

Stability of discrete shock profiles for systems of conservation laws

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Conservation laws and shocks

We consider a one-dimensional scalar conservation law

$$\begin{aligned}\partial_t u + \partial_x f(u) &= 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} &\rightarrow \mathbb{R},\end{aligned}\tag{1}$$

where the flux $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function.

Some of the results that will be presented also hold for systems of conservations laws.

Observation: This type of PDE tends to have solutions with discontinuities.

Overarching goal: When considering a conservative finite difference scheme, understand if it will be able to handle/capture discontinuities of solutions.

We consider two distinct states $u^-, u^+ \in \mathbb{R}^2$ such that:

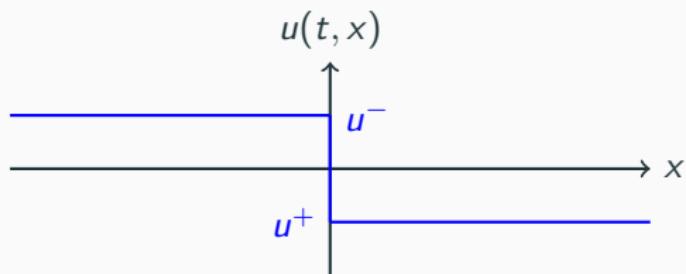
$$f(u^-) = f(u^+), \quad (\text{Rankine-Hugoniot condition})$$

$$f'(u^+) < 0 < f'(u^-). \quad (\text{Lax shock})$$

The function u defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t, x) := \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{else,} \end{cases}$$

which is a weak solution of the scalar conservation law, is known as a steady Lax shock.



Conservative finite difference schemes

We consider a **conservative explicit finite difference scheme**

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n)$$

where for $u = (u_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $j \in \mathbb{Z}$ as

$$\mathcal{N}(u)_j := u_j - \nu (F(\nu; u_{j-p+1}, \dots, u_{j+q}) - F(\nu; u_{j-p}, \dots, u_{j+q-1})).$$

- $u^0 \in \mathbb{R}^{\mathbb{Z}}$: initial condition
- $\mathcal{N} : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}^{\mathbb{Z}}$: nonlinear discrete evolution operator
- $F :]0, +\infty[\times \mathbb{R}^{p+q} \rightarrow \mathbb{R}$: numerical flux
- $p, q \in \mathbb{N} \setminus \{0\}$: integers defining the size of the stencil of the scheme.
- $\nu = \frac{\Delta t}{\Delta x} > 0$: ratio between the time and space steps.

Assumptions:

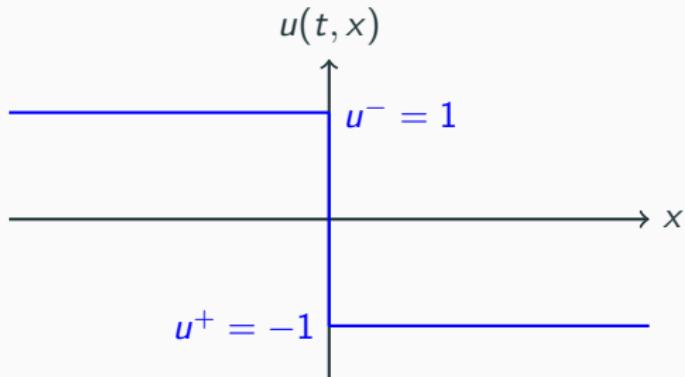
- $\forall u \in \mathbb{R}, \quad F(\nu; u, \dots, u) = f(u)$ (consistency condition)

- For some neighborhood \mathcal{U} of the states u^\pm

$$\forall u \in \mathcal{U}, \quad -q \leq \nu f'(u) \leq p \quad (\text{CFL condition on } \nu)$$

- Linear- ℓ^2 stability for constant states $u \in \mathcal{U}$
- The scheme introduces numerical viscosity. In the present presentation, we consider a first order scheme. This excludes dispersive schemes like for instance the Lax-Wendroff scheme.

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$.



For the numerical scheme, we consider the modified Lax Friedrichs scheme

$$\forall u \in \mathbb{R}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \quad \mathcal{N}(u)_j := \frac{u_{j+1} + u_j + u_{j-1}}{3} - \nu \frac{f(u_{j+1}) - f(u_{j-1})}{2}.$$

Discrete shock profiles

Discrete shock profile (DSP): A discrete shock profile is a solution of the numerical scheme of the form

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^n = \bar{u}(j - s\nu n)$$

where the function $\bar{u} : \mathbb{Z} + s\nu\mathbb{Z} \rightarrow \mathbb{R}$ verifies that

$$\bar{u}(x) \underset{x \rightarrow \pm\infty}{\rightarrow} u^\pm.$$

Stationary discrete shock profiles (SDSP) are sequences $\bar{u} = (\bar{u}_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfy

$$\mathcal{N}(\bar{u}) = \bar{u} \quad \text{and} \quad \bar{u}_j \underset{j \rightarrow \pm\infty}{\rightarrow} u^\pm.$$

