(z)$ or $\kappa_u^2(z)$ is equal to 1 and thus 1 is an eigenvalue of $\mathbb{M}(z)$. We conclude that for $z \in \mathcal{O}\cup\{1\}$, $\kappa_s(z) = \kappa_s(1)$ if and only $z = 1$. We choose

$$b_1 := \frac{1+\kappa_s(1)}{\kappa_s(1)}, \quad b_2 := -\frac{1}{\kappa_s(1)}.$$

so that the Lopatinskii determinant Δ in a neighborhood of 1 is defined by:

$$\Delta(z) := 1 - b_1\kappa_s(z) - b_2\kappa_s(z)^2 = -b_2(\kappa_s(z) - \kappa_s(1))(\kappa_s(z) - 1).$$

Thus, 1 is a simple zero of the Lopatinskii determinant and

$$\forall z \in \mathcal{O}, \quad \ker \mathcal{B} \cap E^s(z) = \{0\}.$$

Thus, Hypothesis 3.2 is satisfied. Furthermore, we observe that

$$\mathcal{B}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - b_1 - b_2 = 0 \in \mathcal{B}E^s(1).$$

It ensues from Theorem 3.1 that the numerical scheme (3.2.1) is ℓ^q-stable for all $q \in [1, +\infty]$. On Figure 3.6, we consider for several choices of $J \in \mathbb{N}\backslash\{0\}$ the initial condition $u_J \in \mathcal{H}_q$ defined by (3.2.5) and compute the solution $(\mathcal{T}^n u_J)_{n\in\mathbb{N}}$ of the numerical scheme (3.2.6). We represent the ratio of $\|\mathcal{T}^n u_J\|_{\mathcal{H}_q}$ and $\|u_J\|_{\mathcal{H}_q}$ which is lower than $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_q)}$. On the left-side of Figure 3.6 and on the right-side of Figure 3.6, we respectively choose

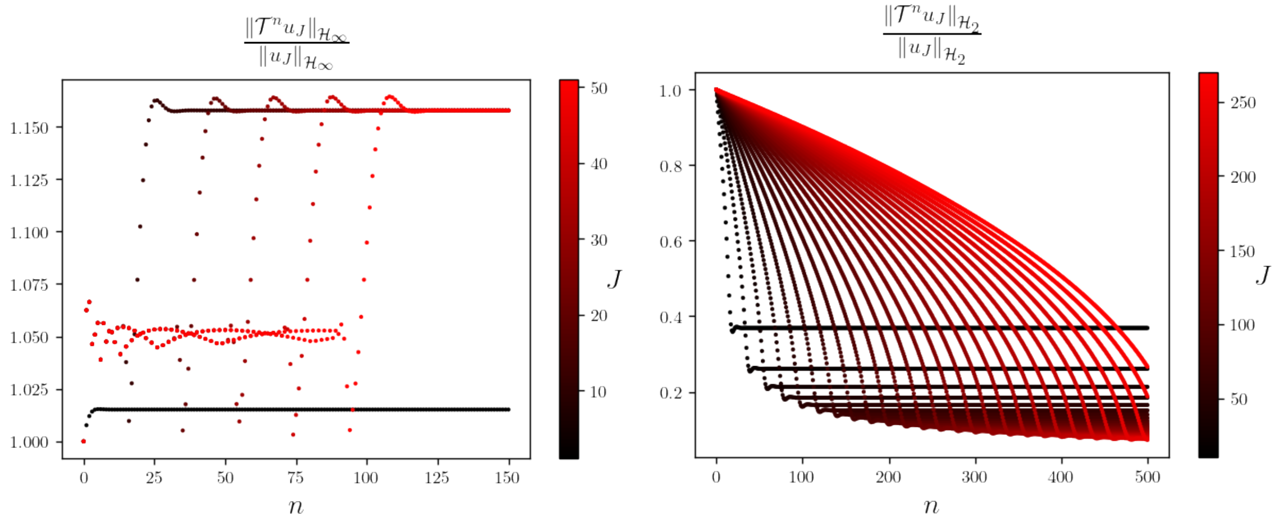

Figure 3.6 – For the O3 scheme (3.2.6), for several choices of integers J, we compute the ratio between $\|\mathcal{T}^n u_J\|_{\mathcal{H}_q}$ and $\|u_J\|_{\mathcal{H}_q}$ depending on n, where the sequence u_J is defined by (3.2.5). For n fixed, this gives a lower bound for $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_q)}$. On the left and right sides, we choose respectively $q = +\infty$ and $q = 2$.

$q = +\infty$ and $q = 2$. The figures support the fact that $\|\mathcal{T}^n\|_{\mathcal{L}(\mathcal{H}_q)}$ can be uniformly bounded in n.

3.3 Proof of Theorem 3.1 using Theorem 3.2

In this section, we will prove Theorem 3.1 whilst assuming that Theorem 3.2 has been proved. Theorem 3.2 allows us to decompose the temporal Green's function $\mathcal{G}(n, j_0, j)$ as follows

$$\forall n, j_0, j \in \mathbb{N}\backslash\{0\}, \quad \mathcal{G}(n, j_0, j) = \mathrm{Err}(n, j_0, j) + \widetilde{\mathcal{G}}(n, j - j_0) + \mathbb{1}_{np \geq j_0}\mathcal{R}^u(j_0, j) + E^\beta_{2\mu}\left(\frac{j_0 + n\alpha}{n^{\frac{1}{2\mu}}}\right)\mathcal{R}^c(j). \quad (3.3.1)$$

Furthermore, for $q \in [1, +\infty]$, $u^0 \in \mathcal{H}_q$ and $n \in \mathbb{N}$, we have that

$$\forall j \in \mathbb{N}\backslash\{0\}, \quad (\mathcal{T}^n u^0)_j = \sum_{j_0 \geq 1} u^0_{j_0}\mathcal{G}(n, j_0, j). \quad (3.3.2)$$

We decompose the operator $\mathcal{T}^n$ in two parts by introducing for all $q \in [1, +\infty]$ and $n \in \mathbb{N}\backslash\{0\}$ the two operators $K_{q,n}, L_{q,n} \in \mathcal{L}(\mathcal{H}_q)$ defined as

$$\forall u^0 \in \mathcal{H}_q, \forall j \in \mathbb{N}\backslash\{0\}, \qquad (K_{q,n}u^0)_j := \sum_{j_0 \geq 1} u^0_{j_0}\left(\mathrm{Err}(n, j_0, j) + \widetilde{\mathcal{G}}(n, j - j_0) + \mathbb{1}_{np \geq j_0}\mathcal{R}^u(j_0, j)\right),$$
$$\forall u^0 \in \mathcal{H}_q, \forall j \in \mathbb{N}\backslash\{0\}, \qquad (L_{q,n}u^0)_j := \sum_{j_0 \geq 1} u^0_{j_0}E^\beta_{2\mu}\left(\frac{j_0 + n\alpha}{n^{\frac{1}{2\mu}}}\right)\mathcal{R}^c(j),$$

Using (3.3.1) and (3.3.2), we have

$$\forall q \in [1, +\infty], \forall n \in \mathbb{N}\backslash\{0\}, \quad \mathcal{T}^n = K_{q,n} + L_{q,n}. \quad (3.3.3)$$

First, we will prove that for all $q \in [1, +\infty]$ the family of operators $(K_{q,n})_{n\in\mathbb{N}\backslash\{0\}}$ is bounded in $\mathcal{L}(\mathcal{H}_q)$. Using (3.3.3), this will obviously imply the ℓ^q-stability of the numerical scheme (3.1.4) for all $q \in [1, +\infty]$ when the boundary layer $\mathcal{R}^c$ is equal to 0.

Then, we will prove that the family of $(L_{1,n})_{n\in\mathbb{N}\backslash\{0\}}$ is also bounded in $\mathcal{L}(\mathcal{H}_1)$. Using (3.3.3), we will have then proved the ℓ^1-stability of the numerical scheme (3.1.4) even when $\mathcal{R}^c$ is not equal to 0

Finally, when the boundary layer $\mathcal{R}^c$ is non zero, we will prove for all $q \in]1, +\infty]$ that there exists a positive constant C such that

$$\forall n \in \mathbb{N}\backslash\{0\}, \quad \|L_{q,n}\|_{\mathcal{L}(\mathcal{H}_q)} \geq Cn^{1-\frac{1}{q}}. \quad (3.3.4)$$

Using (3.3.3), we will thus have proved the existence of a positive constant C such that (3.1.17) is verified and the ℓ^q-instability of the numerical scheme (3.1.4).

Step 1: Boundedness of the family $(K_{q,n})_{n\in\mathbb{N}\backslash\{0\}}$ in $\mathcal{L}(\mathcal{H}_q)$ and ℓ^q-stability when $\mathcal{R}^c=0$

• We consider $q\in[1,+\infty]$ and $u^0\in\mathcal{H}_q$. We also introduce $\tilde{q}\in[1,+\infty]$ the Hölder conjugate of q. Using the estimates (3.1.27) on $\mathrm{Err}(n,j_0,j)$ and Hölder's inequality, there exist two positive constants C,c such that for all $n,j\in\mathbb{N}\backslash\{0\}$

$$\sum_{j_0\geq 1}|\mathrm{Err}(n,j_0,j)||u^0_{j_0}|\leq Ce^{-cj}\left\|u^0\right\|_{\mathcal{H}_q}\left\|\left(\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|n\alpha+j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)_{j_0\geq 1}\right\|_{\ell^{\tilde{q}}}.$$

Furthermore, there exists a constant $C>0$ such that

$$\forall n\in\mathbb{N}\backslash\{0\},\quad \left\|\left(\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|n\alpha+j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)_{j_0\geq 1}\right\|_{\ell^{\tilde{q}}}\leq\frac{C}{n^{\frac{1}{2\mu q}}}.$$

Therefore, there exists a constant $C>0$ such that for all $u^0\in\mathcal{H}_q$ and $n\in\mathbb{N}\backslash\{0\}$, the sequence

$$\left(\sum_{j_0\geq 1}\mathrm{Err}(n,j_0,j)u^0_{j_0}\right)_{j\in\mathbb{N}\backslash\{0\}}$$

belongs to $\mathcal{H}_q$ and

$$\forall n\in\mathbb{N}\backslash\{0\},\quad \left\|\left(\sum_{j_0\geq 1}\mathrm{Err}(n,j_0,j)u^0_{j_0}\right)_{j\in\mathbb{N}\backslash\{0\}}\right\|_{\mathcal{H}_q}\leq\frac{C}{n^{\frac{1}{2\mu q}}}\left\|u^0\right\|_{\mathcal{H}_q}. \tag{3.3.5}$$

• Using the main result of [Tho65], we prove that Hypothesis 3.1 is one of two conditions so that the family $(\widetilde{\mathcal{G}}(n,\cdot))_{n\in\mathbb{N}}$ is bounded in $\ell^1(\mathbb{Z})$. For $q\in[1,+\infty]$, using Young's convolution inequality $\ell^1(\mathbb{Z})*\ell^q(\mathbb{Z})\to\ell^q(\mathbb{Z})$, we can prove the existence of a positive constant $C>0$ such that for all $u^0\in\mathcal{H}_q$ and $n\in\mathbb{N}$, the sequence $\left(\sum_{j_0\geq 1}\widetilde{\mathcal{G}}(n,j-j_0)u^0_{j_0}\right)_{j\in\mathbb{N}\backslash\{0\}}$ belongs to $\mathcal{H}_q$ and

$$\forall n\in\mathbb{N}\backslash\{0\},\quad \left\|\left(\sum_{j_0\geq 1}\widetilde{\mathcal{G}}(n,j-j_0)u^0_{j_0}\right)_{j\in\mathbb{N}\backslash\{0\}}\right\|_{\mathcal{H}_q}\leq C\left\|u^0\right\|_{\mathcal{H}_q}. \tag{3.3.6}$$

• We consider $q\in[1,+\infty]$ and $u^0\in\mathcal{H}_q$. We also introduce $\tilde{q}\in[1,+\infty]$ the Hölder conjugate of q. Using the bounds (3.1.28) on $\mathcal{R}^u$ and Hölder's inequality, we prove that there exist two positive constants C,c independent from q, u^0, n, j_0 and j such that

$$\left|\sum_{j_0\geq 1}\mathbb{1}_{np\geq j_0}\mathcal{R}^u(j_0,j)u^0_{j_0}\right|\leq Ce^{-cj}\left\|u^0\right\|_{\mathcal{H}_q}\left\|\left(e^{-cj_0}\right)_{j_0\geq 1}\right\|_{\ell^{\tilde{q}}}.$$

Therefore, there exists a constant $C>0$ such that for all $u^0\in\mathcal{H}_q$ and $n\in\mathbb{N}\backslash\{0\}$, the sequence

$$\left(\sum_{j_0\geq 1}\mathbb{1}_{np\geq j_0}\mathcal{R}^u(j_0,j)u^0_{j_0}\right)_{j\in\mathbb{N}\backslash\{0\}}$$

belongs to $\mathcal{H}_q$ and

$$\forall n\in\mathbb{N}\backslash\{0\},\quad \left\|\left(\sum_{j_0\geq 1}\mathbb{1}_{np\geq j_0}\mathcal{R}^u(j_0,j)u^0_{j_0}\right)_{j\in\mathbb{N}\backslash\{0\}}\right\|_{\mathcal{H}_q}\leq C\left\|u^0\right\|_{\mathcal{H}_q}. \tag{3.3.7}$$

For $q\in[1,+\infty]$, combining (3.3.5)-(3.3.7), we have proved that the family of operators $(K_{q,n})_{n\in\mathbb{N}\backslash\{0\}}$ is bounded in $\mathcal{L}(\mathcal{H}_q)$. When $\mathcal{B}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T \in \mathcal{B}E^s(1)$, Theorem 3.2 implies that the boundary layer $\mathcal{R}^c$ is equal to 0 and thus that the operator $L_{q,n}$ is equal to 0. We conclude using (3.3.3) that the numerical scheme (3.1.4) is ℓ^q-stable.

Step 2: Boundedness of the family $(L_{1,n})_{n\in\mathbb{N}\backslash\{0\}}$ in $\mathcal{L}(\mathcal{H}_1)$ and proof of the ℓ^1-stability of the numerical scheme (3.1.4)

We consider that $q=1$. Since the function $E^\beta_{2\mu}$ is bounded, the family $\left(E^\beta_{2\mu}\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)\right)_{n,j_0\in\mathbb{N}\backslash\{0\}}$ is bounded. Using the estimates (3.1.28) on $\mathcal{R}^c$, there exist two positive constants C,c such that for all $u^0\in\mathcal{H}_1$, $n,j_0,j\in\mathbb{N}\backslash\{0\}$,

$$\sum_{j_0\geq 1}\left|E^\beta_{2\mu}\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)\mathcal{R}^c(j)u^0_{j_0}\right| \leq Ce^{-cj}\sum_{j_0\geq 1}|u^0_{j_0}| \leq Ce^{-cj}\left\|u^0\right\|_{\mathcal{H}_1}.$$

Therefore, there exists a constant $C>0$ such that for all $u^0\in\mathcal{H}_1$ and $n\in\mathbb{N}$, the sequence

$$L_{1,n}u^0 = \left(\sum_{j_0\geq 1}E^\beta_{2\mu}\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)\mathcal{R}^c(j)u^0_{j_0}\right)_{j\in\mathbb{N}\backslash\{0\}}$$

belongs to $\mathcal{H}_1$ and

$$\left\|L_{1,n}u^0\right\|_{\mathcal{H}_1} \leq C\left\|u^0\right\|_{\mathcal{H}_1}.$$

This implies that the family of operators $(L_{1,n})_{n\in\mathbb{N}\backslash\{0\}}$ is bounded in $\mathcal{L}(\mathcal{H}_1)$. Using (3.3.3), we then immediately conclude that the family of operators $(\mathcal{T}^n)_{n\in\mathbb{N}\backslash\{0\}}$ is bounded in $\mathcal{L}(\mathcal{H}_1)$ and thus that the numerical scheme (3.1.4) is ℓ^1-stable.

Step 3: Proof of (3.3.4) and of the ℓ^q-instability of the numerical scheme (3.1.4) for $q\in]1,+\infty]$ when $\mathcal{R}^c\neq 0$

We fix $q\in]1,+\infty]$. Since the function $E^\beta_{2\mu}$ is continuous, (3.1.24) implies that there exists a constant $M\in\mathbb{R}$ such that

$$\forall x\leq M,\quad E^\beta_{2\mu}(x)\geq\frac{1}{2}. \tag{3.3.8}$$

Let us consider an integer $n\in\mathbb{N}\backslash\{0\}$ such that

$$-\frac{n\alpha}{3}>1 \quad\text{and}\quad \frac{\alpha}{3}n^{\frac{2\mu-1}{2\mu}}\leq M,$$

which is possible since $\alpha<0$. We consider an integer $J\in\mathbb{Z}\cap\left[-\frac{n\alpha}{3},-2\frac{n\alpha}{3}\right]$ and define

$$u_J := \sum_{j_0=1}^{J}\delta_{j_0}\in\mathcal{H}_q.$$

We observe that

$$L_{q,n}u_J = \left(\sum_{j_0=1}^{J}E^\beta_{2\mu}\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)\right)\mathcal{R}^c.$$

Yet, for $j_0\in\{1,\dots,J\}$, we have

$$\frac{n\alpha+j_0}{n^{\frac{1}{2\mu}}}\leq\frac{\alpha}{3}n^{\frac{2\mu-1}{2\mu}}\leq M.$$

Thus, (3.3.8) allows us to conclude that

$$\left\|L_{q,n}u_J\right\|_{\mathcal{H}_q}\geq\frac{J}{2}\left\|\mathcal{R}^c\right\|_{\mathcal{H}_q}.$$

Noticing that $\left\|u_J\right\|_{\mathcal{H}_q}=J^{\frac{1}{q}}$, we conclude that

$$\left\|L_{q,n}\right\|_{\mathcal{L}(\mathcal{H}_q)}\geq\frac{\left\|\mathcal{R}^c\right\|_{\mathcal{H}_q}}{2}J^{1-\frac{1}{q}}\geq\frac{\left\|\mathcal{R}^c\right\|_{\mathcal{H}_q}}{2}\left(-\frac{n\alpha}{3}\right)^{1-\frac{1}{q}}.$$

Thus, when the boundary layer $\mathcal{R}^c$ is a nonzero sequence, the operators $L_{q,n}$ are nonzero operators and there exists a positive constant C such that (3.3.4) is verified. Using (3.3.3), this concludes the proof of the existence of a positive constant C such that (3.1.17) is verified and the ℓ^q-instability of the numerical scheme (3.1.4).

3.4 Spatial Green's function

From now on, we assume that Hypotheses 3.1 and 3.2 are verified and our goal is to prove Theorem 3.2. Since the values of the temporal Green's function $\mathcal{G}(n,j,j_0)$ are independent of q, we consider that we are in the case $q=2$ and we omit the subscript q when we introduce the Banach space $\mathcal{H}$. This section is dedicated to the definition and analysis of the spatial Green's functions defined below by (3.4.9) and (3.4.10) respectively for the operators $\mathcal{T}$ and $\mathscr{L}$.

3.4.1 Resolvent set of the operators $\mathcal{T}$ and $\mathscr{L}$

First, we study the spectrum of the operators $\mathscr{L}$ and $\mathcal{T}$. This information is fundamental to determine the domain of definition of the spatial Green's functions for $\mathcal{T}$ and $\mathscr{L}$. This section will be fairly similar to [CF22, Section 2.1].

First, Wiener's theorem [New75] allows us to conclude that the spectrum of $\mathscr{L}$ is given by:

$$\sigma(\mathscr{L}) = F(\mathbb{S}^1).$$

In particular, this implies that the connected open set $\mathcal{O}$ is included in the resolvent $\rho(\mathscr{L})$. We recall that $\mathcal{O}$ corresponds to the unbounded connected component of $\mathbb{C}\backslash F(\mathbb{S}^1)$ (see Figure 3.1).

We now shift our attention to the study of the spectrum of the operator $\mathcal{T}$. The operator $\mathcal{T}$ is a finite rank perturbation of the Toeplitz operator defined by

$$\begin{array}{cccc} \mathscr{T}: & \ell^2(\mathbb{N}\backslash\{0\}) & \to & \ell^2(\mathbb{N}\backslash\{0\}) \\ & u & \mapsto & \left(\displaystyle\sum_{\substack{l=-r \\ j+l\geq 1}}^{p} a_l u_{j+l}\right)_{j\in\mathbb{N}\backslash\{0\}} \end{array},$$

since the Toeplitz operator $\mathscr{T}$ corresponds to the numerical scheme (3.1.4) where we impose the numerical boundary conditions $u_{1-r}^n = \ldots = u_0^n = 0$. The spectrum of the operator $\mathscr{T}$ has been studied thoroughly, see for instance [Dur64] and [TE05]. Mainly, the resolvent set of the operator $\mathcal{T}$ corresponds to the points on the complex plane that do not belong to the curve $F(\mathbb{S}^1)$ and such that the winding number of the curve $F(\mathbb{S}^1)$ for these points is 0. This is the case for all points in the set $\mathcal{O}$ for instance. Furthermore, the essential spectrum of $\mathscr{T}$ is the curve $F(\mathbb{S}^1)$.

We observe that since the operator $\mathcal{T}$ is a compact perturbation of the operator $\mathscr{T}$, they share the same essential spectrum (see [Con90, Chapter XI, Proposition 4.2]) which is equal to the curve $F(\mathbb{S}^1)$. In particular, 1 belongs to the essential spectrum of the operator $\mathcal{T}$. However, a careful examination of the eigenvalues of $\mathcal{T}$ is still necessary. Using the decompositions (3.1.13) and (3.1.14) of the vector space $\mathbb{C}^{p+r}$, we prove the following lemma:

Lemma 3.4.1. *We have that*

$$\forall z \in \mathcal{O}\cup B_{\widetilde{\varepsilon_0}}(1), \quad \dim\ker\mathcal{B}\cap E^s(z) = \dim\ker(zId-\mathcal{T}), \tag{3.4.1}$$

$$\forall z \in \mathcal{O}, \quad \ker\mathcal{B}\cap E^s(z) = \{0\} \Rightarrow z\in\rho(\mathcal{T}). \tag{3.4.2}$$

Hypothesis 3.2 allows us to conclude that 1 is a simple eigenvalue of $\mathcal{T}$ which lies inside of its essential spectrum. Furthermore, we have

$$\overline{\mathcal{U}}\backslash\{1\}\subset\rho(\mathcal{T}) \quad \text{and} \quad B_{\widetilde{\varepsilon_0}}(1)\cap\mathcal{O}\subset\rho(\mathcal{T}). \tag{3.4.3}$$

Proof The proofs of Assertions (3.4.1) and (3.4.2) use the same method as the proof of [CF22, Lemma 2.1] that we will shortly summarize here. We let the interested reader investigate and adapt the details using [CF22].

The starting point of the proof is to look for a solution $w\in\mathcal{H}$ of the recurrence relation with $z\in\mathbb{C}$ and $f\in\mathcal{H}$:

$$(zId_{\mathcal{H}}-\mathcal{T})w = f. \tag{3.4.4}$$

We introduce the vectors

$$\forall j \geq 1, \quad W_j := \begin{pmatrix} w_{j+p-1} \\ \vdots \\ w_{j-r} \end{pmatrix} \in \mathbb{C}^{p+r}, \quad e = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{p+r},$$

so that we can rewrite (3.4.4) as the dynamical system [1]

$$\begin{cases} \forall j \geq 1, \quad W_{j+1} = \mathbb{M}(z) W_j - \frac{f_j}{a_p} e, \\ \mathcal{B} W_1 = 0, \end{cases} \tag{3.4.5}$$

where the matrices $\mathbb{M}(z)$ and $\mathcal{B}$ are respectively defined by (3.1.12) and (3.1.5).

▶ To prove Assertion (3.4.1), we prove that the linear map

$$\begin{array}{ccc} \ker(zId - \mathcal{T}) & \to & \ker \mathcal{B} \cap E^s(z) \\ w & \mapsto & W_1 = \begin{pmatrix} w_p \\ \vdots \\ w_{1-r} \end{pmatrix} \end{array}$$

is well-defined and invertible. Its inverse will be the linear map

$$\begin{array}{ccc} \ker \mathcal{B} \cap E^s(z) & \to & \ker(zId - \mathcal{T}) \\ W_1 & \mapsto & ((W_j)_p)_{j \geq 1} \end{array}$$

where $(W_j)_{j\geq 1} = (\mathbb{M}(z)^{j-1} W_1)_{j \geq 1}$ is the solution of (3.4.5) for $f = 0$ and $(X)_p$ denotes the pth coefficient of $X \in \mathbb{C}^{p+r}$.

▶ The proof of Assertion (3.4.2) is more technical and relies on the decomposition (3.1.13) of $\mathbb{C}^{p+r}$ using the stable and unstable subspaces $E^s(z)$ and $E^u(z)$ of the matrix $\mathbb{M}(z)$. The idea is to use the hyperbolic dichotomy of $\mathbb{M}(z)$ to carefully integrate the system (3.4.5). This yields the expressions:

$$\forall j \geq 1, \qquad \pi^u(z) W_j = \sum_{k=0}^{+\infty} \frac{f_{j+k}}{a_p} \mathbb{M}(z)^{-1-k} \pi^u(z) e, \tag{3.4.6}$$

$$\mathcal{B}\pi^s(z) W_1 = -\mathcal{B}\pi^u(z) W_1, \tag{3.4.7}$$

$$\forall j \geq 1, \qquad \pi^s(z) W_j = \mathbb{M}(s)^{j-1} \pi^s(z) W_1 - \sum_{k=1}^{j-1} \frac{f_k}{a_p} \mathbb{M}(z)^{j-1-k} \pi^s(z) e. \tag{3.4.8}$$

The exponential decay of the sequences $(\mathbb{M}(z)^j \pi^s(z))_{j\in\mathbb{N}}$ and $(\mathbb{M}(z)^{-j}\pi^u(z))_{j \in \mathbb{N}}$ implies that the series

$$\sum_{k=0}^{+\infty} \frac{f_{j+k}}{a_p} \mathbb{M}(z)^{-1-k} \pi^u(z) e \quad \text{and} \quad \sum_{k=1}^{j-1} \frac{f_k}{a_p} \mathbb{M}(z)^{j-1-k} \pi^s(z) e$$

converge. We observe that $\mathcal{B}_{|E^s(z)}$ is an isomorphism from $E^s(z)$ onto $\mathbb{C}^r$ because of Remark 2. Since $\pi^u(z)W_1$ is defined by (3.4.6), we can deduce $\pi^s(z)W_1$ from (3.4.7) and then construct two sequences $(\pi^u(z)W_j)_{j\geq 1}$ and $(\pi^s(z)W_j)_{j \geq 1}$ which are solutions of (3.4.6)-(3.4.8). This allows us to construct a solution $w \in \mathcal{H}$ of (3.4.4) that depends continuously on f and that we can prove to be unique. □

Now that we have a clearer idea of the localization of the spectrum of the operators $\mathcal{T}$ and $\mathscr{L}$, we can define the spatial Green's functions of those operators.

3.4.2 Definition and estimates of the spatial Green's function

For $j_0 \in \mathbb{N} \backslash \{0\}$ and $z \in \rho(\mathcal{T})$, we define the spatial Green's function $G(z, j_0, \cdot) \in \mathcal{H}$ associated with the operator $\mathcal{T}$ as

$$G(z, j_0, \cdot) := (zId_{\mathcal{H}} - \mathcal{T})^{-1} \delta_{j_0} \tag{3.4.9}$$

1. We use here the fact that $p_b \leq p$. The case $p_b > p$ could be dealt similarly but would use heavier notations.

where δ_{j_0} is defined by (3.1.18). We also define for $j_0 \in \mathbb{N}\backslash\{0\}$ and $z \in \rho(\mathscr{L})$ the spatial Green's function $\widetilde{G}(z,\cdot) \in \ell^2(\mathbb{Z})$ associated with the operator $\mathscr{L}$ as

$$\widetilde{G}(z,\cdot) := (zId_{\ell^2(\mathbb{Z})} - \mathscr{L})^{-1}\widetilde{\delta}. \tag{3.4.10}$$

where $\widetilde{\delta}$ is defined by (3.1.21). We notice that both functions $G(\cdot,j_0,j)$ and $\widetilde{G}(\cdot,j)$ are defined and holomorphic on $\mathcal{O}\cap\rho(\mathcal{T})$ and in particular on $\overline{\mathcal{U}}\backslash\{1\}$ and $B_{\widetilde{\varepsilon}_0}(1)\cap\mathcal{O}$ because of (3.4.3).

The temporal Green's function $\mathcal{G}(n,j_0,j)$ and $\widetilde{\mathcal{G}}(n,j)$ can be expressed using the spatial Green's function $G(\cdot,j_0,j)$ and $\widetilde{G}(\cdot,j)$ we just introduced (see (3.5.6) and (3.5.7) below). To obtain the estimates on $\mathrm{Err}(n,j_0,j)$ expected in Theorem 3.2, we will need to study the difference of those two spatial Green's function defined by

$$\forall z \in \mathcal{O}\cap\rho(\mathcal{T}), \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad R(z,j_0,j) := G(z,j_0,j) - \widetilde{G}(z,j-j_0). \tag{3.4.11}$$

In this section, we prove local uniform estimates of $R(z,j_0,j)$ and meromorphic extensions for $R(\cdot,j_0,j)$ in a neighborhood of 1 with a pole of order 1 at $z=1$. The first step will be to find an alternative expression of $R(z,j_0,j)$.

In the same manner as in the proof of Lemma 3.4.1, we introduce the following vectors for $z \in \mathcal{O}\cap\rho(\mathcal{T})$

$$\forall j, j_0 \in \mathbb{N}\backslash\{0\}, \quad W(z,j_0,j) := \begin{pmatrix} G(z,j_0,j+p-1) \\ \vdots \\ G(z,j_0,j-r) \end{pmatrix} \in \mathbb{C}^{p+r}$$

and

$$\forall j \in \mathbb{Z}, \quad \widetilde{W}(z,j) := \begin{pmatrix} \widetilde{G}(z,j+p-1) \\ \vdots \\ \widetilde{G}(z,j-r) \end{pmatrix} \in \mathbb{C}^{p+r}.$$

Both function $W(z,j_0,j)$ and $\widetilde{W}(z,j)$ depend holomorphically on z. Using the definition (3.1.6) and (3.1.9) of the operators $\mathcal{T}$ and $\mathscr{L}$, we prove that they are solutions of the following systems for $z \in \mathcal{O}\cap\rho(\mathcal{T})$ and $j_0 \in \mathbb{N}\backslash\{0\}$:

$$\begin{cases} \forall j \geq 1, \quad W(z,j_0,j+1) = \mathbb{M}(z)W(z,j_0,j) - \dfrac{\mathbb{1}_{j=j_0}}{a_p}e, \\ \mathcal{B}W(z,j_0,1) = 0. \end{cases}$$

and

$$\forall j \in \mathbb{Z}, \quad \widetilde{W}(z,j+1) = \mathbb{M}(z)\widetilde{W}(z,j) - \frac{\mathbb{1}_{j=0}}{a_p}e.$$

Using the projectors $\pi^s(z)$ and $\pi^u(z)$ introduced via the decomposition (3.1.13) in the same manner as to obtain the equalities (3.4.6)-(3.4.8), we obtain that for $z \in \mathcal{O}\cap\rho(\mathcal{T})$ and $j, j_0 \in \mathbb{N}\backslash\{0\}$

$$\pi^u(z)W(z,j_0,j) = \frac{\mathbb{1}_{j\in]-\infty,j_0]}}{a_p}\mathbb{M}(z)^{j-(j_0+1)}\pi^u(z)e, \tag{3.4.12}$$

$$\mathcal{B}\pi^s(z)W(z,j_0,1) = -\mathcal{B}\pi^u(z)W(z,j_0,1), \tag{3.4.13}$$

$$\pi^s(z)W(z,j_0,j) = \mathbb{M}(z)^{j-1}\pi^s(z)W(z,j_0,1) - \frac{\mathbb{1}_{j\in[j_0+1,+\infty[}}{a_p}\mathbb{M}(z)^{j-(j_0+1)}\pi^s(z)e. \tag{3.4.14}$$

and for $z \in \mathcal{O}\cap\rho(\mathcal{T})$ and $j \in \mathbb{Z}$

$$\pi^u(z)\widetilde{W}(z,j) = \frac{\mathbb{1}_{j\in]-\infty,0]}}{a_p}\mathbb{M}(z)^{j-1}\pi^u(z)e, \tag{3.4.15}$$

$$\pi^s(z)\widetilde{W}(z,j) = -\frac{\mathbb{1}_{j\in[1,+\infty[}}{a_p}\mathbb{M}(z)^{j-1}\pi^s(z)e. \tag{3.4.16}$$

Therefore, summing equalities (3.4.12) and (3.4.14) and using equations (3.4.15) and (3.4.16), we have:

$$\forall z \in \mathcal{O}\cap\rho(\mathcal{T}), \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad R(z,j_0,j) = (\mathbb{M}(z)^{j-1}\pi^s(z)W(z,j_0,1))_p \tag{3.4.17}$$

where $(X)_p$ denotes the pth coefficient of $X \in \mathbb{C}^{p+r}$ and we recall that the remainder $R(z,j_0,j)$ is defined by (3.4.11). The expressions (3.4.11) and (3.4.17) are the starting point for studying $R(z,j_0,j)$.

The following lemma is an equivalent form of [CF22, Lemma 5] and gives local uniform exponential bounds on $R(z, j_0, j)$ far from the spectrum of $\mathcal{T}$.

Lemma 3.4.2. *For $z_0 \in \mathcal{O} \cap \rho(\mathcal{T})$, there exist a radius $\delta > 0$ and two positive constants C, c such that $B_\delta(z_0)$ is contained within $\mathcal{O} \cap \rho(\mathcal{T})$ and*

$$\forall z \in B_\delta(z_0), \forall j, j_0 \geq 1, \quad |R(z, j_0, j)| \leq C \exp(-c(j + j_0)).$$

We observe that the set $\overline{\mathcal{U}} \backslash \{1\}$ is included in $\mathcal{O} \cap \rho(\mathcal{T})$ because of (3.4.3) and Hypothesis 3.1. Thus, Lemma 3.4.2 can be applied on all points of the set $\overline{\mathcal{U}} \backslash \{1\}$.

Proof Since $\mathcal{O} \cap \rho(\mathcal{T})$ is an open set, we can consider $\delta > 0$ such that

$$B_\delta(z_0) \subset \mathcal{O} \cap \rho(\mathcal{T}).$$

Up to considering a smaller radius $\delta > 0$, the families of matrices $(\mathbb{M}(z)^j \pi^s(z))_{j \in \mathbb{N}}$ and $(\mathbb{M}(z)^{-j} \pi^u(z))_{j \in \mathbb{N}}$ decay uniformly exponentially, i.e. there exist two positive constants C, c such that

$$\forall z \in B_\delta(z_0), \forall j \in \mathbb{N}, \qquad \begin{array}{ll} \left\| \mathbb{M}(z)^j \pi^s(z) \right\| & \leq C e^{-cj}, \\ \left\| \mathbb{M}(z)^{-j} \pi^u(z) \right\| & \leq C e^{-cj}. \end{array} \tag{3.4.18}$$

Thus, (3.4.18) implies

$$\forall z \in B_\delta(z_0), \forall j_0, j \in \mathbb{N} \backslash \{0\}, \quad \left| \mathbb{M}(z)^{j-1} \pi^s(z) W(z, j_0, 1) \right| \leq C e^{-c(j-1)} \left| \pi^s(z) W(z, j_0, 1) \right|. \tag{3.4.19}$$

Furthermore, using (3.4.12) and (3.4.18), we prove the existence of a constant $C > 0$ such that

$$\forall z \in B_\delta(z_0), \forall j_0 \in \mathbb{N} \backslash \{0\}, \quad |\pi^u(z) W(z, j_0, 1)| \leq C e^{-cj_0}. \tag{3.4.20}$$

Since the linear map $\mathcal{B}_{|E^s(z)}$ is an isomorphism for all $z \in B_\delta(z_0)$, up to considering a smaller δ, there exists a constant $C > 0$ such that

$$\forall z \in B_\delta(z_0), \forall x \in E^s(z), \quad |\mathcal{B} x| \geq C|x|. \tag{3.4.21}$$

Using (3.4.13), (3.4.20) and (3.4.21), we prove that there exists a positive constant C such that

$$\forall z \in B_\delta(z_0), \forall j_0 \in \mathbb{N} \backslash \{0\}, \quad |\pi^s(z) W(z, j_0, 1)| \leq C e^{-cj_0}. \tag{3.4.22}$$

Combining (3.4.19) and (3.4.22) allows us to conclude that there exist two constants $C, c > 0$ such that

$$\forall z \in B_\delta(z_0), \forall j_0, j \in \mathbb{N} \backslash \{0\}, \quad \left| \mathbb{M}(z)^{j-1} \pi^s(z) W(z, j_0, 1) \right| \leq C e^{-c(j + j_0)}.$$

Using (3.4.17), we conclude the proof. □

We will now study the function $R(z, j_0, j)$ defined by (3.4.11) in a neighborhood of 1 and introduce an equivalent of [CF22, Lemma 6] with some important differences. Let us first recall that the curve $F(\mathbb{S}^1)$ corresponds to the essential spectrum of the operator $\mathcal{T}$ and that 1 is a simple eigenvalue of $\mathcal{T}$. The function $R(z, j_0, j)$ is thus only defined outside the curve $F(\mathbb{S}^1)$ in a neighborhood on 1. The following lemma proves that $R(z, j_0, j)$ can actually be meromorphically extended near 1 with a pole of order 1 at 1. We point out in advance that the radius $\widetilde{\varepsilon}_0$ present in the statement of the following lemma comes from the decomposition (3.1.14) of the vector space $\mathbb{C}^{p+r}$ deduced from the study of the spectrum of the matrix $\mathbb{M}(z)$ in a neighborhood $B_{\widetilde{\varepsilon}_0}(1)$ of 1.

Lemma 3.4.3. *There exists a radius $\widetilde{\varepsilon}_1 \in]0, \widetilde{\varepsilon}_0[$ such that for all $j_0, j \in \mathbb{N} \backslash \{0\}$, the function*

$$z \in \mathcal{O} \cap \rho(\mathcal{T}) \mapsto R(z, j_0, j)$$

can be meromorphically extended on $B_{\widetilde{\varepsilon}_1}(1)$ with a pole of order 1 at 1. Furthermore, there exist some holomorphic complex valued functions $P^u(\cdot, j_0, j)$ and $P^c(\cdot, j_0, j)$ defined on $B_{\widetilde{\varepsilon}_1}(1)$ for all $j_0, j \in \mathbb{N} \backslash \{0\}$ such that :

— *The following equality is satisfied*

$$\forall j_0, j \in \mathbb{N} \backslash \{0\}, \forall z \in B_{\widetilde{\varepsilon}_1}(1) \backslash \{1\}, \quad R(z, j_0, j) = \frac{P^c(z, j_0, j)}{z - 1} + \frac{P^u(z, j_0, j)}{z - 1}. \tag{3.4.23}$$

— *There exist some positive constants C, c independent from z, j_0 and j such that:*

$$\forall z \in B_{\widetilde{\varepsilon}_1}(1), \forall j_0, j \in \mathbb{N} \backslash \{0\}, \quad |P^u(z, j_0, j)| \leq C e^{-cj - cj_0}. \tag{3.4.24}$$

We will notate

$$\forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad \mathcal{R}^u(j_0,j) := P^u(1,j_0,j). \tag{3.4.25}$$

— *The function P^c satisfies the following estimates where C, c are some positive constants independent from z, j_0 and j and $\mathcal{R}^c := (\mathcal{R}^c(j))_{j\in\mathbb{N}\backslash\{0\}}$ is a complex valued family:*

$$\forall z \in B_{\widetilde{\varepsilon_1}}(1), \forall j_0, j \in \mathbb{N}\backslash\{0\}, \qquad \begin{array}{c} |P^c(z,j_0,j)| \leq Ce^{-cj}|\kappa(z)|^{-j_0}, \\ |\mathcal{R}^c(j)| \leq Ce^{-cj}, \\ |P^c(z,j_0,j) - \mathcal{R}^c(j)\kappa(z)^{-j_0}| \leq C|z-1|e^{-cj}|\kappa(z)|^{-j_0}. \end{array} \tag{3.4.26}$$

— *The sequence $\mathcal{R}^c$ satisfies:*

$$\mathcal{R}^c = 0 \Leftrightarrow \mathcal{B}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T \in \mathcal{B}E^s(1). \tag{3.4.27}$$

The two sequences $(\mathcal{R}^c(j))_{j\in\mathbb{N}\backslash\{0\}}$ and $(\mathcal{R}^u(j_0,j))_{j_0,j\in\mathbb{N}\backslash\{0\}}$ will correspond to the same sequences introduced in the statement of Theorem 3.2.

Let us observe that Lemma 3.4.3 and [CF22, Lemma 6] have the same goal but do not state the same result. We recall that in [CF22], the authors suppose that the Lopatinskii determinant Δ does not vanish at 1 and thus that 1 is not an eigenvalue of the operator $\mathcal{T}$. This allows in [CF22, Lemma 6] for an holomorphic extension of the spatial Green's function $G(z,j_0,j)$ on a whole neighborhood of 1.

Proof Using the projectors $\pi^{ss}(z)$, $\pi^c(z)$ and $\pi^{su}(z)$ defined via the decomposition (3.1.14), the equalities (3.4.12)-(3.4.14) imply for $z \in B_{\widetilde{\varepsilon_0}}(1) \cap \mathcal{O}$ and $j_0, j \in \mathbb{N}\backslash\{0\}$

$$\pi^{su}(z)W(z,j_0,j) = \frac{\mathbb{1}_{j\in]-\infty,j_0]}}{a_p}\mathbb{M}(z)^{j-(j_0+1)}\pi^{su}(z)e, \tag{3.4.28}$$

$$\pi^{c}(z)W(z,j_0,j) = \frac{\mathbb{1}_{j\in]-\infty,j_0]}}{a_p}\kappa(z)^{j-(j_0+1)}\pi^{c}(z)e, \tag{3.4.29}$$

$$\mathcal{B}\pi^{ss}(z)W(z,j_0,1) = -\mathcal{B}\left(\pi^{su}(z)W(z,j_0,1) + \pi^c(z)W(z,j_0,1)\right), \tag{3.4.30}$$

$$\pi^{ss}(z)W(z,j_0,j) = \mathbb{M}(z)^{j-1}\pi^{ss}(z)W(z,j_0,1) - \frac{\mathbb{1}_{j\in[j_0+1,+\infty[}}{a_p}\mathbb{M}(z)^{j-(j_0+1)}\pi^{ss}(z)e. \tag{3.4.31}$$

We want to extend meromorphically the function

$$z \in B_{\widetilde{\varepsilon_0}}(1) \cap \mathcal{O} \mapsto \mathbb{M}(z)^{j-1}\pi^{ss}(z)W(z,j_0,1)$$

on $B_{\widetilde{\varepsilon_0}}(1)$ with a pole at 1 of order 1. To do so, we will use equality (3.4.30) and Hypothesis 3.2 to extend meromorphically the function $\pi^{ss}W(\cdot,j_0,1)$ on $B_{\widetilde{\varepsilon_0}}(1)$ with a pole at 1 of order 1. For $z \in B_{\widetilde{\varepsilon_0}}(1)\backslash\{1\}$, since Hypothesis 3.2 implies that $\Delta(z) \neq 0$ where Δ is defined by (3.1.16), the matrix

$$\begin{pmatrix}\mathcal{B}e_1(z) & \dots & \mathcal{B}e_r(z)\end{pmatrix}$$

is invertible and

$$\begin{pmatrix}\mathcal{B}e_1(z) & \dots & \mathcal{B}e_r(z)\end{pmatrix}^{-1} = \frac{\mathrm{com}\begin{pmatrix}\mathcal{B}e_1(z) & \dots & \mathcal{B}e_r(z)\end{pmatrix}^T}{\Delta(z)}.$$

Hypothesis 3.2 states that 1 is a simple zero of Δ. Thus, we can consider the holomorphic function $D : B_{\widetilde{\varepsilon_0}}(1) \to \mathcal{M}_r(\mathbb{C})$ defined by

$$\forall z \in B_{\widetilde{\varepsilon_0}}(1), \quad D(z) = \begin{cases} \frac{z-1}{\Delta(z)}\mathrm{com}\begin{pmatrix}\mathcal{B}e_1(z) & \dots & \mathcal{B}e_r(z)\end{pmatrix}^T & \text{if } z \neq 1, \\ \frac{1}{\Delta'(1)}\mathrm{com}\begin{pmatrix}\mathcal{B}e_1(1) & \dots & \mathcal{B}e_r(1)\end{pmatrix}^T & \text{if } z = 1. \end{cases} \tag{3.4.32}$$

To study the equality (3.4.30), for $z \in B_{\widetilde{\varepsilon_0}}(1)$ and $j_0 \in \mathbb{N}\backslash\{0\}$, we introduce

$$\begin{aligned} V^u(z,j_0) &:= -(e_1(z)\dots e_r(z))D(z)\mathcal{B}\left(\frac{1}{a_p}\mathbb{M}(z)^{-j_0}\pi^{su}(z)e\right), \\ V^c(z,j_0) &:= -(e_1(z)\dots e_r(z))D(z)\mathcal{B}\left(\frac{1}{a_p}\kappa(z)^{-j_0}\pi^{c}(z)e\right). \end{aligned} \tag{3.4.33}$$

We observe that, for all $j_0 \in \mathbb{N}\backslash\{0\}$, the functions $V^u(\cdot,j_0)$ and $V^c(\cdot,j_0)$ are holomorphic on $B_{\widetilde{\varepsilon_0}}(1)$ and for all $z \in B_{\widetilde{\varepsilon_0}}(1)$, $V^u(z,j_0)$ and $V^c(z,j_0)$ belong to $E^{ss}(z)$ since they are linear combinations of $e_1(z), \dots, e_r(z)$. The equality (3.4.30) and the definition (3.4.33) also imply

$$\forall j_0 \in \mathbb{N}\backslash\{0\}, \forall z \in B_{\widetilde{\varepsilon}_0}(1) \cap \mathcal{O}, \quad \mathcal{B}\pi^{ss}(z)W(z,j_0,1) = \mathcal{B}\frac{V^c(z,j_0)+V^u(z,j_0)}{z-1}.$$

Furthermore, since $\Delta(z) \neq 0$ for $z \in B_{\widetilde{\varepsilon}_0}(1) \cap \mathcal{O}$, Hypothesis 3.2 implies that

$$\ker \mathcal{B} \cap E^{ss}(z) = \{0\}.$$

We can then conclude that

$$\forall j_0 \in \mathbb{N}\backslash\{0\}, \forall z \in B_{\widetilde{\varepsilon}_0}(1) \cap \mathcal{O}, \quad \pi^{ss}(z)W(z,j_0,1) = \frac{V^c(z,j_0)+V^u(z,j_0)}{z-1}. \tag{3.4.34}$$

We have thus found a meromorphic extension of the function $\pi^{ss}W(\cdot,j_0,1)$ on $B_{\widetilde{\varepsilon}_0}(1)$ with a pole at 1 of order 1. We are then led to introduce for all $j_0, j \in \mathbb{N}\backslash\{0\}$ the functions $\mathscr{P}^c(\cdot,j_0,j)$ and $\mathscr{P}^u(\cdot,j_0,j)$ defined on $B_{\widetilde{\varepsilon}_0}(1)$ as

$$\forall z \in B_{\widetilde{\varepsilon}_0}(1), \quad \begin{array}{ll} \mathscr{P}^c(z,j_0,j) & := \mathbb{M}(z)^{j-1}V^c(z,j_0), \\ \mathscr{P}^u(z,j_0,j) & := \mathbb{M}(z)^{j-1}V^u(z,j_0). \end{array} \tag{3.4.35}$$

The two functions $\mathscr{P}^c(\cdot,j_0,j)$ and $\mathscr{P}^u(\cdot,j_0,j)$ are both holomorphic on $B_{\widetilde{\varepsilon}_0}(1)$. Since $\kappa(1)=1$, we notice that $V^c(1,j_0)$ does not depend on j_0. We then notate

$$\forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad \mathscr{R}^c(j) := \mathscr{P}^c(1,j_0,j). \tag{3.4.36}$$

We recall that $(X)_p$ denotes the pth coefficient of $X \in \mathbb{C}^{p+r}$. We denote for $z \in B_{\widetilde{\varepsilon}_0}(1)$ and $j, j_0 \in \mathbb{N}\backslash\{0\}$

$$P^c(z,j_0,j) := (\mathscr{P}^c(z,j_0,j))_p, \quad P^u(z,j_0,j) := (\mathscr{P}^u(z,j_0,j))_p, \quad \mathcal{R}^c(j) := (\mathscr{R}^c(j))_p. \tag{3.4.37}$$

Using the equalities (3.4.34) as well as the definition (3.4.35) of the functions $\mathscr{P}^c$ and $\mathscr{P}^u$, we have that for all $j_0, j \in \mathbb{N}\backslash\{0\}$, we can extend the function

$$z \in \mathcal{O} \cap \rho(\mathcal{T}) \mapsto \mathbb{M}(z)^{j-1}\pi^{ss}(z)W(z,j_0,1)$$

meromorphically on $B_{\widetilde{\varepsilon}_0}(1)$ with a pole at 1 of order 1 with the expression

$$\forall j_0, j \in \mathbb{N}\backslash\{0\}, \forall z \in B_{\widetilde{\varepsilon}_0}(1)\backslash\{1\}, \quad \mathbb{M}(z)^{j-1}\pi^{ss}(z)W(z,j_0,1) = \frac{\mathscr{P}^c(z,j_0,j)}{z-1} + \frac{\mathscr{P}^u(z,j_0,j)}{z-1}. \tag{3.4.38}$$

Using (3.4.17), (3.4.37) and (3.4.38) directly imply (3.4.23).

We will now prove the different estimates presented in the statement of Lemma 3.4.3. We start by noticing that there exists a radius $\widetilde{\varepsilon}_1 \in]0, \widetilde{\varepsilon}_0[$ such that the families of matrices $(\mathbb{M}(z)^j\pi^{ss}(z))_{j\in\mathbb{N}}$ and $(\mathbb{M}(z)^{-j}\pi^{su}(z))_{j\in\mathbb{N}}$ uniformly decay exponentially for $z \in B_{\widetilde{\varepsilon}_1}(1)$, i.e. there exist two positive constants C, c such that

$$\forall z \in B_{\widetilde{\varepsilon}_1}(1), \forall j \in \mathbb{N}, \quad \begin{array}{ll} \left\|\mathbb{M}(z)^j\pi^{ss}(z)\right\| & \leq Ce^{-cj}, \\ \left\|\mathbb{M}(z)^{-j}\pi^{su}(z)\right\| & \leq Ce^{-cj}. \end{array} \tag{3.4.39}$$

To study the central component $\mathscr{P}^c$, we will also need the following lemma:

Lemma 3.4.4 ([CF22]). *We consider a holomorphic function M with values in $\mathcal{M}_N(\mathbb{C})$ defined on a open ball $B_\delta(0)$ with $\delta > 0$ and $N \in \mathbb{N}$ such that there exist two positive constants C, c such that*

$$\forall z \in B_\delta(0), \forall j \in \mathbb{N}, \quad \left\|M(z)^j\right\| \leq Ce^{-cj}.$$

Up to considering a smaller radius δ, there exist two new positive constants C, c

$$\forall z_1, z_2 \in B_\delta(0), \forall j \in \mathbb{N}, \quad \left\|M(z_1)^j - M(z_2)^j\right\| \leq C|z_1 - z_2|e^{-cj}.$$

Using inequalities (3.4.39), Lemma 3.4.4 implies that, up to diminishing $\widetilde{\varepsilon}_1$, there exist some positive constants C, c such that

$$\forall z \in B_{\widetilde{\varepsilon}_1}(1), \forall j \in \mathbb{N}, \quad \left\|\mathbb{M}(z)^j\pi^{ss}(z) - \mathbb{M}(1)^j\pi^{ss}(1)\right\| \leq C|z-1|e^{-cj}. \tag{3.4.40}$$

Let us find estimates on $\mathscr{P}^u$. First, we observe that inequality (3.4.39) implies that there exist two positive constant C, c such that

$$\forall z \in B_{\widetilde{\varepsilon_1}}(1), \forall j_0 \in \mathbb{N}\backslash\{0\}, \quad |V^u(z,j_0)| \leq Ce^{-cj_0}. \tag{3.4.41}$$

Since for $z \in B_{\widetilde{\varepsilon_1}}(1)$ and $j_0 \in \mathbb{N}\backslash\{0\}$ we have $V^u(z,j_0) \in E^{ss}(z)$, we notice using the definition (3.4.35) that

$$\forall z \in B_{\widetilde{\varepsilon_1}}(1), \forall j, j_0 \in \mathbb{N}\backslash\{0\}, \quad \mathscr{P}^u(z,j_0,j) = \mathbb{M}(z)^{j-1}\pi^{ss}(z)V^u(z,j_0).$$

Therefore, using the inequalities (3.4.39) and (3.4.41), we can prove that there exist two positive constants C, c such that

$$\forall z \in B_{\widetilde{\varepsilon_1}}(1), \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad |\mathscr{P}^u(z,j_0,j)| \leq Ce^{-cj-cj_0}.$$

We easily deduce (3.4.24).

We now prove the estimates on $\mathscr{P}^c$. We observe that the definition (3.4.33) implies that there exists a positive constant C such that

$$\forall z \in B_{\widetilde{\varepsilon_1}}(1), \forall j_0 \in \mathbb{N}\backslash\{0\}, \quad |V^c(z,j_0)| \leq C|\kappa(z)|^{-j_0}. \tag{3.4.42}$$

Since for $z \in B_{\widetilde{\varepsilon_1}}(1)$ and $j_0 \in \mathbb{N}\backslash\{0\}$ we have $V^c(z,j_0) \in E^{ss}(z)$, we notice using the definition (3.4.35) that

$$\forall z \in B_{\widetilde{\varepsilon_1}}(1), \forall j, j_0 \in \mathbb{N}\backslash\{0\}, \quad \mathscr{P}^c(z,j_0,j) = \mathbb{M}(z)^{j-1}\pi^{ss}(z)V^c(z,j_0).$$

Therefore, using the inequalities (3.4.39), we prove the existence of a constant $C > 0$ such that

$$\forall z \in B_{\widetilde{\varepsilon_1}}(1), \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad \begin{array}{c} |\mathscr{P}^c(z,j_0,j)| \leq Ce^{-cj}|\kappa(z)|^{-j_0}, \\ |\mathscr{R}^c(j)| \leq Ce^{-cj}. \end{array} \tag{3.4.43}$$

Finally, we observe that for $j_0, j \in \mathbb{N}\backslash\{0\}$ and $z \in B_{\widetilde{\varepsilon_1}}(1)$ we have

$$\begin{aligned} &\mathscr{P}^c(z,j_0,j) - \kappa(z)^{-j_0}\mathscr{R}^c(j) \\ &\quad = \kappa(z)^{-j_0}\left(\mathbb{M}(1)^{j-1}\pi^{ss}(1)(e_1(1)\dots e_r(1))D(1)\mathcal{B}\pi^c(1)e - \mathbb{M}(z)^{j-1}\pi^{ss}(z)(e_1(z)\dots e_r(z))D(z)\mathcal{B}\pi^c(z)e\right). \end{aligned}$$

Therefore, the estimates (3.4.39) and (3.4.40) as well as the mean value inequality directly imply that there exist two positive constants C, c such that

$$\forall z \in B_{\widetilde{\varepsilon_1}}(1), \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad |\mathscr{P}^c(z,j_0,j) - \kappa(z)^{-j_0}\mathscr{R}^c(j)| \leq C|z-1|e^{-cj}|\kappa(z)|^{-j_0}.$$

To conclude the proof of Lemma 3.4.3, there remains to prove the condition (3.4.27). First, we need to compute the value of the vector $\pi^c(1)e$. We recall that 1 is a simple eigenvalue of the matrix $\mathbb{M}(1)$, that $\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T$ is an eigenvector of $\mathbb{M}(1)$ associated with 1 and thus that

$$E^c(1) = \mathrm{Span}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T.$$

We also know that there exists a unique eigenvector $L = (l_j)_{j\in\{1,\dots,p+r\}} \in \mathbb{C}^{p+r}$ of $\mathbb{M}(1)^T$ associated with the eigenvalue 1 such that

$$L \cdot \begin{pmatrix}1 & \dots & 1\end{pmatrix}^T = 1 \tag{3.4.44}$$

where the symmetric bilinear form $\cdot$ on $\mathbb{C}^{p+r}$ is defined as [2]

$$\forall X, Y \in \mathbb{C}^{p+r}, \quad X \cdot Y := \sum_{l=1}^{p+r} X_l Y_l.$$

Then, we have that

$$\forall Y \in \mathbb{C}^{p+r}, \quad \pi^c(z)Y = (L\cdot Y)\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T.$$

Thus, applying to the vector e implies that

$$\pi^c(1)e = l_1\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T.$$

2. Observe that this symetric bilinear form is not the Hermitian product on $\mathbb{C}^{p+r}$.

Since L is an eigenvector of $\mathbb{M}(1)^T$ associated with the eigenvalue 1, we have

$$\forall j \in \{1,\dots,p+r\}, \quad l_j = \left(\sum_{l=-r}^{p-j} a_l - \delta_{j\leq p}\right)\frac{l_1}{a_p}. \tag{3.4.45}$$

Since $F'(1) = -\alpha$, the normalization (3.4.44) and the equality (3.4.45) then imply that

$$l_1 = \frac{a_p}{\alpha} \quad \text{and thus} \quad \pi^c(1)e = \frac{a_p}{\alpha}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T. \tag{3.4.46}$$

Using the definitions of (3.4.36), (3.4.33) and (3.4.32) respectively of $\mathscr{R}^c(j)$, the function V^c and the function D, we have that for $j \in \mathbb{N}\backslash\{0\}$

$$\begin{aligned}\mathscr{R}^c(j) &= -\frac{1}{\alpha\Delta'(1)}\mathbb{M}(1)^j(e_1(1)\dots e_r(1))\mathrm{com}\left(\mathcal{B}e_1(1) \quad \dots \quad \mathcal{B}e_r(1)\right)^T\mathcal{B}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T\\ &= -\frac{1}{\alpha\Delta'(1)}\mathbb{M}(1)^j\left(\sum_{k=1}^r \det\left(\mathcal{B}e_1(1) \quad \dots \quad \mathcal{B}e_{k-1}(1) \quad \mathcal{B}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T \quad \mathcal{B}e_{k+1}(1) \quad \dots \quad \mathcal{B}e_r(1)\right)e_k(1)\right).\end{aligned} \tag{3.4.47}$$

• We recall that Hypothesis 3.2 implies that 1 is a simple zero of the Lopatinskii determinant. As a consequence, the vector space $\mathcal{B}E^s(1)$ is of dimension $r-1$. If $\mathcal{B}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T \in \mathcal{B}E^s(1)$, then for all $k \in \{1,\dots,r\}$

$$\det\left(\mathcal{B}e_1(1) \quad \dots \quad \mathcal{B}e_{k-1}(1) \quad \mathcal{B}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T \quad \mathcal{B}e_{k+1}(1) \quad \dots \quad \mathcal{B}e_r(1)\right) = 0$$

and thus $\mathscr{R}^c(j) = 0$ for all $j \in \mathbb{N}\backslash\{0\}$. Using (3.4.37), we deduce the sequence $(\mathcal{R}^c(j))_{j\in\mathbb{N}\backslash\{0\}}$ is equal to 0.

• We now suppose that the sequence $(\mathcal{R}^c(j))_{j\in\mathbb{N}\backslash\{0\}}$ is equal to 0. Since the matrix $\mathbb{M}(1)$ is a companion matrix and $\mathcal{R}^c(j)$ is the pth coefficient of the vector $\mathscr{R}^c(j)$ for all $j \in \mathbb{N}\backslash\{0\}$, the equality (3.4.47) implies that

$$\forall j \geq r+1, \quad \mathscr{R}^c(j) = \begin{pmatrix}\mathcal{R}^c(j+p-1)\\ \vdots \\ \mathcal{R}^c(j-r)\end{pmatrix}.$$

If the sequence $(\mathcal{R}^c(j))_{j\in\mathbb{N}\backslash\{0\}}$ is equal to 0, then the vector $\mathscr{R}^c(j)$ are equal to 0 for all $j \geq r+1$. Since the matrix $\mathbb{M}(1)$ is invertible and the family $(e_k(1))_{k\in\{1,\dots,r\}}$ is linearly independent, the equality (3.4.47) implies that for all $k \in \{1,\dots,r\}$

$$\det\left(\mathcal{B}e_1(1) \quad \dots \quad \mathcal{B}e_{k-1}(1) \quad \mathcal{B}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T \quad \mathcal{B}e_{k+1}(1) \quad \dots \quad \mathcal{B}e_r(1)\right) = 0.$$

Since 1 is a simple zero of the Lopatinskii determinant Δ, there exists an integer $k \in \{1,\dots,r\}$ such that the family $(\mathcal{B}e_j(1))_{j\in\{1,\dots,r\}\backslash\{k\}}$ is linearly independent and spans the whole vector space $\mathcal{B}E^s(1)$. Therefore, the vector $\mathcal{B}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T$ belongs to $\mathcal{B}E^s(1)$. □

3.5 Study of the temporal Green's function

This section is dedicated to the proof of Theorem 3.2. Our goal is to prove the estimate (3.1.27) on the function $\mathrm{Err}(n, j_0, j)$ defined by (3.1.26) for all $n, j_0, j \in \mathbb{N}\backslash\{0\}$. We expect three different behaviors depending on the ratio j_0/n represented on Figure 3.7:

I- The case where j_0 is large compared to n (i.e. $j_0 > np$) will be dealt in Section 3.5.1. For those small times $n < \frac{j_0}{p}$, the numerical boundary condition does not have any impact on the computation of the solution $(\mathcal{G}(n, j_0, \cdot))_{n\in\mathbb{N}}$ of the numerical scheme (3.1.4) with the initial datum $u^0 = \delta_{j_0}$. This will allow us to deduce (3.1.27) in this case.

II- The case where j_0 is small compared to n (i.e. $j_0 < -\frac{n\alpha}{2}$) which will be tackled in Section 3.5.3. It corresponds to the case where the generalized Gaussian wave has already passed the boundary and the boundary layers are almost fully activated.

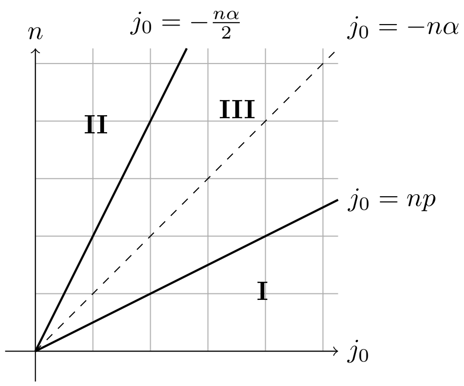

Figure 3.7 – The different sectors for the cases **I**, **II** and **III**.

III- The case where j_0 is close to $-n\alpha$ (i.e. $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$) which will be tackled in Section 3.5.4. The bulk of the proof happens in this section since the limiting estimates occur in this case

Before starting the proof of Theorem 3.2, we will introduce some useful inequalities on the functions $H_{2\mu}^{\beta}$ and $E_{2\mu}^{\beta}$ defined by (3.1.23).

Lemma 3.5.1. *There exist two positive constants C, c such that*

$$\forall x \in \mathbb{R}, \quad |H_{2\mu}^{\beta}(x)| \leq C \exp(-c|x|^{\frac{2\mu}{2\mu-1}}), \tag{3.5.1}$$

$$\forall x \in]0, +\infty[, \quad |E_{2\mu}^{\beta}(x)| \leq C \exp(-c|x|^{\frac{2\mu}{2\mu-1}}), \tag{3.5.2}$$

$$\forall x \in]-\infty, 0[, \quad |1 - E_{2\mu}^{\beta}(x)| \leq C \exp(-c|x|^{\frac{2\mu}{2\mu-1}}). \tag{3.5.3}$$

The interested reader can find a proof of (3.5.1) in [Coe22, Lemma 9] or in [Rob91, Proposition 5.3] for a more general point of view. Inequalities (3.5.2) and (3.5.3) for the function $E_{2\mu}^{\beta}$ are directly deduced by integrating the function $H_{2\mu}^{\beta}$ and using (3.1.24) and (3.5.1).

3.5.1 Case I: j_0 is large compared to n

We consider $n, j_0 \in \mathbb{N}\backslash\{0\}$ such that $j_0 > np$ and we aim to prove the estimate (3.1.27) on $\mathrm{Err}(n, j_0, j)$ in this case for all $j \in \mathbb{N}\backslash\{0\}$. First, we can prove that

$$\forall j \in \mathbb{N}\backslash\{0\}, \quad \mathcal{G}(n, j_0, j) = \widetilde{\mathcal{G}}(n, j - j_0). \tag{3.5.4}$$

This equality translates the fact that for an initial condition $u^0 = \delta_{j_0}$, the solution $\mathcal{G}(n, j_0, j)$ of the numerical scheme (3.1.4) does not see the boundary condition for sufficiently small times n.

Using the definition (3.1.26) of $\mathrm{Err}(n, j_0, j)$ and the equality (3.5.4), we have that

$$\mathrm{Err}(n, j_0, j) = E_{2\mu}^{\beta}\left(\frac{j_0 + n\alpha}{n^{\frac{1}{2\mu}}}\right)\mathcal{R}^c(j).$$

Using (3.4.26) to exponentially bound $\mathcal{R}^c(j)$, we observe that there just remains to prove generalized Gaussian estimates on $E_{2\mu}^{\beta}\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)$ to conclude. Besides, since we have

$$\frac{j_0 + n\alpha}{n^{\frac{1}{2\mu}}} \geq (p+\alpha)n^{\frac{2\mu-1}{2\mu}} > 0$$

and the function $x \in [p+\alpha, +\infty[\mapsto x^{\frac{1}{2\mu-1}} \exp\left(-\frac{c}{2}x^{\frac{2\mu}{2\mu-1}}\right)$ is bounded where c is the positive constant in (3.5.2), we conclude using (3.5.2) that there exists a positive constant $\tilde{C}$ which verify for all $n, j_0 \in \mathbb{N}\backslash\{0\}$ such that $j_0 > np$, we have

$$\left|E_{2\mu}^{\beta}\left(\frac{j_0 + n\alpha}{n^{\frac{1}{2\mu}}}\right)\right| \leq \frac{\tilde{C}}{n^{\frac{1}{2\mu}}} \exp\left(-\frac{c}{2}\left(\frac{|j_0 + n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{3.5.5}$$

This concludes the proof of (3.1.27) when $j_0 > np$.

3.5.2 Inverse Laplace transform

To prove (3.1.27) when $j_0 \leq np$, we will use a representation of the temporal Green's functions $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ using the spatial Green's functions we defined in Section 3.4. Considering a path that surrounds the spectra $\sigma(\mathcal{T})$ and $\sigma(\mathscr{L})$, for instance $\widetilde{\Gamma}_{r_0} = \exp(r_0)\mathbb{S}^1$ with $r_0 \in]0, +\infty[$, the inverse Laplace transform implies that

$$\forall n, j_0, j \in \mathbb{N}\backslash\{0\}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi}\int_{\widetilde{\Gamma}_{r_0}} z^n G(z, j_0, j)dz, \tag{3.5.6}$$

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j \in \mathbb{Z}, \quad \widetilde{\mathcal{G}}(n, j) = \frac{1}{2i\pi}\int_{\widetilde{\Gamma}_{r_0}} z^n \widetilde{G}(z, j)dz, \tag{3.5.7}$$

where the spatial Green's functions G and $\widetilde{G}$ are defined by (3.4.9) and (3.4.10). Using the definition of the function R given by (3.4.11) and the equalities (3.5.6) and (3.5.7), we then obtain that

$$\forall n, j_0, j \in \mathbb{N}\backslash\{0\}, \quad \mathcal{G}(n, j_0, j) - \widetilde{\mathcal{G}}(n, j - j_0) = \frac{1}{2i\pi}\int_{\widetilde{\Gamma}_{r_0}} z^n R(z, j_0, j)dz. \tag{3.5.8}$$

Our goal will be to deform the path $\widetilde{\Gamma}_{r_0}$ to use optimally the estimates we determined in Section 3.4 on the function $R(z, j_0, j)$ while being aware that this function has a pole of order 1 at $z = 1$. We use the change of variables $z = \exp(\tau)$. If we define the path $\Gamma_{r_0} := \{r_0 + it, t \in [-\pi, \pi]\}$ represented on Figure 3.8 and $\mathbf{R}(\tau, j_0, j) := e^\tau R(e^\tau, j_0, j)$, we then have

$$\forall n, j_0, j \in \mathbb{N}\backslash\{0\}, \quad \mathcal{G}(n, j_0, j) - \widetilde{\mathcal{G}}(n, j - j_0) = \frac{1}{2i\pi}\int_{\Gamma_{r_0}} e^{n\tau}\mathbf{R}(\tau, j_0, j)d\tau. \tag{3.5.9}$$

We recall that in Lemma 3.4.3, we present a precise description of the function R in a neighborhood $B_{\widetilde{\varepsilon}_1}(1)$ of 1. We fix a radius $\varepsilon_0^\star \in]0, \pi[$ such that

$$\forall \tau \in B_{\varepsilon_0^\star}(0), \quad e^\tau \in B_{\widetilde{\varepsilon}_1}(1)$$

and such that there exists a holomorphic function $\varpi : B_{\varepsilon_0^\star}(0) \to \mathbb{C}$ which verifies $\varpi(0) = 0$ and

$$\forall \tau \in B_{\varepsilon_0^\star}(0), \quad \exp(\varpi(\tau)) = \kappa(e^\tau).$$

We recall that $\kappa(z)$ is the eigenvalue of the matrix $\mathbb{M}(z)$ such that $\kappa(1) = 1$ and which depends holomorphically on z. We observe that Lemma 3.1.2 implies that

$$\forall \tau \in B_{\varepsilon_0^\star}(0), \quad F(e^{\varpi(\tau)}) = e^\tau. \tag{3.5.10}$$

If we define the holomorphic function φ such that

$$\forall \tau \in \mathbb{C}, \quad \varphi(\tau) := -\frac{\tau}{\alpha} + (-1)^{\mu+1}\frac{\beta}{\alpha^{2\mu+1}}\tau^{2\mu} \tag{3.5.11}$$

where α and β are defined in Hypothesis 3.1, then using the equality (3.5.10) and the asymptotic expansion (3.1.11) of the logarithm of F, we end up proving that there exists a holomorphic function $\xi : B_{\varepsilon_0^\star}(0) \to \mathbb{C}$ such that

$$\forall \tau \in B_{\varepsilon_0^\star}(0), \quad \varpi(\tau) = \varphi(\tau) + \xi(\tau)\tau^{2\mu+1}. \tag{3.5.12}$$

For all $j_0, j \in \mathbb{N}\backslash\{0\}$, we then define the holomorphic functions

$$\forall \tau \in B_{\varepsilon_0^\star}(0), \quad \mathbf{P}^c(\tau, j_0, j) := \frac{\tau}{e^\tau - 1}e^\tau P^c(e^\tau, j_0, j) \quad \text{and} \quad \mathbf{P}^u(\tau, j_0, j) := \frac{\tau}{e^\tau - 1}e^\tau P^u(e^\tau, j_0, j). \tag{3.5.13}$$

Using Lemma 3.4.3, we can prove that for all $j_0, j \in \mathbb{N}\backslash\{0\}$, the function $\mathbf{R}(\cdot, j_0, j)$ can be meromorphically extended on $B_{\varepsilon_0^\star}(0)$ with a pole of order 1 at 0 and that it satisfies the equality

$$\forall \tau \in B_{\varepsilon_0^\star}(0)\backslash\{0\}, \quad \mathbf{R}(\tau, j_0, j) = \frac{\mathbf{P}^c(\tau, j_0, j)}{\tau} + \frac{\mathbf{P}^u(\tau, j_0, j)}{\tau}. \tag{3.5.14}$$

We will now prove a lemma to pass from estimates on the function $R(z, j_0, j)$ in Lemmas 3.4.2 and 3.4.3 to estimates on the function $\mathbf{R}(\tau, j_0, j)$.

Lemma 3.5.2. *There exist two positive constants C, c such that*

$$\forall \tau \in B_{\varepsilon_0^\star}(0), \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad |\boldsymbol{P}^u(\tau, j_0, j)| \leq Ce^{-cj_0 - cj}, \tag{3.5.15}$$

$$\forall \tau \in B_{\varepsilon_0^\star}(0), \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad \left|\boldsymbol{P}^c(\tau, j_0, j) - \mathcal{R}^c(j)e^{-j_0\varpi(\tau)}\right| \leq C|\tau|e^{-cj}\exp\left(-j_0\Re(\varpi(\tau))\right). \tag{3.5.16}$$

Furthermore, for all $\varepsilon \in]0, \varepsilon_0^\star[$, there exists a width $\eta_\varepsilon \in]0, \varepsilon[$ such that if we define

$$\Omega_\varepsilon := \{\tau \in \mathbb{C}, \Re(\tau) \in]-\eta_\varepsilon, \pi], \mathrm{Im}\,(\tau) \in [-\pi, \pi]\} \backslash B_\varepsilon(0)$$

then for all $j_0, j \in \mathbb{N}\backslash\{0\}$ the function $\tau \mapsto \boldsymbol{R}(\tau, j_0, j)$ is holomorphically defined on Ω_ε and there exist two positive constants $C_\varepsilon, c_\varepsilon > 0$ such that

$$\forall \tau \in \Omega_\varepsilon, \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad |\boldsymbol{R}(\tau, j_0, j)| \leq C_\varepsilon e^{-c_\varepsilon j_0 - c_\varepsilon j}. \tag{3.5.17}$$

Proof Inequality (3.5.15) is a direct consequence from (3.4.24). We also observe that the triangular inequality implies

$$\forall \tau \in B_{\varepsilon_\star}(0), \forall j_0, j \in \mathbb{N}\backslash\{0\},$$
$$|\mathbf{P}^c(\tau, j_0, j) - \mathcal{R}^c(j)e^{-j_0\varpi(\tau)}| \leq \left|\frac{\tau}{e^\tau - 1}e^\tau - 1\right| |P^c(e^\tau, j_0, j)| + |P^c(e^\tau, j_0, j) - \mathcal{R}^c(j)\kappa(e^\tau)^{-j_0}|.$$

Therefore, (3.5.16) is a direct consequence from (3.4.26) and the mean value inequality.

We now consider $\varepsilon \in]0, \varepsilon_0^\star[$. The set

$$U_\varepsilon := \{\tau \in \mathbb{C}, \Re(\tau) \in [0, \pi], \mathrm{Im}\,(\tau) \in [-\pi, \pi]\} \backslash B_\varepsilon(0)$$

is compact. Furthermore, for all $\tau_0 \in U_\varepsilon$, since $e^{\tau_0} \in \overline{\mathcal{U}}\backslash\{1\}$, we have thanks to Lemma 3.4.2 the existence of a radius $\delta > 0$ and two positive constants C, c such that $\tau \mapsto \mathbf{R}(\tau, j_0, j)$ is holomorphically defined on $B_\delta(\tau_0)$ and

$$\forall \tau \in B_\delta(\tau_0), \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad |\mathbf{R}(\tau, j_0, j)| \leq Ce^{-c(j+j_0)}.$$

Using a compactness argument, we find a width $\eta_\varepsilon \in]0, \varepsilon[$ such that for all $j_0, j \in \mathbb{N}\backslash\{0\}$ the function $\tau \mapsto \mathbf{R}(\tau, j_0, j)$ is holomorphically defined on Ω_ε and there exist two positive constants $C, c > 0$ such that (3.5.17) is verified. □

The following lemma gives us bounds on the real part of the functions ϖ and φ that will be useful later on, for instance when using (3.5.16).

Lemma 3.5.3. *There exist a radius $\varepsilon_1^\star \in]0, \varepsilon_0^\star[$ and two positive constants A_R, A_I such that*

$$\forall \tau \in \mathbb{C}, \quad \alpha\Re(\varphi(\tau)) \leq -\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}, \tag{3.5.18}$$

$$\forall \tau \in B_{\varepsilon_1^\star}(0), \quad \alpha\Re(\varpi(\tau)) + |\alpha||\xi(\tau)\tau^{2\mu+1}| \leq -\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}. \tag{3.5.19}$$

Proof We start with the proof of (3.5.18). Because of Young's inequality, for $l \in \{1, \dots, 2\mu - 1\}$, we have that for all $\delta > 0$, there exists a constant $C_\delta > 0$ such that for all $\tau \in \mathbb{C}$

$$|\Re(\tau)|^l|\mathrm{Im}\,(\tau)|^{2\mu - l} \leq \delta\mathrm{Im}\,(\tau)^{2\mu} + C_\delta\Re(\tau)^{2\mu}.$$

Furthermore, we have that

$$\alpha\Re(\varphi(\tau)) = -\Re(\tau) + (-1)^{\mu+1}\left(\frac{\Re(\beta)}{\alpha^{2\mu}}\Re(\tau^{2\mu}) - \frac{\mathrm{Im}\,(\beta)}{\alpha^{2\mu}}\mathrm{Im}\,(\tau^{2\mu})\right).$$

Then, for $\delta > 0$, there exists $C_\delta > 0$ such that

$$\alpha\Re(\varphi(\tau)) \leq -\Re(\tau) + \Re(\tau)^{2\mu}\left(\frac{\Re(\beta)}{\alpha^{2\mu}} + C_\delta\right) + \mathrm{Im}\,(\tau)^{2\mu}\left(-\frac{\Re(\beta)}{\alpha^{2\mu}} + \delta\right). \tag{3.5.20}$$

Therefore, by taking δ small enough, we can end the proof of inequality (3.5.18).

We will now prove inequality (3.5.19). Using (3.5.12), we have for $\tau \in B_{\varepsilon_\star}(0)$

$$\alpha\Re(\varpi(\tau)) + |\alpha||\xi(\tau)\tau^{2\mu+1}| \leq \alpha\Re(\varphi(\tau)) + 2|\alpha||\xi(\tau)\tau^{2\mu+1}|. \tag{3.5.21}$$

If we fix a radius $\varepsilon \in]0, \varepsilon_0^\star[$, the function ξ is bounded by some constant $\tilde{C} > 0$ on $B_\varepsilon(0)$. Furthermore, we know there exist two constants $c_1, c_2 > 0$ such that

$$\forall \tau \in \mathbb{C}, \quad |\tau|^{2\mu} \leq c_1\Re(\tau)^{2\mu} + c_2 \mathrm{Im}\,(\tau)^{2\mu}.$$

Thus, using (3.5.20) and (3.5.21), for all radii $\varepsilon_1^\star \in]0, \varepsilon]$ and for all $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\begin{aligned}\forall \tau \in B_{\varepsilon_1^\star}(1), \quad & \alpha\Re(\varpi(\tau)) + |\alpha||\xi(\tau)\tau^{2\mu+1}| \\ & \leq -\Re(\tau) + \Re(\tau)^{2\mu}\left(\frac{\Re(\beta)}{\alpha^{2\mu}} + C_\delta + 2\alpha c_1\tilde{C}\varepsilon_1^\star\right) + \mathrm{Im}\,(\tau)^{2\mu}\left(-\frac{\Re(\beta)}{\alpha^{2\mu}} + \delta + 2\alpha c_2\tilde{C}\varepsilon_1^\star\right).\end{aligned}$$

Taking δ and $\varepsilon_1^\star$ small enough allows us to prove (3.5.19). □

Choice of the radius ε and of the width η

We will now introduce a radius $\varepsilon > 0$ and a width $\eta > 0$ which will satisfy a list of conditions. Those conditions will be used throughout the proof and are centralized here in order to fix the notations.

First, we fix a choice of radius $\varepsilon \in \left]0, \min\left(\varepsilon_1^\star, \left(\frac{1}{2\mu A_R}\right)^{\frac{1}{2\mu-1}}\right)\right[$ where the radius $\varepsilon_1^\star$ is defined in Lemma 3.5.3. This choice for ε will allow us to use the results of Lemmas 3.5.2 and 3.5.3. Furthermore, if we introduce the function

$$\begin{array}{rccc}\Psi: & \mathbb{R} & \to & \mathbb{R} \\ & \tau_p & \mapsto & \tau_p - A_R{\tau_p}^{2\mu}\end{array} \tag{3.5.22}$$

which we will use to define a family of parameterized curve in Section 3.5.4, then the function Ψ is continuous and strictly increasing on $]-\infty, \varepsilon]$.

We now introduce the function

$$\begin{array}{rccc}r_\varepsilon: &]0, \varepsilon[& \to & \mathbb{R} \\ & \eta & \mapsto & \sqrt{\varepsilon^2 - \eta^2}\end{array} \tag{3.5.23}$$

which serves to define the extremities of the curve $-\eta + i\mathbb{R} \cap B_\varepsilon(0)$. We recall that the width η_ε is defined in Lemma 3.5.2. We claim that there exists a width $\eta \in]0, \eta_\varepsilon[$ that we fix for the rest of the paper such that:

— The following inequality is satisfied:

$$\frac{\eta}{2} > A_R\eta^{2\mu}. \tag{3.5.24}$$

— There exists a radius $\varepsilon_\# \in]0, \varepsilon[$ such that if we define

$$l_{extr} := \left(\frac{\Psi(\varepsilon_\#) - \Psi(-\eta)}{A_I}\right)^{\frac{1}{2\mu}}, \tag{3.5.25}$$

then $-\eta + il_{extr} \in B_\varepsilon(0)$.

— For all $n, j_0 \in \mathbb{N}^*$ which verify $-\frac{n\alpha}{2} \leq j_0 \leq np$, we have

$$\left(-\frac{n\alpha}{j_0} - 1\right)(-\eta) + A_R(-\eta)^{2\mu} \leq \frac{A_I}{2} r_\varepsilon(\eta)^{2\mu}. \tag{3.5.26}$$

We introduce the paths Γ_{in}, Γ_{out}, Γ and $\Gamma_{\eta,in}$ that are represented on Figure 3.8 and are defined as

$$\begin{aligned}\Gamma_{out} &:= [-\eta - i\pi, -\eta - ir_\varepsilon(\eta)] \cup [-\eta + ir_\varepsilon(\eta), -\eta + i\pi], \\ \Gamma_{in} &:= \left[-\eta - ir_\varepsilon(\eta), \frac{\varepsilon}{2}\right] \cup \left[\frac{\varepsilon}{2}, -\eta + ir_\varepsilon(\eta)\right], \\ \Gamma &:= \Gamma_{in} \cup \Gamma_{out}, \\ \Gamma_{\eta,in} &:= [-\eta - ir_\varepsilon(\eta), -\eta + ir_\varepsilon(\eta)].\end{aligned}$$

We observe that those paths lie in $\Omega_\varepsilon \cup B_\varepsilon(0)$. Noticing the "$2i\pi$-periodicity" of the function $\mathbf{R}(\cdot, j_0, j)$,

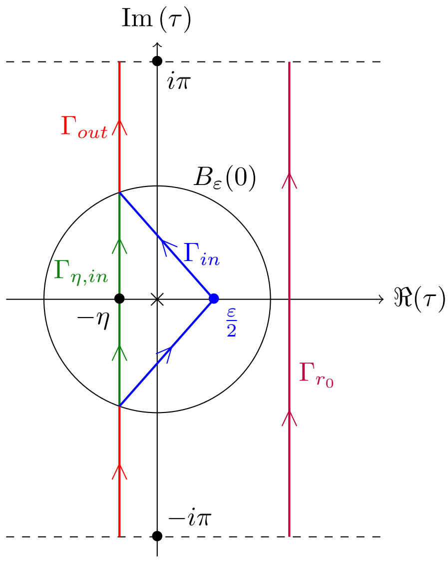

Figure 3.8 – A representation of the path Γ_{r_0} (in purple), Γ_{in} (in blue), Γ_{out} (in red), $\Gamma := \Gamma_{out} \cup \Gamma_{in}$ and $\Gamma_{\eta,in}$ (in green)

Cauchy's formula implies that equality (3.5.9) can be rewritten as

$$\forall n \in \mathbb{N}, \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad \mathcal{G}(n, j_0, j) - \widetilde{\mathcal{G}}(n, j - j_0) = \frac{1}{2i\pi}\int_\Gamma e^{n\tau}\mathbf{R}(\tau, j_0, j)d\tau = \frac{1}{2i\pi}(T^{out} + T^u + T^c) \tag{3.5.27}$$

where

$$T^{out} := \int_{\Gamma_{out}} e^{n\tau}\mathbf{R}(\tau, j_0, j)d\tau, \quad T^u := \int_{\Gamma_{in}} e^{n\tau}\frac{\mathbf{P}^u(\tau, j_0, j)}{\tau}d\tau, \quad T^c := \int_{\Gamma_{in}} e^{n\tau}\frac{\mathbf{P}^c(\tau, j_0, j)}{\tau}d\tau.$$

Thus, to prove (3.1.27) when $j_0 \leq np$, we need to estimate the terms T^{out}, T^u and T^c. We start by proving estimates for T^{out} and T^u.

Proposition 3.1 (Estimate on T^{out} and T^u)**.** *There exist two constants $C, c > 0$ such that*

$$\forall n \in \mathbb{N}, \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad |T^{out}| \leq C\exp(-n\eta - c(j + j_0)).$$

and

$$\forall n \in \mathbb{N}, \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad |T^u - 2i\pi\mathcal{R}^u(j_0, j)| \leq C\exp(-n\eta - c(j + j_0)).$$

Proof We consider $n \in \mathbb{N}$ and $j_0, j \in \mathbb{N}\backslash\{0\}$.

- Since Γ_{out} lies within Ω_ε, using (3.5.17), there exists a positive constant c such that

$$|T^{out}| \lesssim \int_{\Gamma_{out}} \exp(-n\eta)\exp(-cj - cj_0)|d\tau| \lesssim \exp(-n\eta)\exp(-cj - cj_0).$$

- Using the residue theorem and the definition (3.4.25) of $\mathcal{R}^u(j_0, j)$, we have that

$$T^u = \int_{\Gamma_{in}} e^{n\tau}\frac{\mathbf{P}^u(\tau, j_0, j)}{\tau}d\tau = 2i\pi\mathcal{R}^u(j_0, j) + \int_{\Gamma_{\eta,in}} e^{n\tau}\frac{\mathbf{P}^u(\tau, j_0, j)}{\tau}d\tau.$$

Thus, using (3.5.15), there exists a constant $c > 0$ such that

$$|T^u - 2i\pi\mathcal{R}^u(j_0, j)| \leq \int_{\Gamma_{\eta,in}} e^{n\Re(\tau)}\frac{|\mathbf{P}^u(\tau, j_0, j)|}{|\tau|}|d\tau| \lesssim \exp(-n\eta - c(j + j_0)).$$

$\square$

Let us observe that the exponential estimates on the terms T^{out} and T^{u} we just proved can be altered to recover similar generalized Gaussian estimates as in (3.1.27) since there exists a constant $c>0$ such that for all $n, j_0 \in \mathbb{N}\backslash\{0\}$ which verify $j_0 \leq np$, we have

$$-n \leq -c\left(\frac{|j_0+n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}.$$

The same kind of exponential bounds will be encountered regularly in the rest of the proof and the same reasoning will allow us to obtain generalized Gaussian estimates.

Now that have found estimates for the two terms T^{out} and T^{u}, there just remains to study the term T^c. Section 3.5.3 will be dedicated to proving estimates on T^c in the case **II** when j_0 small with regard to $-n\alpha$. Finally, Section 3.5.4 will tackle the study of the term T^c in the case **III** when j_0 is close to $-n\alpha$.

3.5.3 Case II: Estimate for T^c for j_0 small with regard to $-n\alpha$

The main goal of this section is to prove the following proposition.

Proposition 3.2. *There exist two positive constants C, c such that for all $n \in \mathbb{N}$ and $j, j_0 \in \mathbb{N}\backslash\{0\}$ such that $j_0 < -\frac{n\alpha}{2}$, we have*

$$\left|T^c - 2i\pi E^{\beta}_{2\mu}\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)\mathcal{R}^c(j)\right| \leq \frac{Ce^{-cj}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|n\alpha+j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Combining (3.5.27), Propositions 3.1 and 3.2, we then prove that there exist two positive constants C, c such that for all $n \in \mathbb{N}$ and $j_0, j \in \mathbb{N}\backslash\{0\}$ which verify $j_0 < -\frac{n\alpha}{2}$:

$$|\mathrm{Err}(n, j_0, j)| \leq \frac{Ce^{-cj}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|n\alpha+j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{3.5.28}$$

Thus, (3.1.27) is proved when j_0 is small compared to n (Case **II**).

