Stability of discrete shock profiles for systems of conservation laws

Lucas Coeuret

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Institut de Mathématiques de Toulouse (IMT)

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Conservation laws and shocks

We consider a one-dimensional scalar conservation law

$$\begin{aligned} \partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \end{aligned} \tag{CL}$$

where the flux $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

Some of the results that will be presented also hold for systems of conservations laws.

Observation: Generally, we observe discontinuities appearing after finite time for solutions of this type of PDEs.

Overarching goal: When considering a conservative finite difference scheme, understand if it will be able to handle/capture discontinuities of solutions.

We consider two distinct states $u^-, u^+ \in \mathbb{R}^2$ and a velocity $s \in \mathbb{R}$ such that:

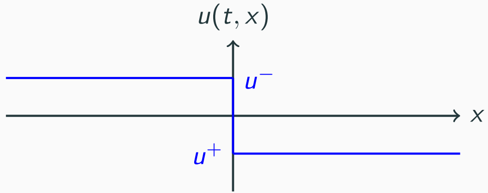
$$f(u^-) - f(u^+) = s(u^- - u^+), \quad \text{(Rankine-Hugoniot condition)}$$
$$f'(u^+) < s < f'(u^-). \quad \text{(Lax shock inequalities)}$$

The function u defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t,x) := \begin{cases} u^- & \text{if } x < st, \\ u^+ & \text{else,} \end{cases}$$

which is a weak solution of the scalar conservation law, is known as a Lax shock.

In this thesis, we focus on steady Lax shock, i.e. $s = 0$.



Conservative finite difference schemes

We introduce a cartesian grid with a space step $\Delta x > 0$ and a time step $\Delta t > 0$.

<u>**Goal:**</u> Compute sequences $u^n := (u_j^n)_{j\in\mathbb{Z}}$ such that u_j^n is close to the solution u on $[n\Delta t, (n+1)\Delta t[\times[j\Delta x, (j+1)\Delta x[$.

<u>**Numerical scheme:**</u> We consider a conservative explicit finite difference scheme:

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n) \qquad \text{(Num. Scheme)}$$

where for $u = (u_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $j \in \mathbb{Z}$ as

$$\mathcal{N}(u)_j := u_j - \nu \left(F\left(\nu; u_{j-p+1}, \dots, u_{j+q}\right) - F\left(\nu; u_{j-p}, \dots, u_{j+q-1}\right)\right).$$

- $u^0 \in \mathbb{R}^{\mathbb{Z}}$: initial condition
- $\mathcal{N} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$: nonlinear discrete evolution operator
- $F :]0, +\infty[\times\mathbb{R}^{p+q} \to \mathbb{R}$: numerical flux
- $p, q \in \mathbb{N}\backslash\{0\}$: integers defining the size of the stencil of the scheme.
- $\nu = \frac{\Delta t}{\Delta x} > 0$: ratio between the time and space steps.

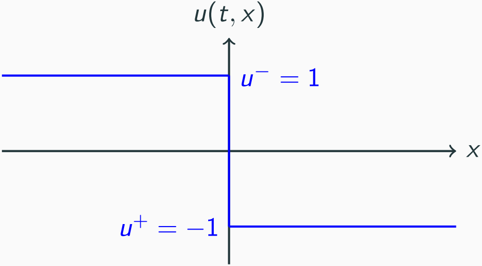
Assumptions on the numerical scheme:

- $\forall u \in \mathbb{R}, \quad F(\nu; u, \dots, u) = f(u)$ (consistency condition)
- For some neighborhood $\mathcal{U}$ of the states $u^{\pm}$

$$\forall u \in \mathcal{U}, \quad -q \leq \nu f'(u) \leq p \quad \text{(CFL condition on } \nu\text{)}$$

- Linear-ℓ^2 stability for constant states $u \in \mathcal{U}$
- The scheme introduces numerical diffusion. This excludes dispersive schemes like for instance the Lax-Wendroff scheme.
  In the present presentation, we consider a first order scheme.

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$.



For the numerical scheme, we consider the modified Lax Friedrichs scheme

$$\forall u \in \mathbb{R}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \quad \mathcal{N}(u)_j := \frac{u_{j+1} + u_j + u_{j-1}}{3} - \nu \frac{f(u_{j+1}) - f(u_{j-1})}{2}.$$

Discrete shock profiles

- Discrete shock profile (DSP): A discrete shock profile is a solution of the numerical scheme of the form

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^n = \overline{u}(j - s\nu n) \tag{1}$$

where the function $\overline{u} : \mathbb{Z} + s\nu\mathbb{Z} \to \mathbb{R}$ verifies that

$$\overline{u}(x) \underset{x \to \pm\infty}{\to} u^{\pm}.$$

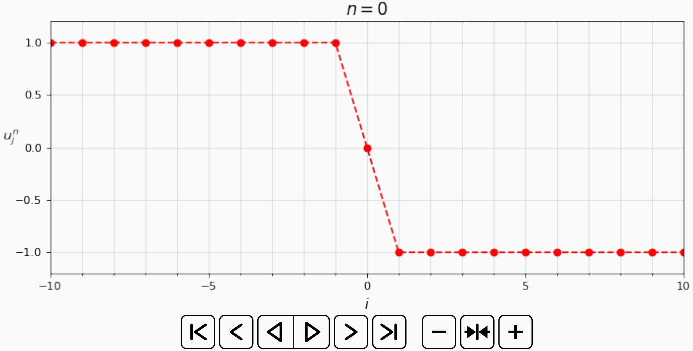
- The particular case of steady Lax shocks, i.e. $s = 0$:

