

Stability of discrete shock profiles for systems of conservation laws

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Conservation laws and shocks

We consider a one-dimensional scalar conservation law

$$\begin{aligned}\partial_t u + \partial_x f(u) &= 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} &\rightarrow \mathbb{R},\end{aligned}$$

where the flux $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function.

Some of the results that will be presented also hold for systems of conservation laws.

Observation: Generally, we observe discontinuities appearing after finite time for solutions of conservation laws. (Example with the Burgers equation $f(u) := \frac{u^2}{2}$)

Overarching goal: When considering a conservative finite difference scheme, understand if it will be able to handle/capture discontinuities of solutions.

We consider two distinct states u^- , $u^+ \in \mathbb{R}^2$ and a velocity $s \in \mathbb{R}$ such that:

$$f(u^-) - f(u^+) = s(u^- - u^+), \quad (\text{Rankine-Hugoniot condition})$$

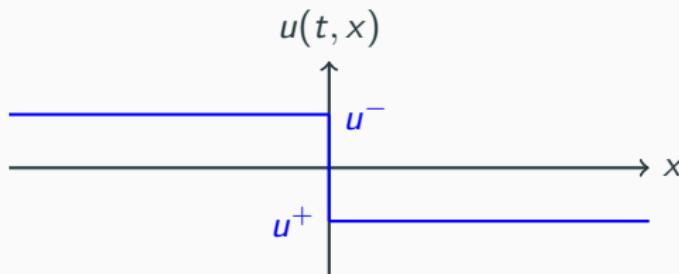
$$f'(u^+) < s < f'(u^-). \quad (\text{Lax shock inequalities})$$

The function u defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t, x) := \begin{cases} u^- & \text{if } x < st, \\ u^+ & \text{else,} \end{cases}$$

which is a weak solution of the scalar conservation law, is known as a [Lax shock](#).

In this thesis, we focus on [steady Lax shock](#), i.e. $s = 0$.



Conservative finite difference schemes

We introduce a cartesian grid with a space step $\Delta x > 0$ and a time step $\Delta t > 0$.

Goal: Compute sequences $u^n := (u_j^n)_{j \in \mathbb{Z}}$ such that u_j^n is close to the solution u on $[n\Delta t, (n+1)\Delta t] \times [(j - \frac{1}{2})\Delta x, (j + \frac{1}{2})\Delta x]$.

We consider a **conservative explicit finite difference scheme**:

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n)$$

where for $u = (u_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $j \in \mathbb{Z}$ as

$$\mathcal{N}(u)_j := u_j - \nu (F(\nu; u_{j-p+1}, \dots, u_{j+q}) - F(\nu; u_{j-p}, \dots, u_{j+q-1})).$$

- $u^0 \in \mathbb{R}^{\mathbb{Z}}$: initial condition
- $\mathcal{N} : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}^{\mathbb{Z}}$: **nonlinear** discrete evolution operator
- $F :]0, +\infty[\times \mathbb{R}^{p+q} \rightarrow \mathbb{R}$: numerical flux
- $p, q \in \mathbb{N} \setminus \{0\}$: integers defining the size of the stencil of the scheme.
- $\nu = \frac{\Delta t}{\Delta x} > 0$: ratio between the time and space steps.

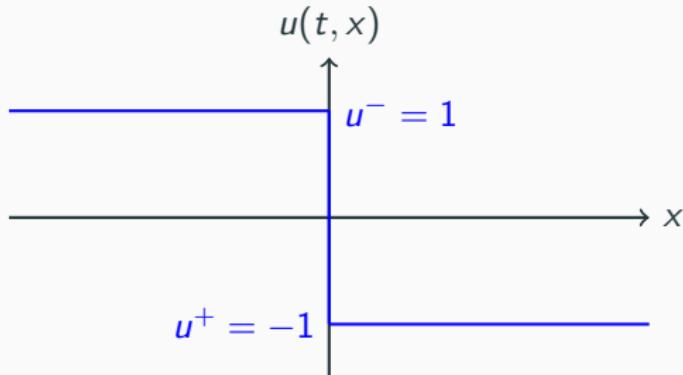
"Usual" assumptions on the numerical scheme:

- $\forall u \in \mathbb{R}, \quad F(\nu; u, \dots, u) = f(u)$ (consistency condition)
- For some neighborhood \mathcal{U} of the states u^\pm
$$\forall u \in \mathcal{U}, \quad -q \leq \nu f'(u) \leq p \quad (\text{CFL condition on } \nu)$$
- Linear- ℓ^2 stability for constant states $u \in \mathcal{U}$

Diffusion assumption on the numerical scheme:

- The scheme introduces numerical viscosity. This excludes dispersive schemes like for instance the Lax-Wendroff scheme.
In the present presentation, we consider a first order scheme.