Proof

Step 1: We decompose T^c in two parts:

$$T^c = T^c_1 + T^c_2 \tag{3.5.29}$$

where

$$T^c_1 := \int_{\Gamma_{in}} e^{n\tau}\frac{\mathbf{P}^c(\tau, j_0, j) - \mathcal{R}^c(j)\exp(-j_0\varpi(\tau))}{\tau}d\tau,$$
$$T^c_2 := \mathcal{R}^c(j)\int_{\Gamma_{in}}\frac{\exp(n\tau - j_0\varpi(\tau))}{\tau}d\tau.$$

We will estimate separately both terms in order to prove the existence of two positive constants C, c such that for all $n \in \mathbb{N}$ and $j, j_0 \in \mathbb{N}\backslash\{0\}$ which verify $j_0 < -n\frac{\alpha}{2}$, we have

$$|T^c - 2i\pi\mathcal{R}^c(j)| \leq Ce^{-c(j+n)}. \tag{3.5.30}$$

- Inequality (3.5.16) implies that the function

$$\tau \in B_\varepsilon(0)\backslash\{0\} \mapsto e^{n\tau}\frac{\mathbf{P}^c(\tau, j_0, j) - \mathcal{R}^c(j)\exp(-j_0\varpi(\tau))}{\tau}$$

can be holomorphically extended on $B_\varepsilon(0)$. Using Cauchy's formula, we then have

$$T^c_1 = \int_{\Gamma_{\eta,in}} e^{n\tau}\frac{\mathbf{P}^c(\tau, j_0, j) - \mathcal{R}^c(j)\exp(-j_0\varpi(\tau))}{\tau}d\tau.$$

Using (3.5.16), there exist two positive constants C, c such that

$$|T^c_1| \leq Ce^{-cj}\int_{\Gamma_{\eta,in}}\exp(n\Re(\tau) - j_0\Re(\varpi(\tau)))|d\tau| = Ce^{-cj-n\eta}\int_{\Gamma_{\eta,in}}\exp(-j_0\Re(\varpi(\tau)))|d\tau|.$$

For $\tau \in \Gamma_{\eta,in}$, using (3.5.19) and the fact that $j_0 < -\frac{n\alpha}{2}$, we have that

$$-j_0 \Re(\varpi(\tau)) \leq \frac{-j_0}{\alpha}\left(-\Re(\tau) + A_R \Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}\right) \leq \frac{n}{2}(\eta + A_R \eta^{2\mu}). \tag{3.5.31}$$

Thus, there exists a new constant $C > 0$ such that

$$|T_1^c| \leq C \exp\left(-cj - n\left(\frac{\eta}{2} - A_R \eta^{2\mu}\right)\right).$$

Therefore, the condition (3.5.24) on η implies that there exist two positive constants C, c such that

$$\forall n \in \mathbb{N}, \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad j_0 < -\frac{n\alpha}{2} \Rightarrow |T_1^c| \leq C \exp(-c(j+n)). \tag{3.5.32}$$

- Using the residue theorem, we have

$$T_2^c - 2i\pi \mathcal{R}^c(j) = \mathcal{R}^c(j) \int_{\Gamma_{\eta,in}} \frac{\exp(n\tau - j_0 \varpi(\tau))}{\tau} d\tau.$$

Therefore, using (3.4.26) to exponentially bound $\mathcal{R}^c(j)$, there exist two positive constants C, c such that

$$|T_2^c - 2i\pi\mathcal{R}^c(j)| \leq Ce^{-cj} \int_{\Gamma_{\eta,in}} \frac{\exp(n\Re(\tau) - j_0\Re(\varpi(\tau)))}{|\tau|}|d\tau| \leq C\frac{\exp(-cj - n\eta)}{\eta} \int_{\Gamma_{\eta,in}} \exp(-j_0 \Re(\varpi(\tau)))|d\tau|.$$

Thus, using (3.5.31), there exists a new constant $C > 0$ independent from n, j_0 and j such that

$$|T_2^c - 2i\pi\mathcal{R}^c(j)| \leq C \exp\left(-cj - n\left(\frac{\eta}{2} - A_R \eta^{2\mu}\right)\right).$$

Therefore, the condition (3.5.24) on η implies that there exist two positive constants C, c such that

$$\forall n \in \mathbb{N}, \forall j_0, j \in \mathbb{N}\backslash\{0\}, \quad j_0 < -\frac{n\alpha}{2} \Rightarrow |T_2^c - 2i\pi\mathcal{R}^c(j)| \leq C \exp(-c(j+n)). \tag{3.5.33}$$

Using (3.5.29), (3.5.32) and (3.5.33), we conclude the proof of (3.5.30).

Step 2: Since we have

$$\frac{j_0 + n\alpha}{n^{\frac{1}{2\mu}}} \leq \frac{\alpha}{2} n^{\frac{2\mu-1}{2\mu}} < 0,$$

and the function $x \in]-\infty, \frac{\alpha}{2}] \mapsto |x|^{\frac{1}{2\mu-1}} \exp\left(-\frac{c}{2}|x|^{\frac{2\mu}{2\mu-1}}\right)$ is bounded where c is the positive constant in (3.5.3), we conclude using (3.5.3) that there exist a positive constants $\tilde{C}$ such that for all $n, j_0 \in \mathbb{N}\backslash\{0\}$ such that $j_0 < -\frac{n\alpha}{2}$, we have

$$\left|1 - E_{2\mu}^{\beta}\left(\frac{j_0 + n\alpha}{n^{\frac{1}{2\mu}}}\right)\right| \leq \frac{\tilde{C}}{n^{\frac{1}{2\mu}}} \exp\left(-\frac{c}{2}\left(\frac{|j_0 + n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{3.5.34}$$

Using (3.5.30), (3.5.34) and the estimate (3.4.26) to exponentially bound $\mathcal{R}^c(j)$, we conclude the proof of Proposition 3.2. □

3.5.4 Case III: Estimate for T^c for j_0 close to $-n\alpha$

The goal of this section will be to study what happens when j_0 is close to $-n\alpha$ and to prove the following proposition:

Proposition 3.3. *There exist two constants $C, c > 0$ such that for all $n, j_0, j \in \mathbb{N}\backslash\{0\}$ such that $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$, we have*

$$\left|T^c - 2i\pi E_{2\mu}^{\beta}\left(\frac{j_0 + n\alpha}{n^{\frac{1}{2\mu}}}\right)\mathcal{R}^c(j)\right| \leq \frac{Ce^{-cj}}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{|n\alpha + j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Combining (3.5.27) and Propositions 3.1 and 3.3, we prove that there exist two constants $C, c > 0$ such that for $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$, we have

$$|\mathrm{Err}(n, j_0, j)| \leq \frac{Ce^{-cj}}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{|n\alpha + j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{3.5.35}$$

Consequently, using the result of Section 3.5.1, (3.5.28) and (3.5.35), we can conclude the proof of Theorem 3.2.

Therefore, there just remains to prove Proposition 3.3. This part of the article requires the finest attention since the limiting estimates of Theorem 3.2 occur here. To prove Proposition 3.3, we will decompose T^c in three parts:

$$T^c = T_1^c + T_2^c + T_{princ}^c \tag{3.5.36}$$

where

$$T_1^c := \int_{\Gamma_{in}} e^{n\tau} \frac{\mathbf{P}^c(\tau, j_0, j) - \mathcal{R}^c(j)\exp(-j_0\varpi(\tau))}{\tau} d\tau, \tag{3.5.37}$$

$$T_2^c := \mathcal{R}^c(j) \int_{\Gamma_{in}} e^{n\tau} \frac{e^{-j_0\varpi(\tau)} - e^{-j_0\varphi(\tau)}}{\tau} d\tau, \tag{3.5.38}$$

$$T_{princ}^c := \mathcal{R}^c(j) \int_{\Gamma_{in}} \frac{e^{n\tau} e^{-j_0\varphi(\tau)}}{\tau} d\tau. \tag{3.5.39}$$

We now summarize the method of proof of Proposition 3.3. In Section 3.5.4, we introduce a family of integration paths Γ_p that are fundamental to optimally use the estimates on the function $\mathbf{R}(\tau, j_0, j)$ we proved in Section 3.4. We then prove in Section 3.5.4 estimates on the terms T_1^c and T_2^c respectively in Propositions 3.4 and 3.5. Finally, Section 3.5.4 is dedicated to the analysis of the term T_{princ}^c. We will change the integration path in the term

$$\int_{\Gamma_{in}} \frac{e^{n\tau} e^{-j_0\varphi(\tau)}}{\tau} d\tau$$

in order to compare it with $E_{2\mu}^{\beta}\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)$ (see Proposition 3.6).

Choice of integration path

We will now follow a strategy developed in [ZH98], which has also been used in [God03; CF23; CF22; Coe22], and introduce a family of parameterized curves.

We recall that we introduced in (3.5.22) the function Ψ defined by

$$\forall \tau_p \in \mathbb{R}, \quad \Psi(\tau_p) := \tau_p - A_R {\tau_p}^{2\mu}.$$

and that we chose ε small enough so that the function Ψ is continuous and strictly increasing on $]-\infty, \varepsilon]$. We can therefore introduce for $\tau_p \in [-\eta, \varepsilon]$ the curve Γ_p defined by

$$\Gamma_p := \left\{\tau \in \mathbb{C}, -\eta \leq \Re(\tau) \leq \tau_p, \quad \Re(\tau) - A_R \Re(\tau)^{2\mu} + A_I \mathrm{Im}\,(\tau)^{2\mu} = \Psi(\tau_p)\right\}.$$

It is a symmetric curve with respect to the axis $\mathbb{R}$ which intersects this axis on the point τ_p. If we introduce $\ell_p = \left(\frac{\Psi(\tau_p) - \Psi(-\eta)}{A_I}\right)^{\frac{1}{2\mu}}$, then $-\eta + i\ell_p$ and $-\eta - i\ell_p$ are the end points of Γ_p. We can also introduce a parametrization of this curve by defining $\gamma_p : [-\ell_p, \ell_p] \to \mathbb{C}$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall t \in [-\ell_p, \ell_p], \quad \mathrm{Im}\,(\gamma_p(t)) = t, \quad \Re(\gamma_p(t)) = h_p(t) := \Psi^{-1}\left(\Psi(\tau_p) - A_I t^{2\mu}\right). \tag{3.5.40}$$

The above parametrization immediately yields that there exists a constant $C > 0$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall t \in [-\ell_p, \ell_p], \quad |h_p'(t)| \leq C. \tag{3.5.41}$$

Also, there exists a constant $c_\star > 0$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall \tau \in \Gamma_p, \quad \Re(\tau) - \tau_p \leq -c_\star \mathrm{Im}\,(\tau)^{2\mu}. \tag{3.5.42}$$

We introduce those integration paths Γ_p because they allow us to use optimally the inequalities (3.5.18) and (3.5.19). For example, if we seek to bound $e^{n\tau - j_0\varpi(\tau)}$ for $\tau \in \Gamma_p \cap B_\varepsilon(0)$, it follows from the equality $\mathrm{sgn}(-j_0) = \mathrm{sgn}(\alpha)$ and the inequalities (3.5.19) and (3.5.42) that

$$\begin{aligned} n\Re(\tau) - j_0\Re(\varpi(\tau)) &\leq n\Re(\tau) + \frac{j_0}{\alpha}\left(\Re(\tau) - A_R\Re(\tau)^{2\mu} + A_I \mathrm{Im}\,(\tau)^{2\mu}\right) \\ &\leq -nc_\star \mathrm{Im}\,(\tau)^{2\mu} + \left(\frac{j_0}{\alpha} + n\right)\tau_p - \frac{j_0}{\alpha} A_R \tau_p^{2\mu}. \end{aligned} \tag{3.5.43}$$

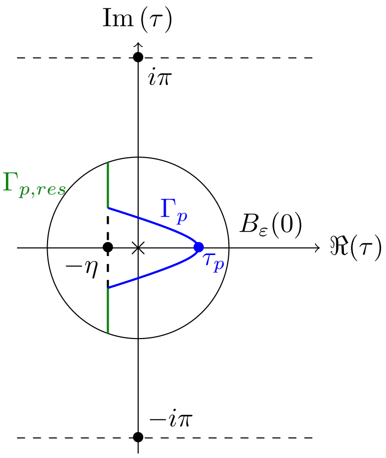

Figure 3.9 – A representation of the path $\Gamma_{p,in}$. It is composed of $\Gamma_{p,res}$ (in green) and Γ_p (in blue).

Such calculations will happen regularly in the following proof. There remains to make an appropriate choice of τ_p depending on n and j_0 that minimizes the right-hand side of the inequality (3.5.43) whilst the paths Γ_p remain within the ball $B_\varepsilon(0)$. We recall that when we fixed our choice of width η, we defined a radius $\varepsilon_\# \in]0,\varepsilon[$ such that $-\eta + il_{extr} \in B_\varepsilon(0)$ where the real number l_{extr} is defined by (3.5.25). This implies that the curve Γ_p associated with $\tau_p = \varepsilon_\#$ intersects the axis $-\eta + i\mathbb{R}$ within $B_\varepsilon(0)$. We let

$$\zeta = \frac{-j_0 - n\alpha}{2\mu n}, \quad \gamma = \frac{-j_0 A_R}{n}, \quad \rho\left(\frac{\zeta}{\gamma}\right) = \operatorname{sgn}\left(\frac{\zeta}{\gamma}\right)\left(\left|\frac{\zeta}{\gamma}\right|\right)^{\frac{1}{2\mu-1}}.$$

Inequality (3.5.43) thus becomes

$$n\Re(\tau) - j_0\Re(\varpi(\tau)) \leq -nc_\star \mathrm{Im}\,(\tau)^{2\mu} - \frac{n}{\alpha}(2\mu\zeta\tau_p - \gamma\tau_p^{2\mu}). \tag{3.5.44}$$

Our limiting estimates will come from the case where ζ is close to 0. We observe that the condition $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$ implies

$$-pA_R \leq \gamma \leq \frac{\alpha}{2}A_R. \tag{3.5.45}$$

Then, we take

$$\tau_p := \begin{cases} \rho\left(\frac{\zeta}{\gamma}\right), & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) \in [-\frac{\eta}{2}, \varepsilon_\#], & \text{(Case } \mathbf{A}\text{)} \\ \varepsilon_\#, & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) > \varepsilon_\#, & \text{(Case } \mathbf{B}\text{)} \\ -\frac{\eta}{2}, & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) < -\frac{\eta}{2}. & \text{(Case } \mathbf{C}\text{)} \end{cases}$$

The case **A** corresponds to the choice to minimize the right-hand side of (3.5.44) since $\rho\left(\frac{\zeta}{\gamma}\right)$ is the unique real root of the polynomial

$$\gamma x^{2\mu-1} = \zeta.$$

The cases **B** and **C** allow the path Γ_p to stay within $B_\varepsilon(0)$.

We now define the paths represented on Figure 3.9:

$$\begin{aligned} \Gamma_{p,res} &:= \{-\eta + it, \quad t \in [-r_\varepsilon(\eta), -\ell_p] \cup [\ell_p, r_\varepsilon(\eta)]\}, \\ \Gamma_{p,in} &:= \Gamma_p \cup \Gamma_{p,res}, \end{aligned}$$

where the function r_ε is defined by (3.5.23).

Finally, before we start to determine the estimates on the terms T_1^c and T_2^c in Section 3.5.4, we are going to introduce some inequalities to simplify the redaction.

Lemma 3.5.4.

— *There exists a constant $C > 0$ such that for all $\tau \in B_\varepsilon(0)$ and $n, j_0 \in \mathbb{N}\backslash\{0\}$ such that $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$, we have*

$$\left|e^{n\tau}\left(e^{-j_0\varpi(\tau)} - e^{-j_0\varphi(\tau)}\right)\right| \leq Cn|\tau|^{2\mu+1}\exp(n\Re(\tau) - j_0(\Re(\varpi(\tau)) - |\xi(\tau)\tau^{2\mu+1}|)). \tag{3.5.46}$$

— *For $n, j_0 \in \mathbb{N}\backslash\{0\}$ and $\tau \in \Gamma_p$, we have*
  — *Case* ***A***: $\rho\left(\frac{\zeta}{\gamma}\right) \in \left[-\frac{\eta}{2}, \varepsilon_\#\right]$

$$n\Re(\tau) - j_0(\Re(\varpi(\tau)) - |\xi(\tau)\tau^{2\mu+1}|) \leq -nc_\star \mathrm{Im}\,(\tau)^{2\mu} - \frac{n\gamma}{\alpha}(2\mu-1)\left(\left|\frac{\zeta}{\gamma}\right|\right)^{\frac{2\mu}{2\mu-1}}. \tag{3.5.47}$$

  — *Case* ***B***: $\rho\left(\frac{\zeta}{\gamma}\right) > \varepsilon_\#$

$$n\Re(\tau) - j_0(\Re(\varpi(\tau)) - |\xi(\tau)\tau^{2\mu+1}|) \leq -\frac{n}{2}(2\mu-1)A_R\varepsilon_\#^{2\mu}. \tag{3.5.48}$$

  — *Case* ***C***: $\rho\left(\frac{\zeta}{\gamma}\right) < -\frac{\eta}{2}$

$$n\Re(\tau) - j_0(\Re(\varpi(\tau)) - |\xi(\tau)\tau^{2\mu+1}|) \leq -\frac{n}{2}(2\mu-1)A_R\left(\frac{\eta}{2}\right)^{2\mu}. \tag{3.5.49}$$

— *For $n, j_0 \in \mathbb{N}\backslash\{0\}$ and $\tau \in \Gamma_{p,res}$, we have in all cases (**A**, **B** and **C**)*

$$n\Re(\tau) - j_0(\Re(\varpi(\tau)) - |\xi(\tau)\tau^{2\mu+1}|) \leq -n\frac{\eta}{2}. \tag{3.5.50}$$

The proof of inequalities (3.5.47)-(3.5.50) mainly rely on the inequalities (3.5.18) and (3.5.19) and calculations similar as those done to obtain (3.5.44). For a complete proof of Lemma 3.5.4, we advice the interested reader to look at the proof of [Coe22, Lemmas 17, 18 and 19] which prove similar inequalities in the context of the study of the temporal Green's function for the Laurent operator. The notation have intentionally been kept quite similar. The only difference is that the proof in [Coe22] are usually done with a positive velocity α.

Estimates for T_1^c and T_2^c

In this section, we prove generalized Gaussian estimates for the terms T_1^c and T_2^c when j_0 is close to $-n\alpha$.

Proposition 3.4. *There exist two positive constants C, c such that*

$$\forall n, j, j_0 \in \mathbb{N}^*, \quad j_0 \in \left[-\frac{n\alpha}{2}, np\right] \Rightarrow |T_1^c| \leq \frac{C}{n^{\frac{1}{2\mu}}}\exp\left(-cj - c\left(\frac{|j_0+n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Proof • Inequality (3.5.16) implies that the function

$$\tau \in B_\varepsilon(0)\backslash\{0\} \mapsto e^{n\tau}\frac{\mathbf{P}^c(\tau, j_0, j) - \mathcal{R}^c(j)\exp(-j_0\varpi(\tau))}{\tau}$$

can be holomorphically extended on $B_\varepsilon(0)$. Therefore, Cauchy's formula implies that

$$T_1^c = \int_{\Gamma_{in,p}} e^{n\tau}\frac{\mathbf{P}^c(\tau, j_0, j) - \mathcal{R}^c(j)\exp(-j_0\varpi(\tau))}{\tau}d\tau.$$

Using (3.5.16), there exist two positive constants C, c such that

$$\begin{aligned}|T_1^c| &\leq Ce^{-cj}\int_{\Gamma_{in,p}}\exp\left(n\Re(\tau) - j_0\Re(\varpi(\tau))\right)|d\tau| \\ &\leq Ce^{-cj}\left(\int_{\Gamma_p}\exp\left(n\Re(\tau) - j_0\Re(\varpi(\tau))\right)|d\tau| + \int_{\Gamma_{res,p}}\exp\left(n\Re(\tau) - j_0\Re(\varpi(\tau))\right)|d\tau|\right).\end{aligned}$$

• Using (3.5.50), we have

$$\int_{\Gamma_{res,p}}\exp\left(n\Re(\tau) - j_0\Re(\varpi(\tau))\right)|d\tau| \leq 2\pi\exp\left(-n\frac{\eta}{2}\right). \tag{3.5.51}$$

• In cases **B** and **C**, using (3.5.48) or (3.5.49) depending on the case, there exist two constants $C, c > 0$ such that

$$\int_{\Gamma_p}\exp\left(n\Re(\tau) - j_0\Re(\varpi(\tau))\right)|d\tau| \leq C\exp(-cn). \tag{3.5.52}$$

• In case **A**, using (3.5.47), we have

$$\int_{\Gamma_p} \exp\left(n\Re(\tau) - j_0\Re(\varpi(\tau))\right)|d\tau| \leq \int_{\Gamma_p} \exp\left(-nc_\star \mathrm{Im}\,(\tau)^{2\mu}\right)|d\tau| \exp\left(-\frac{n\gamma}{\alpha}(2\mu-1)\left(\left|\frac{\zeta}{\gamma}\right|\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Using the parametrization (3.5.40), the inequality (3.5.41) and the change of variables $u = n^{\frac{1}{2\mu}}t$, we have

$$\int_{\Gamma_p} e^{-nc_\star \mathrm{Im}\,(\tau)^{2\mu}}|d\tau| \lesssim \int_{-\ell_p}^{\ell_p} e^{-nc_\star t^{2\mu}}dt \lesssim \frac{1}{n^{\frac{1}{2\mu}}}.$$

Thus,

$$\int_{\Gamma_p} \exp\left(n\Re(\tau) - j_0\Re(\varpi(\tau))\right)|d\tau| \lesssim \frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-\frac{n\gamma}{\alpha}(2\mu-1)\left(\left|\frac{\zeta}{\gamma}\right|\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Lastly, the inequality (3.5.45) implies that we have a constant $c > 0$ independent from j_0 and n such that

$$-\frac{n\gamma}{\alpha}(2\mu-1)\left(\left|\frac{\zeta}{\gamma}\right|\right)^{\frac{2\mu}{2\mu-1}} \leq -c\left(\frac{|n\alpha + j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}$$

so,

$$\int_{\Gamma_p} \exp\left(n\Re(\tau) - j_0\Re(\varpi(\tau))\right)|d\tau| \lesssim \frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{|n\alpha + j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{3.5.53}$$

Combining (3.5.51)-(3.5.53), we conclude the proof of Proposition 3.4. □

Proposition 3.5. *There exist two positive constants C, c such that*

$$\forall n, j, j_0 \in \mathbb{N}^*, \quad j_0 \in \left[-\frac{n\alpha}{2}, np\right] \Rightarrow |T_2^c| \leq \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-cj - c\left(\frac{|j_0 + n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Proof The function

$$\tau \in B_\varepsilon(0) \backslash \{0\} \mapsto e^{n\tau}\frac{e^{-j_0\varpi(\tau)} - e^{-j_0\varphi(\tau)}}{\tau}$$

can be holomorphically extended on $B_\varepsilon(0)$. Therefore, Cauchy's formula implies that

$$T_2^c = \mathcal{R}^c(j)\int_{\Gamma_{in,p}} e^{n\tau}\frac{e^{-j_0\varpi(\tau)} - e^{-j_0\varphi(\tau)}}{\tau}d\tau.$$

Using (3.5.46) and (3.4.26), there exist two positive constant C, c such that

$$|T_2^c| \leq Ce^{-cj}n\int_{\Gamma_{in,p}} |\tau|^{2\mu}\exp(n\Re(\tau) - j_0(\Re(\varpi(\tau)) - |\xi(\tau)\tau^{2\mu+1}|))|d\tau|. \tag{3.5.54}$$

• Using (3.5.50), there exist a constant $C > 0$ independent from j_0 and n such that

$$n\int_{\Gamma_{res,p}} |\tau|^{2\mu}\exp(n\Re(\tau) - j_0(\Re(\varpi(\tau)) - |\xi(\tau)\tau^{2\mu+1}|))|d\tau| \leq Cn\exp\left(-n\frac{\eta}{2}\right). \tag{3.5.55}$$

• In cases **B** and **C**, using (3.5.48) or (3.5.49) depending on the case, there exist two constants $C, c > 0$ such that

$$n\int_{\Gamma_p} |\tau|^{2\mu}\exp(n\Re(\tau) - j_0(\Re(\varpi(\tau)) - |\xi(\tau)\tau^{2\mu+1}|))|d\tau| \leq Cn\exp\left(-cn\right). \tag{3.5.56}$$

• In case **A**, using (3.5.47), we have

$$n\int_{\Gamma_p} |\tau|^{2\mu}\exp(n\Re(\tau) - j_0(\Re(\varpi(\tau)) - |\xi(\tau)\tau^{2\mu+1}|))|d\tau|$$
$$\leq n\int_{\Gamma_p} |\tau|^{2\mu}\exp\left(-nc_\star \mathrm{Im}\,(\tau)^{2\mu}\right)|d\tau| \exp\left(-\frac{n\gamma}{\alpha}(2\mu-1)\left(\left|\frac{\zeta}{\gamma}\right|\right)^{\frac{2\mu}{2\mu-1}}\right).$$

But, the inequality (3.5.45) and the fact that $\rho\left(\frac{\zeta}{\gamma}\right) = \tau_p$ imply

$$-\frac{n\gamma}{\alpha}(2\mu-1)\left(\left|\frac{\zeta}{\gamma}\right|\right)^{\frac{2\mu}{2\mu-1}} \leq -\frac{2\mu-1}{2}A_R n|\tau_p|^{2\mu}.$$

If we introduce $c>0$ small enough, then

$$n\int_{\Gamma_p}|\tau|^{2\mu}\exp(n\Re(\tau)-j_0(\Re(\varpi(\tau))-|\xi(\tau)\tau^{2\mu+1}|))|d\tau| \leq n\int_{\Gamma_p}|\tau|^{2\mu}\exp\left(-nc_\star \mathrm{Im}\,(\tau)^{2\mu}\right)|d\tau|\exp\left(-cn|\tau_p|^{2\mu}\right).$$

Using the parametrization (3.5.40) and the inequality (3.5.41), we have

$$\int_{\Gamma_p}|\tau|^{2\mu}e^{-nc_\star \mathrm{Im}\,(\tau)^{2\mu}}|d\tau| \lesssim \int_{-\ell_p}^{\ell_p}(|\tau_p|^{2\mu}+|t|^{2\mu})e^{-nc_\star t^{2\mu}}dt.$$

The change of variables $u = n^{\frac{1}{2\mu}}t$ and the fact that the function $x\geq 0\mapsto x^{2\mu}\exp\left(-\frac{c}{2}x^{2\mu}\right)$ is bounded imply

$$\begin{cases} \displaystyle\int_{-\ell_p}^{\ell_p}|t|^{2\mu}e^{-nc_\star t^{2\mu}}dt \lesssim \frac{1}{n^{1+\frac{1}{2\mu}}}, \\ \displaystyle\int_{-\ell_p}^{\ell_p}|\tau_p|^{2\mu}e^{-nc_\star t^{2\mu}}dt \lesssim \frac{1}{n^{1+\frac{1}{2\mu}}}\exp\left(\frac{c}{2}n|\tau_p|^{2\mu}\right). \end{cases}$$

Thus,

$$n\int_{\Gamma_p}|\tau|^{2\mu}\exp(n\Re(\tau)-j_0(\Re(\varpi(\tau))-|\xi(\tau)\tau^{2\mu+1}|))|d\tau| \lesssim \frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-\frac{c}{2}n|\tau_p|^{2\mu}\right).$$

Lastly, the inequality (3.5.45) implies that we have a constant $c>0$ independent from j_0 and n such that

$$\frac{c}{2}n|\tau_p|^{2\mu} \geq \tilde{c}\left(\frac{|n\alpha+j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}$$

so,

$$n\int_{\Gamma_p}|\tau|^{2\mu}\exp(n\Re(\tau)-j_0(\Re(\varpi(\tau))-|\xi(\tau)\tau^{2\mu+1}|))|d\tau| \lesssim \frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-\tilde{c}\left(\frac{|n\alpha+j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{3.5.57}$$

Combining (3.5.54)-(3.5.57), we conclude the proof of Proposition 3.5. □

Calculations around T^c_{princ}

There just remains to study the term T^c_{princ} defined by (3.5.39). The end goal of this section is to prove the following proposition.

Proposition 3.6. *There exist two positive constants C, c such that for all $n, j_0\in\mathbb{N}^*$ which verify $-\frac{n\alpha}{2}\leq j_0\leq np$, we have*

$$\left|\int_{\Gamma_{in}}\frac{\exp(n\tau-j_0\varphi(\tau))}{\tau}d\tau - 2i\pi E^\beta_{2\mu}\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)\right| \leq \frac{C}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|j_0+n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

By utilizing (3.5.36) along with Propositions 3.4, 3.5, and 3.6, we complete the demonstration of Proposition 3.3. Subsequently, this concludes the proof of Theorem 3.2.

Thus, there just remains to prove Proposition 3.6. The main idea of the proof is to change the integration path on the term

$$\int_{\Gamma_{in}}\frac{e^{n\tau}e^{-j_0\varphi(\tau)}}{\tau}d\tau.$$

Proof We fix a constant $s>0$ such that for all $n, j_0\in\mathbb{N}^*$ which verify $-\frac{n\alpha}{2}\leq j_0\leq np$, we have

$$\left(-\frac{n\alpha}{j_0}-1\right)s + A_R s^{2\mu} \leq \frac{A_I}{2}r_\varepsilon(\eta)^{2\mu}. \tag{3.5.58}$$

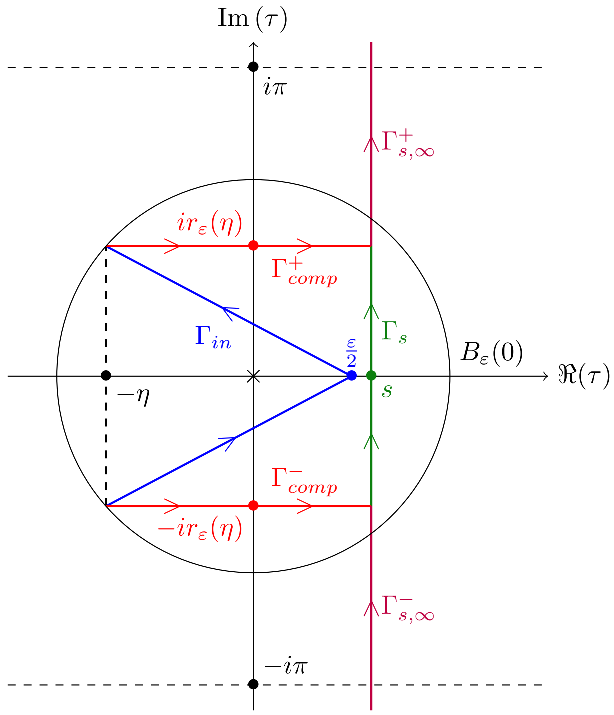

Figure 3.10 – Representation of the paths Γ_s (in green), Γ^+_{comp}, Γ^-_{comp} (both in red), $\Gamma^+_{s,\infty}$, $\Gamma^-_{s,\infty}$ (both in purple) and $\Gamma_{s,\infty} := \Gamma^-_{s,\infty} \cup \Gamma_s \cup \Gamma^+_{s,\infty}$.

We introduce the paths Γ_s, Γ^+_{comp}, Γ^-_{comp}, $\Gamma^+_{s,\infty}$, $\Gamma^-_{s,\infty}$ and $\Gamma_{s,\infty}$ represented on Figure 3.10 and that are defined as

$$\begin{aligned}
\Gamma_s &:= \{s+it, \quad t\in[-r_\varepsilon(\eta), r_\varepsilon(\eta)]\}, & \Gamma^+_{s,\infty} &:= \{s+it, \quad t\in[r_\varepsilon(\eta),+\infty[\},\\
\Gamma^+_{comp} &:= \{t+ir_\varepsilon(\eta), \quad t\in[-\eta,s]\}, & \Gamma^-_{s,\infty} &:= \{s+it, \quad t\in]-\infty,-r_\varepsilon(\eta)]\},\\
\Gamma^-_{comp} &:= \{t-ir_\varepsilon(\eta), \quad t\in[-\eta,s]\}, & \Gamma_{s,\infty} &:= \Gamma^-_{s,\infty}\cup\Gamma_s\cup\Gamma^+_{s,\infty}.
\end{aligned}$$

The proof of Proposition 3.6 is separated in different steps where we will use the different paths we introduced.

• **Step 1:** In this step, we start by proving that there exist two positive constants C, c such that for all $n, j_0 \in \mathbb{N}^*$ which verify $-\frac{n\alpha}{2} \leq j_0 \leq np$, we have

$$\left|\int_{\Gamma_{in}} \frac{\exp(n\tau - j_0\varphi(\tau))}{\tau}d\tau - \int_{\Gamma_s} \frac{\exp(n\tau - j_0\varphi(\tau))}{\tau}d\tau\right| \leq Ce^{-cn}. \tag{3.5.59}$$

Cauchy's formula implies that

$$\left|\int_{\Gamma_{in}} \frac{e^{n\tau-j_0\varphi(\tau)}}{\tau}d\tau - \int_{\Gamma_s} \frac{e^{n\tau-j_0\varphi(\tau)}}{\tau}d\tau\right| \leq \left|\int_{\Gamma^+_{comp}} \frac{e^{n\tau-j_0\varphi(\tau)}}{\tau}d\tau\right| + \left|\int_{\Gamma^-_{comp}} \frac{e^{n\tau-j_0\varphi(\tau)}}{\tau}d\tau\right|. \tag{3.5.60}$$

We need to find estimates for the two terms on the right-hand side. Both terms will be bounded similarly so we will focus on the first one. First, we observe that

$$\left|\int_{\Gamma^+_{comp}} \frac{e^{n\tau-j_0\varphi(\tau)}}{\tau}d\tau\right| \leq \frac{1}{r_\varepsilon(\eta)}\int_{-\eta}^{s} \exp\left(nt - j_0\Re(\varphi(t+ir_\varepsilon(\eta)))\right)dt.$$

Using (3.5.18), we have for $t\in[-\eta,s]$

$$nt - j_0\Re(\varphi(t+ir_\varepsilon(\eta))) \leq \left(n+\frac{j_0}{\alpha}\right)t - \frac{j_0}{\alpha}A_R t^{2\mu} + \frac{j_0}{\alpha}A_I r_\varepsilon(\eta)^{2\mu} = -\frac{j_0}{\alpha}\left(\left(-\frac{n\alpha}{j_0}-1\right)t + A_R t^{2\mu} - A_I r_\varepsilon(\eta)^{2\mu}\right).$$

Since the function

$$t \in [-\eta, s] \mapsto \left(-\frac{n\alpha}{j_0} - 1\right) t + A_R t^{2\mu} - A_I r_\varepsilon(\eta)^{2\mu}$$

is convex, it attains its maximum for $t \in \{-\eta, s\}$. Thus, the conditions (3.5.26) and (3.5.58) on η and s imply that for $t \in [-\eta, s]$

$$nt - j_0 \Re(\varphi(t + i r_\varepsilon(\eta))) \leq \frac{j_0}{2\alpha} A_I r_\varepsilon(\eta)^{2\mu}.$$

Thus, recalling that α is negative and that $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$, we have that

$$\left| \int_{\Gamma^+_{comp}} \frac{e^{n\tau - j_0\varphi(\tau)}}{\tau} d\tau \right| \leq \frac{s+\eta}{r_\varepsilon(\eta)} \exp\left(-\frac{n}{4} A_I r_\varepsilon(\eta)^{2\mu}\right).$$

Using a similar proof to bound the second term in the right-hand side of (3.5.60), we can conclude the proof of (3.5.59).

• **Step 2:** In this second step, we now prove that there exist two positive constants C, c such that for all $n, j_0 \in \mathbb{N}^*$ which verify $-\frac{n\alpha}{2} \leq j_0 \leq np$, we have

$$\left| \int_{\Gamma_s} \frac{\exp(n\tau - j_0\varphi(\tau))}{\tau} d\tau - \int_{\Gamma_{s,\infty}} \frac{\exp(n\tau - j_0\varphi(\tau))}{\tau} d\tau \right| \leq Ce^{-cn}. \tag{3.5.61}$$

We have that

$$\left| \int_{\Gamma_s} \frac{\exp(n\tau - j_0\varphi(\tau))}{\tau} d\tau - \int_{\Gamma_{s,\infty}} \frac{\exp(n\tau - j_0\varphi(\tau))}{\tau} d\tau \right| \leq \left| \int_{\Gamma^+_{s,\infty}} \frac{e^{n\tau - j_0\varphi(\tau)}}{\tau} d\tau \right| + \left| \int_{\Gamma^-_{s,\infty}} \frac{e^{n\tau - j_0\varphi(\tau)}}{\tau} d\tau \right|. \tag{3.5.62}$$

We need to find estimates for the two terms on the right-hand side. Both terms will be bounded similarly so we will focus on the first one. First, we observe that

$$\left| \int_{\Gamma^+_{s,\infty}} \frac{e^{n\tau - j_0\varphi(\tau)}}{\tau} d\tau \right| \leq \frac{1}{r_\varepsilon(\eta)} \int_{r_\varepsilon(\eta)}^{+\infty} \exp\left(ns - j_0 \Re(\varphi(s+it))\right) dt.$$

Using (3.5.18) and (3.5.58), we have for $t \in [r_\varepsilon(\eta), +\infty[$ and $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$

$$ns - j_0\Re(\varphi(s+it)) \leq \left(n + \frac{j_0}{\alpha}\right) s - \frac{j_0}{\alpha} A_R s^{2\mu} + \frac{j_0}{\alpha} A_I t^{2\mu} \leq \frac{j_0}{\alpha} A_I \left(t^{2\mu} - \frac{r_\varepsilon(\eta)^{2\mu}}{2}\right) \leq -\frac{n}{2} A_I \left(t^{2\mu} - \frac{r_\varepsilon(\eta)^{2\mu}}{2}\right).$$

Thus,

$$ns - j_0\Re(\varphi(s+it)) \leq -\frac{n}{4} A_I r_\varepsilon(\eta)^{2\mu} - \frac{n}{2} A_I \left(t^{2\mu} - r_\varepsilon(\eta)^{2\mu}\right) \leq -\frac{n}{4} A_I r_\varepsilon(\eta)^{2\mu} - \frac{1}{2} A_I \left(t^{2\mu} - r_\varepsilon(\eta)^{2\mu}\right).$$

We can then conclude that

$$\left| \int_{\Gamma^+_{s,\infty}} \frac{e^{n\tau - j_0\varphi(\tau)}}{\tau} d\tau \right| \leq \frac{1}{r_\varepsilon(\eta)} \exp\left(-\frac{n}{4} A_I r_\varepsilon(\eta)^{2\mu}\right) \underbrace{\int_{r_\varepsilon(\eta)}^{+\infty} \exp\left(-\frac{1}{2} A_I \left(t^{2\mu} - r_\varepsilon(\eta)^{2\mu}\right)\right) dt}_{<+\infty}.$$

Using a similar proof to bound the second term in the right-hand side of (3.5.62), we can conclude the proof of (3.5.61).

• **Step 3:** We introduce the functions

$$\forall u \in \mathbb{R}, \forall x \in \mathbb{R}, \forall s \in \mathbb{R}, \quad g(u,x,s) := \exp\left(i(u+is)x - \beta(u+is)^{2\mu}\right), \tag{3.5.63}$$

$$\forall x \in \mathbb{R}, \forall s \in]0, +\infty[, \quad \mathcal{F}(x,s) := \int_{-\infty}^{+\infty} \frac{g(u,x,s)}{i(u+is)} du. \tag{3.5.64}$$

We can prove that the function $\mathcal{F}$ verifies

$$\forall s \in]0, +\infty[, \forall x \in \mathbb{R}, \quad -\mathcal{F}(x,s) = 2\pi E^\beta_{2\mu}(x). \tag{3.5.65}$$

For the sake of completeness, we give a proof of (3.5.65) in the Appendix (Section 3.A).

We observe that for $n, j_0 \in \mathbb{N}^*$, if we define $\tilde{s}_{j_0} := \frac{s}{-\alpha}\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}$, we have using the change of variables $\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}} t = \alpha u$ that

$$\begin{aligned}
\int_{\Gamma_{s,\infty}} \frac{\exp(n\tau - j_0\varphi(\tau))}{\tau} d\tau &= i\int_{-\infty}^{+\infty} \frac{\exp\left(\left(n+\frac{j_0}{\alpha}\right)(s+it) + (-1)^{\mu+1}\frac{\beta}{\alpha^{2\mu}}\left(-\frac{j_0}{\alpha}\right)(s+it)^{2\mu}\right)}{s+it} dt \\
&= -i\int_{-\infty}^{+\infty} \frac{\exp\left(i(u+i\tilde{s}_{j_0})\frac{(n\alpha+j_0)}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}} - \beta(u+i\tilde{s}_{j_0})^{2\mu}\right)}{i(u+i\tilde{s}_{j_0})} du \\
&= -i\mathcal{F}\left(\frac{(n\alpha+j_0)}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}}, \tilde{s}_{j_0}\right).
\end{aligned} \tag{3.5.66}$$

The equalities (3.5.65) and (3.5.66) imply that

$$\int_{\Gamma_{s,\infty}} \frac{\exp(n\tau - j_0\varphi(\tau))}{\tau} d\tau = 2i\pi E_{2\pi}^\beta\left(\frac{j_0+n\alpha}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}}\right). \tag{3.5.67}$$

To end the proof of Proposition 3.6, we will prove the existence of two positive constants C, c such that for all $n, j_0 \in \mathbb{N}\backslash\{0\}$ which verify $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$, we have

$$\left|E_{2\mu}^\beta\left(\frac{j_0+n\alpha}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}}\right) - E_{2\mu}^\beta\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)\right| \leq \frac{C}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|j_0+n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{3.5.68}$$

We recall that

$$\forall x \in \mathbb{R}, \quad {E_{2\mu}^\beta}'(x) = -H_{2\mu}^\beta(x).$$

Because of the mean value inequality, the fact that $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$ and (3.5.1), there exists a positive constant $c > 0$ such that

$$\begin{aligned}
\left|E_{2\mu}^\beta\left(\frac{j_0+n\alpha}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}}\right) - E_{2\mu}^\beta\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)\right| &\leq |j_0+n\alpha|\left|\frac{1}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}}\right| \sup_{t\in\left[\frac{j_0+n\alpha}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}}, \frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right]} \left|H_{2\mu}^\beta(t)\right| \\
&\lesssim |j_0+n\alpha|\left|\frac{1}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}}\right| \exp\left(-c\left(\frac{|j_0+n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).
\end{aligned}$$

Furthermore, using once again the mean value inequality and the fact that $j_0 \in \left[-\frac{n\alpha}{2}, np\right]$, we have

$$\left|\frac{1}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}}\right| \lesssim |j_0+n\alpha| \sup_{t\in[j_0,-n\alpha]} \frac{1}{t^{1+\frac{1}{2\mu}}} \lesssim \frac{|j_0+n\alpha|}{n^{1+\frac{1}{2\mu}}}.$$

Therefore,

$$\left|E_{2\mu}^\beta\left(\frac{j_0+n\alpha}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}}\right) - E_{2\mu}^\beta\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)\right| \lesssim \frac{1}{n^{1-\frac{1}{2\mu}}}\left(\frac{|j_0+n\alpha|}{n^{\frac{1}{2\mu}}}\right)^2 \exp\left(-c\left(\frac{|j_0+n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Since the function $x \mapsto x^2\exp\left(-\frac{c}{2}x\right)$ is bounded, we conclude that

$$\left|E_{2\mu}^\beta\left(\frac{j_0+n\alpha}{\left(-\frac{j_0}{\alpha}\right)^{\frac{1}{2\mu}}}\right) - E_{2\mu}^\beta\left(\frac{j_0+n\alpha}{n^{\frac{1}{2\mu}}}\right)\right| \lesssim \frac{1}{n^{1-\frac{1}{2\mu}}}\exp\left(-\frac{c}{2}\left(\frac{|j_0+n\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Since $1-\frac{1}{2\mu} \geq \frac{1}{2\mu}$, we easily conclude the proof of (3.5.68).

Combining (3.5.59), (3.5.61), (3.5.67) and (3.5.68), we can end the proof of Proposition 3.6. □

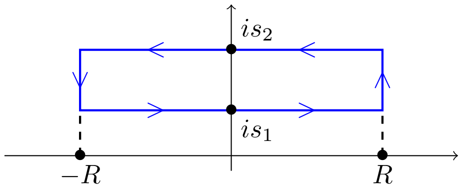

Figure 3.11 – Integrating path to prove (3.A.3).

3.A Appendix of the chapter

Proof of equality (3.5.65)

We recall that (3.5.65) states that

$$\forall s \in]0,+\infty[, \forall x \in \mathbb{R}, \quad -\mathcal{F}(x,s) = 2\pi E^{\beta}_{2\mu}(x).$$

Proof The starting point of the proof will be to prove sharp estimates on the function g defined by (3.5.63). We observe that

$$\forall u,x,s \in \mathbb{R}, \quad |g(u,x,s)| \leq \exp(-sx)\exp\left(-\Re(\beta(u+is)^{2\mu})\right).$$

Using Young's inequality, we prove that there exists a constant $c > 0$ such that

$$\forall u,s \in \mathbb{R}, \quad \Re(\beta(u+is)^{2\mu}) \geq \frac{\Re(\beta)}{2}u^{2\mu} - cs^{2\mu}.$$

Thus, we have

$$\forall u,x,s \in \mathbb{R}, \quad |g(u,x,s)| \leq \exp(-sx+cs^{2\mu})\exp\left(-\frac{\Re(\beta)}{2}u^{2\mu}\right). \tag{3.A.1}$$

We observe that for all $s \in]0,+\infty[$, the function $\mathcal{F}(\cdot,s)$ defined by (3.5.64) is in the class $\mathcal{C}^1$ and

$$\forall x \in \mathbb{R}, \forall s \in]0,+\infty[, \quad \frac{\partial \mathcal{F}}{\partial x}(x,s) = 2\pi H^{\beta}_{2\mu}(x). \tag{3.A.2}$$

Integrating the function $z \mapsto \frac{\exp(izx-\beta z^{2\mu})}{iz}$ on the rectangle depicted in the right-side of Figure 3.11, using the Cauchy formula as well as (3.A.1) and passing to the limit $R \to +\infty$, we prove that

$$\forall s_1,s_2 \in]0,+\infty[, \quad \mathcal{F}(\cdot,s_1) = \mathcal{F}(\cdot,s_2). \tag{3.A.3}$$

Finally, using (3.A.1), there exists $C > 0$ independent from x and s such that

$$\forall x \in \mathbb{R}, \forall s \in]0,+\infty[, \quad |\mathcal{F}(x,s)| \leq C\frac{e^{-sx+cs^{2\mu}}}{s}. \tag{3.A.4}$$

For $x > 0$, optimizing $e^{-sx+cs^{2\mu}}$ with respect to s drives us to choose $s = \left(\frac{x}{2\mu c}\right)^{\frac{1}{2\mu-1}}$ in (3.A.4). Using (3.A.3), we can prove that there exist two constants $C, c > 0$ such that

$$\forall x \in]0,+\infty[, \forall s \in]0,+\infty[, \quad |\mathcal{F}(x,s)| \leq \frac{C}{x^{\frac{1}{2\mu-1}}}\exp(-c|x|^{\frac{2\mu}{2\mu-1}}).$$

Thus,

$$\forall s \in]0,+\infty[, \quad \lim_{x\to+\infty} \mathcal{F}(x,s) = 0. \tag{3.A.5}$$

Using (3.A.2) and (3.A.5), we easily conclude the proof of (3.5.65). □

Chapter 4

Linear stability of discrete shock profiles for systems of conservation laws

This chapter presents the content of the preprint [Coe23].

Abstract of the current chapter

We prove the linear orbital stability of spectrally stable stationary discrete shock profiles for conservative finite difference schemes applied to systems of conservation laws. The proof relies on an accurate description of the pointwise asymptotic behavior of the Green's function associated with those discrete shock profiles, improving on the result of Lafitte-Godillon [God03]. The main novelty of this stability result is that it applies to a fairly large family of schemes that introduce some artificial possibly high-order viscosity. The result is obtained under a sharp spectral assumption rather than by imposing a smallness assumption on the shock amplitude.

Outline of the current chapter

Notations

Throughout this chapter, we define the following sets:

$$\mathbb{U} := \{z \in \mathbb{C}, |z| > 1\}, \quad \mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}, \quad \mathbb{S}^1 := \{z \in \mathbb{C}, |z| = 1\},$$

$$\overline{\mathbb{U}} := \mathbb{S}^1 \cup \mathbb{U}, \quad \overline{\mathbb{D}} := \mathbb{S}^1 \cup \mathbb{D}.$$

For $z \in \mathbb{C}$ and $r > 0$, we let $B(z, r)$ denote the open ball in $\mathbb{C}$ centered at z with radius r. We also introduce the Kronecker symbol $\delta_{i,j}$ which equals 1 if $i = j$ and 0 when $i \neq j$.

For E a Banach space, we denote $\mathcal{L}(E)$ the space of bounded operators acting on E and $\|\cdot\|_{\mathcal{L}(E)}$ the operator norm. For T in $\mathcal{L}(E)$, the notation $\sigma(T)$ stands for the spectrum of the operator T and $\rho(T)$ denotes the resolvent set of T.

We let $\mathcal{M}_{n,k}(\mathbb{C})$ denote the space of complex valued $n \times k$ matrices and we use the notation $\mathcal{M}_n(\mathbb{C})$ when $n = k$. For an element M of $\mathcal{M}_{n,k}(\mathbb{C})$, the notation M^T stands for the transpose of M. For a square matrix M, $\mathrm{com}(M)$ corresponds to the cofactor matrix associated with M.

We use the notation $\lesssim$ to express an inequality up to a multiplicative constant. Eventually, we let C (resp. c) denote some large (resp. small) positive constants that may vary throughout the text (sometimes within the same line). Furthermore, we use the usual Landau notation $O(\cdot)$ to introduce a term uniformly bounded with respect to the argument. For more clarity, we will also occasionally use the notation $O_{\mathbb{C}}(\cdot)$ to precise the fact that the term is a complex scalar and $O_{\mathcal{M}_{1,d}}(\cdot)$ to say that it belongs to $\mathcal{M}_{1,d}(\mathbb{C})$.

We let $\mathrm{Res}(f, a)$ denote the residue of a meromorphic function f at the point a.

4.1 Introduction

4.1.1 Context

A fundamental issue on the subject of systems conservation laws is understanding how discontinuities that can arise in solutions are handled by conservative finite difference schemes. At the center of this question stands the notion of discrete shock profiles which are defined as solutions of the numerical scheme which are traveling waves linking two states and correspond to numerical approximations of shocks. A desirable feature of the numerical scheme should be that stable shock waves for the system of conservation laws should yield stable discrete shock profiles (or a family of them) for the numerical scheme. For a general introduction on the questions of existence and stability of discrete shock profiles, we highly encourage the interested reader to take a close look at [Ser07].

In the present chapter, we will consider conservative finite difference schemes that introduce numerical viscosity and will focus on the study of the discrete shock profiles associated with standing Lax shocks. We assume that there exists a differentiable one-parameter family of discrete shock profiles associated with such a shock. Such an existence result has been proved for instance in [MR79; Mic84] under a weakness assumption on the shock, i.e. when the difference between the two states is sufficiently small. Let us introduce two notions of stability for the family of discrete shock profiles:

— Spectral stability amounts to asking for the operators obtained by linearizing the numerical scheme about the discrete shock profiles to have no unstable or marginally stable eigenvalues except for 1 which is always an eigenvalue because of the existence of the differentiable one-parameter family of discrete shock profiles. Furthermore, we ask for 1 to be a simple eigenvalue of the linearized operator. This corresponds to Hypotheses 4.6 and 4.7 below.

— Nonlinear orbital stability signifies that for initial conditions of the numerical scheme which are small enough perturbations of a discrete shock profile, the solution of the numerical scheme that ensues stays close to the manifold of the discrete shock profiles. This is a stronger stability property.

There are some results surrounding nonlinear stability that have already been proven. Most of them introduce a weakness assumption on the amplitude of the underlying shocks and/or focus on fairly specific schemes or situations. For instance, [LX93b; LX93a; Yin97] focus on proving a nonlinear orbital stability result for discrete shock profiles moving with rational speeds associated with weak Lax shocks for the Lax-Friedrichs scheme. Another example is [Smy90] which also proves a nonlinear orbital stability result on some stationary discrete shock profiles for the Lax-Wendroff scheme without any weakness assumption. Two of the main results that we can point out are the following:

- In [Mic02], Michelson proves the nonlinear orbital stability of the family of discrete shock profiles associated with weak standing Lax shocks for schemes of any odd order under an assumption of stability of the viscous shock profiles associated with some scalar problem.
- In [Jen74], Jennings focuses on the particular case of *monotone* schemes for *scalar* conservation laws. The main results are the existence and uniqueness of continuous one-parameter family of discrete shock profiles with rational speeds and a proof of nonlinear orbital stability for them when they are associated with Lax shocks. In this chapter, no weakness assumption on the associated shocks is introduced.

Similar as for the nonlinear stability theory for viscous shock profiles [ZH98] or for semi-discrete shock profiles [BHR03; Bec+10], we hope to prove that spectrally stable discrete shock profiles verify nonlinear orbital stability. This new result would generalize the previously cited articles by proving a result of nonlinear stability for *systems* of conservation laws, for a fairly large family of finite difference schemes, while avoiding to introduce a weakness assumption on the shocks. This would answer Open problem 5.3 of [Ser07].

Just like in [ZH98; BHR03; Bec+10], proving that spectral stability implies nonlinear orbital stability relies on an accurate description of the Green's function (defined below by (4.1.25)) associated with the operator obtained by linearizing the numerical scheme about the discrete shock profile. The main result of the present chapter provides such an accurate description (see Theorem 4.1 below). We have not yet proven that spectral stability implies nonlinear stability; however, the description of the semi-group associated with the linearized operator deduced from Theorem 4.1 already allows us to prove linear stability and decay estimates for linear perturbations (see Theorem 4.2 below). We will prove the nonlinear orbital stability result in the scalar case in Chapter 5 and hope to consider the general system case in a future paper.

Let us now focus on the study of the Green's function. Theorem 4.1 can be seen as an improvement on the result of [God03] that highly influenced the analysis performed in the present chapter. In [God03; Laf01], Lafitte-Godillon generalizes in the fully discrete setting several tools introduced in [ZH98] that are necessary to study the Green's function for the linearized operator. More precisely, she constructs the Evans function for this problem and introduces in her thesis [Laf01] the notion of geometric dichotomies (an equivalent version of the exponential dichotomies for the discrete dynamical systems). Those tools will be redefined and used intensively in the present chapter. Lafitte-Godillon then attempts to obtain precise estimates on the Green's function of the linearized operator. However, the result of [God03] has two limitations:

- The proof is done specifically for the modified Lax-Friecrichs scheme. This is not a strong limitation as it is quite clear that the content of the paper [God03] can be generalized to a larger class of numerical schemes (at least for odd ordered schemes, just like in the present chapter).
- The estimates on the Green's function proved in [God03, Theorem 1.1] are not sufficient to conclude on the nonlinear stability as they are only local with respect to the initial localization of the Dirac mass associated with the Green's function (the parameter l in [God03, Theorem 1.1] which corresponds to the parameter j_0 in Theorem 4.1). This is a consequence of the analysis on the so-called spatial Green's function (defined below by (4.1.26)) done in [God03] which is not precise enough for linear/nonlinear stability purposes.

In the present chapter, we solve those issues by describing precisely the leading order of the Green's function and proving sharp and *uniform* estimates on the remainder and its discrete derivative. We also consider schemes of any odd order, in particular with only few restrictions on the size of the stencil of the scheme.

4.1.2 Definition of stationary discrete shock profiles (SDSP)

We consider a one-dimensional system of conservation laws

$$\begin{gathered}\partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{U},\end{gathered} \tag{4.1.1}$$

where $d \in \mathbb{N}\backslash\{0\}$ corresponds to the number of unknown $u =: (u_1, \dots, u_d)$ of (4.1.1), the space of states $\mathcal{U}$ is an open set of $\mathbb{R}^d$ and the flux $f : \mathcal{U} \to \mathbb{R}^d$ is a $\mathscr{C}^\infty$ function. We will suppose that the system of conservation laws is hyperbolic, meaning that for all $u \in \mathcal{U}$, the jacobian matrix $df(u)$ is diagonalisable with real eigenvalues.

We fix two states $u^-, u^+ \in \mathcal{U}$ such that

$$f(u^-) = f(u^+). \tag{4.1.2}$$

This is the well-known Rankine-Hugoniot condition which allows to state that the standing shock defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t,x) := \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{else,} \end{cases} \tag{4.1.3}$$

is a weak solution of (4.1.1).

Since the system of conservation laws we consider is hyperbolic at the states $u^\pm$, we introduce the eigenvalues $\boldsymbol{\lambda}_1^\pm, \dots, \boldsymbol{\lambda}_d^\pm \in \mathbb{R}$ and a basis of nonzero eigenvectors $\boldsymbol{r}_1^\pm, \dots, \boldsymbol{r}_d^\pm \in \mathbb{R}^d$ of $df(u^\pm) \in \mathcal{M}_d(\mathbb{R})$ associated with those eigenvalues. We also define the invertible matrix

$$\mathbf{P}^{\pm} := \left(\boldsymbol{r}_1^{\pm} \quad |\dots| \quad \boldsymbol{r}_d^{\pm}\right) \in \mathcal{M}_d(\mathbb{R}) \tag{4.1.4}$$

and the dual basis $\boldsymbol{l}_1^{\pm}, \dots, \boldsymbol{l}_d^{\pm} \in \mathbb{R}^d$ associated with the eigenvectors $\boldsymbol{r}_1^{\pm}, \dots, \boldsymbol{r}_d^{\pm}$:

$$\left(\boldsymbol{l}_1^{\pm} \quad |\dots| \quad \boldsymbol{l}_d^{\pm}\right)^T := \left(\mathbf{P}^{\pm}\right)^{-1}. \tag{4.1.5}$$

The vectors $\boldsymbol{l}_l^{\pm}$ are then eigenvectors of $df(u^{\pm})^T$ associated with the eigenvalues $\boldsymbol{\lambda}_l^{\pm}$. We organize the eigenvalues so that

$$\boldsymbol{\lambda}_1^{\pm} \leq \dots \leq \boldsymbol{\lambda}_d^{\pm}.$$

In this chapter, we focus our attention on Lax shocks.

Hypothesis 4.1 (Lax shock)**.** *We assume that $0 \notin \sigma(df(u^{\pm})) = \left\{\boldsymbol{\lambda}_1^{\pm}, \dots, \boldsymbol{\lambda}_d^{\pm}\right\}$ (i.e. the shock is non-characteristic). Furthermore, we assume that there exists an index $I \in \{1, \dots, d\}$ such that*

$$\begin{aligned} \boldsymbol{\lambda}_I^{+} < 0 < \boldsymbol{\lambda}_{I+1}^{+}, \\ \boldsymbol{\lambda}_{I-1}^{-} < 0 < \boldsymbol{\lambda}_I^{-}, \end{aligned}$$

where we use the convention $\boldsymbol{\lambda}_0^{\pm} := -\infty$ and $\boldsymbol{\lambda}_{d+1}^{\pm} := +\infty$ if $I = 1$ or $I = d$.

We consider a constant $\boldsymbol{\nu} > 0$ and introduce a space step $\Delta x > 0$ and a time step $\Delta t := \boldsymbol{\nu}\Delta x > 0$. The constant $\boldsymbol{\nu}$ then corresponds to the ratio between the space and time steps. We impose the following CFL condition on the constant $\boldsymbol{\nu}$: [1]

$$\forall u \in \mathcal{U}, \quad \boldsymbol{\nu} \min \sigma(df(u)) > -q \quad \text{and} \quad \boldsymbol{\nu} \max \sigma(df(u)) < p. \tag{4.1.6}$$

We introduce the discrete evolution operator $\mathcal{N} : \mathcal{U}^{\mathbb{Z}} \to (\mathbb{R}^d)^{\mathbb{Z}}$ defined for $u = (u_j)_{j\in\mathbb{Z}} \in \mathcal{U}^{\mathbb{Z}}$ as

$$\forall j \in \mathbb{Z}, \quad (\mathcal{N}(u))_j := u_j - \boldsymbol{\nu}\left(F\left(\boldsymbol{\nu}; u_{j-p+1}, \dots, u_{j+q}\right) - F\left(\boldsymbol{\nu}; u_{j-p}, \dots, u_{j+q-1}\right)\right), \tag{4.1.7}$$

where $p, q \in \mathbb{N}\backslash\{0\}$ and the numerical flux $F : (\nu; u_{-p}, \dots, u_{q-1}) \in]0, +\infty[\times \mathcal{U}^{p+q} \to \mathbb{R}^d$ is a $\mathcal{C}^\infty$ function. We are interested in solutions of the conservative one-step explicit finite difference scheme defined by

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n) \tag{4.1.8}$$

where $u^0 \in \mathcal{U}^{\mathbb{Z}}$.

We assume that the numerical scheme satisfies the following consistency condition with regards to the PDE (4.1.1)

$$\forall u \in \mathcal{U}, \quad F(\boldsymbol{\nu}; u, \dots, u) = f(u). \tag{4.1.9}$$

Traveling wave solutions of the numerical scheme (4.1.8) linking two end states of a shock wave are the so-called discrete shock profiles. Since we are considering stationary shocks (4.1.3) in the present chapter, the associated discrete shock profiles will also be stationary and will thus correspond to fixed points of the operator $\mathcal{N}$.

Hypothesis 4.2 (Existence of a stationary discrete shock profile (SDSP))**.** *We suppose that there exists a sequence $\overline{u}^s = (\overline{u}_j^s)_{j\in\mathbb{Z}} \in \mathcal{U}^{\mathbb{Z}}$ that satisfies*

$$\mathcal{N}(\overline{u}^s) = \overline{u}^s \quad \textit{and} \quad \overline{u}_j^s \underset{j\to\pm\infty}{\to} u^{\pm}.$$

Let us point out that in [Ser07], it is proved that the existence of a SDSP implies that the Rankine-Hugoniot condition (4.1.2) is verified. However, the existence of a SDSP for all admissible and physically significant standing shock is not fully answered. Existence results tend to actually prove the existence of a continuous one-parameter family of discrete shock profiles. The main results tackling the issue of existence of SDSP would be [MR79; Mic84; Jen74]:

— In [Jen74], Jennings focuses on discrete shock profiles for monotone conservative schemes applied to scalar conservation laws. In this context, he proves the existence and uniqueness of a continuous one-parameter family of discrete shock profiles associated with shocks of any strength for rational speeds. He also proves nonlinear orbital stability for such DSPs.

1. Up to considering that the space of state $\mathcal{U}$ is a close neighborhood of the SDSP defined underneath in Hypothesis 4.2, we should be able to satisfy such a condition.

— In [MR79], Majda and Ralston tackle the case of system of conservation laws and prove the existence of a continuous one-parameter family of DSPs with rational speeds. They introduce two limitations though: They consider schemes of order 1 (this corresponds to the case where $\mu = 1$ in Hypothesis 4.5 below) and they only consider weak shocks, i.e. shocks where the difference between the two states must be small enough. The result is generalized in [Mic84] for schemes of order 3 (i.e. $\mu = 2$ in Hypothesis 4.5 below).

The following assumption on the convergence of the SDSP $\overline{u}^s$ towards its limit state is important in our work as it it used to construct some of the main tools needed to carry out the analysis (for instance to prove the geometric dichotomy in Section 4.3.3 or for the proof of Lemma 4.4.3).

Hypothesis 4.3 (Exponential convergence of the SDSP towards its limit states)**.** *There exist some constants $C, c > 0$ such that*

$$\forall j \in \mathbb{N}, \qquad \begin{array}{c} |\overline{u}^s_j - u^+| \leq Ce^{-cj}, \\ |\overline{u}^s_{-j} - u^-| \leq Ce^{-cj}. \end{array} \tag{4.1.10}$$

Hypothesis 4.3 can most likely be proved to be a consequence of the shock being non-characteristic (Hypothesis 4.1). We refer to [ZH98, Corollary 1.2] for a proof of this fact in the continuous setting and [Bec+10, Lemma 1.1] in the semi-discrete case.

4.1.3 Linearized scheme about the end states $u^\pm$

Let us now introduce some hypotheses on the end states u^+ and u^- and on the considered numerical scheme. To summarize briefly the main assumptions, we mainly ask for the numerical schemes we consider to introduce numerical viscosity and to have linear ℓ^r-stability at the states u^+ and u^- for any $r \in [1, +\infty]$.

We linearize the discrete evolution operator $\mathcal{N}$ about the constant states u^- and u^+ and thus introduce the bounded operators $\mathscr{L}^\pm$ acting on $\ell^r(\mathbb{Z}, \mathbb{C}^d)$ with $r \in [1, +\infty]$ defined by

$$\forall h \in \ell^r(\mathbb{Z}, \mathbb{C}^d), \forall j \in \mathbb{Z}, \quad (\mathscr{L}^\pm h)_j := \sum_{k=-p}^{q} A_k^\pm h_{j+k}, \tag{4.1.11}$$

where for $k \in \{-p, \dots, q-1\}$, we first define

$$B_k^\pm := \boldsymbol{\nu} \partial_{u_k} F\left(\boldsymbol{\nu}; u^\pm, \dots, u^\pm\right) \in \mathcal{M}_d(\mathbb{C}) \tag{4.1.12}$$

and then for $k \in \{-p, \dots, q\}$, we let

$$A_k^\pm := \left\{ \begin{array}{cc} -B_{q-1}^\pm & \text{if } k = q, \\ B_{-p}^\pm & \text{if } k = -p, \\ \delta_{k,0} Id + B_k^\pm - B_{k-1}^\pm & \text{else.} \end{array} \right. \tag{4.1.13}$$

We start by introducing the following assumption on the matrices $B_k^\pm$ and $A_k^\pm$.

Hypothesis 4.4. *For $k \in \{-p, \dots, q-1\}$, the eigenvectors $\boldsymbol{r}_1^\pm, \dots, \boldsymbol{r}_d^\pm$ of $df(u^\pm)$ are also eigenvectors of the matrices $B_k^\pm$ defined by (4.1.12). If the system of conservation laws (4.1.1) is strictly hyperbolic at the states $u^\pm$, this is equivalent to the matrices $df(u^\pm)$ and $B_k^\pm$ commuting.*

Hypothesis 4.4 implies that the eigenvectors $\boldsymbol{r}_1^\pm, \dots, \boldsymbol{r}_d^\pm$ of $df(u^\pm)$ are also eigenvectors of the matrices $A_k^\pm$ defined by (4.1.13) for $k \in \{-p, \dots, q\}$. Hypothesis 4.4 is fairly usual and is not that far fetched since the consistency condition (4.1.9) links the numerical flux F and the flux f and that, most of the time, the matrices $B_k^\pm$ defined by (4.1.12) are expressed using $df(u^\pm)$. We can then introduce the notation for $k \in \{-p, \dots, q\}$:

$$\begin{pmatrix} \lambda_{1,k}^\pm & & \\ & \ddots & \\ & & \lambda_{d,k}^\pm \end{pmatrix} := {\mathbf{P}^\pm}^{-1} A_k^\pm \mathbf{P}^\pm. \tag{4.1.14}$$

For $l \in \{1, \dots, d\}$, we then define the meromorphic function $\mathcal{F}_l^\pm$ on $\mathbb{C} \backslash \{0\}$ by:

$$\forall \kappa \in \mathbb{C} \backslash \{0\}, \quad \mathcal{F}_l^\pm(\kappa) := \sum_{k=-p}^{q} \lambda_{l,k}^\pm \kappa^k \in \mathbb{C}. \tag{4.1.15}$$

The functions $\mathcal{F}_l^\pm$ allow us to characterize the spectrum of the operators $\mathscr{L}^\pm$. We refer for instance to [Tho65; DS14; RS15; CF22; Coe22] for a study in the scalar case of similar convolution operators as $\mathscr{L}^\pm$. Fourier analysis and in particular the well-known Wiener theorem [New75] imply that

$$\sigma(\mathscr{L}^{\pm}) = \bigcup_{\kappa\in\mathbb{S}^1} \sigma\left(\sum_{k=-p}^{q} \kappa^k A_k^{\pm}\right) = \bigcup_{l=1}^{d} \mathcal{F}_l^{\pm}(\mathbb{S}^1). \tag{4.1.16}$$

The definition (4.1.13) of the matrices $A_k^{\pm}$ and the consistency condition (4.1.9) imply that

$$\sum_{k=-p}^{q} A_k^{\pm} = Id \quad \text{and} \quad \sum_{k=-p}^{q} kA_k^{\pm} = -\boldsymbol{\nu} df(u^{\pm})$$

which translates into having:

$$\forall l \in \{1,\dots,d\}, \quad \mathcal{F}_l^{\pm}(1) = 1 \quad \text{and} \quad \alpha_l^{\pm} := -\mathcal{F}_l^{\pm\prime}(1) = \boldsymbol{\nu}\boldsymbol{\lambda}_l^{\pm} \neq 0 \tag{4.1.17}$$

with $\boldsymbol{\lambda}_l^{\pm}$ defined in Hypothesis 4.1.

The following assumption is linked to the linear ℓ^r-stability of the numerical scheme (4.1.8) at the end states which corresponds to the ℓ^r-power boundedness of the operators $\mathscr{L}^{\pm}$.

Hypothesis 4.5. *For all $l \in \{1,\dots,d\}$, we have:*

$$\forall \kappa \in \mathbb{S}^1 \backslash \{1\}, \quad |\mathcal{F}_l^{\pm}(\kappa)| < 1. \quad \textit{(Dissipativity condition)}$$

Moreover, we suppose that there exists an integer $\mu \in \mathbb{N}\backslash\{0\}$ and for all $l \in \{1,\dots,d\}$, there exists a complex number $\beta_l^{\pm}$ with positive real part such that:

$$\mathcal{F}_l^{\pm}(e^{i\xi}) \underset{\xi\to 0}{=} \exp(-i\alpha_l^{\pm}\xi - \beta_l^{\pm}\xi^{2\mu} + O(|\xi|^{2\mu+1})). \quad \textit{(Diffusivity condition)} \tag{4.1.18}$$

Asking for the ℓ^2-power boundedness of the operator $\mathscr{L}^{\pm}$ is equivalent to asking that $\sigma(\mathscr{L}^{\pm}) \subset \overline{\mathbb{D}}$ (Von Neumann condition). The stronger Hypothesis 4.5 is inspired by the fundamental contribution [Tho65] due to Thomée and has much further consequences, as the asymptotic expansion (4.1.18) assures the ℓ^r-power boundedness of the operator $\mathscr{L}^{\pm}$ for every r in $[1,+\infty]$ (see [Tho65, Theorem 1] which focuses in the scalar case on the ℓ^∞-power boundedness but also studies the ℓ^r-power boundedness as a consequence). The diffusivity condition (4.1.18) can be translated into asking for the numerical scheme $\mathcal{N}$ to introduce numerical viscosity at the end state $u^{\pm}$. Let us point out that this assumption implies that the results that we will prove won't apply for several schemes, in particular even ordered schemes like Lax-Wedroff scheme.

We conclude this section by defining the open set $\mathcal{O}$ which corresponds to the unbounded connected component of $\mathbb{C}\backslash(\sigma(\mathscr{L}^+)\cup\sigma(\mathscr{L}^-))$ represented on Figure 4.1. Hypothesis 4.5 implies that $\overline{\mathbb{U}}\backslash\{1\} \subset \mathcal{O}$.

4.1.4 Linearized scheme about the SDSP $\overline{u}^s$

We now linearize the discrete evolution operator $\mathcal{N}$ about the discrete shock profile $\overline{u}^s$ and thus define the bounded operator $\mathscr{L}$ acting on $\ell^r(\mathbb{Z},\mathbb{C}^d)$ with $r \in [1,+\infty]$ defined by

$$\forall h \in \ell^r(\mathbb{Z},\mathbb{C}^d), \forall j \in \mathbb{Z}, \quad (\mathscr{L}h)_j := \sum_{k=-p}^{q} A_{j,k} h_{j+k}, \tag{4.1.19}$$

where for $j \in \mathbb{Z}$ and $k \in \{-p,\dots,q-1\}$, we first define the matrix:

$$B_{j,k} := \boldsymbol{\nu}\partial_{u_k} F\left(\boldsymbol{\nu}; \overline{u}^s_{j-p},\dots,\overline{u}^s_{j+q-1}\right) \in \mathcal{M}_d(\mathbb{C}) \tag{4.1.20}$$

and for $j \in \mathbb{Z}$ and $k \in \{-p,\dots,q\}$:

$$A_{j,k} := \begin{cases} -B_{j+1,q-1} & \text{if } k = q, \\ B_{j,-p} & \text{if } k = -p, \\ \delta_{k,0} Id + B_{j,k} - B_{j+1,k-1} & \text{otherwise.} \end{cases} \tag{4.1.21}$$

We observe that since the SDSP $(\overline{u}^s_j)_{j\in\mathbb{Z}}$ converges exponentially fast towards its limit states $u^{\pm}$, we have that the matrices $A_{j,k}$ (resp. $B_{j,k}$) converge exponentially fast towards the matrices $A_k^{\pm}$ (resp. $B_k^{\pm}$) defined by (4.1.13) (resp. (4.1.12)) as j tends towards $\pm\infty$.

We will now focus on the spectral properties of the operator $\mathscr{L}$ when it acts on $\ell^2(\mathbb{Z},\mathbb{C}^d)$. The following proposition which localizes the essential spectrum of the operator $\mathscr{L}$ is central.

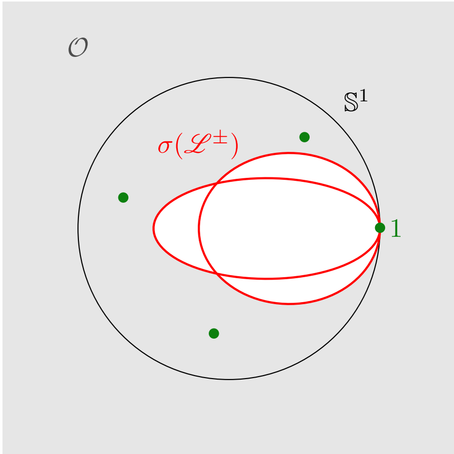

Figure 4.1 – In red, we have the spectrum of the operators $\mathscr{L}^{\pm}$ which corresponds to the union of the curves $\mathcal{F}_l^{\pm}(\mathbb{S}^1)$. Here we chose $d=1$, i.e. one curve corresponds to the spectrum of $\mathscr{L}^+$ and the other corresponds to the spectrum of $\mathscr{L}^-$. In gray, we represent the set $\mathcal{O}$ which corresponds to the unbounded component of $\mathbb{C}\backslash(\sigma(\mathscr{L}^+)\cup\sigma(\mathscr{L}^-))$. The elements of the set $\mathcal{O}$ are either eigenvalues of the operator $\mathscr{L}$ (represented in green) or belong to the resolvent set $\rho(\mathscr{L})$. We know that 1 is an eigenvalue of $\mathscr{L}$ and Hypothesis 4.6 implies that the eigenvalues of $\mathscr{L}$ in $\mathcal{O}$ are located within the open unit disk.

Proposition 4.1. *We have that*

$$\sigma_{ess}(\mathscr{L})\cap\mathcal{O}=\emptyset.$$

Proposition 4.1 allows us to conclude that for $z\in\mathcal{O}$, $zId_{\ell^2}-\mathscr{L}$ is a Fredholm operator of index 0 and thus that z either belongs to the resolvent set of $\mathscr{L}$ or is an eigenvalue of $\mathscr{L}$. Proposition 4.1 is proved for instance in [Ser07, Theorem 4.1] using the so-called geometric dichotomy developped in the thesis of Lafitte-Godillon [Laf01, Section III.1.5]. We will have to reintroduce the geometric dichotomy in Section 4.3 and we will thus provide the reader with the proof of Proposition 4.1 (see Lemma 4.3.8 hereafter).

We will now introduce the spectral stability assumption that we impose on our SDSP $\overline{u}^s$. It can be separated in two parts.

Hypothesis 4.6. *The operator $\mathscr{L}$ has no eigenvalue of modulus equal or larger than* 1 *other than* 1.

Combining Hypothesis 4.6 with Proposition 4.1, we can then conclude that the set $\overline{\mathbb{U}}\backslash\{1\}$ is included in the resolvent set of $\mathscr{L}$.

The second part of the spectral stability assumption and the last hypothesis we will introduce on the spectrum of the operator $\mathscr{L}$ has to do with the so-called Evans function Ev defined later on in the chapter by (4.4.16). This is a complex holomorphic function defined in a neighborhood of 1 that vanishes at the eigenvalues of $\mathscr{L}$. The Evans function plays the role of a characteristic polynomial for the operator $\mathscr{L}$.

We will show that under the previous hypotheses, 1 is an eigenvalue of the operator $\mathscr{L}$ and thus that the Evans function Ev vanishes at 1. This is the consequence of the existence of a differentiable one-parameter family of SDSPs associated with the one we are studying. In continuous and semi-discrete settings as in [ZH98; Bec+10], there is an underlying regular profile that describes the differentiable one-parameter family of traveling waves studied in those papers sd translations of said profile. The one-parameter family then corresponds to an invariance by translation. The derivative of the regular profile then belongs to the kernel of the linearized operator in these settings. In our present fully discrete setting, the existence of such a regular underlying profile that describes the different SDSPs of the one-parameter family is not clear.

Here, we will make a strong hypothesis on the behavior of the Evans function at 1.

Hypothesis 4.7. *We have that* 1 *is a simple zero of the Evans function* Ev *defined below by* (4.4.16)*, i.e.*

$$\mathrm{Ev}(1)=0 \quad \textit{and} \quad \frac{\partial\mathrm{Ev}}{\partial z}(1)\neq 0.$$

We will show that Hypothesis 4.7 implies that 1 is actually a simple eigenvalue of the operator $\mathscr{L}$. More precisely, we will define a sequence $V\in\ell^2(\mathbb{Z},\mathbb{C}^d)\backslash\{0\}$ below in (4.4.30) such that

$$\ker(Id_{\ell^2}-\mathscr{L})=\mathrm{Span}V \tag{4.1.22}$$

and we will prove that this sequence V converges exponentially fast towards 0 at infinity, i.e. there exist two positive constants C, c such that

$$\forall j \in \mathbb{Z}, \quad |V(j)| \leq Ce^{-c|j|}. \tag{4.1.23}$$

Coming back to the above discussion, if there exists a regular profile that allows us to describe the one-parameter family of SDSPs as this profile up to translations, then the sequence V and the derivative of the said profile would be collinear.

We finalize this section by introducing two last (technical) hypotheses.

Hypothesis 4.8. *The matrices $A_{j,-p} = B_{j,-p}$ and $A_{j,q} = -B_{j+1,q-1}$ are invertible for all $j \in \mathbb{Z}$ and the matrices $A^{\pm}_{-p} = B^{\pm}_{-p}$ and $A^{\pm}_{q} = -B^{\pm}_{q-1}$ are also invertible.*

This hypothesis is usually a consequence of the CFL condition (4.1.6). Hypothesis 4.8 serves us in the chapter to express the eigenvalue problem associated with the operators $\mathscr{L}$, $\mathscr{L}^+$ and $\mathscr{L}^-$ as dynamical systems (see Section 4.3.1 consecrated to the so-called "spatial dynamics"). Finally, we impose for the following assumption to be verified.

Hypothesis 4.9. *For all $l \in \{1, \dots, d\}$, the equation*

$$\mathcal{F}^{\pm}_{l}(\kappa) = 1 \tag{4.1.24}$$

has $p+q$ distinct solutions $\kappa \in \mathbb{C}\backslash\{0\}$.

Hypothesis 4.9 will be used to prove that the matrix $M^{\pm}_{l}(1)$ defined by (4.3.8) is diagonalizable with simple eigenvalues. This will allow us to study the eigenvalues and eigenvectors of the matrix $M^{\pm}(1)$ defined by (4.3.1) in Section 4.4. Let us observe that the expression (4.1.15) of $\mathcal{F}^{\pm}_{l}$ implies that searching for solutions $\kappa \in \mathbb{C}^{p+q}\backslash\{0\}$ of (4.1.24) is equivalent to searching for zeroes of the polynomial:

$$\kappa \in \mathbb{C}\backslash\{0\} \mapsto \kappa^p - \sum_{k=-p}^{q} \lambda^{\pm}_{l,k}\kappa^{k+p}.$$

The function above has $p+q$ zeroes counted with multiplicity and Hypothesis 4.9 is then just equivalent to asking for the zeroes of the above function to be simple.

4.1.5 Temporal and spatial Green's functions

For $j_0 \in \mathbb{Z}$, we define the temporal Green's function recursively as

$$\begin{gathered} \mathscr{G}(0, j_0, \cdot) := \delta_{j_0} \\ \forall n \in \mathbb{N}, \quad \mathscr{G}(n+1, j_0, \cdot) := \mathscr{L}\mathscr{G}(n, j_0, \cdot), \end{gathered} \tag{4.1.25}$$

where the sequence δ_{j_0} is defined by

$$\delta_{j_0} := (\delta_{j_0,j} Id)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathcal{M}_d(\mathbb{C})).$$

For $z \in \rho(\mathscr{L})$, we also define the spatial Green's function $G(z, j_0, \cdot)$ as the only element of $\ell^2(\mathbb{Z}, \mathcal{M}_d(\mathbb{C}))$ such that

$$(zId_{\ell^2} - \mathscr{L})G(z, j_0, \cdot) = \delta_{j_0}. \tag{4.1.26}$$

The main consequence of the introduction of the temporal Green's function is that for all $h \in \ell^r(\mathbb{Z}, \mathbb{C}^d)$ with $r \in [1, +\infty]$, we have

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad (\mathscr{L}^n h)_j = \sum_{j_0 \in \mathbb{Z}} \mathscr{G}(n, j_0, j) h_{j_0}. \tag{4.1.27}$$

Thus, a precise description of the temporal Green's function is sufficient to understand the action of the semi-group $(\mathscr{L}^n)_{n\in\mathbb{N}}$ associated with the operator $\mathscr{L}$.

The following lemma proved via a simple recurrence is a direct consequence of the definition (4.1.25) of the temporal Green's function and the finite speed propagation of the linearized scheme.

Lemma 4.1.1. *For all $n \in \mathbb{N}$, $j_0, j \in \mathbb{Z}$, we have that*

$$j - j_0 \notin \{-nq, \dots, np\} \Rightarrow \mathscr{G}(n, j_0, j) = 0.$$

Our goal is now to describe the pointwise asymptotic behavior of the temporal Green's function when $j - j_0 \in \{-nq, \dots, np\}$. Such a description of the Green's function for convolution operators has been the topic of the articles [DS14; RS15; CF22; Coe22]. We will thus try to use similar notations. For instance, we define the functions $H_{2\mu}, E_{2\mu} : \mathbb{R} \to \mathbb{C}$ such that for $\beta \in \mathbb{C}$ with positive real part, we have

$$\begin{aligned} \forall x \in \mathbb{R}, \quad & H_{2\mu}(\beta; x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixu} e^{-\beta u^{2\mu}} du, \\ \forall x \in \mathbb{R}, \quad & E_{2\mu}(\beta; x) := \int_x^{+\infty} H_{2\mu}(\beta; y) dy, \end{aligned} \tag{4.1.28}$$

where we recall that the integer μ is defined in Hypothesis 4.5. Let us point out that Lemma 4.1.2 implies that the function $E_{2\mu}$ is well-defined. We call the functions $H_{2\mu}$ generalized Gaussians and the functions $E_{2\mu}$ generalized Gaussian error functions since for $\mu = 1$, we have

$$\forall x \in \mathbb{R}, \quad H_2(\beta; x) = \frac{1}{\sqrt{4\pi\beta}} e^{-\frac{x^2}{4\beta}}.$$

Noticing that the function $H_{2\mu}$ is an even function and that it is the inverse Fourier transform of $u \mapsto e^{-\beta u^{2\mu}}$, we observe that:

$$\lim_{x \to -\infty} E_{2\mu}(\beta; x) = \int_{-\infty}^{+\infty} H_{2\mu}(\beta; y) dy = 1 \tag{4.1.29a}$$

$$\forall x \in \mathbb{R}, \quad E_{2\mu}(\beta, -x) = 1 - E_{2\mu}(\beta, x). \tag{4.1.29b}$$

The following lemma introduces some useful inequalities on the functions $H_{2\mu}$ and $E_{2\mu}$ defined by (4.1.28).

Lemma 4.1.2. *Let us consider a compact subset A of $\{\beta \in \mathbb{C}, \Re(\beta) > 0\}$ and integers $\mu, m \in \mathbb{N}\backslash\{0\}$. There exist two positive constants C, c such that for all $\beta \in A$*

$$\forall x \in \mathbb{R}, \quad |\partial_x^m H_{2\mu}(\beta; x)| \leq C \exp(-c|x|^{\frac{2\mu}{2\mu-1}}), \tag{4.1.30a}$$

$$\forall x \in]0, +\infty[, \quad |E_{2\mu}(\beta; x)| \leq C \exp(-c|x|^{\frac{2\mu}{2\mu-1}}), \tag{4.1.30b}$$

$$\forall x \in]-\infty, 0[, \quad |1 - E_{2\mu}(\beta; x)| \leq C \exp(-c|x|^{\frac{2\mu}{2\mu-1}}). \tag{4.1.30c}$$

The interested reader can find a proof of (4.1.30a) when the subset A is a one point set in [Coe22, Lemma 9] or in [Rob91, Proposition 5.3] for a more general point of view. By observing that the constants C, c constructed in those proofs depend continuously on β, we can then conclude the proof (4.1.30a) for general sets A. Inequalities (4.1.30b) and (4.1.30c) for the function $E_{2\mu}$ are directly deduced by integrating the function $H_{2\mu}$ and using (4.1.29a) and (4.1.30a).