Stationary discrete shock profiles (SDSP) are sequences $\overline{u} = (\overline{u}_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfy

$$\mathcal{N}(\overline{u}) = \overline{u} \quad \text{and} \quad \overline{u}_j \underset{j \to \pm\infty}{\to} u^{\pm}.$$

Example : We consider the following initial condition (mean of the standing shock on each cell):

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$



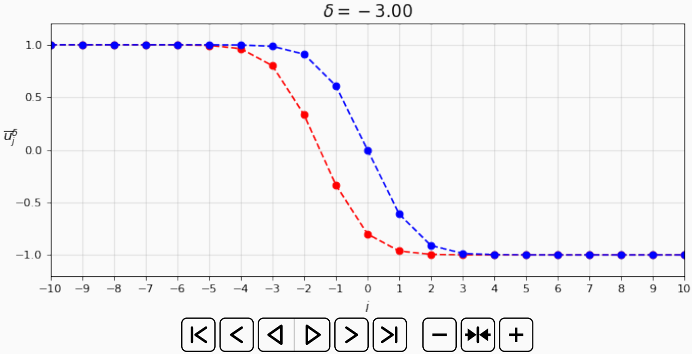
Main goal: Finding conditions on the numerical schemes so that:

stable shock waves for the conservation law	$\Longrightarrow$	stable DSPs for the numerical scheme

This separates the theory surrounding DSPs in two parts:

- Existence of DSPs
- Stability of DSPs

Existence results on SDSPs



In the scalar case, we consider that we can parametrize the SDSPs such that:

$$\forall \delta \in]-\Delta, \Delta[, \quad \sum_{j \in \mathbb{Z}} \overline{u}_j^\delta - \overline{u}_j = \delta.$$

For standing Lax shocks, one aims to have the existence of a differentiable one-parameter family $(\overline{u}^\delta)_{\delta\in]-\Delta,\Delta[}$ of SDSPs.

- Jennings, *Discrete shocks* ('74)
  - **scalar** case
  - conservative **monotone** scheme
  - for shocks satisfying Oleinik's E-condition
- Majda and Ralston, *Discrete Shock Profiles for Systems of Conservation Laws* ('79)
  - **system** case
  - **first order** scheme
  - **weak** Lax shocks
- Michelson, *Discrete shocks for difference approximations to systems of conservation laws* ('84)
  - extension of Majda-Ralston for **third order** scheme
- Different cases: [Smyrlis,90'], [Liu and Yu, 99'], [Serre, 04'] etc...

Nonlinear orbital stability of discrete shock profiles

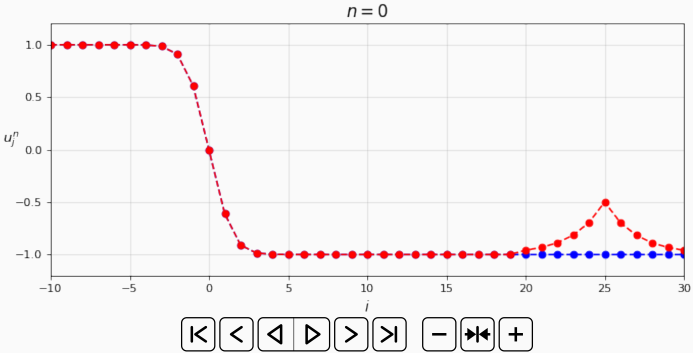
Goal: Find two suitable vector spaces X, Y such that:

There exists a positive constant $\varepsilon > 0$ such that for an initial pertubation $h^0 \in X$ such that:

$$\left\| h^0 \right\|_X < \varepsilon,$$

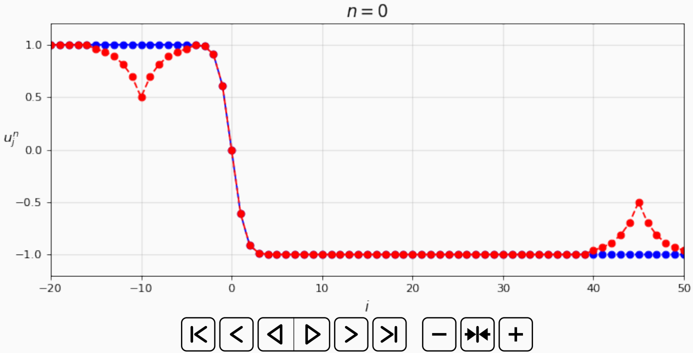
then the solution $(u^n)_{n\in\mathbb{N}}$ of the numerical scheme associated with the initial condition $u^0 := \overline{u} + h^0$ is defined for all time $n \in \mathbb{N}$ and we have:

$$\inf_{\delta \in]-\Delta,\Delta[} \left\| u^n - \overline{u}^\delta \right\|_Y \underset{n\to+\infty}{\to} 0.$$



We have conservation of mass:

$$\forall n \in \mathbb{N}, \quad \sum_{j \in \mathbb{Z}} u_j^n - \overline{u}_j = \sum_{j \in \mathbb{Z}} u_j^0 - \overline{u}_j = \sum_{j \in \mathbb{Z}} h_j^0 .$$



$n = 0$
1.0
0.5
0.0
−0.5
−1.0
u_j^n
−20
−10
0
10
20
30
40
50
j

[Jennings, '74]

- scalar case
- conservative monotone scheme
- nonlinear orbital stability for ℓ^1 perturbations

[Smyrlis, '90]

- scalar case
- stationnary Lax shocks
- Lax-Wendroff scheme
- exponential weight on the initial perturbation

[Liu-Xin, '93]

- system case
- Lax-Friedrichs scheme
- weak Lax shocks
- zero mass (dropped in [Ying, 97']) and polynomial weight on the initial perturbation

[Michelson, '02]

- system case
- weak stationary Lax shocks
- First and third order schemes

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- First and third order schemes

Goal: We want to prove a result of nonlinear orbital stability that works:

- for systems of conservation laws,
- for a fairly large class of numerical schemes,
- with the fewest restrictions possible on the initial perturbations h^0,
- replacing the smallness assumption on the amplitude of the shock with a spectral stability assumption.