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$.



For the numerical scheme, we consider the modified Lax Friedrichs scheme

$$\forall u \in \mathbb{R}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \quad \mathcal{N}(u)_j := \frac{u_{j+1} + u_j + u_{j-1}}{3} - \nu \frac{f(u_{j+1}) - f(u_{j-1})}{2}.$$

Discrete shock profiles

- Discrete shock profile (DSP): A discrete shock profile is a solution of the numerical scheme of the form

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^n = \bar{u}(j - s\nu n)$$

where the function $\bar{u} : \mathbb{Z} + s\nu\mathbb{Z} \rightarrow \mathbb{R}$ verifies that

$$\bar{u}(x) \underset{x \rightarrow \pm\infty}{\rightarrow} u^\pm.$$

- The particular case of steady Lax shocks, i.e. $s = 0$:

Stationary discrete shock profiles (SDSP) are sequences

$\bar{u} = (\bar{u}_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfy

$$\mathcal{N}(\bar{u}) = \bar{u} \quad \text{and} \quad \bar{u}_j \underset{j \rightarrow \pm\infty}{\rightarrow} u^\pm.$$

Example : We consider the following initial condition (mean of the standing shock on each cell):

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$

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Main goal: Finding conditions on the numerical schemes so that:

stable shock waves for
the conservation law \Rightarrow stable DSPs for the
numerical scheme

This separates the theory surrounding DSPs in two parts:

- Existence of DSPs
- Stability of DSPs

Existence results on SDSPs

Existence of a continuous one-parameter family $(\bar{u}^\delta)_{\delta \in]-\Delta, \Delta[}$ of SDSPs around [Jennings '74, Majda-Ralston '79, Michelson '84, ...]

From now on, we denote a reference discrete shock profile:

$$\bar{u} := \bar{u}^0.$$

Existence results on SDSPs

Assumption: The *mass* function M is injective, where:

$$\forall \delta \in]-\Delta, \Delta[, \quad M(\delta) := \sum_{j \in \mathbb{Z}} \overline{u}_j^\delta - \overline{u}_j.$$

Nonlinear orbital stability of discrete shock profiles

Goal: Find two suitable vector spaces X, Y such that:

There exists a positive constant $\varepsilon > 0$ such that for an initial perturbation $\mathbf{h} \in X$ such that:

$$\|\mathbf{h}\|_X < \varepsilon,$$

then the solution $(u^n)_{n \in \mathbb{N}}$ of the numerical scheme associated with the initial condition $u^0 := \bar{u} + \mathbf{h}$ is defined for all time $n \in \mathbb{N}$ and we have:

$$\inf_{\delta \in]-\Delta, \Delta[} \|u^n - \bar{u}^\delta\|_Y \xrightarrow{n \rightarrow +\infty} 0.$$

We have **conservation of mass**:

$$\forall n \in \mathbb{N}, \quad \sum_{j \in \mathbb{Z}} u_j^n - \bar{u}_j = \sum_{j \in \mathbb{Z}} u_j^0 - \bar{u}_j = \sum_{j \in \mathbb{Z}} \mathbf{h}_j.$$

Thus, if $u^n \xrightarrow{n \rightarrow +\infty} \bar{u}^\delta$ in ℓ^1 , then:

$$M(\delta) = \sum_{j \in \mathbb{Z}} \mathbf{h}_j.$$

The case of zero-mass perturbations \mathbf{h} .