We then introduce for $n \in \mathbb{N}\backslash\{0\}$ and $j, j_0 \in \mathbb{Z}$ the functions defined as follows:

— For $j_0 \geq 0$ and $l \in \{1, \dots, d\}$

$$S_l^+(n, j_0, j) := \mathbb{1}_{j \geq 0} \mathbb{1}_{\frac{j-j_0}{\alpha_l^+} \in [\frac{n}{2}, 2n]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+, \frac{n\alpha_l^+ + j_0 - j}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{r}_l^+ {\boldsymbol{l}_l^+}^T, \tag{4.1.31a}$$

— For $j_0 \geq 0$, $l' \in \{1, \dots, I\}$ and $l \in \{I+1, \dots d\}$ (i.e. such that $\alpha_{l'}^+ < 0$ and $\alpha_l^+ > 0$)

$$R_{l',l}^+(n, j_0, j) := \mathbb{1}_{j \geq 0} \mathbb{1}_{\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+} \in [\frac{n}{2}, 2n]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\frac{j}{n\alpha_l^+}\beta_l^+ - \frac{j_0}{n\alpha_{l'}^+}\beta_{l'}^+ \left(\frac{\alpha_l^+}{\alpha_{l'}^+}\right)^{2\mu}, \frac{n\alpha_l^+ + j_0\frac{\alpha_l^+}{\alpha_{l'}^+} - j}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{r}_l^+ {\boldsymbol{l}_{l'}^+}^T, \tag{4.1.31b}$$

— For $j_0 \geq 0$, $l' \in \{1, \dots, I\}$ and $l \in \{1, \dots I-1\}$ (i.e. such that $\alpha_{l'}^+ < 0$ and $\alpha_l^- < 0$)

$$T_{l',l}^+(n, j_0, j) := \mathbb{1}_{j < 0} \mathbb{1}_{\frac{j}{\alpha_l^-} - \frac{j_0}{\alpha_{l'}^+} \in [\frac{n}{2}, 2n]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\frac{j}{n\alpha_l^-}\beta_l^- - \frac{j_0}{n\alpha_{l'}^+}\beta_{l'}^+ \left(\frac{\alpha_l^-}{\alpha_{l'}^+}\right)^{2\mu}, \frac{n\alpha_l^- + j_0\frac{\alpha_l^-}{\alpha_{l'}^+} - j}{n^{\frac{1}{2\mu}}}\right) \boldsymbol{r}_l^- {\boldsymbol{l}_{l'}^+}^T, \tag{4.1.31c}$$

— For $j_0 \geq 0$ and $l' \in \{1, \dots, I\}$ (i.e. such that $\alpha_{l'}^+ < 0$)

$$E_{l'}^+(n, j_0) := E_{2\mu}\left(\beta_{l'}^+, \frac{n\alpha_{l'}^+ + j_0}{n^{\frac{1}{2\mu}}}\right) {\boldsymbol{l}_{l'}^+}^T, \tag{4.1.31d}$$

— For $j_0 < 0$ and $l \in \{1, \dots, d\}$

$$S_l^-(n, j_0, j) := \mathbb{1}_{j \leq 0} \mathbb{1}_{\frac{j-j_0}{\alpha_l^-} \in [\frac{n}{2}, 2n]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu} \left(\beta_l^-, \frac{n\alpha_l^- + j_0 - j}{n^{\frac{1}{2\mu}}} \right) \boldsymbol{r}_l^- \boldsymbol{l}_l^{-T}, \tag{4.1.31e}$$

— For $j_0 < 0$, $l' \in \{I, \dots, d\}$ and $l \in \{1, \dots I-1\}$ (i.e. such that $\alpha_{l'}^- > 0$ and $\alpha_l^- < 0$)

$$R_{l',l}^-(n, j_0, j) := \mathbb{1}_{j \leq 0} \mathbb{1}_{\frac{j}{\alpha_l^-} - \frac{j_0}{\alpha_{l'}^-} \in [\frac{n}{2}, 2n]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu} \left(\frac{j}{n\alpha_l^-} \beta_l^- - \frac{j_0}{n\alpha_{l'}^-} \beta_{l'}^- \left(\frac{\alpha_l^-}{\alpha_{l'}^-} \right)^{2\mu}, \frac{n\alpha_l^- + j_0 \frac{\alpha_l^-}{\alpha_{l'}^-} - j}{n^{\frac{1}{2\mu}}} \right) \boldsymbol{r}_l^- \boldsymbol{l}_{l'}^{-T}, \tag{4.1.31f}$$

— For $j_0 < 0$, $l' \in \{I, \dots, d\}$ and $l \in \{I+1, \dots d\}$ (i.e. such that $\alpha_{l'}^- > 0$ and $\alpha_l^+ > 0$)

$$T_{l',l}^-(n, j_0, j) := \mathbb{1}_{j > 0} \mathbb{1}_{\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^-} \in [\frac{n}{2}, 2n]} \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu} \left(\frac{j}{n\alpha_l^+} \beta_l^+ - \frac{j_0}{n\alpha_{l'}^-} \beta_{l'}^- \left(\frac{\alpha_l^+}{\alpha_{l'}^-} \right)^{2\mu}, \frac{n\alpha_l^+ + j_0 \frac{\alpha_l^+}{\alpha_{l'}^-} - j}{n^{\frac{1}{2\mu}}} \right) \boldsymbol{r}_l^+ \boldsymbol{l}_{l'}^{-T}, \tag{4.1.31g}$$

— For $j_0 < 0$, $l' \in \{I, \dots, d\}$ (i.e. such that $\alpha_{l'}^- > 0$)

$$E_{l'}^-(n, j_0) := E_{2\mu} \left(\beta_{l'}^-, \frac{-n\alpha_{l'}^- - j_0}{n^{\frac{1}{2\mu}}} \right) \boldsymbol{l}_{l'}^{-T}. \tag{4.1.31h}$$

We observe that the functions $S_l^\pm$, $R_{l',l}^\pm$ and $T_{l',l}^\pm$ are valued in $\mathcal{M}_d(\mathbb{C})$ and the functions $E_{l'}^\pm$ are valued in $\mathcal{M}_{1,d}(\mathbb{C})$. Furthermore, the functions introduced above describe different behaviors that will be observed for the temporal Green's function. To be more precise, the main theorem of this chapter is the following description of the temporal Green's function. We recall that the Landau notations O, $O_{\mathcal{M}_{1,d}}$ and $O_\mathbb{C}$ are described in the notations section at the beginning of the chapter and are used respectively to differentiate matrices, line vectors and scalars.

Theorem 4.1. *Let us assume that Hypotheses 4.1-4.9 are verified. Then, there exist:*

— *A sequence $V \in \ell^2(\mathbb{Z}, \mathbb{C}^d) \backslash \{0\}$ (defined below by (4.4.30)) satisfying (4.1.22) and (4.1.23),*

— *Families of complex scalars $(C_{l'}^{E,\pm})_{l'}$, $(C_{l',l}^{R,\pm})_{l',l}$ and $(C_{l',l}^{T,\pm})_{l',l}$,*

— *Families of complex vectors $(P_U^+(j_0))_{j_0 \in \mathbb{N}} \in \mathcal{M}_{1,d}(\mathbb{C})^\mathbb{N}$ and $(P_U^-(j_0))_{j_0 \in -\mathbb{N} \backslash \{0\}} \in \mathcal{M}_{1,d}(\mathbb{C})^{-\mathbb{N} \backslash \{0\}}$ which verify that there exist two positive constants C, c such that:*

$$\begin{aligned} \forall j_0 \in \mathbb{N}, \quad & \left| P_U^+(j_0) \right| \leq C e^{-c|j_0|}, \\ \forall j_0 \in -\mathbb{N} \backslash \{0\}, \quad & \left| P_U^-(j_0) \right| \leq C e^{-c|j_0|}, \end{aligned}$$

such that for all $n \in \mathbb{N} \backslash \{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$ which verify $j - j_0 \in \{-nq, \dots, np\}$, we have that:

$$\begin{aligned} \mathscr{G}(n, j_0, j) = \sum_{l=1}^{d} S_l^+(n, j_0, j) + \sum_{l'=1}^{I} \left[\sum_{l=I+1}^{d} C_{l',l}^{R,+} R_{l',l}^+(n, j_0, j) + \sum_{l=1}^{I-1} C_{l',l}^{T,+} T_{l',l}^+(n, j_0, j) \right] \\ + V(j) \left[\sum_{l'=1}^{I} C_{l'}^{E,+} E_{l'}^+(n, j_0) + P_U^+(j_0) \right] + \mathcal{R}(n, j_0, j) \end{aligned} \tag{4.1.32}$$

where $\mathcal{R}(n, j_0, j)$ is a faster decaying residual of the following form for some constant $c > 0$ independent from n, j_0, j:

- *For $j \geq 0$ such that $j - j_0 \in \{-nq, \dots, np\}$:*

$$\begin{aligned} &\mathcal{R}(n, j_0, j) = O(e^{-cn}) \\ &+ \sum_{l=1}^{d} \exp \left(-c \left(\frac{\left| n - \left(\frac{j-j_0}{\alpha_l^+} \right) \right|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) \left(O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}} \right) + \boldsymbol{r}_l^+ O_{\mathcal{M}_{1,d}} \left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}} \right) + O_\mathbb{C} \left(\frac{1}{n^{\frac{1}{\mu}}} \right) \boldsymbol{r}_l^+ \boldsymbol{l}_l^{+T} \right) \end{aligned}$$

$$+\sum_{l'=1}^{I}\sum_{l=I+1}^{d}\exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+\boldsymbol{r}_l^+O_{\mathcal{M}_{1,d}}\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\boldsymbol{r}_l^+\boldsymbol{l}_{l'}^{+T}\right)$$

$$+\sum_{l'=1}^{I}O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)+\sum_{l=I+1}^{d}\boldsymbol{r}_l^+O_{\mathcal{M}_{1,d}}\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right),$$

- *For $j<0$ such that $j-j_0\in\{-nq,\dots,np\}$:*

$$\mathcal{R}(n,j_0,j)=O(e^{-cn})$$

$$+\sum_{l'=1}^{I}\sum_{l=1}^{I-1}\exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^-}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+\boldsymbol{r}_l^-O_{\mathcal{M}_{1,d}}\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\boldsymbol{r}_l^-\boldsymbol{l}_{l'}^{+T}\right)$$

$$+\sum_{l'=1}^{I}O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)+\sum_{l=1}^{I-1}\boldsymbol{r}_l^-O_{\mathcal{M}_{1,d}}\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j}{\alpha_l^-}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right).$$

The coefficients $C_{l',l}^{R,+},C_{l',l}^{T,+},C_{l'}^{E,+}$ and the sequence P_U^+ are defined by (4.5.29). There is a similar result when $j_0\in-\mathbb{N}\backslash\{0\}$ using the families of complex scalars $(C_{l'}^{E,-})_{l'}$, $(C_{l',l}^{R,-})_{l',l}$ and $(C_{l',l}^{T,-})_{l',l}$, the vectors P_U^- and the functions $E_{l'}^-$, S_l^-, $R_{l',l}^-$ and $T_{l',l}^-$.

Furthermore, we can choose the constant $c>0$ such that the discrete derivative with regards to the parameter j_0 of the temporal Green's function verifies that, for $j_0\in\mathbb{N}$:

- *For $j\geq 0$ such that $j-j_0\in\{-nq-1,\dots,np\}$:*

$$\mathscr{G}(n,j_0,j)-\mathscr{G}(n,j_0-1,j)=V(j)(P_U^+(j_0)-P_U^+(j_0-1))+O(e^{-cn})$$

$$+\sum_{l=1}^{d}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+\boldsymbol{r}_l^+O_{\mathcal{M}_{1,d}}\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\boldsymbol{r}_l^+\boldsymbol{l}_l^{+T}\right)$$

$$+\sum_{l'=1}^{I}\sum_{l=I+1}^{d}\exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+\boldsymbol{r}_l^+O_{\mathcal{M}_{1,d}}\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\boldsymbol{r}_l^+\boldsymbol{l}_{l'}^{+T}\right)$$

$$+\sum_{l'=1}^{I}O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)+\sum_{l=I+1}^{d}\boldsymbol{r}_l^+O_{\mathcal{M}_{1,d}}\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right),\tag{4.1.33a}$$

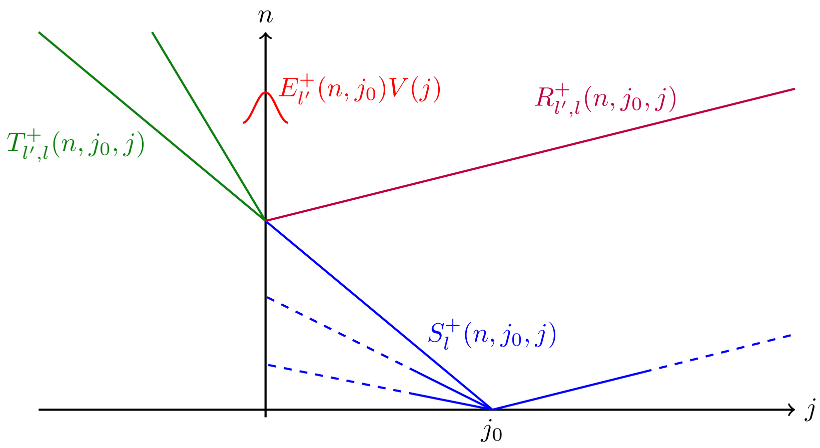

Figure 4.2 – A schematic representation of the result (4.1.32) of Theorem 4.1 on the Green's function $\mathscr{G}(n,j_0,j)$. Here we represent a case where $j_0 \in \mathbb{N}$, $d=4$ and $I=3$. We recall that the integer I is defined by Hypothesis 4.1. We have d generalized Gaussian waves (in blue) arising from the Dirac mass at j_0 which travel along the characteristics of the right state u^+. The ones reaching the shock location are decomposed into reflected waves (in purple), transmitted waves (in green) and the activation of the component of the Green's function along the vector space $\ker(Id_{\ell^2}-\mathscr{L})$ (in red). We only represent this decomposition for one of the incoming waves.

- *For $j<0$ such that $j-j_0 \in \{-nq-1,\dots,np\}$:*

$$\begin{aligned}
&\mathscr{G}(n,j_0,j)-\mathscr{G}(n,j_0-1,j) = V(j)(P_U^+(j_0)-P_U^+(j_0-1)) + O(e^{-cn})\\
&+\sum_{l'=1}^{I}\sum_{l=1}^{I-1}\exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^-}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+\boldsymbol{r}_l^- O_{\mathcal{M}_{1,d}}\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\boldsymbol{r}_l^-\boldsymbol{l}_{l'}^{+T}\right)\\
&+\sum_{l'=1}^{I} O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)+\sum_{l=1}^{I-1}\boldsymbol{r}_l^- O_{\mathcal{M}_{1,d}}\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j}{\alpha_l^-}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right).
\end{aligned} \tag{4.1.33b}$$

A similar description of the discrete derivative of the temporal Green's function holds for $j_0 \in -\mathbb{N}\backslash\{0\}$.

Let us describe more clearly what the result (4.1.32) of Theorem 4.1 conveys for $j_0 \geq 0$. The same description can be done for $j_0<0$ and the result (4.1.33) on the discrete derivative of the Green's function would be interpreted in a similar way. Figure 4.2 is a schematic representation of the result (4.1.32) of Theorem 4.1. The first term on the right hand-side using the function S_l^+ of (4.1.32) corresponds to d generalized Gaussian waves arising from the Dirac mass at j_0 which travel along the characteristics of the right state u^+ (the blue waves in Figure 4.2). The generalized Gaussian behavior of the different waves originates from the smearing effect caused by the diffusivity condition in Hypothesis 4.5 which corresponds to the introduction of artificial viscosity at the states $u^\pm$. These waves correspond to the leading order of the Green's function of the operator $\mathscr{L}^+$ associated with the "right" state u^+. Recalling that we are considering a Lax shock under Hypothesis 4.1, we observe the following distinction:

— The first I generalized Gaussian waves follow the characteristics incoming the shock since $\alpha_l^+<0$ for $l\in\{1,\dots,I\}$ and will eventually reach the shock (located at $j=0$).

— The last $d-I$ generalized Gaussian waves follow the outgoing characteristics with respect to the shock since $\alpha_l^+>0$ for $l\in\{I+1,\dots,d\}$ and travel towards $+\infty$ without interaction with the shock.

When the generalized Gaussian waves following incoming characteristics reach the shock location, we observe that they lead to different behaviors for three new types of waves:

— There are reflected generalized Gaussian waves along the outgoing characteristics of the state u^+ (i.e. the purple waves in Figure 4.2). It corresponds to the second term using the function $R^+_{l',l}$ in (4.1.32). This explains the restriction of $l\in\{I+1,\dots,d\}$ which implies that $\alpha_l^+>0$.

— There are transmitted generalized Gaussian waves along the outgoing characteristics of the state u^- (i.e.

the green waves in Figure 4.2). It corresponds to the third term using the function $T^+_{l',l}$ in (4.1.32). This explains the restriction of $l \in \{1, \dots, I-1\}$ which implies that $\alpha_l^- < 0$.

— Because of the properties of the function $E_{2\mu}$ defined by (4.1.28), we have that the vectors $E^+_{l'}(n, j_0)$ are close to 0 for small times n and converge towards ${\boldsymbol{l}^+_{l'}}^T$ as n tends towards $+\infty$. Thus, the last term in the decomposition (4.1.32) could be described as the progressive construction of the component of the Green's function along the vector subspace $\ker(Id_{\ell^2} - \mathscr{L})$ (i.e. the red wave in Figure 4.2). Each wave with speed $\alpha^+_{l'}$ activates part of this profile as they reach the shock. Since V is an eigenvector of $\mathscr{L}$ for the eigenvalue 1, this part of the Green's function does not decay as n tends towards $+\infty$.

One of the main consequences that can be deduced from Theorem 4.1 corresponds to the so-called linear orbital stability of the stationary discrete shock profile $\overline{u}^s$.

Theorem 4.2. *Let us assume that Hypotheses 4.1-4.9 are verified. For $r_1, r_2 \in [1, +\infty]$ such that $r_1 \leq r_2$, there exists positive constant C such that*

$$\forall h \in \ell^{r_1}(\mathbb{Z}, \mathbb{C}^d), \forall n \in \mathbb{N}, \qquad \min_{V \in \ker(Id_{\ell^2} - \mathscr{L})} \|\mathscr{L}^n h - V\|_{\ell^{r_2}} \leq \frac{C}{n^{\frac{1}{2\mu}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}} \|h\|_{\ell^{r_1}}.$$

Let us point out that such decay estimates are similar to the one associated with the heat equation for instance. This is a consequence of the diffusive nature of the numerical scheme introduced in Hypothesis 4.5.

4.1.6 Plan of the chapter

Firstly, Section 4.2 will be dedicated to the proof of Theorem 4.2 using Theorem 4.1. The main part of the chapter however (from Sections 4.3 to 4.5) will concern the proof of Theorem 4.1:

— In Section 4.3, we will prove Proposition 4.1 which describes the spectrum of the operator $\mathscr{L}$ in the set $\mathcal{O}$ and will allow to define the spatial Green's function defined by (4.1.26) on $\overline{\mathbb{U}} \setminus \{1\}$. We will then prove Proposition 4.2 which implies exponential bounds on the spatial Green's function in the neighborhood of any point of $\overline{\mathbb{U}} \setminus \{1\}$.

— In Section 4.4, we prove that Proposition 4.3 which claims that the spatial Green's function can be meromorphically extended in a neighborhood of 1 through the essential spectrum of the operator $\mathscr{L}$. We will show that it has a pole of order 1 at $z = 1$ and find precise expressions (4.4.34a)-(4.4.34c) on it that will be essential in Section 4.5.

— In Section 4.5, we express the temporal Green's function defined by (4.1.25) with the spatial Green's function using the inverse Laplace Transform. Using the different results proved on the spatial Green's function in Sections 4.3 and 4.4, we will conclude the proof of Theorem 4.1.

Section 4.6 will be dedicated to an example. We will first show that some of the previously presented hypotheses are verified for the modified Lax-Friedrichs scheme and then present a numerical example of the result of Theorem 4.1 when the conservation law we consider is the Burgers equation. Section 4.A is the Appendix and contains the proof of some technical lemmas used throughout the chapter.

The author feels like it is important to point out that the proofs of some lemmas in the present chapter are done in other papers. However, the author feels like they needed to be reproved either to correct mistakes or because the way they are presented here is fairly different from the statement in other papers. For instance, the geometric dichotomy (Lemma 4.3.4 below) uses the same proof with slight variations to [Laf01, Section III.1.5] where it is first introduced.

4.2 Proof of linear stability (Theorem 4.2)

The goal of this section is to prove Theorem 4.2 using the description (4.1.32) of the temporal Green's function obtained in Theorem 4.1. We recall that for $r_1, r_2 \in [1, +\infty]$ such that $r_1 \leq r_2$, we have that for all $n \in \mathbb{N}$ and $h \in \ell^{r_1}(\mathbb{Z}, \mathbb{C}^d)$:

$$\mathscr{L}^n h = \left(\sum_{j_0 \in \mathbb{Z}} \mathscr{G}(n, j_0, j) h_{j_0} \right)_{j \in \mathbb{Z}} \in \ell^{r_2}(\mathbb{Z}, \mathbb{C}^d). \tag{4.2.1}$$

__**Step 1:**__ Let us prove that for all integers $l \in \{1, \dots, d\}$ and for all couples $r_1, r_2 \in [1, +\infty]$ such that $r_1 \leq r_2$, the operators

$$\begin{array}{ccc} \ell^{r_1}(\mathbb{Z}, \mathbb{C}^d) & \to & \ell^{r_2}(\mathbb{Z}, \mathbb{C}^d) \\ h & \mapsto & \left(\sum_{j_0 \in \mathbb{N}} \mathbb{1}_{j - j_0 \in \{-nq, \dots, np\}} S_l^+(n, j_0, j) h_{j_0} \right)_{j \in \mathbb{Z}} \end{array} \tag{4.2.2}$$

for $n \in \mathbb{N}\backslash\{0\}$ are well-defined and that there exists a positive constants C such that

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^{r_1}(\mathbb{Z},\mathbb{C}^d), \quad \left\| \left(\sum_{j_0 \in \mathbb{N}} \mathbb{1}_{j-j_0 \in \{-nq,\dots,np\}} S_l^+(n,j_0,j) h_{j_0} \right)_{j\in\mathbb{Z}} \right\|_{\ell^{r_2}} \leq \frac{C}{n^{\frac{1}{2\mu}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}} \|h\|_{l^{r_1}}. \tag{4.2.3}$$

We fix an integer $l \in \{1,\dots,d\}$. Using (4.1.30a), we have that there exist two positive constants C, c such that

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^\infty(\mathbb{Z},\mathbb{C}^d), \forall j \in \mathbb{Z},$$
$$\left| \sum_{j_0\in\mathbb{N}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} S_l^+(n,j_0,j)h_{j_0} \right| \leq \sum_{j_0\in\mathbb{N}} \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{|n\alpha_l^+ + j_0 - j|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) |h_{j_0}|. \tag{4.2.4}$$

We also observe that there exists a constant $\tilde{C} > 0$ such that

$$\forall n \in \mathbb{N}\backslash\{0\}, \quad \sum_{j\in\mathbb{Z}} \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{|n\alpha_l^+ - j|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) \leq \tilde{C}. \tag{4.2.5}$$

Let us now prove that, for different choices of couples r_1, r_2, the operator defined by (4.2.2) is well-defined and that (4.2.3) is verified.

— Using (4.2.4) and (4.2.5), we have that:

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^\infty(\mathbb{Z},\mathbb{C}^d), \forall j \in \mathbb{Z}, \quad \left| \sum_{j_0\in\mathbb{N}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} S_l^+(n,j_0,j)h_{j_0} \right| \leq \tilde{C} \|h\|_{\ell^\infty}.$$

Thus, for $r_1, r_2 = +\infty$, the operator defined by (4.2.2) is well-defined and (4.2.3) is verified.

— Using (4.2.4), we have that:

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^1(\mathbb{Z},\mathbb{C}^d), \forall j \in \mathbb{Z}, \quad \left| \sum_{j_0\in\mathbb{N}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} S_l^+(n,j_0,j)h_{j_0} \right| \leq \frac{C}{n^{\frac{1}{2\mu}}} \|h\|_{\ell^1}.$$

Thus, for $r_1 = 1$ and $r_2 = +\infty$, the operator defined by (4.2.2) is well-defined and (4.2.3) is verified.

— Using (4.2.4), Fubini-Tonelli's theorem and (4.2.5), we have that for $n \in \mathbb{N}\backslash\{0\}$ and $h \in \ell^1(\mathbb{Z},\mathbb{C}^d)$:

$$\sum_{j\in\mathbb{Z}} \left| \sum_{j_0\in\mathbb{N}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} S_l^+(n,j_0,j)h_{j_0} \right|$$
$$\leq \sum_{j_0\in\mathbb{N}} \left(\sum_{j\in\mathbb{Z}} \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{|n\alpha_l^+ + j_0 - j|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) \right) |h_{j_0}| \leq \tilde{C} \|h\|_{\ell^1}.$$

Thus, for $r_1 = 1$ and $r_2 = 1$, the operator defined by (4.2.2) is well-defined and (4.2.3) is verified.

— Up until now, we have proved that for all couple $(r_1, r_2) \in \{(1,1), (1,+\infty), (+\infty,+\infty)\}$ the operator defined by (4.2.2) is well-defined and that (4.2.3) is verified. Using Riesz-Thorin Theorem, we can then conclude that, for all couple $(r_1, r_2) \in [1,+\infty]^2$ such that $r_1 \leq r_2$, the operator defined by (4.2.2) is well-defined and that (4.2.3) is verified.

Step 2: We claim that, using an identical proof as in Step 1, for all couple $r_1, r_2 \in [1,+\infty]$ such that $r_1 \leq r_2$, the operators from $\ell^{r_1}(\mathbb{Z},\mathbb{C}^d)$ to $\ell^{r_2}(\mathbb{Z},\mathbb{C}^d)$ defined by

$$\begin{aligned}
&h\mapsto\left(\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}R^+_{l',l}(n,j_0,j)h_{j_0}\right)_{j\in\mathbb{Z}} && h\mapsto\left(\sum_{j_0<0}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}R^-_{l',l}(n,j_0,j)h_{j_0}\right)_{j\in\mathbb{Z}}\\
&h\mapsto\left(\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}T^+_{l',l}(n,j_0,j)h_{j_0}\right)_{j\in\mathbb{Z}} && h\mapsto\left(\sum_{j_0<0}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}T^-_{l',l}(n,j_0,j)h_{j_0}\right)_{j\in\mathbb{Z}}\\
&h\mapsto\left(\sum_{j_0<0}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}S^-_{l}(n,j_0,j)h_{j_0}\right)_{j\in\mathbb{Z}} && h\in\mapsto\left(\sum_{j_0\in\mathbb{Z}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\mathcal{R}(n,j_0,j)h_{j_0}\right)_{j\in\mathbb{Z}}
\end{aligned}\tag{4.2.6}$$

for $n\in\mathbb{N}\backslash\{0\}$ are well-defined and satisfy similar bounds as (4.2.3). Let us now introduce the linear operators Ψ_n from $\ell^{r_1}(\mathbb{Z},\mathbb{C}^d)$ to $\ell^{r_2}(\mathbb{Z},\mathbb{C}^d)$ defined by:

$$\begin{aligned}
&\forall n\in\mathbb{N}\backslash\{0\},\forall h\in\ell^{r_1}(\mathbb{Z},\mathbb{C}^d),\forall j\in\mathbb{Z},\\
&\qquad(\Psi_n h)_j:=\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\left(\left(\sum_{l'=1}^{I}C^{E^+}_{l'}E^+_{l'}(n,j_0)+P^+_U(j_0)\right)h_{j_0}\right)V(j)\\
&\qquad\qquad+\sum_{j_0<0}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\left(\left(\sum_{l'=I}^{d}C^{E^-}_{l'}E^-_{l'}(n,j_0)+P^-_U(j_0)\right)h_{j_0}\right)V(j).
\end{aligned}\tag{4.2.7}$$

We recall that the Green's function $\mathscr{G}(n,j_0,j)$ is equal to 0 when $j-j_0\notin\{-nq,\dots,np\}$. Thus, using the equality (4.2.1) and the decomposition (4.1.32) of the Green's function $\mathscr{G}(n,j_0,j)$ when $j-j_0\in\{-nq,\dots,np\}$ given by Theorem 4.1, the previous results (4.2.5) we proved on the operators (4.2.4) and (4.2.6) allow us to prove that for all $r_1,r_2\in[1,+\infty]$ such that $r_1\leq r_2$, we have that there exists a positive constant C such that:

$$\forall h\in\ell^{r_1}(\mathbb{Z},\mathbb{C}^d),\forall n\in\mathbb{N}\backslash\{0\},\quad\|\mathscr{L}^n h-\Psi_n h\|_{\ell^{r_2}}\leq\frac{C}{n^{\frac{1}{2\mu}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}}\|h\|_{\ell^{r_1}}\tag{4.2.8}$$

Step 3: Let us prove that for all integers $l'\in\{1,\dots,I\}$, the operators

$$\begin{array}{ccc}\ell^\infty(\mathbb{Z},\mathbb{C}^d)&\to&\ell^1(\mathbb{Z},\mathbb{C}^d)\\ h&\mapsto&\left(\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\notin\{-nq,\dots,np\}}C^{E,+}_{l'}E^+_{l'}(n,j_0)h_{j_0}V(j)\right)_{j\in\mathbb{Z}}\end{array}\tag{4.2.9}$$

for $n\in\mathbb{N}\backslash\{0\}$ are well-defined and that there exist two positive constants C,c such that

$$\forall n\in\mathbb{N}\backslash\{0\},\forall h\in\ell^\infty(\mathbb{Z},\mathbb{C}^d),\quad\left\|\left(\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\notin\{-nq,\dots,np\}}C^{E,+}_{l'}E^+_{l'}(n,j_0)h_{j_0}V(j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1}\leq Ce^{-cn}\|h\|_{\ell^\infty}.\tag{4.2.10}$$

We recall that (4.1.23) implies that there exist two positive constants C_V,c_V such that:

$$\forall j\in\mathbb{Z},\quad|V(j)|\leq C_Ve^{-c_V|j|}.\tag{4.2.11}$$

Furthermore, for $l'\in\{1,\dots,I\}$, the CFL condition (4.1.6) implies that

$$0<-\alpha^+_{l'}<\frac{q-\alpha^+_{l'}}{2}<q.\tag{4.2.12}$$

We fix an integer $l'\in\{1,\dots,I\}$. For $h\in\ell^\infty(\mathbb{Z},\mathbb{C}^d)$, $n\in\mathbb{N}\backslash\{0\}$ and $j\in\mathbb{Z}$, we have that:

$$\sum_{j_0\in\mathbb{N}}\left|\mathbb{1}_{j-j_0\notin\{-nq,\dots,np\}}C^{E,+}_{l'}E^+_{l'}(n,j_0)h_{j_0}V(j)\right|$$

$$\leq \left|C_{l'}^{E,+}\right| C_V \left[\sum_{j_0 \in \left[0, n\frac{q-\alpha_{l'}^+}{2}\right] \cap \mathbb{N}} \mathbb{1}_{j-j_0 \notin \{-nq,\dots,np\}} e^{-c_V|j|} |E_{l'}^+(n,j_0)||h_{j_0}| \right.$$

$$\left. + \sum_{j_0 \in \left[n\frac{q-\alpha_{l'}^+}{2}, +\infty\right[\cap \mathbb{N}} \mathbb{1}_{j-j_0 \notin \{-nq,\dots,np\}} e^{-c_V|j|} |E_{l'}^+(n,j_0)||h_{j_0}| \right]. \quad (4.2.13)$$

— Using (4.2.12), we observe that there exists a positive constant $c > 0$ such that for all $j_0 \in \left[0, n\frac{q-\alpha_{l'}^+}{2}\right] \cap \mathbb{N}$ and $j \in \mathbb{Z}$ which verify $j - j_0 \notin \{-nq, \dots, np\}$, then:

$$|j| \geq cn.$$

Therefore, since the function $E_{2\mu}$ is bounded, we have that there exists a positive constant C such that for all $h \in \ell^\infty(\mathbb{Z}, \mathbb{C}^d)$ and $n \in \mathbb{N}\backslash\{0\}$:

$$\sum_{j \in \mathbb{Z}} \sum_{j_0 \in \left[0, \frac{q-\alpha_{l'}^+}{2} n\right] \cap \mathbb{N}} \mathbb{1}_{j-j_0 \notin \{-nq,\dots,np\}} e^{-c_V|j|} |E_{l'}^+(n,j_0)||h_{j_0}| \leq C \left(\sum_{j \in \mathbb{Z}} e^{-\frac{c_V}{2}|j|} \right) n e^{-\frac{c_V}{2} cn} \|h\|_{\ell^\infty}.$$

Therefore, there exist two new positive constants C, c such that for all $h \in \ell^\infty(\mathbb{Z}, \mathbb{C}^d)$ and $n \in \mathbb{N}\backslash\{0\}$:

$$\sum_{j \in \mathbb{Z}} \sum_{j_0 \in \left[0, \frac{q-\alpha_{l'}^+}{2} n\right] \cap \mathbb{N}} \mathbb{1}_{j-j_0 \notin \{-nq,\dots,np\}} e^{-c_V|j|} |E_{l'}^+(n,j_0)||h_{j_0}| \leq Ce^{-cn} \|h\|_{\ell^\infty}. \quad (4.2.14)$$

— Using (4.1.30b), there exist two positive constants C, c such that for all $h \in \ell^\infty(\mathbb{Z}, \mathbb{C}^d)$ and $n \in \mathbb{N}\backslash\{0\}$:

$$\sum_{j \in \mathbb{Z}} \sum_{j_0 \in \left[\frac{q-\alpha_{l'}^+}{2} n, +\infty\right[\cap \mathbb{N}} \mathbb{1}_{j-j_0 \notin \{-nq,\dots,np\}} e^{-c_V|j|} |E_{l'}^+(n,j_0)||h_{j_0}|$$

$$\leq C \|h\|_{\ell^\infty} \left(\sum_{j \in \mathbb{Z}} e^{-c_V|j|} \right) \sum_{j_0 \in \left[\frac{q-\alpha_{l'}^+}{2} n, +\infty\right[\cap \mathbb{N}} \exp\left(-c \left(\frac{|n\alpha_{l'}^+ + j_0|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right).$$

Therefore, using (4.2.12), there exist two new positive constants C, c such that for all $h \in \ell^\infty(\mathbb{Z}, \mathbb{C}^d)$ and $n \in \mathbb{N}\backslash\{0\}$:

$$\sum_{j \in \mathbb{Z}} \sum_{j_0 \in \left[\frac{q-\alpha_{l'}^+}{2} n, +\infty\right[\cap \mathbb{N}} \mathbb{1}_{j-j_0 \notin \{-nq,\dots,np\}} e^{-c_V|j|} |E_{l'}^+(n,j_0)||h_{j_0}| \leq Ce^{-cn} \|h\|_{\ell^\infty}. \quad (4.2.15)$$

Thus, combining (4.2.13), (4.2.14) and (4.2.15), we can conclude the proof of (4.2.10).
Step 4: Using a similar proof as in Step 3 for the operators from $\ell^\infty(\mathbb{Z}, \mathbb{C}^d)$ to $\ell^1(\mathbb{Z}, \mathbb{C}^d)$ defined by:

$$h \mapsto \left(\sum_{j_0 < 0} \mathbb{1}_{j-j_0 \notin \{-nq,\dots,np\}} C_{l'}^{E,-} E_{l'}^-(n,j_0) h_{j_0} V(j) \right)_{j \in \mathbb{Z}},$$

$$h \mapsto \left(\sum_{j_0 \in \mathbb{N}} \mathbb{1}_{j-j_0 \notin \{-nq,\dots,np\}} P_U^+(j_0) h_{j_0} V(j) \right)_{j \in \mathbb{Z}},$$

$$h \mapsto \left(\sum_{j_0<0} \mathbb{1}_{j-j_0 \notin \{-nq,\dots,np\}} P_U^-(j_0) h_{j_0} V(j) \right)_{j\in\mathbb{Z}},$$

we can obtain similar bounds as (4.2.10). We can then prove that there exist two positive constants C, c such that:

$$\forall h \in \ell^{r_1}(\mathbb{Z},\mathbb{C}^d), \forall n \in \mathbb{N}\backslash\{0\},$$
$$\left\| \Psi_n h - \left(\sum_{j_0\in\mathbb{N}} \left(\sum_{l'=1}^{I} C_{l'}^{E^+} E_{l'}^+(n,j_0) + P_U^+(j_0) \right) h_{j_0} + \sum_{j_0<0} \left(\sum_{l'=I}^{d} C_{l'}^{E^-} E_{l'}^-(n,j_0) + P_U^-(j_0) \right) h_{j_0} \right) V \right\|_{\ell^{r_2}} \leq \frac{C}{n^{\frac{1}{2\mu}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}} \|h\|_{\ell^{r_1}} \quad (4.2.16)$$

where the operator Ψ_n is defined by (4.2.7). Using (4.2.8) and (4.2.16), we can conclude that there exists a positive constant $C>0$ such that:

$$\forall h \in \ell^{r_1}(\mathbb{Z},\mathbb{C}^d), \forall n \in \mathbb{N}\backslash\{0\},$$
$$\left\| \mathscr{L}^n h - \left(\sum_{j_0\in\mathbb{N}} \left(\sum_{l'=1}^{I} C_{l'}^{E^+} E_{l'}^+(n,j_0) + P_U^+(j_0) \right) h_{j_0} + \sum_{j_0<0} \left(\sum_{l'=I}^{d} C_{l'}^{E^-} E_{l'}^-(n,j_0) + P_U^-(j_0) \right) h_{j_0} \right) V \right\|_{\ell^{r_2}} \leq \frac{C}{n^{\frac{1}{2\mu}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)}} \|h\|_{\ell^{r_1}}.$$

Since the sequence $V \in \ell^1(\mathbb{Z},\mathbb{C}^d)$ defined in Theorem 4.1 verifies that:

$$\ker(Id_{\ell^2} - \mathscr{L}) = \mathrm{Span} V,$$

this allows us to conclude the proof of Theorem 4.2.

4.3 Local exponential bounds on the spatial Green's function for z far from 1

In this section, the goal is twofold:

- In order to determine where the spatial Green's function is defined, we want to study the spectrum of the operator $\mathscr{L}$ in the set $\mathcal{O}$ (i.e. outside of the curves representing the spectrum of the limit operators $\mathscr{L}^{\pm}$). More precisely, we will prove Lemma 4.3.8 below that characterizes the eigenvalues of $\mathscr{L}$ in the set $\mathcal{O}$ and states that there is no essential spectrum of the operator $\mathscr{L}$ which lies in the set $\mathcal{O}$. This result was already proved in [Ser07, Theorem 4.1]. As a direct consequence, we will have proved that the elements of the set $\mathcal{O}$ are either in the resolvent set of the operator $\mathscr{L}$ or are eigenvalues of $\mathscr{L}$. Using Hypothesis 4.6, we can thus deduce that the set $\overline{\mathbb{U}}\backslash\{1\}$ is included in the resolvent set of $\mathscr{L}$ and that the spatial Green's function can be defined in a neighborhood of any point of $\overline{\mathbb{U}}\backslash\{1\}$.
- We will prove Proposition 4.2 below that introduces locally uniform exponential bounds on the spatial Green's function $G(z,j_0,\cdot)$ when z belongs to $\overline{\mathbb{U}}\backslash\{1\}$ and $j_0 \in \mathbb{Z}$. We will see later on in Section 4.4 that the study of the spatial Green's function for z near 1 will require some special care and that it is a more refined analysis of the case where z is in $\overline{\mathbb{U}}\backslash\{1\}$. It might be important to keep in mind that a lot of the tools we will introduce in Section 4.3 will also be useful in Section 4.4 to deal with the case where z is close to 1.

The main ideas of this section will be to characterize the solutions of the eigenvalue problem associated with the operator $\mathscr{L}$ by using solutions of a discrete dynamical system of finite dimension.

We will then define a central tool for our analysis : the *geometric dichotomy* introduced by Lafitte-Godillon in her thesis [Laf01] and based on the exponential dichotomy coined by Coppel in [Cop78]. We will take some time to rewrite the proofs of some lemmas even though most of the ideas can already be found in the previously cited texts. This will motivate the introduction of several notations and will make several crucial bounds precise.

4.3.1 Rewriting the eigenvalue problem as a dynamical system

As we explained, one of our objectives is to study the spectrum of the operator $\mathscr{L}$. In this section, we express the eigenvalue problem $(zId_{\ell^2} - \mathscr{L})u = 0$ as a discrete dynamical system. We will define a few important mathematical objects that will appear often throughout this chapter.

For $z \in \mathbb{C}$, $j \in \mathbb{Z}$ and $k \in \{-p, \dots, q\}$, we define the matrices $\mathbb{A}_{j,k}(z) = z\delta_{k,0}Id - A_{j,k}$ and $\mathbb{A}_k^\pm(z) = z\delta_{k,0}Id - A_k^\pm$. We recall that Hypothesis 4.8 implies that for all $z \in \mathbb{C}$ and $j \in \mathbb{Z}$, the matrices $\mathbb{A}_{j,-p}(z)$, $\mathbb{A}_{j,q}(z)$, $\mathbb{A}_{-p}^\pm(z)$ and $\mathbb{A}_q^\pm(z)$ are invertible and we can thus introduce the matrices

$$\forall j \in \mathbb{Z}, \forall z \in \mathbb{C}, \quad M_j(z) := \begin{pmatrix} -\mathbb{A}_{j,q}(z)^{-1}\mathbb{A}_{j,q-1}(z) & \dots & \dots & -\mathbb{A}_{j,q}(z)^{-1}\mathbb{A}_{j,-p}(z) \\ Id & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & Id & 0 \end{pmatrix} \in \mathcal{M}_{d(p+q)}(\mathbb{C})$$

and

$$\forall z \in \mathbb{C}, \quad M^\pm(z) := \begin{pmatrix} -\mathbb{A}_q^\pm(z)^{-1}\mathbb{A}_{q-1}^\pm(z) & \dots & \dots & -\mathbb{A}_q^\pm(z)^{-1}\mathbb{A}_{-p}^\pm(z) \\ Id & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & Id & 0 \end{pmatrix} \mathcal{M}_{d(p+q)}(\mathbb{C}). \tag{4.3.1}$$

which are well-defined and are furthermore invertible. We observe that for all $z \in \mathbb{C}$ the family of matrices $(M_j(z))_{j\in\mathbb{Z}}$ converges towards $M^\pm(z)$ as j tends towards $\pm\infty$. If we define for every $z \in \mathbb{C}$ and $j \in \mathbb{Z}$ the matrices

$$\mathcal{E}_j^\pm(z) := M_j(z) - M^\pm(z),$$

then Hypothesis 4.3 implies the following lemma.

Lemma 4.3.1. *There exists a constant $\alpha > 0$ such that for every bounded set U of $\mathbb{C}$, there exists a constant $C > 0$ such that*

$$\forall z \in U, \forall j \in \mathbb{N}, \quad \begin{array}{l} |\mathcal{E}_j^+(z)| \leq Ce^{-\alpha j}, \\ |\mathcal{E}_{-j}^-(z)| \leq Ce^{-\alpha j}. \end{array} \tag{4.3.2}$$

Proof We have that

$$\mathcal{E}_j^\pm(z) = \begin{pmatrix} \varepsilon_{j,q-1}^\pm(z) & \dots & \varepsilon_{j,-p}^\pm(z) \\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

where

$$\varepsilon_{j,k}^\pm(z) := \begin{cases} (A_q^\pm)^{-1}A_k^\pm - (A_{j,q})^{-1}A_{j,k} & \text{if } k \in \{-p, \dots, q-1\} \setminus \{0\}, \\ (A_q^\pm)^{-1}A_0^\pm - (A_{j,q})^{-1}A_{j,0} - z((A_q^\pm)^{-1} - (A_{j,q})^{-1}) & \text{if } k = 0. \end{cases}$$

Using Hypothesis 4.3, we conclude the proof of (4.3.2) and observe that the constant α can be taken uniformly on $\mathbb{C}$ but the constant C must depend on z. □

Let us consider $u \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$ such that

$$(zId_{\ell^2} - \mathscr{L})u = 0.$$

If we define for all $j \in \mathbb{Z}$ the vector

$$W_j = \begin{pmatrix} u_{j+q-1} \\ \vdots \\ u_{j-p} \end{pmatrix},$$

we observe that $(W_j)_{j\in\mathbb{Z}} \in \ell^2\left(\mathbb{Z}, \mathbb{C}^{d(p+q)}\right)$ which implies that the vector W_j converges towards 0 as j tends towards $\pm\infty$. We also observe that

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z)W_j. \tag{4.3.3}$$

To study the solutions of the dynamical system (4.3.3), we define the family of fundamental matrices $(X_j(z))_{j\in\mathbb{Z}} \in \mathcal{M}_{d(p+q)}(\mathbb{C})$ defined by

$$\begin{array}{l} \forall j \in \mathbb{Z}, \quad X_{j+1}(z) = M_j(z)X_j(z), \\ X_0(z) = Id. \end{array} \tag{4.3.4}$$

We observe that a solution $(W_j)_{j\in\mathbb{Z}}$ of the dynamical system (4.3.3) thus verifies:

$$\forall j\in\mathbb{Z},\quad W_j = X_j(z)W_0.$$

To find the eigenvalues of the operator $\mathscr{L}$, we thus need to search for the solutions $(W_j)_{j\in\mathbb{Z}}$ of the dynamical system (4.3.3) which belong to ℓ^2 and therefore converge towards 0 when j tends to $\pm\infty$. We thus introduce the following sets for any $z\in\mathbb{C}$:

$$E^\pm(z) := \left\{(W_j)_{j\in\mathbb{Z}}\in\left(\mathbb{C}^{(p+q)d}\right)^{\mathbb{Z}} \text{ solution of (4.3.3) such that } W_j \underset{j\to\pm\infty}{\to} 0\right\}, \tag{4.3.5a}$$

$$E_0^\pm(z) := \left\{W_0\in\mathbb{C}^{(p+q)d},\quad (X_j(z)W_0)_{j\in\mathbb{Z}}\in E^\pm(z)\right\}. \tag{4.3.5b}$$

The sets correspond to the solutions of the dynamical system (4.3.3) which converge towards 0 as j tends towards $\pm\infty$ and to their traces at $j=0$.

4.3.2 Spectral splitting: study of the spectrum of $M^\pm(z)$

Since the matrices $M_j(z)$ converge towards $M^\pm(z)$ as j converges towards $\pm\infty$, the dynamical system (4.3.3) can be considered to be perturbations respectively for $j\in\mathbb{N}$ and $j\in-\mathbb{N}$ of the dynamical systems

$$\forall j\in\mathbb{N},\quad W_{j+1} = M^+(z)W_j, \tag{4.3.6a}$$

$$\forall j\in-\mathbb{N},\quad W_{j+1} = M^-(z)W_j. \tag{4.3.6b}$$

To study the solutions of (4.3.3) which converge towards 0 as j tends to $\pm\infty$, we will study solutions converging towards 0 of the dynamical systems (4.3.6a) and (4.3.6b). This relies on studying the spectrum of the matrices $M^\pm(z)$.

Using the eigenvalues $\lambda_{l,k}^\pm$ of the matrix $A_k^\pm$ defined by (4.1.14), we introduce the scalar quantities:

$$\forall z\in\mathbb{C}, \forall l\in\{1,\dots,d\}, \forall k\in\{-p,\dots,q\},\quad \Lambda_{l,k}^\pm(z) := z\delta_{k,0}-\lambda_{l,k}^\pm, \tag{4.3.7}$$

and the matrices:

$$\forall z\in\mathbb{C}, \forall l\in\{1,\dots,d\},\quad M_l^\pm(z) := \begin{pmatrix} -\Lambda_{l,q}^\pm(z)^{-1}\Lambda_{l,q-1}^\pm(z) & \dots & \dots & -\Lambda_{l,q}^\pm(z)^{-1}\Lambda_{l,-p}^\pm(z)\\ 1 & 0 & \dots & 0\\ 0 & \ddots & \ddots & \vdots\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{4.3.8}$$

Hypothesis 4.8 implies that the matrices $A_{-p}^\pm$ and $A_q^\pm$ are invertible so $\Lambda_{l,-p}^\pm(z), \Lambda_{l,q}^\pm(z)\neq 0$. Thus, the matrices $M_l^\pm(z)$ are well-defined and invertible. We have the following result.

Lemma 4.3.2. *There exist invertible matrices $Q^\pm\in\mathcal{M}_{(p+q)d}(\mathbb{C})$ such that*

$$\forall z\in\mathbb{C},\quad M^\pm(z) = \left(Q^\pm\right)^{-1}\begin{pmatrix} M_1^\pm(z) & & \\ & \ddots & \\ & & M_d^\pm(z)\end{pmatrix} Q^\pm.$$

Proof Recalling the definition (4.1.4) of the matrices $\mathbf{P}^\pm$, we observe that

$$\begin{pmatrix}\mathbf{P}^\pm & & \\ & \ddots & \\ & & \mathbf{P}^\pm\end{pmatrix}^{-1} M^\pm(z)\begin{pmatrix}\mathbf{P}^\pm & & \\ & \ddots & \\ & & \mathbf{P}^\pm\end{pmatrix} = \begin{pmatrix} -\mathbb{D}_q^\pm(z)^{-1}\mathbb{D}_{q-1}^\pm(z) & \dots & \dots & -\mathbb{D}_q^\pm(z)^{-1}\mathbb{D}_{-p}^\pm(z)\\ Id & 0 & \dots & 0\\ 0 & \ddots & \ddots & \vdots\\ 0 & 0 & Id & 0\end{pmatrix} \tag{4.3.9}$$

where

$$\forall k\in\{-p,\dots,q\},\quad \mathbb{D}_k^\pm(z) = \begin{pmatrix}\Lambda_{1,k}^\pm(z) & & \\ & \ddots & \\ & & \Lambda_{d,k}^\pm(z)\end{pmatrix}.$$

Then, in the right hand term's matrix of (4.3.9), by reassembling the first columns of each blocks, then the second columns, ...and then doing the same for the lines, we prove that the matrix $M^{\pm}(z)$ is similar to the block diagonal matrix

$$\begin{pmatrix} M_1^{\pm}(z) & & \\ & \ddots & \\ & & M_d^{\pm}(z) \end{pmatrix}.$$

□

The following lemma is due to Kreiss (see [Kre68]) and describes precisely the spectrum of the matrix $M_l^{\pm}(z)$ as z belongs to $\mathcal{O} \cup \{1\}$ and $l \in \{1, \dots, d\}$.

Lemma 4.3.3 (Spectral Splitting)**.**
— *For $z \in \mathbb{C}$ and $l \in \{1, \dots, d\}$, $\kappa \in \mathbb{C}$ is an eigenvalue of $M_l^{\pm}(z)$ if and only if $\kappa \neq 0$ and*

$$\mathcal{F}_l^{\pm}(\kappa) = z.$$

— *Let $z \in \mathcal{O}$ and $l \in \{1, \dots, d\}$. Then the companion matrix $M_l^{\pm}(z)$ has*
  — *no eigenvalue on $\mathbb{S}^1$,*
  — *p eigenvalues in $\mathbb{D}\backslash\{0\}$ (that we call stable eigenvalues),*
  — *q eigenvalues in $\mathbb{U}$ (that we call unstable eigenvalues).*

— *We also have that*
  — *if $\alpha_l^{\pm} > 0$, $M_l^{\pm}(1)$ has 1 as a simple eigenvalue, $p-1$ eigenvalues in $\mathbb{D}\backslash\{0\}$ and q eigenvalues in $\mathbb{U}$.*
  — *if $\alpha_l^{\pm} < 0$, $M_l^{\pm}(1)$ has 1 as a simple eigenvalue, p eigenvalues in $\mathbb{D}\backslash\{0\}$ and $q-1$ eigenvalues in $\mathbb{U}$.*

Lemma 4.3.3 is proved in [CF22, Lemma 1] (see also [Kre68]). For $z \in \mathcal{O}$, combining the consequences of Lemmas 4.3.2 and 4.3.3, the matrix $M^{\pm}(z)$ only has eigenvalues in $\mathbb{D}$ or $\mathbb{U}$. Also, if we define the space $E^s(M^{\pm}(z))$ (resp. $E^u(M^{\pm}(z))$) which is the strictly stable (resp. strictly unstable) subspace of $M^{\pm}(z)$ corresponding to the subspace spanned by the generalized eigenvectors of $M^{\pm}(z)$ associated with eigenvalues in $\mathbb{D}$ (resp. $\mathbb{U}$), then we have $\dim E^s(M^{\pm}(z)) = dp$, $\dim E^u(M^{\pm}(z)) = dq$ and

$$\mathbb{C}^{(p+q)d} = E^s(M^{\pm}(z)) \oplus E^u(M^{\pm}(z)).$$

We consider $P_s^{\pm}(z)$ and $P_u^{\pm}(z)$ the associated projectors in $\mathbb{C}^{d(p+q)}$. Those projectors can be expressed as contour integrals (see [Kat95]). For instance, we have

$$P_s^{\pm}(z) = \frac{1}{2i\pi} \int_{\gamma} (tId - M^{\pm}(z))^{-1} dt$$

where γ is a simple closed positively oriented contour which surrounds the stable eigenvalues of $M^{\pm}(z)$ and not the unstable ones ($\mathbb{S}^1$ is a good candidate). Therefore, the projectors $P_s^{\pm}(z)$ and $P_u^{\pm}(z)$ depend holomorphically on $z \in \mathcal{O}$.

4.3.3 "Local" geometric dichotomy

The conclusion of the study of the spectrum of the matrices $M^{\pm}(z)$ done in Section 4.3.2 is that the vector space of solutions of (4.3.6a) (resp. (4.3.6b)) converging towards 0 as j tends towards $+\infty$ (resp. $-\infty$) has dimension dp (resp. dq) and can be characterized by using the spectral projector $P_s^{+}(z)$ (resp. $P_u^{-}(z)$). We recall that (4.3.3) is a perturbation of the dynamical systems (4.3.6a) and (4.3.6b). Thus, we could expect for the vector spaces $E^{+}(z)$ and $E_0^{+}(z)$ (resp. $E^{-}(z)$ and $E_0^{-}(z)$) defined by (4.3.5a) and (4.3.5b) to be of dimension dp (resp. dq) and we would want some way to characterize their elements.

In the present section, our goal is to construct projectors which will play for the dynamical system (4.3.3) a similar role as the spectral projectors $P_s^{+}(z)$ and $P_u^{-}(z)$ for the dynamical systems (4.3.6a) and (4.3.6b). This is the aim of the following lemma.

Lemma 4.3.4 (Geometric dichotomy)**.** *For any bounded open set U such that $\overline{U} \subset \mathcal{O}$, there exist two holomorphic functions $Q_U^{\pm} : U \to \mathcal{M}_{(p+q)d}(\mathbb{C})$ such that*
— *For all $z \in U$, $Q_U^{\pm}(z)$ is a projector and we have*

$$\dim \operatorname{Im} Q_U^{+}(z) = \dim \ker Q_U^{-}(z) = dp \quad \textit{and} \quad \dim \operatorname{Im} Q_U^{-}(z) = \dim \ker Q_U^{+}(z) = dq.$$

— *There exist two positive constants C, c such that for all $z \in U$, there holds:*

$$\forall j \geq k \geq 0, \quad \left| X_j(z) Q_U^{+}(z) X_k(z)^{-1} \right| \leq C e^{-c|j-k|}, \tag{4.3.10a}$$

$$\forall k \geq j \geq 0, \quad \left|X_j(z)(Id - Q_U^+(z))X_k(z)^{-1}\right| \leq Ce^{-c|j-k|}, \tag{4.3.10b}$$
$$\forall j \leq k \leq 0, \quad \left|X_j(z)Q_U^-(z)X_k(z)^{-1}\right| \leq Ce^{-c|j-k|}, \tag{4.3.10c}$$
$$\forall k \leq j \leq 0, \quad \left|X_j(z)(Id - Q_U^-(z))X_k(z)^{-1}\right| \leq Ce^{-c|j-k|}. \tag{4.3.10d}$$

Lemma 4.3.4 has been developped in the thesis of Lafitte-Godillon [Laf01, Section III.1.5] and is inspired by the exponential dichotomy discussed by Coppel in [Cop78]. As it is explained in [Cop78], to better understand the meaning of this lemma, it is interesting to see that the inequalities (4.3.10a) and (4.3.10b) imply that for all vector $\xi \in \mathbb{C}^{(p+q)d}$, there holds:

$$\forall j \geq 0, \quad \left|X_j(z)Q_U^+(z)\xi\right| \leq Ce^{-cj}\left|\xi\right|,$$
$$\forall k \geq 0, \quad \left|(Id - Q_U^+(z))\xi\right| \leq Ce^{-ck}\left|X_k(z)\xi\right|.$$

The first inequality implies that there exists a dp-dimensional subspace of solutions $(W_j)_{j\in\mathbb{Z}}$ of the dynamical system (4.3.3) which converge exponentially fast toward 0 at $+\infty$ (and which thus belong to $\ell^2(\mathbb{N})$). The second inequality translates to the fact that there exists a supplementary of the previous subspace of solutions for which the solutions explode exponentially at $+\infty$. Thus, $Q_U^+(z)$ plays a similar role for the dynamical system (4.3.3) as any projector for which the range is $E_s^+(z)$ (for instance the spectral projector $P_s^+(z)$) for the dynamical system (4.3.6a). The same kind of conclusion can be achieved with Q_U^- for the behavior at $-\infty$.

Thus, the construction of those two projectors $Q_U^\pm$ is fundamental to study the solutions $(W_j)_{j\in\mathbb{Z}}$ of (4.3.3) that converge toward 0 as j tends to $\pm\infty$, i.e. the elements of the set $E^\pm(z)$ defined by (4.3.5a). We will see in Lemma 4.3.7 that the projectors $Q_U^\pm(z)$ allow to characterize the elements of the vector spaces $E_0^\pm(z)$.

We give below the proof of Lemma 4.3.4 which is overall the same as in [Laf01, Section III.1.5]. The main modification is that this new iteration of the proof pinpoints more clearly the holomorphicity of the projectors $Q_U^\pm$ and most importantly why the projectors $Q_U^\pm$ we construct depend on the relatively compact subset U of $\mathcal{O}$ which is considered. This last point is not really addressed in [Laf01; God03] and we feel that it is quiet important. For instance, for two sets U_1 and U_2 that satisfy the conditions of Lemma 4.3.4, the construction of the proof of Lemma 4.3.4 does not imply that the projectors $Q_{U_1}^\pm$ and $Q_{U_2}^\pm$ are equal on $U_1 \cap U_2$, even if $U_1 \subset U_2$. Therefore, we cannot immediately construct two functions $Q^\pm$ that are defined on $\mathcal{O}$ which would verify similar properties as $Q_U^\pm$. However, it turns out that the ranges $\operatorname{Im} Q_{U_1}^\pm(z)$ and $\operatorname{Im} Q_{U_2}^\pm(z)$ coincide for $z \in U_1 \cap U_2$. We will prove this fact later on and use it to extend uniformly the geometric dichotomy on a large part of $\mathcal{O}$ (see Lemma 4.3.9 below).

Proof of Lemma 4.3.4

The construction of both functions $Q_U^\pm$ is similar so we focus here on the construction of Q_U^+. The proof will be separated in four steps. In the first step, we will construct the function Q_U^+ using a fixed point argument. The second step will be dedicated to proving that for all $z \in U$, $Q_U^+(z)$ is a projector for which the kernel and the range are respectively of dimension dq and dp. The third and fourth steps concern the proof of the inequalities (4.3.10a) and (4.3.10b).

Step 1: Construction of Q_U^+.

We set for $z \in \mathcal{O}$ that

$$\eta_s^\pm(z) := \max\left\{\ln(|\zeta|), \quad \zeta \in \sigma(M^\pm(z)) \cap \mathbb{D}\right\},$$
$$\eta_u^\pm(z) := \min\left\{\ln(|\zeta|), \quad \zeta \in \sigma(M^\pm(z)) \cap \mathbb{U}\right\}.$$

The functions η_s^+ and η_u^+ are continuous on $\mathcal{O}$ and verify that

$$\forall z \in \mathcal{O}, \quad \eta_s^+(z) < 0 \quad \text{and} \quad \eta_u^+(z) > 0.$$

The set $\overline{U}$ being compact and included in $\mathcal{O}$, there exists a constant c_H such that

$$\max_{z\in\overline{U}} \eta_s^+(z) < -c_H < 0 \quad \text{and} \quad 0 < c_H < \min_{z\in\overline{U}} \eta_u^+(z).$$

We will also ask that $c_H < \alpha$ where α is the constant appearing in (4.3.2). By definition of η_s^+ and η_u^+, there exists a positive constant C_H such that

$$\forall z \in U, \forall j \in \mathbb{N}, \quad \begin{array}{r}\left|M^+(z)^j P_s^+(z)\right| \leq C_H e^{-c_H j},\\ \left|M^+(z)^{-j} P_u^+(z)\right| \leq C_H e^{-c_H j}.\end{array} \tag{4.3.11}$$

Furthermore, using (4.3.2), since U is bounded, there exists a positive constant $C_\mathcal{E}$ such that

$$\forall z \in U, \forall j \in \mathbb{N}, \quad |\mathcal{E}_j^+(z)| \leq C_\mathcal{E} e^{-\alpha j}. \tag{4.3.12}$$

We consider an integer $J \in \mathbb{N}$ and we will make a more precise choice later. We define the Banach space

$$\ell^\infty_J := \left\{ (Y_j)_{j\geq J} \in \mathcal{M}_{(p+q)d}(\mathbb{C})^{\{j\in\mathbb{N}, j\geq J\}}, \quad \sup_{j\geq J} |Y_j| < +\infty \right\}$$

with the norm

$$\|Y\|_{\infty,J} := \sup_{j\geq J} |Y_j|.$$

Furthermore, for $z \in U$, we define the linear map $\varphi(z) \in \mathcal{L}\left(\ell^\infty_J\right)$ and $T(z) : \ell^\infty_J \to \ell^\infty_J$ such that for $Y \in \ell^\infty_J$ and $j \geq J$, we have

$$\left(\varphi(z)Y\right)_j := \sum_{k=J}^{j-1} M^+(z)^{j-1-k} P^+_s(z)\mathcal{E}^+_k(z)Y_k - \sum_{k=j}^{+\infty} M^+(z)^{j-1-k} P^+_u(z)\mathcal{E}^+_k(z)Y_k$$

and

$$T(z)Y := \left(M^+(z)^{j-J} P^+_s(z)\right)_{j\geq J} + \varphi(z)Y.$$

We observe that

$$\forall Y \in \ell^\infty_J, \forall j \geq J, \quad (\varphi(z)Y)_{j+1} = M^+(z)(\varphi(z)Y)_j + \mathcal{E}^+_j(z)Y_j,$$

and thus

$$\forall Y \in \ell^\infty_J, \forall j \geq J, \quad (T(z)Y)_{j+1} = M^+(z)(T(z)Y)_j + \mathcal{E}^+_j(z)Y_j. \tag{4.3.13}$$

Our goal will be to find a fixed point of $T(z)$. It will be a solution of the dynamical system (4.3.3) for $j \geq J$. To do so, we will have to prove that there exists J large enough so that

$$\|\varphi(z)\|_{\mathcal{L}(\ell^\infty_J)} < 1.$$

We begin by proving that the applications $\varphi(z)$ and $T(z)$ are well-defined. We consider $Y \in \ell^\infty_J$ and $j \geq J$. Using (4.3.11) and (4.3.12), we have the estimates:

$$\begin{aligned}
|(\varphi(z)Y)_j| &\leq \sum_{k=J}^{j-1} C_H C_\mathcal{E} e^{-c_H|j-1-k|} e^{-\alpha k}|Y_k| + \sum_{k=j}^{+\infty} C_H C_\mathcal{E} e^{-c_H|j-1-k|} e^{-\alpha k}|Y_k| \\
&\leq \|Y\|_{\infty,J} C_H C_\mathcal{E} e^{-\alpha J} \sum_{k=J}^{+\infty} e^{-c_H|j-1-k|} \\
&\leq \|Y\|_{\infty,J} C_H C_\mathcal{E} e^{-\alpha J} \frac{1+e^{-c_H}}{1-e^{-c_H}}.
\end{aligned}$$

If we set

$$\theta := C_H C_\mathcal{E} e^{-\alpha J} \frac{1+e^{-c_H}}{1-e^{-c_H}}, \tag{4.3.14}$$

then we have just proved that the operator $\varphi(z)$ is well-defined, bounded and

$$\|\varphi(z)\|_{\mathcal{L}(\ell^\infty_J)} \leq \theta.$$

We also observe that (4.3.11) implies that $(M^+(z)^{j-J}P^+_s(z))_{j\geq J} \in \ell^\infty_J$. Therefore, $T(z)$ is well-defined.

Let us choose the integer J large enough so that $\theta < 1$. For $z \in U$, we have that $Id - \varphi(z)$ is invertible. Thus, we can define

$$Y(z) := (Id - \varphi(z))^{-1} \left(M^+(z)^{j-J} P^+_s(z)\right)_{j\geq J}.$$

This sequence $Y(z) \in \ell^\infty_J$ is the only fixed point of $T(z)$ and it depends holomorphically on z. We observe that (4.3.13) implies that

$$\forall z \in U, \forall j \geq J, \quad Y_{j+1}(z) = M_j(z)Y_j(z). \tag{4.3.15}$$

We now define

$$\forall z \in U, \quad Q^+_U(z) := X_J(z)^{-1} Y_J(z) X_J(z). \tag{4.3.16}$$

Since Y depends holomorphically on z and is bounded on U, $Q^+_U(z)$ also depends holomorphically on z for $z \in U$ and is bounded on U.

Step 2: We now show that Q^+_U is a projector.

We are going to prove that for all $z \in U$ the matrix $Q_U^+(z)$ we have just constructed is a projector such that

$$\ker Q_U^+(z) = X_J(z)^{-1} E^u(M^+(z)) \quad \text{and} \quad \dim \operatorname{Im} Q_U^+(z) = dp.$$

By observing that $P_s^+(z)^2 = P_s^+(z)$, we can prove that $(Y_j(z)P_s^+(z))_{j \geq J}$ is another fixed point $T(z)$. Since $Y(z)$ is the only fixed point of $T(z)$ in ℓ_J^∞, we thus have that:

$$Y_J(z)P_s^+(z) = Y_J(z). \tag{4.3.17}$$

Using that $Y(z)$ is a fixed point of $T(z)$, we also have that

$$\begin{aligned} P_s^+(z)Y_J(z) &= P_s^+(z)(T(z)Y(z))_J \\ &= P_s^+(z)\left(P_s^+(z) - \sum_{k=J}^{+\infty} M^+(z)^{J-1-k}P_u^+(z)\mathcal{E}_k^+(z)Y_k(z)\right). \end{aligned}$$

Because $P_s^+(z)$ commutes with $M^+(z)$, $P_s^+(z)^2 = P_s^+(z)$ and $P_s^+(z)P_u^+(z) = 0$, we have proved

$$P_s^+(z) = P_s^+(z)Y_J(z). \tag{4.3.18}$$

Using (4.3.18), we prove that $(Y_j(z)Y_J(z))_{j \geq J}$ is a fixed point $T(z)$. Since $Y(z)$ is the only fixed point of $T(z)$ in ℓ_J^∞, we have in particular that:

$$Y_J(z)^2 = Y_J(z)$$

which means that $Y_J(z)$ is a projector. The definition (4.3.16) of $Q_U^+(z)$ then implies that $Q_U^+(z)$ is a projector. The equalities (4.3.17) and (4.3.18) allow us to prove that $\ker Y_J(z) = \ker P_s^+(z) = E^u(M^+(z))$ which implies that:

$$\operatorname{Im} Q_U^+(z) = X_J(z)^{-1} \operatorname{Im} Y_J(z) \quad \text{and} \quad \ker Q_U^+(z) = X_J(z)^{-1} E^u(M^+(z)).$$

Step 3: We now show that Q_U^+ satisfies the inequalities (4.3.10a) and (4.3.10b).

First, we are going to prove the inequality (4.3.10a) for $j \geq k \geq J$ and the inequality (4.3.10b) for $k \geq j \geq J$. We observe that (4.3.15) implies that

$$\forall z \in U, \forall j \geq J, \quad Y_{j+1}(z) = M_j(z)Y_j(z).$$

and thus

$$\forall z \in U, \forall j \geq J, \quad Y_j(z) = X_j(z)X_J(z)^{-1}Y_J(z) = X_j(z)Q_U^+(z)X_J(z)^{-1}. \tag{4.3.19a}$$

We introduce

$$\forall z \in U, \forall j \geq J, \quad Z_j(z) := X_j(z)X_J(z)^{-1}\left(Id - Y_J(z)\right) = X_j(z)\left(Id - Q_U^+(z)\right)X_J(z)^{-1}. \tag{4.3.19b}$$

We have the following lemma.

Lemma 4.3.5. *We have that*

$$\begin{aligned} \forall j \geq k \geq J, \quad Y_j(z) =& M^+(z)^{j-k}P_s^+(z)Y_k(z) + \sum_{l=k}^{j-1} M^+(z)^{j-1-l}P_s^+(z)\mathcal{E}_l^+(z)Y_l(z) \\ & - \sum_{l=j}^{+\infty} M^+(z)^{j-1-l}P_u^+(z)\mathcal{E}_l^+(z)Y_l(z), \end{aligned}$$

and

$$\begin{aligned} \forall k \geq j \geq J, \quad Z_j(z) =& M^+(z)^{j-k}P_u^+(z)Z_k(z) + \sum_{l=J}^{j-1} M^+(z)^{j-1-l}P_s^+(z)\mathcal{E}_l^+(z)Z_l(z) \\ & - \sum_{l=j}^{k-1} M^+(z)^{j-1-l}P_u^+(z)\mathcal{E}_l^+(z)Z_l(z). \end{aligned}$$

Proof of Lemma 4.3.5

— Since we have that

$$\forall j \geq J, \quad Y_{j+1}(z) = M_j(z)Y_j(z) = (M^+(z) + \mathcal{E}_j^+(z))Y_j(z),$$

using the Duhamel formula, we find that

$$\forall k \geq J, \quad Y_k(z) = M^+(z)^{k-J} Y_J(z) + \sum_{l=J}^{k-1} M^+(z)^{k-1-l} \mathcal{E}_l^+(z) Y_l(z).$$

Knowing that $Y(z)$ is a fixed point of $T(z)$ and that $P_s^+(z) Y_J(z) = P_s^+(z)$, we have for $j \geq k \geq J$

$$\begin{aligned}
&Y_j(z) \\
=&(T(z)Y(z))_j \\
=&M^+(z)^{j-J} P_s^+(z) + \sum_{l=J}^{j-1} M^+(z)^{j-1-l} P_s^+(z) \mathcal{E}_l^+(z) Y_l(z) - \sum_{l=j}^{+\infty} M^+(z)^{j-1-l} P_u^+(z) \mathcal{E}_l^+(z) Y_l(z) \\
=&M^+(z)^{j-k} P_s^+(z) \left(M^+(z)^{k-J} Y_J(z) + \sum_{l=J}^{k-1} M^+(z)^{k-1-l} \mathcal{E}_l^+(z) Y_l(z) \right) \\
&+ \sum_{l=k}^{j-1} M^+(z)^{j-1-l} P_s^+(z) \mathcal{E}_l^+(z) Y_l(z) - \sum_{l=j}^{+\infty} M^+(z)^{j-1-l} P_u^+(z) \mathcal{E}_l^+(z) Y_l(z) \\
=&M^+(z)^{j-k} P_s^+(z) Y_k(z) + \sum_{l=k}^{j-1} M^+(z)^{j-1-l} P_s^+(z) \mathcal{E}_l^+(z) Y_l(z) - \sum_{l=j}^{+\infty} M^+(z)^{j-1-l} P_u^+(z) \mathcal{E}_l^+(z) Y_l(z)
\end{aligned}$$

which corresponds to the statement of Lemma 4.3.5.

— We now turn to the equation of $Z_j(z)$. Since we have that

$$\forall j \geq J, \quad Z_{j+1}(z) = M_j(z) Z_j(z) = (M^+(z) + \mathcal{E}_j^+(z)) Z_j(z),$$

using the Duhamel formula, we find that for $k \geq j \geq J$

$$Z_j(z) = M^+(z)^{j-J} Z_J(z) + \sum_{l=J}^{j-1} M^+(z)^{j-1-l} \mathcal{E}_l^+(z) Z_l(z) \tag{4.3.20}$$

$$Z_k(z) = M^+(z)^{k-j} Z_j(z) + \sum_{l=j}^{k-1} M^+(z)^{k-1-l} \mathcal{E}_l^+(z) Z_l(z). \tag{4.3.21}$$

Using (4.3.20) and knowing that $P_s^+(z) Z_J(z) = P_s^+(z)(Id - Y_J(z)) = 0$, we have that

$$P_s^+(z) Z_j(z) = \sum_{l=J}^{j-1} M^+(z)^{j-1-l} P_s^+(z) \mathcal{E}_l^+(z) Z_l(z).$$

Furthermore, (4.3.21) implies that

$$P_u^+(z) Z_j(z) = M^+(z)^{j-k} P_u^+(z) Z_k(z) - \sum_{l=j}^{k-1} M^+(z)^{j-1-l} P_u^+(z) \mathcal{E}_l^+(z) Z_l(z).$$

We end the proof of Lemma 4.3.5 by observing that $Z_j(z) = P_s^+(z) Z_j(z) + P_u^+(z) Z_j(z)$.

□

We introduce the constant

$$\Theta := \theta \frac{1 - e^{-c_H}}{1 + e^{-c_H}} = C_H C_{\mathcal{E}} e^{-\alpha J} > 0 \tag{4.3.22}$$

where the constant θ is defined by (4.3.14). Using Lemma 4.3.5 and (4.3.11), we obtain that for any vector $\xi \in \mathbb{C}^{(p+q)d}$:

$$\begin{aligned}
\forall j \geq k \geq J, \quad |Y_j(z)\xi| \leq& C_H e^{-c_H(j-k)} |Y_k(z)\xi| + \sum_{l=k}^{+\infty} C_H C_{\mathcal{E}} e^{-c_H|j-1-l|} e^{-\alpha l} |Y_l(z)\xi| \\
\leq& C_H e^{-c_H(j-k)} |Y_k(z)\xi| + \Theta \sum_{l=k}^{+\infty} e^{-c_H|j-1-l|} |Y_l(z)\xi|,
\end{aligned}$$

and similarly:

$$\forall k \geq j \geq J, \quad |Z_j(z)\xi| \leq C_H e^{-c_H(k-j)} |Z_k(z)\xi| + \Theta \sum_{l=J}^{k-1} e^{-c_H|j-1-l|} |Z_l(z)\xi| .$$

The following lemma corresponding to [Laf01, Lemma 1.5.1, Section III.1.5] will allow us to obtain clearer bounds on $|Y_j(z)\xi|$ and $|Z_j(z)\xi|$.

Lemma 4.3.6. *Let us consider positive constants C_H, c_H and Θ such that*

$$\theta := \Theta \frac{1+e^{-c_H}}{1-e^{-c_H}} = \Theta \frac{\cosh\left(\frac{c_H}{2}\right)}{\sinh\left(\frac{c_H}{2}\right)} \in]0,1[. \tag{4.3.23}$$

For any real valued sequence $y \in \ell^\infty(\mathbb{N})$ with non negative coefficients that satisfies :

$$\forall j \in \mathbb{N}, \quad y_j \leq C_H e^{-c_H j} + \Theta \sum_{k=0}^{+\infty} e^{-c_H|j-1-k|} y_k, \tag{4.3.24}$$

we have that:

$$\forall j \in \mathbb{N}, \quad y_j \leq \rho r^j,$$

where

$$r := \cosh(c_H) - 2\sinh\left(\frac{c_H}{2}\right)\sqrt{\cosh^2\left(\frac{c_H}{2}\right) - \theta} \in]e^{-c_H}, 1[\quad \textit{and} \quad \rho := \frac{C_H}{\Theta}(r - e^{-c_H}) > 0. \tag{4.3.25}$$

The proof can be found in the Appendix (Section 4.A). We will now use Lemma 4.3.6 to prove that

$$\forall j \geq k \geq J, \quad |Y_j(z)\xi| \leq \rho r^{j-k} |Y_k(z)\xi|, \tag{4.3.26a}$$

$$\forall k \geq j \geq J, \quad |Z_j(z)\xi| \leq \rho r^{k-j} |Z_k(z)\xi|, \tag{4.3.26b}$$

where r and ρ are defined by (4.3.25).

We consider $k \geq J$. If $Y_k(z)\xi \neq 0$, then by applying Lemma 4.3.6 to the bounded sequence $y := \left(\frac{|Y_{k+j}(z)\xi|}{|Y_k(z)\xi|}\right)_{j\in\mathbb{N}}$, we obtain (4.3.26a) with r and ρ defined by (4.3.25). Else, if $Y_k(z)\xi = 0$, then for $j \geq k$, we have:

$$Y_j(z)\xi = X_j(z)X_k(z)^{-1}Y_k(z)\xi = 0.$$

Thus, (4.3.26a) is also verified in this case.

The proof of (4.3.26b) is similar. If $Z_k(z)\xi \neq 0$, then we apply Lemma 4.3.6 to the sequence y defined by

$$\forall j \in \mathbb{N}, \quad y_j := \begin{cases} \frac{|Z_{k-j}(z)\xi|}{|Z_k(z)\xi|} & \text{if } j \in \{0,\dots,k\}, \\ 0 & \text{else.} \end{cases}$$

This proves (4.3.26b) in this case. If $Z_k(z)\xi = 0$, then since

$$\forall j \in \{J,\dots,k\}, \quad Z_j(z)\xi = X_j(z)X_k(z)^{-1}Z_k(z)\xi = 0,$$

(4.3.26b) is also trivially verified in this case.

Using (4.3.19), (4.3.26a) and (4.3.26b), we proved that:

$$\forall j \geq k \geq J, \quad |X_j(z)Q_U^+(z)X_k(z)^{-1}| \leq \rho r^{j-k} |X_k(z)Q_U^+(z)X_k(z)^{-1}|, \tag{4.3.27a}$$

$$\forall k \geq j \geq J, \quad |X_j(z)(Id - Q_U^+(z))X_k(z)^{-1}| \leq \rho r^{k-j} |X_k(z)(Id - Q_U^+(z))X_k(z)^{-1}|. \tag{4.3.27b}$$

If we prove that the families $(X_k(z)Q_U^+(z)X_k(z)^{-1})_{k\geq J}$ and $(X_k(z)(Id - Q_U^+(z))X_k(z)^{-1})_{k\geq J}$ are uniformly bounded for $z \in U$, we will have proved (4.3.10a) and (4.3.10b) respectively for $j \geq k \geq J$ and $k \geq j \geq J$.

Using Lemma 4.3.5, we prove that for $j \geq J$:

$$P_u^+(z)(Y(z))_j = -\sum_{l=j}^{+\infty} M^+(z)^{j-1-l} P_u^+(z)\mathcal{E}_l^+(z)(Y(z))_l,$$

$$P_s^+(z)(Z(z))_j = \sum_{l=J}^{j-1} M^+(z)^{j-1-l} P_s^+(z)\mathcal{E}_l^+(z)(Z(z))_l.$$

Thus, we have using (4.3.19) that:

$$\begin{aligned} P_u^+(z)X_j(z)Q_U^+(z)X_j(z)^{-1} &= -\sum_{l=j}^{+\infty} M^+(z)^{j-1-l} P_u^+(z)\mathcal{E}_l^+(z)X_l(z)Q_U^+(z)X_j(z)^{-1}, \\ P_s^+(z)X_j(z)(Id - Q_U^+(z))X_j(z)^{-1} &= \sum_{l=J}^{j-1} M^+(z)^{j-1-l} P_s^+(z)\mathcal{E}_l^+(z)X_l(z)(Id - Q_U^+(z))X_j(z)^{-1}. \end{aligned}$$

Using (4.3.11), the definitions (4.3.14), (4.3.22) and (4.3.25) of the constants θ, Θ and ρ, as well as (4.3.27a), we have

$$\begin{aligned} |P_u^+(z)X_j(z)Q_U^+(z)X_j(z)^{-1}| &\leq \Theta \sum_{l=j}^{+\infty} e^{-c_H(l-(j-1))} |X_l(z)Q_U^+(z)X_j(z)^{-1}| \\ &\leq \Theta e^{-c_H}\rho \sum_{l=j}^{+\infty} (re^{-c_H})^{l-j} |X_j(z)Q_U^+(z)X_j(z)^{-1}| \\ &= \Theta \frac{e^{-c_H}}{1-re^{-c_H}} \rho |X_j(z)Q_U^+(z)X_j(z)^{-1}| \\ &= C_H \frac{r-e^{-c_H}}{e^{c_H}-r} |X_j(z)Q_U^+(z)X_j(z)^{-1}|. \end{aligned}$$

Similarly, using (4.3.27b), we have

$$\begin{aligned} |P_s^+(z)X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| &\leq \Theta \sum_{l=J}^{j-1} e^{-c_H(j-1-l)} |X_l(z)(Id - Q_U^+(z))X_j(z)^{-1}| \\ &\leq \Theta e^{c_H}\rho \sum_{l=J}^{j-1} (re^{-c_H})^{j-l} |X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| \\ &\leq \Theta \rho r \frac{1}{1-re^{-c_H}} |X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| \\ &= C_H r e^{c_H} \frac{r-e^{-c_H}}{e^{c_H}-r} |X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}|. \end{aligned}$$

Therefore, if we define $\eta := C_H \frac{e^{c_H}}{e^{c_H}-1}(r - e^{-c_H})$, we have for all $j \geq J$

$$\begin{aligned} |P_u^+(z)X_j(z)Q_U^+(z)X_j(z)^{-1}| &\leq \eta |X_j(z)Q_U^+(z)X_j(z)^{-1}|, \\ |P_s^+(z)X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| &\leq \eta |X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}|. \end{aligned} \tag{4.3.28}$$

Using the definition (4.3.25) of r, we observe that

$$\eta = C_H \frac{e^{c_H}}{e^{c_H}-1} 2\sinh\left(\frac{c_H}{2}\right)\left(\cosh\left(\frac{c_H}{2}\right) - \sqrt{\cosh\left(\frac{c_H}{2}\right)^2 - \theta}\right). \tag{4.3.29}$$

We already supposed that J was taken large enough so that the number θ in (4.3.14) satisfies $\theta < 1$. We will now also suppose that we took J large enough so that θ is close enough to 0 so that the above number η in (4.3.29) satisfies $\eta < \frac{1}{2}$. To conclude this step of the proof, we observe that

$$X_j(z)Q_U^+(z)X_j(z)^{-1} - P_s^+(z) = P_u^+(z)X_j(z)Q_U^+(z)X_j(z)^{-1} - P_s^+(z)X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}$$

and

$$X_j(z)(Id - Q_U^+(z))X_j(z)^{-1} - P_u^+(z) = P_s^+(z)X_j(z)(Id - Q_U^+(z))X_j(z)^{-1} - P_u^+(z)X_j(z)Q_U^+(z)X_j(z)^{-1}.$$

Thus, using (4.3.11) to bound P_s^+ and P_u^+ and (4.3.28), we have

$$|X_j(z)Q_U^+(z)X_j(z)^{-1}| \leq C_H + \eta\left(|X_j(z)Q_U^+(z)X_j(z)^{-1}| + |X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}|\right)$$

and

$$|X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| \leq C_H + \eta \left(|X_j(z)Q_U^+(z)X_j(z)^{-1}| + |X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}|\right).$$

This implies that :

$$\forall j \geq J, \qquad \begin{array}{r}|X_j(z)Q_U^+(z)X_j(z)^{-1}| \leq \dfrac{2C_H}{1-2\eta}, \\ |X_j(z)(Id - Q_U^+(z))X_j(z)^{-1}| \leq \dfrac{2C_H}{1-2\eta}.\end{array}$$

Therefore, we have proved that for all $z \in U$, we have:

$$\forall j \geq k \geq J, \quad |X_j(z)Q_U^+(z)X_k(z)^{-1}| \leq \rho \frac{2C_H}{1-2\eta} r^{j-k}, \tag{4.3.30a}$$

$$\forall k \geq j \geq J, \quad |X_j(z)(Id - Q_U^+(z))X_k(z)^{-1}| \leq \rho \frac{2C_H}{1-2\eta} r^{k-j}. \tag{4.3.30b}$$

Step 4: Q_U^+ satisfies the inequalities (4.3.10a) and (4.3.10b) respectively for all $j \geq k \geq 0$ and $k \geq j \geq 0$
We will only finish the proof of (4.3.10a) since the proof for (4.3.10b) is similar. We have proved (4.3.10a) for $j \geq k \geq J$. We consider a constant $C > 0$ such that

$$\forall z \in U, \qquad \begin{array}{l} C > r^{-J} \max_{j\in\{0,\dots,J-1\}} |X_j(z)Q_U^+(z)X_J(z)^{-1}| \\ C > r^{-J} \max_{j\in\{0,\dots,J-1\}} |X_J(z)Q_U^+(z)X_j(z)^{-1}|. \end{array}$$

This can be done since the projector Q_U^+ defined by (4.3.16) is bounded on U.

— If $j \geq J > k \geq 0$, we have

$$\begin{aligned} |X_j(z)Q_U^+(z)X_k(z)^{-1}| &\leq |X_j(z)Q_U^+(z)X_J(z)^{-1}||X_J(z)Q_U^+(z)X_k(z)^{-1}| \\ &\leq \rho \frac{2C_H}{1-2\eta} r^{j-J} C r^J \\ &\leq C\rho \frac{2C_H}{1-2\eta} r^{j-k}. \end{aligned}$$

— If $J > j \geq k \geq 0$, we have

$$\begin{aligned} |X_j(z)Q_U^+(z)X_k(z)^{-1}| &\leq |X_j(z)Q_U^+(z)X_J(z)^{-1}||X_J(z)Q_U^+(z)X_k(z)^{-1}| \\ &\leq C^2 r^{2J} \\ &\leq C^2 r^{j-k}. \end{aligned}$$

Therefore, there exist two constants $C, c > 0$ such that for all $z \in U$, (4.3.10a) is verified. □

4.3.4 Spectrum of $\mathscr{L}$ and extended geometric dichotomy

Now that we have proved the geometric dichotomy, Lemma 4.3.4 let us go back on studying the vector spaces $E_0^\pm(z)$ which characterize the solutions of the dynamical system (4.3.3) converging towards 0 as j tends towards $\pm\infty$. The previous section about the geometric dichotomy allows us to prove the following lemma.

Lemma 4.3.7. *For any open bounded set U such that $\overline{U} \subset \mathcal{O}$, we have*

$$\forall z \in U, \quad E_0^+(z) = \mathrm{Im}\,(Q_U^+(z)) \quad \text{and} \quad E_0^-(z) = \mathrm{Im}\,(Q_U^-(z)).$$

Therefore, for all $z \in \mathcal{O}$, $\dim E_0^+(z) = dp$ and $\dim E_0^-(z) = dq$. Also, for $W_0 \in \mathbb{C}^{d(p+q)}$, we have that

$$W_0 \in E_0^+(z) \cap E_0^-(z) \Leftrightarrow (X_j(z)W_0)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{C}^{(p+q)d})$$

where $X_j(z)$ is defined by (4.3.4).

Proof We prove the first set equality on $E_0^+(z)$. The second one on $E_0^-(z)$ would be proved similarly.

— For $W_0 \in \mathrm{Im}\,(Q_U^+(z))$, we have for $j \in \mathbb{N}$ using (4.3.10a)

$$X_j(z)W_0 = X_j(z)Q_U^+(z)X_0(z)^{-1}W_0 \underset{j\to+\infty}{\to} 0.$$

Thus, we have that $W_0 \in E_0^+(z)$.

— For $W_0 \in E_0^+(z)$, we have for $j \in \mathbb{N}$ using (4.3.10b)

$$(Id - Q_U^+(z))W_0 = X_0(z)(Id - Q_U^+(z))X_j(z)^{-1}X_j(z)W_0 \underset{j\to+\infty}{\rightarrow} 0.$$

Thus, W_0 belongs to the kernel of $Id - Q_U^+(z)$, i.e. $W_0 \in \mathrm{Im}\,(Q_U^+(z))$.
Therefore, we have proved that

$$E_0^\pm(z) = \mathrm{Im}\,(Q_U^\pm(z)).$$

For $W_0 \in \mathbb{C}^{d(p+q)}$, we immediately have that if the family $(X_j(z)W_0)_{j\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z}, \mathbb{C}^{d(p+q)})$, then W_0 belongs to $E_0^+(z) \cap E_0^-(z)$. We now consider $W_0 \in E_0^+(z) \cap E_0^-(z) = \mathrm{Im}\,(Q_U^+(z)) \cap \mathrm{Im}\,(Q_U^-(z))$. Since we have

$$\forall j \in \mathbb{Z}, \quad X_j(z)W_0 = X_j(z)Q_U^+(z)X_0(z)^{-1}W_0 = X_j(z)Q_U^-(z)X_0(z)^{-1}W_0,$$

the inequalities (4.3.10a) and (4.3.10c) imply that $(X_j(z)W_0)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{C}^{d(p+q)})$. □

Let us now come back to the heart of the matter: the study of the spectrum of the operator $\mathscr{L}$. We introduced the dynamical system (4.3.3) to study the solutions of the eigenvalue problem

$$(zId_{\ell^2} - \mathscr{L})u = 0.$$

The following lemma, for which the main part is proved in [Ser07, Theorem 4.1], is deduced by using the geometric dichotomy.

Lemma 4.3.8. *For $z \in \mathcal{O}$, we have that*

$$\dim \ker(zId_{\ell^2} - \mathscr{L}) = \dim E_0^+(z) \cap E_0^-(z). \tag{4.3.31}$$

Furthermore, $zId_{\ell^2} - \mathscr{L}$ is a Fredholm operator of index 0, i.e.

$$\sigma_{ess}(\mathscr{L}) \cap \mathcal{O} = \emptyset.$$

Before proving Lemma 4.3.8, let us thus introduce the linear map which extracts the center values of a vector of size $d(p+q)$

$$\Pi : \begin{array}{ccc} \mathbb{C}^{d(p+q)} & \to & \mathbb{C}^d \\ (x_j)_{j\in\{1,\dots,d(p+q)\}} & \mapsto & (x_j)_{j\in\{d(q-1)+1,\dots,dq\}} \end{array}. \tag{4.3.32}$$

We now give the proof of Lemma 4.3.8. Let us point out that the proof of the fact that the essential spectrum of $\mathscr{L}$ does not intersect $\mathcal{O}$ is exactly the same proof as in [Ser07, Theorem 4.1].

Proof of Lemma 4.3.8

We consider $z \in \mathcal{O}$ and start by proving the relation (4.3.31).

• For $w \in \ker(zId_{\ell^2} - \mathscr{L})$, if we introduce

$$W_0 := \begin{pmatrix} w_{q-1} \\ \vdots \\ w_{-p} \end{pmatrix} \in \mathbb{C}^{d(p+q)},$$

then we have that

$$\forall j \in \mathbb{Z}, \quad X_j(z)W_0 = \begin{pmatrix} w_{j+q-1} \\ \vdots \\ w_{j-p} \end{pmatrix}.$$

Since w belongs to $\ell^2(\mathbb{Z}, \mathbb{C}^d)$, we have that $W_0 \in E_0^+(z) \cap E_0^-(z)$. This implies that the linear map:

$$\varphi : \begin{array}{ccc} \ker(zId_{\ell^2} - \mathscr{L}) & \to & E_0^+(z) \cap E_0^-(z) \\ w & \mapsto & \begin{pmatrix} w_{q-1} \\ \vdots \\ w_{-p} \end{pmatrix} \end{array}$$

is well-defined.

