Stability of discrete shock profiles for systems of conservation laws

Lucas Coeuret

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Institut de Mathématiques de Toulouse (IMT)

Conservation laws and shocks

We consider a one-dimensional scalar conservation law

$$\begin{aligned} \partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \end{aligned} \tag{1}$$

where the flux $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

Some of the results that will be presented also hold for systems of conservations laws.

Observation: This type of PDE tends to have solutions with discontinuities.

Overarching goal: When considering a conservative finite difference scheme, understand if it will be able to handle/capture discontinuities of solutions.

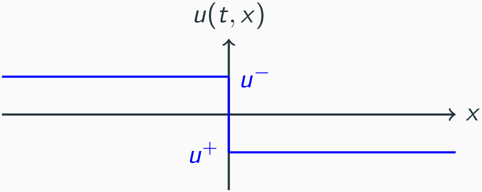
We consider two distinct states $u^-, u^+ \in \mathbb{R}^2$ such that:

$$\begin{aligned} f(u^-) &= f(u^+), \quad \text{(Rankine-Hugoniot condition)} \\ f'(u^+) < 0 &< f'(u^-). \quad \text{(Lax shock)} \end{aligned}$$

The function u defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t,x) := \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{else}, \end{cases}$$

which is a weak solution of the scalar conservation law, is known as a steady Lax shock.



Conservative finite difference schemes

We consider a conservative explicit finite difference scheme

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n)$$

where for $u = (u_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $j \in \mathbb{Z}$ as

$$\mathcal{N}(u)_j := u_j - \nu \left(F\left(\nu; u_{j-p+1}, \ldots, u_{j+q}\right) - F\left(\nu; u_{j-p}, \ldots, u_{j+q-1}\right)\right).$$

- $u^0 \in \mathbb{R}^{\mathbb{Z}}$: initial condition
- $\mathcal{N} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$: nonlinear discrete evolution operator
- $F :]0, +\infty[\times \mathbb{R}^{p+q} \to \mathbb{R}$: numerical flux
- $p, q \in \mathbb{N}\backslash\{0\}$: integers defining the size of the stencil of the scheme.
- $\nu = \frac{\Delta t}{\Delta x} > 0$: ratio between the time and space steps.

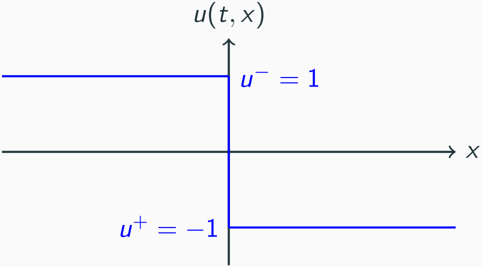
Assumptions:

- $\forall u \in \mathbb{R}, \quad F(\nu; u, \ldots, u) = f(u)$ (consistency condition)
- For some neighborhood $\mathcal{U}$ of the states $u^{\pm}$

$$\forall u \in \mathcal{U}, \quad -q \leq \nu f'(u) \leq p \quad \text{(CFL condition on } \nu\text{)}$$

- Linear-ℓ^2 stability for constant states $u \in \mathcal{U}$
- The scheme introduces numerical viscosity. In the present presentation, we consider a first order scheme. This excludes dispersive schemes like for instance the Lax-Wendroff scheme.

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$.



For the numerical scheme, we consider the modified Lax Friedrichs scheme

$$\forall u \in \mathbb{R}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \quad \mathcal{N}(u)_j := \frac{u_{j+1} + u_j + u_{j-1}}{3} - \nu \frac{f(u_{j+1}) - f(u_{j-1})}{2}.$$

Discrete shock profiles

Discrete shock profile (DSP): A discrete shock profile is a solution of the numerical scheme of the form

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^n = \overline{u}(j - s\nu n)$$

where the function $\overline{u} : \mathbb{Z} + s\nu\mathbb{Z} \to \mathbb{R}$ verifies that

