Wave-front tracking approximation for the 1D Hughes' model

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1 – The 1D Hughes' model

The **1D Hughes' model** aims to represent the evacuation of a corridor (represented by the interval $]-1,1[$), where the population density $\rho = \rho(t,x) \in [0,1]$ satisfies the following scalar conservation law:

$$\partial_t \rho + \partial_x \left(\operatorname{sgn}(x - \xi(t))\rho v(\rho)\right) = 0, \quad t > 0, x \in]-1,1[, \tag{1a}$$

$$\int_{-1}^{\xi(t)} c(\rho(t,x))dx = \int_{\xi(t)}^{1} c(\rho(t,x))dx, \quad t > 0, \tag{1b}$$

$$\rho(t,-1) = \rho(t,1) = 0, \quad t > 0, \tag{1c}$$

$$\rho(0,x) = \overline{\rho}(x), \quad x \in]-1,1[. \tag{1d}$$

where:

- the function $v \in \mathcal{C}^2([0,1],[0,1])$ corresponds to the *speed of the pedestrian*. It is strictly decreasing and satisfies $v(0) = 1$ and $v(1) = 0$. Furthermore, the flux function $f(\rho) := \rho v(\rho)$ must be strictly concave. We notate $\rho_m := \arg\max(f) \in]0,1[$.
- the function $c \in \mathcal{C}^2([0,1],[1,+\infty[)$ is a *cost function*. It is strictly increasing and verifies $c(0) = 1$.
- the function $\xi := \xi(t)$ is called the *turning curve*. It is implicitly determined by the equality (1b) which represents the fact that, for a pedestrian standing at $\xi(t)$, it will cost just as much to decide to go towards the exit 1 or -1.

In the following, we consider for $\alpha \in [0,+\infty[$ that:

$$\forall \rho \in [0,1], \quad v(\rho) := 1 - \rho \quad \text{and} \quad c(\rho) := 1 + \alpha\rho. \tag{2}$$

In this case, the equation (1b) becomes:

$$2\xi(t) + \alpha\left(\int_{-1}^{\xi(t)} \rho(t,x)dx - \int_{\xi(t)}^{1} \rho(t,x)dx\right) = 0. \tag{3}$$

Main technical difficulty: Pedestrians can change directions (see numerical approximations below).

2 – Goals and methodology

The goal on the subject is to prove existence and uniqueness results on the entropic solutions of (1). We would also want to prove that the evacuation happens in finite time and hopefully study the time of evacuation depending on the parameter α.

Usual methods used:

- Wave-front tracking approximation
Amadori, Di Francesco, 12' — Goatin, Mimault, 13' — Amadori, Goatin, Rosini, 14'
- Deterministic many-particle approximation
Di Francesco, Faglioli, Rosini, Russo, 17' — Andreianov, Rosini, Stivaletta, 23'
- Fixed point method
Andreianov, Girard, 24'

3 – Local (in time) construction of the WFT approximation

We fix for $N \in \mathbb{N}$:

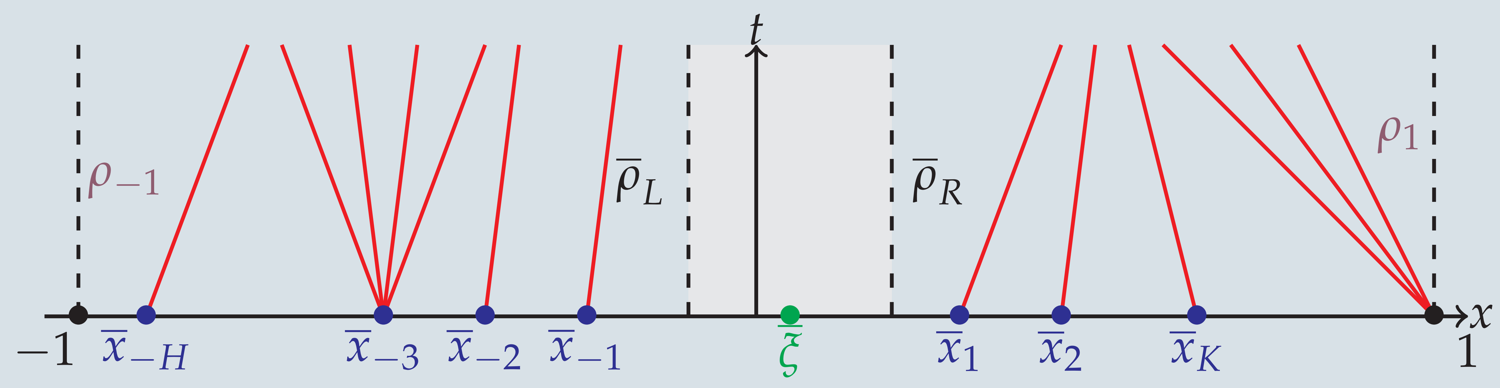
$$\varepsilon := 2^{-N} \quad \text{and} \quad \mathcal{G}^\varepsilon := \{j\varepsilon, \quad j \in \{0,\dots,2^N\}\} \cup \{\rho_m\}.$$

We define f^ε the piecewise linear function that interpolates the points $(\rho, f(\rho))_{\rho \in \mathcal{G}^\varepsilon}$.

Initialisation: We consider a piecewise constant function $\overline{\rho} \in L^1(]-1,1[,[0,1])$ and define $\overline{\xi} \in]-1,1[$ the initial position of the turning curve. We then define the discontinuity points of $\overline{\rho}$ on the left and on the right of the turning curve $\overline{\xi}$:

$$-1 < \overline{x}_{-H} < \ldots < \overline{x}_{-1} < \overline{\xi} < \overline{x}_1 < \ldots < \overline{x}_K < 1.$$

Riemann solvers far from the turning curve:
At each of the discontinuity points $\overline{x}_{-H}, \dots, \overline{x}_{-1}, \overline{x}_1, \dots, \overline{x}_K$ and at the borders -1 and 1, we use "slightly modified" Riemann solvers associated with the fluxes $-f^\varepsilon$ and f^ε depending on which side of the turning curve we are.



We notate $\overline{\rho}_L$ and $\overline{\rho}_R$ the values taken by $\overline{\rho}$ on the left and on the right of the turning curve $\overline{\xi}$. We also introduce ρ_{-1} and ρ_1 the values at the borders -1 and 1.

Riemann solver at the turning curve:
We want to construct the function ρ and the turning curve ξ for small times so that, for ρ_ξ^- and ρ_ξ^+ being the value of ρ on either side of ξ, we have:

$$\left(\rho_\xi^+ - \rho_\xi^-\right)\frac{d\xi}{dt}(t) = f(\rho_\xi^+) + f(\rho_\xi^-), \tag{4a}$$

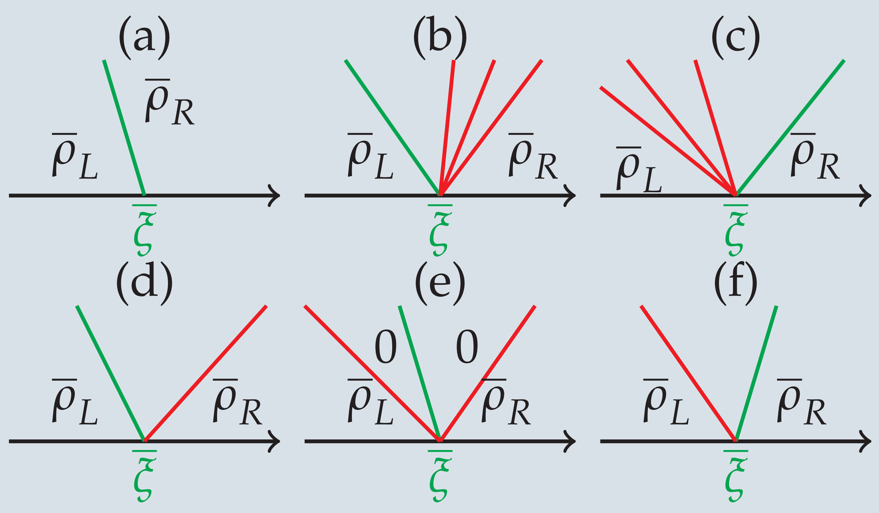
$$\frac{d\xi}{dt}(t)\left(2 + \alpha(\rho_\xi^- + \rho_\xi^+)\right) = \alpha\left(f(\rho_\xi^+) - f(\rho_\xi^-) + f(\rho_{-1}) - f(\rho_1)\right). \tag{4b}$$