• We consider $W_0 \in E_0^+(z) \cap E_0^-(z)$ and define for $j \in \mathbb{Z}$

$$w_j := \Pi(X_j(z)W_0) \in \mathbb{C}^d$$

where the operator Π is defined by (4.3.32). Lemma 4.3.7 implies that the sequence $w := (w_j)_{j\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z},\mathbb{C}^d)$. Furthermore, since $(X_j(z)W_0)_{j\in\mathbb{Z}}$ is a solution of (4.3.3), we have that

$$(zId_{\ell^2} - \mathscr{L})w = 0.$$

Therefore, the linear map:

$$\begin{array}{cccc} \psi: & E_0^+(z)\cap E_0^-(z) & \to & \ker(zId_{\ell^2}-\mathscr{L}) \\ & W_0 & \mapsto & (\Pi(X_j(z)W_0))_{j\in\mathbb{Z}} \end{array}$$

is also well-defined and we can easily verify the solutions:

$$\varphi\circ\psi = Id_{E_0^+(z)\cap E_0^-(z)} \quad \text{and} \quad \psi\circ\varphi = Id_{\ker(zId_{\ell^2}-\mathscr{L})}.$$

This concludes the proof of (4.3.31).

We now focus on the second part of Lemma 4.3.8 which consists in proving that for any $z\in\mathcal{O}$ we have that $zId_{\ell^2}-\mathscr{L}$ is a Fredholm operator of index 0. This part of the proof is the same as [Ser07, Theorem 4.1]. Our first goal is to prove that $zId_{\ell^2}-\mathscr{L}$ is a Fredholm operator. We have already that

$$\dim\ker(zId_{\ell^2}-\mathscr{L}) = \dim E_0^+(z)\cap E_0^-(z) < +\infty.$$

There remains to prove that $\operatorname{Im}(zId_{\ell^2}-\mathscr{L})$ is closed and that

$$\operatorname{codim}\operatorname{Im}(zId_{\ell^2}-\mathscr{L}) < +\infty.$$

We now fix a bounded open neighborhood U of $z\in\mathcal{O}$ such that $\overline{U}\subset\mathcal{O}$. We consider $h\in\ell^2(\mathbb{Z},\mathbb{C}^d)$. The sequence h belongs to the range of $zId_{\ell^2}-\mathscr{L}$ if and only if there exists $v\in\ell^2(Z,\mathbb{C}^d)$ such that, if we define the vectors:

$$\forall j\in\mathbb{Z},\quad W_j = \begin{pmatrix} v_{j+q-1} \\ \vdots \\ v_{j-p}\end{pmatrix}\in\mathbb{C}^{d(p+q)},\quad H_j = \begin{pmatrix}\mathbb{A}_{j,q}(z)^{-1}h_j \\ 0 \\ \vdots \\ 0\end{pmatrix}\in\mathbb{C}^{d(p+q)},$$

then there holds the recurrence relations

$$\forall j\in\mathbb{Z},\quad W_{j+1} = M_j(z)W_j + H_j. \tag{4.3.33}$$

For $j\geq 0$, we define

$$Z_j^+ := \sum_{k=0}^{j} X_j(z)Q_U^+(z)X_k(z)^{-1}H_{k-1} - \sum_{k=j+1}^{+\infty} X_j(z)(Id - Q_U^+(z))X_k(z)^{-1}H_{k-1},$$

where the matrix $Q_U^+(z)$ is defined in Lemma 4.3.4. Those vectors Z_j^+ are well-defined and verify that

$$\forall j\geq 0,\quad Z_{j+1}^+ = M_j(z)Z_j^+ + H_j.$$

Furthermore, using inequalities (4.3.10a) and (4.3.10b), there exist two positive constants C, c such that

$$\forall j\in\mathbb{N},\quad |Z_j^+|\leq C\sum_{k=0}^{+\infty} e^{-c|j-k|}|H_{k-1}|.$$

Using Young's convolution inequality, the sequence $(Z_j^+)_{j\in\mathbb{N}}$ belongs to $\ell^2(\mathbb{N})$ and satisfies the estimates

$$\left(\sum_{j\geq 0}|Z_j^+|^2\right)^{\frac{1}{2}} \leq C\left(\sum_{j\in\mathbb{Z}}|H_j|^2\right)^{\frac{1}{2}}$$

where the positive C is independent of h. At this stage, we have found a particular solution of (4.3.33) on $\mathbb{N}$ and any sequence $(\tilde{Z}_j)_{j\geq 0}$ solution of (4.3.33) on $\mathbb{N}$ can be written as:

$$\forall j\geq 0,\quad \tilde{Z}_j = Z_j^+ + X_j(z)V^+$$

where $V^+\in E_0^+(z)$. We prove in a similar way that the sequences $(\tilde{Z}_j)_{j\leq 0}\in\ell^2(-\mathbb{N})$ solution of (4.3.33) on $-\mathbb{N}$

are the sequences defined by

$$\forall j \leq 0, \quad \tilde{Z}_j = Z_j^- + X_j(z)V^-$$

where V^- is a vector in the finite dimension space $E_0^-(z)$ and the vectors Z_j^- are defined by:

$$\forall j \leq 0, \quad Z_j^- := \sum_{k=-\infty}^{j} X_j(z)(Id - Q_U^-(z))X_k(z)^{-1}H_{k-1} - \sum_{k=j+1}^{1} X_j(z)Q_U^-(z)X_k(z)^{-1}H_{k-1}.$$

We also have that

$$\left(\sum_{j\leq 0}|Z_j^-|^2\right)^{\frac{1}{2}} \leq C\left(\sum_{j\in\mathbb{Z}}|H_j|^2\right)^{\frac{1}{2}}$$

where the positive constant C is independent from h. Using all those information, we conclude that a sequence h belongs to the range of $zId_{\ell^2} - \mathscr{L}$ if and only if there exists a couple of vectors $(V^+, V^-) \in E_0^+(z) \times E_0^-(z)$ such that

$$Z_0^+ - Z_0^- = V^- - V^+.$$

If we now define the bounded operator

$$\nu : h \in \ell^2(\mathbb{Z}, \mathbb{C}^d) \mapsto Z_0^+ - Z_0^- \in \mathbb{C}^{d(p+q)} \tag{4.3.34}$$

and

$$\varphi : (V^+, V^-) \in E_0^+(z) \times E_0^-(z) \mapsto V^- - V^+ \in \mathbb{C}^{d(p+q)} \tag{4.3.35}$$

which is an operator from a finite dimension vector space to another finite dimension vector space, then we have proved that

$$\mathrm{Im}\,(zId_{\ell^2} - \mathscr{L}) = \nu^{-1}(\mathrm{Im}\,\varphi).$$

Therefore, the range $\mathrm{Im}\,(zId_{\ell^2} - \mathscr{L})$ is closed.

We now want to prove that $\mathrm{codim}\,\mathrm{Im}\,(zId_{\ell^2} - \mathscr{L}) < +\infty$. We consider $N \geq 1$ such that there exists $(h_1, \dots, h_N)$ a linearly independent family of $\ell^2(\mathbb{Z}, \mathbb{C}^d)$ such that

$$\mathrm{Im}\,(zId_{\ell^2} - \mathscr{L}) \cap Span(h_1, \dots, h_N) = \{0\}.$$

We are going to prove that the family $(\nu(h_1), \dots, \nu(h_N))$ is linearly independent in $\mathbb{C}^{d(p+q)}$ and therefore that $N \leq d(p+q)$. Here and below, the operator ν is the one defined in (4.3.34). We consider scalars $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ such that

$$0 = \sum_{i=1}^{N} \lambda_i \nu(h_i) = \nu\left(\sum_{i=1}^{N} \lambda_i h_i\right).$$

We therefore have that

$$\nu\left(\sum_{i=1}^{N} \lambda_i h_i\right) \in \mathrm{Im}\,\varphi$$

with φ defined in (4.3.35) and thus

$$\sum_{i=1}^{N} \lambda_i h_i \in \mathrm{Im}\,(zId_{\ell^2} - \mathscr{L}) \cap Span(h_1, \dots, h_N).$$

This implies that

$$\sum_{i=1}^{N} \lambda_i h_i = 0$$

and the linear independency of $(h_1, \dots, h_N)$ allows us to conclude that $\lambda_1 = \ldots = \lambda_N = 0$. We have thus proved that $zId_{\ell^2} - \mathscr{L}$ is a Fredholm operator for all $z \in \mathcal{O}$. We also know that, since $\mathcal{O}$ is unbounded, there exists $z \in \mathcal{O}$ such that $zId_{\ell^2} - \mathscr{L}$ is an isomorphism. The set $\mathcal{O}$ being connected and by continuity of the Fredholm index, the statement of the lemma is true. □

We now introduce the sets

$$\mathcal{O}_\rho := \mathcal{O} \cap \rho(\mathscr{L}) \quad \text{and} \quad \mathcal{O}_\sigma := \mathcal{O} \cap \sigma(\mathscr{L}).$$

Lemma 4.3.8 implies that the set $\mathcal{O}_\sigma$ only contains eigenvalues of $\mathscr{L}$. Then, because of Hypothesis 4.6, we have that

$$\overline{\mathbb{U}}\backslash\{1\} \subset \mathcal{O}_\rho.$$

We also observe that Lemma 4.3.7 gives us the dimension of the subspaces $E_0^{\pm}(z)$. Then (4.3.31) implies that

$$\forall z \in \mathcal{O}_\rho, \quad E_0^+(z) \oplus E_0^-(z) = \mathbb{C}^{(p+q)d}. \tag{4.3.36}$$

Thus, for $z \in \mathcal{O}_\rho$, we can define the unique projector $Q(z)$ from $\mathbb{C}^{d(p+q)}$ to $E_0^+(z)$ such that

$$\operatorname{Im} Q(z) = E_0^+(z) \quad \text{and} \quad \ker Q(z) = E_0^-(z).$$

The function $z \in \mathcal{O}_\rho \mapsto Q(z)$ is holomorphic (see [Kat95]). We will now prove that the function Q is fundamental to the study of (4.3.3) by extending the geometric dichotomy. The following lemma is once again very much inspired by [Laf01, Section III.1.5] and [Cop78].

Lemma 4.3.9 (Extended geometric dichotomy). *For any bounded open set U such that $\overline{U} \subset \mathcal{O}_\rho$, there exist two positive constants $C, c > 0$ such that for all $z \in U$, the projector $Q(z)$ associated with the decomposition* (4.3.36) *satisfies:*

$$\forall j \geq k, \quad \left|X_j(z)Q(z)X_k(z)^{-1}\right| \leq Ce^{-c|j-k|}, \tag{4.3.37a}$$
$$\forall k \geq j, \quad \left|X_j(z)(Id - Q(z))X_k(z)^{-1}\right| \leq Ce^{-c|j-k|}. \tag{4.3.37b}$$

Proof We begin by assuming that we proved the existence of two constants $C, c > 0$ such that for all $z \in U$

$$\forall j \geq k \geq 0, \quad \left|X_j(z)Q(z)X_k(z)^{-1}\right| \leq Ce^{-c|j-k|}, \tag{4.3.38a}$$
$$\forall k \geq j \geq 0, \quad \left|X_j(z)(Id - Q(z))X_k(z)^{-1}\right| \leq Ce^{-c|j-k|}, \tag{4.3.38b}$$
$$\forall k \leq j \leq 0, \quad \left|X_j(z)Q(z)X_k(z)^{-1}\right| \leq Ce^{-c|j-k|}, \tag{4.3.38c}$$
$$\forall j \leq k \leq 0, \quad \left|X_j(z)(Id - Q(z))X_k(z)^{-1}\right| \leq Ce^{-c|j-k|}. \tag{4.3.38d}$$

Then, we observe that to prove the assertion (4.3.37a), there would only remain to prove (4.3.37a) in the case where $j \geq 0 \geq k$. Using (4.3.38a) and (4.3.38c), we have

$$\left|X_j(z)Q(z)X_k(z)^{-1}\right| \leq \left|X_j(z)Q(z)X_0(z)^{-1}\right|\left|X_0(z)Q(z)X_k(z)^{-1}\right| \leq C^2e^{-c(j-k)}.$$

Hence, the assertion (4.3.37a) follows from (4.3.38a) and (4.3.38c). Similarly, (4.3.37b) follows from (4.3.38b) and (4.3.38d).

Therefore, there only remains to prove the existence of $C, c > 0$ such that (4.3.38a)-(4.3.38d) are true for all $z \in U$. We will prove (4.3.38a) and (4.3.38c). The proof for (4.3.38b) and (4.3.38d) can be dealt with similarly. First, we need to consider a bounded open set V such that $\overline{V} \subset \mathcal{O}_\rho$ and $\overline{U} \subset V$ and the projector Q_V^+ provided by Lemma 4.3.4. This will be useful later on to bound the difference $Q_V^+ - Q$. For $z \in U$, Lemma 4.3.7 implies that $E_0^+(z) = \operatorname{Im} Q_V^+(z) = \operatorname{Im} Q(z)$, i.e.

$$Q_V^+(z)Q(z) = Q(z) \quad \text{and} \quad Q(z)Q_V^+(z) = Q_V^+(z).$$

This allows us to prove that

$$Q_V^+(z) - Q(z) = Q_V^+(z)(Q_V^+(z) - Q(z))(Id - Q_V^+(z)).$$

Therefore, for $j, k \in \mathbb{N}$, we have

$$X_j(z)(Q_V^+(z) - Q(z))X_k(z)^{-1} = X_j(z)Q_V^+(z)X_0(z)^{-1}(Q_V^+(z) - Q(z))X_0(z)(Id - Q_V^+(z))X_k(z)^{-1}.$$

Thus, because of the inequalities (4.3.10a) and (4.3.10b), we have the estimate

$$\left|X_j(z)(Q_V^+(z) - Q(z))X_k(z)^{-1}\right| \leq C^2e^{-c(j+k)}|Q_V^+(z) - Q(z)|. \tag{4.3.39}$$

Using the inequalities (4.3.10a), (4.3.10b) and (4.3.39), we can thus prove that

$$\forall j \geq k \geq 0, \quad |X_j(z)Q(z)X_k(z)^{-1}| \leq Ce^{-c(j-k)} + C^2e^{-c(j+k)}|Q_V^+(z) - Q(z)|$$

and
$$\forall k \geq j \geq 0, \quad |X_j(z)(Id - Q(z))X_k(z)^{-1}| \leq Ce^{-c(k-j)} + C^2e^{-c(j+k)}|Q_V^+(z) - Q(z)|.$$
Since $z \in V \mapsto |Q_V^+(z) - Q(z)|$ is continuous and $\overline{U} \subset V$, we can uniformly bound $|Q_V^+(z) - Q(z)|$ for $z \in U$. We can then deduce the existence of two positive constants C, c to verify the inequalities (4.3.38a) and (4.3.38c). $\square$

4.3.5 Bounds on the spatial Green's function far from 1

For $z \in \mathcal{O}_\rho$ and $j_0 \in \mathbb{Z}$, since z is in the resolvent set of $\mathscr{L}$, the spatial Green's function $G(z, j_0, \cdot)$ defined by (4.1.26) is well-defined. We observe that the function $z \in \mathcal{O}_\rho \mapsto G(z, j_0, \cdot)$ is holomorphic.

We consider $\mathbf{e} \in \mathbb{C}^d$. We then observe that the vector valued sequence $G(z, j_0, \cdot)\mathbf{e}$ belongs to $\ell^2(\mathbb{Z}, \mathbb{C}^d)$ and satisfies:
$$zG(z, j_0, \cdot)\mathbf{e} - \mathscr{L}G(z, j_0, \cdot)\mathbf{e} = \delta_{j_0}\mathbf{e},$$
i.e.
$$\forall j \in \mathbb{Z}, \quad \sum_{k=-p}^{q} \mathbb{A}_{j,k}(z)G(z, j_0, j+k)\mathbf{e} = \delta_{j_0,j}\mathbf{e}.$$
Thus, we have that
$$\forall j \in \mathbb{Z}, \quad W(z, j_0, j+1, \mathbf{e}) = M_j(z)W(z, j_0, j, \mathbf{e}) - \begin{pmatrix} A_{j,q}^{-1}\delta_{j_0,j}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.3.40}$$
where $W(z, j_0, j, \mathbf{e}) := \begin{pmatrix} G(z, j_0, j+q-1)\mathbf{e} \\ \vdots \\ G(z, j_0, j-p)\mathbf{e} \end{pmatrix}$. We will now prove the following proposition using the extended geometric dichotomy (Lemma 4.3.9).

Proposition 4.2 (Bounds far from 1)**.** *For U a bounded open set such that $\overline{U} \subset \mathcal{O}_\rho$, there exist two constants $C, c > 0$ such that*
$$\forall z \in U, \forall \boldsymbol{e} \in \mathbb{C}^d, \forall j, j_0 \in \mathbb{Z}, \quad |W(z, j_0, j, \mathbf{e})| \leq C|\mathbf{e}|e^{-c|j-j_0|}.$$

In particular, the result of Proposition 4.2 holds true in a neighborhood of any point $z \in \overline{\mathbb{U}} \backslash \{1\}$. A direct consequence of Proposition 4.2 on the spatial Green's function (4.1.26) is that for any U a bounded open set such that $\overline{U} \subset \mathcal{O}_\rho$, there exist two constants $C, c > 0$ such that
$$\forall z \in U, \forall j, j_0 \in \mathbb{Z}, \quad |G(z, j_0, j)| \leq Ce^{-c|j-j_0|}.$$

Proof of Proposition 4.2

We consider $z \in U$, $j, j_0 \in \mathbb{Z}$ and $\mathbf{e} \in \mathbb{C}^d$ such that $|\mathbf{e}| \leq 1$. The equality (4.3.40) implies the following results:

— We have
$$\forall j \geq j_0 + 1, \quad W(z, j_0, j+1, \mathbf{e}) = M_j(z)W(z, j_0, j, \mathbf{e}),$$
i.e.
$$\forall j \geq j_0 + 1, \quad W(z, j_0, j, \mathbf{e}) = X_j(z)X_{j_0+1}(z)^{-1}W(z, j_0, j_0+1, \mathbf{e}). \tag{4.3.41}$$
Also, since $G(z, j_0, \cdot)\mathbf{e} \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$, we have that $X_{j_0+1}(z)^{-1}W(z, j_0, j_0+1, \mathbf{e}) \in E_0^+(z)$.

— We have
$$\forall j \leq j_0 - 1, \quad W(z, j_0, j+1, \mathbf{e}) = M_j(z)W(z, j_0, j, \mathbf{e}),$$
i.e.
$$\forall j \leq j_0, \quad W(z, j_0, j, \mathbf{e}) = X_j(z)X_{j_0}(z)^{-1}W(z, j_0, j_0, \mathbf{e}). \tag{4.3.42}$$
Also, since $G(z, j_0, \cdot)\mathbf{e} \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$, we have that $X_{j_0}(z)^{-1}W(z, j_0, j_0, \mathbf{e}) \in E_0^-(z)$.

— We have
$$W(z, j_0, j_0+1, \mathbf{e}) = M_{j_0}(z)W(z, j_0, j_0, \mathbf{e}) - \begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

i.e.

$$X_{j_0+1}(z)^{-1}W(z,j_0,j_0+1,\mathbf{e}) - X_{j_0}(z)^{-1}W(z,j_0,j_0,\mathbf{e}) = -X_{j_0+1}(z)^{-1}\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $Q(z)$ is the projection on $E_0^+(z)$ with respect to $E_0^-(z)$, we have that

$$X_{j_0+1}(z)^{-1}W(z,j_0,j_0+1,\mathbf{e}) = -Q(z)X_{j_0+1}(z)^{-1}\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$X_{j_0}(z)^{-1}W(z,j_0,j_0,\mathbf{e}) = (Id - Q(z))X_{j_0+1}(z)^{-1}\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using (4.3.41) and (4.3.42), we thus have the formulas:

$$\forall j \geq j_0+1, \quad W(z,j_0,j,\mathbf{e}) = -X_j(z)Q(z)X_{j_0+1}(z)^{-1}\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.3.43a}$$

$$\forall j \leq j_0, \quad W(z,j_0,j,\mathbf{e}) = X_j(z)(Id - Q(z))X_{j_0+1}(z)^{-1}\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.3.43b}$$

We now apply the inequalities (4.3.37a) and (4.3.37b) and obtain:

$$\forall z \in U, \forall j, j_0 \in \mathbb{Z}, \quad |W(z,j_0,j,\mathbf{e})| \leq Ce^{-c|j-(j_0+1)|}\left|\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right|.$$

The result of Proposition 4.2 follows. □

4.4 Extension of the spatial Green's function near 1

The analysis of the spatial Green's function done in the previous section does not hold near 1. The first reason is that we can prove that 1 is an eigenvalue of $\mathscr{L}$ and the curves describing the spectrum of the limit operators $\mathscr{L}^\pm$ in (4.1.16) should belong to the essential spectrum of the operator $\mathscr{L}$. Thus, the definition of the spatial Green's function breaks down near 1. The second reason is that the matrices $M^\pm(1)$ have central eigenvalues equal to 1 as explained in Lemma 4.3.3, thus the geometric dichotomy will not work near 1. To circumvent these issues, the strategy will be to refine the analysis of (4.3.3) near 1 by finding a particular basis of $E_0^\pm(z)$ and using this basis to express the spatial Green's function. In some sense, it amounts at using the projections on a basis of solutions of (4.3.3) rather than the projection associated with the geometric dichotomy. This amounts to adapt in a fully discrete setting the same strategy as in [ZH98; MZ02; BHR03; Bec+10] which tackle continuous or semi-discrete problems.

4.4.1 Right and left eigenvectors of $M^\pm(z)$ for z near 1

To study the spatial Green's function for z near 1, we will need to study the solutions of the dynamical system (4.3.3) with more accuracy. The first step is to have a better understanding of the eigenvalues and eigenvectors of $M^\pm(z)$ when z is close to 1.

First, let us make some observations on the eigenvalues of $M_l^\pm(1)$ defined by (4.3.8) for $l \in \{1, \dots, d\}$. Using Lemma 4.3.3, we know that the eigenvalues $\kappa \in \mathbb{C}\backslash\{0\}$ of $M_l^\pm(1)$ are the solutions of

$$\mathcal{F}_l^\pm(\kappa) = 1.$$

Hypothesis 4.9 allows us to conclude that the matrix $M_l^\pm(1)$ only has simple eigenvalues. Furthermore, Lemma 4.3.3 implies that 1 is a simple eigenvalue of $M_l^\pm(1)$ and that the rest of the eigenvalues are in $\mathbb{D}\backslash\{0\}$ or $\mathbb{U}$ and we know the number of eigenvalues in each set depending on the sign of $\alpha_l^\pm$. Thus, we can define a family $(\underline{\zeta}_m^\pm)_{m\in\{1,\dots,d(p+q)\}} \in \mathbb{C}^{d(p+q)}$ such that

$$\forall l \in \{1, \dots, d\}, \quad \sigma(M_l^\pm(1)) = \left\{\underline{\zeta}_l^\pm, \underline{\zeta}_{l+d}^\pm, \dots, \underline{\zeta}_{l+(p+q-1)d}^\pm\right\}.$$

Furthermore, using Hypothesis 4.1 to determine the sign of $\alpha_l^\pm$ defined by (4.1.17) and Lemma 4.3.3, we can index them in order to have the following fact.

— For all $l \in \{1, \dots, I\}$, since $\alpha_l^+ < 0$, we choose

$$\underline{\zeta}_l^+, \dots, \underline{\zeta}_{l+d(p-1)}^+ \in \mathbb{D}, \quad \underline{\zeta}_{l+dp}^+ = 1, \quad \underline{\zeta}_{l+d(p+1)}^+, \dots, \underline{\zeta}_{l+d(p+q-1)}^+ \in \mathbb{U}.$$

— For all $l \in \{I+1, \dots, d\}$, since $\alpha_l^+ > 0$, we choose

$$\underline{\zeta}_l^+, \dots, \underline{\zeta}_{l+d(p-2)}^+ \in \mathbb{D}, \quad \underline{\zeta}_{l+d(p-1)}^+ = 1, \quad \underline{\zeta}_{l+dp}^+, \dots, \underline{\zeta}_{l+d(p+q-1)}^+ \in \mathbb{U}.$$

— For all $l \in \{1, \dots, I-1\}$, since $\alpha_l^- < 0$, we choose

$$\underline{\zeta}_l^-, \dots, \underline{\zeta}_{l+d(p-1)}^- \in \mathbb{D}, \quad \underline{\zeta}_{l+dp}^- = 1, \quad \underline{\zeta}_{l+d(p+1)}^-, \dots, \underline{\zeta}_{l+d(p+q-1)}^- \in \mathbb{U}.$$

— For all $l \in \{I, \dots, d\}$, since $\alpha_l^- > 0$, we choose

$$\underline{\zeta}_l^-, \dots, \underline{\zeta}_{l+d(p-2)}^- \in \mathbb{D}, \quad \underline{\zeta}_{l+d(p-1)}^- = 1, \quad \underline{\zeta}_{l+dp}^-, \dots, \underline{\zeta}_{l+d(p+q-1)}^- \in \mathbb{U}.$$

We indexed the eigenvalues to separate the stable, central and unstable eigenvalues of the matrices $M_l^\pm(1)$. More precisely, we observe that if we introduce the sets

$$\begin{aligned} I_{ss}^+ &:= \{1, \dots, d(p-1)+I\}, & I_{ss}^- &:= \{1, \dots, d(p-1)+I-1\}, \\ I_{cs}^+ &:= \{d(p-1)+I+1, \dots, dp\}, & I_{cs}^- &:= \{d(p-1)+I, \dots, dp\}, \\ I_{cu}^+ &:= \{dp+1, \dots, dp+I\}, & I_{cu}^- &:= \{dp+1, \dots, dp+I-1\}, \\ I_{su}^+ &:= \{dp+I+1, \dots, d(p+q)\}, & I_{su}^- &:= \{dp+I, \dots, d(p+q)\}, \end{aligned}$$

then we have that

$$\begin{aligned} \forall m \in I_{ss}^\pm, &\quad \underline{\zeta}_m^\pm \in \mathbb{D}, \\ \forall m \in I_{cs}^\pm \cup I_{cu}^\pm, &\quad \underline{\zeta}_m^\pm = 1, \\ \forall m \in I_{su}^\pm, &\quad \underline{\zeta}_m^\pm \in \mathbb{U}. \end{aligned}$$

Since those are simple eigenvalues of $M_l^\pm(1)$, we are able to extend them holomorphically in a neighborhood of 1. We consider $\delta_0 > 0$ a radius such that for each $m = l + (k-1)d \in \{1, \dots, d(p+q)\}$ with $k \in \{1, \dots, p+q\}$ and $l \in \{1, \dots, d\}$, there exists a holomorphic function $\zeta_m^\pm : B(1, \delta_0) \to \mathbb{C}$ such that $\zeta_m^\pm(1) = \underline{\zeta}_m^\pm$ and for all $z \in B(1, \delta_0)$, $\zeta_m^\pm(z)$ is a simple eigenvalue of $M_l^\pm(z)$. We will also separate the different types of eigenvalues by assuming that we chose δ_0 small enough so that there exists a constant $c_* > 0$ such that for all $z \in B(1, \delta_0)$

$$\forall m \in I_{ss}^\pm, \quad |\zeta_m^\pm(z)| \leq \exp(-2c_*), \tag{4.4.1a}$$

$$\forall m \in I_{cs}^\pm \cup I_{cu}^\pm, \quad \exp(-c_*) \leq |\zeta_m^\pm(z)| \leq \exp(c_*) \tag{4.4.1b}$$

$$\forall m \in I_{su}^\pm, \quad \exp(2c_*) \leq |\zeta_m^\pm(z)|. \tag{4.4.1c}$$

When we will study the temporal Green's function $\mathscr{G}(n, j_0, j)$ later on in Section 4.5, we will have to bound terms of the form

$$|\zeta_m^\pm(z)|^j |\zeta_{m'}^\pm(z)|^{-j_0}.$$

The inequalities (4.4.1a)-(4.4.1c) will allow us in a lot of cases to obtain exponential bounds for some of those terms.

Using Lemma 4.3.2, we thus have a complete description of the eigenvalues of $M^{\pm}(z)$ for z in a neighborhood of 1. The following lemma also allows us to introduce a basis of eigenvectors for the matrices $M^{\pm}(z)$.

Lemma 4.4.1. *For $m = l + (k-1)d \in \{1, \dots, d(p+q)\}$ with $k \in \{1, \dots, p+q\}$ and $l \in \{1, \dots, d\}$, the vector*

$$R_m^{\pm}(z) := \begin{pmatrix} \zeta_m^{\pm}(z)^{q-1} \boldsymbol{r}_l^{\pm} \\ \vdots \\ \zeta_m^{\pm}(z)^{-p} \boldsymbol{r}_l^{\pm} \end{pmatrix} \in \mathbb{C}^{d(p+q)} \tag{4.4.2}$$

is an eigenvector of $M^{\pm}(z)$ associated with the eigenvalue $\zeta_m^{\pm}(z)$. Furthermore, for all $z \in B(1, \delta_0)$, the family $(R_m^{\pm}(z))_{m \in \{1,\dots,d(p+q)\}}$ is a basis of $\mathbb{C}^{d(p+q)}$.

Proof Let us start by proving that the vector $R_m^{\pm}(z)$ defined by (4.4.2) is an eigenvector of $M^{\pm}(z)$ associated with the eigenvalue $\zeta_m^{\pm}(z)$. We have that $\zeta_m^{\pm}(z)$ is an eigenvalue of $M_l^{\pm}(z)$ so Lemma 4.3.3 implies that

$$\mathcal{F}_l^{\pm}(\zeta_m^{\pm}(z)) = z.$$

We use the definition (4.3.7) of the functions $\Lambda_{l,k}^{\pm}$ and the definition (4.1.15) of the function $\mathcal{F}_l^{\pm}$ to prove that

$$\begin{aligned} -\sum_{k=-p}^{q-1} \mathbb{A}_q^{\pm}(z)^{-1} \mathbb{A}_k^{\pm}(z) \zeta_m^{\pm}(z)^k \boldsymbol{r}_l^{\pm} &= -\sum_{k=-p}^{q-1} \Lambda_{l,q}^{\pm}(z)^{-1} \Lambda_{l,k}^{\pm}(z) \zeta_m^{\pm}(z)^k \boldsymbol{r}_l^{\pm} \\ &= \left(\zeta_m^{\pm}(z)^q + \Lambda_{l,q}^{\pm}(z)^{-1} \left(z - \mathcal{F}_l^{\pm}(\zeta_m^{\pm}(z)) \right) \right) \boldsymbol{r}_l^{\pm} \\ &= \zeta_m^{\pm}(z)^q \boldsymbol{r}_l^{\pm}. \end{aligned}$$

This allows us to conclude that the vector $R_m^{\pm}(z)$ is an eigenvector of $M^{\pm}(z)$ associated with the eigenvalue $\zeta_m^{\pm}(z)$.

We now consider $z \in B(1, \delta_0)$ and a family of complex numbers $(\lambda_m)_{m \in \{1,\dots,d(p+q)\}}$ such that:

$$0 = \sum_{m=1}^{d(p+q)} \lambda_m R_m^{\pm}(z).$$

Separating the blocks of coefficients of size d in the previous equality and observing that the family $(\boldsymbol{r}_l^{\pm})_{l \in \{1,\dots,d\}}$ is linearly independent, we have for all $l \in \{1, \dots, d\}$:

$$\forall j \in \{-p, \dots, q-1\}, \quad 0 = \sum_{k=1}^{p+q} \lambda_{l+(k-1)d} \zeta_{l+(k-1)d}^{\pm}(z)^j.$$

We have that, for each integer $k \in \{1, \dots, p+q\}$, $\zeta_{l+(k-1)d}^{\pm}(z)$ is a simple eigenvalue of $M_l^{\pm}(z)$ for all $z \in B(1, \delta_0)$. Therefore, the complex values $(\zeta_{l+(k-1)d}^{\pm}(z))_{k \in \{1,\dots,p+q\}}$ are distinct and thus

$$\forall k \in \{1, \dots, p+q\}, \quad \lambda_{l+d(k-1)} = 0.$$

Since this is true for all $l \in \{1, \dots, d\}$, we proved that the family $(R_m^{\pm}(z))_{m \in \{1,\dots,d(p+q)\}}$ is linearly independent and is thus a basis of $\mathbb{C}^{d(p+q)}$. $\square$

Thus, we have a characterization of the eigenvalues and eigenvectors of $M^{\pm}(z)$ for $z \in B(1, \delta_0)$. Lemma 4.3.3 implies that, for all $z \in \mathcal{O} \cap B(1, \delta_0)$, we have that $|\zeta_m^{\pm}(z)| < 1$ for $m \in \{1, \dots, dp\}$ and $|\zeta_m^{\pm}(z)| > 1$ for $m \in \{dp+1, \dots, d(p+q)\}$. Thus, for $z \in \mathcal{O} \cap B(1, \delta_0)$

$$E^s(M^{\pm}(z)) = \operatorname{Span}\left\{ R_m^{\pm}(z), \quad m \in \{1, \dots, dp\} \right\}$$

and

$$E^u(M^{\pm}(z)) = \operatorname{Span}\left\{ R_m^{\pm}(z), \quad m \in \{dp+1, \dots, d(p+q)\} \right\}.$$

This equality implies that we can extend holomorphically the definitions of $E^s(M^{\pm}(z))$ and $E^u(M^{\pm}(z))$ for $z \in B(1, \delta_0)$.

We now conclude this section by studying the dual basis associated with the basis $(R_m^{\pm}(z))_{m \in \{1,\dots,d(p+q)\}}$.

We introduce the invertible matrix

$$\forall z \in B(1,\delta_0), \quad N^{\pm,\infty}(z) := \begin{pmatrix} R_1^\pm(z) & | \dots | & R_{d(p+q)}^\pm(z) \end{pmatrix} \in \mathcal{M}_{d(p+q)}(\mathbb{C}) \tag{4.4.3}$$

and the vectors $L_1^\pm(z), \dots, L_{d(p+q)}^\pm(z) \in \mathbb{C}^{d(p+q)}$ defined by

$$\forall z \in B(1,\delta_0), \quad \begin{pmatrix} L_1^\pm(z) & | \dots | & L_{d(p+q)}^\pm(z) \end{pmatrix}^T := N^{\pm,\infty}(z)^{-1}. \tag{4.4.4}$$

We observe that

$$\forall z \in B(1,\delta_0), \forall m, \widetilde{m} \in \{1, \dots, d(p+q)\}, \quad L_m^\pm(z)^T R_{\widetilde{m}}^\pm(z) = \delta_{m,\widetilde{m}} \tag{4.4.5}$$

and

$$\forall z \in B(1,\delta_0), \forall m \in \{1, \dots, d(p+q)\}, \quad M^\pm(z)^T L_m^\pm(z) = \zeta_m^\pm(z) L_m^\pm(z). \tag{4.4.6}$$

We will now prove the following lemma which gives a more precise description of the vectors $L_m^\pm(z)$ of the dual basis.

Lemma 4.4.2. *We consider $m = l + (k-1)d \in \{1, \dots, d(p+q)\}$ with $k \in \{1, \dots, p+q\}$ and $l \in \{1, \dots, d\}$. For all $z \in \mathbb{C}$, there exist coefficients $x_1^\pm(z), \dots, x_{p+q}^\pm(z) \in \mathbb{C}$ such that*

$$L_m^\pm(z) := \begin{pmatrix} x_1^\pm(z) \boldsymbol{l}_l^\pm \\ \vdots \\ x_{p+q}^\pm(z) \boldsymbol{l}_l^\pm \end{pmatrix}. \tag{4.4.7}$$

Furthermore, we have

$$\forall z \in B(1,\delta_0), \quad x_1^\pm(z) = \lambda_{l,q}^\pm \frac{d\zeta_m^\pm}{dz}(z) \tag{4.4.8}$$

where $\lambda_{l,q}^\pm$ is defined by (4.1.14).

In the proof of Lemma 4.4.2, we also find the expressions of the coefficients $x_2^\pm(z), \dots, x_{p+q}^\pm(z)$ but, contrarily to $x_1^\pm(z)$, they will not be used later on in the chapter.

Proof The proof of Lemma 4.4.2 uses calculations similar to those done at the end of [Coe22, Lemma 2.4]. We consider $z \in B(1,\delta_0)$. We begin by introducing the vectors $\boldsymbol{x}_1^\pm(z), \dots, \boldsymbol{x}_{p+q}^\pm(z) \in \mathbb{C}^d$ defined by

$$\begin{pmatrix} \boldsymbol{x}_1^\pm(z) \\ \vdots \\ \boldsymbol{x}_{p+q}^\pm(z) \end{pmatrix} := L_m^\pm(z).$$

We consider $\widetilde{l} \in \{1, \dots, d\} \setminus \{l\}$. Using the definition (4.4.2) and the linear independence of the vectors $R_{\widetilde{m}}^\pm(z)$, we have that

$$\mathrm{Span}\left\{ R_{\widetilde{l}+(\widetilde{k}-1)d}^\pm(z), \quad \widetilde{k} \in \{1, \dots, p+q\} \right\} = \left\{ \begin{pmatrix} y_1 \boldsymbol{r}_{\widetilde{l}}^\pm \\ \vdots \\ y_{p+q} \boldsymbol{r}_{\widetilde{l}}^\pm \end{pmatrix}, \quad y_1, \dots, y_{p+q} \in \mathbb{C} \right\}.$$

Using (4.4.5), we can then prove that:

$$\forall y_1, \dots, y_{p+q} \in \mathbb{C}, \quad \sum_{j=1}^{p+q} y_j \boldsymbol{x}_j^\pm(z)^T \boldsymbol{r}_{\widetilde{l}}^\pm = 0$$

and thus:

$$\forall j \in \{1, \dots, p+q\}, \quad \boldsymbol{x}_j^\pm(z)^T \boldsymbol{r}_{\widetilde{l}}^\pm = 0.$$

Since this is true for all $\widetilde{l} \in \{1, \dots, d\} \setminus \{l\}$, we have that $\boldsymbol{x}_j^\pm(z) \in \mathrm{Span}\, \boldsymbol{l}_l^\pm$ for all $j \in \{1, \dots, p+q\}$.

Now that we know that we can express the vector $L_m^\pm(z)$ as (4.4.7), let us prove (4.4.8). Using (4.1.14) and the definitions (4.1.5) and (4.3.7) respectively of the vectors $\boldsymbol{l}_l^\pm$ and of the functions $\Lambda_{l,k}^\pm$, we have looking at the j-th block of size d of (4.4.6) that:

$$\forall j \in \{1, \dots, p+q-1\}, \quad \zeta_m^\pm(z) x_j^\pm(z) = x_{j+1}^\pm(z) - \Lambda_{l,q}^\pm(z)^{-1} \Lambda_{l,q-j}^\pm(z) x_1^\pm(z),$$

and:

$$\zeta_m^\pm(z)x_{p+q}^\pm(z) = -\Lambda_{l,q}^\pm(z)^{-1}\Lambda_{l,-p}^\pm(z)x_1^\pm(z).$$

Thus, we have:

$$\forall j \in \{1,\dots,p+q\}\,, \quad x_j^\pm(z) = -\left(\sum_{k=-p}^{q-j} \frac{\Lambda_{l,k}^\pm(z)}{\zeta_m^\pm(z)^{q-j-k+1}}\right)\Lambda_{l,q}^\pm(z)^{-1}x_1^\pm(z).$$

We now have an expression of each $x_j^\pm(z)$ depending on $x_1^\pm(z)$. We also recall that

$$\boldsymbol{l}_l^{\pm\,T}\boldsymbol{r}_l^\pm = 1.$$

Using the expressions (4.4.2) and (4.4.7) respectively of the vectors $R_m^\pm(z)$ and $L_m^\pm(z)$ as well as (4.4.5), we have

$$1 = L_m^\pm(z)^T R_m^\pm(z) = \sum_{j=1}^{p+q} x_j^\pm(z)\zeta_m^\pm(z)^{q-j} = -\left(\sum_{j=1}^{p+q}\sum_{k=-p}^{q-j}\Lambda_{l,k}^\pm(z)\zeta_m^\pm(z)^{k-1}\right)\Lambda_{l,q}^\pm(z)^{-1}x_1^\pm(z).$$

Using the definitions (4.3.7) and (4.1.15) of the functions $\Lambda_{l,k}^\pm$ and $\mathcal{F}_l^\pm$, we have

$$\begin{aligned}
1 &= -\left(\sum_{k=-p}^{q}(q-k)\Lambda_{l,k}^\pm(z)\zeta_m^\pm(z)^{k-1}\right)\Lambda_{l,q}^\pm(z)^{-1}x_1^\pm(z)\\
&= -\left(q\zeta_m^\pm(z)^{-1}z - \sum_{k=-p}^{q}(q-k)\lambda_{l,k}^\pm\zeta_m^\pm(z)^{k-1}\right)\Lambda_{l,q}^\pm(z)^{-1}x_1^\pm(z)\\
&= -\left(q\zeta_m^\pm(z)^{-1}(z-\mathcal{F}_l^\pm(\zeta_m^\pm(z))) + \frac{d\mathcal{F}_l^\pm}{d\kappa}(\zeta_m^\pm(z))\right)\Lambda_{l,q}^\pm(z)^{-1}x_1^\pm(z).
\end{aligned}$$

We observe that since $\zeta_m^\pm(z)$ is an eigenvalue $M_l^\pm(z)$, Lemma 4.3.3 allows us to prove that

$$\mathcal{F}_l^\pm(\zeta_m^\pm(z)) = z \quad \text{and} \quad \frac{d\zeta_m^\pm}{dz}(z)\frac{d\mathcal{F}_l^\pm}{d\kappa}(\zeta_m^\pm(z)) = 1.$$

Thus, since $\Lambda_{l,q}^\pm(z) = -\lambda_{l,q}^\pm$, we have that

$$1 = \left(\frac{d\zeta_m^\pm}{dz}(z)\lambda_{l,q}^\pm\right)^{-1}x_1^\pm(z)$$

and we deduce (4.4.8). □

4.4.2 Choice of a precise basis of $E_0^\pm(z)$ for z near 1

Now that we have a better understanding of the spectrum of $M^\pm(z)$, we are going to prove a lemma that is quite similar to the geometric dichotomy. This lemma corresponds to [God03, Lemma 3.1], itself inspired by [ZH98, Proposition 3.1].

Lemma 4.4.3. *There exist a radius $\delta_1 \in]0,\delta_0]$ and two constants $C, c > 0$ such that for all $m \in \{1,\dots,d(p+q)\}$ and $z \in B(1,\delta_1)$, there exists a sequence $(V_m^\pm(z,j))_{j\in\mathbb{Z}} \in \mathbb{C}^{(p+q)d^{\mathbb{Z}}}$ such that :*

- *For all $j \in \mathbb{Z}$, the function $V_m^\pm(\cdot,j)$ is holomorphic on $B(1,\delta_1)$.*
- *The functions $z \in B(1,\delta_1) \mapsto (V_m^+(z,j))_{j\in\mathbb{N}} \in \ell^\infty(\mathbb{N},\mathbb{C}^{d(p+q)})$ and $z \in B(1,\delta_1) \mapsto (V_m^-(z,j))_{j\in-\mathbb{N}} \in \ell^\infty(-\mathbb{N},\mathbb{C}^{d(p+q)})$ are holomorphic. Furthermore, up to considering a smaller radius δ_1, those functions and their derivatives are bounded on $B(1,\delta_1)$.*
- *For $z \in B(1,\delta_1)$, if we define $W_m^\pm(z,j) := \zeta_m^\pm(z)^j V_m^\pm(z,j)$ for all $j \in \mathbb{Z}$, then $W_m^\pm(z,\cdot)$ is a solution of (4.3.3), i.e.*

$$\forall j \in \mathbb{Z}, \quad W_m^\pm(z,j+1) = M_j(z)W_m^\pm(z,j).$$

- *We have*

$$\forall z \in B(1,\delta_1), \quad \begin{array}{ll} \forall j \in \mathbb{N}, & |V_m^+(z,j) - R_m^+(z)| \leq Ce^{-c|j|},\\ \forall j \in -\mathbb{N}, & |V_m^-(z,j) - R_m^-(z)| \leq Ce^{-c|j|}.\end{array}$$

The proof of this lemma is quite similar to the construction of $Q_U^\pm(z)$ in Lemma 4.3.4 and is fairly based on the proof of [ZH98, Proposition 3.1].

Proof We will focus on the construction of $(V_m^+(z,j))_{j\in\mathbb{Z}}$ for an integer $m \in \{1,\dots,d(p+q)\}$. Because of (4.3.2), we have a constant $C>0$ such that

$$\forall j\in\mathbb{N}, \forall z\in B(1,\delta_0), \quad |\mathcal{E}_j^+(z)| \leq Ce^{-\alpha j}.$$

We fix $\Delta := \frac{\alpha}{4}$ and define the sets

$$\begin{aligned} I_m^s &= \left\{\nu\in\{1,\dots,d(p+q)\}, \quad |\zeta_\nu^+(1)| < |\zeta_m^+(1)|e^{-\Delta}\right\}, \\ I_m^u &= \left\{\nu\in\{1,\dots,d(p+q)\}, \quad |\zeta_\nu^+(1)| \geq |\zeta_m^+(1)|e^{-\Delta}\right\}. \end{aligned}$$

Because the functions ζ_ν^+ depend holomorphicaly on z in $B(1,\delta_0)$, there exists $\delta_1\in]0,\delta_0[$ such that

$$\forall z\in B(1,\delta_1), \quad \begin{array}{ll} \forall \nu\in I_m^s, & |\zeta_\nu^+(z)| < |\zeta_m^+(z)|e^{-\Delta}, \\ \forall\nu\in I_m^u, & |\zeta_\nu^+(z)| > |\zeta_m^+(z)|e^{-\frac{3}{2}\Delta}. \end{array} \tag{4.4.9}$$

We define for $z\in B(1,\delta_1)$

$$\begin{aligned} E_m^s(z) &:= Span\left(R_\nu^+(z), \quad \nu\in I_m^s\right), \\ E_m^u(z) &:= Span\left(R_\nu^+(z), \quad \nu\in I_m^u\right). \end{aligned}$$

We have that

$$\mathbb{C}^{(p+q)d} = E_m^s(z)\oplus E_m^u(z).$$

We define $P_m^s(z)$ and $P_m^u(z)$ the projectors defined by this decomposition of $\mathbb{C}^{(p+q)d}$. They depend holomorphically on z and commute with $M^+(z)$. Because of (4.4.9), there exists a constant $C>0$ such that

$$\forall z\in B(1,\delta_1),\forall j\in\mathbb{N}, \quad \begin{array}{l} \left|\left(\zeta_m^+(z)^{-1}M^+(z)\right)^j P_m^s(z)\right| \leq C\exp\left(-\frac{\Delta}{2}j\right), \\ \left|\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{-j} P_m^u(z)\right| \leq C\exp\left(2\Delta j\right). \end{array} \tag{4.4.10}$$

We consider $J\in\mathbb{N}$ and we will make a more precise choice later. For $z\in B(1,\delta_1)$, we define the linear map $\varphi(z)\in\mathcal{L}\left(\ell^\infty\left(\{j\in\mathbb{Z}, j\geq J\},\mathbb{C}^{(p+q)d}\right)\right)$ such that for $Y\in\ell^\infty\left(\{j\in\mathbb{Z},j\geq J\},\mathbb{C}^{(p+q)d}\right)$ and $j\geq J$, we have:

$$\begin{aligned} \left(\varphi(z)Y\right)_j := & \sum_{k=J}^{j-1}\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-1-k} P_m^s(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)Y_k \\ & -\sum_{k=j}^{+\infty}\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-1-k} P_m^u(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)Y_k. \end{aligned} \tag{4.4.11}$$

Using the inequalities (4.4.10), we have that:

$$\begin{aligned} \sum_{k=J}^{j-1}\left|\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-1-k} P_m^s(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)Y_k\right| &\lesssim \|Y\|_\infty\sum_{k=J}^{j-1}e^{-\frac{\Delta}{2}(j-k)}e^{-\alpha k} \\ &\lesssim \|Y\|_\infty e^{-\frac{\Delta}{2}j}\sum_{k=J}^{+\infty}e^{\left(\frac{\Delta}{2}-\alpha\right)k} \\ &\lesssim \|Y\|_\infty e^{-\frac{\Delta}{2}(j-J)}e^{-\alpha J} \end{aligned} \tag{4.4.12}$$

and :

$$\begin{aligned} \sum_{k=j}^{+\infty}\left|\left(\zeta_m^+(z)^{-1}M^+(z)\right)^{j-1-k} P_m^u(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)Y_k\right| &\lesssim \|Y\|_\infty e^{-\alpha j}\sum_{k=j}^{+\infty}e^{(2\Delta-\alpha)(k-j)} \\ &\lesssim \|Y\|_\infty e^{-\alpha j}. \end{aligned} \tag{4.4.13}$$

We have thus proved that the linear map $\varphi(z)$ is well-defined and that there exists a constant $C>0$ independent from J such that

$$\forall z\in B(1,\delta_1), \quad \|\varphi(z)\|_{\mathcal{L}(\ell^\infty)} \leq Ce^{-\alpha J}.$$

We can then choose J large enough so that there exists a constant $C \in]0,1[$ such that

$$\forall z \in B(1,\delta_1), \quad \|\varphi(z)\|_{\mathcal{L}(\ell^\infty)} \leq C < 1.$$

Furthermore, φ depends holomorphically on z. We can thus define for $z \in B(1,\delta_1)$

$$V(z) := (Id - \varphi(z))^{-1} \left(R_m^+(z)\right)_{j \geq J} \in \ell^\infty \left(\{j \in \mathbb{Z}, j \geq J\}, \mathbb{C}^{(p+q)d}\right)$$

which depends holomorphically on z. We have that:

$$\forall z \in B(1,\delta_1), \quad V(z) = \left(R_m^+(z)\right)_{j \geq J} + \varphi(z)V(z).$$

Thus, for $z \in B(1,\delta_1)$ and $j \geq J$, we find that:

$$\begin{aligned}
V_{j+1}(z) = R_m^+(z) &+ \sum_{k=J}^{j} \left(\zeta_m^+(z)^{-1} M^+(z)\right)^{j-k} P_m^s(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)V_k(z) \\
&- \sum_{k=j+1}^{+\infty} \left(\zeta_m^+(z)^{-1} M^+(z)\right)^{j-k} P_m^u(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)V_k(z) \\
&= \zeta_m^+(z)^{-1}M^+(z)\left(R_m^+(z) + \sum_{k=J}^{j} \left(\zeta_m^+(z)^{-1} M^+(z)\right)^{j-1-k} P_m^s(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)V_k(z)\right. \\
&\left. - \sum_{k=j+1}^{+\infty} \left(\zeta_m^+(z)^{-1} M^+(z)\right)^{j-1-k} P_m^u(z)\zeta_m^+(z)^{-1}\mathcal{E}_k^+(z)V_k(z)\right) \\
&= \zeta_m^+(z)^{-1}M^+(z)\left(R_m^+(z) + (\varphi(z)V(z))_j + \left(\zeta_m^+(z)^{-1}M^+(z)\right)^{-1}\zeta_m^+(z)^{-1}\mathcal{E}_j^+(z)V_j(z)\right) \\
&= \zeta_m^+(z)^{-1}M^+(z)V_j(z) + \zeta_m^+(z)^{-1}\mathcal{E}_j^+(z)V_j(z) \\
&= \zeta_m^+(z)^{-1}M_j(z)V_j(z).
\end{aligned}$$

Thus, for $j \geq J$, we have

$$V_j(z) = \zeta_m^+(z)^{J-j}X_j(z)X_J(z)^{-1}V_J(z).$$

We define for $z \in B(1,\delta_1)$ and $j \in \mathbb{Z}$

$$V_m^+(z,j) := \zeta_m^+(z)^{J-j}X_j(z)X_J(z)^{-1}V_J(z)$$

and

$$W_m^+(z,j) := \zeta_m^+(z)^j V_m^+(z,j).$$

The two first points of the statement of Lemma 4.4.3 are easily proved from the previous observations. There remains to prove the inequalities in the third point of Lemma 4.4.3. For $z \in B(1,\delta_1)$ and $j \geq J$, we have using (4.4.11)-(4.4.13)

$$\left|V_m^+(z,j) - R_m^+(z)\right| = |(\varphi(z)V(z))_j| \lesssim e^{-\frac{\Delta}{2}j} + e^{-\alpha j}.$$

□

We recall that for $z \in \mathcal{O}_\rho \cap B(1,\delta_1)$, we have for $m \in \{1,\ldots,dp\}$ that

$$|\zeta_m^+(z)| < 1.$$

Therefore, $(W_m^+(z,0))_{m\in\{1,\ldots,dp\}}$ is a family of elements of $E_0^+(z) = \operatorname{Im} Q(z)$ for $z \in \mathcal{O}_\rho \cap B(1,\delta_1)$. In the same way, we prove that for all $z \in \mathcal{O}_\rho \cap B(1,\delta_1)$, $(W_m^-(z,0))_{m\in\{dp+1,\ldots,d(p+q)\}}$ is a family of elements of $E_0^-(z) = \ker Q(z)$. We are going to prove the following lemma.

Lemma 4.4.4. *For all $z \in B(1,\delta_1)$ and $j \in \mathbb{Z}$, $(W_m^+(z,j))_{m\in\{1\ldots,d(p+q)\}}$ and $(W_m^-(z,j))_{m\in\{1\ldots,d(p+q)\}}$ are bases of $\mathbb{C}^{d(p+q)}$. The same is then also true for the families $(V_m^+(z,j))_{m\in\{1\ldots,d(p+q)\}}$ and $(V_m^-(z,j))_{m\in\{1\ldots,d(p+q)\}}$.*

Proof We will write the proof for the family of vectors $(W_m^+(z,j))_{m\in\{1\ldots,d(p+q)\}}$. We consider $z \in B(1,\delta_1)$ and $j \in \mathbb{Z}$ such that the family of vectors $(W_m^+(z,j))_{m\in\{1,\ldots,d(p+q)\}}$ is not linearly independent. We can then introduce a family $(c_m)_{m\in\{1,\dots,d(p+q)\}}\in\mathbb{C}^{d(p+q)}\backslash\{0\}$ such that

$$0=\sum_{m=1}^{d(p+q)}c_mW_m^+(z,j). \tag{4.4.14}$$

Since the sequences $(W_m^+(z,j))_{j\in\mathbb{Z}}$ are solutions of (4.3.3), (4.4.14) is verified for all $j\in\mathbb{Z}$. We define

$$\begin{aligned}
I_n &:= \{m\in\{1,\dots,d(p+q)\}, \quad c_m\neq 0\}\neq\emptyset,\\
R &:= \max_{m\in I_n}|\zeta_m^+(z)|>0,\\
I_R &:= \mathrm{argmax}_{m\in I_n}|\zeta_m^+(z)|\neq\emptyset.
\end{aligned}$$

Using (4.4.14), we obtain

$$\begin{aligned}
0 &= \sum_{m\in I_n}c_m\frac{W_m^+(z,j)}{R^j}\\
&= \sum_{m\in I_n}c_m\left(\frac{\zeta_m^+(z)}{R}\right)^jR_m^+(z)+\sum_{m\in I_n}c_m\frac{W_m^+(z,j)-\zeta_m^+(z)^jR_m^+(z)}{R^j}.
\end{aligned}$$

— Using Lemma 4.4.3, there exist two positive constants C,c such that we have for $m\in I_n$ and $j\in\mathbb{N}$

$$\left|\frac{W_m^+(z,j)-\zeta_m^+(z)^jR_m^+(z)}{R^j}\right|\leq Ce^{-cj}\left(\frac{|\zeta_m^+(z)|}{R}\right)^j\leq Ce^{-cj}.$$

Thus,

$$\frac{W_m^+(z,j)-\zeta_m^+(z)^jR_m^+(z)}{R^j}\underset{j\to+\infty}{\to}0.$$

— For $m\in I_n\backslash I_R$, we have

$$\left(\frac{\zeta_m^+(z)}{R}\right)^jR_m^+(z)\underset{j\to+\infty}{\to}0.$$

Thus, we have that

$$\sum_{m\in I_R}c_m\left(\frac{\zeta_m^+(z)}{R}\right)^jR_m^+(z)\underset{j\to+\infty}{\to}0.$$

Since $I_R\neq\emptyset$, we fix $m_0\in I_R$. Because of Lemma 4.4.1, the projection of the previous expression on $\mathrm{Span}(R_{m_0}^+(z))$ along $\mathrm{Span}(R_m^+(z),m\neq m_0)$ implies that

$$c_{m_0}\left(\frac{\zeta_{m_0}^+(z)}{R}\right)^j\underset{j\to+\infty}{\to}0.$$

But, m_0 belongs to I_R so $|\zeta_{m_0}^+(z)|=R$. This implies that $c_{m_0}=0$. However, $m_0\in I_R\subset I_n$ so $c_{m_0}\neq 0$. This is a contradiction. □

For $z\in\mathcal{O}_\rho\cap B(1,\delta_1)$, we recall that

$$\dim E_0^+(z)=dp \quad\text{and}\quad \dim E_0^-(z)=dq.$$

Thus, Lemma 4.4.4 implies that the family $(W_m^+(z,0))_{m\in\{1,\dots,dp\}}$ (resp. $(W_m^-(z,0))_{m\in\{dp+1,\dots,d(p+q)\}}$) is a basis of $E_0^+(z)$ (resp. $E_0^-(z)$). We can then extend holomorphically the subspaces $E_0^+(z)$ and $E_0^-(z)$ on the whole ball $B(1,\delta_1)$ as

$$\forall z\in B(1,\delta_1),\quad E_0^+(z):=\mathrm{Span}\left(W_m^+(z,0)\right)_{m\in\{1,\dots,dp\}}\quad\text{and}\quad E_0^-(z):=\mathrm{Span}\left(W_m^-(z,0)\right)_{m\in\{dp+1,\dots,d(p+q)\}}. \tag{4.4.15}$$

We will also define

$$\begin{aligned}
I_{ss}&:=I_{ss}^+, & I_{cs}&:=I_{cs}^+,\\
I_{cu}&:=I_{cu}^-, & I_{su}&:=I_{su}^-.
\end{aligned}$$

4.4.3 Definition of the Evans function

In this section, we are going to define an Evans function and prove that 1 is a simple eigenvalue of the operator $\mathscr{L}$ when it acts on $\ell^2(\mathbb{Z},\mathbb{C}^d)$.

An important part of the study of the spatial Green's function far from 1 was dedicated to introduce the projection Q of the geometric dichotomy. The main ingredient of the introduction of $Q(z)$ has been to understand when $E_0^+(z)$ and $E_0^-(z)$ are supplementary, as it allowed via Lemma 4.3.8 to conclude on which elements of $\mathcal{O}$ where eigenvalues of the operator $\mathscr{L}$ and which elements of $\mathcal{O}$ are in the resolvent set of the operator $\mathscr{L}$. For z near 1, because of (4.4.15), studying whether the vector subspace $E_0^+(z)$ and $E_0^-(z)$ are supplementary comes down to knowing when $\left(W_1^+(z,0),\dots,W_{dp}^+(z,0),W_{dp+1}^-(z,0),\dots,W_{d(p+q)}^-(z,0)\right)$ is a basis of $\mathbb{C}^{d(p+q)}$. We define the Evans function as

$$\forall z\in B(1,\delta_1),\quad \mathrm{Ev}(z):=\det(W_1^+(z,0),\dots,W_{dp}^+(z,0),W_{dp+1}^-(z,0),\dots,W_{d(p+q)}^-(z,0)). \tag{4.4.16}$$

The function Ev is holomorphic on $B(1,\delta_1)$. Furthermore, for $z\in\mathcal{O}\cap B(1,\delta_1)$, (4.3.31) and the set equality (4.4.15) imply that the function Ev vanishes when z is an eigenvalue of the operator $\mathscr{L}$. Thus, Hypothesis 4.6 implies that the function Ev is not uniformly equal to 0. We will now prove the following lemma which links the behavior at $z=1$ of the Evans function Ev, the eigenspace associated with the eigenvalue 1 for the operator $\mathscr{L}$ and the vector subspace

$$\mathrm{Span}\left(W_m^+(1,0),m\in I_{ss}\right)\cap\mathrm{Span}\left(W_m^-(1,0),m\in I_{su}\right).$$

Lemma 4.4.5. *We have that* $\mathrm{Ev}(1)=0$, 1 *is a simple eigenvalue of the operator* $\mathscr{L}$ *and*

$$\dim\mathrm{Span}\left(W_m^+(1,0),m\in I_{ss}\right)\cap\mathrm{Span}\left(W_m^-(1,0),m\in I_{su}\right)=1. \tag{4.4.17}$$

Furthermore, if we consider a vector $V_0\in\mathrm{Span}\left(W_m^+(1,0),m\in I_{ss}\right)\cap\mathrm{Span}\left(W_m^-(1,0),m\in I_{su}\right)\setminus\{0\}$ *then we have that:*

$$\ker(Id_{\ell^2}-\mathscr{L})=\mathrm{Span}(\Pi(X_j(z)V_0))_{j\in\mathbb{Z}} \tag{4.4.18}$$

where the operator Π *defined by* (4.3.32) *is the linear map which extracts the center values of a vector of size* $d(p+q)$.

Proof The proof is separated in several steps.

- **Step 1:** We start by proving that

$$\dim\mathrm{Span}\left(W_m^+(1,0),m\in I_{ss}\right)\cap\mathrm{Span}\left(W_m^-(1,0),m\in I_{su}\right)\geq 1.$$

We consider $m\in\{1,\dots,d(p+q)\}$. For all $j\in\mathbb{Z}$, we define ${W_m^\pm}^{-p}(1,j),\dots,{W_m^\pm}^{q-1}(1,j)\in\mathbb{C}^d$ such that

$$W_m^\pm(1,j)=:\begin{pmatrix}{W_m^\pm}^{q-1}(1,j)\\ \vdots\\ {W_m^\pm}^{-p}(1,j)\end{pmatrix}$$

and we notate $u_m^\pm(j):={W_m^\pm}^0(1,j)$. Since $W_m^\pm$ satisfies

$$\forall j\in\mathbb{Z},\quad W_m^\pm(1,j+1)=M_j(1)W_m^\pm(1,j),$$

we have

$$\forall j\in\mathbb{Z},\forall k\in\{-p,\dots,q-2\},\quad {W_m^\pm}^k(1,j+1)={W_m^\pm}^{k+1}(1,j) \tag{4.4.19}$$

and

$$\forall j\in\mathbb{Z},{W_m^\pm}^{q-1}(1,j+1)=-\mathbb{A}_{j,q}(1)^{-1}\left(\sum_{k=-p}^{q-1}\mathbb{A}_{j,k}(1){W_m^\pm}^k(1,j)\right). \tag{4.4.20}$$

The equality (4.4.19) implies that

$$\forall j\in\mathbb{Z},\forall k\in\{-p,\dots,q-1\},\quad {W_m^\pm}^k(1,j)=u_m^\pm(j+k).$$

We then obtain using (4.4.20)

$$\sum_{k=-p}^{q} \mathbb{A}_{j,k}(1)u_m^{\pm}(j+k)=0.$$

Using the definition (4.1.21) of $A_{j,k}$, we have that

$$\forall j\in\mathbb{Z},\quad \sum_{k=-p}^{q-1} B_{j+1,k}u_m^{\pm}(j+1+k)=\sum_{k=-p}^{q-1} B_{j,k}u_m^{\pm}(j+k).$$

Therefore, the sequence $\left(\sum_{k=-p}^{q-1} B_{j,k}u_m^{\pm}(j+k)\right)_{j\in\mathbb{Z}}$ is constant.

— If $m\in I_{ss}^{\pm}$, since $\zeta_m^{\pm}(1)=\underline{\zeta}_m^{\pm}\in\mathbb{D}$, Lemma 4.4.3 implies that

$$W_m^{\pm}(1,j)\underset{j\to+\infty}{\to}0.$$

Therefore, $u_m^{\pm}(j)$ tends to 0 as j tends to $+\infty$.

— If $m\in I_{su}^{\pm}$, since $\zeta_m^{-}(1)=\underline{\zeta}_m^{-}\in\mathbb{U}$, Lemma 4.4.3 implies that

$$W_m^{\pm}(1,j)\underset{j\to-\infty}{\to}0.$$

Therefore, $u_m^{\pm}(j)$ tends to 0 as j tends to $-\infty$.

Therefore, the two previous points and the fact that the sequence $\left(\sum_{k=-p}^{q-1} B_{j,k}u_m^{\pm}(j+k)\right)_{j\in\mathbb{Z}}$ is constant imply that for $m\in I_{ss}^{\pm}\cup I_{su}^{\pm}$

$$\forall j\in\mathbb{Z},\quad \sum_{k=-p}^{q-1} B_{j,k}u_m^{\pm}(j+k)=0.$$

Therefore, using Hypothesis 4.8 to invert the matrix $B_{j,-p}$, we have:

$$\forall j\in\mathbb{Z},\quad u_m^{\pm}(j-p)=D_j\begin{pmatrix}u_m^{\pm}(j+q-1)\\ \vdots\\ u_m^{\pm}(j-p+1)\end{pmatrix}$$

where

$$D_j:=\begin{pmatrix}-B_{j,q}^{-1}B_{j,q-1} & \dots & -B_{j,q}^{-1}B_{j,-p+1}\end{pmatrix}\in\mathcal{M}_{d,d(p+q-1)}(\mathbb{C}).$$

This implies that

$$\forall m\in I_{ss}^{\pm}\cup I_{su}^{\pm},\quad W_m^{\pm}(1,0)\in\left\{\begin{pmatrix}V\\ D_0V\end{pmatrix},\quad V\in\mathbb{C}^{d(p+q-1)}\right\}$$

and thus

$$\operatorname{Span}\left(W_m^{+}(1,0),m\in I_{ss}\right)\cup\operatorname{Span}\left(W_m^{-}(1,0),m\in I_{su}\right)\subset\left\{\begin{pmatrix}V\\ D_0V\end{pmatrix},\quad V\in\mathbb{C}^{d(p+q-1)}\right\}.$$

Also,

$$\dim\left\{\begin{pmatrix}V\\ D_0V\end{pmatrix},\quad V\in\mathbb{C}^{d(p+q-1)}\right\}=d(p+q-1),$$

and Hypothesis 4.1 implies that

$$\#I_{ss}\cup I_{su}=d(p+q-1)+1.$$

Therefore, since Lemma 4.4.4 implies that the families $(W_m^{+}(1,0))_{m\in I_{ss}}$ and $(W_m^{-}(1,0))_{m\in I_{su}}$ are both linearly independent, this allows us to conclude that

$$\dim\operatorname{Span}\left(W_m^{+}(1,0),m\in I_{ss}\right)\cap\operatorname{Span}\left(W_m^{-}(1,0),m\in I_{su}\right)\geq 1. \tag{4.4.21}$$

• **Step 2:** Let us now prove (4.4.17). The inequality (4.4.21) allows us to conclude that $\mathrm{Ev}(1)=0$. We recall that Hypothesis 4.7 implies that

$$\frac{\partial\mathrm{Ev}}{\partial z}(1)\neq 0.$$

Therefore, using (4.4.21) and the expression of the Evans function Ev, we easily deduce (4.4.17).

• **Step 3:** We will now prove (4.4.18) and thus that 1 is a simple eigenvalue of $\mathscr{L}$. We consider a vector $V_0 \in \mathrm{Span}\,(W_m^+(1,0), m \in I_{ss}) \cap \mathrm{Span}\,(W_m^-(1,0), m \in I_{su}) \setminus \{0\}$. We then obviously have that $\left(\Pi\,(X_j(1)V_0)\right)_{j\in\mathbb{Z}}$ has exponential decay as j tends towards $+\infty$ and $-\infty$ and thus belongs to $\ell^2(\mathbb{Z},\mathbb{C}^d)$. Furthermore, since $(X_j(1)V_0)_{j\in\mathbb{Z}}$ is a solution of (4.3.3) for $z=1$, we have that $\left(\Pi\,(X_j(1)V_0)\right)_{j\in\mathbb{Z}}$ belongs to $\ker(Id_{\ell^2}-\mathscr{L})$. Thus,

$$\mathrm{Span}\left(\Pi\,(X_j(1)V_0)\right)_{j\in\mathbb{Z}} \subset \ker(Id_{\ell^2}-\mathscr{L}).$$

We now consider $w \in \ker\left(Id_{\ell^2}-\mathscr{L}\right)$ and define

$$W_0 := \begin{pmatrix} w_{q-1} \\ \vdots \\ w_{-p} \end{pmatrix} \in \mathbb{C}^{d(p+q)}.$$

We then have that

$$\forall j \in \mathbb{Z}, \quad W_j := X_j(1)W_0 = \begin{pmatrix} w_{j+q-1} \\ \vdots \\ w_{j-p} \end{pmatrix}.$$

If we introduce a family of complex scalars $(c_m)_{m\in\{1,\dots,d(p+q)\}} \in \mathbb{C}^{d(p+q)}$ such that:

$$W_0 = \sum_{m=1}^{d(p+q)} c_m W_m^+(1,0), \tag{4.4.22}$$

then, since W_m^+ are solutions of (4.3.3), we have that:

$$\forall j \in \mathbb{Z}, \quad W_j = \sum_{m=1}^{d(p+q)} c_m W_m^+(1,j).$$

We will now use a strategy similar to the proof of Lemma 4.4.4 to prove that

$$\forall m \in \{1,\dots,d(p+q)\} \setminus I_{ss}, \quad c_m = 0. \tag{4.4.23}$$

We introduce:

$$\begin{aligned} I_n &:= \{m \in \{1,\dots,d(p+q)\}, \quad c_m \neq 0\} \neq \emptyset, \\ R &:= \max_{m\in I_n} |\zeta_m^+(z)| > 0, \\ I_R &:= \mathrm{argmax}_{m\in I_n} |\zeta_m^+(z)| \neq \emptyset. \end{aligned}$$

Let us assume that the assertion (4.4.23) is false. Then, I_R is not an empty set and the scalar R is larger or equal to 1. We have that for all $j \in \mathbb{Z}$:

$$\frac{W_j}{R^j} = \sum_{m\in I_R} c_m \left(\frac{\zeta_m^+(z)}{R}\right)^j R_m^+(z) + \sum_{m\in I_n\setminus I_R} c_m \left(\frac{\zeta_m^+(z)}{R}\right)^j R_m^+(z) + \sum_{m\in I_n} c_m \frac{W_m^+(z,j) - \zeta_m^+(z)^j R_m^+(z)}{R^j}. \tag{4.4.24}$$

Since W_j converges towards 0 as j tends towards $+\infty$ and R is larger or equal than 1, we have that:

$$\frac{W_j}{R^j} \underset{j\to+\infty}{\to} 0.$$

Furthermore, using similar ideas as in the proof of Lemma 4.4.4, we have that:

$$\sum_{m\in I_n\setminus I_R} c_m \left(\frac{\zeta_m^+(z)}{R}\right)^j R_m^+(z) + \sum_{m\in I_n} c_m \frac{W_m^+(z,j) - \zeta_m^+(z)^j R_m^+(z)}{R^j} \underset{j\to+\infty}{\to} 0.$$

Thus, (4.4.24) implies that:

$$\sum_{m\in I_R} c_m \left(\frac{\zeta_m^+(z)}{R}\right)^j R_m^+(z) \underset{j\to+\infty}{\to} 0.$$

Since I_R is not empty, projecting the equality above on $\mathrm{Span}(R_{m_0}^+(z))$ along $\mathrm{Span}(R_m^+(z), m \neq m_0)$ for some

$m_0 \in I_R$ implies that

$$c_{m_0}\left(\frac{\zeta_{m_0}^+(z)}{R}\right)^j \underset{j\to+\infty}{\to} 0.$$

But, m_0 belongs to I_R so $|\zeta_{m_0}^+(z)| = R$. This implies that $c_{m_0} = 0$. However, $m_0 \in I_R \subset I_n$ so $c_{m_0} \neq 0$. This is a contradiction which implies that (4.4.23) has to be verified. Then, using (4.4.22), we have that

$$W_0 \in \mathrm{Span}(W_m^+(1,0), m \in I_{ss}).$$

Furthermore, using a similar proof with the family $(W_m^-(1,0))_{m\in\{1,\dots,d(p+q)\}}$, we have that W_0 belongs to $\mathrm{Span}(W_m^+(1,0), m \in I_{ss}) \cap \mathrm{Span}(W_m^-(1,0), m \in I_{su}) = \mathrm{Span} V_0$. Therefore, since for all $j \in \mathbb{Z}$ we have $w_j = \Pi(W_j) = \Pi(X_j(1)W_0)$, we conclude that the sequence w belongs to $\mathrm{Span}\left(\Pi\left(X_j(1)V_0\right)\right)_{j\in\mathbb{Z}}$. We can thus finally verify (4.4.18). □

First, as a consequence of Lemma 4.4.5, since the Evans function Ev is holomorphic on $B(1,\delta_1)$ and not uniformly equal to 0, the equality $\mathrm{Ev}(1) = 0$ implies that we can consider δ_1 small enough so that the Evans function Ev only vanishes at $z = 1$.

Our new goal for the rest of this section will be to use Lemma 4.4.5 to introduce below two new bases $(\Phi_m(z,0))_{m\in\{1,\dots,dp\}}$ and $(\Phi_m(z,0))_{m\in\{dp+1,\dots,d(p+q)\}}$ in (4.4.27) for the vector spaces $E_0^+(z)$ and $E_0^-(z)$ more suitable for the study of the spatial Green's function when z is close to 1. We will also define in (4.4.31c) below a new Evans function D^Φ associated with this new choice of bases which will share the same properties as Ev.

The equality (4.4.17) of Lemma 4.4.5 implies that there exist two non zero families of complex numbers $(\theta_{s,m})_{m\in I_{ss}}$ and $(\theta_{u,m})_{m\in I_{su}}$ such that

$$\sum_{m\in I_{ss}} \theta_{s,m} W_m^+(1,0) = \sum_{m\in I_{su}} \theta_{u,m} W_m^-(1,0). \tag{4.4.25}$$

In the rest of the chapter, we fix the choice of families of coefficients $\theta_{s,m}$ and $\theta_{u,m}$. Even if we have to reindex the eigenvalues $\zeta_m^\pm$, we will assume that $\theta_{s,1}, \theta_{u,d(p+q)} \neq 0$. Furthermore, we also define

$$\theta_{s,m} := 0 \text{ for } m \in \{1,\dots,d(p+q)\}\setminus I_{ss} \quad \text{and} \quad \theta_{u,m} := 0 \text{ for } m \in \{1,\dots,d(p+q)\}\setminus I_{su}. \tag{4.4.26}$$

We define for $m \in \{1,\dots,d(p+q)\}$

$$\forall z \in B(1,\delta_1), \forall j \in \mathbb{Z}, \quad \Phi_m(z,j) := \begin{cases} \sum_{m\in I_{ss}} \theta_{s,m} W_m^+(z,j), & \text{if } m = 1, \\ W_m^+(z,j), & \text{if } m \in \{2,\dots,dp\}, \\ W_m^-(z,j), & \text{if } m \in \{dp+1,\dots,d(p+q)-1\}, \\ \sum_{m\in I_{su}} \theta_{u,m} W_m^-(z,j), & \text{if } m = d(p+q). \end{cases} \tag{4.4.27}$$

Since $\theta_{s,1}, \theta_{u,d(p+q)} \neq 0$, we have that $(\Phi_m(z,0))_{m\in\{1,\dots,dp\}}$ and $(\Phi_m(z,0))_{m\in\{dp+1,\dots,d(p+q)\}}$ are respectively bases of $E_0^+(z)$ and $E_0^-(z)$. Equality (4.4.25) implies that $\Phi_1(1,0) = \Phi_{d(p+q)}(1,0)$ and since $\Phi_1(1,\cdot)$ and $\Phi_{d(p+q)}(1,\cdot)$ are solutions of (4.3.3) for $z = 1$, we have

$$\forall j \in \mathbb{Z}, \quad \Phi_1(1,j) = \Phi_{d(p+q)}(1,j). \tag{4.4.28}$$

Furthermore, using the expression of $\Phi_1(z,j)$ and $\Phi_{d(p+q)}(z,j)$ as well as Lemma 4.4.3 and inequalities (4.4.1a) and (4.4.1c), we prove that there exists a positive constant C such that

$$\begin{aligned} \forall z \in B(1,\delta_1), \forall j \in \mathbb{N}, \quad & |\Phi_1(z,j)| \leq Ce^{-2c_*|j|}, \\ \forall z \in B(1,\delta_1), \forall j \in -\mathbb{N}, \quad & \left|\Phi_{d(p+q)}(z,j)\right| \leq Ce^{-2c_*|j|}. \end{aligned} \tag{4.4.29}$$

If we define

$$\forall j \in \mathbb{Z}, \quad V(j) = \Pi(\Phi_1(1,j)) = \Pi(\Phi_{d(p+q)}(1,j)), \tag{4.4.30}$$

then using (4.4.18) because

$$\Phi_1(1,0) = \Phi_{d(p+q)}(1,0) \in \mathrm{Span}\left(W_m^+(1,0), m \in I_{ss}\right) \cap \mathrm{Span}\left(W_m^-(1,0), m \in I_{su}\right) \setminus \{0\},$$

we have that V is a sequence in $\ell^2(\mathbb{Z},\mathbb{C}^d)\setminus\{0\}$ such that (4.1.22) and (4.1.23) are verified. It will correspond to the sequence V in Theorem 4.1.

If we summarize, for $z \in B(1,\delta_1)$, we have five families to describe the solutions of the dynamical system (4.3.3):

— The bases $\left(\Phi_1(z,j), W_2^+(z,j), \dots, W_{d(p+q)}^+(z,j)\right)$ and $\left(W_1^+(z,j), W_2^+(z,j), \dots, W_{d(p+q)}^+(z,j)\right)$ for which we know the asymptotic behavior when j tends to $+\infty$ thanks to Lemma 4.4.3.

— The bases

$$\left(W_1^-(z,j), \dots, W_{d(p+q)-1}^-(z,j), \Phi_{d(p+q)}(z,j)\right) \quad \text{and} \quad \left(W_1^-(z,j), \dots, W_{d(p+q)-1}^-(z,j), W_{d(p+q)}^-(z,j)\right)$$

for which we know the asymptotic behavior when j tends to $-\infty$ thanks to Lemma 4.4.3.

— The family $(\Phi_m(z,j))_{m\in\{1,\dots,d(p+q)\}}$ which is linked to the solutions of the dynamical system (4.3.3) which tend towards 0 as j tends towards $+\infty$ or $-\infty$, at least when $z\in\mathcal{O}$. It is a basis of $\mathbb{C}^{d(p+q)}$ if and only if $E_0^+(z)\oplus E_0^-(z) = \mathbb{C}^{d(p+q)}$.

We introduce a few more notations. For $z\in B(1,\delta_1)$ and $j\in\mathbb{Z}$, we define

$$\mathcal{G}^\Phi(z,j) := \left(\Phi_1(z,j)|\dots|\Phi_{d(p+q)}(z,j)\right), \qquad D^\Phi(z) := \det(\mathcal{G}^\Phi(z,0)), \tag{4.4.31a}$$

$$\mathcal{G}^+(z,j) := \left(\Phi_1(z,j)|W_2^+(z,j)|\dots|W_{d(p+q)}^+(z,j)\right), \qquad D^+(z) := \det(\mathcal{G}^+(z,0)), \tag{4.4.31b}$$

$$\mathcal{G}^-(z,j) := \left(W_1^-(z,j)|\dots|W_{d(p+q)-1}^-(z,j)|\Phi_{d(p+q)}(z,j)\right), \qquad D^-(z) := \det(\mathcal{G}^-(z,0)), \tag{4.4.31c}$$

$$\widetilde{\mathcal{G}}^+(z,j) := \left(W_1^+(z,j)|W_2^+(z,j)|\dots|W_{d(p+q)}^+(z,j)\right), \tag{4.4.31d}$$

$$\widetilde{\mathcal{G}}^-(z,j) := \left(W_1^-(z,j)|W_2^-(z,j)|\dots|W_{d(p+q)}^-(z,j)\right). \tag{4.4.31e}$$

Those functions are holomorphic on $B(1,\delta_1)$. Let us now conclude the section with a few observations:

- We observe using the definition (4.4.27) of Φ_1 and $\Phi_{d(p+q)}$ that for $z\in B(1,\delta_1)$ and $j\in\mathbb{Z}$

$$\mathcal{G}^+(z,j) = \widetilde{\mathcal{G}}^+(z,j)\begin{pmatrix} \theta_{s,1} & & (0) & \\ \theta_{s,2} & 1 & & \\ \vdots & & \ddots & \\ \theta_{s,d(p+q)} & (0) & & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{G}^-(z,j) = \widetilde{\mathcal{G}}^-(z,j)\begin{pmatrix} 1 & & (0) & \theta_{u,1} \\ & \ddots & & \vdots \\ & & 1 & \theta_{u,d(p+q)-1} \\ & (0) & & \theta_{u,d(p+q)} \end{pmatrix}. \tag{4.4.32}$$

- Using (4.4.27) and (4.4.26), we have for $z\in B(1,\delta_1)$

$$\mathcal{G}^\Phi(z,0) = \left(W_1^+(z,0)|\dots|W_{dp}^+(z,0)|W_{dp+1}^-(z,0)|\dots|W_{d(p+q)}^-(z,0)\right)\begin{pmatrix} \theta_{s,1} & & & & \theta_{u,1} \\ \theta_{s,2} & 1 & & (0) & \theta_{u,2} \\ \vdots & & \ddots & & \vdots \\ \theta_{s,d(p+q)-1} & (0) & & 1 & \theta_{u,d(p+q)-1} \\ \theta_{s,d(p+q)} & & & & \theta_{u,d(p+q)} \end{pmatrix}$$

and thus since $\theta_{s,d(p+q)}, \theta_{u,1} = 0$, we have that:

$$D^\Phi(z) = \theta_{s,1}\theta_{u,d(p+q)}\mathrm{Ev}(z).$$

The function D^Φ thus shares the same properties as the Evans function Ev, i.e. the function D^Φ is holomorphic on $B(1,\delta_1)$, vanishes only at $z=1$ and 1 is a simple zero of D^Φ. We will thus also call D^Φ Evans function.

- For $z\in B(1,\delta_1)\backslash\{1\}$, the function D^Φ does not vanish at z and thus $(\Phi_m(z,0))_{m\in\{1,\dots,d(p+q)\}}$ is a basis. We can define for $m\in\{1,\dots,d(p+q)\}$ the projector $\Pi_m(z)$ projecting on the vector space $\mathrm{Span}(\Phi_m(z,0))$ along $\mathrm{Span}(\Phi_\nu(z,0))_{\nu\in\{1,\dots,d(p+q)\}\backslash\{m\}}$. We observe that

$$\Pi_m(z) = \mathcal{G}^\Phi(z,0)P_m\mathcal{G}^\Phi(z,0)^{-1} \tag{4.4.33}$$

where $P_m = (\delta_{i,m}\delta_{j,m})_{i,j\in\{1,\dots,d(p+q)\}}\in\mathcal{M}_{d(p+q)}(\mathbb{C})$. The function Π_m is holomorphic on $B(1,\delta_1)\backslash\{1\}$.

4.4.4 Behavior of the spatial Green's function near 1

Now that we have introduced all the tools necessary, our goal is to prove the following proposition which implies that the spatial Green's function can be meromorphically extended near 1 and which gives expressions of the spatial Green's function using the solutions $W_m^\pm$ and Φ_m of the dynamical system (4.3.3) defined previously respectively in Lemma 4.4.3 and (4.4.27).

Proposition 4.3. *We consider $j_0, j\in\mathbb{Z}$ and $\boldsymbol{e}\in\mathbb{C}^d$. The function $W(\cdot, j_0, j, \boldsymbol{e})$ can be meromorphically extended on the whole ball $B(1,\delta_1)$ with a pole of order 1 at 1. Furthermore, using the functions $\widetilde{\mathscr{C}}_{m'}^+$ and $\widetilde{g}_{m',m}^+$*

defined below respectively by (4.4.38b) and (4.4.42), we have that for $j_0 \in \mathbb{N}$, $j \in \mathbb{Z}$, $\boldsymbol{e} \in \mathbb{C}^d$ and $z \in B(1,\delta_1)\backslash\{1\}$:

— *For $j \geq j_0+1$:*

$$W(z,j_0,j,\boldsymbol{e}) = -\sum_{m=1}^{dp} \widetilde{\mathscr{C}}_m^+(z,j_0,\boldsymbol{e})W_m^+(z,j) - \sum_{m=2}^{dp}\sum_{m'=dp+1}^{d(p+q)} \widetilde{g}_{m',m}^+(z)\widetilde{\mathscr{C}}_{m'}^+(z,j_0,\boldsymbol{e})W_m^+(z,j) \\ - \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}_{m',1}^+(z)\widetilde{\mathscr{C}}_{m'}^+(z,j_0,\boldsymbol{e})\Phi_1(z,j). \quad (4.4.34a)$$

— *For $j \in \{0,\dots,j_0\}$:*

$$W(z,j_0,j,\boldsymbol{e}) = \sum_{m=dp+1}^{d(p+q)} \widetilde{\mathscr{C}}_m^+(z,j_0,\boldsymbol{e})W_m^+(z,j) - \sum_{m=2}^{dp}\sum_{m'=dp+1}^{d(p+q)} \widetilde{g}_{m',m}^+(z)\widetilde{\mathscr{C}}_{m'}^+(z,j_0,\boldsymbol{e})W_m^+(z,j) \\ - \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}_{m',1}^+(z)\widetilde{\mathscr{C}}_{m'}^+(z,j_0,\boldsymbol{e})\Phi_1(z,j). \quad (4.4.34b)$$

— *For $j < 0$:*

$$W(z,j_0,j,\boldsymbol{e}) = \sum_{m=dp+1}^{d(p+q)-1}\sum_{m'=dp+1}^{d(p+q)} \widetilde{g}_{m',m}^+(z)\widetilde{\mathscr{C}}_{m'}^+(z,j_0,\boldsymbol{e})W_m^-(z,j) \\ + \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}_{m',d(p+q)}^+(z)\widetilde{\mathscr{C}}_{m'}^+(z,j_0,\boldsymbol{e})\Phi_{d(p+q)}(z,j). \quad (4.4.34c)$$

Similar expressions can be deduced when $j_0 \in -\mathbb{N}\backslash\{0\}$ using the functions $\widetilde{\mathscr{C}}_{m'}^-$ and $\widetilde{g}_{m',m}^-$.

To enter into more details, in Section 4.4.4, we will use the families of solutions of the dynamical system (4.3.3) previously introduced to decompose the expressions (4.3.43a) and (4.3.43b) proved on the spatial Green's function in Section 4.3 and obtain the expressions (4.4.37) below of the spatial Green's function on $B(1,\delta_1)\cap\mathcal{O}_\rho$. Since the sequence $W_m^\pm$ are holomorphic on $B(1,\delta_1)$, this will allow us to extend the spatial Green's function meromorphically on the whole ball $B(1,\delta_1)$ with a pole of order 1 at 1. The computations performed in this section will be fairly inspired by [God03, Section 3] and the expression (4.4.37) corresponds to the result of [God03, Proposition 3.1]. However, our goal is to improve the expression (4.4.37) to obtain the result (4.4.34). Those calculations will be performed in Sections 4.4.4 and 4.4.4.

Meromorphic extension of the spatial Green's function near 1

We observe that for $z \in \mathcal{O}_\rho \cap B(1,\delta_1)$, we have that $Q(z)$ is the projector on $E_0^+(z)$ along $E_0^-(z)$ and thus

$$Q(z) = \sum_{m=1}^{dp} \Pi_m(z) \quad \text{and} \quad Id - Q(z) = \sum_{m=dp+1}^{(p+q)d} \Pi_m(z)$$

where the projectors $\Pi_m(z)$ are defined by (4.4.33). Since the equalities (4.3.43a) and (4.3.43b) are still verified for $z \in \mathcal{O}_\rho \cap B(1,\delta_1)$, if we define for $m \in \{1,\dots,d(p+q)\}$ the functions ν_m:

$$\forall z \in B(1,\delta_1)\backslash\{1\}, \forall j_0, j \in \mathbb{Z}, \forall \mathbf{e} \in \mathbb{C}^d, \quad \nu_m(z,j_0,j,\mathbf{e}) := X_j(z)\Pi_m(z)X_{j_0+1}(z)^{-1}\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

then, we have that for $z \in \mathcal{O}_\rho \cap B(1,\delta_1)$

$$\forall j \geq j_0+1, \quad W(z,j_0,j,\mathbf{e}) = -\sum_{m=1}^{dp} \nu_m(z,j_0,j,\mathbf{e}), \quad (4.4.35a)$$

$$\forall j \leq j_0, \quad W(z, j_0, j, \mathbf{e}) = \sum_{m=dp+1}^{(p+q)d} \nu_m(z, j_0, j, \mathbf{e}). \tag{4.4.35b}$$

Since the right hand terms of (4.4.35) are holomorphic on $B(1, \delta_1) \backslash \{1\}$, we can extend holomorphically the function $W(\cdot, j_0, j, \mathbf{e})$ on $B(1, \delta_1) \backslash \{1\}$.