The spectral stability assumption corresponds to additional information on the point spectrum of the linearization about the SDSP $\overline{u}$.

Technique: Adaptation of the ideas "à la Zumbrun" which study the stability of traveling waves for parabolic PDEs.

$$\textbf{Spectral stability} \underset{\textit{Chapter 4}}{\Longrightarrow} \textbf{Linear stability} \underset{\textit{Chapter 5}}{\Longrightarrow} \textbf{Nonlinear stability}$$

Chapter 4: Spectral stability implies linear stability

The content of this chapter is contained in [C. '23] and is an extension of the result of [Godillon '03] and [Serre '07].

Quick overview of the content:

- Based on [Zumbrun and Howard, '98] and [Mascia and Zumbrun,'02,'03].
- Translate the spectral information on the linearization about the wave into estimates on the semi-group associated with it
- Main difficulty: No spectral gap $\oplus$ Presence of an eigenvalue of modulus 1 in the essential spectrum
- Solution: Pointwise description of the Green's function of the linearized operator using spatial dynamics

Linearization of the numerical scheme about the SDSP

- The linear operators $\mathcal{L}^{\pm} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}^{\pm} h)_j := \sum_{k=-p}^{q} a_k^{\pm} h_{j+k}$$

correspond to the linearization of the numerical scheme $\mathcal{N}$ about the constant states $u^{\pm}$.

- We define the linear operator $\mathcal{L} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ obtained by linearizing $\mathcal{N}$ about $\overline{u}$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L} h)_j := \sum_{k=-p}^{q} a_{j,k} h_{j+k},$$

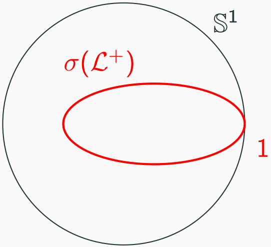
with $a_{j,k} \to a_k^{\pm}$ as $j \to \pm\infty$.

Observation on the spectrum of $\mathcal{L}^{\pm}$

- The spectrum of $\mathcal{L}^{\pm}$ is given by:

$$\sigma(\mathcal{L}^{\pm}) := \left\{ \sum_{k=-p}^{q} a_k^{\pm} e^{i\xi k}, \xi \in \mathbb{R} \right\}.$$

- The consistency condition implies that $1 \in \sigma(\mathcal{L}^{\pm})$.



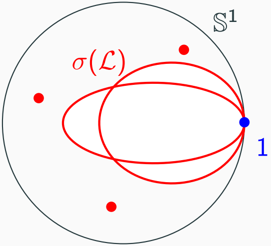
- The linear ℓ^2-stability implies:

$$\forall \kappa \in \mathbb{S}^1 \backslash \{1\}, \quad \left| \sum_{k=-p}^{q} a_k^{\pm} \kappa^k \right| < 1. \quad (\textit{Dissipativity condition})$$

- The diffusivity condition states there exist two complex constants $\beta^{\pm}$ with positive real parts such that:

$$\sum_{k=-p}^{q} a_k^{\pm} e^{i\xi k} \underset{\xi \to 0}{=} \exp\left(-i\xi\nu f'(u^{\pm}) - \beta^{\pm}\xi^2 + o(\xi^2)\right) \quad (\textit{Diffusion})$$

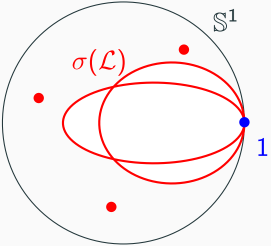
for a first order scheme.



Observation on the spectrum of $\mathcal{L}$

- The curves $\sigma(\mathcal{L}^+)$ and $\sigma(\mathcal{L}^-)$ belong to the essential spectrum of $\mathcal{L}$.
- Outside of those essential spectrum curves, the spectrum $\sigma(\mathcal{L})$ is only composed of eigenvalues (represented with red points).
- 1 is an eigenvalue of the operator $\mathcal{L}$:

$$"\mathcal{N}(\overline{u}^\delta) = \overline{u}^\delta \Longrightarrow \left.\frac{\partial \overline{u}^\delta}{\partial \delta}\right|_{\delta=0} \in \ker(Id - \mathcal{L})."$$



Spectral stability assumption

- We construct a so-called Evans function. We assume that 1 is a simple zero of the Evans function. As a consequence, 1 is a simple eigenvalue of the operator $\mathcal{L}$:

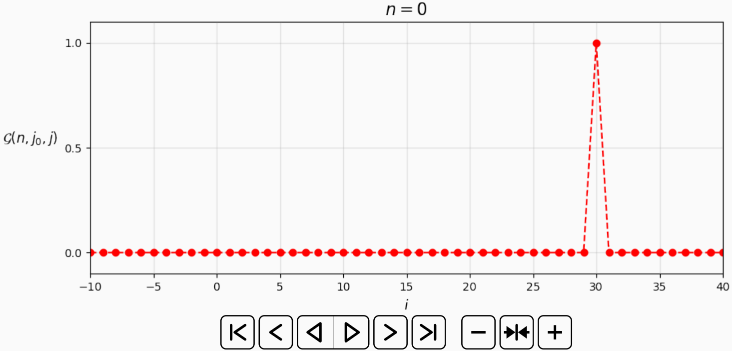
$$" \ker(Id - \mathcal{L}) = \mathrm{Span}\left(\left.\frac{\partial \overline{u}^{\delta}}{\partial \delta}\right|_{\delta=0} \right)." $$

- The operator $\mathcal{L}$ has no other eigenvalue of modulus equal or larger than 1.