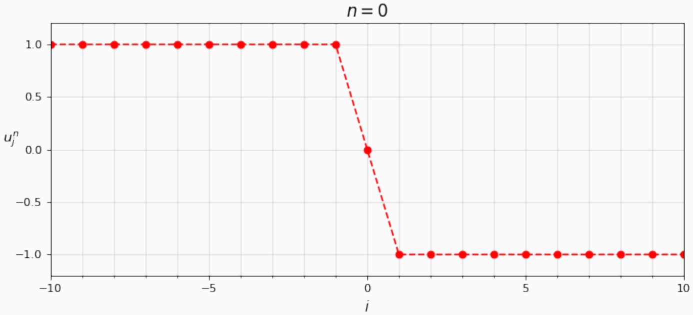
$$\overline{u}(x) \underset{x \to \pm\infty}{\to} u^{\pm}.$$

Stationary discrete shock profiles (SDSP) are sequences $\overline{u} = (\overline{u}_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfy

$$\mathcal{N}(\overline{u}) = \overline{u} \quad \text{and} \quad \overline{u}_j \underset{j \to \pm\infty}{\to} u^{\pm}.$$

Example : We consider the initial condition (mean of the standing shock on each cell $\left[(j-\frac{1}{2})\Delta x,(j+\frac{1}{2})\Delta x\right[$)

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$



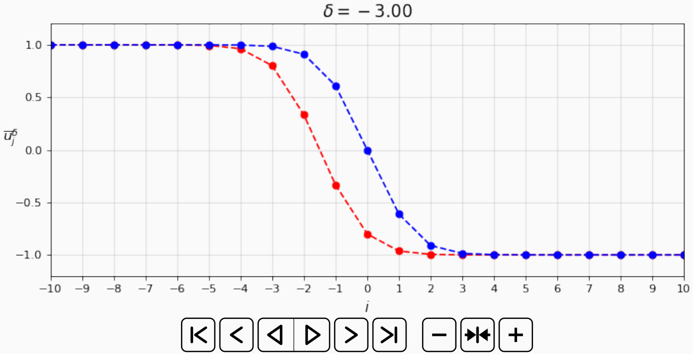
Main goal: Finding conditions on the numerical schemes so that:

stable shock waves for the PDE $\Rightarrow$ stable DSPs for the numerical scheme

This separates the theory surrounding DSPs in two parts:

- Existence of DSPs
- Stability of DSPs

Example : We consider the same initial condition u^0 as before but add a mass δ at $j = 0$. We look at the limit of the solution of the numerical scheme.



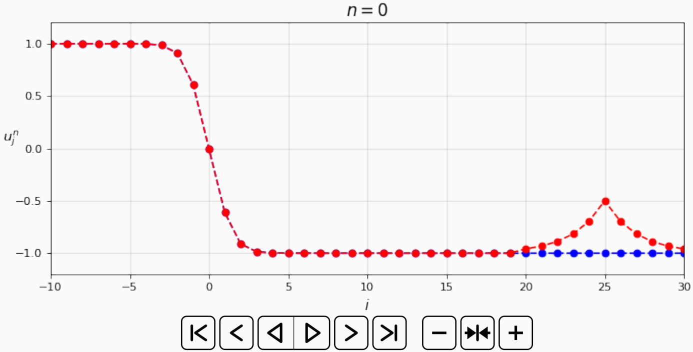
For standing Lax shocks, one aims to have the existence of a differentiable one-parameter family $(\overline{u}^\delta)_{\delta\in]-\varepsilon,\varepsilon[}$ of SDSPs.

- Jennings, *Discrete shocks* ('74)
    - **scalar** case
    - conservative **monotone** scheme
    - for shocks satisfying Oleinik's E-condition
- Majda and Ralston, *Discrete Shock Profiles for Systems of Conservation Laws* ('79)
    - **system** case
    - **first order** scheme
    - **weak** Lax shocks
- Michelson, *Discrete shocks for difference approximations to systems of conservation laws* ('84)
    - extension of Majda-Ralston for **third order** scheme
- Different cases: [Smyrlis,90'], [Liu and Yu, 99'], [Serre, 04'] etc...

Stability of discrete shock profiles

The end goal would be the nonlinear orbital stability of those DSPs:

For small admissible perturbations h, prove that the solution u^n of the numerical scheme for the initial condition $u^0 = \overline{u} + h$ converges towards the set of translations of the DSP $\{\overline{u}^\delta, \delta \in] - \varepsilon, \varepsilon[\}$.