We observe that (4.4.33) implies for $z \in B(1, \delta_1) \backslash \{1\}$ and $m \in \{1, \ldots, (p+q)d\}$,

$$\nu_m(z, j_0, j, \mathbf{e}) = \frac{\mathcal{G}^\Phi(z, j)}{D^\Phi(z)} P_m \mathrm{com}(\mathcal{G}^\Phi(z, 0))^T X_{j_0+1}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If we define

$$\forall z \in B(1, \delta_1), \forall j_0 \in \mathbb{Z}, \forall \mathbf{e} \in \mathbb{C}^d, \quad \begin{pmatrix} \widehat{D}_1(z, j_0, \mathbf{e}) \\ \vdots \\ \widehat{D}_{d(p+q)}(z, j_0, \mathbf{e}) \end{pmatrix} := \mathrm{com}(\mathcal{G}^\Phi(z, 0))^T X_{j_0+1}(z)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.4.36}$$

then the functions $\widehat{D}_m(\cdot, j_0, \mathbf{e})$ is holomorphic on $B(1, \delta_1)$ and we have:

$$\forall z \in B(1, \delta_1) \backslash \{1\}, \forall j_0, j \in \mathbb{Z}, \forall \mathbf{e} \in \mathbb{C}^d, \quad \nu_m(z, j_0, j, \mathbf{e}) = \frac{\widehat{D}_m(z, j_0, \mathbf{e})}{D^\Phi(z)} \Phi_m(z, j).$$

Therefore, (4.4.35a) and (4.4.35b) can be rewritten for $z \in B(1, \delta_1) \backslash \{1\}$ as

$$\forall j \geq j_0 + 1, \quad W(z, j_0, j, \mathbf{e}) = -\sum_{m=1}^{dp} \frac{\widehat{D}_m(z, j_0, \mathbf{e})}{D^\Phi(z)} \Phi_m(z, j), \tag{4.4.37a}$$

$$\forall j \leq j_0, \quad W(z, j_0, j, \mathbf{e}) = \sum_{m=dp+1}^{(p+q)d} \frac{\widehat{D}_m(z, j_0, \mathbf{e})}{D^\Phi(z)} \Phi_m(z, j). \tag{4.4.37b}$$

Thus, recalling that 1 is a simple zero of the Evans function D^Φ, the expressions (4.4.37a) and (4.4.37b) allow us to conclude that the spatial Green's function has been meromorphically extended on $B(1, \delta_1) \backslash \{1\}$ with a pole of order 1 at 1.

Decomposing the function $\widehat{D}_m$

Now that we have found the meromorphic extension of the spatial Green's function near 1 using the family $(\Phi_m(z, j))_{m \in \{1, \ldots, d(p+q)\}}$, we will use the other families $\left(W_1^\pm(z, j), \ldots, W_{d(p+q)}^\pm(z, j)\right)$ for which we know precisely the behavior as j tends towards $\pm\infty$ to find the new improved expressions (4.4.34) of the spatial Green's function of Proposition 4.3. This will rely on finding precise expressions for $\widehat{D}_m$ defined by (4.4.36).

We begin this section by introducing the vectors

$$\forall z \in B(1, \delta_1), \forall j_0 \in \mathbb{Z}, \forall \mathbf{e} \in \mathbb{C}^d, \quad \mathscr{C}^\pm(z, j_0, \mathbf{e}) = \begin{pmatrix} \mathscr{C}_1^\pm(z, j_0, \mathbf{e}) \\ \vdots \\ \mathscr{C}_{d(p+q)}^\pm(z, j_0, \mathbf{e}) \end{pmatrix} := \mathcal{G}^\pm(z, j_0 + 1)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.4.38a}$$

$$\forall z \in B(1, \delta_1), \forall j_0 \in \mathbb{Z}, \forall \mathbf{e} \in \mathbb{C}^d, \quad \widetilde{\mathscr{C}}^\pm(z, j_0, \mathbf{e}) = \begin{pmatrix} \widetilde{\mathscr{C}}_1^\pm(z, j_0, \mathbf{e}) \\ \vdots \\ \widetilde{\mathscr{C}}_{d(p+q)}^\pm(z, j_0, \mathbf{e}) \end{pmatrix} := \widetilde{\mathcal{G}}^\pm(z, j_0 + 1)^{-1} \begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.4.38b}$$

Using (4.4.32) and (4.4.26), we obtain that for $z \in B(1, \delta_1)$, $j_0 \in \mathbb{Z}$ and $\mathbf{e} \in \mathbb{C}^d$

$$\widetilde{\mathscr{C}}_1^+(z, j_0, \mathbf{e}) = \theta_{s,1} \mathscr{C}_1^+(z, j_0, \mathbf{e}), \tag{4.4.39a}$$

$$\forall m \in I_{ss} \backslash \{1\}, \quad \widetilde{\mathscr{C}}_m^+(z, j_0, \mathbf{e}) = \mathscr{C}_m^+(z, j_0, \mathbf{e}) + \theta_{s,m} \mathscr{C}_1^+(z, j_0, \mathbf{e}), \tag{4.4.39b}$$

$$\forall m \in \{1, \dots, d(p+q)\} \backslash I_{ss}, \quad \widetilde{\mathscr{C}}_m^+(z, j_0, \mathbf{e}) = \mathscr{C}_m^+(z, j_0, \mathbf{e}), \tag{4.4.39c}$$

$$\forall m \in \{1, \dots, d(p+q)\} \backslash I_{su}, \quad \widetilde{\mathscr{C}}_m^-(z, j_0, \mathbf{e}) = \mathscr{C}_m^-(z, j_0, \mathbf{e}), \tag{4.4.39d}$$

$$\forall m \in I_{su} \backslash \{d(p+q)\}, \quad \widetilde{\mathscr{C}}_m^-(z, j_0, \mathbf{e}) = \mathscr{C}_m^-(z, j_0, \mathbf{e}) + \theta_{u,m} \mathscr{C}_{d(p+q)}^-(z, j_0, \mathbf{e}), \tag{4.4.39e}$$

$$\widetilde{\mathscr{C}}_{d(p+q)}^-(z, j_0, \mathbf{e}) = \theta_{u,d(p+q)} \mathscr{C}_{d(p+q)}^-(z, j_0, \mathbf{e}). \tag{4.4.39f}$$

In Section 4.4.4 and more precisely in Lemma 4.4.7 below, we will prove estimates to bound $\widetilde{\mathscr{C}}_m^\pm$. However, the functions $\widetilde{\mathscr{C}}_m^\pm$ will be put on the side for now and will naturally reappear later on in Section 4.4.4 using (4.4.39). For now, we will mainly focus on properties linked to the functions $\mathscr{C}_m^\pm$.

We introduce the matrices

$$\forall z \in B(1, \delta_1), \quad \mathcal{M}^\pm(z) := \mathcal{G}^\pm(z, 0)^{-1} \mathcal{G}^\Phi(z, 0) \quad \text{and} \quad \left(g_{m',m}^\pm(z)\right)_{(m',m) \in \{1, \dots, d(p+q)\}} := \operatorname{com}\left(\mathcal{M}^\pm(z)\right). \tag{4.4.40}$$

Using the definition (4.4.36) of $\widehat{D}_m$, we have that:

$$\forall z \in B(1, \delta_1), \forall j_0 \in \mathbb{Z}, \forall \mathbf{e} \in \mathbb{C}^d, \quad \begin{pmatrix} \widehat{D}_1(z, j_0, \mathbf{e}) \\ \vdots \\ \widehat{D}_{d(p+q)}(z, j_0, \mathbf{e}) \end{pmatrix} := D^\pm(z) \operatorname{com}(\mathcal{M}^\pm(z))^T \mathcal{G}^\pm(z, j_0, \mathbf{e})$$

and thus:

$$\forall m \in \{1, \dots, d(p+q)\}, \forall z \in B(1, \delta_1), \forall j_0 \in \mathbb{Z}, \forall \mathbf{e} \in \mathbb{C}^d, \quad \widehat{D}_m(z, j_0, \mathbf{e}) = D^\pm(z) \sum_{m'=1}^{d(p+q)} g_{m',m}^\pm(z) \mathscr{C}_{m'}^\pm(z, j_0, \mathbf{e}). \tag{4.4.41}$$

Looking at (4.4.37), we are interested in studying the quotient of $\widehat{D}_m(\cdot, j_0, \mathbf{e})$ and D^Φ. In order to have lighter expressions later on, we also introduce the functions $\widetilde{g}_{m',m}^\pm$ defined by:

$$\forall m, m' \in \{1, \dots, d(p+q)\}, \forall z \in B(1, \delta_1) \backslash \{1\}, \quad \widetilde{g}_{m',m}^\pm(z) := D^\pm(z) \frac{g_{m',m}^\pm(z)}{D^\Phi(z)}. \tag{4.4.42}$$

The function $\widetilde{g}_{m',m}^\pm$ is meromorphic on $B(1, \delta_1) \backslash \{1\}$ with a pole of order at most 1 at 1 since 1 is a simple zero of the Evans function D^Φ. If $g_{m',m}^\pm(1) = 0$, it can thus be extended holomorphically on the whole ball $B(1, \delta_1)$.

This decomposition of the functions $\widehat{D}_m$ will be used in Section 4.4.4 with (4.4.37) to obtain a better expression of the spatial Green's function. We end this section by proving the following lemma using (4.4.28).

Lemma 4.4.6.

1. *For $m' \in \{1, \dots, dp\}$ and $m \in \{1, \dots, d(p+q)\}$, we have*
$$\forall z \in B(1, \delta_1), \quad g_{m',m}^+(z) = \delta_{m',m} \frac{D^\Phi(z)}{D^+(z)}.$$
2. *For $m \notin \{1, d(p+q)\}$ and $m' \in \{1, \dots, d(p+q)\}$, we have*
$$g_{m',m}^+(1) = 0.$$
3. *For $m' \in \{dp+1, \dots, d(p+q)\}$ and $m \in \{1, \dots, d(p+q)\}$, we have*
$$\forall z \in B(1, \delta_1), \quad g_{m',m}^-(z) = \delta_{m',m} \frac{D^\Phi(z)}{D^-(z)}.$$
4. *For $m \notin \{1, d(p+q)\}$ and $m' \in \{1, \dots, d(p+q)\}$, we have*
$$g_{m',m}^-(1) = 0.$$
5. *We have*
$$g_{1,1}^+(1) = g_{1,d(p+q)}^+(1) = 0 \quad \text{and} \quad g_{d(p+q),d(p+q)}^-(1) = g_{d(p+q),1}^-(1) = 0.$$
6. *For $m' \in \{1, \dots, d(p+q)\}$, we have*
$$g_{m',1}^+(1) = -g_{m',d(p+q)}^+(1) \quad \text{and} \quad g_{m',1}^-(1) = -g_{m',d(p+q)}^-(1). \tag{4.4.43}$$

Lemma 4.4.6 determines the couple of indexes $m, m' \in \{1, \dots, d(p+q)\}$ such that $g^{\pm}_{m',m}$ is equal to 0. This allows us to remove many terms in (4.4.41). It also determines the couple of indexes $m, m' \in \{1, \dots, d(p+q)\}$ such that $g^{\pm}_{m',m}$ vanishes at $z = 1$ which implies that the function $\widetilde{g}^{\pm}_{m',m}$ defined by (4.4.42) can be holomorphically extended on the whole ball $B(1, \delta_1)$.

Proof We will focus on proving the statements involving $g^{+}_{m',m}$ since every statement involving $g^{-}_{m',m}$ will have similar proofs. We observe that the definition (4.4.27) of $\Phi_m(z, 0)$ implies that if we define for $z \in B(1, \delta_1)$ and $m \in \{1, \dots, d(p+q)\}$

$$C^+_m(z) := \mathcal{G}^+(z, 0)^{-1}\Phi_m(z, 0)$$

then, for $m \in \{1, \dots, dp\}$, we have

$$C^+_m(z) = e_m$$

where $(e_j)_{j\in\{1,\dots,d(p+q)\}}$ is the canonical basis of $\mathbb{C}^{d(p+q)}$. Thus,

$$\mathcal{M}^+(z) = \left(\begin{array}{c|c|c|c} \frac{I_{dp}}{0} & C^+_{dp+1}(z) & \dots & C^+_{d(p+q)}(z) \end{array} \right). \tag{4.4.44}$$

For $m, m' \in \{1, \dots, d(p+q)\}$, we recall that $g^+_{m',m}(z)$ is the (m', m)-cofactor of the matrix above.

We observe that

$$\mathrm{com}(\mathcal{M}^+(z))^T \mathcal{M}^+(z) = \frac{D^{\Phi}(z)}{D^+(z)} I_{d(p+q)}. \tag{4.4.45}$$

Looking at the first dp columns of $\mathcal{M}^+(z)$ in (4.4.44) and using (4.4.45), we then conclude that the first dp lines of $\mathrm{com}(\mathcal{M}^+(z))$ are equal to

$$\left(\begin{array}{c|c} \frac{D^{\Phi}(z)}{D^+(z)} I_{dp} & 0 \end{array} \right)$$

which implies Point (i).

The equality (4.4.28) also implies that:

$$C^+_{d(p+q)}(1) = e_1 = C^+_1(1).$$

and thus:

$$\mathcal{M}^+(1) = \left(\begin{array}{c|c|c|c|c} \frac{I_{dp}}{0} & C^+_{dp+1}(1) & \dots & C^+_{d(p+q)-1}(1) & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array} \right).$$

Points (ii) and (v) are then easily deduced by equality between the first and last columns of the matrix above.

There just remains to prove Point (vi). We observe that

$$\mathcal{M}^+(1)\mathrm{com}(\mathcal{M}^+(1))^T = \frac{D^{\Phi}(1)}{D^+(1)} = 0.$$

Looking at the coefficient at the first line and m'-th column, we have that

$$g^+_{m',1}(1) + \sum_{m=dp+1}^{d(p+q)-1} (C^+_m(1))_1 g^+_{m',m}(1) + g^+_{m',d(p+q)}(1) = 0.$$

Using Point (ii), we easily conclude the proof of Point (vi). $\square$

Final expression of the spatial Green's function

We will now prove the expressions (4.4.34) on the spatial Green's function when j_0 is larger than 0 (i.e. the location of the initial perturbation δ_{j_0} is on the right of the shock). The case where $j_0 < 0$ would be handled similarly and give another expression of the spatial Green's function on $B(1, \delta_1)$ that would be necessary to prove a decomposition of the temporal Green's function similar to (4.1.32) when $j_0 < 0$.

Case where $j_0 \geq 0$ and $j \geq j_0 + 1$:

Using (4.4.37a) and (4.4.41), we have

$$W(z, j_0, j, \mathbf{e}) = -\sum_{m=1}^{dp} \sum_{m'=1}^{d(p+q)} \widetilde{g}^+_{m',m}(z)\mathscr{C}^+_{m'}(z, j_0, \mathbf{e})\Phi_m(z, j).$$

Point (i) of Lemma 4.4.6 then implies that

$$W(z,j_0,j,\mathbf{e}) = -\sum_{m=1}^{dp} \mathscr{C}_m^+(z,j_0,\mathbf{e})\Phi_m(z,j) - \sum_{m=1}^{dp}\sum_{m'=dp+1}^{d(p+q)} \widetilde{g}_{m',m}^+(z)\mathscr{C}_{m'}^+(z,j_0,\mathbf{e})\Phi_m(z,j).$$

Using the definition (4.4.27) of Φ_1, we then have that

$$\begin{aligned} W(z,j_0,j,\mathbf{e}) = -\theta_{s,1}\mathscr{C}_1^+(z,j_0,\mathbf{e})W_1^+(z,j) - \sum_{m\in I_{ss}\setminus\{1\}} (\mathscr{C}_m^+(z,j_0,\mathbf{e}) + \theta_{s,m}\mathscr{C}_1^+(z,j_0,\mathbf{e}))W_m^+(z,j) \\ - \sum_{m\in I_{cs}} \mathscr{C}_m^+(z,j_0,\mathbf{e})W_m^+(z,j) - \sum_{m=1}^{dp}\sum_{m'=dp+1}^{d(p+q)} \widetilde{g}_{m',m}^+(z)\mathscr{C}_{m'}^+(z,j_0,\mathbf{e})\Phi_m(z,j). \end{aligned}$$

Using (4.4.39a)-(4.4.39c) which link the functions $\mathscr{C}_m^\pm$ and $\widetilde{\mathscr{C}}_m^\pm$ and the definition (4.4.27) of Φ_m for $m \in \{2,\dots,dp\}$, we obtain that (4.4.34a) is verified, i.e. :

$$\begin{aligned} W(z,j_0,j,\mathbf{e}) = -\sum_{m=1}^{dp} \widetilde{\mathscr{C}}_m^+(z,j_0,\mathbf{e})W_m^+(z,j) - \sum_{m=2}^{dp}\sum_{m'=dp+1}^{d(p+q)} \widetilde{g}_{m',m}^+(z)\widetilde{\mathscr{C}}_{m'}^+(z,j_0,\mathbf{e})W_m^+(z,j) \\ - \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}_{m',1}^+(z)\widetilde{\mathscr{C}}_{m'}^+(z,j_0,\mathbf{e})\Phi_1(z,j). \end{aligned}$$

Remark 4. If $m \in \{2,\dots,dp\}$ and $m' \in \{dp+1,\dots,d(p+q)\}$, we have that $g_{m',m}^+(1) = 0$ because of Lemma 4.4.6. Thus, the function

$$z \in B(1,\delta_1)\setminus\{1\} \mapsto \widetilde{g}_{m',m}^+(z)\widetilde{\mathscr{C}}_{m'}^+(z,j_0,\mathbf{e})\Phi_m(z,j)$$

can be holomorphically extended on the whole ball $B(1,\delta_1)$.

Case where $j_0 \geq 0$ and $j \in \{0,\dots,j_0\}$:

Let us consider $m \in \{dp+1,\dots,d(p+q)\}$. Using the respective definitions (4.4.31a), (4.4.31b) and (4.4.40) of the matrices $\mathcal{G}^\Phi$, $\mathcal{G}^+$ and $\mathcal{M}^+$, we have the following expression of the vector $\Phi_m(z,j)$ depending on the family $(W_k^+(z,j))_{k\in\{1,\dots,d(p+q)\}}$:

$$\Phi_m(z,j) = \mathcal{G}^+(z,j)(\mathcal{M}^+(z)_{k,m})_{k\in\{1,\dots,d(p+q)\}} = \mathcal{M}^+(z)_{1,m}\Phi_1(z,j) + \sum_{k=2}^{d(p+q)} \mathcal{M}^+(z)_{k,m}W_k^+(z,j).$$

Thus, since $m \in \{dp+1,\dots,d(p+q)\}$, using (4.4.41) and the fact that Lemma 4.4.6 implies $g_{m',m}^+ = 0$ for $m' \in \{1,\dots,dp\}$ we have

$$\begin{aligned} &\frac{\widehat{D}_m(z,j_0,\mathbf{e})}{D^\Phi(z)}\Phi_m(z,j) \\ &\qquad = \frac{D^+(z)}{D^\Phi(z)} \sum_{m'=dp+1}^{d(p+q)} g_{m',m}^+(z)\mathscr{C}_{m'}^+(z,j_0,\mathbf{e}) \left(\mathcal{M}^+(z)_{1,m}\Phi_1(z,j) + \sum_{k=2}^{d(p+q)} \mathcal{M}^+(z)_{k,m}W_k^+(z,j)\right). \end{aligned}$$

Using (4.4.37b), we then have that:

$$\begin{aligned} &W(z,j_0,j,\mathbf{e}) \\ &= \sum_{m'=dp+1}^{d(p+q)} \left[\left(\sum_{m=dp+1}^{d(p+q)} \mathcal{M}^+(z)_{1,m}g_{m',m}^+(z)\right)\Phi_1(z,j) + \sum_{k=2}^{d(p+q)}\left(\sum_{m=dp+1}^{d(p+q)} \mathcal{M}^+(z)_{k,m}g_{m',m}^+(z)\right)W_k^+(z,j)\right] \\ &\qquad \frac{D^+(z)}{D^\Phi(z)}\mathscr{C}_{m'}^+(z,j_0,\mathbf{e}). \end{aligned} \tag{4.4.46}$$

Let us find expressions for the sums $\sum_{m=dp+1}^{d(p+q)} \mathcal{M}^+(z)_{k,m}g_{m',m}^+(z)$ when $m' \in \{dp+1,\dots,d(p+q)\}$ and $k \in \{1,\dots,d(p+q)\}$. We recall that $g_{m',m}^+(z)$ is the (m',m)-cofactor of the matrix $\mathcal{M}^+(z)$. Furthermore, by definition (4.4.40) of the matrix $\mathcal{M}^+$, we have that

$$\mathcal{M}^+(z)\mathrm{com}(\mathcal{M}^+(z))^T = \frac{D^\Phi(z)}{D^+(z)} Id. \tag{4.4.47}$$

Thus, by observing (4.4.44) implies that for $k \in \{dp+1, \dots, d(p+q)\}$ and $m \in \{1, \dots, dp\}$ we have $\mathcal{M}^+(z)_{k,m} = 0$, we conclude looking at the k-th line and m'-th column of (4.4.47) that

$$\forall k \in \{dp+1, \dots, d(p+q)\}, \forall m' \in \{dp+1, \dots, d(p+q)\}, \quad \sum_{m=dp+1}^{d(p+q)} \mathcal{M}^+(z)_{k,m} g^+_{m',m}(z) = \frac{D^\Phi(z)}{D^+(z)} \delta_{k,m'}. \tag{4.4.48}$$

Furthermore, (4.4.44) implies that

$$\forall k \in \{1, \dots, dp\}, \forall m \in \{1, \dots, dp\}, \quad \mathcal{M}^+(z)_{k,m} = \delta_{k,m}.$$

Thus, looking once again at the k-th line and m'-th column of (4.4.47), we have

$$\forall k \in \{1, \dots, dp\}, \forall m' \in \{dp+1, \dots, d(p+q)\}, \quad \sum_{m=dp+1}^{d(p+q)} \mathcal{M}^+(z)_{k,m} g^+_{m',m}(z) = -g^+_{m',k}(z). \tag{4.4.49}$$

We finally conclude using (4.4.39a)-(4.4.39c) which links the functions $\mathscr{C}^\pm_m$ and $\widetilde{\mathscr{C}}^\pm_m$ and combining (4.4.46), (4.4.48) and (4.4.49) that (4.4.34b) is verified, i.e.:

$$\begin{aligned} W(z, j_0, j, \mathbf{e}) = \sum_{m=dp+1}^{d(p+q)} \widetilde{\mathscr{C}}^+_m(z, j_0, \mathbf{e}) W^+_m(z, j) - \sum_{m=2}^{dp} \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^+_{m',m}(z) \widetilde{\mathscr{C}}^+_{m'}(z, j_0, \mathbf{e}) W^+_m(z, j) \\ - \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^+_{m',1}(z) \widetilde{\mathscr{C}}^+_{m'}(z, j_0, \mathbf{e}) \Phi_1(z, j). \end{aligned}$$

Remark 5. We observe that some terms in (4.4.34b) are equal to terms of (4.4.34a). We will see later on that they contribute to the reflected waves in the decomposition of Theorem 4.1.

Case where $j_0 \geq 0$ and $j < 0$:
Using (4.4.37b) and (4.4.41),

$$W(z, j_0, j, \mathbf{e}) = \sum_{m=dp+1}^{d(p+q)} \sum_{m'=1}^{d(p+q)} \widetilde{g}^+_{m',m}(z) \mathscr{C}^+_{m'}(z, j_0, \mathbf{e}) \Phi_m(z, j).$$

Lemma 4.4.6 implies that

$$\forall m \in \{dp+1, \dots, d(p+q)\}, \forall m' \in \{1, \dots, dp\}, \quad g^+_{m',m} = 0.$$

Thus, using (4.4.39a)-(4.4.39c) which links the functions $\mathscr{C}^\pm_m$ and $\widetilde{\mathscr{C}}^\pm_m$, we have that (4.4.34c) is verified, i.e.:

$$\begin{aligned} W(z, j_0, j, \mathbf{e}) = \sum_{m=dp+1}^{d(p+q)-1} \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^+_{m',m}(z) \widetilde{\mathscr{C}}^+_{m'}(z, j_0, \mathbf{e}) W^-_m(z, j) \\ + \sum_{m'=dp+1}^{d(p+q)} \widetilde{g}^+_{m',d(p+q)}(z) \widetilde{\mathscr{C}}^+_{m'}(z, j_0, \mathbf{e}) \Phi_{d(p+q)}(z, j). \end{aligned}$$

Remark 6. If $m \in \{dp+1, \dots, d(p+q)-1\}$ and $m' \in \{dp+1, \dots, d(p+q)\}$, we have that $g^+_{m',m}(1) = 0$. Thus, the function

$$z \in B(1, \delta_1) \backslash \{1\} \mapsto \widetilde{g}^+_{m',m}(z) \widetilde{\mathscr{C}}^+_{m'}(z, j_0, \mathbf{e}) \Phi_m(z, j)$$

can be holomorphically extended on the whole ball $B(1, \delta_1)$.

Useful estimates

In this section, we will introduce the necessary observations to properly bound the terms appearing in the decomposition of the spatial Green's function of Section 4.4.4. We will in particular introduce a new expression of the functions $\widetilde{\mathscr{C}}_m^\pm(z,j_0,\mathbf{e})$, prove that they roughly act like $\zeta_m^\pm(z)^{-j_0}$ and determine their behavior as j_0 tends towards $\pm\infty$.

For $z\in B(1,\delta_1)$ and $j\in\mathbb{Z}$, we recall that Lemma 4.4.4 implies that $(V_m^\pm(z,j))_{m\in\{1,\dots,d(p+q)\}}$ is a basis of $\mathbb{C}^{d(p+q)}$. Thus, we can define for $z\in B(1,\delta_1)$, $j_0\in\mathbb{Z}$ and $\mathbf{e}\in\mathbb{C}^d$

$$N^\pm(z,j_0):=\left(V_1^\pm(z,j_0+1)\quad\middle|\dots\middle|\quad V_{d(p+q)}^\pm(z,j_0+1)\right)\quad\text{and}\quad\begin{pmatrix}\Delta_1^\pm(z,j_0,\mathbf{e})\\ \vdots\\ \Delta_{d(p+q)}^\pm(z,j_0,\mathbf{e})\end{pmatrix}:=N^\pm(z,j_0)^{-1}\begin{pmatrix}A_{j_0,q}^{-1}\mathbf{e}\\0\\ \vdots\\0\end{pmatrix}. \tag{4.4.50}$$

We observe that (4.4.38b) implies that for all $z\in B(1,\delta_1)$, $j_0\in\mathbb{Z}$ and $\mathbf{e}\in\mathbb{C}^d$, we have

$$\sum_{m=1}^{d(p+q)}\zeta_m^\pm(z)^{j_0+1}\widetilde{\mathscr{C}}_m^\pm(z,j_0,\mathbf{e})V_m^\pm(z,j_0+1)=\begin{pmatrix}A_{j_0,q}^{-1}\mathbf{e}\\0\\ \vdots\\0\end{pmatrix}.$$

Thus, we have that for $m\in\{1,\dots,d(p+q)\}$, $z\in B(1,\delta_1)$, $j_0\in\mathbb{Z}$ and $\mathbf{e}\in\mathbb{C}^d$

$$\widetilde{\mathscr{C}}_m^\pm(z,j_0,\mathbf{e})=\zeta_m^\pm(z)^{-j_0-1}\Delta_m^\pm(z,j_0,\mathbf{e}). \tag{4.4.51}$$

We now prove the following lemma which gives us the asymptotic behavior of $\Delta_m^\pm(z,j_0,\mathbf{e})$.

Lemma 4.4.7. *There exist a radius $\delta_2\in]0,\delta_1]$ and two constants $C,c>0$ such that for all $z\in B(1,\delta_2)$, $m=l+(k-1)d\in\{1,\dots,d(p+q)\}$ with $k\in\{1,\dots,p+q\}$ and $l\in\{1,\dots,d\}$ and $\boldsymbol{e}\in\mathbb{C}^d$, we have*

$$\forall j\in\mathbb{N},\quad \left|V_m^+(z,j)-V_m^+(1,j)\right|\leq C|z-1|, \tag{4.4.52a}$$
$$\forall j\in-\mathbb{N},\quad \left|V_m^-(z,j)-V_m^-(1,j)\right|\leq C|z-1|, \tag{4.4.52b}$$
$$\forall j\in\mathbb{N},\quad |\Phi_1(z,j)-\Phi_1(1,j)|\leq C|z-1|e^{-\frac{3c_*}{2}|j|}, \tag{4.4.52c}$$
$$\forall j\in-\mathbb{N},\quad \left|\Phi_{d(p+q)}(z,j)-\Phi_{d(p+q)}(1,j)\right|\leq C|z-1|e^{-\frac{3c_*}{2}|j|}, \tag{4.4.52d}$$
$$\forall j_0\in\mathbb{N},\quad \left|\Delta_m^+(z,j_0,\boldsymbol{e})-\frac{d\zeta_m^+}{dz}(z)\boldsymbol{l}_l^{+T}\boldsymbol{e}\right|\leq C|\boldsymbol{e}|e^{-c|j_0|}, \tag{4.4.52e}$$
$$\forall j_0\in-\mathbb{N},\quad \left|\Delta_m^-(z,j_0,\boldsymbol{e})-\frac{d\zeta_m^-}{dz}(z)\boldsymbol{l}_l^{-T}\boldsymbol{e}\right|\leq C|\boldsymbol{e}|e^{-c|j_0|}, \tag{4.4.52f}$$
$$\forall j_0\in\mathbb{N},\quad \left|\Delta_m^+(z,j_0,\boldsymbol{e})\right|\leq C|\boldsymbol{e}|, \tag{4.4.52g}$$
$$\forall j_0\in-\mathbb{N},\quad \left|\Delta_m^-(z,j_0,\boldsymbol{e})\right|\leq C|\boldsymbol{e}|. \tag{4.4.52h}$$
$$\forall j_0\in\mathbb{N},\quad \left|\Delta_m^+(z,j_0,\boldsymbol{e})-\Delta_m^+(1,j_0,\boldsymbol{e})\right|\leq C|z-1||\boldsymbol{e}|, \tag{4.4.52i}$$
$$\forall j_0\in-\mathbb{N},\quad \left|\Delta_m^-(z,j_0,\boldsymbol{e})-\Delta_m^-(1,j_0,\boldsymbol{e})\right|\leq C|z-1||\boldsymbol{e}|, \tag{4.4.52j}$$

where c_* is the positive constant in (4.4.1).

Proof The first two inequalities are direct consequences from the fact that $z\in B(1,\delta_1)\mapsto\frac{\partial V_m^\pm}{\partial z}(z,\cdot)\in\ell^\infty(\pm\mathbb{N},\mathbb{C}^{d(p+q)})$ is bounded (see Lemma 4.4.3).

We will prove (4.4.52c). The proof of (4.4.52d) would be similar. Using the definition (4.4.27) of Φ_1, we conclude that we only have to prove that for all $m\in I_{ss}$ that there exists a constant $C>0$ and a radius $\delta_2\in]0,\delta_1]$ such that

$$\forall z\in B(1,\delta_2),\forall j\in\mathbb{N},\quad \left|W_m^+(z,j)-W_m^+(1,j)\right|\leq C|z-1|e^{-\frac{3c_*}{2}|j|}.$$

We observe that for $z\in B(1,\delta_1)$ and $j\in\mathbb{N}$, we have

$$\left|W_m^+(z,j)-W_m^+(1,j)\right|\leq\left|\zeta_m^+(z)\right|^j\left|V_m^+(z,j)-V_m^+(1,j)\right|+\left|V_m^+(1,j)\right|\left|\zeta_m^+(z)^j-\zeta_m^+(1)^j\right|.$$

Using (4.4.52a), (4.4.1a) and the fact that $\frac{d\zeta_m^+}{dz}$ is bounded on $B(1,\delta_2)$ for $\delta_2 \in]0,\delta_2[$, we easily conclude the proof of (4.4.52c).

We will focus on (4.4.52e) as (4.4.52f) would be proved in a similar way. We observe that Lemma 4.4.2, (4.1.14) and (4.1.5) imply that:

$$L_m^+(z)^T \begin{pmatrix} {A_q^+}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_{l,q}^+ \frac{d\zeta_m^+}{dz}(z) {\boldsymbol{l}_l^+}^T {A_q^+}^{-1}\mathbf{e} = \frac{d\zeta_m^+}{dz}(z){\boldsymbol{l}_l^+}^T\mathbf{e}.$$

Thus, $\Delta_m^+(z,j_0,\mathbf{e}) - \frac{d\zeta_m^+}{dz}(z){\boldsymbol{l}_l^+}^T\mathbf{e}$ is the m-th coefficient of the vector

$$N^+(z,j_0)^{-1}\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - N^{+,\infty}(z)^{-1}\begin{pmatrix} {A_q^+}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the matrix $N^{+,\infty}$ is defined by (4.4.3). We then just have to find bounds for this difference of vectors. We have

$$\begin{aligned} & N^+(z,j_0)^{-1}\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - N^{+,\infty}(z)^{-1}\begin{pmatrix} {A_q^\pm}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ =& N^+(z,j_0)^{-1}\left(N^{+,\infty}(z) - N^+(z,j_0)\right)N^{+,\infty}(z)^{-1}\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + N^{+,\infty}(z)^{-1}\begin{pmatrix} (A_{j_0,q}^{-1} - {A_q^+}^{-1})\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned} \tag{4.4.53}$$

We wish to bound each term in the right-hand side of the equality (4.4.53). Let us start by looking at the first term. Lemma 4.4.3 implies that the functions $N^+(\cdot,j_0)^{-1}$ are bounded on $B(1,\delta_1)$ and that the bound can considered to be uniform for $j_0 \in \mathbb{N}$. The function $N^{+,\infty}(\cdot)^{-1}$ is also bounded on $B(1,\delta_1)$. Since $A_{j_0,q}^{-1}$ converges towards ${A_q^+}^{-1}$ as j_0 converges towards $+\infty$, we also have that the family of matrices $(A_{j_0,q}^{-1})_{j_0\in\mathbb{N}}$ is bounded. Finally, using Lemma 4.4.3, we have that there exist two constants $C,c>0$ such that

$$\forall z \in B(1,\delta_1), \forall j_0 \in \mathbb{N}, \quad \left|N^{+,\infty}(z) - N^+(z,j_0)\right| \leq Ce^{-cj_0}.$$

Thus, there exists another constant $C>0$ such that

$$\forall z \in B(1,\delta_1), \forall \mathbf{e} \in \mathbb{C}^d, \forall j_0 \in \mathbb{N}, \quad \left| N^+(z,j_0)^{-1}\left(N^{+,\infty}(z) - N^+(z,j_0)\right)N^{+,\infty}(z)^{-1}\begin{pmatrix} A_{j_0,q}^{-1}\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right| \leq C|\mathbf{e}|e^{-cj_0}.$$

We now focus on the second term. The function $N^{\pm,\infty}(\cdot)^{-1}$ is bounded on $B(1,\delta_1)$. Furthermore, Hypothesis 4.3 allows us to determine that there exist two constants $C,c>0$ such that

$$\forall j_0 \in \mathbb{N}, \quad \left|A_{j_0,q}^{-1} - {A_q^+}^{-1}\right| \leq Ce^{-cj_0}.$$

Therefore, there exists a new constant $C>0$ such that

$$\forall z \in B(1,\delta_1), \forall \mathbf{e} \in \mathbb{C}^d, \forall j_0 \in \mathbb{N}, \quad \left| N^{+,\infty}(z)^{-1}\begin{pmatrix} (A_{j_0,q}^{-1} - {A_q^+}^{-1})\mathbf{e} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right| \leq C|\mathbf{e}|e^{-cj_0}.$$

We can then conclude the proof of (4.4.52e).

We observe that (4.4.52g) and (4.4.52h) are direct consequences of (4.4.52e) and (4.4.52f).

There remains to prove (4.4.52i) as (4.4.52j) would be proved similarly. We observe that for $z \in B(1,\delta_2)$ and $j_0 \in \mathbb{N}$, we have

$$N^+(z,j_0)^{-1} - N^+(1,j_0)^{-1} = N^+(z,j_0)^{-1}\left(N^+(1,j_0) - N^+(z,j_0)\right)N^+(1,j_0)^{-1}.$$

Using (4.4.52a) and the observations above which claimed that $N^+(z,j_0)^{-1}$ is bounded uniformly for $z \in B(1,\delta_2)$ and $j_0 \in \mathbb{N}$, we have that there exists a positive constant C such that

$$\forall z \in B(1,\delta_2), \forall j_0 \in \mathbb{N}, \quad |N^+(z,j_0)^{-1} - N^+(1,j_0)^{-1}| \leq C|z-1|.$$

The definition (4.4.50) and the fact that the family of matrices $(A_{j_0,q}^{-1})_{j_0\in\mathbb{N}}$ is bounded imply that (4.4.52i) is verified for some constant $C>0$. □

4.5 Temporal Green's function and proof of Theorem 4.1

The previous Sections 4.3 and 4.4 served respectively to describe the spatial Green's function far from 1 and near 1. Our objective is now to focus on the core of the chapter: the study of temporal Green's function and the proof of Theorem 4.1. In the present section, we will express the temporal Green's function with the spatial Green's function using functional analysis. We will then use the different results of the previous sections (mainly Propositions 4.2 and 4.3) to obtain the result of Theorem 4.1. Just as when we proved Proposition 4.3 on the spatial Green's function near 1, the proof of Theorem 4.1 will be done whilst assuming that j_0 is larger than 0 to obtain (4.1.32) and (4.1.33). The case where $j_0<0$ would be handled similarly and would necessitate to prove expressions of the spatial Green's function on $B(1,\delta_1)$ similar to (4.4.34a)-(4.4.34c) when $j_0<0$.

4.5.1 Link between the spatial and temporal Green's function

First, we recall that in Sections 4.3 and 4.4, we studied the vectors $W(z,j_0,j,\mathbf{e})$ defined in Section 4.3.5 which are composed of several components of the spatial Green's function. The inverse Laplace transform implies that if we introduce a path $\widetilde{\Gamma}$ that surrounds the spectrum $\sigma(\mathscr{L})$, for instance $\widetilde{\Gamma}_r := r\mathbb{S}^1$ where $r>1$, then we have

$$\mathscr{G}(n,j_0,j)\mathbf{e} = \frac{1}{2i\pi}\int_{\widetilde{\Gamma}_r} z^n G(z,j_0,j)\mathbf{e}dz = \frac{1}{2i\pi}\int_{\widetilde{\Gamma}_r} z^n \Pi(W(z,j_0,j,\mathbf{e}))dz$$

where Π is the linear application defined by (4.3.32) which extracts the center values of a large vector. We consider the change of variables $z=\exp(\tau)$. If we define $\Gamma_r = \{r+it, t\in[-\pi,\pi]\}$, then we have

$$\mathscr{G}(n,j_0,j)\mathbf{e} = \frac{1}{2i\pi}\int_{\Gamma_r} e^{n\tau}e^{\tau}\Pi(W(e^\tau,j_0,j,\mathbf{e}))d\tau. \tag{4.5.1}$$

The goal will now be to use Cauchy's formula and/or the residue theorem to modify our choice of path Γ and to use at best the properties we proved on the spatial Green's function in Propositions 4.2 and 4.3.

In Proposition 4.3 of Section 4.4, we proved that we can meromorphically extend the spatial Green's function on a ball $B(1,\delta_1)$ with a pole of order 1 at 1 and have found the decompositions (4.4.34). We also introduced an even smaller ball $B(1,\delta_2)$ on which we have more precise bounds (Lemma 4.4.7) that will help us later on in the proof. We consider a radius $\varepsilon_0^\star \in]0,\pi[$ such that

$$\forall \tau \in B(0,\varepsilon_0^\star), \quad e^\tau \in B(1,\delta_2).$$

Lemma 4.5.1. *For all radii $\varepsilon \in]0,\varepsilon_0^\star[$, there exists a width $\eta_\varepsilon>0$ such that if we define*

$$\Omega_\varepsilon := \{\tau\in\mathbb{C}, \quad \Re\tau \in]-\eta_\varepsilon,\pi], \operatorname{Im}\tau\in[-\pi,\pi]\} \cup B(0,\varepsilon),$$

then for all $j,j_0\in\mathbb{Z}$ and $\boldsymbol{e}\in\mathbb{C}^d$, the function $\tau\mapsto W(e^\tau,j_0,j,\boldsymbol{e})$ is meromorphically defined on $\Omega_\varepsilon\backslash\{0\}$ with a pole of order 1 at 0 and there exist two positive constants $C_\varepsilon, c_\varepsilon$ such that

$$\forall \tau\in\Omega_\varepsilon\backslash B(0,\varepsilon), \forall j,j_0\in\mathbb{Z}, \forall \boldsymbol{e}\in\mathbb{C}^d, \quad |W(e^\tau,j_0,j,\boldsymbol{e})| \leq C_\varepsilon|\boldsymbol{e}|e^{-c_\varepsilon|j-j_0|}. \tag{4.5.2}$$

Defining this width η_ε is important for the following calculations since we have defined a set Ω_ε on which we can change the path of integration of (4.5.1) using the residue theorem. Furthermore, for $\tau\in\Omega_\varepsilon$, either $\tau\in B(0,\varepsilon)$ which implies that $e^\tau\in B(1,\delta_2)$ and that we can thus use the decomposition (4.4.34) of the spatial

Green's function we obtained in Proposition 4.3, or $\tau \notin B(0,\varepsilon)$ and we can use (4.5.2) to obtain exponential bounds on the spatial Green's function.

Proof The proof is identical as [Coe24, Lemma 5.2] and will thus not be detailled. It just relies on observing that for any radius $\varepsilon \in]0,\varepsilon_0^\star[$, the results of Section 4.4 imply that for all $j_0, j \in \mathbb{Z}$ and $\mathbf{e} \in \mathbb{C}^d$, the function $\tau \mapsto W(e^\tau, j_0, j, \mathbf{e})$ is meromorphically extended on $B(0,\varepsilon)\backslash\{0\}$ with a pole of order 1 at 0. We then use Proposition 4.2 on a neighborhood of each point of the set

$$U_\varepsilon := \{\tau \in \mathbb{C}, \quad \Re\tau \in [0,\pi], \mathrm{Im}\,\tau \in [-\pi,\pi]\} \backslash B(0,\varepsilon)$$

and conclude via a compactness argument on the existence of a width $\eta_\varepsilon \in]0,\varepsilon[$ and of two positive constants $C_\varepsilon, c_\varepsilon$ such that (4.5.2) is verified. □

Let us observe that for all $m \in I_{cs}^\pm \cup I_{cu}^\pm$, we have that the function $\zeta_m^\pm$ (which we recall are defined in Section 4.4.1 and are eigenvalues of the matrices $M^\pm(z)$) is holomorphic and $\zeta_m^\pm(1) = 1$. Therefore, there exists a radius $\varepsilon_1^\star \in]0,\varepsilon_0^\star[$ so that for all $m \in I_{cs}^\pm \cup I_{cu}^\pm$ that we write as $m = l + (k-1)d$ with $k \in \{1,\dots,p+q\}$ and $l \in \{1,\dots,d\}$, there exists an holomorphic function $\varpi_l^\pm : B(0,\varepsilon_1^\star) \to \mathbb{C}$ such that $\varpi_l^\pm(0) = 0$ and

$$\forall \tau \in B(0,\varepsilon_1^\star), \quad \zeta_m^\pm(e^\tau) = \exp(\varpi_l^\pm(\tau)). \tag{4.5.3}$$

Since $\zeta_m^\pm(e^\tau)$ is an eigenvalue of $M_l^\pm(e^\tau)$, Lemma 4.3.3 implies that

$$\forall \tau \in B(0,\varepsilon_1^\star), \quad \mathcal{F}_l^\pm(e^{\varpi_l^\pm(\tau)}) = e^\tau.$$

If we define the holomorphic function

$$\begin{array}{cccc} \varphi_l^\pm : & \mathbb{C} & \to & \mathbb{C} \\ & \tau & \mapsto & -\frac{\tau}{\alpha_l^\pm} + (-1)^{\mu+1} \frac{\beta_l^\pm}{{\alpha_l^\pm}^{2\mu+1}} \tau^{2\mu} \end{array}, \tag{4.5.4}$$

then, up to considering $\varepsilon_1^\star$ to be slightly smaller, the asymptotic expansion (4.1.18) implies that there exists a bounded holomorphic function $\xi_l^\pm : B(0,\varepsilon_1^\star) \to \mathbb{C}$ such that

$$\forall \tau \in B(0,\varepsilon_1^\star), \quad \varpi_l^\pm(\tau) = \varphi_l^\pm(\tau) + \tau^{2\mu+1}\xi_l^\pm(\tau). \tag{4.5.5}$$

Lemma 4.5.2. *There exists a radius $\varepsilon_2^\star \in]0,\varepsilon_1^\star[$ and two positive constants A_R, A_I such that for all $l \in \{1,\dots,d\}$*

$$\forall \tau \in \mathbb{C}, \qquad \alpha_l^\pm \Re(\varphi_l^\pm(\tau)) \leq -\Re(\tau) + A_R \Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}, \tag{4.5.6}$$

$$\forall \tau \in B(0,\varepsilon_2^\star), \qquad \alpha_l^\pm \Re(\varpi_l^\pm(\tau)) + |\alpha_l^\pm \xi_l^\pm(\tau)\tau^{2\mu+1}| \leq -\Re(\tau) + A_R \Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}. \tag{4.5.7}$$

The proof is identical as [Coe24, Lemma 5.3] and will thus not be detailed here.

Choice of the radius ε and of the width η

We will now fix choices for a radius $\varepsilon > 0$ and a width $\eta > 0$ which will satisfy a list of conditions. Those conditions will be centralized here in order to fix the notations and are especially important to prove some technical lemmas in Section 4.5.2. We will try to indicate at best where those conditions are used.

First, we fix a choice of radius $\varepsilon \in \left]0, \min\left(\varepsilon_2^\star, \left(\frac{1}{2\mu A_R}\right)^{\frac{1}{2\mu-1}}\right)\right[$ where the radius $\varepsilon_2^\star$ is defined in Lemma 4.5.2. This choice for ε will allow us to use the results of Lemmas 4.5.1 and 4.5.2. Furthermore, if we introduce the function

$$\begin{array}{cccc} \Psi : & \mathbb{R} & \to & \mathbb{R} \\ & \tau_p & \mapsto & \tau_p - A_R {\tau_p}^{2\mu} \end{array} \tag{4.5.8}$$

which we will use to define a family of parameterized curve later on in Lemma 4.5.4, then the function Ψ is continuous and strictly increasing on $]-\infty,\varepsilon]$. This conclusion on the function Ψ will be essential in the proof of Lemma 4.5.4 to construct the path Γ appearing in (4.5.17c).

We now introduce the function

$$\begin{array}{cccc} r_\varepsilon : &]0,\varepsilon[& \to & \mathbb{R} \\ & \eta & \mapsto & \sqrt{\varepsilon^2 - \eta^2} \end{array} \tag{4.5.9}$$

which serves to define the extremities of the curve $-\eta + i\mathbb{R} \cap B(0,\varepsilon)$. We recall that we defined a width η_ε in Lemma 4.5.1. We claim that there exists a width $\eta \in]0,\eta_\varepsilon[$ that we fix for the rest of the chapter such that:

— The following inequality is satisfied:

$$\frac{\eta}{2} > A_R \eta^{2\mu}. \tag{4.5.10}$$

It is used for instance in the proof of Lemma 4.5.4.

— We have

$$\eta + A_R\eta^{2\mu} - \frac{A_I}{2} r_\varepsilon(\eta)^{2\mu} < 0. \tag{4.5.11}$$

It is quite clear that we can choose η small enough to satisfy this condition since, when η tends towards 0, the first two terms on the left hand side converge towards 0 and the third converges towards $-\frac{A_I}{2}\varepsilon$. The condition (4.5.11) is used in Lemma 4.5.4 to prove (4.5.17b). A consequence of (4.5.11) is that

$$\forall n \in \mathbb{N}, \forall x \in \left[\frac{n}{2}, 2n\right], \forall t \in [-\eta, \eta], \quad (n-x)t + xA_R t^{2\mu} - x\frac{A_I r_\varepsilon(\eta)^{2\mu}}{2} \leq 0. \tag{4.5.12}$$

Indeed, using the convexity with regards to t of the left hand side of (4.5.12), we have that

$$(n-x)t + xA_R t^{2\mu} - x\frac{A_I r_\varepsilon(\eta)^{2\mu}}{2} \leq |n-x|\eta + xA_R\eta^{2\mu} - x\frac{A_I r_\varepsilon(\eta)^{2\mu}}{2}.$$

We observe that $n \in \left[\frac{x}{2}, 2x\right]$ and thus, using (4.5.11), we have

$$(n-x)t + xA_R t^{2\mu} - x\frac{A_I r_\varepsilon(\eta)^{2\mu}}{2} \leq x\left(\eta + A_R\eta^{2\mu} - \frac{A_I r_\varepsilon(\eta)^{2\mu}}{2}\right) \leq 0.$$

The consequence (4.5.12) of (4.5.11) will be used in the proof of Lemma 4.5.6.

— There exists a radius $\varepsilon_\# \in]0, \varepsilon[$ such that if we define

$$l_{extr} := \left(\frac{\Psi(\varepsilon_\#) - \Psi(-\eta)}{A_I}\right)^{\frac{1}{2\mu}}, \tag{4.5.13}$$

then $-\eta + il_{extr} \in B(0, \varepsilon)$. It is used in the proof of Lemma 4.5.4.

We introduce the paths $\Gamma_{out}(\eta)$, $\Gamma_{in}^{\pm}(\eta)$, $\Gamma_{in}^{0}(\eta)$, $\Gamma_{in}(\eta)$, $\Gamma(\eta)$, $\Gamma_d(\eta)$ represented on Figure 4.3 and defined as

$$\begin{aligned}
\Gamma_{out}(\eta) &:= [-\eta - i\pi, -\eta - ir_\varepsilon(\eta)] \cup [-\eta + ir_\varepsilon(\eta), -\eta + i\pi], \\
\Gamma_{in}^{\pm}(\eta) &:= [-\eta \pm ir_\varepsilon(\eta), \eta \pm ir_\varepsilon(\eta)], \\
\Gamma_{in}^{0}(\eta) &:= [\eta - ir_\varepsilon(\eta), \eta + ir_\varepsilon(\eta)], \\
\Gamma_{in}(\eta) &:= \Gamma_{in}^{-}(\eta) \cup \Gamma_{in}^{0}(\eta) \cup \Gamma_{in}^{+}(\eta) \\
\Gamma(\eta) &:= \Gamma_{in}(\eta) \cup \Gamma_{out}(\eta), \\
\Gamma_d(\eta) &:= [-\eta - ir_\varepsilon(\eta), -\eta + ir_\varepsilon(\eta)].
\end{aligned} \tag{4.5.14}$$

We observe that those paths lie in Ω_ε. Using the Cauchy formula and acknowledging the "$2i\pi$"-periodicity of $\tau \mapsto W(e^\tau, j_0, j, \mathbf{e})$, we can prove via the equality (4.5.1) that

$$\begin{aligned}
\mathscr{G}(n, j_0, j)\mathbf{e} &= \frac{1}{2i\pi}\int_{\Gamma(\eta)} e^{n\tau}e^\tau \Pi(W(e^\tau, j_0, j, \mathbf{e}))d\tau \\
&= \frac{1}{2i\pi}\int_{\Gamma_{out}(\eta)} e^{n\tau}e^\tau \Pi(W(e^\tau, j_0, j, \mathbf{e}))d\tau + \frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau \Pi(W(e^\tau, j_0, j, \mathbf{e}))d\tau.
\end{aligned} \tag{4.5.15}$$

Lemma 4.5.3. *There exist two positive constants C, c such that for all $n \in \mathbb{N}$, $j_0, j \in \mathbb{Z}$ and $\mathbf{e} \in \mathbb{C}^d$ we have that*

$$\left|\frac{1}{2i\pi}\int_{\Gamma_{out}(\eta)} e^{n\tau}e^\tau \Pi(W(e^\tau, j_0, j, \mathbf{e}))d\tau\right| \leq C|\mathbf{e}|e^{-n\eta}.$$

Proof The conclusion of the lemma directly follows from (4.5.2) and the definition of $\Gamma_{out}(\eta)$ which implies that

$$\forall \tau \in \Gamma_{out}(\eta), \quad |e^{n\tau}| = e^{-n\eta}.$$

□

The equality (4.5.15) and the sharp exponential bounds on

$$\frac{1}{2i\pi}\int_{\Gamma_{out}(\eta)} e^{n\tau}e^\tau \Pi(W(e^\tau, j_0, j, \mathbf{e}))d\tau,$$

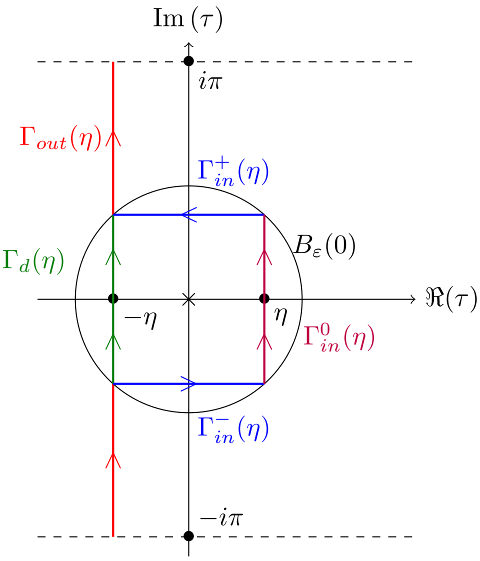

Figure 4.3 – A representation of the path described in (4.5.14): $\Gamma_{out}(\eta)$ (in red), $\Gamma_{in}^{\pm}(\eta)$ (in blue), $\Gamma_{in}^{0}(\eta)$ (in purple) and $\Gamma_d(\eta)$ (in green)

we just proved imply that there just remains to handle the term

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\Pi(W(e^\tau, j_0, j, \mathbf{e}))d\tau \tag{4.5.16}$$

to obtain the description (4.1.32) of the temporal Green's function expected in Theorem 4.1. We recall that $\Gamma_{in}(\eta)$ is a path that lies inside the set $B(0,\varepsilon)$ by construction and that we chose the radius $\varepsilon\in]0,\varepsilon_2^*[$ to be small enough so that

$$\forall \tau\in B(0,\varepsilon),\quad e^\tau\in B(1,\delta_2).$$

Thus, recalling that we consider $j_0\geq 0$, we can use the expressions (4.4.34a), (4.4.34b) and (4.4.34c) to decompose the integral (4.5.16) into different terms depending on the position of j with respect to 0 and j_0. Our new goal can now be separated in two parts:
- In Section 4.5.2, we will prove Lemmas 4.5.7-4.5.11 that will allow to bound the several terms that can appear when decomposing the integral (4.5.16) using (4.4.34).
- In Section 4.5.3, we conclude the proof of the decomposition 4.1.32 of the temporal Green's function using the previously proved Lemmas 4.5.3 and 4.5.7-4.5.11.
- In Section 4.5.4, we adapt the results of Sections 4.5.2 and 4.5.3 to prove the decomposition (4.1.33) on the discrete derivative of the temporal Green's function and thus conclude the proof of Theorem 4.1.

4.5.2 Decomposition of the integral within $B(0,\varepsilon)$

This section will be mainly devoted to the proof of Lemmas 4.5.7-4.5.11 that will allow to study each term that can appear in the decomposition of the integral (4.5.16) using the expressions (4.4.34a), (4.4.34b) and (4.4.34c). However, we are first going to need to introduce a few more technical lemmas that will be used several time throughout the rest of the chapter.

Gaussian estimates

First and foremost, we define the set X of the paths going from $-\eta - ir_\varepsilon(\eta)$ to $-\eta + ir_\varepsilon(\eta)$ whilst remaining in $B(0,\varepsilon)$. We observe in particular that $\Gamma_d(\eta),\Gamma_{in}(\eta)\in X$.

Lemma 4.5.4. *We consider an integer $k\in\mathbb{N}$.*
- *There exist two positive constants C, c such that for all $n\in\mathbb{N}\backslash\{0\}$ and $x\in\left[0,\frac{n}{2}\right]$*

$$\int_{\Gamma_d(\eta)}|\tau|^k\exp\left(n\Re(\tau)+x\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|\leq Ce^{-cn}. \tag{4.5.17a}$$

— *There exist two positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$ and $x \in [2n, +\infty[$*

$$\int_{\Gamma_{in}(\eta)} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau| \leq Ce^{-cn}. \tag{4.5.17b}$$

— *There exist two positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$ and $x \in \left[\frac{n}{2}, 2n\right]$, there exists a path $\Gamma \in X$ such that*

$$\int_{\Gamma} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau| \leq \frac{C}{n^{\frac{k+1}{2\mu}}} \exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{4.5.17c}$$

Lemma 4.5.4 will allow us to obtain generalized Gaussian bounds for several terms throughout the proof of Theorem 4.1. The inequalities (4.5.17a)-(4.5.17c) separate different cases depending on x. An important point to observe is that the path Γ appearing in (4.5.17c) depends on n and x whereas the constants C, c are uniform.

The way Lemma 4.5.4 will be used is to first observe that the integral of some holomorphic function over some path of X is equal by Cauchy's formula to the integral of the same function over any path of X. We then prove that the integrand can be well bounded and use the result of Lemma 4.5.4. The proof of Lemma 4.5.4 can be adapted from [CF22; CF23; Coe22; Coe24] and will be done in the Appendix (Section 4.A).

Lemma 4.5.5. *There exists a constant $C > 0$ such that for all $l \in \{1, \dots, d\}$, $n \in \mathbb{N}\backslash\{0\}$ and $x \in [0, 2n]$*

$$\left|e^{x\alpha_l^\pm \varpi_l^\pm(\tau)} - e^{x\alpha_l^\pm \varphi_l^\pm(\tau)}\right| \leq Cn|\tau|^{2\mu+1}\exp\left(x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right).$$

The proof of Lemma 4.5.5 is similar to the proof of [Coe22, Lemma 16] and will not be detailed here. We recall that the functions $\varpi_l^\pm$ and $\varphi_l^\pm$ respectively defined by (4.5.3) and (4.5.4) are linked by the equality (4.5.5). Lemma 4.5.5 will allow us to "extract" the principal part $\varphi_l^\pm$ of the function $\varpi_l^\pm$. This principal part will then appear in terms that can be studied using the following lemma.

Lemma 4.5.6. *We consider $l, l' \in \{1, \dots, d\}$ and $?, ?' \in \{-, +\}$. There exist two constants $C, c > 0$ such that for all $n \in \mathbb{N}\backslash\{0\}$, we have:*

— *For $x, y \in [0, +\infty[$ such that $x + y \in \left[\frac{n}{2}, 2n\right]$ and $\Gamma \in X$*

$$\left|\frac{1}{2i\pi}\int_\Gamma \exp\left(n\tau + x\alpha_l^?\varphi_l^?(\tau) + y\alpha_{l'}^{?'}\varphi_{l'}^{?'}(\tau)\right)d\tau - \frac{|\alpha_l^?|}{n^{\frac{1}{2\mu}}}H_{2\mu}\left(\frac{x}{n}\beta_l^? + \frac{y}{n}\beta_{l'}^{?'}\left(\frac{\alpha_l^?}{\alpha_{l'}^{?'}}\right)^{2\mu}; \frac{\alpha_l^?(n-(x+y))}{n^{\frac{1}{2\mu}}}\right)\right| \leq Ce^{-cn}. \tag{4.5.18a}$$

— *For $x \in \left[\frac{n}{2}, 2n\right]$ and $\Gamma \in X$*

$$\left|\frac{1}{2i\pi}\int_\Gamma \exp\left(n\tau + x\alpha_l^?\varphi_l^?(\tau)\right)d\tau - \frac{|\alpha_l^?|}{n^{\frac{1}{2\mu}}}H_{2\mu}\left(\beta_l^?; \frac{\alpha_l^?(n-x)}{n^{\frac{1}{2\mu}}}\right)\right| \leq \frac{C}{n^{\frac{1}{\mu}}}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{4.5.18b}$$

— *For $x \in \left[\frac{n}{2}, 2n\right]$,*

$$\left|\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^?\varphi_l^?(\tau)\right)}{\tau}d\tau - E_{2\mu}\left(\beta_l^?; \frac{-|\alpha_l^?|(n-x)}{n^{\frac{1}{2\mu}}}\right)\right| \leq \frac{C}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{4.5.18c}$$

The proof of Lemma 4.5.6 is a summary of calculations performed in [Coe22; Coe24] and will be done in the Appendix (Section 4.A). Let us observe that there is no condition on the paths of integration in (4.5.18a) and (4.5.18b). However, since the integrand is only meromorphic in (4.5.18a), we only consider the path $\Gamma_{in}(\eta)$.

Outgoing and incoming waves

We will start by looking at the outgoing and incoming waves by proving the following lemma.

Lemma 4.5.7. *We consider $m \in \{1, \dots, d(p+q)\}$ and write it as $m = l + (k-1)d$ with $k \in \{1, \dots, p+q\}$ and $l \in \{1, \dots, d\}$. There exists a constant $c > 0$ such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0, j \in \mathbb{N}$ such that $j - j_0 \in \{-nq, \dots, np\}$ and $\boldsymbol{e} \in \mathbb{C}^d$ we have:*

- *If $m\in I_{cs}^+\cup I_{cu}^+$ and $\frac{j-j_0}{\alpha_l^+}\in\left[\frac{n}{2},2n\right]$, we have that:*

$$-\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_{in}(\eta)}e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^+(e^\tau,j))d\tau-S_l^+(n,j_0,j)\boldsymbol{e}$$
$$=\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{|\boldsymbol{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{|\boldsymbol{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)\boldsymbol{r}_l^++O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right){\boldsymbol{l}_l^+}^T\boldsymbol{e}\boldsymbol{r}_l^+\right). \quad (4.5.19a)$$

- *If $m\in I_{cs}^+\cup I_{cu}^+$, $\frac{j-j_0}{\alpha_l^+}\notin\left[\frac{n}{2},2n\right]$ and $\frac{j-j_0}{\alpha_l^+}\geq 0$, we have that:*

$$-\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_{in}(\eta)}e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^+(e^\tau,j))d\tau-S_l^+(n,j_0,j)\boldsymbol{e}=O\left(|\boldsymbol{e}|e^{-cn}\right). \quad (4.5.19b)$$

- *If $m\in I_{ss}^+$ and $j\geq j_0+1$ or if $m\in I_{su}^+$ and $j\in\{0,\dots,j_0\}$, we have that:*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)}e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^+(e^\tau,j))d\tau=O\left(|\boldsymbol{e}|e^{-cn}\right). \quad (4.5.19c)$$

Proof • We start by proving (4.5.19a). The proofs of (4.5.19b) and (4.5.19c) will be done afterwards as they are fairly less complicated. We consider $m\in I_{cs}^+\cup I_{cu}^+$ such that $\frac{j-j_0}{\alpha_l^+}\in\left[\frac{n}{2},2n\right]$.

Using the expressions of W_m^+ and $\widetilde{\mathscr{C}}_m^+$ given respectively by Lemma 4.4.3 and (4.4.51), we have using Cauchy's formula that for any $\Gamma\in X$

$$\int_{\Gamma_{in}(\eta)}e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\mathbf{e})\Pi(W_m^+(e^\tau,j))d\tau=\int_{\Gamma}e^{n\tau}\zeta_m^+(e^\tau)^{j-j_0-1}e^{\tau}\Delta_m^+(e^\tau,j_0,\mathbf{e})\Pi(V_m^+(e^\tau,j))d\tau. \quad (4.5.20)$$

We recall that the function ϖ_l^+ is defined by (4.5.3). Using Cauchy's formula once again, the definition (4.1.31a) of the function S_l^+ and (4.5.3) since the index m belongs to $I_{cs}^+\cup I_{cu}^+$, we then have that

$$-\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_{in}(\eta)}e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\mathbf{e})\Pi(W_m^+(e^\tau,j))d\tau-S_l^+(n,j_0,j)\mathbf{e}=E_1+E_2\boldsymbol{r}_l^++(E_3+E_4+E_5)\,{\boldsymbol{l}_l^+}^T\mathbf{e}\boldsymbol{r}_l^+ \quad (4.5.21)$$

where E_1 is a vector and $E_2,\dots,E_5$ are complex scalars defined by

$$\begin{aligned}
E_1&:=-\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_1}e^{n\tau}e^{(j-j_0-1)\varpi_l^+(\tau)}e^{\tau}\Delta_m^+(e^\tau,j_0,\mathbf{e})\Pi(V_m^+(e^\tau,j)-R_m^+(e^\tau))d\tau,\\
E_2&:=-\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_2}e^{n\tau}e^{(j-j_0-1)\varpi_l^+(\tau)}e^{\tau}\left(\Delta_m^+(e^\tau,j_0,\mathbf{e})-\frac{d\zeta_m^+}{dz}(e^\tau){\boldsymbol{l}_l^+}^T\mathbf{e}\right)d\tau,\\
E_3&:=-\frac{\operatorname{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_3}e^{n\tau}e^{(j-j_0)\varpi_l^+(\tau)}\left(e^{\tau}\zeta_m^+(e^\tau)^{-1}\frac{d\zeta_m^+}{dz}(e^\tau)+\frac{1}{\alpha_l^+}\right)d\tau,\\
E_4&:=\frac{1}{2i\pi|\alpha_l^+|}\int_{\Gamma_4}e^{n\tau}\left(e^{(j-j_0)\varpi_l^+(\tau)}-e^{(j-j_0)\varphi_l^+(\tau)}\right)d\tau,\\
E_5&:=\frac{1}{2i\pi|\alpha_l^+|}\int_{\Gamma_5}e^{n\tau}e^{(j-j_0)\varphi_l^+(\tau)}d\tau-\frac{1}{n^{\frac{1}{2\mu}}}H_{2\mu}\left(\beta_l^+;\frac{n\alpha_l^++j_0-j}{n^{\frac{1}{2\mu}}}\right),
\end{aligned}$$

and $\Gamma_1,\dots,\Gamma_5$ are paths belonging to the set X defined at the beginning of Section 4.5.2. We just have to prove correct bounds on the terms $E_1,\dots,E_5$ appearing in (4.5.21) to obtain (4.5.19a). In particular, we will use good choices of paths $\Gamma_1,\dots,\Gamma_5$ to optimize the bounds using Lemma 4.5.4.

▶ Using (4.4.52g), (4.5.7) and Lemma 4.4.3 which claims that the vectors $V_m^+(z,j)$ converge exponentially fast towards $R_m^+(e^\tau)$, we have that there exist two positive constants C,c independent from n, j_0, j and $\mathbf{e}$ such that

$$|E_1|\leq Ce^{-c|j|}|\mathbf{e}|\int_{\Gamma_1}\exp\left(n\Re(\tau)+\left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ Using (4.4.52e) and (4.5.7), we have that there exist two positive constants C,c independent from n, j_0,

j and $\mathbf{e}$ such that

$$|E_2| \leq Ce^{-c|j_0|}|\mathbf{e}| \int_{\Gamma_2} \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ We notice that $\zeta_m^+(1) = 1$ and $\frac{d\zeta_m^+}{dz}(1) = -\frac{1}{\alpha_l^+}$. Using a Taylor expansion and (4.5.7), we have that there exists a positive constant C independent from n, j_0 and j such that

$$|E_3| \leq C\int_{\Gamma_3} |\tau| \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ Since $\frac{j-j_0}{\alpha_l^+} \in \left[\frac{n}{2}, 2n\right]$, we can use Lemma 4.5.5 and prove that there exists a positive constant C independent from n, j_0 and j such that

$$|E_4| \leq Cn\int_{\Gamma_4} |\tau|^{2\mu+1} \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using Lemma 4.5.4 which gives a good choices of path $\Gamma_1, \dots, \Gamma_4 \in X$ depending on n, j_0 and j to handle the integrals in the terms above as well as Lemma 4.5.6 to take care of the term E_5, there exist new constants $C, c > 0$ independent from n, j_0, j and $\mathbf{e}$ such that

$$|E_1| \leq \frac{Ce^{-c|j|}|\mathbf{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right), \quad |E_2| \leq \frac{Ce^{-c|j_0|}|\mathbf{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right),$$

$$|E_3| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right), \quad |E_4| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right),$$

$$|E_5| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

We have thus obtained (4.5.19a).

• We now focus on (4.5.19b). We thus consider $m \in I_{cs}^+$ and $j \geq j_0 + 1$ or $m \in I_{cu}^+$ and $j \in \{0, \dots, j_0\}$ such that $\frac{j-j_0}{\alpha_l^+} \notin \left[\frac{n}{2}, 2n\right]$. We observe that in particular, $S_l^+(n, j_0, j) = 0$. Using (4.5.20), Lemma 4.4.3 which claims that the vectors $V_m^+(z, j)$ are uniformly bounded for $z \in B(1, \delta_1)$ and $j \in \mathbb{N}$, (4.4.52g) to bound $\Delta_m^+(e^\tau, j_0, \mathbf{e})$ and (4.5.7), there exists a positive constant C independent from n, j_0, j and $\mathbf{e}$ such that for all $\Gamma \in X$

$$\begin{aligned}\left|-\frac{\mathrm{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \mathbf{e})\Pi(W_m^+(e^\tau, j))d\tau - S_l^+(n, j_0, j)\mathbf{e}\right| \\ \leq C|\mathbf{e}|\int_\Gamma \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.\end{aligned}$$

We then use Lemma 4.5.4 to prove that there exist two new positive constants C, c independent from n, j_0, j and $\mathbf{e}$ such that

$$\left|-\frac{\mathrm{sgn}(\alpha_l^+)}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{\mathscr{C}}_m^+(e^\tau, j_0, \mathbf{e})\Pi(W_m^+(e^\tau, j))d\tau - S_l^+(n, j_0, j)\mathbf{e}\right| \leq C|\mathbf{e}|e^{-cn}.$$

• We now focus on (4.5.19c). We will consider the case where the integer m belongs to I_{ss}^+ and $j \geq j_0 + 1$. The second case considered in (4.5.19c) would be handled similarly. Using (4.5.20) with $\Gamma = \Gamma_d(\eta)$, Lemma 4.4.3 which claims that the vectors $V_m^+(z, j)$ are uniformly bounded for $z \in B(1, \delta_1)$ and $j \in \mathbb{N}$, (4.4.52g) to bound $\Delta_m^+(e^\tau, j_0, \mathbf{e})$ and (4.4.1a) whilst noticing that $j \geq j_0 + 1$, there exists a positive constant C independent from n, j_0, j and $\mathbf{e}$ such that

$$\begin{aligned}\left|\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\mathbf{e})\Pi(W_m^+(e^\tau,j))d\tau\right| &\leq C|\mathbf{e}|e^{-2c_*|j-j_0|}\int_{\Gamma_d(\eta)} e^{n\Re(\tau)}|d\tau| \\ &\leq 2r_\varepsilon(\eta)C|\mathbf{e}|e^{-n\eta-2c_*|j-j_0|}.\end{aligned}$$

□

Reflected waves

We now look at the reflected waves.

Lemma 4.5.8. *We consider $m\in\{2,\dots,dp\}$ and $m'\in\{dp+1,\dots,d(p+q)\}$ and write them as $m=l+(k-1)d$ and $m'=l'+(k'-1)d$ with $k,k'\in\{1,\dots,p+q\}$ and $l,l'\in\{1,\dots,d\}$. There exists a positive constant c such that for all $n\in\mathbb{N}\backslash\{0\}$, $j_0,j\in\mathbb{N}$ such that $j-j_0\in\{-nq,\dots,np\}$ and $\boldsymbol{e}\in\mathbb{C}^d$, we have:*

- *If $m\in I_{cs}^+$, $m'\in I_{cu}^+$ and $\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\in\left[\frac{n}{2},2n\right]$, we have that:*

$$\begin{aligned}&-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^+(e^\tau,j))d\tau-\frac{\alpha_l^+}{\alpha_{l'}^+}\widetilde{g}_{m',m}^+(1)R_{l',l}^+(n,j_0,j)\boldsymbol{e}\\ &\quad=\exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{|\boldsymbol{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{|\boldsymbol{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)\boldsymbol{r}_l^+ +O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\boldsymbol{l}_{l'}^{+\,T}\boldsymbol{e}\boldsymbol{r}_l^+\right).\end{aligned}\tag{4.5.22a}$$

- *If $m\in I_{cs}^+$, $m'\in I_{cu}^+$ and $\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\notin\left[\frac{n}{2},2n\right]$, we have that:*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^+(e^\tau,j))d\tau-\frac{\alpha_l^+}{\alpha_{l'}^+}\widetilde{g}_{m',m}^+(1)R_{l',l}^+(n,j_0,j)\boldsymbol{e}=O(|\boldsymbol{e}|e^{-cn}).\tag{4.5.22b}$$

- *If $m\in I_{ss}^+$, $m'\in I_{cu}^+$ and $-\frac{j_0}{\alpha_{l'}^+}\in\left[\frac{n}{2},2n\right]$, we have that:*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^+(e^\tau,j))d\tau=O\left(\frac{|\boldsymbol{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right).\tag{4.5.22c}$$

- *If $m\in I_{ss}^+$, $m'\in I_{cu}^+$ and $-\frac{j_0}{\alpha_{l'}^+}\notin\left[\frac{n}{2},2n\right]$, we have that:*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^+(e^\tau,j))d\tau=O\left(|\boldsymbol{e}|e^{-cn}\right).\tag{4.5.22d}$$

- *If $m\in I_{cs}^+$, $m'\in I_{su}^+$ and $\frac{j}{\alpha_l^+}\in\left[\frac{n}{2},2n\right]$, we have that:*

$$\begin{aligned}&-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^+(e^\tau,j))d\tau\\ &\qquad=O(|\boldsymbol{e}|e^{-cn})+O_{\mathbb{C}}\left(\frac{|\boldsymbol{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)\boldsymbol{r}_l^+.\end{aligned}\tag{4.5.22e}$$

- *If $m\in I_{cs}^+$, $m'\in I_{su}^+$ and $\frac{j}{\alpha_l^+}\notin\left[\frac{n}{2},2n\right]$, we have that:*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^+(e^\tau,j))d\tau=O\left(|\boldsymbol{e}|e^{-cn}\right).\tag{4.5.22f}$$

- *If $m \in I_{ss}^+$, $m' \in I_{su}^+$, we have that:*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\mathbf{e})\Pi(W^+_m(e^\tau,j))d\tau = O\left(|\mathbf{e}|e^{-cn}\right). \tag{4.5.22g}$$

We observe that since we consider $m \in \{2,\dots,dp\}$ and $m' \in \{dp+1,\dots,d(p+q)\}$, Lemma 4.4.6 implies that $\widetilde{g}^+_{m',m}$ can be holomorphically extended on the whole ball $B(1,\delta_1)$ and thus the term $\widetilde{g}^+_{m',m}(1)$ is well defined.

Proof • We start by proving (4.5.22a). We consider that $m \in I^+_{cs}$, $m' \in I^+_{cu}$ and $\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha^+_{l'}} \in \left[\frac{n}{2}, 2n\right]$.

Since the function $\widetilde{g}^+_{m',m}$ can be holomorphically extended on the whole ball $B(1,\varepsilon)$, using the expressions of W^+_m and $\widetilde{\mathscr{C}}^+_{m'}$ given respectively by Lemma 4.4.3 and (4.4.51), we have using Cauchy's formula that for any $\Gamma \in X$

$$\int_{\Gamma_{in}(\eta)} e^{\tau}\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\mathbf{e})\Pi(W^+_m(e^\tau,j))d\tau$$
$$= \int_\Gamma e^{n\tau}\widetilde{g}^+_{m',m}(e^\tau)\zeta^+_m(e^\tau)^j\zeta^+_{m'}(e^\tau)^{-j_0-1}e^\tau\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(V^+_m(e^\tau,j))d\tau. \tag{4.5.23}$$

Using Cauchy's formula once again, the definition (4.1.31b) of the function $R^+_{l',l}$ and (4.5.3) since the index m belongs to I^+_{cs} and m' belongs to I^+_{cu}, we then have that

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\mathbf{e})\Pi(W^+_m(e^\tau,j))d\tau - \frac{\alpha^+_l}{\alpha^+_{l'}}\widetilde{g}^+_{m',m}(1)R^+_{l',l}(n,j_0,j)\mathbf{e}$$
$$= E_1 + E_2\boldsymbol{r}^+_l + (E_3+E_4+E_5+E_6)\,{\boldsymbol{l}^+_{l'}}^T\mathbf{e}\boldsymbol{r}^+_l \tag{4.5.24}$$

where E_1 is a vector and $E_2,\dots,E_6$ are complex scalars defined by

$$E_1 := -\frac{1}{2i\pi}\int_{\Gamma_1} e^{n\tau}\widetilde{g}^+_{m',m}(e^\tau)e^{j\varpi^+_l(\tau)}e^{-(j_0+1)\varpi^+_{l'}(\tau)}e^\tau\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(V^+_m(e^\tau,j)-R^+_m(e^\tau))d\tau,$$
$$E_2 := -\frac{1}{2i\pi}\int_{\Gamma_2} e^{n\tau}\widetilde{g}^+_{m',m}(e^\tau)e^{j\varpi^+_l(\tau)}e^{-(j_0+1)\varpi^+_{l'}(\tau)}e^\tau\left(\Delta^+_{m'}(e^\tau,j_0,\mathbf{e}) - \frac{d\zeta^+_{m'}}{dz}(e^\tau){\boldsymbol{l}^+_{l'}}^T\mathbf{e}\right)d\tau,$$
$$E_3 := -\frac{1}{2i\pi}\int_{\Gamma_3} e^{n\tau}e^{j\varpi^+_l(\tau)}e^{-j_0\varpi^+_{l'}(\tau)}\left(\widetilde{g}^+_{m',m}(e^\tau)e^\tau\zeta^+_{m'}(e^\tau)^{-1}\frac{d\zeta^+_{m'}}{dz}(e^\tau) + \frac{\widetilde{g}^+_{m',m}(1)}{\alpha^+_{l'}}\right)d\tau,$$
$$E_4 := \frac{\widetilde{g}^+_{m',m}(1)}{2i\pi\alpha^+_{l'}}\int_{\Gamma_4} e^{n\tau}\left(e^{j\varpi^+_l(\tau)} - e^{j\varphi^+_l(\tau)}\right)e^{-j_0\varpi^+_{l'}(\tau)}d\tau,$$
$$E_5 := \frac{\widetilde{g}^+_{m',m}(1)}{2i\pi\alpha^+_{l'}}\int_{\Gamma_5} e^{n\tau}e^{j\varphi^+_l(\tau)}\left(e^{-j_0\varpi^+_{l'}(\tau)} - e^{-j_0\varphi^+_{l'}(\tau)}\right)d\tau,$$

$$E_6 := \frac{\widetilde{g}^+_{m',m}(1)}{2i\pi\alpha^+_{l'}}\int_{\Gamma_5} e^{n\tau}e^{j\varphi^+_l(\tau)}e^{-j_0\varphi^+_{l'}(\tau)}d\tau$$
$$-\frac{\alpha^+_l}{\alpha^+_{l'}}\widetilde{g}^+_{m',m}(1)\frac{1}{n^{\frac{1}{2\mu}}}H_{2\mu}\left(\frac{j}{n\alpha^+_l}\beta^+_l - \frac{j_0}{n\alpha^+_{l'}}\beta^+_{l'}\left(\frac{\alpha^+_l}{\alpha^+_{l'}}\right)^{2\mu}, \frac{n\alpha^+_l + j_0\frac{\alpha^+_l}{\alpha^+_{l'}} - j}{n^{\frac{1}{2\mu}}}\right),$$

and $\Gamma_1,\dots,\Gamma_6$ are paths belonging to the set X. We just have to prove bounds on the terms $E_1,\dots,E_6$. In particular, we will use good choices of paths $\Gamma_1,\dots,\Gamma_6$ to optimize the bounds using Lemma 4.5.4.

► Using (4.4.52g) to bound $\Delta^+_{m'}$, (4.5.7) and Lemma 4.4.3 which claims that the vectors $V^+_m(z,j)$ converge exponentially fast towards $R^+_m(e^\tau)$, we have that there exist two positive constants C,c independent from n, j_0, j and $\mathbf{e}$ such that

$$|E_1| \leq Ce^{-c|j|}|\mathbf{e}|\int_{\Gamma_1}\exp\left(n\Re(\tau) + \left(\frac{j}{\alpha^+_l} - \frac{j_0}{\alpha^+_{l'}}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

► Using (4.4.52e) and (4.5.7), we have that there exist two positive constants C,c independent from n, j_0,

j and $\mathbf{e}$ such that

$$|E_2| \leq Ce^{-c|j_0|}|\mathbf{e}| \int_{\Gamma_2} \exp\left(n\Re(\tau) + \left(\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ We notice that $\zeta_{m'}^+(1) = 1$ and $\frac{d\zeta_{m'}^+}{dz}(1) = -\frac{1}{\alpha_l^+}$. Using a Taylor expansion and (4.5.7), we have that there exists a positive constant C independent from n, j_0 and j such that

$$|E_3| \leq C \int_{\Gamma_3} |\tau| \exp\left(n\Re(\tau) + \left(\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

▶ We observe that $\frac{j}{\alpha_l^+}$ and $-\frac{j_0}{\alpha_{l'}^+}$ are positive and $\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+} \in \left[\frac{n}{2}, 2n\right]$. Thus, we have that $\frac{j}{\alpha_l^+} \in [0, 2n]$. We can then use Lemma 4.5.5 and (4.5.7) to prove that there exists a positive constant C independent from n, j_0 and j such that

$$|E_4| \leq Cn \int_{\Gamma_4} |\tau|^{2\mu+1} \exp\left(n\Re(\tau) + \left(\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

Furthermore, we also have that and $-\frac{j_0}{\alpha_{l'}^+} \in [0, 2n]$. We can also use Lemma 4.5.5 and (4.5.6) to prove that there exists a positive constant C independent from n, j_0 and j such that

$$|E_5| \leq Cn \int_{\Gamma_5} |\tau|^{2\mu+1} \exp\left(n\Re(\tau) + \left(\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using Lemma 4.5.4 which gives a good choices of path $\Gamma_1, \dots, \Gamma_5 \in X$ depending on n, j_0 and j to handle the integrals in the terms above as well as Lemma 4.5.6 to take care of the term E_6, there exist new constants $C, c > 0$ independent from n, j_0, j and $\mathbf{e}$ such that

$$|E_1| \leq \frac{Ce^{-c|j|}|\mathbf{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right),$$

$$|E_2| \leq \frac{Ce^{-c|j_0|}|\mathbf{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right),$$

$$|E_3| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right), \quad |E_4| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right),$$

$$|E_5| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right), \quad |E_6| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n - \left(\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

We have thus obtained (4.5.22a).

• We now focus on (4.5.22b). We consider that $m \in I_{cs}^+$, $m' \in I_{cu}^+$ and $\frac{j}{\alpha_l^+} - \frac{j_0}{\alpha_{l'}^+} \notin \left[\frac{n}{2}, 2n\right]$. We observe that in particular, $R_{l',l}^+(n, j_0, j) = 0$. Using (4.5.23), Lemma 4.4.3 which claims that the vectors $V_m^+(z, j)$ are uniformly bounded for $z \in B(1, \delta_1)$ and $j \in \mathbb{N}$, (4.4.52g) to bound $\Delta_{m'}^+$ and (4.5.7), there exists a positive constant C independent from n, j_0, j and $\mathbf{e}$ such that for all $\Gamma \in X$

$$\left|-\frac{1}{2i\pi} \int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}_{m',m}^+(e^\tau) \mathscr{C}_m^+(e^\tau, j_0, \mathbf{e}) \Pi(W_m^+(e^\tau, j)) d\tau - \frac{\alpha_l^+}{\alpha_{l'}^+} \widetilde{g}_{m',m}^+(1) R_{l',l}^+(n, j_0, j)\mathbf{e}\right|$$

$$\leq C|\mathbf{e}|\int_\Gamma \exp\left(n\Re(\tau)+\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

We then use Lemma 4.5.4 to prove that there exist two new positive constants C,c independent from n, j_0, j and $\mathbf{e}$ such that

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\mathbf{e})\Pi(W^+_m(e^\tau,j))d\tau-\frac{\alpha_l^+}{\alpha_{l'}^+}\widetilde{g}^+_{m',m}(1)R^+_{l',l}(n,j_0,j)\mathbf{e}\right|\leq C|\mathbf{e}|e^{-cn}.$$

- We now focus on (4.5.22c) and (4.5.22d). We consider that $m\in I^+_{ss}$, $m'\in I^+_{cu}$. Using (4.5.23), (4.4.1a) to bound ζ^+_m, (4.5.3) since the index m' belongs to I^+_{cu}, Lemma 4.4.3 which claims that the vectors $V^+_m(z,j)$ are uniformly bounded for $z\in B(1,\delta_1)$ and $j\in\mathbb{N}$, (4.4.52g) to bound $\Delta^+_{m'}$ and (4.5.7), there exists a positive constant C independent from n, j_0, j and $\mathbf{e}$ such that for all $\Gamma\in X$

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\mathbf{e})\Pi(W^+_m(e^\tau,j))d\tau\right|$$
$$\leq Ce^{-2c_\star|j|}|\mathbf{e}|\int_\Gamma \exp\left(n\Re(\tau)+\left(-\frac{j_0}{\alpha_{l'}^+}\right)\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

We observe that $-\frac{j_0}{\alpha_{l'}^+}$ is positive since $\alpha_{l'}^+<0$ because m' belongs to I^+_{cu}. Using (4.5.17c) when $-\frac{j_0}{\alpha_{l'}^+}\in\left[\frac{n}{2},2n\right]$ and (4.5.17a) and (4.5.17b) else, we end up proving (4.5.22c) and (4.5.22d).

- We now focus on (4.5.22e) and (4.5.22f). We consider that $m\in I^+_{cs}$, $m'\in I^+_{su}$. Using (4.5.23), (4.5.3) since the index m belongs to I^+_{cs} and Cauchy's formula, we have that

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\mathbf{e})\Pi(W^+_m(e^\tau,j))d\tau=E_1+E_2\boldsymbol{r}^+_l$$

where the vector E_1 and the complex scalar E_2 are defined by

$$E_1:=\int_{\Gamma_1} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)e^{j\varpi^+_l(\tau)}\zeta^+_{m'}(e^\tau)^{-j_0-1}\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(V^+_m(e^\tau,j)-R^+_m(e^\tau))d\tau$$
$$E_2:=\int_{\Gamma_2} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)e^{j\varpi^+_l(\tau)}\zeta^+_{m'}(e^\tau)^{-j_0-1}\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})d\tau$$

where Γ_1,Γ_2 are paths belonging to the set X. Using (4.4.1c) to bound $\zeta^+_{m'}$, Lemma 4.4.3 which claims that the vectors $V^+_m(z,j)$ converge exponentially fast towards $R^+_m(e^\tau)$, (4.4.52g) to bound $\Delta^+_{m'}$ and (4.5.7), we prove that there exist two positive constants C,c independent from n, j_0, j, $\mathbf{e}$, Γ_1 and Γ_2 such that

$$|E_1|\leq C|\mathbf{e}|e^{-c|j|}e^{-c|j_0|}\int_{\Gamma_1}\exp\left(n\Re(\tau)+\frac{j}{\alpha_l^+}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|,$$
$$|E_2|\leq C|\mathbf{e}|e^{-c|j_0|}\int_{\Gamma_2}\exp\left(n\Re(\tau)+\frac{j}{\alpha_l^+}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

Whilst observing that $\frac{j}{\alpha_l^+}$ is positive since m belongs to I^+_{cs}, when $\frac{j}{\alpha_l^+}\notin\left[\frac{n}{2},2n\right]$, Lemma 4.5.4 allows us to prove exponential bounds with regard to n on the terms E_1 and E_2 and to thus immediately conclude the proof of (4.5.22f). When $\frac{j}{\alpha_l^+}\in\left[\frac{n}{2},2n\right]$, Lemma 4.5.4 allows us to choose Γ_1 and Γ_2 depending on n, j_0, j and $\mathbf{e}$ so that there exist new constants $C,c>0$ independent from n, j_0, j and $\mathbf{e}$ such that

$$|E_1|\leq\frac{C|\mathbf{e}|e^{-c|j|}e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\qquad|E_2|\leq\frac{C|\mathbf{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Since $j\in\left[\frac{n}{2},2n\right]$, we have that there exist two other constants $C,c>0$ independent from n, j_0, j and $\mathbf{e}$ such that

$$|E_1|\leq C|\mathbf{e}|e^{-cn}.$$

This allows us to conclude (4.5.22e).