We define the Green's function associated to $\mathcal{L}$:

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j \in \mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Green's function associated to the operator $\mathcal{L}$ for $j_0 = 30$



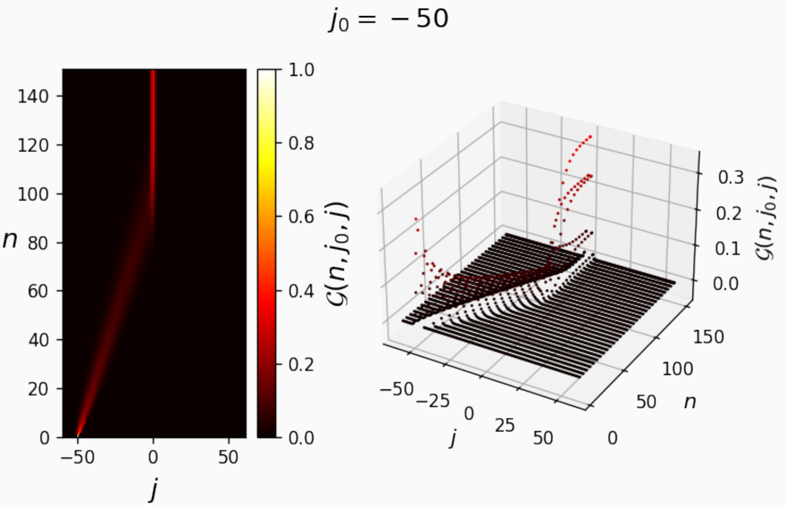
Theorem [C. 23']

Under some more precise assumptions, there exist a positive constant c, an element V of $\ker(Id - \mathcal{L})$ and an (explicit) function $E : \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$
\begin{aligned}
&\mathcal{G}(n, j_0, j)\\
=&E\left(\frac{nf'(u^+)\nu + j_0}{\sqrt{n}}\right) V(j) \quad \text{(Excited eigenvector)}\\
+&\mathbb{1}_{j\in\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\frac{\left|nf'(u^+)\nu - (j - j_0)\right|^2}{n}\right)\right) \quad \text{(Gaussian wave)}\\
+&\mathbb{1}_{j\in-\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\frac{\left|nf'(u^+)\nu + j_0\right|^2}{n}\right) e^{-c|j|}\right) \quad \text{(Exponential residual)}\\
+&O(e^{-cn-c|j-j_0|})
\end{aligned}
$$

where $E(x) \underset{x\to-\infty}{\longrightarrow} 1$ and $E(x) \underset{x\to+\infty}{\longrightarrow} 0$.

There is a similar result for $j_0 \in -\mathbb{N}$.


$j_0 = -50$
140
120
100
80
60
40
20
0
n
−50
0
50
j
1.0
0.8
0.6
0.4
0.2
0.0
$\mathcal{G}(n, j_0, j)$
0.3
0.2
0.1
0.0
$\mathcal{G}(n, j_0, j)$
150
100
50
0
n
−50
−25
0
25
50
j

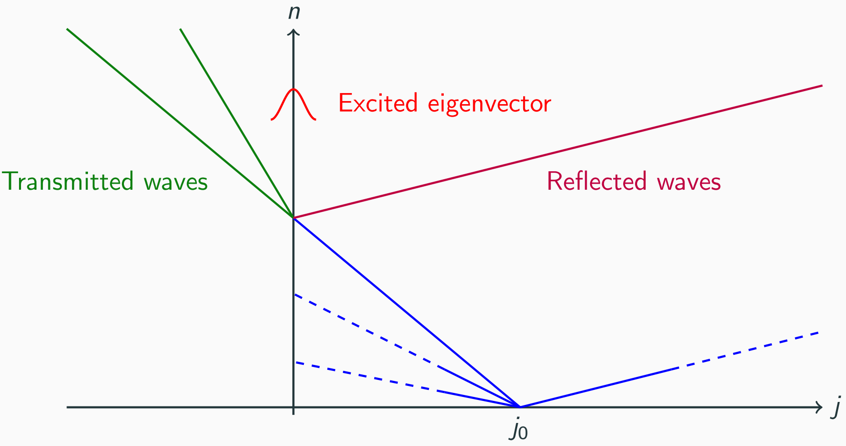
Some precisions on the result:

- We actually extract the leading behavior of the Gaussian wave of the Green's function.
- There is also such a description of the discrete derivative of the Green's function:

$$\mathcal{G}(n, j_0 - 1, j) - \mathcal{G}(n, j_0, j).$$

- The result is proved for a large family of higher order diffusive schemes.
- The result is proved in the case of systems of conservation laws.

Result on the Green's function in the case of systems of conservation laws



Brief idea of the proof

Using the inverse Laplace tranform, we have:

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi} \int_{\Gamma} z^n G(z, j_0, j) dz$$

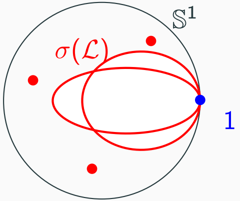
where the spatial Green's function $G(z, j_0, j)$ is defined by:

$$\forall z \notin \sigma(\mathcal{L}), \forall j_0, j \in \mathbb{Z}, \quad G(z, j_0, j) := \left((zId - \mathcal{L})^{-1} \delta_{j_0} \right)_j$$

and the integration path Γ surrounds the spectrum $\sigma(\mathcal{L})$.