[Jennings, '74]

- scalar case
- conservative monotone scheme
- nonlinear orbital stability for ℓ^1 perturbations

[Liu-Xin, '93]

- system case
- Lax-Friedrichs scheme
- weak Lax shocks
- zero mass (dropped in [Ying, 97']) and polynomial weight on the initial perturbation

[Smyrlis, '90]

- scalar case
- stationnary Lax shocks
- Lax-Wendroff scheme
- exponential weight on the initial perturbation

[Michelson, '02]

- system case
- weak stationary Lax shocks
- First and third order schemes

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Goal: We want to prove a result of nonlinear orbital stability that works:

- for **systems** of conservation laws,
- for a **fairly large class** of numerical schemes,
- with **the fewest restrictions** possible on the initial perturbations **h** ,
- replacing the **smallness assumption** on the amplitude of the shock with a **spectral stability assumption** on the discrete shock profile.

The **spectral stability assumption** corresponds to additional information on the point spectrum of the linearization about the SDSP \bar{u} .

Technique: Adaptation of the ideas "à la Zumbrun" which study the stability of traveling waves for parabolic PDEs.

Spectral stability \Rightarrow Linear stability \Rightarrow Nonlinear stability

Spectral stability implies linear stability

The content of this part is contained in [C. '23] and is an extension of the result of [Godillon '03] and [Serre '07].

Quick overview of the content:

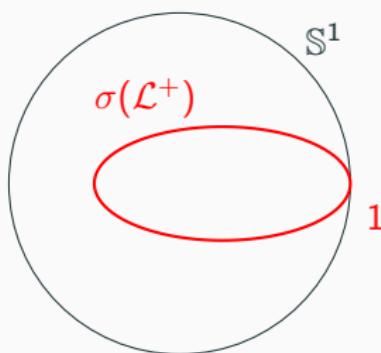
- Based on [Zumbrun and Howard, '98] and [Mascia and Zumbrun, '02, '03].
- Translate the **spectral information on the linearization** about the wave into **estimates on the semi-group** associated with it
- Main difficulty: No spectral gap \oplus Presence of an **eigenvalue of modulus 1 in the essential spectrum**
- Solution: Pointwise description of the **Green's function** of the linearized operator using spatial dynamics

Linearization of the numerical scheme about the SDSP

- The linear operators $\mathcal{L}^\pm : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ defined by:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}^\pm h)_j := \sum_{k=-p}^q a_k^\pm h_{j+k}$$

correspond to the linearization of the numerical scheme \mathcal{N} about the constant states u^\pm . The coefficients a_k^\pm are expressed using $\partial_{u_l} F(\nu, u^\pm, \dots, u^\pm)$.



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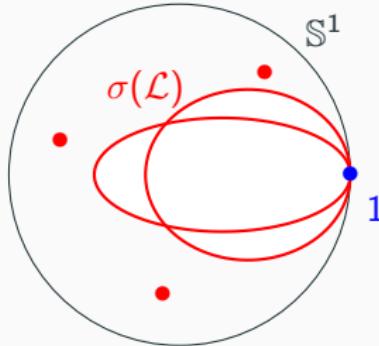
correspond to the linearization of the numerical scheme \mathcal{N} about the constant states u^\pm . The coefficients a_k^\pm are expressed using $\partial_{u_l} F(\nu, u^\pm, \dots, u^\pm)$.

- We define the linear operator $\mathcal{L} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ obtained by linearizing \mathcal{N} about \bar{u} :

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L} h)_j := \sum_{k=-p}^q a_{j,k} h_{j+k},$$

with the coefficients $a_{j,k}$ are expressed using $\partial_{u_l} F(\nu, \bar{u}_{j-p}, \dots, \bar{u}_{j+q-1})$.

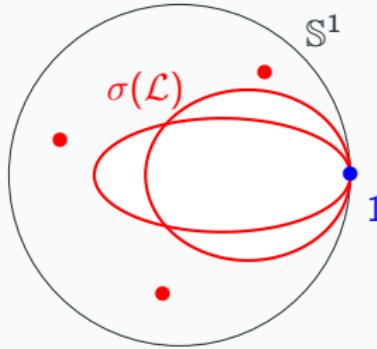
We have that $a_{j,k} \rightarrow a_k^\pm$ as $j \rightarrow \pm\infty$ exponentially fast.