The equality (4a) is a Rankine-Hugoniot condition at the turning curve and the condition (4b) ensures that ξ is indeed the turning curve.

Depending on the value of:

$$\alpha(f(\rho_{-1}) - f(\rho_1))$$

there is a valid construction of ρ to satisfy the required properties, possibly with additional discontinuities arising from $\overline{\xi}$.



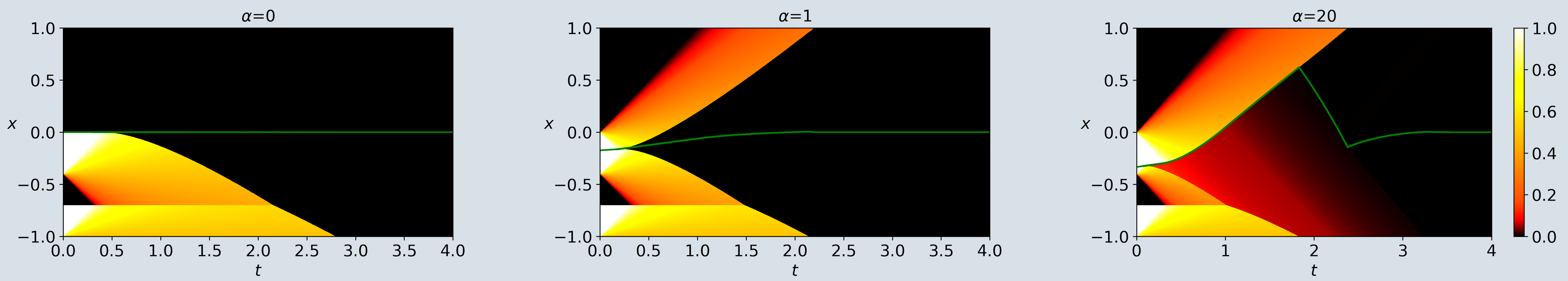
Main technical difficulty:
Introduction of new waves and of states that might not belong to the grid $\mathcal{G}^\varepsilon$.

This procedure is iterated at each interaction time between fronts, the turning curve and borders to extend it for larger times.