Goal:

- Study the spatial Green's function $G(z, j_0, j)$
- Find a suitable choice of path Γ



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$$\forall z \notin \sigma(\mathcal{L}), \forall j_0, j \in \mathbb{Z}, \quad G(z, j_0, j) := \left((zId - \mathcal{L})^{-1} \delta_{\mathbf{j_0}}\right)_j .$$

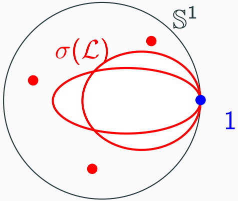
Properties of $G(z, j_0, j)$:

- **Local exponential bounds:** For all z_0 outside of $\sigma(\mathcal{L})$:

$$|G(z, j_0, j)| \leq Ce^{-c|j - \mathbf{j_0}|}$$

with z in a neighborhood U of z_0 and two constants C, c.

- **Meromorphical extension** of $G(\cdot, j_0, j)$ near 1 through the essential spectrum with a pole of order 1 at 1



We rewrite the eigenvalue problem:

$$(zId - \mathcal{L})u = 0$$

as a discrete dynamical system:

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z)W_j. \qquad \text{(Dyn. Syst.)}$$

We are interested in solutions of the dynamical system that tend towards 0 as j tends to $+\infty$ or $-\infty$:

$$E_0^{\pm}(z) := \left\{ W_0, \quad W_j \underset{j \to \pm\infty}{\to} 0 \right\}.$$

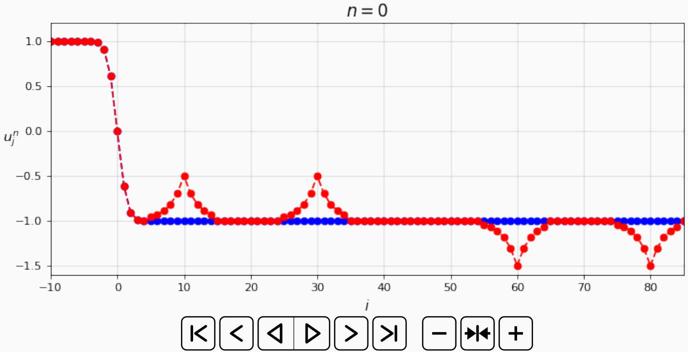
We essentially characterize the elements of the vector spaces $E_0^{\pm}(z)$ in two ways:

•**Geometric dichotomy** (projectors) • **Jost solutions** (basis)

Several uses:

- Analysis of the essential spectrum of $\mathcal{L}$
- Characterization of the point spectrum $\mathcal{L}$ and construction of the Evans function.
- Expression of the spatial Green's function $G(z, j_0, j)$.

Chapter 5: Linear stability implies nonlinear stability



In a continuous setting:

Shock tracking techniques based on [Zumbrun '00]

We consider the solution u for an initial condition $u^0(x) := \overline{u}(x) + h^0(x)$. We want to prove the decay of:

$$v(t,x) := u(t, x + \delta(t)) - \overline{u}(x)$$

for a good choice of co-moving frame parametrized by a function δ.

For $h \in \ell^\infty(\mathbb{Z})$ and $n \in \mathbb{N}$, we have:

$$\forall j \in \mathbb{Z}, \quad (\mathcal{L}^n h)_j = \sum_{j_0 \in \mathbb{Z}} \mathcal{G}(n, j_0, j) h_{j_0}.$$

Also:

$$\begin{aligned}
&\mathcal{G}(n, j_0, j) \\
=&E\left(\frac{nf'(u^+)\nu + j_0}{\sqrt{n}}\right) V(j) \quad \text{(Excited eigenvector)} \\
+&\mathbb{1}_{j\in\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu - (j-j_0)|^2}{n}\right)\right)\right) \quad \text{(Gaussian wave)} \\
+&\mathbb{1}_{j\in-\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu + j_0|^2}{n}\right)\right) e^{-c|j|}\right) \quad \text{(Exponential residual)} \\
+&O(e^{-cn-c|j-j_0|})
\end{aligned}$$

In a continuous setting:

Shock tracking stability based on [Zumbrun '00]

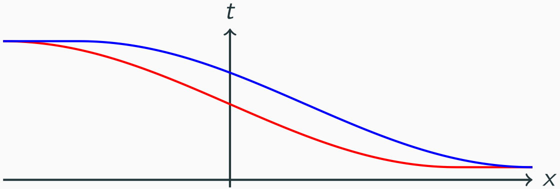
We want to prove the decay of:

$$v(t,x) := u(t, x + \delta(t)) - \overline{u}(x)$$

for a good choice of co-moving frame parametrized by a function δ.

Difficulties for the shock tracking technique in the fully discrete setting:

- Restriction of translations on the grid
- "Linearization and translation do not commute in the discrete setting"



Idea: We will use the limit SDSP towards which the solution converges.

We consider an initial perturbation h^0 such that:

$$\sum_{j\in\mathbb{Z}} h_j^0 = 0. \quad \text{(Zero-mass assumption)}$$

For the solution of the numerical scheme $(u^n)_{n\in\mathbb{N}}$ with $u^0 := \overline{u} + h^0$, then the sequence u^n should converge towards $\overline{u}$.

We thus want to estimate $h^n := u^n - \overline{u}$ in some suitable norm.

Main calculations: For $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, we have that:

$$u_j^{n+1} = \mathcal{N}(u^n)_j \quad \Rightarrow \quad h_j^{n+1} = (\mathcal{L}h^n)_j - Q(h^n)_{j+1} + Q(h^n)_j \tag{2}$$

where $Q(h^n)_j$ is a quadratic remainder.