- There only remains to prove (4.5.22g). We observe using (4.5.23) with $\Gamma=\Gamma_d(\eta)\in X$, (4.4.1a) and

(4.4.1c) to bound ζ_m^+ and $\zeta_{m'}^+$, (4.4.52g) to bound $\Delta_{m'}^+$ and Lemma 4.4.3 which claims that the vectors $V_m^+(z,j)$ are uniformly bounded for $z\in B(1,\delta_1)$ and $j\in\mathbb{N}$, we have that there exists a positive constant C independent from n, j_0, j and $\mathbf{e}$ such that

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\mathbf{e})\Pi(W_m^+(e^\tau,j))d\tau\right|\leq C|\mathbf{e}|e^{-2c_*(|j|+|j_0|)-n\eta}.$$

□

Transmitted waves

We now look at the transmitted waves.

Lemma 4.5.9. *We consider $m\in\{dp+1,\dots,d(p+q)-1\}$ and $m'\in\{dp+1,\dots,d(p+q)\}$ and write them as $m=l+(k-1)d$ and $m'=l'+(k'-1)d$ with $k,k'\in\{1,\dots,p+q\}$ and $l,l'\in\{1,\dots,d\}$. There exists a positive constant c such that for all $n\in\mathbb{N}\backslash\{0\}$, $j_0\in\mathbb{N}$, $j\in-\mathbb{N}$ such that $j-j_0\in\{-nq,\dots,np\}$ and $\boldsymbol{e}\in\mathbb{C}^d$, we have:*

- *If $m\in I_{cu}^-$, $m'\in I_{cu}^+$ and $\frac{j}{\alpha_l^-}-\frac{j_0}{\alpha_{l'}^+}\in\left[\frac{n}{2},2n\right]$, we have that:*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^-(e^\tau,j))d\tau-\frac{\alpha_l^-}{\alpha_{l'}^+}\widetilde{g}^+_{m',m}(1)T^+_{l',l}(n,j_0,j)\boldsymbol{e}$$
$$=\exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^-}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{|\boldsymbol{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{|\boldsymbol{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)\boldsymbol{r}_l^-+O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right){\boldsymbol{l}_{l'}^+}^T\boldsymbol{e}\boldsymbol{r}_l^-\right).\tag{4.5.25a}$$

- *If $m\in I_{cu}^-$, $m'\in I_{cu}^+$ and $\frac{j}{\alpha_l^-}-\frac{j_0}{\alpha_{l'}^+}\notin\left[\frac{n}{2},2n\right]$, we have that:*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^-(e^\tau,j))d\tau-\frac{\alpha_l^-}{\alpha_{l'}^+}\widetilde{g}^+_{m',m}(1)T^+_{l',l}(n,j_0,j)\boldsymbol{e}=O\left(|\boldsymbol{e}|e^{-cn}\right).\tag{4.5.25b}$$

- *If $m\in I_{su}^-$, $m'\in I_{cu}^+$ and $-\frac{j_0}{\alpha_{l'}^+}\in\left[\frac{n}{2},2n\right]$, we have that:*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^-(e^\tau,j))d\tau=O\left(\frac{e^{-c|j|}|\boldsymbol{e}|}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right).\tag{4.5.25c}$$

- *If $m\in I_{su}^-$, $m'\in I_{cu}^+$ and $-\frac{j_0}{\alpha_{l'}^+}\notin\left[\frac{n}{2},2n\right]$, we have that:*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^-(e^\tau,j))d\tau=O\left(|\boldsymbol{e}|e^{-cn}\right).\tag{4.5.25d}$$

- *If $m\in I_{cu}^-$, $m'\in I_{su}^+$ and $\frac{j}{\alpha_l^-}\in\left[\frac{n}{2},2n\right]$, we have that:*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^-(e^\tau,j))d\tau$$
$$=O(|\boldsymbol{e}|e^{-cn})+O_{\mathbb{C}}\left(\frac{|\boldsymbol{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j}{\alpha_l^-}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)\boldsymbol{r}_l^-.\tag{4.5.25e}$$

- *If $m\in I_{cu}^-$, $m'\in I_{su}^+$ and $\frac{j}{\alpha_l^-}\notin\left[\frac{n}{2},2n\right]$, we have that:*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\boldsymbol{e})\Pi(W_m^-(e^\tau,j))d\tau=O\left(|\boldsymbol{e}|e^{-cn}\right).\tag{4.5.25f}$$

- *If $m \in I_{su}^-$, $m' \in I_{su}^+$, we have that:*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^{+}_{m',m}(e^\tau)\widetilde{\mathscr{C}}^{+}_{m}(e^\tau,j_0,\boldsymbol{e})\Pi(W^-_m(e^\tau,j))d\tau = O\left(|\boldsymbol{e}|e^{-cn}\right). \tag{4.5.25g}$$

Just like in the case of the reflected waves, since we consider $m \in \{dp+1,\dots,d(p+q)-1\}$ and $m' \in \{dp+1,\dots,d(p+q)\}$, Lemma 4.4.6 implies that $\widetilde{g}^+_{m',m}$ can be holomorphically extended on the whole ball $B(1,\delta_1)$ and thus the term $\widetilde{g}^+_{m',m}(1)$ is well defined.

Proof The proof of Lemma 4.5.9 is sensibly the same one as for Lemma 4.5.8 so the proof is left to the reader. Let us just point out that in order to prove (4.5.25a), we have using Cauchy's formula that

$$\begin{aligned}\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^{+}_{m',m}(e^\tau)\widetilde{\mathscr{C}}^{+}_{m}(e^\tau,j_0,\mathbf{e})\Pi(W^-_m(e^\tau,j))d\tau - \frac{\alpha_l^-}{\alpha_{l'}^+}\widetilde{g}^+_{m',m}(1)T^+_{l',l}(n,j_0,j)\mathbf{e}\\ = E_1 + E_2\boldsymbol{r}_l^- + (E_3+E_4+E_5+E_6)\,{\boldsymbol{l}_{l'}^+}^T\mathbf{e}\boldsymbol{r}_l^-\end{aligned}$$

where E_1 is a vector and $E_2,\dots,E_6$ are complex scalars defined by

$$\begin{aligned}
E_1 &:= \frac{1}{2i\pi}\int_{\Gamma_1} e^{n\tau}\widetilde{g}^+_{m',m}(e^\tau)e^{j\varpi_l^-(\tau)}e^{-(j_0+1)\varpi_{l'}^+(\tau)}e^{\tau}\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(V^-_m(e^\tau,j)-R^-_m(e^\tau))d\tau,\\
E_2 &:= \frac{1}{2i\pi}\int_{\Gamma_2} e^{n\tau}\widetilde{g}^+_{m',m}(e^\tau)e^{j\varpi_l^-(\tau)}e^{-(j_0+1)\varpi_{l'}^+(\tau)}e^{\tau}\left(\Delta^+_{m'}(e^\tau,j_0,\mathbf{e}) - \frac{d\zeta^+_{m'}}{dz}(e^\tau){\boldsymbol{l}_{l'}^+}^T\mathbf{e}\right)d\tau,\\
E_3 &:= \frac{1}{2i\pi}\int_{\Gamma_3} e^{n\tau}e^{j\varpi_l^-(\tau)}e^{-j_0\varpi_{l'}^+(\tau)}\left(\widetilde{g}^+_{m',m}(e^\tau)e^\tau\zeta^+_{m'}(e^\tau)^{-1}\frac{d\zeta^+_{m'}}{dz}(e^\tau) + \frac{\widetilde{g}^+_{m',m}(1)}{\alpha^+_{l'}}\right)d\tau,\\
E_4 &:= -\frac{\widetilde{g}^+_{m',m}(1)}{2i\pi\alpha^+_{l'}}\int_{\Gamma_4} e^{n\tau}\left(e^{j\varpi_l^-(\tau)} - e^{j\varphi_l^-(\tau)}\right)e^{-j_0\varpi_{l'}^+(\tau)}d\tau\\
E_5 &:= -\frac{\widetilde{g}^+_{m',m}(1)}{2i\pi\alpha^+_{l'}}\int_{\Gamma_5} e^{n\tau}e^{j\varphi_l^-(\tau)}\left(e^{-j_0\varpi_{l'}^+(\tau)} - e^{-j_0\varphi_{l'}^+(\tau)}\right)d\tau,\\
E_6 &:= -\frac{\widetilde{g}^+_{m',m}(1)}{2i\pi\alpha^+_{l'}}\int_{\Gamma_5} e^{n\tau}e^{j\varphi_l^-(\tau)}e^{-j_0\varphi_{l'}^+(\tau)}d\tau\\
&\qquad - \frac{\alpha_l^-}{\alpha_{l'}^+}\widetilde{g}^+_{m',m}(1)\frac{1}{n^{\frac{1}{2\mu}}}H_{2\mu}\left(\frac{j}{n\alpha_l^-}\beta_l^- - \frac{j_0}{n\alpha^+_{l'}}\beta^+_{l'}\left(\frac{\alpha_l^-}{\alpha_{l'}^+}\right)^{2\mu}, \frac{n\alpha_l^- + j_0\frac{\alpha_l^-}{\alpha^+_{l'}} - j}{n^{\frac{1}{2\mu}}}\right),
\end{aligned}$$

and $\Gamma_1,\dots,\Gamma_6$ are paths belonging to the set X. We just have to prove bounds on the terms $E_1,\dots,E_6$ just like in the proof of (4.5.22a). □

Unstable excited mode

Lemma 4.5.10. *There exist two positive constants C, c such that for all $m' \in I^+_{su}$, $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}$, $j \in \mathbb{Z}$ such that $j - j_0 \in \{-nq,\dots,np\}$ and $\boldsymbol{e} \in \mathbb{C}^d$, we have*

- *For $j \in \mathbb{N}$, we have that:*

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^{+}_{m',1}(e^\tau)\widetilde{\mathscr{C}}^{+}_{m'}(e^\tau,j_0,\boldsymbol{e})\Pi(\Phi_1(e^\tau,j))d\tau + \zeta^+_{m'}(1)^{-j_0-1}\Delta^+_{m'}(1,j_0,\boldsymbol{e})\mathrm{Res}(\widetilde{g}^+_{m'},1)V(j)\right| \leq C|\boldsymbol{e}|e^{-cn}. \tag{4.5.26a}$$

- *For $j \in -\mathbb{N}$, we have that:*

$$\left|\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^{+}_{m',d(p+q)}(e^\tau)\widetilde{\mathscr{C}}^{+}_{m'}(e^\tau,j_0,\boldsymbol{e})\Pi(\Phi_{d(p+q)}(e^\tau,j))d\tau + \zeta^+_{m'}(1)^{-j_0-1}\Delta^+_{m'}(1,j_0,\boldsymbol{e})\mathrm{Res}(\widetilde{g}^+_{m'},1)V(j)\right| \leq C|\boldsymbol{e}|e^{-cn}. \tag{4.5.26b}$$

We recall that the sequence V is defined by (4.4.30).

Proof We are going to prove (4.5.26a). We consider $m' \in I^+_{su}$, $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}$. For $j \in \mathbb{N}$, using the residue theorem and the equality (4.4.51), we have

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(\Phi_1(e^\tau,j))d\tau + \zeta^+_{m'}(1)^{-j_0-1}\Delta^+_{m'}(1,j_0,\mathbf{e})\mathrm{Res}(\widetilde{g}^+_{m',1},1)V(j)$$
$$= -\frac{1}{2i\pi}\int_{\Gamma_d(\eta)} e^{n\tau}\zeta^+_{m'}(e^\tau)^{-j_0-1}e^\tau\widetilde{g}^+_{m',1}(e^\tau)\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(\Phi_1(e^\tau,j))d\tau.$$

Using (4.4.1c) to bound $\zeta^+_{m'}$, (4.4.52g) to handle the term $\Delta^+_{m'}$ and (4.4.29) to bound Φ_1, there exists another positive constant C independent from n, j_0, j and $\mathbf{e}$ such that

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_d(\eta)} e^{n\tau}\zeta^+_{m'}(e^\tau)^{-j_0-1}e^\tau\widetilde{g}^+_{m',1}(e^\tau)\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(\Phi_1(e^\tau,j))d\tau\right|$$
$$\leq C|\mathbf{e}|\int_{\Gamma_d(\eta)} e^{n\Re(\tau)}|d\tau| \leq 2r_\varepsilon(\eta)C|\mathbf{e}|e^{-n\eta}.$$

We thus obtain (4.5.26a). The proof of (4.5.26b) is fairly similar and is thus left to the reader. We observe that the definition (4.4.42) of $\widetilde{g}^\pm_{m',m}$ and (4.4.43) imply:

$$\mathrm{Res}(\widetilde{g}^+_{m',d(p+q)},1) = -\mathrm{Res}(\widetilde{g}^+_{m',1},1).$$

□

Central excited mode

Lemma 4.5.11. *We consider $m' \in I^+_{cu}$ and write it as $m' = l' + (k'-1)d$ with $k' \in \{1,\dots,p+q\}$ and $l' \in \{1,\dots,d\}$. There exists a positive constant c such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}$, $j \in \mathbb{Z}$ such that $j - j_0 \in \{-nq,\dots,np\}$ and $\boldsymbol{e} \in \mathbb{C}^d$, we have:*

- *For $-\frac{j_0}{\alpha^+_{l'}} \in \left[\frac{n}{2}, 2n\right]$ and $j \geq 0$, we have that:*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\boldsymbol{e})\Pi(\Phi_1(e^\tau,j))d\tau - \frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\alpha^+_{l'}}E^+_{l'}(n,j_0)\boldsymbol{e}V(j) =$$
$$O\left(\frac{|\boldsymbol{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+_l}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right) + O\left(|\boldsymbol{e}|e^{-cn}\right). \quad (4.5.27a)$$

- *For $-\frac{j_0}{\alpha^+_{l'}} \notin \left[\frac{n}{2}, 2n\right]$ and $j \geq 0$, we have that:*

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\boldsymbol{e})\Pi(\Phi_1(e^\tau,j))d\tau - \frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\alpha^+_{l'}}E^+_{l'}(n,j_0)\boldsymbol{e}V(j) =$$
$$O(|\boldsymbol{e}|e^{-cn}). \quad (4.5.27b)$$

- *For $-\frac{j_0}{\alpha^+_{l'}} \in \left[\frac{n}{2}, 2n\right]$ and $j < 0$, we have that:*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',d(p+q)}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\boldsymbol{e})\Pi(\Phi_{d(p+q)}(e^\tau,j))d\tau - \frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\alpha^+_{l'}}E^+_{l'}(n,j_0)\boldsymbol{e}V(j) =$$
$$O\left(\frac{|\boldsymbol{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+_l}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right) + O\left(|\boldsymbol{e}|e^{-cn}\right). \quad (4.5.27c)$$

- *For $-\frac{j_0}{\alpha^+_{l'}} \notin \left[\frac{n}{2}, 2n\right]$ and $j < 0$, we have that:*

$$\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',d(p+q)}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(\Phi_{d(p+q)}(e^\tau,j))d\tau - \frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\alpha^+_{l'}}E^+_{l'}(n,j_0)\boldsymbol{e}V(j) = O(|\boldsymbol{e}|e^{-cn}). \quad (4.5.27\text{d})$$

We recall once again that the sequence V is defined by (4.4.30).

Proof We will focus on proving (4.5.27a) and (4.5.27b) as the proof of (4.5.27c) and (4.5.27d) would be similar whilst observing that the equality (4.4.28) is verified and that the definition (4.4.42) of $\widetilde{g}^\pm_{m',m}$ and (4.4.43) imply:

$$\mathrm{Res}(\widetilde{g}^+_{m',d(p+q)},1) = -\mathrm{Res}(\widetilde{g}^+_{m',1},1).$$

• **Proof of** (4.5.27a)**:**
Using Cauchy's formula, the equality (4.4.51) on $\widetilde{\mathscr{C}}^+_{m'}$ and (4.5.3) since m' belongs to I^+_{cu}, we have:

$$-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(\Phi_1(e^\tau,j))d\tau - \frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\alpha^+_{l'}}E^+_{l'}(n,j_0)\mathbf{e}\Pi\left(\Phi_1(1,j)\right)$$
$$= E_1 + E_2\Pi(\Phi(1,j)) + (E_3+E_4+E_5)\Delta^+_{m'}(1,j_0,\mathbf{e})\Pi(\Phi(1,j)) + E_6\Pi(\Phi(1,j)) \quad (4.5.28)$$

where

$$E_1 := -\frac{1}{2i\pi}\int_{\Gamma_1} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)e^{-(j_0+1)\varpi^+_{l'}(\tau)}\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(\Phi_1(e^\tau,j)-\Phi_1(1,j))d\tau,$$
$$E_2 := -\frac{1}{2i\pi}\int_{\Gamma_2} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)e^{-(j_0+1)\varpi^+_{l'}(\tau)}\left(\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})-\Delta^+_{m'}(1,j_0,\mathbf{e})\right)d\tau,$$
$$E_3 := -\frac{1}{2i\pi}\int_{\Gamma_3} e^{n\tau}e^{-j_0\varpi^+_{l'}(\tau)}\left(e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)\zeta^+_{m'}(e^\tau)^{-1} - \frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\tau}\right)d\tau,$$
$$E_4 := -\frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{2i\pi}\int_{\Gamma_4} e^{n\tau}\frac{e^{-j_0\varpi^+_{l'}(\tau)}-e^{-j_0\varphi^+_{l'}(\tau)}}{\tau}d\tau,$$
$$E_5 := -\mathrm{Res}(\widetilde{g}^+_{m',1},1)\left(\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}\frac{e^{-j_0\varphi^+_{l'}(\tau)}}{\tau}d\tau - E_{2\mu}\left(\beta^+_{l'};\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}}\right)\right),$$
$$E_6 := -\mathrm{Res}(\widetilde{g}^+_{m',1},1)E_{2\mu}\left(\beta^+_{l'},\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}}\right)\left(\Delta^+_{m'}(1,j_0,\mathbf{e}) + \frac{{\boldsymbol{l}^+_{l'}}^T\mathbf{e}}{\alpha^+_{l'}}\right),$$

and $\Gamma_1,\dots,\Gamma_4\in X$. Let us observe that, since the function $\widetilde{g}^+_{m',1}$ has a simple pole of order 1 at 1, we have the right to use the Cauchy's formula for the first 4 terms as the functions inside the integrals can be holomorphically extended on the whole ball $B(1,\varepsilon)$.

► Using (4.4.52g) to bound $\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})$, (4.4.52c) to bound $\widetilde{g}^+_{m',1}(e^\tau)\Pi\left(\Phi_1(e^\tau,j)-\Phi_1(1,j)\right)$ and (4.5.7), there exists a constant $C>0$ independent from n, j_0, j, $\mathbf{e}$ and Γ_1 such that

$$|E_1| \leq C|\mathbf{e}|e^{-\frac{3c_*}{2}|j|}\int_{\Gamma_1}\exp\left(n\Re(\tau) - \frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

► Using (4.4.52i) to bound $\widetilde{g}^+_{m',1}(e^\tau)\left(\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})-\Delta^+_{m'}(1,j_0,\mathbf{e})\right)$ and (4.5.7), there exists a constant $C>0$ independent from n, j_0, j, $\mathbf{e}$ and Γ_2 such that

$$|E_2| \leq C|\mathbf{e}|\int_{\Gamma_2}\exp\left(n\Re(\tau) - \frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

► Using (4.5.7), there exists a constant $C>0$ independent from n, j_0, j, $\mathbf{e}$ and Γ_3 such that

$$|E_3| \leq C\int_{\Gamma_3}\exp\left(n\Re(\tau) - \frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

► Using Lemma 4.5.2, there exists a constant $C>0$ independent from n, j_0, j, $\mathbf{e}$ and Γ_4 such that

$$|E_4| \leq Cn\int_{\Gamma_4}|\tau|^{2\mu}\exp\left(n\Re(\tau) - \frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using Lemma 4.5.4 which gives a good choices of path $\Gamma_1, \ldots, \Gamma_4 \in X$ depending on n, j_0 and j to handle the integrals in the terms above, there exist new constants $C, c > 0$ independent from n, j_0, j and $\mathbf{e}$ such that

$$|E_1| \leq \frac{Ce^{-c|j|}|\mathbf{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right), \qquad |E_2| \leq \frac{C|\mathbf{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right),$$

$$|E_3| \leq \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right), \qquad |E_4| \leq \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

There remains to bound the terms E_5 and E_6. We use (4.5.18c) of Lemma 4.5.6 to bound the term E_5 and obtain the existence of two positive constants C, c such that:

$$|E_5| \leq \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

We observe that (4.5.3) and the asymptotic expansion (4.5.5) imply that

$$\frac{d\zeta_{m'}^+}{dz}(1) = -\frac{1}{\alpha_{l'}^+}.$$

Then, using (4.4.52e) and the fact that the function $E_{2\mu}(\beta_{l'}^+;\cdot)$ is bounded to handle E_6, we can also consider that we chose the constants C, c so that:

$$|E_6| \leq C|\mathbf{e}|e^{-c|j_0|}.$$

Since $-\frac{j_0}{\alpha_{l'}^+} \in \left[\frac{n}{2}, 2n\right]$, we can change the constants C, c such that:

$$|E_6| \leq C|\mathbf{e}|e^{-cn}.$$

Using (4.4.29) and (4.4.52g) to bound $\Phi_1(1,j)$ and $\Delta_{m'}^+$ in (4.5.28), we can conclude the proof of (4.5.27a).

• **Proof of** (4.5.27b)**:**

We will separate this proof into two parts.

▶ Let us assume that $-\frac{j_0}{\alpha_l^+} \in \left[0, \frac{n}{2}\right]$. We recall that $\widetilde{g}_{m',1}^+$ is a meromorphic function with a pole of order 1 at 1. Using the Residue Theorem, we have

$$\begin{aligned}
-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',1}^+(e^\tau)\widetilde{\mathscr{C}}_{m'}^+(e^\tau, j_0, \mathbf{e})\Pi(\Phi_1(e^\tau, j))d\tau - \frac{\operatorname{Res}(\widetilde{g}_{m',1}^+, 1)}{\alpha_{l'}^+}E_{l'}^+(n, j_0)\mathbf{e}\Pi\left(\Phi_1(1,j)\right) = \\
-\frac{1}{2i\pi}\int_{\Gamma_d(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',1}^+(e^\tau)e^{-(j_0+1)\varpi_{l'}^+(\tau)}\Delta_{m'}^+(e^\tau, j_0, \mathbf{e})\Pi(\Phi_1(e^\tau, j))d\tau \\
-\frac{\operatorname{Res}(\widetilde{g}_{m',1}^+, 1)}{\alpha_{l'}^+}\left(1 - E_{2\mu}\left(\beta_{l'}^+; \frac{n\alpha_{l'}^+ + j_0}{n^{\frac{1}{2\mu}}}\right)\right) {\boldsymbol{l}_{l'}^+}^T\mathbf{e}\Pi\left(\Phi_1(1,j)\right).
\end{aligned}$$

We need to obtain exponential bounds on both terms on the right hand side of the equality above. First, we observe that since $-\frac{j_0}{\alpha_{l'}^+}$ belongs to $\left[0, \frac{n}{2}\right]$, we have

$$\frac{n\alpha_{l'}^+ + j_0}{n^{\frac{1}{2\mu}}} \leq \frac{\alpha_{l'}^+}{2}n^{\frac{2\mu}{2\mu-1}}.$$

We have that $\alpha_{l'}^+ < 0$ since m' belongs to I_{cu}^+ and thus, using (4.1.30c) and (4.4.29), we have that there exist two positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0, j \in \mathbb{N}$, $\mathbf{e} \in \mathbb{C}^d$

$$\left|\frac{\operatorname{Res}(\widetilde{g}_{m',1}^+, 1)}{\alpha_{l'}^+}\left(1 - E_{2\mu}\left(\beta_{l'}^+; \frac{n\alpha_{l'}^+ + j_0}{n^{\frac{1}{2\mu}}}\right)\right) {\boldsymbol{l}_{l'}^+}^T\mathbf{e}\Pi\left(\Phi_1(1,j)\right)\right| \leq C|\mathbf{e}|e^{-c|j|}e^{-cn}.$$

We now observe that using (4.4.52g), (4.4.29) and (4.5.7), we can prove that there exist two positive constants C, c such that such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0, j \in \mathbb{N}$, $\mathbf{e} \in \mathbb{C}^d$

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_d(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)e^{-(j_0+1)\varpi^+_{l'}(\tau)}\Delta^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(\Phi_1(e^\tau,j))d\tau\right|$$
$$\leq C|\mathbf{e}|e^{-c|j|}\int_{\Gamma_d(\eta)}\exp\left(n\Re(\tau)-\frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using (4.5.17a), we can find exponential bounds for the integral in the right hand term above. This allows us to conclude the proof of (4.5.27b) when $-\frac{j_0}{\alpha^+_{l'}}$ belongs to $\left[0,\frac{n}{2}\right]$.

► Let us assume that $-\frac{j_0}{\alpha^+_{l'}}\in[2n,+\infty[$. Since $-\frac{j_0}{\alpha^+_{l'}}$ belongs to $[2n,+\infty[$, we have

$$\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}}\geq-\alpha^+_{l'}n^{\frac{2\mu}{2\mu-1}}.$$

We have that $\alpha^+_{l'}<0$ since m' belongs to I^+_{cu} and thus, using (4.1.30b) and (4.4.29), we have that there exist two positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0, j \in \mathbb{N}$, $\mathbf{e} \in \mathbb{C}^d$

$$\left|\frac{\mathrm{Res}(\widetilde{g}^+_{m',1},1)}{\alpha^+_{l'}}E_{2\mu}\left(\beta^+_{l'};\frac{n\alpha^+_{l'}+j_0}{n^{\frac{1}{2\mu}}}\right)\boldsymbol{l}^{+\,T}_{l'}\mathbf{e}\Pi\left(\Phi_1(1,j)\right)\right|\leq C|\mathbf{e}|e^{-c|j|}e^{-cn}.$$

Furthermore, using (4.4.52g), (4.4.29) and (4.5.7), we can prove that there exist two positive constants C, c such that such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0, j \in \mathbb{N}$, $\mathbf{e} \in \mathbb{C}^d$

$$\left|-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi(\Phi_1(e^\tau,j))d\tau\right|$$
$$\leq C|\mathbf{e}|e^{-c|j|}\int_{\Gamma_{in}(\eta)}\exp\left(n\Re(\tau)-\frac{j_0}{\alpha^+_{l'}}\left(-\Re(\tau)+A_R\Re(\tau)^{2\mu}-A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using (4.5.17b), we can find exponential bounds for the integral in the right hand term above. This allows us to conclude the proof of (4.5.27b) when $-\frac{j_0}{\alpha^+_{l'}}$ belongs to $[2n,+\infty[$. □

4.5.3 Proof of the decomposition (4.1.32) of the temporal Green's function

Let us now conclude the proof of the decomposition (4.1.32) of the temporal Green's function. We will give details on the case when $j\geq j_0+1$. The cases when $j\in\{0,\dots,j_0\}$ and $j\leq-1$ would be handled similarly using (4.4.34b) and (4.4.34c) rather than (4.4.34a).

Using (4.5.15), Lemma 4.5.3 and (4.4.34a), there exists a constant $c>0$ such that for $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}$, $j\geq j_0+1$ and $\mathbf{e} \in \mathbb{C}^d$ which verify $j-j_0\in\{-nq,\dots,np\}$:

$$\begin{aligned}\mathscr{G}(n,j_0,j)\mathbf{e} =&\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\Pi(W(e^\tau,j_0,j,\mathbf{e}))d\tau+\frac{1}{2i\pi}\int_{\Gamma_{out}(\eta)} e^{n\tau}e^{\tau}\Pi(W(e^\tau,j_0,j,\mathbf{e}))d\tau\\
=&\sum_{m\in I^+_{ss}\cup I^+_{cs}}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{\mathscr{C}}^+_m(e^\tau,j_0,\mathbf{e})\Pi\left(W^+_m(e^\tau,j)\right)d\tau\right)\\
&+\sum_{m\in I^+_{ss}\cup I^+_{cs}\backslash\{1\}}\sum_{m'\in I^+_{cu}\cup I^+_{su}}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',m}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi\left(W^+_m(e^\tau,j)\right)d\tau\right)\\
&+\sum_{m'\in I^+_{cu}\cup I^+_{su}}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}^+_{m',1}(e^\tau)\widetilde{\mathscr{C}}^+_{m'}(e^\tau,j_0,\mathbf{e})\Pi\left(\Phi_1(e^\tau,j)\right)d\tau\right)+O(|\mathbf{e}|e^{-cn}).\end{aligned}$$

Thus, using Lemmas 4.5.7-4.5.11, there exists a positive constant c such that we have:

$\mathscr{G}(n,j_0,j)\mathbf{e}$

$$
\begin{aligned}
&= \sum_{l=I+1}^{d} S_l^+(n,j_0,j)\mathbf{e} + \sum_{l'=1}^{I}\sum_{l=I+1}^{d} C_{l',l}^{R,+} R_{l',l}^+(n,j_0,j)\mathbf{e} + \left(\sum_{l'=1}^{I} C_{l'}^{E,+} E_{l'}^+(n,j_0)\mathbf{e} + P_U^+(j_0)\right) V(j) \\
&\quad + \sum_{l=I+1}^{d} \exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{|\mathbf{e}|\left(e^{-c|j|}+e^{-c|j_0|}\right)}{n^{\frac{1}{2\mu}}}\right) + O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right) \boldsymbol{l}_l^{+T}\mathbf{e}\boldsymbol{r}_l^+\right) \\
&\quad + \sum_{l'=1}^{I}\sum_{l=I+1}^{d} \exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{|\mathbf{e}|\left(e^{-c|j|}+e^{-c|j_0|}\right)}{n^{\frac{1}{2\mu}}}\right) + O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right) \boldsymbol{l}_{l'}^{+T}\mathbf{e}\boldsymbol{r}_l^+\right) \\
&\quad + \sum_{l'=1}^{I} O\left(\frac{|\mathbf{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right) + \sum_{l=I+1}^{d} O_{\mathbb{C}}\left(\frac{|\mathbf{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)\boldsymbol{r}_l^+ \\
&\quad + O(|\mathbf{e}|e^{-cn}).
\end{aligned}
$$

The coefficients in the formula above and Theorem 4.1 have the following expressions:

— The coefficients $C_{l',l}^{R,+}$ are given by:

$$\forall l' \in \{1,\dots,I\}, \forall l \in \{I+1,\dots,d\}, \quad C_{l',l}^{R,+} := \frac{\alpha_l^+}{\alpha_{l'}^+}\widetilde{g}_{m',m}^+(1), \tag{4.5.29a}$$

where $m' = l' + dp \in I_{cu}^+$ and $m = l + d(p-1) \in I_{cs}^+$.

— The coefficients $C_{l',l}^{T,+}$ are given by:

$$\forall l' \in \{1,\dots,I\}, \forall l \in \{1,\dots,I-1\}, \quad C_{l',l}^{T,+} := \frac{\alpha_l^-}{\alpha_{l'}^+}\widetilde{g}_{m',m}^+(1), \tag{4.5.29b}$$

where $m' = l' + dp \in I_{cu}^+$ and $m = l + dp \in I_{cu}^-$.

— The coefficients $C_{l'}^{E,+}$ are given by:

$$\forall l' \in \{1,\dots,I\}, \quad C_{l'}^{E,+} := \frac{\mathrm{Res}(\widetilde{g}_{m',1}^+, 1)}{\alpha_{l'}^+}, \tag{4.5.29c}$$

where $m' = l' + dp \in I_{cu}^+$,

— The matrices $P_U^+(j_0)$ are defined by:

$$\forall j_0 \in \mathbb{Z}, \forall \mathbf{e} \in \mathbb{C}^d, \quad P_U^+(j_0)\mathbf{e} := \sum_{m' \in I_{su}^+} \left(-\zeta_{m'}^+(1)^{-j_0-1}\Delta_{m'}^+(1,j_0,\mathbf{e})\mathrm{Res}(\widetilde{g}_{m',1}^+, 1)\right). \tag{4.5.29d}$$

Thus, observing that $S_l^+(n,j_0,j)$ for $l \in \{1,\dots,I\}$ and $T_{l',l}^+(n,j_0,j)$ for $l' \in \{1,\dots,I\}$ and $l \in \{1,\dots,I-1\}$ are equal to 0 for j larger than j_0+1, we have obtained the decomposition (4.1.32) when $j \geq j_0+1$. The proof for $j \in \{0,\dots,j_0\}$ and $j<0$ would be handled similarly.

4.5.4 Proof of the decomposition (4.1.33) of discrete derivative of the temporal Green's function

To obtain the decomposition (4.1.33) of discrete derivative of the temporal Green's function, we must follow a similar path as for the proof of the decomposition (4.1.32) of the temporal Green's function. First, we need some additional lemmas (Lemmas 4.5.12-4.5.15) which are discrete derivative version of Lemmas 4.5.7-4.5.11.

Lemma 4.5.12 (Discrete derivative version of Lemma 4.5.7 on outgoing and incoming waves)**.**

We consider $m \in \{1,\dots,d(p+q)\}$ and write it as $m = l+(k-1)d$ with $k \in \{1,\dots,p+q\}$ and $l \in \{1,\dots,d\}$. There exists a constant $c>0$ such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}\backslash\{0\}$, $j \in \mathbb{N}$ such that $j - j_0 \in \{-nq,\dots,np\}$ and $\boldsymbol{e} \in \mathbb{C}^d$ we have:

- *If $m \in I_{cs}^+ \cup I_{cu}^+$ and $\frac{j-j_0}{\alpha_l^+} \in \left[\frac{n}{2}, 2n\right]$, we have that:*

$$\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\left(\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})-\widetilde{\mathscr{C}}_m^+(e^\tau,j_0-1,\boldsymbol{e})\right)\Pi(W_m^+(e^\tau,j))d\tau$$

$$= \exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha_l^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{|\boldsymbol{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{|\boldsymbol{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)\boldsymbol{r}_l^+ + O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\boldsymbol{l}_l^{+T}\boldsymbol{e}\boldsymbol{r}_l^+\right). \quad (4.5.30a)$$

- *If $m\in I_{cs}^+\cup I_{cu}^+$, $\frac{j-j_0}{\alpha_l^+}\notin\left[\frac{n}{2},2n\right]$ and $\frac{j-j_0}{\alpha_l^+}\geq 0$, we have that:*

$$\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\left(\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})-\widetilde{\mathscr{C}}_m^+(e^\tau,j_0-1,\boldsymbol{e})\right)\Pi(W_m^+(e^\tau,j))d\tau = O\left(|\boldsymbol{e}|e^{-cn}\right). \quad (4.5.30b)$$

Lemma 4.5.13 (Discrete derivative version of Lemma 4.5.8 on reflected waves)**.**
We consider $m\in\{2,\dots,dp\}$ and $m'\in\{dp+1,\dots,d(p+q)\}$ and write them as $m=l+(k-1)d$ and $m'=l'+(k'-1)d$ with $k,k'\in\{1,\dots,p+q\}$ and $l,l'\in\{1,\dots,d\}$. There exists a positive constant c such that for all $n\in\mathbb{N}\backslash\{0\}$, $j_0\in\mathbb{N}\backslash\{0\}$, $j\in\mathbb{N}$ such that $j-j_0\in\{-nq,\dots,np\}$ and $\boldsymbol{e}\in\mathbb{C}^d$, we have:
- *If $m\in I_{cs}^+$, $m'\in I_{cu}^+$ and $\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\in\left[\frac{n}{2},2n\right]$, we have that:*

$$\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\left(\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})-\widetilde{\mathscr{C}}_m^+(e^\tau,j_0-1,\boldsymbol{e})\right)\Pi(W_m^+(e^\tau,j))d\tau$$

$$= \exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{|\boldsymbol{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{|\boldsymbol{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)\boldsymbol{r}_l^+ + O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\boldsymbol{l}_{l'}^{+T}\boldsymbol{e}\boldsymbol{r}_l^+\right).$$

(4.5.31a)

- *If $m\in I_{cs}^+$, $m'\in I_{cu}^+$ and $\frac{j}{\alpha_l^+}-\frac{j_0}{\alpha_{l'}^+}\notin\left[\frac{n}{2},2n\right]$, we have that:*

$$\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\left(\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})-\widetilde{\mathscr{C}}_m^+(e^\tau,j_0-1,\boldsymbol{e})\right)\Pi(W_m^+(e^\tau,j))d\tau = O(|\boldsymbol{e}|e^{-cn}). \quad (4.5.31b)$$

Lemma 4.5.14 (Discrete derivative version of Lemma 4.5.9 on transmitted waves)**.**
We consider $m\in\{dp+1,\dots,d(p+q)-1\}$ and $m'\in\{dp+1,\dots,d(p+q)\}$ and write them as $m=l+(k-1)d$ and $m'=l'+(k'-1)d$ with $k,k'\in\{1,\dots,p+q\}$ and $l,l'\in\{1,\dots,d\}$. There exists a positive constant c such that for all $n\in\mathbb{N}\backslash\{0\}$, $j_0\in\mathbb{N}\backslash\{0\}$, $j\in-\mathbb{N}$ such that $j-j_0\in\{-nq,\dots,np\}$ and $\boldsymbol{e}\in\mathbb{C}^d$, we have:
- *If $m\in I_{cu}^-$, $m'\in I_{cu}^+$ and $\frac{j}{\alpha_l^-}-\frac{j_0}{\alpha_{l'}^+}\in\left[\frac{n}{2},2n\right]$, we have that:*

$$\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\left(\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})-\widetilde{\mathscr{C}}_m^+(e^\tau,j_0-1,\boldsymbol{e})\right)\Pi(W_m^-(e^\tau,j))d\tau$$

$$= \exp\left(-c\left(\frac{\left|n-\left(\frac{j}{\alpha_l^-}-\frac{j_0}{\alpha_{l'}^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{|\boldsymbol{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+O_{\mathbb{C}}\left(\frac{|\boldsymbol{e}|e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)\boldsymbol{r}_l^- + O_{\mathbb{C}}\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\boldsymbol{l}_{l'}^{+T}\boldsymbol{e}\boldsymbol{r}_l^-\right).$$

(4.5.32a)

- *If $m\in I_{cu}^-$, $m'\in I_{cu}^+$ and $\frac{j}{\alpha_l^-}-\frac{j_0}{\alpha_{l'}^+}\notin\left[\frac{n}{2},2n\right]$, we have that:*

$$\int_{\Gamma_{in}(\eta)} e^{n\tau}e^{\tau}\widetilde{g}_{m',m}^+(e^\tau)\left(\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\boldsymbol{e})-\widetilde{\mathscr{C}}_m^+(e^\tau,j_0-1,\boldsymbol{e})\right)\Pi(W_m^-(e^\tau,j))d\tau = O\left(|\boldsymbol{e}|e^{-cn}\right). \quad (4.5.32b)$$

Lemma 4.5.15 (Discrete derivative version of Lemma 4.5.11 on the central excited mode)**.**
We consider $m'\in I_{cu}^+$ and write it as $m'=l'+(k'-1)d$ with $k'\in\{1,\dots,p+q\}$ and $l'\in\{1,\dots,d\}$. There exists a positive constant c such that for all $n\in\mathbb{N}\backslash\{0\}$, $j_0\in\mathbb{N}\backslash\{0\}$, $j\in\mathbb{Z}$ such that $j-j_0\in\{-nq,\dots,np\}$ and $\boldsymbol{e}\in\mathbb{C}^d$, we have:
- *For $-\frac{j_0}{\alpha_{l'}^+}\in\left[\frac{n}{2},2n\right]$ and $j\geq 0$, we have that:*

$$\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^{+}_{m',1}(e^{\tau}) \left(\widetilde{\mathscr{C}}^{+}_{m'}(e^{\tau}, j_0, \boldsymbol{e}) - \widetilde{\mathscr{C}}^{+}_{m'}(e^{\tau}, j_0 - 1, \boldsymbol{e}) \right) \Pi(\Phi_1(e^{\tau}, j)) d\tau =$$
$$O\left(\frac{|\boldsymbol{e}| e^{-c|j|}}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{\left| n + \frac{j_0}{\alpha_l^+} \right|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) \right) + O\left(|\boldsymbol{e}| e^{-cn} \right). \quad (4.5.33a)$$

- *For* $-\frac{j_0}{\alpha^+_{l'}} \notin \left[\frac{n}{2}, 2n\right]$ *and* $j \geq 0$, *we have that:*

$$\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^{+}_{m',1}(e^{\tau}) \left(\widetilde{\mathscr{C}}^{+}_{m'}(e^{\tau}, j_0, \boldsymbol{e}) - \widetilde{\mathscr{C}}^{+}_{m'}(e^{\tau}, j_0 - 1, \boldsymbol{e}) \right) \Pi(\Phi_1(e^{\tau}, j)) d\tau = O(|\boldsymbol{e}| e^{-cn}). \quad (4.5.33b)$$

- *For* $-\frac{j_0}{\alpha^+_{l'}} \in \left[\frac{n}{2}, 2n\right]$ *and* $j < 0$, *we have that:*

$$\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^{+}_{m',d(p+q)}(e^{\tau}) \left(\widetilde{\mathscr{C}}^{+}_{m'}(e^{\tau}, j_0, \boldsymbol{e}) - \widetilde{\mathscr{C}}^{+}_{m'}(e^{\tau}, j_0 - 1, \boldsymbol{e}) \right) \Pi(\Phi_{d(p+q)}(e^{\tau}, j)) d\tau =$$
$$O\left(\frac{|\boldsymbol{e}| e^{-c|j|}}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{\left| n + \frac{j_0}{\alpha_l^+} \right|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) \right) + O\left(|\boldsymbol{e}| e^{-cn} \right). \quad (4.5.33c)$$

- *For* $-\frac{j_0}{\alpha^+_{l'}} \notin \left[\frac{n}{2}, 2n\right]$ *and* $j < 0$, *we have that:*

$$\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \widetilde{g}^{+}_{m',d(p+q)}(e^{\tau}) \left(\widetilde{\mathscr{C}}^{+}_{m'}(e^{\tau}, j_0, \boldsymbol{e}) - \widetilde{\mathscr{C}}^{+}_{m'}(e^{\tau}, j_0 - 1, \boldsymbol{e}) \right) \Pi(\Phi_{d(p+q)}(e^{\tau}, j)) d\tau = O(|\boldsymbol{e}| e^{-cn}). \quad (4.5.33d)$$

Let us now give proofs of Lemmas 4.5.12-4.5.15. The proofs will be fairly similar to those of Lemmas 4.5.7-4.5.11 with some adjustments.

Proof of Lemmas 4.5.12, 4.5.13 and 4.5.14

The proofs of Lemmas 4.5.12, 4.5.13 and 4.5.14 are fairly similar. We will prove in details the result of Lemma 4.5.12 and the other Lemmas will be left to the reader.

We consider $m \in I^+_{cs} \cup I^+_{cu}$. We recall that ϖ^+_l is defined using (4.5.3). Using Cauchy's formula and the expressions of W^+_m and $\widetilde{\mathscr{C}}^+_m$ given respectively by Lemma 4.4.3 and (4.4.51), we have that:

$$\int_{\Gamma_{in}(\eta)} e^{n\tau} e^{\tau} \left(\widetilde{\mathscr{C}}^{+}_{m}(e^{\tau}, j_0, \mathbf{e}) - \widetilde{\mathscr{C}}^{+}_{m}(e^{\tau}, j_0 - 1, \mathbf{e}) \right) \Pi(W^+_m(e^{\tau}, j)) d\tau = E_1 + E_2 \boldsymbol{r}^+_l + E_3 {\boldsymbol{l}^+_l}^T \mathbf{e} \boldsymbol{r}^+_l$$

where the vector E_1 and the scalars E_2 and E_3 are defined as:

$$E_1 := \int_{\Gamma_1} e^{n\tau} e^{(j-j_0)\varpi^+_l(\tau)} e^{\tau} \left(e^{-\varpi^+_l(\tau)} \Delta^+_m(e^{\tau}, j_0, \mathbf{e}) - \Delta^+_m(e^{\tau}, j_0 - 1, \mathbf{e}) \right) \Pi\left(V^+_m(e^{\tau}, j) - R^+_m(e^{\tau}) \right) d\tau,$$

$$E_2 := \int_{\Gamma_2} e^{n\tau} e^{(j-j_0)\varpi^+_l(\tau)} e^{\tau} \left(e^{-\varpi^+_l(\tau)} \left(\Delta^+_m(e^{\tau}, j_0, \mathbf{e}) - \frac{d\zeta^+_m}{dz}(e^{\tau}) {\boldsymbol{l}^+_l}^T \mathbf{e} \right) - \left(\Delta^+_m(e^{\tau}, j_0 - 1, \mathbf{e}) - \frac{d\zeta^+_m}{dz}(e^{\tau}) {\boldsymbol{l}^+_l}^T \mathbf{e} \right) \right) d\tau,$$

$$E_3 := \int_{\Gamma_3} e^{n\tau} e^{(j-j_0)\varpi^+_l(\tau)} e^{\tau} \left(e^{-\varpi^+_l(\tau)} - 1 \right) \frac{d\zeta^+_m}{dz}(e^{\tau}) d\tau,$$

and Γ_1, Γ_2 and Γ_3 are paths belonging to the set X.

► Using (4.5.7) to bound the function ϖ^+_l and Lemma 4.4.3 to bound $V^+_m(e^{\tau}, j) - R^+_m(e^{\tau})$, we prove that there exist two positive constants C, c independent from n, j_0, j and $\mathbf{e}$ such that:

$$|E_1| \leq C e^{-c|j|} |\mathbf{e}| \int_{\Gamma_1} \exp\left(n\Re(\tau) + \left(\frac{j - j_0}{\alpha^+_l} \right) \left(-\Re(\tau) + A_R \Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu} \right) \right) |d\tau|.$$

► Using (4.5.7) to bound the function ϖ^+_l and (4.4.52e), we prove that there exist two positive constants

C, c independent from n, j_0, j and $\mathbf{e}$ such that:

$$|E_2| \leq Ce^{-c|j_0|}|\mathbf{e}| \int_{\Gamma_2} \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

► Using (4.5.7) to bound the function ϖ_l^+ and a Taylor expansion, we have that there exist two positive constants C, c independent from n, j_0, j and $\mathbf{e}$ such that

$$|E_3| \leq C \int_{\Gamma_3} |\tau| \exp\left(n\Re(\tau) + \left(\frac{j-j_0}{\alpha_l^+}\right)\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau|.$$

Using Lemma 4.5.4 which gives a good choices of path $\Gamma_1, \Gamma_2, \Gamma_3 \in X$ depending on n, j_0 and j to handle the integrals in the terms, there exist new constants $C, c > 0$ independent from n, j_0, j and $\mathbf{e}$ such that if $\frac{j-j_0}{\alpha_l^+} \in \left[\frac{n}{2}, 2n\right]$:

$$|E_1| \leq \frac{Ce^{-c|j|}|\mathbf{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j-j_0}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \quad \text{and} \quad |E_2| \leq \frac{Ce^{-c|j_0|}|\mathbf{e}|}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j-j_0}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)$$

$$|E_3| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j-j_0}{\alpha_l^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)$$

and if $\frac{j-j_0}{\alpha_l^+} \notin \left[\frac{n}{2}, 2n\right]$:

$$|E_1| \leq C|\mathbf{e}|e^{-cn}, \quad |E_2| \leq C|\mathbf{e}|e^{-cn} \quad \text{and} \quad |E_3| \leq Ce^{-cn}.$$

We have thus obtained the results (4.5.30). □

Proof of Lemma 4.5.15

We consider $m' \in I_{cu}^+$ and write it as $m' = l' + (k'-1)d$ with $k' \in \{1, \dots, p+q\}$ and $l' \in \{1, \dots, d\}$. We start by proving that there exist two positive constants C, c such that:

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j_0 \in \mathbb{Z}, \quad \left|E_{2\mu}\left(\beta_{l'}^+; \frac{n\alpha_{l'}^+ + j_0 - 1}{n^{\frac{1}{2\mu}}}\right) - E_{2\mu}\left(\beta_{l'}^+; \frac{n\alpha_{l'}^+ + j_0}{n^{\frac{1}{2\mu}}}\right)\right|$$
$$\leq \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n + \frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{4.5.34}$$

Since $-H_{2\mu}(\beta_{l'}^+; \cdot)$ is the derivative of the function $E_{2\mu}(\beta_{l'}^+; \cdot)$, using (4.1.30a), there exist two positive constants C, c such that:

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j_0 \in \mathbb{Z}, \quad \left|E_{2\mu}\left(\beta_{l'}^+; \frac{n\alpha_{l'}^+ + j_0 - 1}{n^{\frac{1}{2\mu}}}\right) - E_{2\mu}\left(\beta_{l'}^+; \frac{n\alpha_{l'}^+ + j_0}{n^{\frac{1}{2\mu}}}\right)\right|$$
$$\leq \frac{C}{n^{\frac{1}{2\mu}}} \sup_{t \in \left[\frac{n\alpha_{l'}^+ + j_0 - 1}{n^{\frac{1}{2\mu}}}, \frac{n\alpha_{l'}^+ + j_0}{n^{\frac{1}{2\mu}}}\right]} \exp\left(-c|t|^{\frac{2\mu}{2\mu-1}}\right).$$

Then, there exist two new positive constants C, c such that (4.5.34) is verified. Using the definition (4.1.31d) of $E_{l'}^+$ and the bound (4.1.23) of $V(j)$, the inequality (4.5.34) implies that there exist two positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}\backslash\{0\}$, $j \in \mathbb{Z}$ such that $j - j_0 \in \{-nq, \dots, np\}$ and $\mathbf{e} \in \mathbb{C}^d$, we have that:

- For $-\frac{j_0}{\alpha_{l'}^+} \in \left[\frac{n}{2}, 2n\right]$:

$$\left|\left(E_{l'}^+(n,j_0) - E_{l'}^+(n,j_0-1)\right)\mathbf{e}V(j)\right| \leq \frac{C|\mathbf{e}|e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha_{l'}^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{4.5.35a}$$

- For $-\frac{j_0}{\alpha_{l'}^+} \notin \left[\frac{n}{2}, 2n\right]$:

$$\left|\left(E_{l'}^+(n,j_0) - E_{l'}^+(n,j_0-1)\right)\mathbf{e}V(j)\right| \leq C|\mathbf{e}|e^{-cn}. \tag{4.5.35b}$$

Thus, using (4.5.35) and the result of Lemma 4.5.11 for j_0 and j_0-1, we conclude the proof of Lemma 4.5.15. □

There just remains to conclude the proof of the decomposition (4.1.33) of the discrete derivative of the temporal Green's function. We will give details on the case when $j \geq j_0+1$. The cases when $j \in \{0, \dots, j_0\}$ and $j<0$ would be handled similarly using (4.4.34b) and (4.4.34c) rather than (4.4.34a)

Using (4.5.15), Lemma 4.5.3 and (4.4.34a), there exists a constant $c>0$ such that for $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}\backslash\{0\}$, $j \geq j_0+1$ and $\mathbf{e} \in \mathbb{C}^d$ which verify $j-j_0 \in \{-nq, \dots, np\}$:

$$\begin{aligned}
&(\mathscr{G}(n,j_0,j) - \mathscr{G}(n,j_0-1,j))\,\mathbf{e} \\
=&\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\Pi(W(e^\tau,j_0,j,\mathbf{e}))d\tau - \frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\Pi(W(e^\tau,j_0-1,j,\mathbf{e}))d\tau \\
&+\frac{1}{2i\pi}\int_{\Gamma_{out}(\eta)} e^{n\tau}e^\tau\Pi(W(e^\tau,j_0,j,\mathbf{e}))d\tau - \frac{1}{2i\pi}\int_{\Gamma_{out}(\eta)} e^{n\tau}e^\tau\Pi(W(e^\tau,j_0-1,j,\mathbf{e}))d\tau \\
=&\sum_{m\in I_{cs}^+}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\left(\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\mathbf{e}) - \widetilde{\mathscr{C}}_m^+(e^\tau,j_0-1,\mathbf{e})\right)\Pi\left(W_m^+(e^\tau,j)\right)d\tau\right) \\
&+\sum_{m\in I_{ss}^+}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\left(\widetilde{\mathscr{C}}_m^+(e^\tau,j_0,\mathbf{e}) - \widetilde{\mathscr{C}}_m^+(e^\tau,j_0-1,\mathbf{e})\right)\Pi\left(W_m^+(e^\tau,j)\right)d\tau\right) \\
&+\sum_{m\in I_{cs}^+}\sum_{m'\in I_{cu}^+}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}_{m',m}^+(e^\tau)\left(\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\mathbf{e}) - \widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0-1,\mathbf{e})\right)\Pi\left(W_m^+(e^\tau,j)\right)d\tau\right) \\
&+\sum_{m\in I_{ss}^+\backslash\{1\}}\sum_{m'\in I_{cu}^+}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}_{m',m}^+(e^\tau)\left(\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\mathbf{e}) - \widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0-1,\mathbf{e})\right)\Pi\left(W_m^+(e^\tau,j)\right)d\tau\right) \\
&+\sum_{m\in I_{cs}^+}\sum_{m'\in I_{su}^+}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}_{m',m}^+(e^\tau)\left(\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\mathbf{e}) - \widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0-1,\mathbf{e})\right)\Pi\left(W_m^+(e^\tau,j)\right)d\tau\right) \\
&+\sum_{m\in I_{ss}^+\backslash\{1\}}\sum_{m'\in I_{su}^+}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}_{m',m}^+(e^\tau)\left(\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\mathbf{e}) - \widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0-1,\mathbf{e})\right)\Pi\left(W_m^+(e^\tau,j)\right)d\tau\right) \\
&+\sum_{m'\in I_{cu}^+}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}_{m',1}^+(e^\tau)\left(\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\mathbf{e}) - \widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0-1,\mathbf{e})\right)\Pi\left(\Phi_1(e^\tau,j)\right)d\tau\right) \\
&+\sum_{m'\in I_{su}^+}\left(-\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} e^{n\tau}e^\tau\widetilde{g}_{m',1}^+(e^\tau)\left(\widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0,\mathbf{e}) - \widetilde{\mathscr{C}}_{m'}^+(e^\tau,j_0-1,\mathbf{e})\right)\Pi\left(\Phi_1(e^\tau,j)\right)d\tau\right) + O(|\mathbf{e}|e^{-cn}).
\end{aligned}$$

Using Lemmas 4.5.7-4.5.11 to bound the blue terms and 4.5.12-4.5.15 to bound the green terms, we obtain the decomposition (4.1.33) for $j \geq j_0+1$. The decomposition (4.1.33) for $j \in \{0,\dots,j_0\}$ and $j<0$ would be handled similarly.

4.6 Numerical example

In the present section, we will prove that several hypotheses of the chapter are verified under some CFL condition for the modified Lax-Friedrichs scheme. We will then apply the modified Lax-Friedrichs scheme to

Burgers equation and numerically observe the result of Theorem 4.1 in the scalar case (i.e. $d = 1$). Let us observe that there exists an example of numerical simulation for the gas dynamics system ($d = 3$) and for the modified Lax-Friedrichs scheme in [God03, Section 5].

Some hypotheses verified for the modified Lax-Friedrichs scheme

Let us start by considering a one-dimensional system of conservation laws (4.1.1) with $\mathcal{U} = \mathbb{R}^d$ and two states u^- and u^+ in $\mathbb{R}^d$ which satisfy the Rankine-Hugoniot condition (4.1.2) and Hypothesis 4.1 (i.e. it is a Lax shock). We now consider as an example the modified Lax-Friedrichs scheme for which the numerical flux is defined by:

$$\forall \nu \in]0,+\infty[, \forall u_{-1}, u_0 \in \mathbb{R}^d, \quad F(\nu; u_{-1}, u_0) := \frac{f(u_{-1}) + f(u_0)}{2} + D(u_{-1} - u_0)$$

where D is a positive constant. The integer p and q are then both equal to 1. We immediately observe that the consistency condition (4.1.9) is verified. We consider a constant $\boldsymbol{\nu} > 0$ which corresponds to the ratio between the space and time steps and which satisfies the CFL condition (4.1.6). The discrete evolution operator $\mathcal{N}$ is then defined for $u \in (\mathbb{R}^d)^{\mathbb{Z}}$ by:

$$\forall j \in \mathbb{Z}, \quad (\mathcal{N}(u))_j := u_j - \boldsymbol{\nu}\left(\frac{f(u_{j+1}) - f(u_{j-1})}{2} + D\left(-u_{j+1} + 2u_j - u_{j-1}\right)\right). \tag{4.6.1}$$

Let us prove that Hypothesis 4.4 and part of Hypothesis 4.8 on the matrices $B_k^\pm$ and $A_k^\pm$ are verified under a precise condition on the constant D. We have that:

$$B_{-1}^\pm = \boldsymbol{\nu}\left(\frac{1}{2}df(u^\pm) + DId\right) \quad \text{and} \quad B_0^\pm = \boldsymbol{\nu}\left(\frac{1}{2}df(u^\pm) - DId\right).$$

Hypothesis 4.4 is immediately verified. Furthermore, if we assume that:

$$\forall l \in \{1, \dots, d\}, \quad |\boldsymbol{\lambda}_l^\pm| < 2D$$

then Hypothesis 4.8 is partially verified.

We will now prove that Hypothesis 4.5 on the spectrum of the operator $\mathscr{L}^\pm$ is verified under conditions on $\boldsymbol{\nu}$ and D. For $l \in \{1, \dots, d\}$, we have that the functions $\mathcal{F}_l^\pm$ defined by (4.1.15) verify:

$$\forall \kappa \in \mathbb{C}\backslash\{0\}, \quad \mathcal{F}_l^\pm(\kappa) = (1 - 2D\boldsymbol{\nu}) + D\boldsymbol{\nu}\left(\kappa + \kappa^{-1}\right) - \frac{\boldsymbol{\lambda}_l^\pm \boldsymbol{\nu}}{2}\left(\kappa - \kappa^{-1}\right)$$

and thus:

$$\forall \xi \in \mathbb{R}, \quad \mathcal{F}_l^\pm\left(e^{i\xi}\right) = (1 - 2D\boldsymbol{\nu}) + 2D\boldsymbol{\nu}\cos(\xi) - i\boldsymbol{\lambda}_l^\pm \boldsymbol{\nu}\sin(\xi).$$

Therefore, under the assumption that:

$$\forall l \in \{1, \dots, d\}, \quad |\boldsymbol{\lambda}_l^\pm|\boldsymbol{\nu} < 2D\boldsymbol{\nu} < 1,$$

then, we have that:

$$\forall \xi \in \mathbb{R}, \quad \left|\mathcal{F}_l^\pm\left(e^{i\xi}\right)\right|^2 = (1 - 2D\boldsymbol{\nu})^2 + 4D\boldsymbol{\nu}(1 - 2D\boldsymbol{\nu})\cos(\xi) + (\boldsymbol{\lambda}_l^\pm \boldsymbol{\nu})^2 + \left((2D\boldsymbol{\nu})^2 - (\boldsymbol{\lambda}_l^\pm \boldsymbol{\nu})^2\right)\cos^2(\xi) \leq 1$$

with an equality if and only if $\xi \in 2\pi\mathbb{Z}$. Furthermore, the asymptotic expansion (4.1.18) is verified for $\mu = 1$ and we have that:

$$\forall l \in \{1, \dots, d\}, \quad \beta_l^\pm := \frac{2D\boldsymbol{\nu} - (\boldsymbol{\lambda}_l^\pm \boldsymbol{\nu})^2}{2} \in]0, +\infty[.$$

Thus, Hypothesis 4.5 is verified.

Let us conclude this section by proving that Hypothesis 4.9 is verified. We consider $l \in \{1, \dots, d\}$. For $\kappa \in \mathbb{C}\backslash\{0\}$, we have that:

$$\mathcal{F}_l^\pm(\kappa) = 1 \Leftrightarrow \left(D\boldsymbol{\nu} - \frac{\boldsymbol{\lambda}_l^\pm \boldsymbol{\nu}}{2}\right)\kappa^2 - 2D\boldsymbol{\nu}\kappa + \left(D\boldsymbol{\nu} + \frac{\boldsymbol{\lambda}_l^\pm \boldsymbol{\nu}}{2}\right) = 0.$$

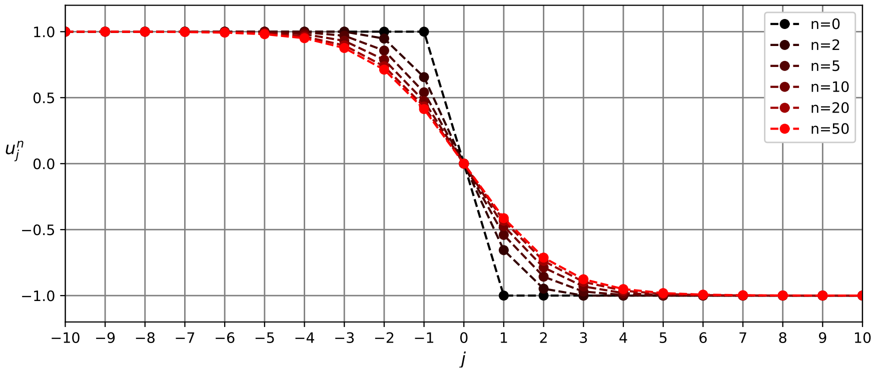

Figure 4.4 – Solution u^n of the numerical scheme (4.6.1) for the initial condition (4.6.2)

The equation on the right-hand side above has two solutions:

$$1 \quad \text{and} \quad \frac{2D\boldsymbol{\nu} + \boldsymbol{\lambda}_l^\pm \boldsymbol{\nu}}{2D\boldsymbol{\nu} - \boldsymbol{\lambda}_l^\pm \boldsymbol{\nu}} \neq 1.$$

Thus, Hypothesis 4.9 is verified.

The last hypotheses remaining to prove depend on the existence of a SDSP associated with the shock we consider.

Application for the Burgers equation

In the present section, we consider the scalar ($d = 1$) conservation law (4.1.1) where the flux f is defined by:

$$\forall u \in \mathbb{R}, \quad f(u) := \frac{u^2}{2}.$$

This corresponds to the so-called Burgers equation. For the shock, we consider the states

$$u^- = 1 \quad \text{and} \quad u^+ = -1.$$

The Rankine-Hugoniot condition (4.1.2) and Hypothesis 4.1 are verified. Thus, the shock associated with the states u^- and u^+ is a stationary Lax shock.

For the numerical scheme, we consider the modified Lax-Friedrichs scheme defined above by (4.6.1) with the positive constants D and $\boldsymbol{\nu}$ satisfying:

$$\boldsymbol{\nu} < 2D\boldsymbol{\nu} < 1.$$

The section above implies that several of the hypotheses we introduced in the chapter are verified. From now on, we choose $\boldsymbol{\nu} = 0.3$ and $D = \frac{1}{3\boldsymbol{\nu}}$.

We want to know whether Hypothesis 4.2 is verified, i.e. if there exists a stationary discrete shock profile associated with the shock we are considering. In Figure 4.4, we display the solution u^n of the numerical scheme (4.1.8) for the initial condition

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases} \tag{4.6.2}$$

We observe that the solution u^n seems to converge in time (quite fast) towards a fixed point $\overline{u}^s$ of the discrete evolution operator $\mathscr{N}$. Furthermore, it seems as if there exist two positive constants C, c such that:

$$\forall j \in \mathbb{N}, \qquad \begin{array}{r} |\overline{u}_j^s - u^+| = |\overline{u}_j^s + 1| \leq Ce^{-c|j|}, \\ |\overline{u}_{-j}^s - u^-| = |\overline{u}_{-j}^s - 1| \leq Ce^{-c|j|}. \end{array}$$

Hypotheses 4.2 and 4.3 would then be verified. We can then compute the Green's function $\mathscr{G}(n, j_0, j)$ defined by (4.1.25) associated with the operator $\mathscr{L}$. In Figures 4.5 and 4.6, we display the Green's function $\mathscr{G}(n, j_0, j)$ for $j_0 = 50$ and -30. The behavior displayed by the Green's function fits the result of Theorem 4.1. For small times, the Green's function resembles a Gaussian wave traveling along the characteristics of the conservation law (4.1.1). Since we are considering Lax shocks, the waves are traveling towards the shock location at $j = 0$. When they reach the shock location, there only remains a fixed solution of the operator $\mathscr{L}$. This coincides with

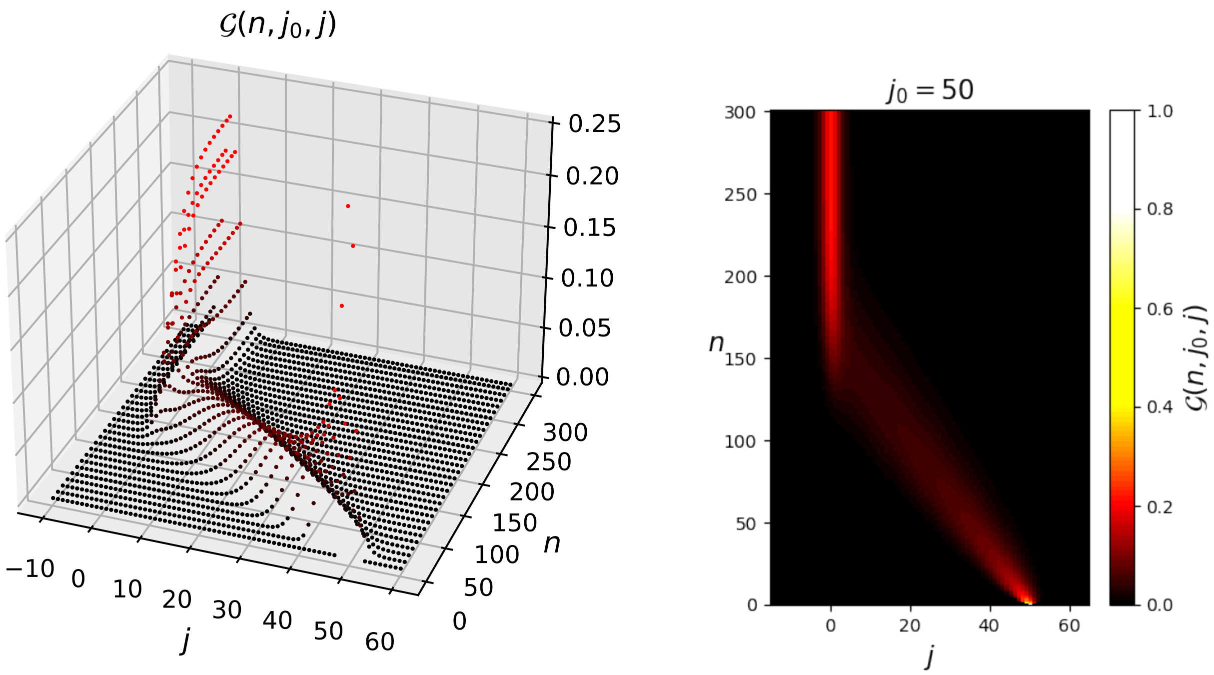

Figure 4.5 – Representation of the Green's function $\mathscr{G}(n, j_0, j)$ for $j_0 = 50$

the fact that the terms $E_{l'}^{+}(n, j_0)$ and $E_{l'}^{-}(n, j_0)$ converge towards 1 as n tends towards $+\infty$.

4.A Appendix of the chapter

Proof of Lemma 4.3.6

We define recursively

$$\forall j \in \mathbb{N}, \quad z_j^0 := y_j$$

and

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{N}, \quad z_j^{n+1} := C_H e^{-c_H j} + \Theta \sum_{k=0}^{+\infty} e^{-c_H|j-1-k|} z_k^n. \tag{4.A.1}$$

We then prove recursively that for all $n \in \mathbb{N}$, the sequence z^n is bounded, has non negative coefficients and

$$\|z^n\|_\infty \leq \frac{C_H}{1-\theta} \quad \text{and} \quad \forall j \in \mathbb{N}, \quad z_j^n \leq z_j^{n+1}.$$

Indeed, this property is obviously true for $n = 0$ using the inequality (4.3.24) since $z^0 = y$. We now consider $n \in \mathbb{N}$ for which the property is verified we will prove that the property for $n+1$ is satisfied. We first observe using the equality (4.A.1) that for all $j \in \mathbb{N}$ the coefficient z_j^{n+1} is non negative and

$$z_j^{n+1} \leq C_H + \theta \|z^n\|_\infty .$$

Thus, we have that $z^{n+1} \in \ell^\infty(\mathbb{N})$ and

$$\left\|z^{n+1}\right\|_\infty \leq \frac{C_H}{1-\theta}.$$

Finally, we observe that using the equality (4.A.1) for n and $n+1$, we have

$$z_j^{n+2} - z_j^{n+1} = \Theta \sum_{k=0}^{+\infty} e^{-c_H|j-1-k|} \underbrace{(z_k^{n+1} - z_k^n)}_{\geq 0} \geq 0.$$

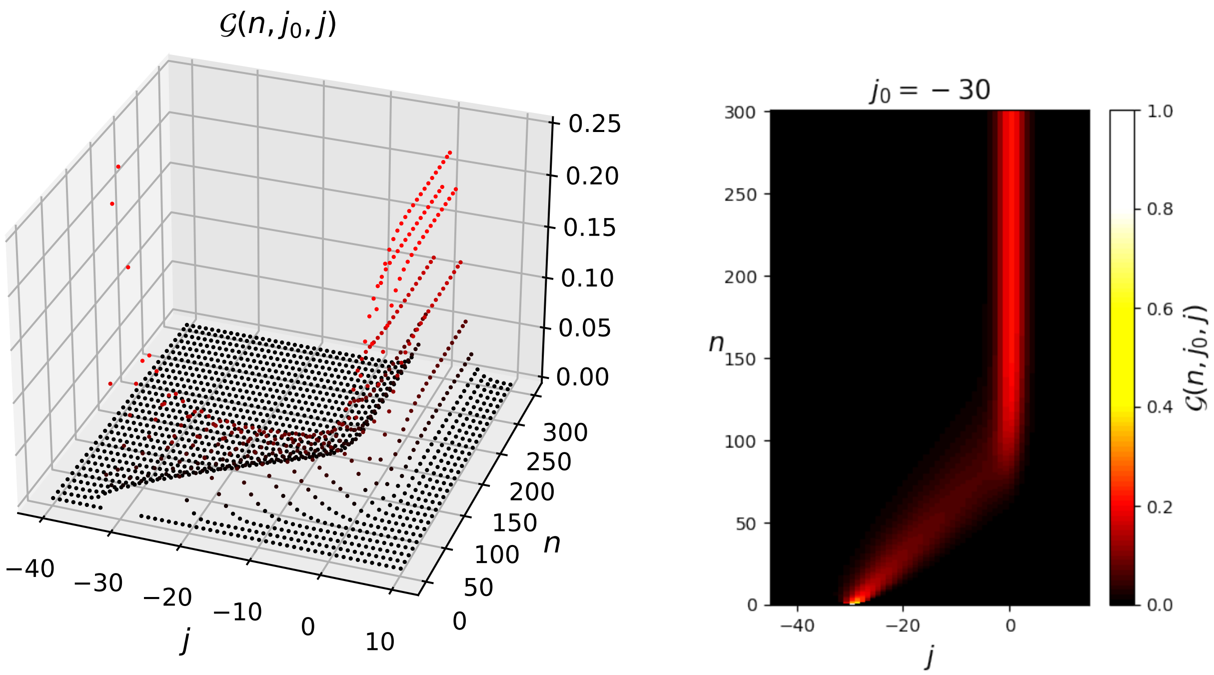

Figure 4.6 – Representation of the Green's function $\mathscr{G}(n, j_0, j)$ for $j_0 = -30$

This concludes the recurrence. We now observe that, for $p, q \in \mathbb{N}\setminus\{0\}$, using the equality (4.A.1), we have

$$z_j^p - z_j^q = \Theta \sum_{k=0}^{+\infty} e^{-c_H|j-1-k|}(z_k^{p-1} - z_k^{q-1}).$$

This implies that

$$\|z^p - z^q\|_\infty \le \theta \left\|z^{p-1} - z^{q-1}\right\|_\infty .$$

Thus, we have that

$$\forall p \ge q \ge 0, \quad \|z^p - z^q\|_\infty \le \theta^q \left\|z^{p-q} - y\right\|_\infty \le \theta^q \frac{2C_H}{1-\theta}.$$

Since $\theta < 1$, the sequence $(z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $\ell^\infty(\mathbb{N})$, thus it converges towards a sequence $z^\infty \in \ell^\infty(\mathbb{N})$. Since we have $y_j \le z_j^n$ for all $n, j \in \mathbb{N}$, we obviously have

$$\forall j \in \mathbb{N}, \quad y_j \le z_j^\infty.$$

Also, the equality (4.A.1) implies that

$$\forall j \in \mathbb{N}, \quad z_j^\infty = C_H e^{-c_H j} + \Theta \sum_{k=0}^{+\infty} e^{-c_H|j-1-k|} z_k^\infty. \tag{4.A.2}$$

Thus, there just remain to prove that there exists only one bounded sequence that satisfies (4.A.2) and that it has the form $z^\infty = (\rho r^j)_{j\in\mathbb{N}}$ where ρ and r satisfies the properties we expected.

We write (4.A.2) for j, $j+1$ and $j+2$ and reassemble the terms z_j^∞, z_{j+1}^∞ and z_{j+2}^∞ on the left side. We then have

$$\begin{aligned} &\left(1 - e^{-c_H}\Theta\right) z_j^\infty - e^{-2c_H}\Theta z_{j+1}^\infty - e^{-3c_H}\Theta z_{j+2}^\infty \\ &\qquad = C_H e^{-c_H j} + \Theta\left(\sum_{k=0}^{j-1} e^{-c_H|j-1-k|} z_k^\infty + \sum_{k=j+3}^{+\infty} e^{-c_H|j-1-k|} z_k^\infty\right) \end{aligned} \tag{4.A.3a}$$

$$-\Theta z_j^\infty + \left(1 - e^{-c_H}\Theta\right) z_{j+1}^\infty - e^{-2c_H}\Theta z_{j+2}^\infty$$

$$= e^{-c_H} C_H e^{-c_H j} + \Theta \left(e^{-c_H} \sum_{k=0}^{j-1} e^{-c_H |j-1-k|} z_k^\infty + e^{c_H} \sum_{k=j+3}^{+\infty} e^{-c_H |j-1-k|} z_k^\infty \right) \quad (4.A.3b)$$

$$- e^{-c_H} \Theta z_j^\infty - \Theta z_{j+1}^\infty + \left(1 - e^{-c_H} \Theta\right) z_{j+2}^\infty$$
$$= e^{-2c_H} C_H e^{-c_H j} + \Theta \left(e^{-2c_H} \sum_{k=0}^{j-1} e^{-c_H |j-1-k|} z_k^\infty + e^{2c_H} \sum_{k=j+3}^{+\infty} e^{-c_H |j-1-k|} z_k^\infty \right). \quad (4.A.3c)$$

We consider three scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{cases} \alpha + \beta e^{-c_H} + \gamma e^{-2c_H} & = 0 \\ \alpha + \beta e^{c_H} + \gamma e^{2c_H} & = 0. \end{cases}$$

A solution is $\alpha = \gamma = 1$ and $\beta = -\frac{e^{2c_H} - e^{-2c_H}}{e^{c_H} - e^{-c_H}} = -\frac{\sinh(2c_H)}{\sinh(c_H)} = -2\cosh(c_H)$. We then have that

$$\begin{aligned} \alpha - \Theta\left(\alpha e^{-c_H} + \beta + \gamma e^{-c_H}\right) &= 1 - \theta \frac{e^{\frac{c_H}{2}} - e^{-\frac{c_H}{2}}}{e^{\frac{c_H}{2}} + e^{-\frac{c_H}{2}}} \left(2e^{-c_H} - 2\cosh(c_H)\right) \\ &= 1 + 2\theta \sinh(c_H) \frac{\sinh\left(\frac{c_H}{2}\right)}{\cosh\left(\frac{c_H}{2}\right)} \\ &= 1 + 4\theta \sinh^2\left(\frac{c_H}{2}\right). \end{aligned}$$

Multiplying the equalities (4.A.3) respectively by α, β, γ and summing them, we obtain that

$$\forall j \in \mathbb{N}, \quad z_{j+2}^\infty - 2\cosh(c_H) z_{j+1}^\infty + \left(1 + 4\theta \sinh^2\left(\frac{c_H}{2}\right)\right) z_j^\infty = 0.$$

We are thus led to study the polynomial

$$P := X^2 - 2\cosh(c_H) X + \left(1 + 4\theta \sinh^2\left(\frac{c_H}{2}\right)\right). \quad (4.A.4)$$

Its discriminant is

$$\Delta = 4\sinh^2(c_H) - 16\theta \sinh^2\left(\frac{c_H}{2}\right) = 16 \sinh^2\left(\frac{c_H}{2}\right) \left(\cosh^2\left(\frac{c_H}{2}\right) - \theta\right) > 0.$$

Its roots are

$$r_\pm := \cosh(c_H) \pm 2\sinh\left(\frac{c_H}{2}\right) \sqrt{\cosh^2\left(\frac{c_H}{2}\right) - \theta}.$$

We observe that evaluating the polynomial P at 0 and 1 gives us

$$P(0) = 1 + 4\theta \sinh^2\left(\frac{c_H}{2}\right) > 0$$

and

$$P(1) = 2 - 2\cosh(c_H) + 4\theta \sinh^2\left(\frac{c_H}{2}\right) = -4(1-\theta)\sinh^2\left(\frac{c_H}{2}\right) < 0.$$

Thus, $r_+ > 1$ and $r := r_- \in]0, 1[$. We have that $\rho, \tilde{\rho} \in \mathbb{R}$ such that

$$\forall j \in \mathbb{N}, \quad z_j^\infty = \rho r^j + \tilde{\rho} r_+^j.$$

Since the sequence z^∞ is bounded, we have that $\tilde{\rho} = 0$, i.e.

$$\forall j \in \mathbb{N}, \quad z_j^\infty = \rho r^j.$$

Let us now compute the value of ρ. Using the equality (4.A.2), we obtain for $j \in \mathbb{Z}$:

$$\rho r^j = C_H e^{-c_H j} + \Theta \rho \sum_{k=0}^{+\infty} e^{-c_H |j-1-k|} r^k$$

$$
\begin{aligned}
&= C_H e^{-c_H j} + \Theta\rho \left(e^{-c_H(j-1)} \frac{(re^{c_H})^j - 1}{re^{c_H} - 1} + e^{c_H(j-1)} \frac{(re^{-c_H})^j}{1 - re^{-c_H}} \right) \\
&= e^{-c_H j} \left(C_H - \Theta\rho \frac{e^{c_H}}{re^{c_H} - 1} \right) + r^j \Theta\rho \left(\frac{e^{c_H}}{re^{c_H} - 1} + \frac{e^{-c_H}}{1 - re^{-c_H}} \right).
\end{aligned}
$$

Using the definition (4.3.23) of θ and the fact that r is a root of the polynomial P defined by (4.A.4), we have that:

$$
\begin{aligned}
\Theta \left(\frac{e^{c_H}}{re^{c_H} - 1} + \frac{e^{-c_H}}{1 - re^{-c_H}} \right) &= \theta \frac{\sinh\left(\frac{c_H}{2}\right)}{\cosh\left(\frac{c_H}{2}\right)} \frac{2\sinh(c_H)}{-r^2 + 2r\cosh(c_H) - 1} \\
&= \theta \frac{\sinh\left(\frac{c_H}{2}\right)}{\cosh\left(\frac{c_H}{2}\right)} \frac{2\sinh(c_H)}{4\theta \sinh^2\left(\frac{c_H}{2}\right)} \\
&= 1.
\end{aligned}
$$

Thus, we obtain:

$$
\rho r^j = e^{-c_H j} \left(C_H - \Theta\rho \frac{e^{c_H}}{re^{c_H} - 1} \right) + \rho r^j
$$

i.e.

$$
C_H = \Theta\rho \frac{e^{c_H}}{re^{c_H} - 1}.
$$

Therefore, the scalar ρ verifies:

$$
\rho = \frac{C_H}{\Theta}(r - e^{-c_H}).
$$

To conclude, we observe that $r > e^{-c_H}$. Indeed, we have that:

$$
\begin{aligned}
r &= \cosh(c_H) - 2\sinh\left(\frac{c_H}{2}\right)\sqrt{\cosh^2\left(\frac{c_H}{2}\right) - \theta} \\
&> \cosh(c_H) - 2\sinh\left(\frac{c_H}{2}\right)\cosh\left(\frac{c_H}{2}\right) \\
&= \cosh(c_H) - \sinh(c_H) \\
&= e^{-c_H}.
\end{aligned}
$$

Therefore, r belongs to the interval $]e^{-c_H}, 1[$ and ρ is positive.

Proof of Lemma 4.5.4

- **Proof of** (4.5.17a)

We consider $n \in \mathbb{N}\backslash\{0\}$ and $x \in \left[0, \frac{n}{2}\right]$. Noticing that $\Gamma_d(\eta) \subset B(0,\varepsilon)$ and using (4.5.10), we have

$$
\begin{aligned}
&\int_{\Gamma_d(\eta)} |\tau|^k \exp\left(n\Re(\tau) + x(-\Re(\tau) + A_R \Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu})\right) |d\tau| \\
\leq & \varepsilon^k \int_{-r_\varepsilon(\eta)}^{r_\varepsilon(\eta)} \exp\left(-(n-x)\eta + xA_R\eta^{2\mu} - xA_I t^{2\mu}\right) dt \\
\leq & 2r_\varepsilon(\eta)\varepsilon^k \exp\left(-\frac{n}{2}\left(\eta - A_R\eta^{2\mu}\right)\right) \\
\leq & 2r_\varepsilon(\eta)\varepsilon^k \exp\left(-\frac{n\eta}{4}\right).
\end{aligned}
$$

- **Proof of** (4.5.17b)**:**

We consider $n \in \mathbb{N}\backslash\{0\}$ and $x \in [2n, +\infty[$. We will separate the integral on the path $\Gamma_{in}(\eta)$ using the paths $\Gamma_{in}^0(\eta)$ and $\Gamma_{in}^\pm(\eta)$ introduced in (4.5.14).

▶ Noticing that $\Gamma_d(\eta) \subset B(0,\varepsilon)$ and that $n - x \leq -\frac{x}{2}$, we have using condition (4.5.10)

$$
\begin{aligned}
&\int_{\Gamma_{in}^0(\eta)} |\tau|^k \exp\left(n\Re(\tau) + x(-\Re(\tau) + A_R \Re(\tau)^{2\mu} - A_I \mathrm{Im}\,(\tau)^{2\mu})\right) |d\tau| \\
\leq & \varepsilon^k \int_{-r_\varepsilon(\eta)}^{r_\varepsilon(\eta)} \exp\left((n-x)\eta + xA_R\eta^{2\mu} - xA_I t^{2\mu}\right) dt \\
\leq & 2r_\varepsilon(\eta)\varepsilon^k \exp\left(-x\left(\frac{\eta}{2} - A_R\eta^{2\mu}\right)\right)
\end{aligned}
$$

$$\leq 2r_\varepsilon(\eta)\varepsilon^k \exp\left(-2n\left(\frac{\eta}{2} - A_R\eta^{2\mu}\right)\right).$$

We have proved exponential bounds on this first term.

▶ We have that using (4.5.11) and that $x \geq 2n$

$$\begin{aligned}
&\int_{\Gamma_{in}^{\pm}(\eta)} |\tau|^k \exp\left(n\Re(\tau) + x(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu})\right)|d\tau| \\
\leq &\varepsilon^k \int_{-\eta}^{\eta} \exp\left((n-x)t + xA_R t^{2\mu} - xA_I r_\varepsilon(\eta)^{2\mu}\right) dt \\
\leq &2\eta\varepsilon^k \exp\left((x-n)\eta + xA_R\eta^{2\mu} - xA_I r_\varepsilon(\eta)^{2\mu}\right) \\
\leq &2\eta\varepsilon^k \exp\left(x\left(\eta + A_R\eta^{2\mu} - A_I r_\varepsilon(\eta)^{2\mu}\right)\right) \\
\leq &2\eta\varepsilon^k \exp\left(-A_I r_\varepsilon(\eta)^{2\mu} n\right).
\end{aligned}$$

We have proved exponential bounds on this second term.

We can then easily conclude the proof of (4.5.17b)

• **Proof of (4.5.17c):**

We consider $n \in \mathbb{N}\backslash\{0\}$ and $x \in \left[\frac{n}{2}, 2n\right]$. We start by observing that

$$\forall x \in \left[\frac{n}{2}, 2n\right], \quad \left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}} \leq n. \tag{4.A.5}$$

Thus, obtaining exponential bounds on certain terms when $x \in [\frac{n}{2}, 2n]$ would also allow to conclude on the proof of (4.5.17c).

We will now follow a strategy developed in [ZH98] in a continuous setting, which has also been used in [God03; CF22; CF23; Coe22; Coe24] in the discrete case, and introduce a family of parameterized curves.

We recall that we introduced in (4.5.8) the function Ψ defined by

$$\forall \tau_p \in \mathbb{R}, \quad \Psi(\tau_p) := \tau_p - A_R{\tau_p}^{2\mu}.$$

and that we chose ε small enough so that the function Ψ is continuous and strictly increasing on $]-\infty, \varepsilon]$. We can therefore introduce for $\tau_p \in [-\eta, \varepsilon]$ the curve Γ_p defined by

$$\Gamma_p := \left\{\tau \in \mathbb{C}, -\eta \leq \Re(\tau) \leq \tau_p, \quad \Re(\tau) - A_R\Re(\tau)^{2\mu} + A_I\mathrm{Im}\,(\tau)^{2\mu} = \Psi(\tau_p)\right\}.$$

It is a symmetric curve with respect to the axis $\mathbb{R}$ which intersects this axis on the point τ_p. If we introduce $\ell_p = \left(\frac{\Psi(\tau_p)-\Psi(-\eta)}{A_I}\right)^{\frac{1}{2\mu}}$, then $-\eta + i\ell_p$ and $-\eta - i\ell_p$ are the end points of Γ_p. We can also introduce a parametrization of this curve by defining $\gamma_p : [-\ell_p, \ell_p] \to \mathbb{C}$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall t \in [-\ell_p, \ell_p], \quad \mathrm{Im}\,(\gamma_p(t)) = t, \quad \Re(\gamma_p(t)) = h_p(t) := \Psi^{-1}\left(\Psi(\tau_p) - A_I t^{2\mu}\right). \tag{4.A.6}$$

The above parametrization immediately yields that there exists a constant $C > 0$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall t \in [-\ell_p, \ell_p], \quad |h_p'(t)| \leq C. \tag{4.A.7}$$

Also, there exists a constant $c_p > 0$ such that

$$\forall \tau_p \in [-\eta, \varepsilon], \forall \tau \in \Gamma_p, \quad \Re(\tau) - \tau_p \leq -c_p\mathrm{Im}\,(\tau)^{2\mu}. \tag{4.A.8}$$

For $\tau \in \Gamma_p$, it follows from (4.A.8) that

$$n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right) \leq -nc_p\mathrm{Im}\,(\tau)^{2\mu} + (n-x)\,\tau_p + xA_R\tau_p^{2\mu}. \tag{4.A.9}$$

There remains to make an appropriate choice of τ_p depending on n and x that minimizes the right-hand side of the inequality (4.A.9) whilst the paths Γ_p have to remain within the ball $B(0, \varepsilon)$. We recall that when we fixed our choice of width η, we defined a radius $\varepsilon_\# \in]0, \varepsilon[$ such that $-\eta + il_{extr} \in B(0, \varepsilon)$ where the real number l_{extr} is defined by (4.5.13). This implies that the curve Γ_p associated with $\tau_p = \varepsilon_\#$ intersects the axis $-\eta + i\mathbb{R}$ within $B(0, \varepsilon)$. We let

$$\zeta = \frac{x-n}{2\mu n}, \quad \gamma = \frac{xA_R}{n}, \quad \rho\left(\frac{\zeta}{\gamma}\right) = \mathrm{sgn}(\zeta)\left(\frac{|\zeta|}{\gamma}\right)^{\frac{1}{2\mu-1}}.$$

We observe that the condition $x \geq \frac{n}{2}$ implies
$$\gamma \geq \frac{A_R}{2}. \tag{4.A.10}$$
Coming back to inequality (4.A.9), we now have
$$n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right) \leq -nc_p\mathrm{Im}\,(\tau)^{2\mu} + n(\gamma\tau_p^{2\mu} - 2\mu\zeta\tau_p). \tag{4.A.11}$$
Our limiting estimates will come from the case where ζ is close to 0.