Observation on the spectrum of \mathcal{L}

- The curves $\sigma(\mathcal{L}^+)$ and $\sigma(\mathcal{L}^-)$ belong to the essential spectrum of \mathcal{L} .
- Outside of those essential spectrum curves, the spectrum $\sigma(\mathcal{L})$ is only composed of eigenvalues (represented with red points).
- 1 is an eigenvalue of the operator \mathcal{L} :

$$\text{"}\mathcal{N}(\bar{u}^\delta) = \bar{u}^\delta \implies \left. \frac{\partial \bar{u}^\delta}{\partial \delta} \right|_{\delta=0} \in \ker(Id - \mathcal{L}).\text{"}$$



Spectral stability assumption

- We construct a so-called Evans function. We assume that 1 is a **simple** zero of the Evans function. As a consequence, 1 is a **simple** eigenvalue of the operator \mathcal{L} :

$$\text{" } \ker(Id - \mathcal{L}) = \text{Span} \left(\frac{\partial \bar{u}^\delta}{\partial \delta} \Big|_{\delta=0} \right) \text{".}$$

- The operator \mathcal{L} has no other eigenvalue of modulus equal or larger than 1 .

We define the **Green's function** associated to \mathcal{L} :

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j \in \mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Green's function associated to the operator \mathcal{L} for $j_0 = 30$

Theorem [C. 23']

Under some more precise assumptions, there exist a positive constant c , an element V of $\ker(Id - \mathcal{L})$ and an (explicit) function $E : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $n \in \mathbb{N} \setminus \{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$\begin{aligned}\mathcal{G}(n, j_0, j) &= E \left(\frac{nf'(u^+) \nu + j_0}{\sqrt{n}} \right) V(j) \quad (\text{Excited eigenvector}) \\ &+ \mathbf{1}_{j \in \mathbb{N}} O \left(\frac{1}{\sqrt{n}} \exp \left(-c \frac{|nf'(u^+) \nu - (j - j_0)|^2}{n} \right) \right) \quad (\text{Gaussian wave}) \\ &+ \mathbf{1}_{j \in -\mathbb{N}} O \left(\frac{1}{\sqrt{n}} \exp \left(-c \frac{|nf'(u^+) \nu + j_0|^2}{n} \right) e^{-c|j|} \right) \quad (\text{Exponential residual}) \\ &+ O(e^{-cn - c|j - j_0|})\end{aligned}$$

where $E(x) \xrightarrow[x \rightarrow -\infty]{} 1$ and $E(x) \xrightarrow[x \rightarrow +\infty]{} 0$.

There is a similar result for $j_0 \in -\mathbb{N}$.

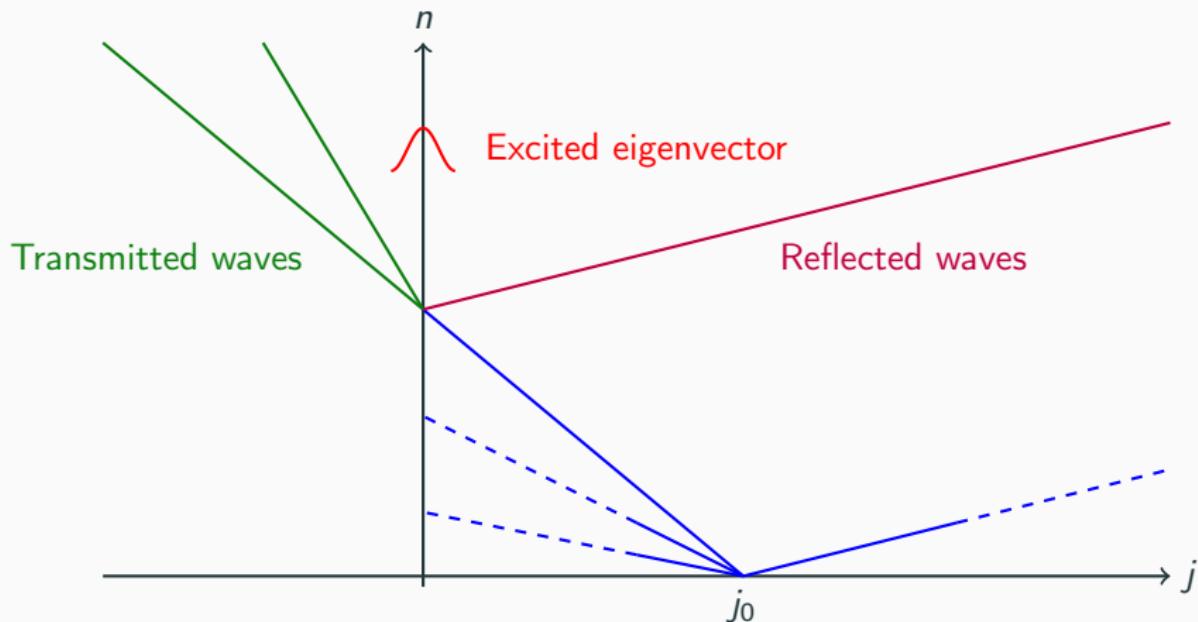
Some precisions on the result:

- We actually extract the leading behavior of the Gaussian wave of the Green's function.
- There is also such a description of the discrete derivative of the Green's function:

$$\mathcal{G}(n, j_0 - 1, j) - \mathcal{G}(n, j_0, j).$$

- The result is proved for a large family of higher order diffusive schemes.
- The result is proved in the case of systems of conservation laws.

Result on the Green's function in the case of systems of conservation laws



Brief idea of the proof

Using the inverse Laplace transform, we have:

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi} \int_{\Gamma} z^n \mathcal{G}(z, j_0, j) dz$$

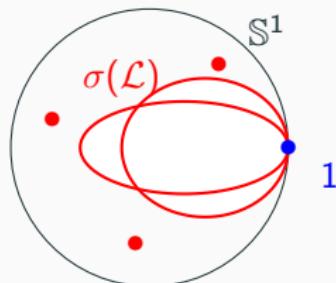
where the spatial Green's function $\mathcal{G}(z, j_0, j)$ is defined by:

$$\forall z \notin \sigma(\mathcal{L}), \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(z, j_0, j) := ((zId - \mathcal{L})^{-1} \delta_{j_0})_j$$

and the integration path Γ surrounds the spectrum $\sigma(\mathcal{L})$.

Goal:

- Meromorphic extension of the spatial Green's function $\mathcal{G}(\cdot, j_0, j)$ near 1
- Find a suitable choice of path Γ



For $h \in \ell^\infty(\mathbb{Z})$ and $n \in \mathbb{N}$, we have:

$$\forall j \in \mathbb{Z}, \quad (\mathcal{L}^n h)_j = \sum_{j_0 \in \mathbb{Z}} \mathcal{G}(n, j_0, j) h_{j_0}.$$

Also, for $j_0 \in \mathbb{N}$ (a similar description exists for $j_0 \in -\mathbb{N}$):

$$\begin{aligned} & \mathcal{G}(n, j_0, j) \\ &= E \left(\frac{nf'(u^+) \nu + j_0}{\sqrt{n}} \right) V(j) \quad (\text{Excited eigenvector}) \\ &+ \mathbb{1}_{j \in \mathbb{N}} O \left(\frac{1}{\sqrt{n}} \exp \left(-c \left(\frac{|nf'(u^+) \nu - (j - j_0)|^2}{n} \right) \right) \right) \quad (\text{Gaussian wave}) \\ &+ \mathbb{1}_{j \in -\mathbb{N}} O \left(\frac{1}{\sqrt{n}} \exp \left(-c \left(\frac{|nf'(u^+) \nu + j_0|^2}{n} \right) \right) e^{-c|j|} \right) \quad (\text{Exponential residual}) \\ &+ O(e^{-cn - c|j - j_0|}) \end{aligned}$$

The description of the Green's function allows to prove decay estimates on the semigroup $(\mathcal{L}^n)_{n \in \mathbb{N}}$.

Linear stability implies nonlinear orbital stability

We consider an initial perturbation $\mathbf{h} \in \ell^1$ such that there exists $\delta \in]-\Delta, \Delta[$ satisfying:

$$M(\delta) = \sum_{j \in \mathbb{Z}} \mathbf{h}_j.$$

For the solution of the numerical scheme $(u^n)_{n \in \mathbb{N}}$ with $u^0 := \bar{u} + \mathbf{h}$, then the sequence u^n should converge towards \bar{u}^δ .

We thus want to estimate $h^n := u^n - \bar{u}^\delta$ in some suitable norm.