Thus, we have:

$$\forall n \in \mathbb{N}, \quad h^{n+1} = \mathcal{L}h^n + (Id - \mathcal{T})Q(h^n) \tag{3}$$

where the shift operator $\mathcal{T}$ is defined by:

$$\mathcal{T} : (h_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \mapsto (h_{j+1})_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}.$$

Finally, using Duhamel's formula:

$$\boxed{\forall n \in \mathbb{N}, \quad h^n := \mathcal{L}^n h^0 + \sum_{m=0}^{n-1} \mathcal{L}^{n-1-m}(Id - \mathcal{T})Q(h^m).}$$

We have for $h \in \ell^\infty(\mathbb{Z})$, $n \in \mathbb{N}$ and $j \in \mathbb{Z}$:

$$\mathcal{L}^n h = \sum_{j_0 \in \mathbb{Z}} \mathcal{G}(n, j_0, j) h_{j_0},$$

$$\mathcal{L}^n(Id - \mathcal{T})h = \sum_{j_0 \in \mathbb{Z}} \left(\mathcal{G}(n, j_0 - 1, j) - \mathcal{G}(n, j_0, j)\right) h_{j_0}.$$

The polynomially weighted norms

For $r \in [1, +\infty]$ and $\gamma \in [0, +\infty[$, we define the polynomial-weighted spaces ℓ^r_γ:

$$\ell^r_\gamma := \{(h_j)_{j \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad ((1 + |j|)^\gamma h_j)_{j \in \mathbb{Z}} \in \ell^r(\mathbb{Z})\}$$

with the norm:

$$\forall h \in \ell^r_\gamma, \quad \|h\|_{\ell^r_\gamma} = \left\| ((1 + |j|)^\gamma h_j)_{j \in \mathbb{Z}} \right\|_{\ell^r}.$$

A nonlinear stability result

Theorem

Let us assume that the same assumptions are verified (and thus especially the spectral stability assumption). We consider a constant $\mathbf{p} \in [0, +\infty[$. There exist two constants $\varepsilon, C \in [0, +\infty[$ such that, if we consider a initial perturbation $h^0 \in \mathbb{R}^{\mathbb{Z}}$ such that:

$$\left\|h^0\right\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}} < \varepsilon \quad \text{(polynomial weight condition)}$$

$$\sum_{j\in\mathbb{Z}} h_j^0 = 0 \quad \text{(zero mass perturbation)}$$

then for all $n \in \mathbb{N}\backslash\{0\}$:

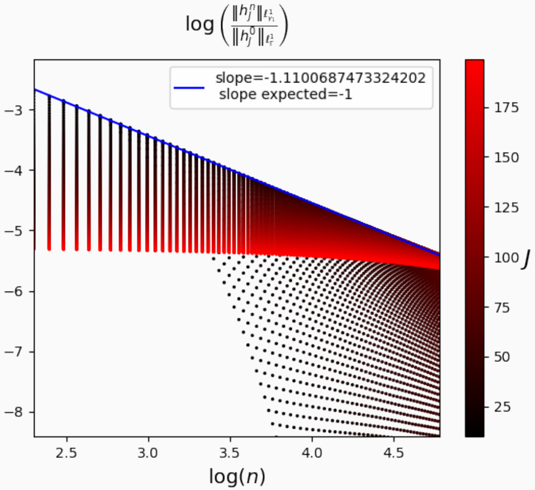
$$\left\|u^n - \overline{u}\right\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})}} \leq \frac{C}{n^{\mathbf{p}}} \left\|h^0\right\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}},$$

$$\left\|u^n - \overline{u}\right\|_{\ell^{\infty}_{\max(\mathbf{1},\mathbf{p})}} \leq \frac{C}{n^{\mathbf{p}+\frac{\mathbf{1}}{\mathbf{2}}}} \left\|h^0\right\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}}.$$

We display the value of:

$$\ln\left(\frac{\|h^n\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})}}}{\|h^0\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}}}\right) \quad \text{and} \quad \ln\left(\frac{\|h^n\|_{\ell^{\infty}_{\max(\mathbf{1},\mathbf{p})}}}{\|h^0\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}}}\right)$$

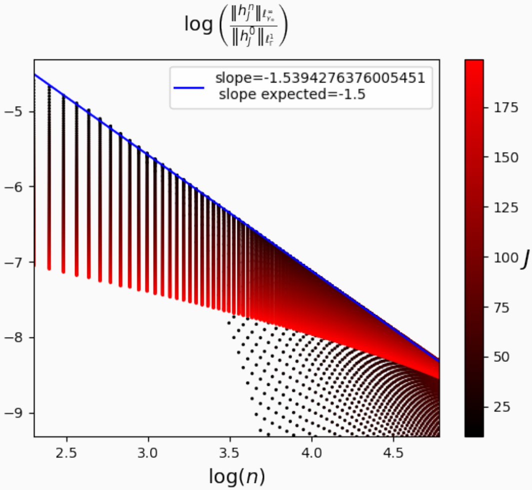
for $\mathbf{p} = 1$ and $h^0_J := \delta_J - \delta_0$.



We display the value of:

$$\ln\left(\frac{\|h^n\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})}}}{\|h^0\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}}}\right) \quad \text{and} \quad \ln\left(\frac{\|h^n\|_{\ell^{\infty}_{\max(\mathbf{1},\mathbf{p})}}}{\|h^0\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}}}\right)$$

for $\mathbf{p} = 1$ and $h^0_J := \delta_J - \delta_0$.