Then, we take
$$\tau_p := \begin{cases} \rho\left(\frac{\zeta}{\gamma}\right), & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) \in [-\frac{\eta}{2}, \varepsilon_\#], \quad (\text{Case } \mathbf{A}) \\ \varepsilon_\#, & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) > \varepsilon_\#, \quad (\text{Case } \mathbf{B}) \\ -\frac{\eta}{2}, & \text{if } \rho\left(\frac{\zeta}{\gamma}\right) < -\frac{\eta}{2}. \quad (\text{Case } \mathbf{C}) \end{cases}$$
The case **A** corresponds to the choice to minimize the right-hand side of (4.A.11) since $\rho\left(\frac{\zeta}{\gamma}\right)$ is the unique real root of the polynomial
$$\gamma X^{2\mu-1} = \zeta.$$
The cases **B** and **C** allow the path Γ_p to stay within $B(0,\varepsilon)$.

We now define the paths:
$$\begin{aligned}\Gamma_{p,res} &:= \{-\eta + it, \quad t \in [-r_\varepsilon(\eta), -\ell_p] \cup [\ell_p, r_\varepsilon(\eta)]\}, \\ \Gamma_{p,in} &:= \Gamma_p \cup \Gamma_{p,res},\end{aligned}$$
where the function r_ε is defined by (4.5.9). We observe that $\Gamma_{p,in}$ belongs to the set of paths X. We will decompose the integral
$$\int_{\Gamma_{p,in}} |\tau|^k \exp(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right))|d\tau|$$
using the paths Γ_p and $\Gamma_{p,res}$ and we will then bound each term.

▶ Let us assume that x and n are such that we are in Case **A**. Since $\tau_p = \rho\left(\frac{\zeta}{\gamma}\right)$ is the unique root of $\gamma X^{2\mu-1} - \zeta$, we have:
$$\gamma\tau_p^{2\mu} - 2\mu\zeta\tau_p = -(2\mu-1)\gamma\tau_p^{2\mu} \leq 0. \tag{4.A.12}$$
Thus, the inequality (4.A.11) becomes for $\tau \in \Gamma_p$
$$n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right) \leq -nc_p\mathrm{Im}\,(\tau)^{2\mu} - (2\mu-1)\gamma n\tau_p^{2\mu}.$$
Therefore, we have
$$\begin{aligned}\int_{\Gamma_p} &|\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau| \\ &\leq \int_{\Gamma_p} |\tau|^k \exp\left(-nc_p\mathrm{Im}\,(\tau)^{2\mu}\right)|d\tau| \exp\left(-(2\mu-1)\gamma n\tau_p^{2\mu}\right).\end{aligned}$$
Using the parametrization (4.A.6) and the inequality (4.A.7), we have that
$$\int_{\Gamma_p} |\tau|^k \exp\left(-nc_p\mathrm{Im}\,(\tau)^{2\mu}\right)|d\tau| \lesssim \int_{-\ell_p}^{-\ell_p} (|\tau_p|^k + t^k)e^{-nc_pt^{2\mu}}dt.$$
The change of variables $u = n^{\frac{1}{2\mu}}t$ and the fact that the functions $y \geq 0 \mapsto y^k \exp\left(-\frac{2\mu-1}{2}\gamma y^{2\mu}\right)$ are uniformly bounded with respect to $\gamma \geq \frac{A_R}{2}$ imply
$$\begin{aligned}\int_{-\ell_p}^{-\ell_p} |t|^k e^{-nc_pt^{2\mu}}dt &\lesssim \frac{1}{n^{\frac{k+1}{2\mu}}} \\ \int_{-\ell_p}^{-\ell_p} |\tau_p|^k e^{-nc_pt^{2\mu}}dt &\lesssim \frac{1}{n^{\frac{k+1}{2\mu}}}\exp\left(\frac{2\mu-1}{2}\gamma n\tau_p^{2\mu}\right).\end{aligned}$$

Thus,

$$\int_{\Gamma_p} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau| \lesssim \frac{1}{n^{\frac{k+1}{2\mu}}}\exp\left(-\frac{2\mu-1}{2}\gamma n\tau_p^{2\mu}\right).$$

Furthermore, since we are in the Case **A**

$$-\frac{2\mu-1}{2}\gamma n\tau_p^{2\mu} = -\frac{2\mu-1}{2(2\mu A_R)^{\frac{2\mu}{2\mu-1}}}A_R\left(\frac{|n-x|}{x^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}.$$

Therefore, there exist two positive constants C, c independent from n and x such that if we are in Case **A**,

$$\int_{\Gamma_p} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau| \leq \frac{C}{n^{\frac{k+1}{2\mu}}}\exp\left(-c\left(\frac{|n-x|}{x^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Since $x \in [\frac{n}{2}, 2n]$, this gives us two new constants C, c independent from n and x such that if we are in Case **A** and $x \in [\frac{n}{2}, 2n]$, then as expected

$$\int_{\Gamma_p} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau| \leq \frac{C}{n^{\frac{k+1}{2\mu}}}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

▶ Let us assume that x and n are such that we are in Case **B**. Since $\tau_p = \varepsilon_\# < \rho\left(\frac{\zeta}{\gamma}\right)$, we have

$$-\zeta \leq -\gamma\varepsilon_\#^{2\mu-1}$$

and thus using (4.A.10)

$$\gamma\tau_p^{2\mu} - 2\mu\zeta\tau_p \leq -(2\mu-1)\gamma\varepsilon_\#^{2\mu} \leq -\frac{2\mu-1}{2}A_R\varepsilon_\#^{2\mu}. \tag{4.A.13}$$

Therefore, the inequality (4.A.11) becomes for $\tau \in \Gamma_p$

$$\begin{aligned} n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right) &\leq -nc_p\mathrm{Im}\,(\tau)^{2\mu} - \frac{2\mu-1}{2}A_Rn\varepsilon_\#^{2\mu} \\ &\leq -\frac{2\mu-1}{2}\varepsilon_\#^{2\mu}A_Rn. \end{aligned}$$

We conclude that there exist two positive constants C, c independent from n and x such that if we are in Case **B**,

$$\int_{\Gamma_p} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau| \leq Ce^{-cn}.$$

Using (4.A.5) if necessary, we obtain the bound expected in the statement of the lemma.

▶ Let us assume that x and n are such that we are in Case **C**. Since $\tau_p = -\frac{\eta}{2} > \rho\left(\frac{\zeta}{\gamma}\right)$, we have

$$\zeta \leq -\gamma\left(\frac{\eta}{2}\right)^{2\mu-1}$$

and thus using (4.A.10)

$$\gamma\tau_p^{2\mu} - 2\mu\zeta\tau_p = \gamma\left(\frac{\eta}{2}\right)^{2\mu} + 2\mu\zeta\frac{\eta}{2} \leq -(2\mu-1)\gamma\left(\frac{\eta}{2}\right)^{2\mu} \leq -\frac{2\mu-1}{2}A_R\left(\frac{\eta}{2}\right)^{2\mu}. \tag{4.A.14}$$

Therefore, the inequality (4.A.11) becomes for $\tau \in \Gamma_p$

$$\begin{aligned} n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right) &\leq -nc_p\mathrm{Im}\,(\tau)^{2\mu} - \frac{2\mu-1}{2}A_Rn\left(\frac{\eta}{2}\right)^{2\mu} \\ &\leq -\frac{2\mu-1}{2}\left(\frac{\eta}{2}\right)^{2\mu}A_Rn. \end{aligned}$$

We then conclude that there exist two positive constants C, c independent from n and x such that if we are in Case **C**,

$$\int_{\Gamma_p} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right)|d\tau| \leq Ce^{-cn}.$$

Using (4.A.5) if necessary, we obtain the bound expected in the statement of the lemma.

▶ We recall that $-\eta \pm \ell_p$ belongs to Γ_p. For $\tau \in \Gamma_{p,res}$, we have that

$$\Re(\tau) = -\eta \quad \text{and} \quad |\mathrm{Im}\,(\tau)| \geq \ell_p.$$

Thus,

$$\begin{aligned} n\Re(\tau) + x(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}) &\leq -n\eta + x(\eta + A_R\eta^{2\mu} - A_I\ell_p^{2\mu}) \\ &\leq -n\eta + x(-\tau_p + A_R\tau_p^{2\mu}) \\ &\leq -n(\eta + \tau_p) + n(\gamma\tau_p^{2\mu} - 2\mu\zeta\tau_p). \end{aligned}$$

In each cases **A**, **B** and **C**, we have that

$$\eta + \tau_p \geq \frac{\eta}{2} \quad \text{and} \quad \gamma\tau_p^{2\mu} - 2\mu\zeta\tau_p \leq 0.$$

Therefore, for all $\tau \in \Gamma_{p,res}$,

$$n\Re(\tau) + x(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}) \leq -n\frac{\eta}{2}.$$

We then conclude that

$$\int_{\Gamma_{p,res}} |\tau|^k \exp\left(n\Re(\tau) + x\left(-\Re(\tau) + A_R\Re(\tau)^{2\mu} - A_I\mathrm{Im}\,(\tau)^{2\mu}\right)\right) |d\tau| \leq 2\pi\varepsilon^k e^{-n\frac{\eta}{2}}.$$

Using (4.A.5) if necessary, we obtain the bound expected in the statement of the lemma.

Combining all the results we encountered, we easily conclude the proof of (4.5.17c).

Proof of Lemma 4.5.6

We will prove the statement of Lemma 4.5.6 with $? = ?' = +$ in order to alleviate the notations.

• **Proof of (4.5.18a):**

▶ We start by defining the paths

$$\Gamma_0 := \{it, t \in [-r_\varepsilon(\eta), r_\varepsilon(\eta)]\} \quad \text{and} \quad \Gamma_{comp}^{\pm} := \{t \pm ir_\varepsilon(\eta), t \in [-\eta, 0]\}.$$

For all $\Gamma \in X$, Cauchy's formula implies that

$$\begin{aligned} &\left| \int_\Gamma \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau - \int_{\Gamma_0} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau \right| \\ \leq &\left| \int_{\Gamma_{comp}^+} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau \right| + \left| \int_{\Gamma_{comp}^-} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau \right|. \end{aligned}$$

We observe that (4.5.6) implies that

$$\begin{aligned} &\left| \int_{\Gamma_{comp}^{\pm}} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau \right| \\ &\qquad \leq \int_{-\eta}^{0} \exp\left((n - (x+y))t + (x+y)A_Rt^{2\mu} - (x+y)A_Ir_\varepsilon(\eta)^{2\mu}\right) dt. \end{aligned}$$

Using (4.5.12) since $t \in [-\eta, 0]$ and $x + y \in \left[\frac{n}{2}, 2n\right]$, we have that

$$(n - (x+y))t + (x+y)A_Rt^{2\mu} - (x+y)A_Ir_\varepsilon(\eta)^{2\mu} \leq -\frac{A_Ir_\varepsilon(\eta)^{2\mu}}{4}n.$$

Combining the observations above, we have thus proved that for all path $\Gamma \in X$, $n \in \mathbb{N}\backslash\{0\}$, $x, y \in [0, +\infty[$ such that $x + y \in \left[\frac{n}{2}, 2n\right]$

$$\left| \int_\Gamma \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau - \int_{\Gamma_0} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau \right|$$

$$\leq 2\eta \exp\left(-\frac{A_I r_\varepsilon(\eta)^{2\mu}}{4}n\right). \quad (4.A.15)$$

► Since $x+y \geq \frac{n}{2} \geq \frac{1}{2}$, we observe that

$$\int_{r_\varepsilon(\eta)}^{+\infty} \exp\left(-(x+y)A_I t^{2\mu}\right) dt \leq \exp\left(-\frac{A_I r_\varepsilon(\eta)^{2\mu}}{4}n\right) \int_{r_\varepsilon(\eta)}^{+\infty} \exp\left(-\frac{A_I t^{2\mu}}{4}\right) dt.$$

Therefore, if we introduce the path

$$\Gamma_0^\infty := \{it, t \in \mathbb{R}\}$$

then, using (4.5.6), the integral

$$\int_{\Gamma_0^\infty} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau$$

is defined and we have that

$$\left|\int_{\Gamma_0} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau - \int_{\Gamma_0^\infty} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau\right|$$
$$\leq 2\int_{r_\varepsilon(\eta)}^{+\infty} \exp\left(-\frac{A_I t^{2\mu}}{4}\right) dt \exp\left(-\frac{A_I r_\varepsilon(\eta)^{2\mu}}{4}n\right). \quad (4.A.16)$$

► Using the change of variables $u = \frac{n^{\frac{1}{2\mu}}}{\alpha_l^+}t$, we obtain

$$\frac{1}{2i\pi}\int_{\Gamma_0^\infty} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau) + y\alpha_{l'}^+\varphi_{l'}^+(\tau)\right) d\tau = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(it(n-(x+y)) - \left(\frac{x\beta_l^+}{{\alpha_l^+}^{2\mu}} + \frac{y\beta_{l'}^+}{{\alpha_{l'}^+}^{2\mu}}\right)t^{2\mu}\right) dt$$
$$= \frac{|\alpha_l^+|}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\frac{x}{n}\beta_l^+ + \frac{y}{n}\beta_{l'}^+\left(\frac{\alpha_l^+}{\alpha_{l'}^+}\right)^{2\mu}; \frac{\alpha_l^+(n-(x+y))}{n^{\frac{1}{2\mu}}}\right).$$

Combining (4.A.15), (4.A.16) and the observation above, we obtain the inequality (4.5.18a).

• **Proof of (4.5.18b):**

We observe using the change of variables $u = \frac{x^{\frac{1}{2\mu}}}{\alpha_l^+}t$ that

$$\frac{1}{2i\pi}\int_{\Gamma_0^\infty} \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right) d\tau = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(it(n-x) - \frac{x\beta_l^+}{{\alpha_l^+}^{2\mu}}t^{2\mu}\right) dt$$
$$= \frac{|\alpha_l^+|}{x^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right).$$

Combining (4.A.15), (4.A.16) and the observation above, we have proved that there exist two constants $C, c > 0$ such that

$$\forall x \in \left[\frac{n}{2}, 2n\right], \quad \left|\frac{1}{2i\pi}\int_\Gamma \exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right) d\tau - \frac{|\alpha_l^+|}{x^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right)\right| \leq Ce^{-cn}. \quad (4.A.17)$$

Using (4.A.5), we can obtain the same generalized Gaussian bound as the one expected in (4.5.18b).

► We observe that

$$\frac{1}{x^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) - \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right)$$
$$= \frac{1}{x^{\frac{1}{2\mu}}}\left(H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) - H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right)\right) + H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right)\left(\frac{1}{x^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}}\right). \quad (4.A.18)$$

We want to prove generalized Gaussian bounds for the two terms on the right hand side of (4.A.18). Applying the mean value inequality and (4.1.30a), we have that there exist two constants $C, c > 0$ such that for all

$x \in [\frac{n}{2}, 2n]$

$$\left|\frac{1}{x^{\frac{1}{2\mu}}}\left(H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) - H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right)\right)\right| \\ \leq \frac{C}{n^{\frac{1}{2\mu}}}|n-x|\left|\frac{1}{x^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}}\right| \exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Since $x \in \left[\frac{n}{2}, 2n\right]$, we also have using the mean value inequality that

$$\left|\frac{1}{x^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}}\right| \leq \frac{|n-x|}{2\mu} \sup_{t\in[x,n]} \frac{1}{|t|^{1+\frac{1}{2\mu}}} \leq \frac{2^{\frac{1}{2\mu}}}{\mu}\frac{|n-x|}{n^{1+\frac{1}{2\mu}}}. \tag{4.A.19}$$

Therefore, since the function $y \mapsto y^2 \exp\left(-\frac{c}{2}y^{\frac{2\mu}{2\mu-1}}\right)$ is bounded, there exist two new constants $C, c > 0$ such that for all $x \in [\frac{n}{2}, 2n]$

$$\left|\frac{1}{x^{\frac{1}{2\mu}}}\left(H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) - H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right)\right)\right| \leq \frac{C}{n}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{4.A.20}$$

We have thus proved generalized Gaussian bounds for the first term of the right hand side in (4.A.18). We now focus on the second term. Using (4.1.30a), (4.A.19) and the fact that, for any constant $c > 0$, the function $y \mapsto y \exp\left(-cy^{\frac{2\mu}{2\mu-1}}\right)$ is bounded, we have that there exist two constants $C, c > 0$ such that

$$\left|H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right)\left(\frac{1}{x^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}}\right)\right| \leq \frac{C}{n}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \tag{4.A.21}$$

Combining (4.A.18), (4.A.20) and (4.A.21), we have proved generalized Gaussian bounds for the difference

$$\frac{1}{x^{\frac{1}{2\mu}}}H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{x^{\frac{1}{2\mu}}}\right) - \frac{1}{n^{\frac{1}{2\mu}}}H_{2\mu}\left(\beta_l^+; \frac{\alpha_l^+(n-x)}{n^{\frac{1}{2\mu}}}\right).$$

With (4.A.17), we easily conclude the proof of (4.5.18b).

• **Proof of (4.5.18c):**

▶ We observe that (4.5.6) implies that

$$\left|\int_{\Gamma_{in}^{\pm}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau}d\tau\right| \leq \frac{1}{r_\varepsilon(\eta)}\int_{-\eta}^{\eta}\exp\left((n-x)t + xA_Rt^{2\mu} - xA_Ir_\varepsilon(\eta)^{2\mu}\right)dt.$$

Using (4.5.12) since $t \in [-\eta, \eta]$ and $x \in \left[\frac{n}{2}, 2n\right]$, we have that

$$(n-x)t + xA_Rt^{2\mu} - xA_Ir_\varepsilon(\eta)^{2\mu} \leq -\frac{A_Ir_\varepsilon(\eta)^{2\mu}}{4}n.$$

Using the observations above and Cauchy's formula, we have thus proved that for all $n \in \mathbb{N}\backslash\{0\}$, $x \in \left[\frac{n}{2}, 2n\right]$ and paths $\Gamma \in X$

$$\left|\int_{\Gamma_{in}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau}d\tau - \int_{\Gamma_{in}^0(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau}d\tau\right| \leq \frac{4\eta}{r_\varepsilon(\eta)}\exp\left(-\frac{A_Ir_\varepsilon(\eta)^{2\mu}}{4}n\right). \tag{4.A.22}$$

▶ Since $x \geq \frac{n}{2} \geq \frac{1}{2}$, we observe that

$$\int_{r_\varepsilon(\eta)}^{+\infty} \frac{\exp\left((n-x)\eta + xA_R\eta^{2\mu} - xA_It^{2\mu}\right)}{|\eta + it|}dt \\ \leq \underbrace{\frac{1}{\eta}\int_{r_\varepsilon(\eta)}^{+\infty}\exp\left(-\frac{A_It^{2\mu}}{8}\right)dt}_{<+\infty}\exp\left((n-x)\eta + xA_R\eta^{2\mu} - x\frac{3}{4}A_Ir_\varepsilon(\eta)^{2\mu}\right).$$

Furthermore, using (4.5.12) since $x \in \left[\frac{n}{2}, 2n\right]$, we have that

$$\exp\left((n-x)\eta + xA_R\eta^{2\mu} - x\frac{3}{4}A_I r_\varepsilon(\eta)^{2\mu}\right) \leq \exp\left(-\frac{A_I r_\varepsilon(\eta)^{2\mu}}{8}n\right).$$

Therefore, if we introduce the path

$$\Gamma_{in}^{\infty}(\eta) := \{\eta + it, t \in \mathbb{R}\}$$

then, using (4.5.6), the integral

$$\int_{\Gamma_{in}^{\infty}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau} d\tau$$

is defined and we have that

$$\left|\int_{\Gamma_{in}^{0}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau} d\tau - \int_{\Gamma_{in}^{\infty}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau} d\tau\right|$$
$$\leq \frac{2}{\eta}\int_{r_\varepsilon(\eta)}^{+\infty} \exp\left(-\frac{A_I t^{2\mu}}{8}\right) dt \exp\left(-\frac{A_I r_\varepsilon(\eta)^{2\mu}}{8}n\right). \quad (4.A.23)$$

► We observe that using the change of variables $t = -\frac{|\alpha_l^+|}{x^{\frac{1}{2\mu}}}u$, we have

$$\frac{1}{2i\pi}\int_{\Gamma_{in}^{\infty}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau} d\tau = \frac{1}{2i\pi}\int_{-\infty}^{+\infty} \frac{\exp\left(i(n-x)(t-i\eta) - x\frac{\beta_l^+}{\alpha_l^{+2\mu}}(\eta+it)^{2\mu}\right)}{t-i\eta} dt$$
$$= -\frac{1}{2i\pi}\int_{-\infty}^{+\infty} \frac{\exp\left(i\frac{-|\alpha_l^+|(n-x)}{x^{\frac{1}{2\mu}}}\left(u + i\frac{x^{\frac{1}{2\mu}}\eta}{|\alpha_l^+|}\right) - \beta_l^+\left(u + i\frac{x^{\frac{1}{2\mu}}\eta}{|\alpha_l^+|}\right)^{2\mu}\right)}{u + i\frac{x^{\frac{1}{2\mu}}\eta}{|\alpha_l^+|}} du.$$

Furthermore, we can prove that

$$\forall s \in]0, +\infty[, \forall x \in \mathbb{R}, \quad -\frac{1}{2i\pi}\int_{-\infty}^{+\infty} \frac{\exp\left(ix(u+is) - \beta_l^+(u+is)^{2\mu}\right)}{u+is} du = E_{2\mu}\left(\beta_l^+; x\right).$$

The proof is done in [Coe24, (5.65)]. Therefore,

$$\frac{1}{2i\pi}\int_{\Gamma_{in}^{\infty}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau} d\tau = E_{2\mu}\left(\beta_l^+; \frac{-|\alpha_l^+|(n-x)}{x^{\frac{1}{2\mu}}}\right).$$

Combining this observation with (4.A.22), (4.A.23) and (4.A.5), we have that there exist two positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$ and $x \in \left[\frac{n}{2}, 2n\right]$

$$\left|\frac{1}{2i\pi}\int_{\Gamma_{in}(\eta)} \frac{\exp\left(n\tau + x\alpha_l^+\varphi_l^+(\tau)\right)}{\tau} d\tau - E_{2\mu}\left(\beta_l^+; \frac{-|\alpha_l^+|(n-x)}{x^{\frac{1}{2\mu}}}\right)\right| \leq \frac{C}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

► We notice that $\partial_x E_{2\mu}\left(\beta_l^+; \cdot\right) = -H_{2\mu}\left(\beta_l^+; \cdot\right)$. Therefore, we have using the mean value inequality and (4.1.30a) that there exist two positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$ and $x \in \left[\frac{n}{2}, 2n\right]$

$$\left|E_{2\mu}\left(\beta_l^+; \frac{-|\alpha_l^+|(n-x)}{x^{\frac{1}{2\mu}}}\right) - E_{2\mu}\left(\beta_l^+; \frac{-|\alpha_l^+|(n-x)}{n^{\frac{1}{2\mu}}}\right)\right| \leq C|n-x|\left|\frac{1}{x^{\frac{1}{2\mu}}} - \frac{1}{n^{\frac{1}{2\mu}}}\right| \exp\left(-c\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Using (4.A.19) and the fact that $y \mapsto y^2 \exp\left(-\frac{c}{2}y^{\frac{2\mu}{2\mu-1}}\right)$ is bounded, we have that there exist two new positive constants C, c such that for all $n \in \mathbb{N}\backslash\{0\}$ and $x \in \left[\frac{n}{2}, 2n\right]$

$$\left|E_{2\mu}\left(\beta_l^+; \frac{-|\alpha_l^+|(n-x)}{x^{\frac{1}{2\mu}}}\right) - E_{2\mu}\left(\beta_l^+; \frac{-|\alpha_l^+|(n-x)}{n^{\frac{1}{2\mu}}}\right)\right| \leq \frac{C}{n^{1-\frac{1}{2\mu}}}\exp\left(-\frac{c}{2}\left(\frac{|n-x|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

This allows us to conclude the proof of (4.5.18c).

Chapter 5

Nonlinear orbital stability of discrete shock profiles for scalar conservation laws

Outline of the current chapter

Notations

In addition to the notation of the previous chapter, we introduce for $\gamma \in [0, +\infty[$ the polynomial-weighted spaces ℓ^r_γ and their norms, as well as the space $\mathbb{E}_\gamma$ of zero-mass elements of ℓ^1_γ :

$$\forall r \in [1, +\infty], \quad \ell^r_\gamma := \left\{ (h_j)_{j\in\mathbb{Z}} \in \mathbb{C}^\mathbb{Z}, \quad \left((1+|j|)^\gamma h_j\right)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z}) \right\}, \tag{5.0.1a}$$

$$\forall r \in [1, +\infty], \forall h \in \ell^r_\gamma, \quad \|h\|_{\ell^r_\gamma} := \left\| \left((1+|j|)^\gamma h_j\right)_{j\in\mathbb{Z}} \right\|_{\ell^r}, \tag{5.0.1b}$$

$$\mathbb{E}_\gamma := \left\{ h \in \ell^1_\gamma, \quad \sum_{j\in\mathbb{Z}} h_j = 0 \right\}. \tag{5.0.1c}$$

Let us observe that in this chapter, the sequences are not vector valued but only complex valued since we will work on discrete shock profiles for *scalar* conservation laws.

5.1 Introduction

5.1.1 Position of the problem

The present chapter is a direct continuation of the previous one. The main goal is to prove Theorem 5.1 below which claims that, in the scalar case (i.e. $d = 1$ in Chapter 4), under Hypotheses 4.1-4.9, the stationary discrete shock profile (SDSP) $\overline{u}^s$ introduced in Hypothesis 4.2 is nonlinearly asymptotically stable with respect to small polynomial-weighted zero-mass perturbations. To detail the previous statement, we recall that the SDSP $\overline{u}^s$ is a fixed point of the discrete evolution operator $\mathcal{N}$ defined by (4.1.7) and is thus a stationary solution of the numerical scheme (4.1.8) that we recall here:

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n), \tag{5.1.1}$$

where $u^0 \in \mathcal{U}^{\mathbb{Z}}$. The claim of Theorem 5.1 is that for a positive constant γ large enough and a small enough perturbation h of mass zero (i.e. $\sum_{j\in\mathbb{Z}} h_j = 0$) in the polynomial-weighted space $\mathbb{E}_\gamma$ defined by (5.0.1c), the solution $(u^n)_{n\in\mathbb{N}}$ of the numerical scheme (5.1.1) initialized using $u^0 := \overline{u}^s + h$ is well-defined for all time $n \in \mathbb{N}$ and will converge in the ℓ^r-norm towards the SDSP $\overline{u}^s$ as n tends towards $+\infty$ for all $r \in [1, +\infty]$. Furthermore, we determine some convergence rate and prove that stronger choices of weights γ imply faster convergence of the sequence u^n towards $\overline{u}^s$ as n tends towards $+\infty$.

Theorem 5.1 is proved using the results of Chapter 4 on the Green's function associated with the operator $\mathscr{L}$ obtained by linearizing the discrete evolution operator $\mathcal{N}$ about the SDSP $\overline{u}^s$. Since the results of Chapter 4 hold in the system case, we hope that we could extend the result of Theorem 5.1 in the system case. We also point out that the main result of this chapter, though it is proved for zero-mass perturbations, could be extended, in our opinion, for general nonzero mass perturbation, see Paragraph 5.1.4 below.

Let us briefly compare the result of Theorem 5.1 with the nonlinear orbital stability results of [Jen74; Smy90; Mic02]:

— In [Jen74, Theorem 2], Jennings proves a nonlinear orbital stability result that applies for moving discrete shock profiles with rational speed associated with Lax shocks for perturbations in $\ell^1(\mathbb{Z})$ without any zero-mass assumption. However, it is asked that the initial condition $u^0 := \overline{u}^s + h$ must stay within the interval defined by the two states of the shock, and, more importantly, the result only applies to *monotone* schemes, a specific family of order 1 schemes. Theorem 5.1 applies to a larger family of schemes, in particular to higher order schemes, and more importantly our result might be generalized in the system case.

— The result [Smy90, Theorem 4.1] applies in the scalar case, for the Lax-Wendroff scheme and for discrete shock profiles associated with stationary Lax shocks. It proves a result of nonlinear orbital stability for exponentially-weighted perturbations. Though Theorem 5.1 does not apply for the Lax-Wendroff scheme (this scheme does not satisfy Hypothesis 4.5), we observe that we impose less stringent weights on the initial condition. We pass from exponential to polynomial weights for the initial perturbation. This is due to the careful analysis of the Green's function performed in Chapter 4. The spectral configuration that we deal with is more subtle than the one in [Smy90] where the exponential weight yields a spectral gap.

— The strength of the result of [Mic02] is that it is a nonlinear orbital stability result that applies in the system case to SDSPs for a large family of odd order numerical schemes. However, it imposes a weakness assumption on the considered shock, meaning that the amplitude of the shock must be fairly small. This assumption is dropped in Theorem 5.1 and replaced with a spectral stability assumption that is sharper.

5.1.2 Improving the results of Chapter 4 in the scalar case

We consider the scalar case, i.e. $d = 1$ in the notations of Chapter 4. As a consequence, we will slightly modify some of the notations used in the previous chapter. For instance, since we are in the scalar case, the jacobian matrices $df(u^\pm)$ are now scalars $f'(u^\pm)$ and each of the left and right states $u^\pm$ only have one characteristic field associated with them. We will thus adapt the notations used in Chapter 4 in this case and drop the indices of some constants. For instance, we define:

$$\boldsymbol{\lambda}^\pm := f'(u^\pm), \quad \alpha^\pm := \boldsymbol{\nu}\boldsymbol{\lambda}^\pm \quad \text{and} \quad \boldsymbol{\nu} = \frac{\Delta t}{\Delta x}.$$

Hypothesis 4.1 then implies that the characteristics associated with the states u^+ and u^- are entering the shock and thus that:

$$\alpha^+ < 0 \quad \text{and} \quad \alpha^- > 0.$$

Let us start to discuss on some improvements we can make in the scalar case on the results presented in

Chapter 4 surrounding the temporal Green's function $\mathscr{G}(n,j_0,j)$ defined by (4.1.25). Since there is only one characteristic for each state u^+ and u^- and that both characteristics are incoming with respect to the shock, the description of the temporal Green's function $\mathscr{G}(n,j_0,j)$ given by Theorem 4.1 is far simpler. More precisely, there exist some constants $c>0$ and $C^{E,+}\in\mathbb{C}$ such that for $j_0\in\mathbb{N}$, $n\in\mathbb{N}\backslash\{0\}$ and $j\in\mathbb{Z}$ such that $j-j_0\in\{-nq,\dots,np\}$:

$$\mathscr{G}(n,j_0,j)=\left(C^{E,+}E_{2\mu}\left(\beta^+;\frac{n\alpha^++j_0}{n^{\frac{1}{2\mu}}}\right)+P_U^+(j_0)\right)V(j)+\mathbb{1}_{j\geq 0}O\left(\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)$$
$$+O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)+O(e^{-cn}) \quad (5.1.2)$$

where the function $E_{2\mu}$ and the constant β^+ are respectively defined by (4.1.28) and (4.1.18) and the sequences P_U^+ and V are defined in Theorem 4.1. We observe that there are no reflected or transmitted waves and only one generalized Gaussian wave entering the shock and activating the profile V. A similar description of the temporal Green's function holds for $j_0\in-\mathbb{N}\backslash\{0\}$.

Let us now present an improvement of (5.1.2) in this scalar case.

Proposition 5.1. *In the scalar case $d=1$, under Hypotheses 4.1-4.9, the two following assertions are verified:*

— *The sequences P_U^+ and P_U^- defined in Theorem 4.1 are equal to zero.*

— *We have that the sequence V defined in Theorem 4.1 satisfies:*

$$\sum_{j\in\mathbb{Z}}V(j)\neq 0 \quad \text{and} \quad C^{E,+}=C^{E,-}=\frac{1}{\sum_{j\in\mathbb{Z}}V(j)}.$$

We will denote the constants $C^{E,+}$ and $C^{E,-}$ as C^E from now on.

This proposition will be proved in Section 5.3. Let us here give a brief overview of the main ideas of the proof. It relies on the conservative nature of the numerical scheme we consider. Using the decomposition (5.1.2) of the Green's function $\mathscr{G}(n,j_0,\cdot)$ for $j_0\in\mathbb{N}$, the Green's function converges formally towards the sequence $\left(C^{E,+}+P_U^+(j_0)\right)V$ as n tends towards $+\infty$. Since the scheme is conservative, the mass of the Green's function $\mathscr{G}(n,j_0,\cdot)$ is constantly equal to 1 for all $n\in\mathbb{N}$ and this is thus also true for the sequence $\left(C^{E,+}+P_U^+(j_0)\right)V$, which will allow us to conclude.

The result of Proposition 5.1 gives us more insight on the activation of the profile V presented in the description (5.1.2) of the Green's function. More precisely, the fact that the sequences P_U^+ and P_U^- are equal to zero implies that the fast modes do not activate the profile V. The activation is only caused by the slow modes (i.e. the generalized Gaussian wave following the characteristic and entering the shock).

Let us point out that Proposition 5.1 can be generalized in the system case under an additional hypothesis (Hypothesis 5.1 below) similar to the well-known Liu-Majda condition. This will be discussed in the Paragraph 5.1.4.

Using Proposition 5.1, the decompositions (4.1.32) and (4.1.33) on the Green's function and their discrete derivative with regards to the parameter j_0 are thus improved in the scalar case. We have that there exists a constant $c>0$ such that for $j_0\in\mathbb{N}$, $n\in\mathbb{N}\backslash\{0\}$ and $j\in\mathbb{Z}$ such that $j-j_0\in\{-nq,\dots,np\}$, the decomposition (4.1.32) on the Green's function gives:

$$\mathscr{G}(n,j_0,j)=C^{E}E_{2\mu}\left(\beta^+;\frac{n\alpha^++j_0}{n^{\frac{1}{2\mu}}}\right)V(j)+\mathbb{1}_{j\geq 0}O\left(\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)$$
$$+O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)+O(e^{-cn}) \quad (5.1.3a)$$

where the sequence V is defined in Theorem 4.1, and for $j_0\in\mathbb{N}$, $n\in\mathbb{N}\backslash\{0\}$ and $j\in\mathbb{Z}$ such that $j-j_0\in\{-nq-1,\dots,np\}$, the decomposition (4.1.33) on the discrete derivative Green's function gives:

$$\mathscr{G}(n,j_0,j)-\mathscr{G}(n,j_0-1,j)$$
$$=\mathbb{1}_{j\geq 0}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\left(O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\right)+O\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\right)+O\left(\frac{1}{n^{\frac{1}{\mu}}}\right)\right)$$

$$+O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)+O(e^{-cn}). \quad (5.1.3b)$$

Similar decompositions hold for $j_0 \in -\mathbb{N}\backslash\{0\}$.

These decompositions (and thus Proposition 5.1) are fundamental to prove the following proposition which presents sharp bounds on the operators $\mathscr{L}^n$ and $\mathscr{L}^n(Id-\mathcal{T})$ acting on the polynomially weighted spaces defined by (5.0.1) where the shift operator $\mathcal{T}$ is defined by:

$$\mathcal{T}:(h_j)_{j\in\mathbb{Z}}\in\mathbb{C}^{\mathbb{Z}}\mapsto(h_{j+1})_{j\in\mathbb{Z}}\in\mathbb{C}^{\mathbb{Z}} \quad (5.1.4)$$

and the operator $\mathscr{L}$ defined by (4.1.19) corresponds to the linearized operator of the numerical scheme about the discrete shock profile $\overline{u}^s$.

Proposition 5.2. *For any $0\leq\gamma_1\leq\gamma_2$, there exists a constant $C_{\mathscr{L}}(\gamma_1,\gamma_2)>0$ such that we have the following estimates on the semigroup $(\mathscr{L}^n)_{n\in\mathbb{N}}$:*

$$\forall n\in\mathbb{N}\backslash\{0\},\forall h\in\mathbb{E}_{\gamma_2},\quad \|\mathscr{L}^n h\|_{\ell^1_{\gamma_1}}\leq\frac{C_{\mathscr{L}}(\gamma_1,\gamma_2)}{n^{\gamma_2-\gamma_1}}\|h\|_{\ell^1_{\gamma_2}}, \quad (5.1.5a)$$

$$\forall n\in\mathbb{N}\backslash\{0\},\forall h\in\mathbb{E}_{\gamma_2},\quad \|\mathscr{L}^n h\|_{\ell^\infty_{\gamma_1}}\leq C_{\mathscr{L}}(\gamma_1,\gamma_2)\left(\frac{1}{n^{\gamma_2-\gamma_1+\frac{1}{2\mu}}}+\frac{1}{n^{\gamma_2}}\right)\|h\|_{\ell^1_{\gamma_2}}, \quad (5.1.5b)$$

and the following estimates on $(\mathscr{L}^n(Id-\mathcal{T}))_{n\in\mathbb{N}}$:

$$\forall n\in\mathbb{N},\forall h\in\ell^1_{\gamma_2},\quad \|\mathscr{L}^n(Id-\mathcal{T})h\|_{\ell^1_{\gamma_1}}\leq\frac{C_{\mathscr{L}}(\gamma_1,\gamma_2)}{(n+1)^{\gamma_2-\gamma_1+\frac{1}{2\mu}}}\|h\|_{\ell^1_{\gamma_2}}, \quad (5.1.5c)$$

$$\forall n\in\mathbb{N},\forall h\in\ell^1_{\gamma_2},\quad \|\mathscr{L}^n(Id-\mathcal{T})h\|_{\ell^\infty_{\gamma_1}}\leq C_{\mathscr{L}}(\gamma_1,\gamma_2)\left(\frac{1}{(n+1)^{\gamma_2-\gamma_1+\frac{1}{\mu}}}+\frac{1}{(n+1)^{\gamma_2+\frac{1}{2\mu}}}\right)\|h\|_{\ell^1_{\gamma_2}}, \quad (5.1.5d)$$

$$\forall n\in\mathbb{N},\forall h\in\ell^\infty_{\gamma_2},\quad \|\mathscr{L}^n(Id-\mathcal{T})h\|_{\ell^\infty_{\gamma_1}}\leq C_{\mathscr{L}}(\gamma_1,\gamma_2)\left(\frac{1}{(n+1)^{\gamma_2-\gamma_1+\frac{1}{2\mu}}}+\frac{1}{(n+1)^{\gamma_2}}\right)\|h\|_{\ell^\infty_{\gamma_2}}. \quad (5.1.5e)$$

Furthermore, if we assume that $\gamma_1\geq\frac{1}{2\mu}$, the constant $C_{\mathscr{L}}(\gamma_1,\gamma_2)$ can be considered large enough so that (5.1.5b), (5.1.5d) and (5.1.5e) give us:

$$\forall n\in\mathbb{N}\backslash\{0\},\forall h\in\mathbb{E}_{\gamma_2},\quad \|\mathscr{L}^n h\|_{\ell^\infty_{\gamma_1}}\leq\frac{C_{\mathscr{L}}(\gamma_1,\gamma_2)}{n^{\gamma_2-\gamma_1+\frac{1}{2\mu}}}\|h\|_{\ell^1_{\gamma_2}}, \quad (5.1.5f)$$

$$\forall n\in\mathbb{N},\forall h\in\ell^1_{\gamma_2},\quad \|\mathscr{L}^n(Id-\mathcal{T})h\|_{\ell^\infty_{\gamma_1}}\leq\frac{C_{\mathscr{L}}(\gamma_1,\gamma_2)}{(n+1)^{\gamma_2-\gamma_1+\frac{1}{\mu}}}\|h\|_{\ell^1_{\gamma_2}}, \quad (5.1.5g)$$

$$\forall n\in\mathbb{N},\forall h\in\ell^\infty_{\gamma_2},\quad \|\mathscr{L}^n(Id-\mathcal{T})h\|_{\ell^\infty_{\gamma_1}}\leq\frac{C_{\mathscr{L}}(\gamma_1,\gamma_2)}{(n+1)^{\gamma_2-\gamma_1+\frac{1}{2\mu}}}\|h\|_{\ell^\infty_{\gamma_2}}. \quad (5.1.5h)$$

Proposition 5.2 can be seen as an improvement of the linear stability result (Theorem 4.2) of Chapter 4 and will play a central role on proving the main result of this chapter, that is Theorem 5.1.

5.1.3 Main result of the chapter: nonlinear stability in the scalar case for zero-mass perturbations

The main result of this chapter is the following theorem. We recall that the polynomially-weighted spaces and the norms appearing in Theorem 5.1 are defined by (5.0.1).

Theorem 5.1. *We assume that Hypotheses 4.1-4.9 are verified and consider a constant $\boldsymbol{p}\in[0,+\infty[$. Then, there exist two constants $\varepsilon, C_0\in[0,+\infty[$ such that, if we consider a zero-mass initial perturbation $h^0\in\mathbb{E}_{\boldsymbol{p}+\max(1,\boldsymbol{p})}$ such that:*

$$\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\max(1,\boldsymbol{p})}}<\varepsilon$$

then the solution $(u^n)_{n\in\mathbb{N}}$ of the numerical scheme (5.1.1) initialized with the initial condition $u^0:=\overline{u}^s+h^0$ is well-defined for all $n\in\mathbb{N}$. Furthermore, if we define

$$\forall n\in\mathbb{N}\backslash\{0\},\quad h^n:=u^n-\overline{u}^s,$$

then for all $r \in [1, +\infty]$:

$$\forall n \in \mathbb{N} \backslash \{0\}, \quad \|h^n\|_{\ell^r_{\max(1,\boldsymbol{p})}} \leq \frac{C_0}{n^{\boldsymbol{p}+\frac{1}{2\mu}\left(1-\frac{1}{r}\right)}} \left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\max(1,\boldsymbol{p})}}. \tag{5.1.6}$$

In particular, we have that $u^n \underset{n\to+\infty}{\to} \overline{u}^s$ in the vector space $\ell^r_{\max(1,\boldsymbol{p})}$ except for $r = 1$ and $\boldsymbol{p} = 0$. The estimates (5.1.6) are also true for the usual ℓ^r-norms.

Theorem 5.1 partially answers [Ser07, Open Question 5.3] by proving the nonlinear stability of the SDSP $\overline{u}^s$ under the same assumptions as in Chapter 4 in the scalar case and for zero-mass perturbations. There is also hope to extend the result for nonzero mass perturbations and, more importantly, in the system case as will be discussed below in Section 5.1.4. Since we impose for the initial perturbation h^0 to belong to $\mathbb{E}_{\max(1,\boldsymbol{p})+\boldsymbol{p}}$, the polynomial-weight we impose on h^0 increases with $\boldsymbol{p}$. However, the decay rate (5.1.6) also improves as $\boldsymbol{p}$ increases. This means that the stronger the polynomial-weight we impose on the initial perturbation is, the better the decay rate in (5.1.6) is. However, we point out that it is more than likely that the estimate (5.1.6) is not sharp and could be improved. The proof of Theorem 5.1 presented in this chapter makes nonoptimal choices at times. More precisely, we think that we could prove a similar decay rate with less stringent polynomial weights on the initial perturbation but this is left for further study.

Theorem 5.1 and its proof can be seen to some extent as an adaptation in the fully discrete setting of the nonlinear orbital stability result for semi-discrete shock profile [BHR03, Theorem 5.1] which is itself inspired by the papers [ZH98; MZ02]. To briefly summarize the main idea of the proof, we will use a proof by induction. For all $n \in \mathbb{N}$, we will find a way to express the perturbation h^{n+1} using the perturbations h^m and the operators $\mathscr{L}^m$ and $\mathscr{L}^m(Id - \mathcal{T})$ for $m \in \{0, \dots n\}$. The decomposition of the Green's function deduced in Chapter 4 allowed us to prove sharp bounds on those operators (Proposition 5.2). This will allow us to prove inequalities on the sequences h^n by induction.

5.1.4 A few discussions on the results of Chapter 5

In this section, we allow ourselves to discuss fairly quickly about some interesting limitations and possible improvements on Theorem 5.1 and about a quite "unintended" consequence of Proposition 5.1.

Parametrization by excess mass of spectrally stable SDSPs

Let us assume that there exists a continuously differentiable family of SDSPs $(\overline{u}^s_\delta)_{\delta\in]-\varepsilon,\varepsilon[}$ associated with the shock of endstates u^+ and u^-. We also assume that each $\overline{u}^s_\delta$ verifies Hypotheses 4.1-4.9. Let us discuss on the following issue: how can the SDSPs be parametrized (in the scalar case)? We define the differentiable function M:

$$\forall \delta \in]-\varepsilon, \varepsilon[, \quad M(\delta) := \sum_{j\in\mathbb{Z}} \overline{u}^s_{\delta,j} - \overline{u}^s_{0,j}$$

which can be done thanks to Hypothesis 4.3. The interested reader can draw immediate links between the function M and the so-called Y function that is presented in [Ser07]. We claim that we can actually reparametrize the family $(\overline{u}^s_\delta)_{\delta\in]-\varepsilon,\varepsilon[}$ so that:

$$\forall \delta \in]-\varepsilon, \varepsilon[, \quad M(\delta) = \delta$$

i.e. the different SDSPs $\overline{u}^s_\delta$ are parametrized by their mass excess with regards to some reference SDSP $\overline{u}^s_0$. For instance, a fairly similar parametrization has been proved for the SDSPs of the Lax-Wendroff scheme in [Smy90, Theorem 2.1]. Proposition 5.1 allows us to prove such a result. Indeed, the sequence V appearing in Theorem 4.1 (when it is applied to the SDSP $\overline{u}^s_\delta$) is collinear to the sequence $\partial\overline{u}^s_\delta/\partial\delta$. Thus, the result of Proposition 5.1 implies that:

$$\forall \delta \in]-\varepsilon, \varepsilon[, \quad \sum_{j\in\mathbb{Z}} \left(\frac{\partial \overline{u}^s_\delta}{\partial\delta}\right)_j \neq 0$$

and thus

$$\forall \delta \in]-\varepsilon, \varepsilon[, \quad M'(\delta) \neq 0.$$

This would allow us to conclude. This parametrization is essential when discussing about proving nonlinear orbital stability for nonzero mass perturbations as will be discussed below.

The "Shock tracking" issue

We think that there is an interesting limitation of Theorem 5.1 compared to [BHR03, Theorem 5.1] that needs to be pointed out. In Theorem 5.1, we prove decay estimates on the difference between the solution

u^n and the SDSP $\overline{u}^s$. However, in [BHR03, Theorem 5.1], the decay estimates the authors prove are between the solution $v(t)$ of their semi-discrete problem and a well-chosen translation $u(t+\tilde{p}(t))$ of the profile they are considering. Furthermore, this translation depends on time, i.e. they track the position of the shock in time that minimizes the distance to $v(t)$. This is possible due to the continuous in time nature of their problem and the fact they are considering only moving semi-discrete shock profiles. This allows them to not impose any polynomial weight on the initial perturbations, which could be a desirable feature in our framework too.

However, in the fully discrete setting and for the stationary case considered in the present chapter, we have not found a way to adapt what we will here call "shock tracking" techniques. There are several hurdles that we can encounter when trying. For instance, in some fully continuous setting like [ZH98; MZ02], we point out that the linearized operator about a translated profile equals the linearized operator about the original profile conjugated with some translation operators, i.e. translations and linearization commute. This does not hold in the fully discrete setting (or only for translations in $\mathbb{Z}$) and this can create some issues when trying to adapt this shock tracking technique.

In the present chapter, we have overcome those issues by always searching for estimates on $u^n - \overline{u}^s$ and by imposing polynomial weights on the initial perturbation that are not present in [BHR03, Theorem 5.1]. We are interested in finding adaptations of this shock tracking technique in the fully discrete setting.

Discussion on the extension of Theorem 5.1 for nonzero mass perturbations

As stated above, when considering how to extend Theorem 5.1 to nonzero mass perturbations, finding an adaptation of these shock tracking ideas would be the optimal way to solve this issue, though it seems far too complicated for now. Let us pursue another path.

We will use the same notations as in the above Paragraph "Parametrization by excess mass of spectrally stable SDSPs" above. Let us start by writing a formal proof of how an extension of Theorem 5.1 for nonzero mass perturbations could work. We consider an initial perturbation $h \in \ell^1_\gamma$ for some large enough γ. We define:

$$\delta := \sum_{j\in\mathbb{Z}} h_j.$$

Then, under some smallness assumption on h, if we consider the solution $(u^n)_{n\in\mathbb{N}}$ of the numerical scheme (5.1.1) with the initial condition $u^0 := \overline{u}^s_0 + h$, we can expect that the sequence u^n should converge towards $\overline{u}^s_\delta$ due to the conservative nature of the numerical scheme we consider. To prove it, we observe that:

$$u^0 = \overline{u}^s_0 + h = \overline{u}^s_\delta + (h + \overline{u}^s_0 - \overline{u}^s_\delta).$$

Furthermore, the sequence $h + \overline{u}^s_0 - \overline{u}^s_\delta$ belongs to $\mathbb{E}_\gamma$. Indeed, its mass is null and using the fact that h belongs to ℓ^1_γ and Hypothesis 4.3 for both SDSPs $\overline{u}^s_0$ and $\overline{u}^s_\delta$, we have that the sequence $h + \overline{u}^s_0 - \overline{u}^s_\delta$ also belongs to ℓ^1_γ. Using these facts and Theorem 5.1 for $\overline{u}^s_\delta$, we should be able to conclude.

There is only one limit to the above reasoning: It would require some uniformity of the parameter R and ε appearing in Theorem 5.1 with respect to δ. Under the assumption that the description of the Green's function of Theorem 4.1 is uniform with respect to δ, which seems fairly reasonable, the estimates in Proposition 5.2 can also be assumed to be uniform with respects to δ (we would obviously replace the operator $\mathscr{L}$ by the operator $\mathscr{L}_\delta$ obtained by linearizing the numerical scheme about $\overline{u}^s_\delta$). The much desired proof of nonlinear orbital stability for nonzero mass perturbations could then be handled in two ways, where either seem fine in our opinion:

- Either, we use the new uniform version of Proposition 5.2 to prove a version of Theorem 5.1 (i.e. a nonlinear stability result for zero-mass perturbations) that holds for all $\overline{u}^s_\delta$ and where the parameter ε and R are uniform with respect to δ. Following the proof described above would then allow us to conclude on the extension of Theorem 5.1 for nonzero mass perturbations.
- Or, we use the new uniform version of Proposition 5.2 and the knowledge we have obtained above in order to adapt the proof of Theorem 5.1 to prove immediately a nonlinear orbital stability that holds for nonzero mass perturbations.

Discussion on the extension of Theorem 5.1 in the system case

Let us discuss slightly on what would need to be adapted to prove an extension of Theorem 5.1 in the system case. The main difference will come in the introduction of an additional hypothesis:

Hypothesis 5.1. *We assume that:*

$$\det\left(\boldsymbol{r}_1^-,\dots,\boldsymbol{r}_{I-1}^-,\sum_{j\in\mathbb{Z}}V(j),\boldsymbol{r}_{I+1}^+,\dots,\boldsymbol{r}_d^+\right)\neq 0$$

where the sequence V is defined by Theorem 4.1 and the integer I by Hypothesis 4.1.

This assumption can be seen as a modification of the well-known Majda-Liu condition (where $\sum_{j\in\mathbb{Z}}V(j)$ would be replaced by the difference of the states of the shock $u^- - u^+$). This Majda-Liu condition is often introduced in nonlinear orbital stability results ([ZH98, (1.24)], [BHR03, (3.59)] for instance). Adding this hypothesis allows us to prove an extension of Proposition 5.1 in the system case: We can prove that the sequences P_U^+ and P_U^- are equal to zero and we can obtain certain equalities on the coefficients $C_{l'}^{E,+}$ and $C_{l'}^{E,-}$. As a consequence, just like in the present chapter, we obtain improved expressions on the Green's function and its discrete derivative compared to Chapter 4.

The only remaining thing to do would then be to find the right spaces and norms to obtain sharp and useful bounds on the operators $\mathscr{L}^n$ and $\mathscr{L}^n(Id-\mathcal{T})$ where the operator $\mathcal{T}$ is just the adaptation of (5.1.4) in the system case. We will need to take into account that, in the system case, there are two "regimes" :

- A part of the mass of the initial perturbation will follow the characteristics entering the shock and reach the shock at some point. We want to impose some polynomial weights on this part and study them with norms with polynomial weights.
- A part of the mass of the initial perturbation will just follow characteristics that are not entering the shock. For those, we wish to avoid to study them with norms having polynomial weights. We can expect them be constant in the ℓ^1-norm and decay at a rate $\frac{1}{n^{\frac{1}{2\mu}}}$ in the ℓ^∞-norm.

Once we find the right way of defining these spaces and norms, we should be able to extend Proposition 5.2 and Theorem 5.1 in the system case.

5.1.5 Plan of the chapter

The chapter is separated in three parts. First, in Section 5.2, we will assume that Propositions 5.1 and 5.2 are already proved and focus on the main matter of this chapter, the proof of Theorem 5.1. In Section 5.3, we will then prove Proposition 5.1 which improves the result of Chapter 4. Finally, in Section 5.4, we use the decompositions (5.1.3a) and (5.1.3b) of the Green's function and its discrete derivative deduced using Proposition 5.1 to prove the estimates on the operators $\mathscr{L}^n$ and $\mathscr{L}^n(Id-\mathcal{T})$ claimed in Proposition 5.2.

5.2 Proof of Theorem 5.1: Nonlinear stability for zero-mass perturbation

5.2.1 Necessary preliminary observations

The main goal of this section is the proof of Theorem 5.1. We will consider that Propositions 5.1 and 5.2 have been proved. Their proofs are respectively presented in Sections 5.3 and 5.4 below. Let us first start by introducing some useful lemmas and constants that will appear in the proof.

Definition of a neighborhood of the states of the SDSP $\overline{u}^s$

An important part of the proof of Theorem 5.1 will rely on proving that, for small enough initial perturbations h^0, if we define the initial condition $u^0 := \overline{u}^s + h^0$, the sequences u^n constructed using the numerical scheme

$$\forall n\in\mathbb{N},\quad u^{n+1} := \mathcal{N}(u^n). \tag{5.2.1}$$

are actually defined for all $n\in\mathbb{N}$. This is non trivial as there could be a time $n\in\mathbb{N}$ and an integer $j\in\mathbb{Z}$ such that $u_j^n\notin\mathcal{U}$ and thus the solution u^n of the numerical scheme would have left the domain of definition $\mathcal{U}^\mathbb{Z}$ of the operator $\mathcal{N}$. We recall that we do not make any monotonicity assumption on the numerical scheme we consider here. We thus introduce a radius $\delta>0$ such that:

$$\bigcup_{j\in\mathbb{Z}}\overline{B(\overline{u}_j^s,\delta)}\cup\overline{B(u^+,\delta)}\cup\overline{B(u^-,\delta)}\subset\mathcal{U}. \tag{5.2.2}$$

The set defined in (5.2.2) is a neighborhood of the states of the SDSP $\overline{u}^s$, which is also included in the space of states $\mathcal{U}$ of the system of conservation laws (4.1.1). This set and the radius δ can be defined since $\mathcal{U}$ is an open set and the set

$$\left\{\overline{u}^s_j, j\in\mathbb{Z}\right\}\cup\left\{u^+,u^-\right\}$$

is compact. The definition (5.2.2) of the radius δ implies that for $h\in\ell^\infty(\mathbb{Z},\mathbb{C})$ such that $\|h\|_{\ell^\infty}<\delta$, we have that:

$$\forall j\in\mathbb{Z},\quad \overline{u}^s_j+h_j\in\mathcal{U},$$

i.e. the perturbation h is small enough so that the elements of the sequence $\overline{u}^s+h$ remain in the domain of definition $\mathcal{U}$ of the numerical scheme. Therefore, coming back to our initial issue of constructing the solution $(u^n)_{n\in\mathbb{N}}$ of the numerical scheme (5.2.1), if for some integer $n\in\mathbb{N}$ we can define the sequence u^n and that

$$\|u^n-\overline{u}^s\|_{\ell^\infty}<\delta$$

then, using the definition (5.2.2) of the radius δ, we can construct the sequence u^{n+1} using (5.2.1). This will give us later on a way to prove recursively that the solution $(u^n)_{n\in\mathbb{N}}$ of the numerical scheme (5.2.1) is well-defined up to any time $n\in\mathbb{N}$.

Decomposition near the SDSP $\overline{u}^s$ of the operator $\mathcal{N}$ in a linear and nonlinear part

Let us consider a sequence $h\in\ell^\infty(\mathbb{Z},\mathbb{C})$ such that $\|h\|_{\ell^\infty}<\delta$ where the radius δ is defined by (5.2.2). We recall that (5.2.2) implies that the elements of the sequence $\overline{u}^s+h$ belong to the space of states $\mathcal{U}$. Using the definition (4.1.7) of the nonlinear operator $\mathcal{N}$, we thus have that for all $j\in\mathbb{Z}$:

$$\begin{aligned}\mathcal{N}(\overline{u}^s+h)_j&=\overline{u}^s_j+h_j\\&\quad-\boldsymbol{\nu}\left(F\left(\boldsymbol{\nu};\overline{u}^s_{j-p+1}+h_{j-p+1},\dots,\overline{u}^s_{j+q}+h_{j+q}\right)-F\left(\boldsymbol{\nu};\overline{u}^s_{j-p}+h_{j-p},\dots,\overline{u}^s_{j+q-1}+h_{j+q-1}\right)\right).\end{aligned}\tag{5.2.3}$$

Let us now introduce the sequence $Q(h)\in\mathbb{C}^\mathbb{Z}$ defined by:

$$\forall j\in\mathbb{Z},\quad Q(h)_j:=\boldsymbol{\nu}F(\boldsymbol{\nu};\overline{u}^s_{j-p}+h_{j-p},\dots,\overline{u}^s_{j+q-1}+h_{j+q-1})-\boldsymbol{\nu}F(\boldsymbol{\nu};\overline{u}^s_{j-p},\dots,\overline{u}^s_{j+q-1})-\sum_{k=-p}^{q-1}B_{j,k}h_{j+k}\tag{5.2.4}$$

where the real numbers $B_{j,k}$ defined by (4.1.20) are equal to

$$B_{j,k}:=\boldsymbol{\nu}\partial_{u_k}F(\boldsymbol{\nu};\overline{u}^s_{j-p},\dots,\overline{u}^s_{j+q-1}).$$

We observe that, since the sequence $\overline{u}^s$ is a fixed point of the nonlinear evolution operator $\mathcal{N}$ defined by (4.1.7), the sequence

$$\left(F(\boldsymbol{\nu};\overline{u}^s_{j-p},\dots,\overline{u}^s_{j+q-1})\right)_{j\in\mathbb{Z}}$$

is constant. Thus, the equality (5.2.3) can be rewritten as:

$$\forall j\in\mathbb{Z},\quad \mathcal{N}(\overline{u}^s+h)_j=\overline{u}^s_j+(\mathscr{L}h)_j-Q(h)_{j+1}+Q(h)_j,\tag{5.2.5}$$

where the operator $\mathscr{L}$ is defined by (4.1.19) and corresponds to the linearization of $\mathcal{N}$ about the SDSP $\overline{u}^s$. The sequence $Q(h)$ should be thought of as a nonlinear quadratic remainder term. Recalling that the shift operator $\mathcal{T}$ is defined by (5.1.4), the equality (5.2.5) implies that:

$$\forall h\in\ell^\infty(\mathbb{Z},\mathbb{C}),\quad \|h\|_{\ell^\infty}<\delta\quad\Rightarrow\quad \mathcal{N}(\overline{u}^s+h)=\overline{u}^s+\mathscr{L}h+(Id-\mathcal{T})Q(h).\tag{5.2.6}$$

The following lemma allows us to obtain sharp and useful bounds for the function Q. Lemma 5.2.1 below will be proved in the Appendix of the chapter (Section 5.A). We recall that the vector spaces ℓ^1_γ and ℓ^∞_γ for $\gamma\in[0,+\infty[$ are defined by (5.0.1).

Lemma 5.2.1. *Let us consider $\gamma\in[0,+\infty[$. There exists a constant $C_Q(\gamma)>0$ such that for $h\in\ell^1_\gamma$ such that*

$$\|h\|_{\ell^\infty}<\delta,$$

then the sequence $Q(h)$ belongs to $\ell^1_{2\gamma}$ and:

$$\|Q(h)\|_{\ell^1_{2\gamma}}\leq C_Q(\gamma)\,\|h\|_{\ell^1_\gamma}\,\|h\|_{\ell^\infty_\gamma},\tag{5.2.7a}$$

$$\|Q(h)\|_{\ell^\infty_{2\gamma}}\leq C_Q(\gamma)\|h\|_{\ell^\infty_\gamma}{}^2.\tag{5.2.7b}$$

The introduction of the function Q and of the estimates of Lemma 5.2.1 will be central in the proof of

Theorem 5.1. Indeed, let us assume that the solution $(u^n)_{n\in\mathbb{N}}$ of the numerical scheme (5.2.1) is well-defined for all $n \in \mathbb{N}$ (which, we recall, is nontrivial and would need to be proved). Assume furthermore that, for all $n \in \mathbb{N}$, the sequence $h^n := u^n - \overline{u}^s$ satisfies:

$$\|h^n\|_{\ell^\infty} < \delta.$$

Then, using the equality (5.2.6), the equality :

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n)$$

can be rewritten as:

$$\forall n \in \mathbb{N}, \quad h^{n+1} = \mathscr{L}h^n + (Id - \mathcal{T})Q(h^n).$$

Therefore, we have the following expression for the sequences h^n using Duhamel's formula:

$$\forall n \in \mathbb{N}, \quad h^n = \mathscr{L}^n h^0 + \sum_{m=0}^{n-1} \mathscr{L}^{n-1-m}(Id - \mathcal{T})Q(h^m).$$

It is quite apparent in this expression how one could use the estimates of Proposition 5.2 on the families of operators $(\mathscr{L}^n)_{n\in\mathbb{N}}$ and $(\mathscr{L}^n(Id-\mathcal{T}))_{n\in\mathbb{N}}$ as well as the estimates of Lemma 5.2.1 on the function Q to hopefully obtain decay estimates on the sequences h^n. The proof of Theorem 5.1 will essentially use those calculations while taking into account that we need to prove the definition of the solution $(u^n)_{n\in\mathbb{N}}$.

Useful technical lemma

We finally introduce a useful technical lemma that will be used in the proof of Theorem 5.1. It is a discrete version of [Xin92, Lemma 2.3].

Lemma 5.2.2. *We consider constants $a, b, c \in [0, +\infty[$ such that:*

$$\begin{aligned} c &\leq a + b - 1 && \text{if } a \in [0,1[, \\ c &< b && \text{if } a = 1, \\ c &\leq b && \text{if } a > 1. \end{aligned}$$

There exists a constant $C_I(a,b,c) > 0$ such that, for all $n \in \mathbb{N}$, we have that:

$$\sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{m^a(n+1-m)^b} \leq \frac{C_I(a,b,c)}{(n+1)^c}.$$

The proof is quite immediate and will be given in the Appendix (Section 5.A).

5.2.2 Proof of Theorem 5.1 by induction

We start the proof of Theorem 5.1 by fixing the some of the constants that will appear. Let us from now on fix the constant $\boldsymbol{p} \in [0, +\infty[$. We will introduce the parameter:

$$\gamma := \max(1, \boldsymbol{p}) \geq 1 \tag{5.2.8}$$

which will clarify the choice of weighted spaces that we will consider later on. In particular, as stated in Theorem 5.1, the initial condition h^0 we will consider will belong to the vector space $\mathbb{E}_{\boldsymbol{p}+\gamma}$ defined by (5.0.1).

We introduce the constant:

$$C_0 := C_{\mathscr{L}}(\gamma, \boldsymbol{p} + \gamma) + 1 \tag{5.2.9}$$

where the constant $C_{\mathscr{L}}(\gamma, \boldsymbol{p} + \gamma)$ is defined in Proposition 5.2. This constant C_0 will correspond to the one appearing in Theorem 5.1. Finally, there remains to define the parameter $\varepsilon > 0$ that appears in Theorem 5.1. We consider ε to be small enough to satisfy a few conditions stated below:

- We consider that

$$C_0\varepsilon < \delta \tag{5.2.10a}$$

where δ is defined by (5.2.2). In particular, since $C_0 > 1$, we have:

$$\varepsilon < \delta. \tag{5.2.10b}$$

Let us briefly show an example of how this condition will be used. If we consider an initial perturbation

$h^0 \in \mathbb{E}_{\boldsymbol{p}+\gamma}$ such that:

$$\|h^0\|_{\ell^1_{\boldsymbol{p}+\gamma}} < \varepsilon$$

as stated in Theorem 5.1, then we observe that the condition (5.2.10b) on ε implies that:

$$\|h^0\|_{\ell^\infty} \leq \|h^0\|_{\ell^1} \leq \|h^0\|_{\ell^1_{\boldsymbol{p}+\gamma}} < \varepsilon < \delta.$$

Therefore, using the definition (5.2.2) of the constant δ, we have that the elements of the sequence $u^0 := \overline{u}^s + h^0$ belong to $\mathcal{U}$ and we can thus define the sequence $u^1 := \mathcal{N}(u^0)$. Similar ideas using the conditions (5.2.10) will allow us to prove that we can construct the solution $(u^n)_{n\in\mathbb{N}}$ of the numerical scheme (5.2.1) up to any time.

• We recall that the constants $C_{\mathscr{L}}$, C_Q and C_I appear respectively in Proposition 5.2, Lemma 5.2.1 and Lemma 5.2.2 and that the constant C_0 is defined by (5.2.9). We define the following constants C_1 and C_2 which will appear at some point in the proof of Theorem 5.1:

$$\begin{aligned} C_1 := {} & C_{\mathscr{L}}(\gamma, 2(\boldsymbol{p}+\gamma))C_Q(\boldsymbol{p}+\gamma) \\ & + C_0^2 C_{\mathscr{L}}(\gamma, 2\gamma) C_Q(\gamma)\left(C_I\left(\gamma+\frac{1}{2\mu}, 2\boldsymbol{p}+\frac{1}{2\mu}, \boldsymbol{p}\right) + C_I\left(2\boldsymbol{p}+\frac{1}{2\mu}, \gamma+\frac{1}{2\mu}, \boldsymbol{p}\right)\right) \end{aligned} \tag{5.2.11}$$

and

$$\begin{aligned} C_2 := {} & C_{\mathscr{L}}(\gamma, 2(\boldsymbol{p}+\gamma))C_Q(\boldsymbol{p}+\gamma) \\ & + C_0^2 C_{\mathscr{L}}(\gamma, 2\gamma) C_Q(\gamma)\left(C_I\left(2\boldsymbol{p}+\frac{1}{2\mu}, \gamma+\frac{1}{\mu}, \boldsymbol{p}+\frac{1}{2\mu}\right) + C_I\left(\gamma+\frac{1}{2\mu}, 2\boldsymbol{p}+\frac{1}{\mu}, \boldsymbol{p}+\frac{1}{2\mu}\right)\right). \end{aligned} \tag{5.2.12}$$

The constants C_1 and C_2 depend solely on $\boldsymbol{p}$ since γ is defined by (5.2.8). The condition we impose on the parameter ε is that

$$\varepsilon C_1 < 1 \quad \text{and} \quad \varepsilon C_2 < 1.$$

In particular, using the definition (5.2.9) of the constant C_0, we have that

$$C_{\mathscr{L}}(\gamma, \boldsymbol{p}+\gamma) + \varepsilon C_1 < C_0, \tag{5.2.13a}$$

$$C_{\mathscr{L}}(\gamma, \boldsymbol{p}+\gamma) + \varepsilon C_2 < C_0. \tag{5.2.13b}$$

Let us now start with the proof of Theorem 5.1. We consider an initial perturbation $h^0 \in \mathbb{E}_{\boldsymbol{p}+\gamma}$ such that:

$$\|h^0\|_{\ell^1_{\boldsymbol{p}+\gamma}} < \varepsilon \tag{5.2.14}$$

and we define the initial condition $u^0 := \overline{u}^s + h^0$. The proof of Theorem 5.1 will be done by induction. We state for $n \in \mathbb{N}$ the following assertion:

Assertion $\mathbf{P}(n)$: The sequences u^m constructed using the numerical scheme (5.2.1) are well-defined for all $m \in \{0, \dots, n\}$. Furthermore, if we define the sequences $h^m := u^m - \overline{u}^s$ for $m \in \{0, \dots, n\}$, then we have that:

$$\forall m \in \{0, \dots, n\}, \quad \|h^m\|_{\ell^\infty} < \delta, \tag{5.2.15a}$$

$$\forall m \in \{1, \dots, n\}, \quad \|h^m\|_{\ell^1_\gamma} \leq \frac{C_0}{m^{\boldsymbol{p}}}\|h^0\|_{\ell^1_{\boldsymbol{p}+\gamma}}, \tag{5.2.15b}$$

$$\forall m \in \{1, \dots, n\}, \quad \|h^m\|_{\ell^\infty_\gamma} \leq \frac{C_0}{m^{\boldsymbol{p}+\frac{1}{2\mu}}}\|h^0\|_{\ell^1_{\boldsymbol{p}+\gamma}}. \tag{5.2.15c}$$

We observe that if the assertions $\mathbf{P}(n)$ are true for all $n \in \mathbb{N}$, then Theorem 5.1 follows immediately. Indeed, by interpolation, for $r \in [1, +\infty]$, we have using (5.2.15b) and (5.2.15c):

$$\forall n \in \mathbb{N}\backslash\{0\}, \quad \|h^n\|_{\ell^r_\gamma} \leq \|h^n\|_{\ell^1_\gamma}^{\frac{1}{r}} \|h^n\|_{\ell^\infty_\gamma}^{1-\frac{1}{r}} \leq \frac{C_0}{n^{\boldsymbol{p}+\frac{1}{2\mu}\left(1-\frac{1}{r}\right)}}\|h^0\|_{\ell^1_{\boldsymbol{p}+\gamma}}.$$

We thus obtain the result of Theorem 5.1.

▶ **Initialization step:**

Concerning the initialization step, the only point that needs to be proved is that (5.2.15a) for $m = 0$ holds, i.e. that:

$$\left\|h^0\right\|_{\ell^\infty} < \delta.$$

The condition (5.2.14) on the initial perturbation h^0 and the condition (5.2.10b) on ε imply that:

$$\left\|h^0\right\|_{\ell^\infty} \leq \left\|h^0\right\|_{\ell^1} \leq \left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}} < \varepsilon < \delta.$$

Thus, the assertion $\mathbf{P}(0)$ is verified.

► **Induction step:**

We consider $n \in \mathbb{N}$ such that $\mathbf{P}(n)$ is true. Let us prove that $\mathbf{P}(n+1)$ is also true. The first step is to prove why u^{n+1} is well-defined. Using (5.2.15a) for $m = n$ and the definition (5.2.2) of the constant δ, we have that the elements of the sequence $u^n = \overline{u}^s + h^n$ belong to $\mathcal{U}$ and we can thus define

$$u^{n+1} := \mathcal{N}(u^n).$$

The sequence u^{n+1} is well-defined.

We can from now on define the sequence $h^{n+1} := u^{n+1} - \overline{u}^s$. To prove that $\mathbf{P}(n+1)$ is true, there just remains to prove the inequality (5.2.15a), (5.2.15b) and (5.2.15c) for $m = n+1$.

Before starting with the proofs of (5.2.15a)-(5.2.15c) for $m = n+1$, we will need to make a slight observation. The inequality (5.2.15a) for $m \in \{0, \dots, n\}$ implies that we can use the equality (5.2.6) to rewrite:

$$\forall m \in \{0, \dots, n\}, \quad u^{m+1} = \mathcal{N}(u^m)$$

as:

$$\forall m \in \{0, \dots, n\}, \quad h^{m+1} = \mathscr{L}h^m + (Id - \mathcal{T})Q(h^m).$$

Using Duhamel's formula, we then have that:

$$h^{n+1} = \mathscr{L}^{n+1}h^0 + \mathscr{L}^n(Id - \mathcal{T})Q(h^0) + \sum_{m=1}^{n} \mathscr{L}^{n-m}(Id - \mathcal{T})Q(h^m). \tag{5.2.16}$$

This expression of the sequence h^{n+1} will be central to prove (5.2.15b) and (5.2.15c) for $m = n+1$.

We will start by proving (5.2.15b) for $m = n+1$. We will then focus on the proof of (5.2.15c) for $m = n+1$ which will be fairly similar. Finally, we will conclude with the proof of (5.2.15a) for $m = n+1$ which is actually a consequence of (5.2.15c) (or even (5.2.15b)) for $m = n+1$.

• **Proof of (5.2.15b) for $m = n+1$:**

We want to find bounds on the sequence h^{n+1} in ℓ^1_γ. We will prove bounds on the different terms appearing in the expression (5.2.16).

⋆ *Estimates on $\mathscr{L}^{n+1}h^0$ in ℓ^1_γ:*

Using the estimate (5.1.5a) of Proposition 5.2 with $\gamma_1 = \gamma$ and $\gamma_2 = \boldsymbol{p} + \gamma$, we have that:

$$\left\|\mathscr{L}^{n+1}h^0\right\|_{\ell^1_\gamma} \leq \frac{C_{\mathscr{L}}(\gamma, \boldsymbol{p}+\gamma)}{(n+1)^{\boldsymbol{p}}} \left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}. \tag{5.2.17}$$

⋆ *Estimates on $\mathscr{L}^n(Id - \mathcal{T})Q(h^0)$ in ℓ^1_γ:*

Using the estimate (5.1.5c) of Proposition 5.2 with $\gamma_1 = \gamma$ and $\gamma_2 = 2(\boldsymbol{p} + \gamma)$, we have that:

$$\left\|\mathscr{L}^n(Id - \mathcal{T})Q(h^0)\right\|_{\ell^1_\gamma} \leq \frac{C_{\mathscr{L}}(\gamma, 2(\boldsymbol{p}+\gamma))}{(n+1)^{2\boldsymbol{p}+\gamma+\frac{1}{2\mu}}} \left\|Q(h^0)\right\|_{\ell^1_{2(\boldsymbol{p}+\gamma)}}.$$

Thus, using the inequality (5.2.7a) of Lemma 5.2.1 and noticing that $2\boldsymbol{p} + \gamma + \frac{1}{2\mu} \geq \boldsymbol{p}$, we have that:

$$\left\|\mathscr{L}^n(Id - \mathcal{T})Q(h^0)\right\|_{\ell^1_\gamma} \leq \frac{C_{\mathscr{L}}(\gamma, 2(\boldsymbol{p}+\gamma))C_Q(\boldsymbol{p}+\gamma)}{(n+1)^{\boldsymbol{p}}} \left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}} \left\|h^0\right\|_{\ell^\infty_{\boldsymbol{p}+\gamma}}.$$

Let us observe that our choice of polynomial weight is not optimal here and could be improved in future works to improve the statement of Theorem 5.1. Finally, while observing that the condition (5.2.14) on the initial perturbation h^0 implies that:

$$\left\|h^0\right\|_{\ell^\infty_{\boldsymbol{p}+\gamma}} \leq \left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}} < \varepsilon, \tag{5.2.18}$$

we thus have that:

$$\left\|\mathscr{L}^n(Id-\mathcal{T})Q(h^0)\right\|_{\ell^1_\gamma} \leq \frac{\varepsilon C_{\mathscr{L}}(\gamma, 2(\boldsymbol{p}+\gamma))C_Q(\boldsymbol{p}+\gamma)}{(n+1)^{\boldsymbol{p}}}\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}. \tag{5.2.19}$$

⋆ *Estimates on $\sum_{m=1}^n \mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m)$ in ℓ^1_γ:*

Using the estimate (5.1.5c) of Proposition 5.2 with $\gamma_1 = \gamma$ and $\gamma_2 = 2\gamma$, we have that:

$$\left\|\sum_{m=1}^n \mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m)\right\|_{\ell^1_\gamma} \leq \sum_{m=1}^n \frac{C_{\mathscr{L}}(\gamma, 2\gamma)}{(n+1-m)^{\gamma+\frac{1}{2\mu}}}\left\|Q(h^m)\right\|_{\ell^1_{2\gamma}}. \tag{5.2.20}$$

Using the inequality (5.2.7a) of Lemma 5.2.1, the inequalities (5.2.15b) and (5.2.15c) on the sequence h^m and finally the inequality (5.2.14), we have that:

$$\forall m \in \{1,\dots,n\}, \quad \left\|Q(h^m)\right\|_{\ell^1_{2\gamma}} \leq C_Q(\gamma)\left\|h^m\right\|_{\ell^1_\gamma}\left\|h^m\right\|_{\ell^\infty_\gamma} \leq \frac{C_Q(\gamma){C_0}^2}{m^{2\boldsymbol{p}+\frac{1}{2\mu}}}\left\|h^0\right\|^2_{\ell^1_{\boldsymbol{p}+\gamma}} \leq \frac{\varepsilon C_Q(\gamma){C_0}^2}{m^{2\boldsymbol{p}+\frac{1}{2\mu}}}\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}. \tag{5.2.21}$$

Combining (5.2.20) and (5.2.21), we have that:

$$\left\|\sum_{m=1}^n \mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m)\right\|_{\ell^1_\gamma} \leq \varepsilon C_{\mathscr{L}}(\gamma,2\gamma)C_Q(\gamma){C_0}^2\left(\sum_{m=1}^n \frac{1}{(n+1-m)^{\gamma+\frac{1}{2\mu}}m^{2\boldsymbol{p}+\frac{1}{2\mu}}}\right)\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}. \tag{5.2.22}$$

There remains to bound the sum:

$$\sum_{m=1}^n \frac{1}{(n+1-m)^{\gamma+\frac{1}{2\mu}}m^{2\boldsymbol{p}+\frac{1}{2\mu}}}.$$

We recall that the constant γ is defined by (5.2.8). For $a = 2\boldsymbol{p}+\frac{1}{2\mu}$, $b = \gamma+\frac{1}{2\mu}$ and $c = \boldsymbol{p}$, we have that $a+b-1 \geq a \geq \boldsymbol{p}$ and $b > \boldsymbol{p}$. Thus, using Lemma 5.2.2 since every possible case is covered, we have that:

$$\sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor} \frac{1}{(n+1-m)^{\gamma+\frac{1}{2\mu}}m^{2\boldsymbol{p}+\frac{1}{2\mu}}} \leq \frac{C_I\left(2\boldsymbol{p}+\frac{1}{2\mu}, \gamma+\frac{1}{2\mu}, \boldsymbol{p}\right)}{(n+1)^{\boldsymbol{p}}}. \tag{5.2.23a}$$

Similarly, using Lemma 5.2.2 for $a = \gamma+\frac{1}{2\mu}$, $b = 2\boldsymbol{p}+\frac{1}{2\mu}$ and $c = \boldsymbol{p}$, we have that:

$$\sum_{m=\lfloor\frac{n+1}{2}\rfloor+1}^{n} \frac{1}{(n+1-m)^{\gamma+\frac{1}{2\mu}}m^{2\boldsymbol{p}+\frac{1}{2\mu}}} \leq \frac{C_I\left(\gamma+\frac{1}{2\mu}, 2\boldsymbol{p}+\frac{1}{2\mu}, \boldsymbol{p}\right)}{(n+1)^{\boldsymbol{p}}}. \tag{5.2.23b}$$

Combining (5.2.23a) and (5.2.23b), we have that:

$$\sum_{m=1}^{n} \frac{1}{(n+1-m)^{\gamma+\frac{1}{2\mu}}m^{2\boldsymbol{p}+\frac{1}{2\mu}}} \leq \frac{C_I\left(2\boldsymbol{p}+\frac{1}{2\mu}, \gamma+\frac{1}{2\mu}, \boldsymbol{p}\right)+C_I\left(\gamma+\frac{1}{2\mu}, 2\boldsymbol{p}+\frac{1}{2\mu}, \boldsymbol{p}\right)}{(n+1)^{\boldsymbol{p}}}. \tag{5.2.23c}$$

Therefore, combining (5.2.22) and (5.2.23c), we obtain:

$$\begin{aligned}&\left\|\sum_{m=1}^n \mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m)\right\|_{\ell^1_\gamma} \\ &\quad\leq \frac{\varepsilon C_{\mathscr{L}}(\gamma,2\gamma)C_Q(\gamma){C_0}^2\left(C_I\left(2\boldsymbol{p}+\frac{1}{2\mu}, \gamma+\frac{1}{2\mu}, \boldsymbol{p}\right)+C_I\left(\gamma+\frac{1}{2\mu}, 2\boldsymbol{p}+\frac{1}{2\mu}, \boldsymbol{p}\right)\right)}{(n+1)^{\boldsymbol{p}}}\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}.\end{aligned} \tag{5.2.24}$$

Let us observe once again that our choice of polynomial weight is not optimal here and could be improved in future works to improve the statement of Theorem 5.1.

⋆ *Combining the estimates in ℓ^1_γ:*

Using the equality (5.2.16) on the sequence h^{n+1} and combining the inequalities (5.2.17) on $\mathscr{L}^{n+1}h^0$, (5.2.19) on $\mathscr{L}^n(Id-\mathcal{T})Q(h^0)$ and (5.2.24) on $\sum_{m=1}^n \mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m)$, we have that:

$$\left\|h^{n+1}\right\|_{\ell^1_\gamma} \leq \left\|\mathscr{L}^{n+1}h^0\right\|_{\ell^1_\gamma} + \left\|\mathscr{L}^n(Id-\mathcal{T})Q(h^0)\right\|_{\ell^1_\gamma} + \left\|\sum_{m=1}^{n}\mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m)\right\|_{\ell^1_\gamma} \leq \frac{C_{\mathscr{L}}(\gamma,\boldsymbol{p}+\gamma)+\varepsilon C_1}{(n+1)^{\boldsymbol{p}}}\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}$$

where the constant C_1 is defined by (5.2.11). Using the condition (5.2.13a), we thus obtain (5.2.15b) for $m=n+1$.

• **Proof of** (5.2.15c) **for** $m=n+1$**:**

The proof of (5.2.15c) for $m=n+1$ essentially works similarly as the proof of (5.2.15b) for $m=n+1$. We want to find bounds on the sequence h^{n+1} in ℓ^∞_γ. We will prove bounds on the different terms appearing in the expression (5.2.16).

⋆ *Estimates on $\mathscr{L}^{n+1}h^0$ in ℓ^∞_γ:*

Using the estimate (5.1.5f) of Proposition 5.2 with $\gamma_1=\gamma\geq 1$ and $\gamma_2=\boldsymbol{p}+\gamma$, we have that:

$$\left\|\mathscr{L}^{n+1}h^0\right\|_{\ell^\infty_\gamma} \leq \frac{C_{\mathscr{L}}(\gamma,\boldsymbol{p}+\gamma)}{(n+1)^{\boldsymbol{p}+\frac{1}{2\mu}}}\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}. \tag{5.2.25}$$

⋆ *Estimates on $\mathscr{L}^n(Id-\mathcal{T})Q(h^0)$ in ℓ^∞_γ:*

Using the estimate (5.1.5g) of Proposition 5.2 with $\gamma_1=\gamma\geq 1$ and $\gamma_2=2(\boldsymbol{p}+\gamma)\geq\gamma_1$, we have that:

$$\left\|\mathscr{L}^n(Id-\mathcal{T})Q(h^0)\right\|_{\ell^\infty_\gamma} \leq \frac{C_{\mathscr{L}}(\gamma,2(\boldsymbol{p}+\gamma))}{(n+1)^{2\boldsymbol{p}+\gamma+\frac{1}{\mu}}}\left\|Q(h^0)\right\|_{\ell^1_{2(\boldsymbol{p}+\gamma)}}.$$

Thus, using the inequality (5.2.7a) of Lemma 5.2.1 and noticing that $2\boldsymbol{p}+\gamma+\frac{1}{\mu}\geq\boldsymbol{p}+\frac{1}{2\mu}$, we have that:

$$\left\|\mathscr{L}^n(Id-\mathcal{T})Q(h^0)\right\|_{\ell^\infty_\gamma} \leq \frac{C_{\mathscr{L}}(\gamma,2(\boldsymbol{p}+\gamma))C_Q(\boldsymbol{p}+\gamma)}{(n+1)^{\boldsymbol{p}+\frac{1}{2\mu}}}\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}\left\|h^0\right\|_{\ell^\infty_{\boldsymbol{p}+\gamma}}.$$

Finally, using (5.2.18), we have that:

$$\left\|\mathscr{L}^n(Id-\mathcal{T})Q(h^0)\right\|_{\ell^\infty_\gamma} \leq \frac{\varepsilon\, C_{\mathscr{L}}(\gamma,2(\boldsymbol{p}+\gamma))C_Q(\boldsymbol{p}+\gamma)}{(n+1)^{\boldsymbol{p}+\frac{1}{2\mu}}}\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}. \tag{5.2.26}$$

⋆ *Estimates on $\sum_{m=1}^n\mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m)$ in ℓ^∞_γ:*

Using the estimates (5.1.5g) and (5.1.5h) of Proposition 5.2 with $\gamma_1=\gamma\geq 1$ and $\gamma_2=2\gamma\geq\gamma_1$, we have that:

$$\begin{aligned}&\left\|\sum_{m=1}^{n}\mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m)\right\|_{\ell^\infty_\gamma}\\ &\qquad\leq \sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{C_{\mathscr{L}}(\gamma,2\gamma)}{(n+1-m)^{\gamma+\frac{1}{\mu}}}\left\|Q(h^m)\right\|_{\ell^1_{2\gamma}} + \sum_{m=\lfloor\frac{n+1}{2}\rfloor+1}^{n}\frac{C_{\mathscr{L}}(\gamma,2\gamma)}{(n+1-m)^{\gamma+\frac{1}{2\mu}}}\left\|Q(h^m)\right\|_{\ell^\infty_{2\gamma}}.\end{aligned} \tag{5.2.27}$$

Using the inequalities (5.2.7a) and (5.2.7b) of Lemma 5.2.1, the inequalities (5.2.15b) and (5.2.15c) on the sequence h^m and finally the inequality (5.2.14), we have that:

$$\forall m\in\{1,\dots,n\},\quad \left\|Q(h^m)\right\|_{\ell^1_{2\gamma}} \leq C_Q(\gamma)\left\|h^m\right\|_{\ell^1_\gamma}\left\|h^m\right\|_{\ell^\infty_\gamma} \leq \frac{C_Q(\gamma){C_0}^2}{m^{2\boldsymbol{p}+\frac{1}{2\mu}}}\left\|h^0\right\|^2_{\ell^1_{\boldsymbol{p}+\gamma}} \leq \frac{\varepsilon C_Q(\gamma){C_0}^2}{m^{2\boldsymbol{p}+\frac{1}{2\mu}}}\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}, \tag{5.2.28a}$$

$$\forall m\in\{1,\dots,n\},\quad \left\|Q(h^m)\right\|_{\ell^\infty_{2\gamma}} \leq C_Q(\gamma)\left\|h^m\right\|^2_{\ell^\infty_\gamma} \leq \frac{C_Q(\gamma){C_0}^2}{m^{2\boldsymbol{p}+\frac{1}{\mu}}}\left\|h^0\right\|^2_{\ell^1_{\boldsymbol{p}+\gamma}} \leq \frac{\varepsilon C_Q(\gamma){C_0}^2}{m^{2\boldsymbol{p}+\frac{1}{\mu}}}\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}. \tag{5.2.28b}$$

Combining (5.2.27), (5.2.28a) and (5.2.28b), we have that:

$$\left\|\sum_{m=1}^{n}\mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m)\right\|_{\ell^\infty_\gamma} \leq \varepsilon\, C_{\mathscr{L}}(\gamma,2\gamma)C_Q(\gamma){C_0}^2\Upsilon_n\left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}} \tag{5.2.29}$$

where the constant Υ_n is defined by:

$$\Upsilon_n := \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{(n+1-m)^{\gamma+\frac{1}{\mu}} m^{2\boldsymbol{p}+\frac{1}{2\mu}}} + \sum_{m=\lfloor \frac{n+1}{2} \rfloor+1}^{n} \frac{1}{(n+1-m)^{\gamma+\frac{1}{2\mu}} m^{2\boldsymbol{p}+\frac{1}{\mu}}} \tag{5.2.30}$$

There remains to bound the constant Υ_n. We recall that the parameter γ is defined by (5.2.8). Using Lemma 5.2.2 for $a = 2\boldsymbol{p} + \frac{1}{2\mu}$, $b = \gamma + \frac{1}{\mu}$ and $c = \boldsymbol{p} + \frac{1}{2\mu}$, we have that:

$$\sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{(n+1-m)^{\gamma+\frac{1}{\mu}} m^{2\boldsymbol{p}+\frac{1}{2\mu}}} \leq \frac{C_I\left(2\boldsymbol{p}+\frac{1}{2\mu}, \gamma+\frac{1}{\mu}, \boldsymbol{p}+\frac{1}{2\mu}\right)}{(n+1)^{\boldsymbol{p}+\frac{1}{2\mu}}}. \tag{5.2.31a}$$

Similarly, using Lemma 5.2.2 for $a = \gamma + \frac{1}{2\mu}$, $b = 2\boldsymbol{p} + \frac{1}{\mu}$ and $c = \boldsymbol{p} + \frac{1}{2\mu}$, we have that:

$$\sum_{m=\lfloor \frac{n+1}{2} \rfloor+1}^{n} \frac{1}{(n+1-m)^{\gamma+\frac{1}{2\mu}} m^{2\boldsymbol{p}+\frac{1}{\mu}}} \leq \frac{C_I\left(\gamma+\frac{1}{2\mu}, 2\boldsymbol{p}+\frac{1}{\mu}, \boldsymbol{p}+\frac{1}{2\mu}\right)}{(n+1)^{\boldsymbol{p}+\frac{1}{2\mu}}}. \tag{5.2.31b}$$

Combining (5.2.31a) and (5.2.31b), we have that:

$$\Upsilon_n \leq \frac{C_I\left(2\boldsymbol{p}+\frac{1}{2\mu}, \gamma+\frac{1}{\mu}, \boldsymbol{p}+\frac{1}{2\mu}\right) + C_I\left(\gamma+\frac{1}{2\mu}, 2\boldsymbol{p}+\frac{1}{\mu}, \boldsymbol{p}+\frac{1}{2\mu}\right)}{(n+1)^{\boldsymbol{p}+\frac{1}{2\mu}}}. \tag{5.2.31c}$$

Therefore, combining (5.2.29) and (5.2.31c), we have that:

$$\begin{aligned} &\left\| \sum_{m=1}^{n} \mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m) \right\|_{\ell^\infty_\gamma} \\ &\leq \frac{\varepsilon C_{\mathscr{L}}(\gamma, 2\gamma) C_Q(\gamma) {C_0}^2 \left(C_I\left(2\boldsymbol{p}+\frac{1}{2\mu}, \gamma+\frac{1}{\mu}, \boldsymbol{p}+\frac{1}{2\mu}\right) + C_I\left(\gamma+\frac{1}{2\mu}, 2\boldsymbol{p}+\frac{1}{\mu}, \boldsymbol{p}+\frac{1}{2\mu}\right)\right)}{(n+1)^{\boldsymbol{p}+\frac{1}{2\mu}}} \left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}}. \end{aligned} \tag{5.2.32}$$

⋆ *Combining the estimates in ℓ^∞_γ:*

Using the equality (5.2.16) on the sequence h^{n+1} and combining the inequalities (5.2.25) on $\mathscr{L}^{n+1}h^0$, (5.2.26) on $\mathscr{L}^n(Id-\mathcal{T})Q(h^0)$ and (5.2.32) on $\sum_{m=1}^{n} \mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m)$, we have that:

$$\begin{aligned} \left\|h^{n+1}\right\|_{\ell^\infty_\gamma} \leq \left\|\mathscr{L}^{n+1}h^0\right\|_{\ell^\infty_\gamma} + \left\|\mathscr{L}^n(Id-\mathcal{T})Q(h^0)\right\|_{\ell^\infty_\gamma} + \left\| \sum_{m=1}^{n} \mathscr{L}^{n-m}(Id-\mathcal{T})Q(h^m) \right\|_{\ell^\infty_\gamma} \\ \leq \frac{C_{\mathscr{L}}(\gamma, \boldsymbol{p}+\gamma) + \varepsilon C_2}{(n+1)^{\boldsymbol{p}+\frac{1}{2\mu}}} \left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}} \end{aligned}$$

where the constant C_2 is defined by (5.2.12). Using the condition (5.2.13b), we thus obtain (5.2.15c) for $m = n+1$.