The polynomially weighted norms

For $r \in [1, +\infty]$ and $\gamma \in [0, +\infty[$, we define the polynomial-weighted spaces ℓ_γ^r :

$$\ell_\gamma^r := \{(h_j)_{j \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad ((1 + |j|)^\gamma h_j)_{j \in \mathbb{Z}} \in \ell^r(\mathbb{Z})\}$$

with the norm:

$$\forall h \in \ell_\gamma^r, \quad \|h\|_{\ell_\gamma^r} = \left\| ((1 + |j|)^\gamma h_j)_{j \in \mathbb{Z}} \right\|_{\ell^r}.$$

Theorem [C. '24]

Let us assume that the same assumptions are verified (and thus especially the **spectral stability assumption**). We consider a constant $\mathbf{p} > 0$. There exist two constants $\varepsilon, C \in [0, +\infty[$ such that, if we consider a initial perturbation $\mathbf{h} \in \ell_{\mathbf{p}+\max(\frac{1}{2}, \mathbf{p})}^1$ such that:

$$\|\mathbf{h}\|_{\ell_{\mathbf{p}+\max(\frac{1}{2}, \mathbf{p})}^1} < \varepsilon \quad (\text{polynomial weight condition}),$$

then there exists $\delta \in]-\Delta, \Delta[$ such that:

$$M(\delta) = \sum_{j \in \mathbb{Z}} \mathbf{h}_j$$

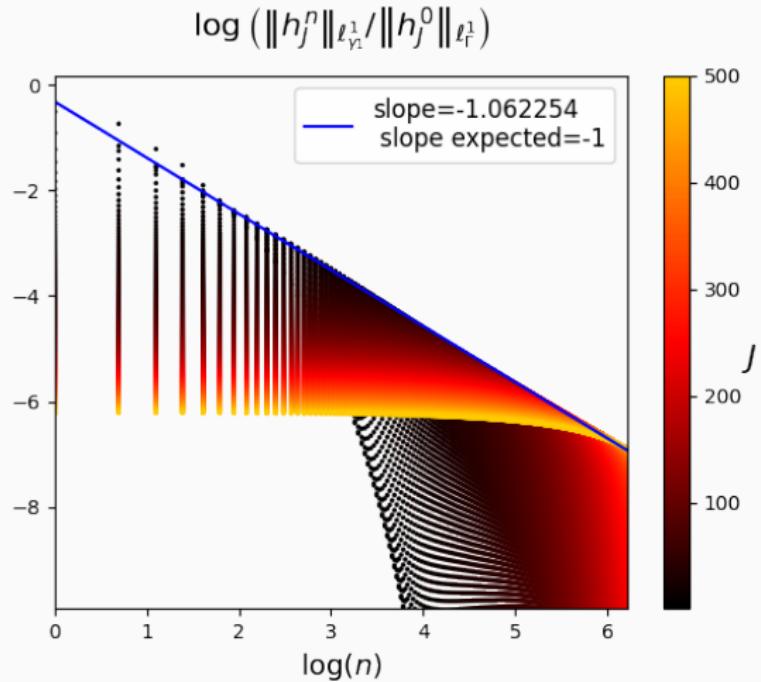
Furthermore, for the solution $(u^n)_n$ of the numerical scheme initialized with $u^0 = \bar{u}$, we have that for all $n \in \mathbb{N} \setminus \{0\}$, the sequence $h^n = u^n - \bar{u}^\delta$ satisfies:

$$\|h^n\|_{\ell_{\mathbf{p}}^1} \leq \frac{C}{n^{\mathbf{p}}} \|\mathbf{h}\|_{\ell_{\mathbf{p}+\max(\frac{1}{2}, \mathbf{p})}^1} \quad \text{and} \quad \|h^n\|_{\ell_{\mathbf{p}}^\infty} \leq \frac{C}{n^{\mathbf{p}+\frac{1}{2}}} \|\mathbf{h}\|_{\ell_{\mathbf{p}+\max(\frac{1}{2}, \mathbf{p})}^1}.$$

We display the value of:

$$\ln \left(\frac{\|h^n\|_{\ell_p^1}}{\|\mathbf{h}\|_{\ell^1_{p+\max(\frac{1}{2}, p)}}} \right) \quad \text{and} \quad \ln \left(\frac{\|h^n\|_{\ell_p^\infty}}{\|\mathbf{h}\|_{\ell^1_{p+\max(\frac{1}{2}, p)}}} \right)$$

for $p = 1$ and $\mathbf{h}_J := \delta_J - \delta_0$.