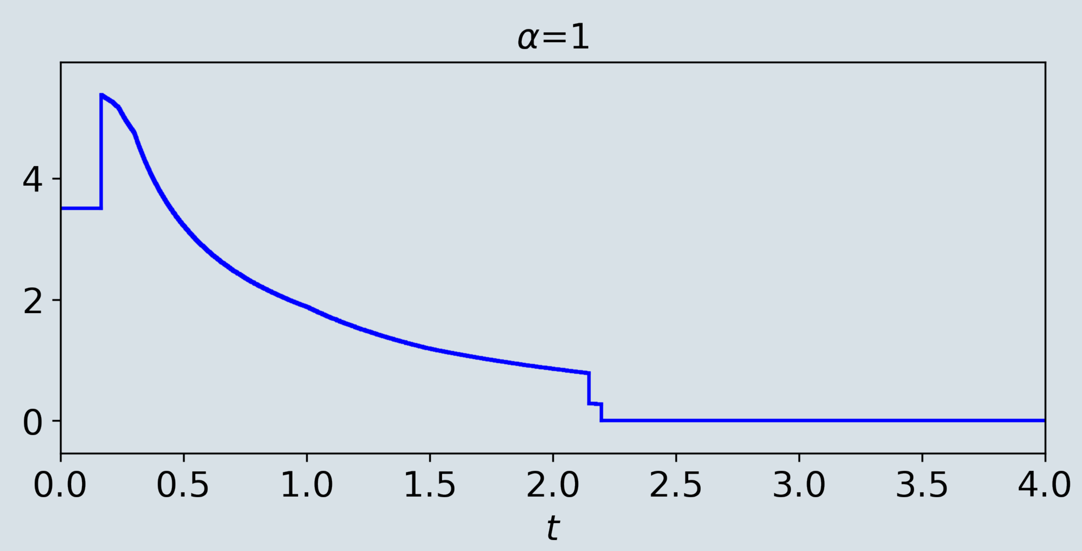
4 – Numerical results using the wave-front tracking approximation

We consider the choices of functions v and c defined by (2) and the initial condition:

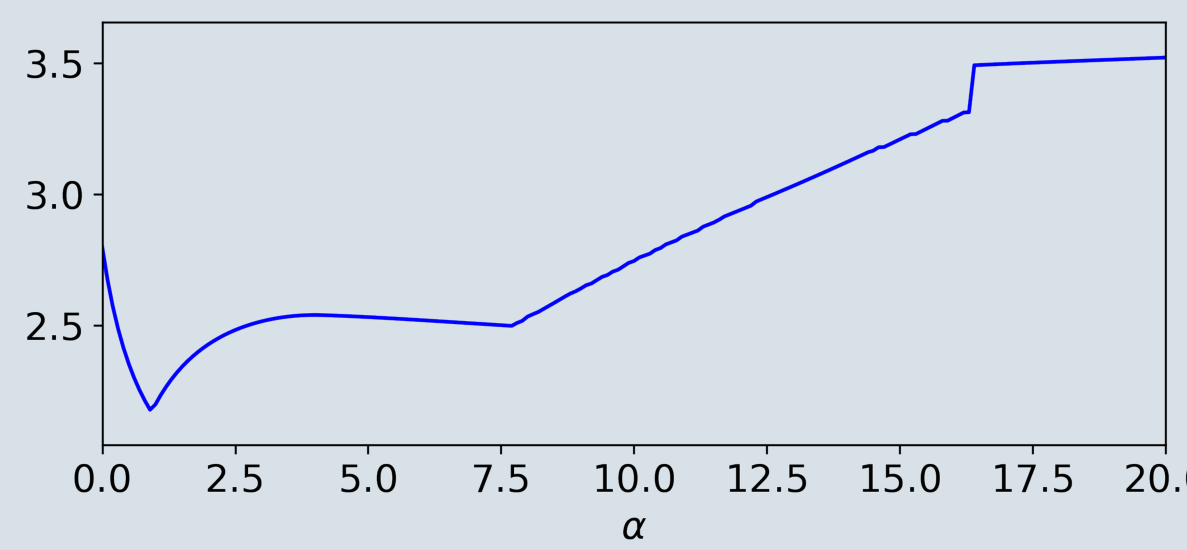
$$\forall x \in]-1,1[, \quad \overline{\rho}(x) = \begin{cases} 1 & \text{if } x < -0.7 \text{ or } -0.4 < x < 0, \\ 0 & \text{else.} \end{cases}$$



Representation of the approximation ρ with the turning curve ξ in green obtained by the wave-front tracking for α equal respectively to 0, 1 and 20 and for $\varepsilon = 2^{-10}$.



Representation of the total variation of the constructed approximation for $\alpha = 1$.



Representation of the time of evacuation depending on α.