Conclusion/ Perspective / Open questions

About the Green's function theorem:

- Bounds on Green's function uniform in j_0
- Proved for a large family of schemes
- The result is proved for systems

About the nonlinear stabilty theorem:

- The result is not **yet** proved for systems
- The "Shock tracking issue" creates limitations

Other Perspectives:

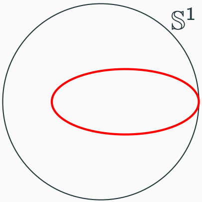
- Proving a more general nonlinear orbital stability result
- Existence of spectrally stable SDSPs?
- What can we say for moving shocks (with rational speed) and/or for under/over-compressive shocks?
- What can we say for dispersive schemes? (Lax-Wendroff for instance)
- Study of the stability for multi-dimensional conservation laws

Thank you for your attention!

On the result of Chapter 2

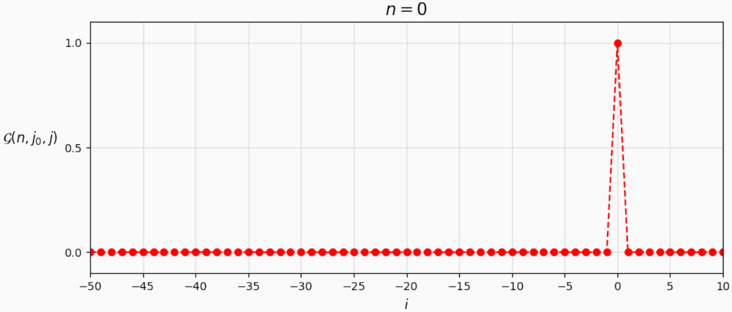
Transport equation on $\mathbb{R}$:

$$\forall t \geq 0, \forall x \in \mathbb{R},\ \partial_t u + v\partial_x u = 0.$$



(Diffusive) Numerical scheme:

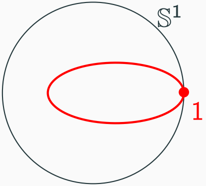
$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z},\ u_j^{n+1} := \sum_{k\in\mathbb{Z}} a_k u_{j+k}^n.$$



On the result of Chapter 3

Transport equation on $\mathbb{R}_+$ with $v < 0$:

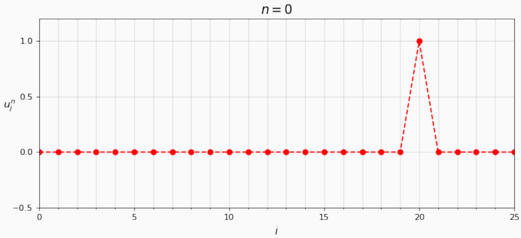
$$\forall t, x \geq 0,\ \partial_t u + v \partial_x u = 0.$$



(Diffusive) Numerical scheme:

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{N} \setminus \{0\}, u_j^{n+1} = \sum_{k=-r}^{p} a_k u_{j+k}^n,$$

$$\forall n \in \mathbb{N}, \forall j \in \{1-r, \ldots, 0\}, u_j^n = \sum_{k=1}^{p_b} b_{k,j} u_k^n.$$



For $h \in \ell^\infty(\mathbb{Z})$ and $n \in \mathbb{N}$, we have:

$$\forall j \in \mathbb{Z}, \quad (\mathcal{L}^n h)_j = \sum_{j_0 \in \mathbb{Z}} h_{j_0} \mathcal{G}(n, j_0, j).$$

Lemma

For any $0 \leq \gamma \leq \Gamma$, there exists a constant $C > 0$ such that:

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \mathbb{E}_\Gamma, \quad \|\mathcal{L}^n h\|_{\ell^1_\gamma} \leq \frac{C}{n^{\Gamma-\gamma}} \|h\|_{\ell^1_\Gamma},$$

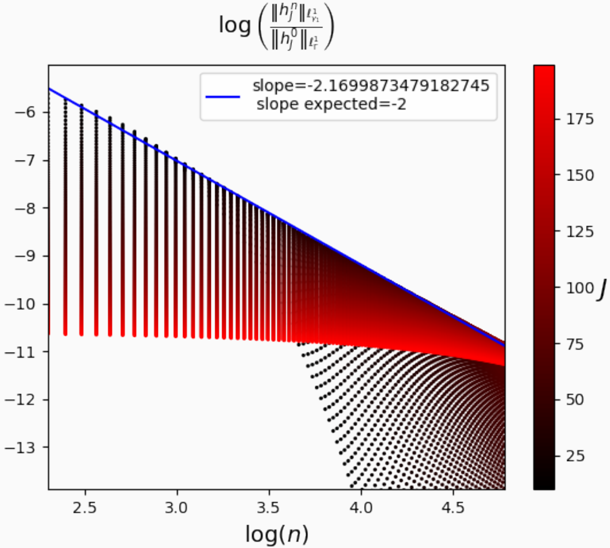
$$\forall n \in \mathbb{N}\backslash\{0\}, \forall h \in \mathbb{E}_\Gamma, \quad \|\mathcal{L}^n h\|_{\ell^\infty_\gamma} \leq \frac{C}{n^{\Gamma-\gamma+\min\left(\gamma, \frac{1}{2\mu}\right)}} \|h\|_{\ell^1_\Gamma},$$

$$\forall n \in \mathbb{N}, \forall h \in \ell^1_\Gamma, \quad \|\mathcal{L}^n (Id - \mathcal{T}) h\|_{\ell^1_\gamma} \leq \frac{C}{(n+1)^{\Gamma-\gamma+\frac{1}{2\mu}}} \|h\|_{\ell^1_\Gamma},$$