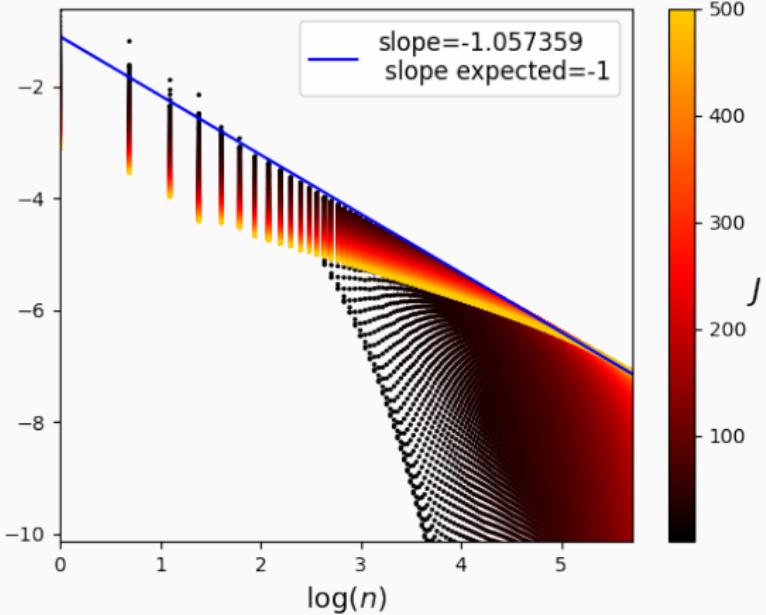


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for $p = 1$ and $\mathbf{h}_J := \delta_J - \delta_0$.

$$\log \left(\|h_J^n\|_{\ell_{\gamma_e}^\infty} / \|h_J^0\|_{\ell_r^1} \right)$$



We introduce the linearization L^δ of the numerical scheme about \bar{u}^δ (on which I have *not necessarily made any spectral stability assumption*).

Main calculations: For $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, we have that:

$$u_j^{n+1} = \mathcal{N}(u^n)_j \quad \Rightarrow \quad h_j^{n+1} = (L^\delta h^n)_j - Q(h^n)_{j+1} + Q(h^n)_j \quad (1)$$

where $Q(h^n)_j$ is a quadratic remainder.

Thus, we have:

$$\forall n \in \mathbb{N}, \quad h^{n+1} = L^\delta h^n + (Id - \mathcal{T})Q(h^n) \quad (2)$$

where the shift operator \mathcal{T} is defined by:

$$\mathcal{T} : (h_j)_{j \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \mapsto (h_{j+1})_{j \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}.$$

Finally, using Duhamel's formula:

$$\forall n \in \mathbb{N}, \quad h^n := (L^\delta)^n h^0 + \sum_{m=0}^{n-1} (L^\delta)^{n-1-m} (Id - \mathcal{T})Q(h^m).$$

We introduce the linearization L^δ of the numerical scheme about \bar{u}^δ (on which I have *not necessarily made any spectral stability assumption*).

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$$\begin{aligned} \forall n \in \mathbb{N}, \quad h^n := & \mathcal{L}^n h^0 + \sum_{m=0}^{n-1} \mathcal{L}^{n-1-m} (L^\delta - \mathcal{L}) h^m + \\ & \sum_{m=0}^{n-1} \mathcal{L}^{n-1-m} (Id - \mathcal{T}) Q(h^m). \end{aligned}$$

Conclusion/ Perspective / Open questions

About the Green's function theorem:

- Bounds on Green's function uniform in j_0
- Proved for a large family of schemes
- The result **is proved** for systems

About the nonlinear stability theorem:

- The result **is not yet proved** for systems

Other Perspectives:

- What can we say for moving shocks (with rational speed) and/or for under/over-compressive shocks?
- What can we say for dispersive schemes? (Lax-Wendroff for instance)
- Existence of spectrally stable SDSPs?
- Study of the stability for multi-dimensional conservation laws

Thank you for your attention!

Brief idea of the proof

Using the inverse Laplace transform, we have:

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi} \int_{\Gamma} z^n \mathcal{G}(z, j_0, j) dz$$

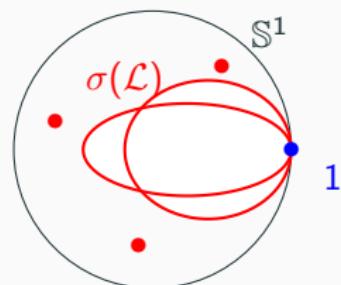
where the spatial Green's function $\mathcal{G}(z, j_0, j)$ is defined by:

$$\forall z \notin \sigma(\mathcal{L}), \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(z, j_0, j) := ((zId - \mathcal{L})^{-1} \delta_{j_0})_j$$

and the integration path Γ surrounds the spectrum $\sigma(\mathcal{L})$.