- Jennings, *Discrete shocks* ('74)
    - scalar case
    - conservative monotone scheme
    - nonlinear orbital stability for ℓ^1 perturbations
- Liu-Xin, *L^1-stability of stationary discrete shocks*, ('93)
    - system case
    - Lax-Friedrichs scheme
    - weak Lax shocks
    - zero mass perturbation (dropped in [Ying, 97'])
- Michelson, *Stability of discrete shocks for difference approximations to systems of conservation laws*, ('02)
    - system case
    - weak Lax shocks
    - First and third order schemes
- Different cases: [Smyrlis, 90'], [Liu-Yu, 99'], etc...

Goal: Can we prove a result of nonlinear orbital stabilty that works:

- in the system case,
- for a fairly large class of numerical schemes,
- replacing the smallness assumption on the amplitude of the shock with a spectral stability assumption.

This correponds to the Open question 5.3 of *Discrete shock profiles: Existence and stability* ('07) by Serre.

Technique: Adaptation of the ideas of Zumbrun and Howard in *Pointwise semigroup methods and stability of viscous shock waves* ('98) which study traveling waves for parabolic PDEs.

A simple example

Spectral stability $\Rightarrow$ **Linear stability** $\Rightarrow$ **Nonlinear stability**

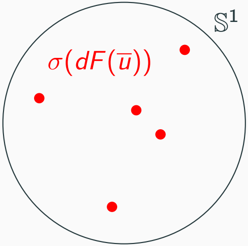
We consider a smooth function $F : \mathbb{R}^n \to \mathbb{R}^n$ and a fixed point $\overline{u} \in \mathbb{R}$ of this function.

$$\forall n \in \mathbb{N}, \quad u^{n+1} = F(u^n).$$

A simple example

Spectral stability $\Rightarrow$ **Linear stability** $\Rightarrow$ **Nonlinear stability**

Spectrum of the jacobian $dF(\overline{u})$



A simple example

Spectral stability $\Rightarrow$ **Linear stability** $\Rightarrow$ **Nonlinear stability**

There exist two constants $C, c > 0$ such that:

$$\forall n \in \mathbb{N}, \quad \|dF(\overline{u})^n\| \leq Ce^{-cn}.$$

A simple example

Spectral stability $\Rightarrow$ **Linear stability** $\Rightarrow$ **Nonlinear stability**

We consider $h^0 \in \mathbb{R}$ and define $u^0 := \overline{u} + h^0$ and the solution $(u^n)_{n\in\mathbb{N}}$ of the dynamic system. We define

$$\forall n \in \mathbb{N}, \quad h^n := u^n - \overline{u}.$$

We have

$$\forall n \in \mathbb{N}, \quad h^{n+1} = dF(\overline{u})h^n + Q(h^n)$$

where $Q(h) := F(\overline{u}+h) - F(\overline{u}) - dF(\overline{u})h$. Using Duhamel's formula, we have:

$$\forall n \in \mathbb{N}, \quad h^n = dF(\overline{u})^n h^0 + \sum_{k=0}^{n-1} dF(\overline{u})^{n-1-k} Q(h^k).$$

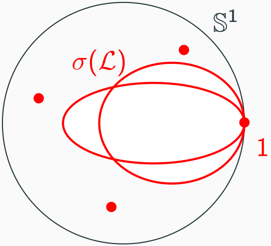
Linearization of the numerical scheme about the SDSP

We define the bounded operator $\mathcal{L} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ obtained by linearizing $\mathcal{N}$ about $\overline{u}$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}h)_j := \sum_{k=-p}^{q} a_{j,k} h_{j+k},$$

with $a_{j,k} \to a_k^{\pm}$ as $j \to \pm\infty$.

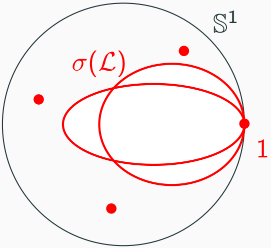
The coefficient $a_{j,k}$ are expressed using the partial derivatives $\partial_k F(\nu; \overline{u}_{j-p}, \dots, \overline{u}_{j+q-1})$.