• **Proof of** (5.2.15a) **for** $m = n+1$**:**

We observe that the equality (5.2.15c) for $m = n+1$, the condition (5.2.14) on the initial perturbation h^0 and the condition (5.2.10a) on ε imply that:

$$\left\|h^{n+1}\right\|_{\ell^\infty} \leq \left\|h^{n+1}\right\|_{\ell^\infty_\gamma} \leq \frac{C_0}{(n+1)^{\boldsymbol{p}+\frac{1}{2\mu}}} \left\|h^0\right\|_{\ell^1_{\boldsymbol{p}+\gamma}} < C_0\varepsilon < \delta.$$

Therefore, we have (5.2.15a) for $m = n+1$.

We have thus proved that $\mathbf{P}(n+1)$ is true and have concluded the induction step. Therefore, by induction, $\mathbf{P}(n)$ is true for all $n \in \mathbb{N}$ and this allows us to conclude the proof of Theorem 5.1.

5.3 Proof of Proposition 5.1: Improvement on the result of Chapter 4 in the scalar case

This section will be dedicated to the proof of Proposition 5.1 which allows to obtain the decompositions (5.1.3a) and (5.1.3b) on the Green's function and its discrete derivative. Since we are considering a conservative finite difference scheme, we start by proving the following lemma.

Lemma 5.3.1. *For all $h\in\ell^1(\mathbb{Z})$, we have:*

$$\sum_{j\in\mathbb{Z}}(\mathscr{L}h)_j=\sum_{j\in\mathbb{Z}}h_j.$$

Proof of Lemma 5.3.1
We recall that the operator $\mathscr{L}$ is defined by (4.1.19) and has the form:

$$\forall h\in\ell^\infty(\mathbb{Z}),\quad (\mathscr{L}h)_j:=h_j+\sum_{k=-p}^{q-1}B_{j,k}h_{j+k}-\sum_{k=-p}^{q-1}B_{j+1,k}h_{j+1+k}$$

where the constants $B_{j,k}$ are defined by (4.1.20). Since the constants $B_{j,k}$ are uniformly bounded for $j\in\mathbb{Z}$ and $k\in\{-q,\dots,p-1\}$, we have for $h\in\ell^1(\mathbb{Z})$:

$$\sum_{j\in\mathbb{Z}}(\mathscr{L}h)_j=\sum_{j\in\mathbb{Z}}\left(h_j+\sum_{k=-p}^{q-1}B_{j,k}h_{j+k}-\sum_{k=-p}^{q-1}B_{j+1,k}h_{j+1+k}\right)=\sum_{j\in\mathbb{Z}}h_j$$

and the conclusion follows. □

Using Lemma 5.3.1, the definition (4.1.25) of the temporal Green's function $\mathscr{G}(n,j_0,j)$ as well as Lemma 4.1.1, we obtain the following equality:

$$\forall n\in\mathbb{N},\forall j_0\in\mathbb{Z},\qquad \sum_{j=j_0-nq}^{j_0+np}\mathscr{G}(n,j_0,j)=1. \tag{5.3.1}$$

Let us now fix $j_0\in\mathbb{N}$. Since $\alpha^+<0$, using the definition (4.1.28) of the function $E_{2\mu}$ and the equality (4.1.29a), we have that:

$$\sum_{j=j_0-nq}^{j_0+np}\left(C^{E,+}E_{2\mu}\left(\beta^+;\frac{n\alpha^++j_0}{n^{\frac{1}{2\mu}}}\right)+P_U^+(j_0)\right)V(j)\underset{n\to+\infty}{\to}\left(C^{E,+}+P_U^+(j_0)\right)\left(\sum_{j\in\mathbb{Z}}V(j)\right). \tag{5.3.2}$$

Furthermore, for any constant $c>0$ (and in particular the one appearing in (5.1.2)), we have:

$$\sum_{j=j_0-nq}^{j_0+np}O\left(e^{-cn}\right)=O\left(ne^{-cn}\right)\underset{n\to+\infty}{\to}0, \tag{5.3.3}$$

$$\sum_{j=j_0-nq}^{j_0+np}O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)=O\left(\frac{1}{n^{\frac{1}{2\mu}}}\right)\underset{n\to+\infty}{\to}0. \tag{5.3.4}$$

where we recall that the Landau notation O is uniform with respect to n, j_0 and j. Finally, we observe that

$$\forall n\in\mathbb{N}\cap\left[-\frac{2j_0}{\alpha^+},+\infty\right[,\forall j\in\mathbb{N},\quad \frac{n}{2}\leq n-\left(\frac{j-j_0}{\alpha^+}\right)$$

and thus there exists a constant $\widetilde{c}>0$ such that:

$$\sum_{j=j_0-nq}^{j_0+np}\mathbb{1}_{j\geq0}O\left(\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)=\sum_{j=j_0-nq}^{j_0+np}\mathbb{1}_{j\geq0}O(e^{-\widetilde{c}n})\underset{n\to+\infty}{\to}0. \tag{5.3.5}$$

Therefore, combining the decomposition (5.1.2) of the Green's function $\mathscr{G}(n,j_0,j)$ and (5.3.2)-(5.3.5), we prove that:

$$\sum_{j=j_0-nq}^{j_0+np} \mathscr{G}(n,j_0,j) \underset{n\to+\infty}{\to} \left(C^{E,+} + P_U^+(j_0)\right)\left(\sum_{j\in\mathbb{Z}} V(j)\right).$$

Using (5.3.1), we then conclude that:

$$\forall j_0 \in \mathbb{N}, \quad \left(C^{E,+} + P_U^+(j_0)\right)\left(\sum_{j\in\mathbb{Z}} V(j)\right) = 1.$$

We then immediately conclude that

$$\sum_{j\in\mathbb{Z}} V(j) \neq 0$$

and that the sequence $P_U^+ = (P_U^+(j_0))_{j_0\in\mathbb{N}}$ is constant. Since the statement of Theorem 4.1 claims that the sequence P_U^+ decays towards 0 at $+\infty$, it is equal to 0 and we have:

$$C^{E,+} = \frac{1}{\sum_{j\in\mathbb{Z}} V(j)}.$$

We get the same conclusion on P_U^- and $C^{E,-}$ when considering $j_0 \in -\mathbb{N}\backslash\{0\}$ and the decomposition of the Green's function $\mathscr{G}(n,j_0,j)$ associated with it.

5.4 Proof of Proposition 5.2: Estimates on the operators $\mathscr{L}^n$ and $\mathscr{L}^n(Id - \mathcal{T})$

The proof of Proposition 5.2 can be seen as a fairly longer version of the proof of Theorem 4.2 of Chapter 4. It will be separated in two sections: Section 5.4.1 will be dedicated to proving the estimates (5.1.5a) and (5.1.5b) on the operator $\mathscr{L}^n$ and Section 5.4.2 will tackle the estimates (5.1.5c) and (5.1.5d) on the operator $\mathscr{L}^n(Id - \mathcal{T})$. Before beginning with the proofs, let us make some useful observations.

— We will fix the constant c that appears in the decompositions (5.1.3a) and (5.1.3b) of the temporal Green's function and its discrete derivative. This will allow us to introduce the constant $\tilde{c} > 0$ defined by:

$$\tilde{c} := \frac{c}{2^{\frac{2\mu}{\mu-1}}}. \tag{5.4.1}$$

Then, we observe that :

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{N} \cap \left[0, -\frac{n\alpha^+}{2}\right], \forall j \in \mathbb{N},$$
$$\exp\left(-c\left(\frac{\left|n - \frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \leq \exp\left(-c\left(\frac{\left|n + \frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) \leq \exp(-\tilde{c}n). \tag{5.4.2}$$

where the constant $\tilde{c}$ is defined by (5.4.1).

— We have that there exists a constant $C > 0$ such that:

$$\forall n \in \mathbb{N}\backslash\{0\}, \quad \sum_{j\in\mathbb{Z}} \frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) < C \tag{5.4.3}$$

where the constant c has been fixed as stated above.

— We also observe that, for any given parameter $\gamma \in [0, +\infty[$, there exists a positive constant C such that:

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \forall j \in \mathbb{Z}, \quad j - j_0 \in \{-nq, \dots, np\} \quad \Rightarrow \quad (1 + |j|)^\gamma \leq C(1 + |j_0|)^\gamma (1 + n)^\gamma. \tag{5.4.4}$$

5.4.1 Proof of the estimates (5.1.5a) and (5.1.5b) on the operator $\mathscr{L}^n$

We first observe that (4.1.27) and Lemma 4.1.1 imply that:

$$\forall n\in\mathbb{N},\forall h\in\ell^\infty(\mathbb{Z}),\quad \mathscr{L}^n h=\left(\sum_{j_0\in\mathbb{Z}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\mathscr{G}(n,j_0,j)h_{j_0}\right)_{j\in\mathbb{Z}}. \tag{5.4.5}$$

Let us recall here that, in the introduction of the present chapter, we proved using Proposition 5.1 an improved decomposition (5.1.3a) of the Green's function in the scalar case. We have that there exists a constant $c>0$ such that for $j_0\in\mathbb{Z}$, $n\in\mathbb{N}\backslash\{0\}$ and $j\in\mathbb{Z}$ such that $j-j_0\in\{-nq,\dots,np\}$, the Green's function can be decomposed in 4 parts:

$$\forall j_0\in\mathbb{N},\quad \mathscr{G}(n,j_0,j)=E^+(n,j_0)V(j)+D^+(n,j_0,j)+R^+(n,j_0,j)+O(e^{-cn}), \tag{5.4.6a}$$
$$\forall j_0\in-\mathbb{N}\backslash\{0\},\quad \mathscr{G}(n,j_0,j)=E^-(n,j_0)V(j)+D^-(n,j_0,j)+R^-(n,j_0,j)+O(e^{-cn}), \tag{5.4.6b}$$

where the sequence V defined in Theorem 4.1 is an eigenvector of the operator $\mathscr{L}$ associated with the eigenvalue 1 and the other terms are defined as follows:

— The terms $E^\pm(n,j_0)$ correspond to the activation of the eigenvalue 1 of the operator $\mathscr{L}$ and is defined by:

$$\forall j_0\in\mathbb{N},\quad E^+(n,j_0):=C^E E_{2\mu}\left(\beta^+;\frac{n\alpha^++j_0}{n^{\frac{1}{2\mu}}}\right), \tag{5.4.7a}$$
$$\forall j_0\in-\mathbb{N}\backslash\{0\},\quad E^-(n,j_0):=C^E E_{2\mu}\left(\beta^-;\frac{-n\alpha^--j_0}{n^{\frac{1}{2\mu}}}\right). \tag{5.4.7b}$$

— The term $D^+(n,j_0,j)$ (resp. $D^-(n,j_0,j)$) corresponds to the diffusion wave which is incoming with respect to the shock and which is associated with the characteristic field of the right (resp. left) state :

$$\forall j_0\in\mathbb{N},\quad D^+(n,j_0,j)=\mathbb{1}_{j\geq 0}O\left(\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right), \tag{5.4.8a}$$
$$\forall j_0\in-\mathbb{N}\backslash\{0\},\quad D^-(n,j_0,j)=\mathbb{1}_{j\leq 0}O\left(\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^-}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right), \tag{5.4.8b}$$

where the term O is uniform with respect to n,j_0 and j.

— The terms $R^\pm(n,j_0,j)$ correspond to a remainder term for the activation of the eigenvector associated with the eigenvalue 1 of the operator $\mathscr{L}$:

$$\forall j_0\in\mathbb{N},\quad R^+(n,j_0,j)=O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right), \tag{5.4.9a}$$
$$\forall j_0\in-\mathbb{N}\backslash\{0\},\quad R^-(n,j_0,j)=O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^-}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right), \tag{5.4.9b}$$

where the term O is uniform with respect to n,j_0 and j.

Applying the decompositions (5.4.6) of the Green's function in the right-hand side term of (5.4.5), we have that for $h\in\ell^\infty(\mathbb{Z})$, $n\in\mathbb{N}\backslash\{0\}$ and $j\in\mathbb{Z}$:

$$\begin{aligned}(\mathscr{L}^n h)_j=T_D^+(h,n,j)+T_D^-(h,n,j)+T_R^+(h,n,j)+T_R^-(h,n,j)+T_E(h,n,j)&\\ +\sum_{j_0\in\mathbb{Z}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}O(e^{-cn})h_{j_0}&\end{aligned} \tag{5.4.10}$$

where the terms $T_D^\pm(h,n,j)$, $T_R^\pm(h,n,j)$ and $T_E(h,n,j)$ are respectively defined by:

$$T_D^+(h,n,j):=\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}D^+(n,j_0,j)h_{j_0}, \tag{5.4.11a}$$

$$T_D^-(h,n,j) := \sum_{j_0\in -\mathbb{N}\backslash\{0\}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} D^-(n,j_0,j)h_{j_0}, \tag{5.4.11b}$$

$$T_R^+(h,n,j) := \sum_{j_0\in \mathbb{N}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} R^+(n,j_0,j)h_{j_0}, \tag{5.4.11c}$$

$$T_R^-(h,n,j) := \sum_{j_0\in -\mathbb{N}\backslash\{0\}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} R^-(n,j_0,j)h_{j_0}, \tag{5.4.11d}$$

$$T_E(h,n,j) := \left(\sum_{j_0\in \mathbb{N}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} E^+(n,j_0)h_{j_0} + \sum_{j_0\in -\mathbb{N}\backslash\{0\}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} E^-(n,j_0)h_{j_0}\right) V(j). \tag{5.4.11e}$$

In the following lemma, we will prove sharp estimates on the terms that appear in the right-hand side of (5.4.10).

Lemma 5.4.1. *For $0\leq\gamma_1\leq\gamma_2$, there exists a constant $C>0$ such that:*

• ***First terms of the decomposition (the diffusion waves):***

$$\forall n\in\mathbb{N}\backslash\{0\}, \forall h\in\ell^1_{\gamma_2}, \quad \left\|\left(T_D^\pm(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} \leq \frac{C}{n^{\gamma_2-\gamma_1}}\|h\|_{\ell^1_{\gamma_2}}, \tag{5.4.12a}$$

$$\forall n\in\mathbb{N}\backslash\{0\}, \forall h\in\ell^\infty_{\gamma_2}, \quad \left\|\left(T_D^\pm(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}} \leq \frac{C}{n^{\gamma_2-\gamma_1}}\|h\|_{\ell^\infty_{\gamma_2}} \tag{5.4.12b}$$

$$\forall n\in\mathbb{N}\backslash\{0\}, \forall h\in\ell^1_{\gamma_2}, \quad \left\|\left(T_D^\pm(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}} \leq \frac{C}{n^{\gamma_2-\gamma_1+\frac{1}{2\mu}}}\|h\|_{\ell^1_{\gamma_2}}, . \tag{5.4.12c}$$

• ***Second term of the decomposition (the activation remainder):***

$$\forall n\in\mathbb{N}\backslash\{0\}, \forall h\in\ell^\infty_{\gamma_2}, \quad \left\|\left(T_R^\pm(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} \leq \frac{C}{n^{\gamma_2}}\|h\|_{\ell^\infty_{\gamma_2}}, \tag{5.4.13a}$$

$$\forall n\in\mathbb{N}\backslash\{0\}, \forall h\in\ell^1_{\gamma_2}, \quad \left\|\left(T_R^\pm(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} \leq \frac{C}{n^{\gamma_2+\frac{1}{2\mu}}}\|h\|_{\ell^1_{\gamma_2}}. \tag{5.4.13b}$$

• ***Third term of the decomposition:***

$$\forall n\in\mathbb{N}\backslash\{0\}, \forall h\in\ell^1_{\gamma_2}, \quad \left\|\left(\sum_{j_0\in\mathbb{Z}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} O(e^{-cn})h_{j_0}\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} \leq Ce^{-\frac{c}{2}n}\|h\|_{\ell^1_{\gamma_2}}, \tag{5.4.14a}$$

$$\forall n\in\mathbb{N}\backslash\{0\}, \forall h\in\ell^\infty_{\gamma_2}, \quad \left\|\left(\sum_{j_0\in\mathbb{Z}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} O(e^{-cn})h_{j_0}\right)_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}} \leq Ce^{-\frac{c}{2}n}\|h\|_{\ell^\infty_{\gamma_2}}. \tag{5.4.14b}$$

• ***Fourth term of the decomposition (associated with the eigenvalue 1 of $\mathscr{L}$):***

$$\forall n\in\mathbb{N}\backslash\{0\}, \forall h\in\mathbb{E}_{\gamma_2}, \quad \left\|\left(T_E(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} \leq \frac{C}{n^{\gamma_2}}\|h\|_{\ell^1_{\gamma_2}}. \tag{5.4.15}$$

We point out that we exhibited estimates of some of the terms (for instance the second or fourth term) only in the $\ell^1_{\gamma_1}$-norm. Indeed, we can obtain estimates in the $\ell^\infty_{\gamma_1}$-norm by observing that:

$$\forall h\in\ell^1_{\gamma_1}, \quad \|h\|_{\ell^\infty_{\gamma_1}} \leq \|h\|_{\ell^1_{\gamma_1}}.$$

Combining the results of Lemma 5.4.1 and the equality (5.4.10), we can immediately obtain the inequalities (5.1.5a) and (5.1.5b) on the semi-group $(\mathscr{L}^n)_{n\in\mathbb{N}}$. The result of Lemma 5.4.1 is actually slightly more complete than what is necessary to prove (5.1.5a) and (5.1.5b). This completeness will be useful in the Section 5.4.2 tackling the proofs of (5.1.5c) and (5.1.5d).

Proof of Lemma 5.4.1

We fix $0\leq\gamma_1\leq\gamma_2$. We will separate the proofs of the different estimates claimed in Lemma 5.4.1.

▶ **Proof of the estimates (5.4.12) on the first term (diffusion waves):**

In this part, we will only prove the estimates for the sequence $\left(T_D^+(h,n,j)\right)_{j\in\mathbb{Z}}$. Everything can then easily be extended for the sequence $\left(T_D^-(h,n,j)\right)_{j\in\mathbb{Z}}$.

First, using the expressions (5.4.11a) of $T_D^+(h,n,j)$ and (5.4.8a) of $D^+(n,j_0,j)$, we observe that for $n \in \mathbb{N}\backslash\{0\}$, $h \in \ell^\infty_{\gamma_2}$ and $j \in \mathbb{Z}$, we have:

$$\begin{aligned}
&(1+|j|)^{\gamma_1}\left|T_D^+(h,n,j)\right| \\
=&(1+|j|)^{\gamma_1}\left|\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\mathbb{1}_{j\geq 0}O\left(\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)h_{j_0}\right| \\
\lesssim&\mathbb{1}_{j\geq 0}\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}(1+|j|)^{\gamma_1}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|
\end{aligned} \tag{5.4.16}$$

where the notations $\lesssim$, introduced in the "Notations" paragraph of the introduction, is used to ease the reading and describes inequalities up to a multiplicative constant independent from the parameters h, n and j.

• *Proof of the estimate* (5.4.12a)

We consider $h \in \ell^1_{\gamma_2}$ and $n \in \mathbb{N}\backslash\{0\}$. To prove (5.4.12a), we want to find estimates on the sum:

$$\left\|\left(T_D^+(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} = \sum_{j\in\mathbb{Z}}(1+|j|)^{\gamma_1}\left|T_D^+(h,n,j)\right|.$$

We observe that the inequality (5.4.16) implies that we only need to find bounds on

$$\sum_{j\in\mathbb{Z}}\mathbb{1}_{j\geq 0}\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}(1+|j|)^{\gamma_1}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|. \tag{5.4.17a}$$

We will decompose the sum (5.4.17a) with respect to j according to various regimes of $j - j_0$.

⋆ For $n \in \mathbb{N}\backslash\{0\}$ and $h \in \ell^1_{\gamma_2}$, we have using the inequality (5.4.4) and the fact that $\alpha^+ < 0$:

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\mathbb{1}_{j\geq 0}\sum_{j_0=0}^{j}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}(1+|j|)^{\gamma_1}\frac{1}{n^{\frac{1}{2\mu}}}\underbrace{\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)}_{\leq\exp(-cn)}|h_{j_0}| \\
&\lesssim \sum_{j_0\in\mathbb{N}}\underbrace{\sum_{j\geq j_0}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}}_{\lesssim n}\frac{(1+n)^{\gamma_1}}{n^{\frac{1}{2\mu}}}\exp(-cn)(1+|j_0|)^{\gamma_1}|h_{j_0}| \\
&\lesssim \exp\left(-\frac{c}{2}n\right)\|h\|_{\ell^1_{\gamma_1}} \\
&\lesssim \exp\left(-\frac{c}{2}n\right)\|h\|_{\ell^1_{\gamma_2}}.
\end{aligned} \tag{5.4.17b}$$

⋆ For $n \in \mathbb{N}\backslash\{0\}$ and $h \in \ell^1_{\gamma_2}$, we have using the inequality (5.4.2) and the fact that $\gamma_1 \leq \gamma_2$:

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\mathbb{1}_{j\geq 0}\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\underbrace{(1+|j|)^{\gamma_1}}_{\leq(1+|j_0|)^{\gamma_1}}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}| \\
&\lesssim \sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\underbrace{\sum_{j=0}^{j_0-1}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}}_{\lesssim n}\frac{1}{n^{\frac{1}{2\mu}}}\exp(-\tilde{c}n)(1+|j_0|)^{\gamma_1}|h_{j_0}| \\
&\lesssim \exp\left(-\frac{\tilde{c}}{2}n\right)\|h\|_{\ell^1_{\gamma_1}} \\
&\lesssim \exp\left(-\frac{\tilde{c}}{2}n\right)\|h\|_{\ell^1_{\gamma_2}}.
\end{aligned} \tag{5.4.17c}$$

$\star$ For $n \in \mathbb{N}\backslash\{0\}$ and $h \in \ell^1_{\gamma_2}$, we have using the inequality (5.4.3):

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \mathbb{1}_{j\geq 0} \sum_{j_0 \geq j+1} \mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} \underbrace{(1+|j|)^{\gamma_1}}_{\leq (1+|j_0|)^{\gamma_1}} \frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}| \\
&\quad \lesssim \sum_{j_0\in\mathbb{N}} \mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[} \underbrace{\left(\sum_{j=0}^{j_0-1} \frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)}_{\lesssim 1} \frac{(1+|j_0|)^{\gamma_2}}{|j_0|^{\gamma_2-\gamma_1}}|h_{j_0}| \qquad (5.4.17d)\\
&\quad \lesssim \frac{1}{n^{\gamma_2-\gamma_1}} \|h\|_{\ell^1_{\gamma_2}}.
\end{aligned}$$

Using (5.4.16) and (5.4.17b)-(5.4.17d) allows us to conclude the proof of (5.4.12a).

• *Proof of the estimate (5.4.12b)*

The proof of (5.4.12b) is fairly similar to (5.4.12a) with slight modifications made precise below. For $n \in \mathbb{N}\backslash\{0\}$ and $h \in \ell^\infty_{\gamma_2}$, we want to prove bounds

$$\left\|\left(T^+_D(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}} = \sup_{j\in\mathbb{Z}}(1+|j|)^{\gamma_1}\left|T^+_D(h,n,j)\right|.$$

The inequality (5.4.16) thus leads us to decompose and study the sum for $j \in \mathbb{Z}$

$$\mathbb{1}_{j\geq 0}\sum_{j_0\in\mathbb{N}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}(1+|j|)^{\gamma_1}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|. \qquad (5.4.18a)$$

$\star$ For $n \in \mathbb{N}\backslash\{0\}$, $h \in \ell^\infty_{\gamma_2}$ and $j \in \mathbb{N}$, we have using (5.4.4):

$$\begin{aligned}
&\sum_{j_0=0}^{j} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}(1+|j|)^{\gamma_1}\frac{1}{n^{\frac{1}{2\mu}}}\underbrace{\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)}_{\leq \exp(-cn)}|h_{j_0}| \\
&\lesssim \underbrace{\sum_{j_0=0}^{j}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}}_{\lesssim n}\frac{(1+n)^{\gamma_1}}{n^{\frac{1}{2\mu}}}\exp(-cn)\underbrace{(1+|j_0|)^{\gamma_1}|h_{j_0}|}_{\leq\|h\|_{\ell^\infty_{\gamma_2}}} \qquad (5.4.18b)\\
&\lesssim \exp\left(-\frac{c}{2}n\right)\|h\|_{\ell^\infty_{\gamma_2}}.
\end{aligned}$$

$\star$ For $n \in \mathbb{N}\backslash\{0\}$, $h \in \ell^\infty_{\gamma_2}$ and $j \in \mathbb{N}$, we have using the inequality (5.4.2):

$$\begin{aligned}
&\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\underbrace{(1+|j|)^{\gamma_1}}_{\leq(1+|j_0|)^{\gamma_1}}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}| \\
&\quad\lesssim \underbrace{\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}}_{\lesssim n}\frac{1}{n^{\frac{1}{2\mu}}}\exp(-\tilde{c}n)\underbrace{(1+|j_0|)^{\gamma_1}|h_{j_0}|}_{\leq\|h\|_{\ell^\infty_{\gamma_2}}} \qquad (5.4.18c)\\
&\quad\lesssim \exp\left(-\frac{\tilde{c}}{2}n\right)\|h\|_{\ell^\infty_{\gamma_2}}.
\end{aligned}$$

⋆ For $n\in\mathbb{N}\backslash\{0\}$, $h\in\ell^\infty_{\gamma_2}$ and $j\in\mathbb{N}$, we have using (5.4.3):

$$\begin{aligned}
&\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\underbrace{(1+|j|)^{\gamma_1}}_{\leq(1+|j_0|)^{\gamma_1}}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|\\
&\lesssim\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\left(\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)\frac{(1+|j_0|)^{\gamma_2}}{|j_0|^{\gamma_2-\gamma_1}}|h_{j_0}|\\
&\lesssim\underbrace{\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\left(\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)}_{\lesssim 1}\frac{1}{n^{\gamma_2-\gamma_1}}\|h\|_{\ell^\infty_{\gamma_2}}\\
&\lesssim\frac{1}{n^{\gamma_2-\gamma_1}}\|h\|_{\ell^\infty_{\gamma_2}}.
\end{aligned}\tag{5.4.18d}$$

Using (5.4.16) and (5.4.18b)-(5.4.18d) allows us to conclude the proof of (5.4.12b).

• *Proof of the estimate* (5.4.12c)

The proof of (5.4.12c) is essentially the same one as for (5.4.12b). Indeed, we still need to find estimates for (5.4.18a) but this time for $h\in\ell^1_{\gamma_2}$ and $n\in\mathbb{N}\backslash\{0\}$. Let us observe that we have the following inequality

$$\forall h\in\ell^1_{\gamma_2},\quad \|h\|_{\ell^\infty_{\gamma_2}}\leq\|h\|_{\ell^1_{\gamma_2}}.\tag{5.4.19a}$$

This inequality allows us to immediately adapt some intermediate results in the proof of (5.4.12b), which applied to the case where h belonged to $\ell^\infty_{\gamma_2}$, for the proof of (5.4.12c), where h belonged to $\ell^1_{\gamma_2}$. This will be explained below.

⋆ For $n\in\mathbb{N}\backslash\{0\}$, $h\in\ell^1_{\gamma_2}$ and $j\in\mathbb{N}$, we have using (5.4.18b) and (5.4.19a):

$$\begin{aligned}
&\sum_{j_0=0}^{j}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}(1+|j|)^{\gamma_1}\underbrace{\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)}_{\leq\exp(-cn)}|h_{j_0}|\\
&\qquad\leq\exp(-cn)\|h\|_{\ell^1_{\gamma_1}}\leq\exp(-cn)\|h\|_{\ell^1_{\gamma_2}}.
\end{aligned}\tag{5.4.19b}$$

⋆ For $n\in\mathbb{N}\backslash\{0\}$, $h\in\ell^1_{\gamma_2}$ and $j\in\mathbb{N}$, we have using (5.4.18c) and (5.4.19a):

$$\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\underbrace{(1+|j|)^{\gamma_1}}_{\leq(1+|j_0|)^{\gamma_1}}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|\lesssim\exp\left(-\frac{\tilde{c}}{2}n\right)\|h\|_{\ell^1_{\gamma_2}}.\tag{5.4.19c}$$

⋆ For $n\in\mathbb{N}\backslash\{0\}$, $h\in\ell^1_{\gamma_2}$ and $j\in\mathbb{N}$, we have:

$$\begin{aligned}
&\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\underbrace{(1+|j|)^{\gamma_1}}_{\leq(1+|j_0|)^{\gamma_1}}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|\\
&\lesssim\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\frac{1}{n^{\frac{1}{2\mu}}}\frac{(1+|j_0|)^{\gamma_2}}{|j_0|^{\gamma_2-\gamma_1}}|h_{j_0}|\\
&\lesssim\frac{1}{n^{\gamma_2-\gamma_1+\frac{1}{2\mu}}}\|h\|_{\ell^1_{\gamma_2}}.
\end{aligned}\tag{5.4.19d}$$

Using (5.4.16) and (5.4.19b)-(5.4.19d) allows us to conclude the proof of (5.4.12c).

▶ **Proof of the estimates (5.4.13) on the second term:**

In this part, we will only prove the estimates for the sequence $\left(T_R^+(h,n,j)\right)_{j\in\mathbb{Z}}$. Everything can then easily be extended for the sequence $\left(T_R^-(h,n,j)\right)_{j\in\mathbb{Z}}$.

First, using the expressions (5.4.11c) of $T_R^+(h,n,j)$ and (5.4.9a) of $R^+(n,j_0,j)$, we observe that for $n\in\mathbb{N}\backslash\{0\}$

and $h \in \ell^\infty_{\gamma_2}$, we have:

$$\begin{aligned}\left\|\left(T_R^+(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} &= \sum_{j\in\mathbb{Z}}(1+|j|)^{\gamma_1}\left|T_R^+(h,n,j)\right| \\ &= \sum_{j\in\mathbb{Z}}(1+|j|)^{\gamma_1}\left|\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)h_{j_0}\right| \\ &\lesssim \sum_{j_0\in\mathbb{N}}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|.\end{aligned} \tag{5.4.20}$$

Thus, to prove the estimates (5.4.13), we just need to find bounds for $n\in\mathbb{N}\backslash\{0\}$ on:

$$\sum_{j_0\in\mathbb{N}}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|$$

when $h\in\ell^\infty_{\gamma_2}$ and when $h\in\ell^1_{\gamma_2}$.

• *Proof of the estimate (5.4.13a)*

⋆ We observe using (5.4.2) that for $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^\infty_{\gamma_2}$:

$$\begin{aligned}&\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}| \\ &\qquad\lesssim \frac{n}{n^{\frac{1}{2\mu}}}e^{-\tilde{c}n}\|h\|_{\ell^\infty}\lesssim e^{-\frac{\tilde{c}}{2}n}\|h\|_{\ell^\infty}\lesssim e^{-\frac{\tilde{c}}{2}n}\|h\|_{\ell^\infty_{\gamma_2}}.\end{aligned} \tag{5.4.21a}$$

⋆ We observe that for $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^\infty_{\gamma_2}$, we have using (5.4.3) that:

$$\begin{aligned}&\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}| \\ &\leq \sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\frac{(1+|j_0|)^{\gamma_2}}{|j_0|^{\gamma_2}}|h_{j_0}| \\ &\lesssim \frac{\|h\|_{\ell^\infty_{\gamma_2}}}{n^{\gamma_2}}.\end{aligned} \tag{5.4.21b}$$

Thus, using (5.4.20),(5.4.21a) and (5.4.21b), we can prove (5.4.13a).

• *Proof of the estimate (5.4.13b)*

⋆ First, using (5.4.19a) and (5.4.21a), we observe that for $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^1_{\gamma_2}$:

$$\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|\lesssim e^{-\frac{\tilde{c}}{2}n}\|h\|_{\ell^\infty_{\gamma_2}}\lesssim e^{-\frac{\tilde{c}}{2}n}\|h\|_{\ell^1_{\gamma_2}}. \tag{5.4.22a}$$

⋆ For $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^1_{\gamma_2}$, we have that:

$$\begin{aligned}\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}| &\leq \sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\frac{1}{n^{\frac{1}{2\mu}}}\frac{(1+|j_0|)^{\gamma_2}}{|j_0|^{\gamma_2}}|h_{j_0}| \\ &\lesssim \frac{\|h\|_{\ell^1_{\gamma_2}}}{n^{\gamma_2+\frac{1}{2\mu}}}.\end{aligned} \tag{5.4.22b}$$

Thus, using (5.4.20),(5.4.22a) and (5.4.22b), we can prove (5.4.13b).

▶ **Proof of the estimates (5.4.14) on the third term:**

For $h\in\ell^1_{\gamma_2}$ and $n\in\mathbb{N}\backslash\{0\}$, using (5.4.4), we have that:

$$\begin{aligned}\sum_{j\in\mathbb{Z}}(1+|j|)^{\gamma_1}\left|\sum_{j_0\in\mathbb{Z}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}O(e^{-cn})h_{j_0}\right| &\lesssim (1+n)^{\gamma_1}e^{-cn}\sum_{j_0\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\right)(1+|j_0|)^{\gamma_1}|h_{j_0}|\\ &\lesssim n(1+n)^{\gamma_1}e^{-cn}\|h\|_{\ell^1_{\gamma_2}}\\ &\lesssim e^{-\frac{c}{2}n}\|h\|_{\ell^1_{\gamma_2}}.\end{aligned}$$

We thus obtain (5.4.14a).

Similarly, for $h\in\ell^\infty_{\gamma_2}$ and $n\in\mathbb{N}\backslash\{0\}$ and $j\in\mathbb{Z}$, using (5.4.4), we have that:

$$\begin{aligned}(1+|j|)^{\gamma_1}\left|\sum_{j_0\in\mathbb{Z}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}O(e^{-cn})h_{j_0}\right| &\lesssim (1+n)^{\gamma_1}e^{-cn}\underbrace{\sum_{j_0\in\mathbb{Z}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}}_{\lesssim n}\underbrace{(1+|j_0|)^{\gamma_1}|h_{j_0}|}_{\leq\|h\|_{\ell^\infty_{\gamma_1}}\leq\|h\|_{\ell^\infty_{\gamma_2}}}\\ &\lesssim n(1+n)^{\gamma_1}e^{-cn}\|h\|_{\ell^\infty_{\gamma_2}}\\ &\lesssim e^{-\frac{c}{2}n}\|h\|_{\ell^\infty_{\gamma_2}}.\end{aligned}$$

We can then deduce (5.4.14b).

▶ **Proof of the estimates (5.4.15) on the fourth term:**

We start the proof of (5.4.15) with an observation. Using the definition (5.4.11e) of $T_E(h,n,j)$, for $n\in\mathbb{N}\backslash\{0\}$ and $h\in\mathbb{E}_{\gamma_2}$, we have:

$$\begin{aligned}&\left\|(T_E(h,n,j))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\\ &=\left\|\left(\left(\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}E^+(n,j_0)h_{j_0}+\sum_{j_0\in-\mathbb{N}\backslash\{0\}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}E^-(n,j_0)h_{j_0}\right)V(j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\\ &\leq\left\|\left(\left(\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\notin\{-nq,\dots,np\}}E^+(n,j_0)h_{j_0}+\sum_{j_0\in-\mathbb{N}\backslash\{0\}}\mathbb{1}_{j-j_0\notin\{-nq,\dots,np\}}E^-(n,j_0)h_{j_0}\right)V(j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\\ &\quad+\left\|\left(\left(\sum_{j_0\in\mathbb{N}}E^+(n,j_0)h_{j_0}+\sum_{j_0\in-\mathbb{N}\backslash\{0\}}E^-(n,j_0)h_{j_0}\right)V(j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\end{aligned}$$

Thus, we have that for all $n\in\mathbb{N}\backslash\{0\}$ and $h\in\mathbb{E}_{\gamma_2}$:

$$\begin{aligned}&\left\|(T_E(h,n,j))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\\ &\leq\left\|\left(\left(\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\notin\{-nq,\dots,np\}}E^+(n,j_0)h_{j_0}+\sum_{j_0\in-\mathbb{N}\backslash\{0\}}\mathbb{1}_{j-j_0\notin\{-nq,\dots,np\}}E^-(n,j_0)h_{j_0}\right)V(j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\\ &\quad+\left|\sum_{j_0\in\mathbb{N}}E^+(n,j_0)h_{j_0}+\sum_{j_0\in-\mathbb{N}\backslash\{0\}}E^-(n,j_0)h_{j_0}\right|\|V\|_{\ell^1_{\gamma_1}}.\end{aligned}\tag{5.4.23}$$

Furthermore, using a similar proof as for (4.2.10), we can prove that there exist two positive constants $C,c>0$ such that:

$$\begin{aligned}&\forall n\in\mathbb{N}\backslash\{0\},\forall h\in\ell^\infty(\mathbb{Z}),\\ &\left\|\left(\left(\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\notin\{-nq,\dots,np\}}E^+(n,j_0)h_{j_0}+\sum_{j_0\in-\mathbb{N}\backslash\{0\}}\mathbb{1}_{j-j_0\notin\{-nq,\dots,np\}}E^-(n,j_0)h_{j_0}\right)V(j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\\ &\qquad\leq Ce^{-cn}\|h\|_{\ell^\infty}.\end{aligned}\tag{5.4.24}$$

Combining (5.4.23) and (5.4.24), we have that for all $n \in \mathbb{N}\backslash\{0\}$ and $h \in \mathbb{E}_{\gamma_2}$:

$$\left\| (T_E(h,n,j))_{j\in\mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} \leq Ce^{-cn} \|h\|_{\ell^1_{\gamma_2}} + \left| \sum_{j_0\in\mathbb{N}} E^+(n,j_0)h_{j_0} + \sum_{j_0\in-\mathbb{N}\backslash\{0\}} E^-(n,j_0)h_{j_0} \right| \|V\|_{\ell^1_{\gamma_1}}. \tag{5.4.25}$$

Therefore, if we prove that there exists a constant $\widetilde{C} > 0$ such that:

$$\forall h \in \mathbb{E}_{\gamma_2}, \forall n \in \mathbb{N}\backslash\{0\}, \quad \left| \sum_{j_0\in\mathbb{N}} E^+(n,j_0)h_{j_0} + \sum_{j_0\in-\mathbb{N}\backslash\{0\}} E^-(n,j_0)h_{j_0} \right| \leq \frac{\widetilde{C}}{n^{\gamma_2}} \|h\|_{\ell^1_{\gamma_2}}, \tag{5.4.26}$$

then, combining (5.4.25) and (5.4.26) allows us to conclude on the proof of (5.4.15). Therefore, there just remains to prove (5.4.26).

Using the definition (5.4.7) of the terms $E^{\pm}$ and the zero-mass assumption on h, we have that:

$$\begin{aligned}
&\left| \sum_{j_0\in\mathbb{N}} E^+(n,j_0)h_{j_0} + \sum_{j_0\in-\mathbb{N}\backslash\{0\}} E^-(n,j_0)h_{j_0} \right| \\
=&|C^E| \left| \sum_{j_0\in\mathbb{N}} E_{2\mu}\left(\beta^+; \frac{n\alpha^+ + j_0}{n^{\frac{1}{2\mu}}}\right) h_{j_0} + \sum_{j_0\in-\mathbb{N}\backslash\{0\}} E_{2\mu}\left(\beta^-; \frac{-n\alpha^- - j_0}{n^{\frac{1}{2\mu}}}\right) h_{j_0} - \underbrace{\sum_{j_0\in\mathbb{Z}} h_{j_0}}_{=0} \right| \qquad (5.4.27a)\\
\leq&|C^E| \left(\sum_{j_0\in\mathbb{N}} \left| E_{2\mu}\left(\beta^+; \frac{n\alpha^+ + j_0}{n^{\frac{1}{2\mu}}}\right) - 1 \right| |h_{j_0}| + \sum_{j_0\in-\mathbb{N}\backslash\{0\}} \left| E_{2\mu}\left(\beta^-; \frac{-n\alpha^- - j_0}{n^{\frac{1}{2\mu}}}\right) - 1 \right| |h_{j_0}| \right).
\end{aligned}$$

We now need to prove estimates on the terms on the right-hand side of (5.4.27a). We will decompose the sums with respect to j_0 according to various regimes depending on n.

First, using the inequality (4.1.30c) on the function $E_{2\mu}$, we have that there exists a positive constant c such that for all $n \in \mathbb{N}\backslash\{0\}$ and $h \in \mathbb{E}_{\gamma_2}$:

$$\begin{aligned}
&\sum_{j_0\in\mathbb{N}} \mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[} \left| E_{2\mu}\left(\beta^+; \frac{n\alpha^+ + j_0}{n^{\frac{1}{2\mu}}}\right) - 1 \right| |h_{j_0}| \\
&\qquad \lesssim \sum_{j_0\in\mathbb{N}} \mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[} \frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{\left|n + \frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) |h_{j_0}| \\
&\qquad \lesssim \underbrace{\sum_{j_0\in\mathbb{N}} \mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}}_{\lesssim n} \frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-\frac{c}{2^{\frac{2\mu}{2\mu-1}}} n \right) \|h\|_{\ell^\infty} \\
&\qquad \lesssim e^{-\frac{c}{8}n} \|h\|_{\ell^1_{\gamma_2}}
\end{aligned} \tag{5.4.27b}$$

and similarly:

$$\sum_{j_0\in-\mathbb{N}\backslash\{0\}} \mathbb{1}_{j_0\in\left]-\frac{n\alpha^-}{2},0\right]} \left| E_{2\mu}\left(\beta^-; \frac{-n\alpha^- - j_0}{n^{\frac{1}{2\mu}}}\right) - 1 \right| |h_{j_0}| \lesssim e^{-\frac{c}{8}n} \|h\|_{\ell^1_{\gamma_2}}. \tag{5.4.27c}$$

Furthermore, for all $n \in \mathbb{N}\backslash\{0\}$ and $h \in \mathbb{E}_{\gamma_2}$:

$$\begin{aligned}
&\sum_{j_0\in\mathbb{N}} \mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[} \underbrace{\left| E_{2\mu}\left(\beta^+; \frac{n\alpha^+ + j_0}{n^{\frac{1}{2\mu}}}\right) - 1 \right|}_{\leq 2} |h_{j_0}| \\
&\qquad \lesssim \sum_{j_0\in\mathbb{N}} \mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[} \frac{(1+|j_0|)^{\gamma_2}}{|j_0|^{\gamma_2}} |h_{j_0}| \lesssim \frac{\|h\|_{\ell^1_{\gamma_2}}}{n^{\gamma_2}}
\end{aligned} \tag{5.4.27d}$$

and similarly:

$$\sum_{j_0\in-\mathbb{N}\backslash\{0\}}\mathbb{1}_{j_0\in\left]-\infty,-\frac{n\alpha^-}{2}\right]}\underbrace{\left|E_{2\mu}\left(\beta^-;\frac{-n\alpha^--j_0}{n^{\frac{1}{2\mu}}}\right)-1\right|}_{\leq 2}|h_{j_0}|\lesssim\frac{\|h\|_{\ell^1_{\gamma_2}}}{n^{\gamma_2}}. \tag{5.4.27e}$$

Thus, combining (5.4.27a)-(5.4.27e), we conclude the proof of (5.4.26) and thus of (5.4.15). Thus, this concludes the proof of Lemma 5.4.1. ☐

5.4.2 Proof of the estimates (5.1.5c) and (5.1.5d) on the operator $\mathscr{L}^n(Id-\mathcal{T})$

The methodology to prove the estimates (5.1.5c) and (5.1.5d) on the operators $\mathscr{L}^n(Id-\mathcal{T})$ will be exactly the same as for the proof of the estimates (5.1.5a) and (5.1.5b) on the operators $\mathscr{L}^n$

Using the definition (5.1.4) of the operator $\mathcal{T}$ and the equality (5.4.5) above, we have that for $n\in\mathbb{N}$, $h\in\ell^\infty(\mathbb{Z})$ and for $j\in\mathbb{Z}$:

$$(\mathscr{L}^n(Id-\mathcal{T})h)_j=\sum_{j_0\in\mathbb{Z}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}\mathscr{G}(n,j_0,j)(h_{j_0+1}-h_{j_0}).$$

Thus, we have that:

$$\forall n\in\mathbb{N},\forall h\in\ell^\infty(\mathbb{Z}),$$
$$\mathscr{L}^n(Id-\mathcal{T})h=\left(\sum_{j_0\in\mathbb{Z}}\mathbb{1}_{j-j_0\in\{-nq-1,\dots,np\}}\left(\mathscr{G}(n,j_0-1,j)-\mathscr{G}(n,j_0,j)\right)h_{j_0}\right)_{j\in\mathbb{Z}}. \tag{5.4.28}$$

We recall that, just like for the Green's function, in the introduction of this chapter, we have proved using the Proposition 5.1 an improved decomposition (5.1.3b) of the discrete derivative of the Green's function in the scalar case. We have that there exists a constant $c>0$ such that for $j_0\in\mathbb{Z}$, $n\in\mathbb{N}\backslash\{0\}$ and $j\in\mathbb{Z}$ such that $j-j_0\in\{-nq,\dots,np\}$, the discrete derivative of the Green's function verifies:

$$\begin{aligned}\forall j_0\in\mathbb{N},\quad \mathscr{G}(n,j_0,j)-\mathscr{G}(n,j_0-1,j)=D_1^+(n,j_0,j)+D_2^+(n,j_0,j)\\ +D_3^+(n,j_0,j)+R^+(n,j_0,j)+O(e^{-cn}),\end{aligned} \tag{5.4.29a}$$

$$\begin{aligned}\forall j_0\in-\mathbb{N}\backslash\{0\},\quad \mathscr{G}(n,j_0,j)-\mathscr{G}(n,j_0-1,j)=D_1^-(n,j_0,j)+D_2^-(n,j_0,j)\\ +D_3^-(n,j_0,j)+R^-(n,j_0,j)+O(e^{-cn}),\end{aligned} \tag{5.4.29b}$$

where:

— The terms $D_1^\pm(n,j_0,j)$, $D_2^\pm(n,j_0,j)$ and $D_3^\pm(n,j_0,j)$ correspond to the remainder of the diffusion waves:

$$\forall j_0\in\mathbb{N},\quad D_1^+(n,j_0,j)=\mathbb{1}_{j\geq0}O\left(\frac{1}{n^{\frac{1}{\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right), \tag{5.4.30a}$$

$$\forall j_0\in-\mathbb{N}\backslash\{0\},\quad D_1^-(n,j_0,j)=\mathbb{1}_{j\leq0}O\left(\frac{1}{n^{\frac{1}{\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^-}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right), \tag{5.4.30b}$$

$$\forall j_0\in\mathbb{N},\quad D_2^+(n,j_0,j)=\mathbb{1}_{j\geq0}O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right), \tag{5.4.31a}$$

$$\forall j_0\in-\mathbb{N}\backslash\{0\},\quad D_2^-(n,j_0,j)=\mathbb{1}_{j\leq0}O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^-}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right), \tag{5.4.31b}$$

$$\forall j_0\in\mathbb{N},\quad D_3^+(n,j_0,j)=\mathbb{1}_{j\geq 0}O\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^+}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right), \tag{5.4.32a}$$

$$\forall j_0\in-\mathbb{N}\backslash\{0\},\quad D_3^-(n,j_0,j)=\mathbb{1}_{j\leq 0}O\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\left(\frac{j-j_0}{\alpha^-}\right)\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right). \tag{5.4.32b}$$

— The terms $R^\pm(n,j_0,j)$ correspond to a remainder term for the activation of the eigenvector associated to the eigenvalue 1 of the operator $\mathscr{L}$:

$$\forall j_0\in\mathbb{N},\quad R^+(n,j_0,j)=O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right), \tag{5.4.33a}$$

$$\forall j_0\in-\mathbb{N}\backslash\{0\},\quad R^-(n,j_0,j)=O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n+\frac{j_0}{\alpha^-}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right). \tag{5.4.33b}$$

Applying the decompositions (5.4.29) of the discrete derivative of the Green's function in the right-hand side term of (5.4.28), we have that for $h\in\ell^\infty(\mathbb{Z})$, $n\in\mathbb{N}\backslash\{0\}$ and $j\in\mathbb{Z}$:

$$\begin{aligned}(\mathscr{L}^n(Id-\mathcal{T})h)_j=&\,T_{D,1}^+(h,n,j)+T_{D,2}^+(h,n,j)+T_{D,3}^+(h,n,j)\\&+T_{D,1}^-(h,n,j)+T_{D,2}^-(h,n,j)+T_{D,3}^-(h,n,j)+T_R^+(h,n,j)+T_R^-(h,n,j)\\&+\sum_{j_0\in\mathbb{Z}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}O(e^{-cn})h_{j_0}\end{aligned} \tag{5.4.34}$$

where the terms $T_{D,k}^\pm(h,n,j)$ for $k\in\{1,2,3\}$ and $T_R^\pm(h,n,j)$ are respectively defined by:

$$T_{D,k}^+(h,n,j):=\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq-1,\dots,np\}}D_k^+(n,j_0,j)h_{j_0}, \tag{5.4.35a}$$

$$T_{D,k}^-(h,n,j):=\sum_{j_0\in-\mathbb{N}\backslash\{0\}}\mathbb{1}_{j-j_0\in\{-nq-1,\dots,np\}}D_k^-(n,j_0,j)h_{j_0}, \tag{5.4.35b}$$

$$T_R^+(h,n,j):=\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq-1,\dots,np\}}R^+(n,j_0,j)h_{j_0}, \tag{5.4.35c}$$

$$T_R^-(h,n,j):=\sum_{j_0\in-\mathbb{N}\backslash\{0\}}\mathbb{1}_{j-j_0\in\{-nq-1,\dots,np\}}R^-(n,j_0,j)h_{j_0}. \tag{5.4.35d}$$

The following lemma is analogous to Lemma 5.4.1. We will prove sharp estimates on the terms that appear in the right-hand side of (5.4.34).

Lemma 5.4.2. *For $0\leq\gamma_1\leq\gamma_2$, there exists a constant $C>0$ such that:*

• ***First term of the decomposition:***

$$\forall n\in\mathbb{N}\backslash\{0\},\forall h\in\ell^1_{\gamma_2},\quad\left\|\left(T_{D,1}^\pm(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\leq\frac{C}{n^{\gamma_2-\gamma_1+\frac{1}{2\mu}}}\|h\|_{\ell^1_{\gamma_2}}, \tag{5.4.36a}$$

$$\forall n\in\mathbb{N}\backslash\{0\},\forall h\in\ell^1_{\gamma_2},\quad\left\|\left(T_{D,1}^\pm(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}\leq\frac{C}{n^{\gamma_2-\gamma_1+\frac{1}{\mu}}}\|h\|_{\ell^1_{\gamma_2}}, \tag{5.4.36b}$$

$$\forall n\in\mathbb{N}\backslash\{0\},\forall h\in\ell^\infty_{\gamma_2},\quad\left\|\left(T_{D,1}^\pm(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}\leq\frac{C}{n^{\gamma_2-\gamma_1+\frac{1}{2\mu}}}\|h\|_{\ell^\infty_{\gamma_2}}. \tag{5.4.36c}$$

• ***Second term of the decomposition:***

$$\forall n\in\mathbb{N}\backslash\{0\},\forall h\in\ell^1_{\gamma_2},\quad\left\|\left(T_{D,2}^\pm(h,n,j)\right)_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\leq\frac{C}{n^{\gamma_2+\frac{1}{2\mu}}}\|h\|_{\ell^1_{\gamma_2}}, \tag{5.4.37a}$$

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^\infty_{\gamma_2}, \quad \left\| \left(T^\pm_{D,2}(h,n,j) \right)_{j\in\mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} \leq \frac{C}{n^{\gamma_2}} \|h\|_{\ell^\infty_{\gamma_2}}. \tag{5.4.37b}$$

• ***Third term of the decomposition:*** *There exists a constant $\hat{c} > 0$ such that:*

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^1_{\gamma_2}, \quad \left\| \left(T^\pm_{D,3}(h,n,j) \right)_{j\in\mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} \leq Ce^{-\hat{c}n} \|h\|_{\ell^1_{\gamma_2}}, \tag{5.4.38a}$$

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^\infty_{\gamma_2}, \quad \left\| \left(T^\pm_{D,3}(h,n,j) \right)_{j\in\mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} \leq Ce^{-\hat{c}n} \|h\|_{\ell^\infty_{\gamma_2}}. \tag{5.4.38b}$$

• ***Fourth term of the decomposition:***

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^1_{\gamma_2}, \quad \left\| \left(T^\pm_{R}(h,n,j) \right)_{j\in\mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} \leq \frac{C}{n^{\gamma_2+\frac{1}{2\mu}}} \|h\|_{\ell^1_{\gamma_2}}, \tag{5.4.39a}$$

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^\infty_{\gamma_2}, \quad \left\| \left(T^\pm_{R}(h,n,j) \right)_{j\in\mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} \leq \frac{C}{n^{\gamma_2}} \|h\|_{\ell^\infty_{\gamma_2}}. \tag{5.4.39b}$$

• ***Fifth term of the decomposition:***

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^1_{\gamma_2}, \quad \left\| \left(\sum_{j_0\in\mathbb{Z}} \mathbb{1}_{j-j_0\in\{-nq-1,\dots,np\}} O(e^{-cn}) h_{j_0} \right)_{j\in\mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} \leq Ce^{-\frac{c}{2}n} \|h\|_{\ell^1_{\gamma_2}}, \tag{5.4.40a}$$

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \ell^\infty_{\gamma_2}, \quad \left\| \left(\sum_{j_0\in\mathbb{Z}} \mathbb{1}_{j-j_0\in\{-nq-1,\dots,np\}} O(e^{-cn}) h_{j_0} \right)_{j\in\mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} \leq Ce^{-\frac{c}{2}n} \|h\|_{\ell^\infty_{\gamma_2}}. \tag{5.4.40b}$$

Combining the results of Lemma 5.4.2 and the equality (5.4.34), we can immediately obtain the inequalities (5.1.5c) and (5.1.5d) on the family of operators $(\mathscr{L}^n(Id - \mathcal{T}))_{n\in\mathbb{N}}$. There just remains to conclude on the proof of Lemma 5.4.2.

Proof of Lemma 5.4.2

Let us observe that the estimates (5.4.39) for the fourth term and (5.4.40) for the fifth term have already been proved in Lemma 5.4.1. Furthermore, the term $D_1^\pm$ defined by (5.4.30) corresponds to the term $D^\pm$ defined by (5.4.8) with a factor $\frac{1}{n^{\frac{1}{\mu}}}$ rather than $\frac{1}{n^{\frac{1}{2\mu}}}$. Thus, the estimates (5.4.36) for the first term of Lemma 5.4.2 corresponds to the estimates (5.4.12) for the first term of Lemma 5.4.1. There will only remain to prove (5.4.37) and (5.4.38).

▶ **Proof of the estimates (5.4.37) on the second term:**

The proof of (5.4.37) is fairly similar to (5.4.13). For $n \in \mathbb{N}\backslash\{0\}$ and $h \in \ell^\infty_{\gamma_2}$, using the expressions (5.4.35a) of $T^+_{D,2}(h,n,j)$ and (5.4.31a) of $D_2^+(n,j_0,j)$, we have that:

$$\begin{aligned}
&\left\| \left(T^+_{D,2}(h,n,j) \right)_{j\in\mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} \\
&= \sum_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \left| \sum_{j_0\in\mathbb{N}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} \mathbb{1}_{j\geq 0} O\left(\frac{e^{-c|j|}}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{\left| n - \frac{j-j_0}{\alpha^+} \right|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) \right) h_{j_0} \right| \\
&\lesssim \sum_{j\in\mathbb{N}} \sum_{j_0\in\mathbb{N}} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} e^{-\frac{c}{2}|j|} \frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{\left| n - \frac{j-j_0}{\alpha^+} \right|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) |h_{j_0}|.
\end{aligned} \tag{5.4.41a}$$

We will separate the sum on the right-hand side of (5.4.41a).

• For $n \in \mathbb{N}\backslash\{0\}$ and $h \in \ell^\infty_{\gamma_2}$, we have that:

$$\sum_{j\in\mathbb{N}} \sum_{j_0=0}^{j} \mathbb{1}_{j-j_0\in\{-nq,\dots,np\}} e^{-\frac{c}{2}|j|} \frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c \left(\frac{\left| n - \frac{j-j_0}{\alpha^+} \right|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right) |h_{j_0}|$$

$$\leq \sum_{j\in\mathbb{N}} e^{-\frac{c}{2}|j|}\underbrace{\left(\sum_{j_0=0}^{j}\mathbb{1}_{j-j_0\in\{-nq,\ldots,np\}}\right)}_{\lesssim n}\frac{1}{n^{\frac{1}{2\mu}}}e^{-cn}\|h\|_{\ell^\infty}\lesssim e^{-\frac{c}{2}n}\|h\|_{\ell^\infty_{\gamma_2}}. \tag{5.4.41b}$$

- For $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^\infty_{\gamma_2}$, we have using (5.4.2) that:

$$\sum_{j\in\mathbb{N}}\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\mathbb{1}_{j-j_0\in\{-nq,\ldots,np\}}e^{-\frac{c}{2}|j|}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|$$
$$\leq \sum_{j\in\mathbb{N}} e^{-\frac{c}{2}|j|}\underbrace{\left(\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\mathbb{1}_{j-j_0\in\{-nq,\ldots,np\}}\right)}_{\lesssim n}\frac{1}{n^{\frac{1}{2\mu}}}e^{-\tilde{c}n}\|h\|_{\ell^\infty}\lesssim e^{-\frac{\tilde{c}}{2}n}\|h\|_{\ell^\infty_{\gamma_2}}. \tag{5.4.41c}$$

- For $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^\infty_{\gamma_2}$, we have using (5.4.3) that:

$$\sum_{j\in\mathbb{N}}\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\mathbb{1}_{j-j_0\in\{-nq,\ldots,np\}}e^{-\frac{c}{2}|j|}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|$$
$$\leq\sum_{j\in\mathbb{N}}e^{-\frac{c}{2}|j|}\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\underbrace{\frac{(1+|j_0|)^{\gamma_2}}{|j_0|^{\gamma_2}}|h_{j_0}|}_{\lesssim\frac{\|h\|_{\ell^\infty_{\gamma_2}}}{n^{\gamma_2}}} \tag{5.4.41d}$$
$$\lesssim\frac{\|h\|_{\ell^\infty_{\gamma_2}}}{n^{\gamma_2}}.$$

Thus, combining (5.4.41a)-(5.4.41d), we obtain (5.4.37a). There remains to prove (5.4.37b). To prove it, we observe that for $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^1_{\gamma_2}$, we have that:

$$\sum_{j\in\mathbb{N}}\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\mathbb{1}_{j-j_0\in\{-nq,\ldots,np\}}e^{-\frac{c}{2}|j|}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|$$
$$\leq\frac{1}{n^{\frac{1}{2\mu}}}\sum_{j\in\mathbb{N}}e^{-\frac{c}{2}|j|}\underbrace{\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\frac{(1+|j_0|)^{\gamma_2}}{|j_0|^{\gamma_2}}|h_{j_0}|}_{\lesssim\frac{\|h\|_{\ell^1_{\gamma_2}}}{n^{\gamma_2}}}\lesssim\frac{\|h\|_{\ell^1_{\gamma_2}}}{n^{\gamma_2+\frac{1}{2\mu}}}. \tag{5.4.41e}$$

Thus, combining (5.4.41a)-(5.4.41c) and (5.4.41e), we obtain (5.4.37b).

▶ **Proof of the estimates (5.4.38) on the third term:**

The proof of (5.4.38) is fairly similar to the proof of (5.4.12). Using the expressions (5.4.35a) of $T^+_{D,3}(h,n,j)$ and (5.4.32a) of $D^+_3(n,j_0,j)$, we observe that for $n\in\mathbb{N}\backslash\{0\}$, $h\in\ell^\infty_{\gamma_2}$ and $j\in\mathbb{Z}$, we have:

$$\begin{aligned}&(1+|j|)^{\gamma_1}\left|T^+_{D,3}(h,n,j)\right|\\ =&(1+|j|)^{\gamma_1}\left|\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\ldots,np\}}\mathbb{1}_{j\geq0}O\left(\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)h_{j_0}\right|\\ \lesssim&\mathbb{1}_{j\geq0}\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\ldots,np\}}(1+|j|)^{\gamma_1}\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|\end{aligned} \tag{5.4.42}$$

- *Proof of the estimate (5.4.38a)*

We consider $h \in \ell^1_{\gamma_2}$ and $n \in \mathbb{N}\backslash\{0\}$. To prove (5.4.38a), we want to find estimates on the sum:

$$\sum_{j\in\mathbb{Z}}(1+|j|)^{\gamma_1}\left|T^+_{D,3}(h,n,j)\right|.$$

We observe that the inequality (5.4.42) implies that we only need to find bounds on

$$\sum_{j\in\mathbb{Z}}\mathbb{1}_{j\geq 0}\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}(1+|j|)^{\gamma_1}\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|. \tag{5.4.43a}$$

We will decompose the sum (5.4.43a) with respect to j according to various regimes of $j-j_0$.
⋆ For $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^1_{\gamma_2}$, we have using the inequality (5.4.4) and the fact that $\alpha^+<0$:

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\mathbb{1}_{j\geq 0}\sum_{j_0=0}^{j}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}(1+|j|)^{\gamma_1}\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}&\underbrace{\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)}_{\leq\exp(-cn)}|h_{j_0}|\\ &\lesssim\sum_{j_0\in\mathbb{N}}\underbrace{\sum_{j\geq j_0}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}}_{\lesssim n}\frac{(1+n)^{\gamma_1}}{n^{\frac{1}{2\mu}}}\exp(-cn)(1+|j_0|)^{\gamma_1}|h_{j_0}|\\ &\lesssim\exp\left(-\frac{c}{2}n\right)\|h\|_{\ell^1_{\gamma_2}}.\end{aligned}\tag{5.4.43b}$$

⋆ For $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^1_{\gamma_2}$, we have using the inequality (5.4.2) and the fact that $\gamma_1\leq\gamma_2$:

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\mathbb{1}_{j\geq 0}\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}&\underbrace{(1+|j|)^{\gamma_1}}_{\leq(1+|j_0|)^{\gamma_1}}\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|\\ &\lesssim\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[0,-\frac{n\alpha^+}{2}\right[}\underbrace{\sum_{j=0}^{j_0-1}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}}_{\lesssim n}\frac{1}{n^{\frac{1}{2\mu}}}\exp(-\tilde{c}n)(1+|j_0|)^{\gamma_1}|h_{j_0}|\\ &\lesssim\exp\left(-\frac{\tilde{c}}{2}n\right)\|h\|_{\ell^1_{\gamma_1}}\\ &\lesssim\exp\left(-\frac{\tilde{c}}{2}n\right)\|h\|_{\ell^1_{\gamma_2}}.\end{aligned}\tag{5.4.43c}$$

⋆ There exists a constant $\hat{c}>0$ such that, for $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^1_{\gamma_2}$, we have using the inequality (5.4.3):

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\mathbb{1}_{j\geq 0}\sum_{j_0\geq j+1}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\mathbb{1}_{j-j_0\in\{-nq,\dots,np\}}&\underbrace{(1+|j|)^{\gamma_1}}_{\leq(1+|j_0|)^{\gamma_1}}\frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)|h_{j_0}|\\ &\lesssim\sum_{j_0\in\mathbb{N}}\mathbb{1}_{j_0\in\left[-\frac{n\alpha^+}{2},+\infty\right[}\frac{e^{-c|j_0|}}{|j_0|^{\gamma_2-\gamma_1}}\underbrace{\left(\sum_{j=0}^{j_0-1}\frac{1}{n^{\frac{1}{2\mu}}}\exp\left(-c\left(\frac{\left|n-\frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)}_{\lesssim 1}(1+|j_0|)^{\gamma_2}|h_{j_0}|\\ &\lesssim e^{-\hat{c}n}\|h\|_{\ell^1_{\gamma_2}}.\end{aligned}\tag{5.4.43d}$$

Using (5.4.42) and (5.4.43b)-(5.4.43d) allows us to conclude the proof of (5.4.38a).
• *Proof of the estimate* (5.4.38b)
The proof of (5.4.38b) is fairly similar to (5.4.38a) with slight modifications made precise below. For $n\in\mathbb{N}\backslash\{0\}$ and $h\in\ell^\infty_{\gamma_2}$, we want to prove bounds

$$(1+|j|)^{\gamma_1}\left|T^+_{D,3}(h,n,j)\right|$$

for $j \in \mathbb{Z}$. The inequality (5.4.42) thus leads us to decompose and study the sum

$$\mathbb{1}_{j\geq 0} \sum_{j_0 \in \mathbb{N}} \mathbb{1}_{j-j_0 \in \{-nq,\dots,np\}} (1+|j|)^{\gamma_1} \frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) |h_{j_0}|. \tag{5.4.44a}$$

⋆ For $n \in \mathbb{N}\backslash\{0\}$, $h \in \ell^\infty_{\gamma_2}$ and $j \in \mathbb{N}$, we have using (5.4.4):

$$\begin{aligned}
&\sum_{j_0=0}^{j} \mathbb{1}_{j-j_0 \in \{-nq,\dots,np\}} (1+|j|)^{\gamma_1} \frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}} \underbrace{\exp\left(-c\left(\frac{\left|n - \frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)}_{\leq \exp(-cn)} |h_{j_0}| \\
&\lesssim \underbrace{\sum_{j_0=0}^{j} \mathbb{1}_{j-j_0 \in \{-nq,\dots,np\}}}_{\lesssim n} \frac{(1+n)^{\gamma_1}}{n^{\frac{1}{2\mu}}} \exp(-cn) \underbrace{(1+|j_0|)^{\gamma_1}|h_{j_0}|}_{\leq \|h\|_{\ell^\infty_{\gamma_2}}} \\
&\lesssim \exp\left(-\frac{c}{2}n\right) \|h\|_{\ell^\infty_{\gamma_2}}.
\end{aligned} \tag{5.4.44b}$$

⋆ For $n \in \mathbb{N}\backslash\{0\}$, $h \in \ell^\infty_{\gamma_2}$ and $j \in \mathbb{N}$, we have using the inequality (5.4.2):

$$\begin{aligned}
&\sum_{j_0 \geq j+1} \mathbb{1}_{j_0 \in \left[0, -\frac{n\alpha^+}{2}\right[} \mathbb{1}_{j-j_0 \in \{-nq,\dots,np\}} \underbrace{(1+|j|)^{\gamma_1}}_{\leq (1+|j_0|)^{\gamma_1}} \frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) |h_{j_0}| \\
&\qquad \lesssim \underbrace{\sum_{j_0 \geq j+1} \mathbb{1}_{j_0 \in \left[0, -\frac{n\alpha^+}{2}\right[} \mathbb{1}_{j-j_0 \in \{-nq,\dots,np\}}}_{\lesssim n} \frac{1}{n^{\frac{1}{2\mu}}} \exp(-\tilde{c}n) \underbrace{(1+|j_0|)^{\gamma_1}|h_{j_0}|}_{\leq \|h\|_{\ell^\infty_{\gamma_2}}} \\
&\qquad \lesssim \exp\left(-\frac{\tilde{c}}{2}n\right) \|h\|_{\ell^\infty_{\gamma_2}}.
\end{aligned} \tag{5.4.44c}$$

⋆ There exists a constant $\hat{c} > 0$ such that, for $n \in \mathbb{N}\backslash\{0\}$, $h \in \ell^\infty_{\gamma_2}$ and $j \in \mathbb{N}$, we have using (5.4.3):

$$\begin{aligned}
&\sum_{j_0 \geq j+1} \mathbb{1}_{j_0 \in \left[-\frac{n\alpha^+}{2}, +\infty\right[} \mathbb{1}_{j-j_0 \in \{-nq,\dots,np\}} \underbrace{(1+|j|)^{\gamma_1}}_{\leq (1+|j_0|)^{\gamma_1}} \frac{e^{-c|j_0|}}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) |h_{j_0}| \\
&\quad \lesssim \sum_{j_0 \in \mathbb{N}} \mathbb{1}_{j_0 \in \left[-\frac{n\alpha^+}{2}, +\infty\right[} \frac{e^{-c|j_0|}}{|j_0|^{\gamma_2-\gamma_1}} \left(\frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right) (1+|j_0|)^{\gamma_2}|h_{j_0}| \\
&\quad \lesssim e^{-\hat{c}n} \underbrace{\sum_{j_0 \in \mathbb{N}} \mathbb{1}_{j_0 \in \left[-\frac{n\alpha^+}{2}, +\infty\right[} \left(\frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{\left|n - \frac{j-j_0}{\alpha^+}\right|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right)}_{\lesssim 1} \|h\|_{\ell^\infty_{\gamma_2}} \\
&\quad \lesssim \frac{1}{n^{\gamma_2-\gamma_1}} \|h\|_{\ell^\infty_{\gamma_2}}.
\end{aligned} \tag{5.4.44d}$$

Using (5.4.42) and (5.4.44b)-(5.4.44d) allows us to conclude the proof of (5.4.38b). □

5.A Appendix of the chapter

Proof of Lemma 5.2.1

Let us recall the statement of Lemma 5.2.1:

Lemma. *Let us consider $\gamma \in [0, +\infty[$. There exists a constant $C_Q(\gamma) > 0$ such that for $h \in \ell^1_\gamma$ such that*

$$\|h\|_{\ell^\infty} < \delta,$$

then the sequence $Q(h)$ belongs to $\ell^1_{2\gamma}$ and:

$$\|Q(h)\|_{\ell^1_{2\gamma}} \leq C_Q(\gamma)\,\|h\|_{\ell^1_\gamma}\,\|h\|_{\ell^\infty_\gamma}\,, \tag{5.A.1a}$$

$$\|Q(h)\|_{\ell^\infty_{2\gamma}} \leq C_Q(\gamma)\|h\|_{\ell^\infty_\gamma}{}^2. \tag{5.A.1b}$$

Proof We recall that the set

$$\left\{\overline{u}^s_j, j\in\mathbb{Z}\right\}\cup\left\{u^+,u^-\right\}$$

is compact. Therefore, using Taylor's Theorem, there exists a constant $C>0$ such that:

$$\forall j\in\mathbb{Z}, \forall h_{-p},\dots,h_{q-1}\in B(0,\delta),$$

$$\left|\boldsymbol{\nu}F\left(\boldsymbol{\nu};\overline{u}^s_{j-p}+h_{-p},\dots,\overline{u}^s_{j+q-1}+h_{q-1}\right)-\boldsymbol{\nu}F\left(\boldsymbol{\nu};\overline{u}^s_{j-p},\dots,\overline{u}^s_{j+q-1}\right)-\sum_{k=-p}^{q-1}B_{j,k}h_k\right|\leq C\left(\sum_{k=-p}^{q-1}|h_k|\right)^2. \tag{5.A.2}$$

Let us fix a choice of constant $\gamma\in[0,+\infty[$. We consider $h\in\ell^1_\gamma$ such that:

$$\left\|h^1\right\|_{\ell^\infty}<\delta.$$

We observe that there exists a constant $\widetilde{C}>0$ such that:

$$\forall j\in\mathbb{Z},\forall k\in\{-p,\dots,q-1\}\,,\quad (1+|j|)^\gamma\leq\widetilde{C}(1+|j+k|)^\gamma.$$

Therefore, using (5.A.2), we have that for all $j\in\mathbb{Z}$:

$$(1+|j|)^{2\gamma}|Q(h)_j|\leq C(1+|j|)^{2\gamma}\left(\sum_{k=-p}^{q-1}|h_{j+k}|\right)^2\leq C\widetilde{C}^2\left(\sum_{k=-p}^{q-1}(1+|j+k|)^\gamma|h_{j+k}|\right)^2. \tag{5.A.3}$$

Thus, we have that:

$$\forall j\in\mathbb{Z},\quad (1+|j|)^{2\gamma}|Q(h)_j|\leq C\widetilde{C}^2(p+q)^2\,\|h\|^2_{\ell^\infty_\gamma}\,.$$

We thus obtain (5.A.1b). By summing (5.A.3) for all $j\in\mathbb{Z}$, we can also obtain (5.A.1a). □

Proof of Lemma 5.2.2

Let us recall the statement of Lemma 5.2.2.

Lemma. *We consider constants $a,b,c\in[0,+\infty[$ such that:*

$$\begin{aligned} c\leq a+b-1 &\quad \text{if } a\in[0,1[,\\ c<b &\quad \text{if } a=1,\\ c\leq b &\quad \text{if } a>1.\end{aligned}$$

There exists a constant $C_I(a,b,c)>0$ such that for all $n\in\mathbb{N}$, we have that:

$$\sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{1}{m^a(n+1-m)^b}\leq\frac{C_I(a,b,c)}{(n+1)^c}.$$

Proof The proof is fairly immediate. First, we observe that for $n\in\mathbb{N}$:

$$\sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{1}{m^a(n+1-m)^b}=\frac{1}{(n+1)^{a+b-1}}\left(\frac{1}{n+1}\sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{1}{\left(\frac{m}{n+1}\right)^a\left(1-\left(\frac{m}{n+1}\right)\right)^b}\right).$$

Since for $m \in \left\{1, \ldots, \lfloor \frac{n+1}{2} \rfloor\right\}$, we have that for $n \in \mathbb{N}$:

$$\frac{1}{\left(1-\left(\frac{m}{n+1}\right)\right)^b} \leq 2^b,$$

we thus have

$$\sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{m^a(n+1-m)^b} \leq \frac{2^b}{(n+1)^{a+b-1}} \left(\frac{1}{n+1} \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{\left(\frac{m}{n+1}\right)^a} \right).$$

We observe that:

$$\forall n \in \mathbb{N}, \forall m \in \left\{2, \ldots, \lfloor \frac{n+1}{2} \rfloor\right\}, \quad \frac{1}{n+1} \frac{1}{\left(\frac{m}{n+1}\right)^a} \leq \int_{\frac{m-1}{n+1}}^{\frac{m}{n+1}} \frac{dt}{t^a}.$$

Thus, we have :

$$\forall n \in \mathbb{N}, \quad \frac{1}{n+1} \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{\left(\frac{m}{n+1}\right)^a} \leq (n+1)^{a-1} + \int_{\frac{1}{n+1}}^{\frac{1}{2}} \frac{dt}{t^a}.$$

We can then easily conclude the proof by separating the cases $a \in [0, 1[$, $a = 1$ and $a > 1$. $\square$

Chapter 6

Perspectives

In the present chapter, we will present some questions which we deem natural and interesting to investigate after this PhD. The list of questions presented is far from being exhaustive.

1) **Extending the result of Chapter 5 for systems of conservation laws.**

This problem is of course the most obvious direction to follow in the continuation to this thesis. In Chapter 5, we prove a nonlinear stability result for spectrally stable stationary discrete shock profiles of *scalar* conservation laws. We wish to find a complete answer to the [Ser07, Open Question 5.3] by proving a generalization of the result of Chapter 5 in the case of *systems* of conservation laws. Essentially, we make the following observations:

— Chapter 4 (i.e. the article [Coe23]) gives us the tools necessary to obtain such a result, that is to say a precise description of the Green's function of spectrally stable stationary discrete shock profiles for *systems* of conservation laws.

— Chapter 5 gives us guidelines to follow to obtain such a result, i.e. we would need to adapt the method of Chapter 5 to the system case. As explained in Section 5.1.4 of Chapter 5, those adaptations are however nontrivial.

2) **Proving that spectral stability implies nonlinear orbital stability for discrete shock profiles associated with stationary undercompressive and overcompressive shocks.**

In [God03], the result presented also holds for the Green's function of operators obtained by linearizing the numerical scheme about discrete shock profiles associated with stationary undercompressive and overcompressive shocks and not only Lax shocks. A subsequent question is to see how the result of Chapter 4 (i.e. the article [Coe23]) can be extended to those cases and as a consequence if a nonlinear stability property can be proved for discrete shock profiles associated with such shocks. The undercompressive case in particular should be at the same time fairly easier to handle and quite surprising as discrete shock profiles associated with those shocks are isolated. Overall, the main modifications compared to Chapter 4 would appear in the analysis of the so-called spatial Green's function near the interest point 1 (Section 4.4) and would follow ideas already introduced in [God03].

3) **Proving that spectral stability implies nonlinear orbital stability for moving discrete shock profiles with rational speeds.**

Another interesting question is to try to generalize the analysis on the Green's function of SDSPs associated with stationary Lax shocks performed in Chapter 4 and 5 for moving discrete shock profiles with rational speeds. In continuous and semi-discrete settings, such a generalization for moving traveling waves is quite often easy as considering a moving frame allows to come back to the stationary case. However, when considering the fully discrete case, such an approach is not usually available. Following the ideas introduced in the introduction of [Ser07, Section 4], it should be possible to come back to the analysis performed on stationary discrete shock profiles by studying iterations of the discrete evolution operator associated with the numerical scheme but at the cost of a far more difficult numerical flux with an increased number of variables.

4) **Studying the impact of the ratio between the time and space steps on estimates of Green's function associated with finite difference schemes.**

In the papers [DS14; CF22; CF23; Coe22; Coe24] and Chapters 2 and 3, the estimates on the Green's function associated with the finite difference scheme on the transport equation on the whole line or half-line hold for only one choice of ratio ν between the time and space steps. Having uniform estimates with regards to the ratio ν (which must satisfy a CFL condition) would allow to obtain stability estimates that are uniform with regards to this ratio ν like in [Des08]. Such a result allows for instance to have some leeway in the choice of the time and space steps when concretely using numerical schemes.

Let us observe that this question has far more impact since similar issues exist in different settings. For instance, in [Bec+10] which investigates the nonlinear orbital stability of semi-discrete approximations of Lax shocks for systems of conservation laws, questions on the dependence of the estimates obtained on the Green's function with regards to the space step are also raised.

5) **Generalize the stability analysis and Green's function estimates of [DS14; CF22; CF23; Coe22; Coe24; Coe23] to dispersive finite difference schemes.**

As explained throughout the different chapters of this thesis, the result proved are focused on the study of finite difference schemes that introduce numerical viscosity (in the sense of the fundamental contribution [Tho65] of Thomée) as it allowed to use proofs based on the techniques introduced in [ZH98] to study the Green's functions associated with linearizations of the discrete evolution operators of those numerical schemes. However, several schemes, like for instance the well-known Lax-Wendroff or Beam-Warming schemes, introduce dispersive behaviors rather than diffusive ones. An interesting direction to follow would be to generalize the proofs of the results of Chapters 2 and 3 as well as of [CF22; CF23; Coe22; Coe24; Coe23] for this family of schemes. Such a generalization seems achievable since the diffusivity condition we impose on the numerical scheme is used fairly late in the proofs and the technical requirements to extend the analysis to the dispersive case do not seem unachievable. The proof relies on the use of the inverse Laplace transform to express the Green's function using the resolvent of the operator. Summarized briefly, looking at the dispersive case would only require making a new appropriate choice of integration path. The analysis of the integrand would be exactly similar as the one done in the Chapters 2 and 3 for diffusive schemes. The first goal would be to improve/generalize results on the Green's function of dispersive scheme applied to the transport equation on the whole line like [RS15; Cou22].

6) **Proving that discrete shock profiles associated with weak stationary Lax shocks satisfy a precise necessary condition for spectral stability linked with the Evans function.**

In Chapters 4 and 5, we assume the spectral stability of the discrete shock profile we consider to prove its nonlinear orbital stability. However, results on spectral stability of discrete shock profiles are fairly rare. It is expected that discrete shock profiles associated with weak Lax shocks are spectrally stable and even nonlinearly stable, just like for the viscous shock profiles (see [Liu85]). Let us recall that spectral stability relies on proving that the linearized operator about the discrete shock profile has no eigenvalue of modulus equal or larger than 1 other than 1 and that the well-known Evans function, that was also introduced in Chapter 4, must have a simple zero at 1. To investigate this question, we need to construct the Evans function just like in Chapter 4. It is a holomorphic function that acts like a characteristic polynomial for the linearized operator, i.e. it vanishes at eigenvalues of the operator. Since the Evans function is real valued on $]1, +\infty[$, a well-known necessary condition of spectral stability is that the sign of the Evans function near 1 and at $+\infty$ must be the same. Proving this necessary condition for discrete shock profiles associated with weak stationary Lax shocks corresponds to [Ser07, Open Question 5.1]. In [Ser07], Serre summarizes known calculations on the subject.

7) **Generalize the results of my previous papers in the multidimensional case and study the carbuncle phenomenon.**

This last question is relatively open to interpretation as there are several interesting directions that can be investigated. For instance, several articles such as [RS17; Ran23] extended the results of [RS15] on convolution powers indexed in $\mathbb{Z}$ to the multidimensional lattice $\mathbb{Z}^d$. Several new difficulties appear in the multidimensional setting and generalizing the result of [Coe22] in this case would be interesting and quite challenging. Another more ambitious direction this time related to discrete shock profiles would be to study the occurrence of possible spectral instabilities for discrete shock profiles in the multidimensional case. Proving this complex result would be a possible explanation for the so-called carbuncle phenomenon which corresponds to numerical instabilities observed in the solutions of shock capturing schemes. Those instabilities have been observed numerically quite thoroughly but a fully fleshed out theory on the subject is still lacking.

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Table des matières

Titre : Stabilité de profils de chocs totalement discrets pour les systèmes de lois de conservation

Mots clés : Schémas aux différences finies, Stabilité, Systèmes de lois de conservation, Profils de choc

Résumé : Cette thèse aborde l'analyse de la stabilité des profils de chocs totalement discrets pour les systèmes de lois de conservation. Ces profils correspondent à l'approximation d'ondes progressives discontinues par des schémas aux différences finies conservatifs. De telles solutions discontinues apparaissent naturellement dans l'étude des systèmes de lois de conservation qui peuvent modéliser de nombreuses situations physiques comme par exemple la dynamique des gaz.

L'étude des profils de choc totalement discrets se divise essentiellement en deux axes, le premier étant de construire de tels profils discrets et donc de prouver leur existence, et le second étant d'étudier leur stabilité. L'objectif principal de cette thèse est d'approfondir cette seconde direction. De nombreux résultats existants sur la stabilité des profils de chocs totalement discrets introduisent des hypothèses contraignantes, telles que la restriction aux lois de conservation scalaires ou encore le fait d'imposer que les discontinuités approchées soient de faible amplitude. Les résultats de cette thèse visent à ouvrir la voie vers des résultats de stabilité non linéaire qui traiterait de systèmes de lois de conservation et non pas seulement de lois scalaires, et qui remplacerait l'hypothèse de faible amplitude des discontinuités par une hypothèse spectrale sur le linéarisé du schéma autour du profil de choc discret considéré.

Au niveau des résultats obtenues, dans un premier temps, la thèse se focalise sur l'obtention d'estimées de décroissance fines sur le linéarisé du schéma aux niveaux de solutions particulières. On se concentrera d'abord sur le linéarisé au niveau des solutions constantes avant de passer au cas plus compliqué du linéarisé au niveau des profils de chocs totalement discrets. D'un point de vue spectral, l'analyse du problème des chocs fait apparaitre une valeur propre plongée dans le spectre essentiel. Il en résulte de nouveaux termes dans l'analyse de la fonction de Green du schéma linéarisé et on détaille les propriétés de décroissance de chacun de ces termes. Dans une dernière partie, on utilise les estimations obtenues sur l'opérateur linéarisé pour établir un argument de stabilité non linéaire.

Title: Stability of discrete shock profiles for systems of conservation laws

Key words: Finite difference schemes, Stability, Systems of conservation laws, Shock profiles

Abstract: This thesis deals with the stability analysis of discrete shock profiles for systems of conservation laws. These profiles correspond to approximations of discontinuous traveling waves by conservative finite difference schemes. Such discontinuous solutions appear naturally in the study of conservation law systems, which can model many physical situations, such as gas dynamics.

The study of discrete shock profiles is essentially divided into two directions, the first one focusing on the construction of such discrete profiles and thus on the proof of their existence, and the second one studying their stability. The main objective of this thesis is to investigate this second direction. Many existing results on the stability of discrete shock profiles introduce constraining hypotheses, such as the restriction to scalar conservation laws or the requirement that the approximated discontinuities should be of small amplitude. The results of this thesis aim to pave the way towards nonlinear stability results that would deal with systems of conservation laws and not just scalar laws, and that would replace the smallness assumption on the amplitude of the discontinuities by a spectral assumption on the linearization of the numerical scheme about the discrete shock profile under consideration.

In terms of the results obtained, the thesis initially focuses on obtaining sharp decay estimates for the linearization of the numerical scheme about particular solutions. We will first focus on the linearization about constant solutions before moving on to the more complicated case of the linearization about discrete shock profiles. From a spectral point of view, the analysis of the shock problem implies the existence of an eigenvalue located within the essential spectrum. This results in new terms in the analysis of the Green's function of the linearized scheme and decay properties of each of these terms will be presented. In a final section, we use the estimates obtained on the linearized operator to establish a nonlinear stability argument.