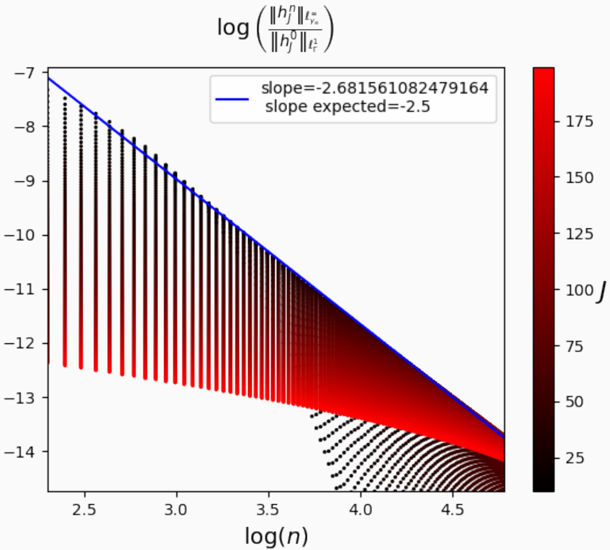
$$\forall n \in \mathbb{N}, \forall h \in \ell^1_\Gamma, \quad \|\mathcal{L}^n (Id - \mathcal{T}) h\|_{\ell^\infty_\gamma} \leq \frac{C}{(n+1)^{\Gamma-\gamma+\frac{1}{2\mu}+\min\left(\gamma, \frac{1}{2\mu}\right)}} \|h\|_{\ell^1_\Gamma},$$

$$\forall n \in \mathbb{N}, \forall h \in \ell^\infty_\Gamma, \quad \|\mathcal{L}^n (Id - \mathcal{T}) h\|_{\ell^\infty_\gamma} \leq \frac{C}{(n+1)^{\Gamma-\gamma+\min\left(\gamma, \frac{1}{2\mu}\right)}} \|h\|_{\ell^\infty_\Gamma}.$$

$\mathbf{p} = 2$

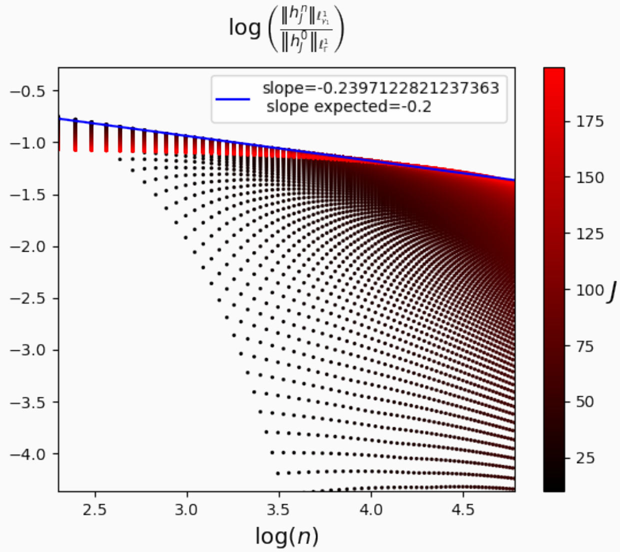

log $\left(\frac{\|h_j^n\|_{\ell^1_{\gamma_1}}}{\|h_j^0\|_{\ell^1_{\Gamma}}}\right)$
slope=-2.1699873479182745
slope expected=-2
−6
−7
−8
−9
−10
−11
−12
−13
2.5
3.0
3.5
4.0
4.5
log(n)
175
150
125
100
75
50
25
J

$\mathbf{p} = 2$



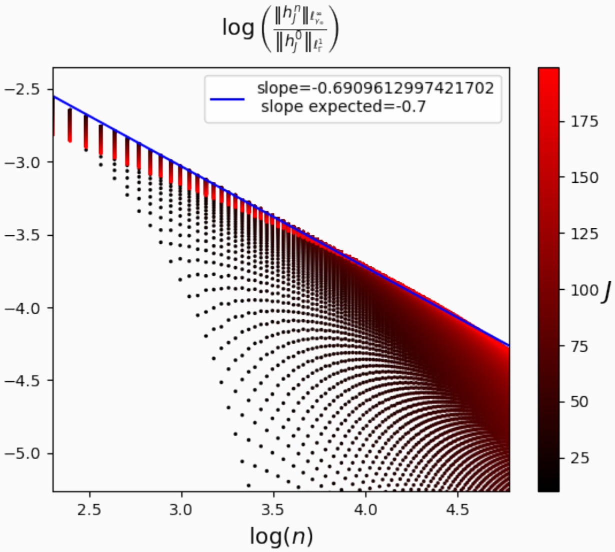
$\log\left(\frac{\|h_j^n\|_{\ell^\infty_{\gamma_\infty}}}{\|h_j^0\|_{\ell^1_r}}\right)$
slope=-2.681561082479164
slope expected=-2.5
−7
−8
−9
−10
−11
−12
−13
−14
2.5
3.0
3.5
4.0
4.5
log(n)
175
150
125
100
75
50
25
J

$\mathbf{p} = 0.2$



$\log\left(\frac{\|h_J^n\|_{\ell^1_{\gamma_1}}}{\|h_J^0\|_{\ell^1_{\gamma}}}\right)$
slope=-0.2397122821237363
slope expected=-0.2
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0
2.5
3.0
3.5
4.0
4.5
log(n)
175
150
125
100
75
50
25
J

$\mathbf{p} = 0.2$



$\log\left(\frac{\|h_J^n\|_{\ell^\infty_{\gamma_\infty}}}{\|h_J^0\|_{\ell^1_\Gamma}}\right)$
slope=-0.6909612997421702
slope expected=-0.7
−2.5
−3.0
−3.5
−4.0
−4.5
−5.0
2.5
3.0
3.5
4.0
4.5
$\log(n)$
175
150
125
100
J
75
50
25