Observation on the spectrum of $\mathcal{L}$

- There are curves of essential spectrum corresponding to the spectras of the linearized operators about the constant states u^+ and u^-.
- Outside of those essential spectrum curves, the spectrum is only composed of eigenvalues.
- 1 is an eigenvalue of the operator $\mathcal{L}$:

$$\text{"}\left(\forall \delta \in]-\varepsilon, \varepsilon[, \quad \mathcal{N}(\overline{u}^\delta) = \overline{u}^\delta\right) \Rightarrow \left.\frac{\partial \overline{u}^\delta}{\partial \delta}\right|_{\delta=0} \in \ker(Id - \mathcal{L})\text{."}$$



Spectral stability assumption

- In the article, we construct a so-called Evans function. We assume that 1 is a simple zero of the Evans function. As a consequence, 1 is a simple eigenvalue of the operator $\mathcal{L}$:

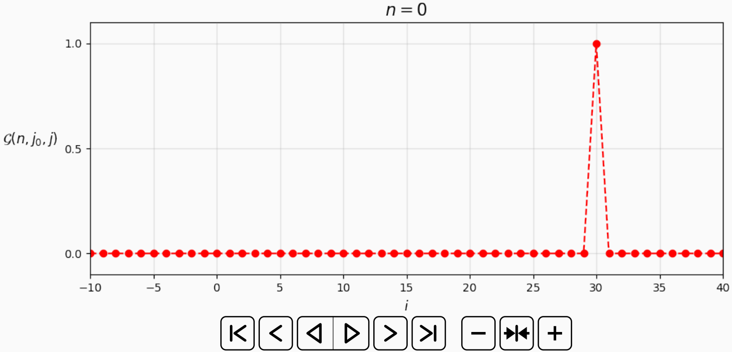
$$\text{" } \ker(Id - \mathcal{L}) = \mathrm{Span}\left(\left.\frac{\partial \overline{u}^{\delta}}{\partial \delta}\right|_{\delta=0} \right) \text{."}$$

- The operator $\mathcal{L}$ has no other eigenvalue of modulus equal or larger than 1.

We define the Green's function associated to $\mathcal{L}$:

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j \in \mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Green's function associated to the operator $\mathcal{L}$ for $j_0 = 30$



Theorem [C. 23']

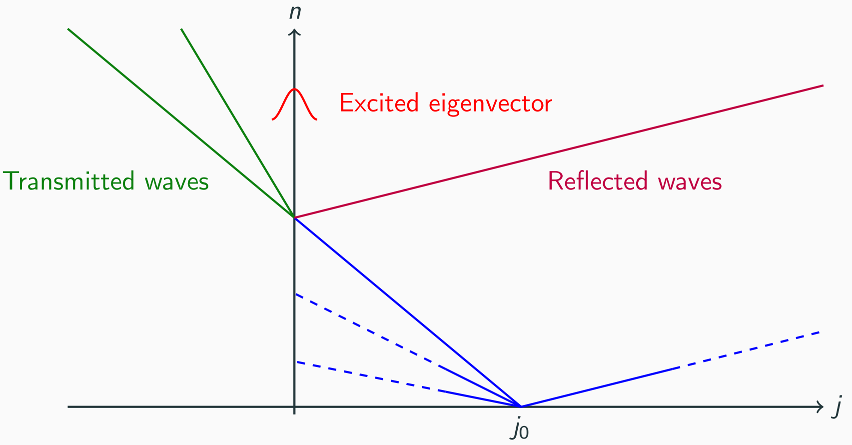
Under some more precise assumptions, there exist a positive constant c, an element V of $\ker(Id - \mathcal{L})$ and an (explicit) function $E : \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$
\begin{aligned}
&\mathcal{G}(n, j_0, j) \\
=&E\left(\frac{nf'(u^+)\nu + j_0}{\sqrt{n}}\right) V(j) \quad \text{(Excited eigenvector)} \\
+&\mathbb{1}_{j \in \mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu - (j - j_0)|^2}{n}\right)\right)\right) \quad \text{(Gaussian wave)} \\
+&\mathbb{1}_{j \in -\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu + j_0|^2}{n}\right)\right) e^{-c|j|}\right) \quad \text{(Exponential residual)} \\
+&O(e^{-cn - c|j - j_0|})
\end{aligned}
$$

where $E(x) \underset{x \to -\infty}{\to} 1$ and $E(x) \underset{x \to +\infty}{\to} 0$.

There is a similar result for $j_0 \in -\mathbb{N}$.

Result on the Green's function in the case of systems of conservation laws



Improvement of *Green's function pointwise estimates for the modified Lax-Friedrichs scheme* ('03) by Lafitte solving the following limitations:

- Relaxation of the restriction to the modified Lax-Friedrichs scheme.
- The description of the Green's function was **not uniform** in j_0.