Goal:

- Study the spatial Green's function $\mathcal{G}(z, j_0, j)$
- Find a suitable choice of path Γ



Brief idea of the proof

Using the inverse Laplace transform, we have:

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi} \int_{\Gamma} z^n \mathcal{G}(z, j_0, j) dz$$

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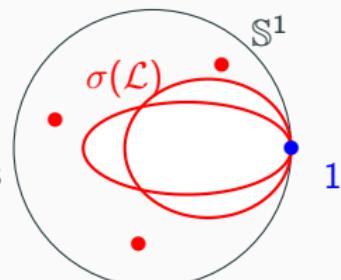
Properties of $\mathcal{G}(z, j_0, j)$:

- **Local exponential bounds:** For all z_0 outside of $\sigma(\mathcal{L})$:

$$|\mathcal{G}(z, j_0, j)| \leq Ce^{-c|j-j_0|}$$

with z in a neighborhood U of z_0 and two constants C, c .

- **Meromorphical extension** of $\mathcal{G}(\cdot, j_0, j)$ near 1 through the essential spectrum with a pole of order 1 at 1



We rewrite the eigenvalue problem:

$$(zId - \mathcal{L})u = 0$$

as a discrete dynamical system:

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z)W_j. \quad (\text{Dyn. Syst.})$$

We are interested in solutions of the dynamical system that tend towards 0 as j tends to $+\infty$ or $-\infty$:

$$E_0^\pm(z) := \left\{ W_0, \quad W_j \underset{j \rightarrow \pm\infty}{\rightarrow} 0 \right\}.$$

We essentially characterize the elements of the vector spaces $E_0^\pm(z)$ in two ways:

- **Geometric dichotomy** (projectors) • **Jost solutions** (basis)

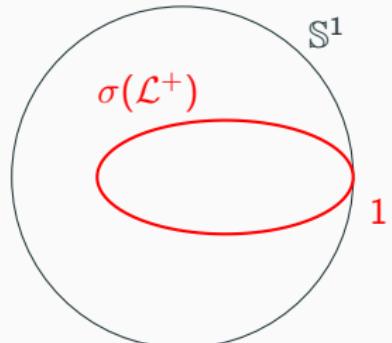
Several uses:

- Analysis of the essential spectrum of \mathcal{L}
- Characterization of the point spectrum \mathcal{L} and construction of the Evans function.
- Expression of the spatial Green's function $G(z, j_0, j)$.

Observation on the spectrum of \mathcal{L}^\pm

- The spectrum of \mathcal{L}^\pm is given by:

$$\sigma(\mathcal{L}^\pm) := \left\{ \sum_{k=-p}^q a_k^\pm e^{i\xi k}, \xi \in \mathbb{R} \right\}.$$



- The consistency condition implies that $1 \in \sigma(\mathcal{L}^\pm)$.
- The linear ℓ^2 -stability implies:

$$\forall \kappa \in \mathbb{S}^1 \setminus \{1\}, \quad \left| \sum_{k=-p}^q a_k^\pm \kappa^k \right| < 1. \quad (\text{Dissipativity condition})$$

- The diffusivity condition states there exist two complex constants β^\pm with positive real parts such that:

$$\sum_{k=-p}^q a_k^\pm e^{i\xi k} \underset{\xi \rightarrow 0}{=} \exp(-i\xi \nu f'(u^\pm) - \beta^\pm \xi^2 + o(\xi^2)) \quad (\text{Diffusion})$$

for a **first order scheme**.

In a continuous setting:

Shock tracking stability based on [Zumbrun '00]

We want to prove the decay of:

$$v(t, x) := u(t, x + \delta(t)) - \bar{u}(x)$$

for a good choice of co-moving frame parametrized by a function δ .

Difficulties for the shock tracking technique in the fully discrete setting:

- Restriction of translations on the grid
- "Linearization and translation do not commute in the discrete setting"