Example : We consider the initial condition (mean of the standing shock on each cell $[(j - \frac{1}{2})\Delta x, (j + \frac{1}{2})\Delta x[$)

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$

Main goal: Finding conditions on the numerical schemes so that:
stable shock waves for the PDE \Rightarrow stable DSPs for the numerical scheme

This separates the theory surrounding DSPs in two parts:

- Existence of DSPs
- Stability of DSPs

Existence results on SDSPs

Example : We consider the same initial condition u^0 as before but add a mass δ at $j = 0$. We look at the limit of the solution of the numerical scheme.

For standing Lax shocks, one aims to have the existence of a differentiable one-parameter family $(\bar{u}^\delta)_{\delta \in]-\varepsilon, \varepsilon[}$ of SDSPs.

- Jennings, *Discrete shocks* ('74)
 - scalar case
 - conservative **monotone** scheme
 - for shocks satisfying Oleinik's E-condition
- Majda and Ralston, *Discrete Shock Profiles for Systems of Conservation Laws* ('79)
 - system case
 - first order scheme
 - weak Lax shocks
- Michelson, *Discrete shocks for difference approximations to systems of conservation laws* ('84)
 - extension of Majda-Ralston for **third order** scheme
- Different cases: [Smyrlis,90'], [Liu and Yu, 99'], [Serre, 04'] etc...

Stability of discrete shock profiles

The end goal would be the [nonlinear orbital stability](#) of those DSPs:

For **small admissible perturbations** h , prove that the solution u^n of the numerical scheme for the initial condition $u^0 = \bar{u} + h$ **converges** towards the set of translations of the DSP $\{\bar{u}^\delta, \delta \in]-\varepsilon, \varepsilon[\}\}$.

- Jennings, *Discrete shocks* ('74)
 - scalar case
 - conservative monotone scheme
 - nonlinear orbital stability for ℓ^1 perturbations
- Liu-Xin, *L¹-stability of stationary discrete shocks*, ('93)
 - system case
 - Lax-Friedrichs scheme
 - weak Lax shocks
 - zero mass perturbation (dropped in [Ying, 97'])
- Michelson, *Stability of discrete shocks for difference approximations to systems of conservation laws*, ('02)
 - system case
 - weak Lax shocks
 - First and third order schemes
- Different cases: [Smyrlis, 90'], [Liu-Yu, 99'], etc...

Goal: Can we prove a result of nonlinear orbital stability that works:

- in the **system** case,
- for a **fairly large class** of numerical schemes,
- replacing the **smallness assumption** on the amplitude of the shock with a **spectral stability assumption**.

Technique: Adaptation of the ideas of Zumbrun and Howard in
Pointwise semigroup methods and stability of viscous shock waves ('98)
which study traveling waves for parabolic PDEs.

Spectral stability \Rightarrow **Linear stability** \Rightarrow **Nonlinear stability**

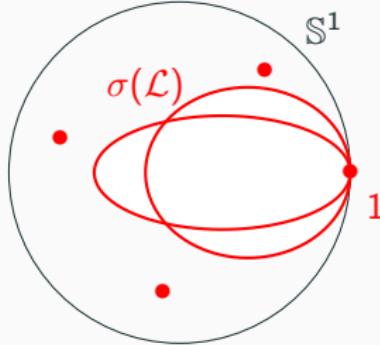
Linearization of the numerical scheme about the SDSP

We define the bounded operator $\mathcal{L} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ obtained by linearizing \mathcal{N} about \bar{u} :

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}h)_j := \sum_{k=-p}^q a_{j,k} h_{j+k},$$

with $a_{j,k} \rightarrow a_k^\pm$ as $j \rightarrow \pm\infty$.

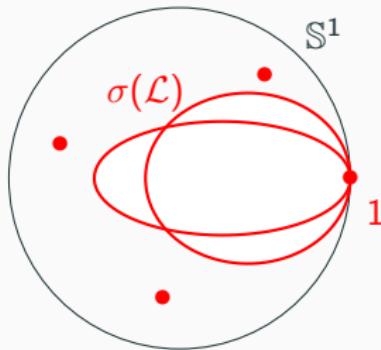
The coefficient $a_{j,k}$ are expressed using the partial derivatives $\partial_k F(\nu; \bar{u}_{j-p}, \dots, \bar{u}_{j+q-1})$.