For $h \in\in \ell^\infty(\mathbb{Z})$ and $n \in \mathbb{N}$, we have:

$$\forall j \in \mathbb{Z}, \quad (\mathscr{L}^n h)_j = \sum_{j_0 \in \mathbb{Z}} h_{j_0} \mathcal{G}(n, j_0, j).$$

The polynomially weighted norms

For $r \in [0, +\infty]$ and $\gamma \in [0, +\infty[$, we define the polynomial-weighted spaces ℓ^r_γ:

$$\ell^r_\gamma := \{(h_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad ((1+|j|)^\gamma h_j)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z})\}$$

and

$$\forall h \in \ell^r_\gamma, \quad \|h\|_{\ell^r_\gamma} = \left\| ((1+|j|)^\gamma h_j)_{j\in\mathbb{Z}} \right\|_{\ell^r}.$$

A nonlinear stability result

Theorem [C., upcoming]
Let us assume that the same assumptions are verified (and thus especially the spectral stability assumption). We consider a constant $\mathbf{p} \in [0, +\infty[$. There exist two constants $\varepsilon, C \in [0, +\infty[$ such that, if we consider a initial perturbation $h^0 \in \mathbb{R}^{\mathbb{Z}}$ such that:

$$\left\| h^0 \right\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}} < \varepsilon \quad \text{(polynomial weight condition)}$$

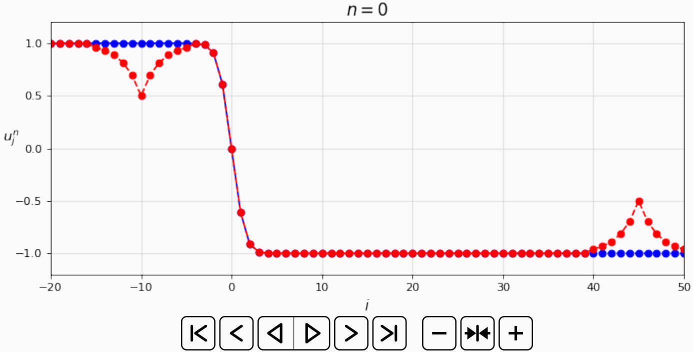
$$\sum_{j\in\mathbb{Z}} h^0_j = 0 \quad \text{(zero mass perturbation)}$$

then for all $n \in \mathbb{N}\backslash\{0\}$:

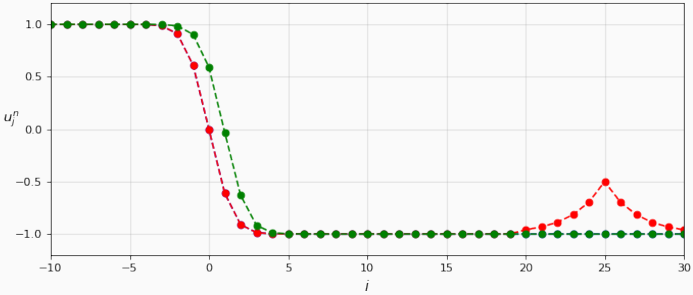
$$\left\| u^n - \overline{u} \right\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})}} \leq \frac{C}{n^{\mathbf{p}}} \left\| h^0 \right\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}},$$

$$\left\| u^n - \overline{u} \right\|_{\ell^{\infty}_{\max(\mathbf{1},\mathbf{p})}} \leq \frac{C}{n^{\mathbf{p}+\frac{\mathbf{1}}{\mathbf{2}\mu}}} \left\| h^0 \right\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}}.$$

The "Shock tracking issue"



Losing the zero mass assumption



Conclusion/ Perspective / Open questions

About the Green's function theorem:

- Bounds on Green's function uniform in j_0
- Proved for a large family of schemes
- The result is proved for systems

About the nonlinear stabilty theorem:

- The result is not **yet** proved for systems
- The "Shock tracking issue" creates limitations

Other Perspectives:

- Proving a more general nonlinear orbital stability result
- Existence of spectrally stable SDSPs?
- What can we say for moving shocks (with rational speed) and/or for under/over-compressive shocks?
- What can we say for dispersive schemes? (Lax-Wendroff for instance)
- Study of the stability for multi-dimensional conservation laws

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