Observation on the spectrum of \mathcal{L}

- There are curves of essential spectrum corresponding to the spectra of the linearized operators about the constant states u^+ and u^- .
- Outside of those essential spectrum curves, the spectrum is only composed of eigenvalues.
- 1 is an eigenvalue of the operator \mathcal{L} :

$$\text{“} (\forall \delta \in]-\varepsilon, \varepsilon[, \quad \mathcal{N}(\bar{u}^\delta) = \bar{u}^\delta) \Rightarrow \left. \frac{\partial \bar{u}^\delta}{\partial \delta} \right|_{\delta=0} \in \ker(Id - \mathcal{L}). \text{”}$$



Spectral stability assumption

- In the article, we construct a so-called Evans function. We assume that 1 is a simple zero of the Evans function. As a consequence, 1 is a simple eigenvalue of the operator \mathcal{L} :

$$\text{" } \ker(Id - \mathcal{L}) = \text{Span} \left(\frac{\partial \bar{u}^\delta}{\partial \delta} \Big|_{\delta=0} \right). \text{"}$$

- The operator \mathcal{L} has no other eigenvalue of modulus equal or larger than 1.

We define the **Green's function** associated to \mathcal{L} :

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j \in \mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Green's function associated to the operator \mathcal{L} for $j_0 = 30$

Theorem [C. 23']

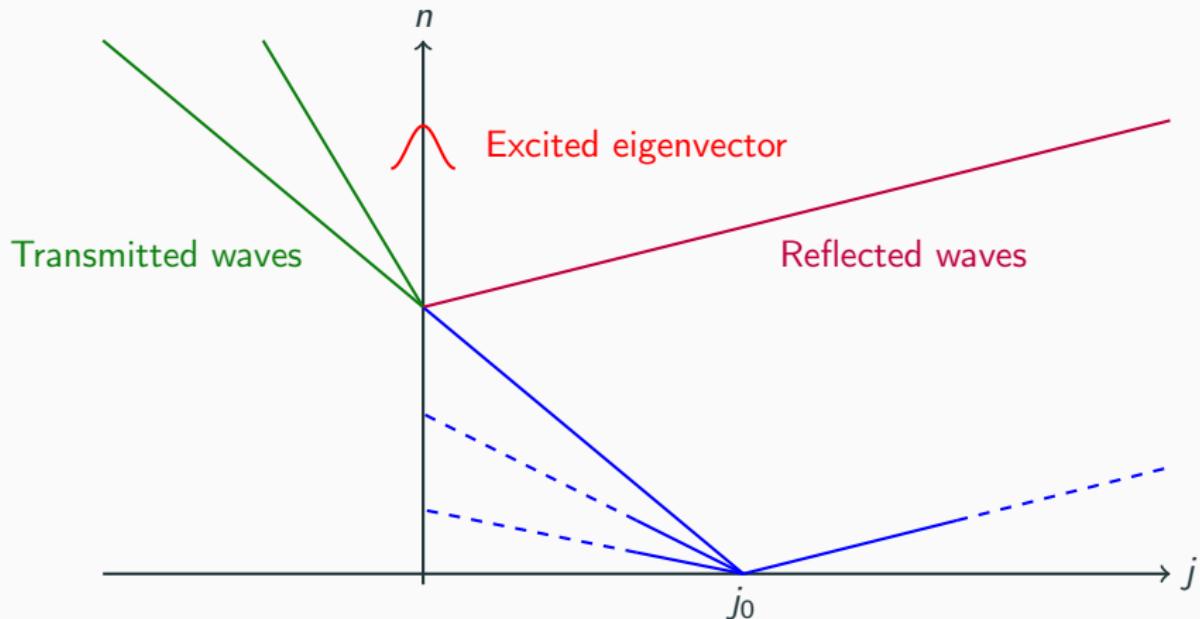
Under some more precise assumptions, there exist a positive constant c , an element V of $\ker(Id - \mathcal{L})$ and an (explicit) function $E : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $n \in \mathbb{N} \setminus \{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$\begin{aligned} & \mathcal{G}(n, j_0, j) \\ &= E\left(\frac{nf'(u^+)\nu + j_0}{\sqrt{n}}\right) V(j) \quad (\text{Excited eigenvector}) \\ &+ \mathbf{1}_{j \in \mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu - (j - j_0)|^2}{n}\right)\right)\right) \quad (\text{Gaussian wave}) \\ &+ \mathbf{1}_{j \in -\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|nf'(u^+)\nu + j_0|^2}{n}\right)\right) e^{-c|j|}\right) \quad (\text{Exponential residual}) \\ &+ O(e^{-cn - c|j - j_0|}) \end{aligned}$$

where $E(x) \xrightarrow{x \rightarrow -\infty} 1$ and $E(x) \xrightarrow{x \rightarrow +\infty} 0$.

There is a similar result for $j_0 \in -\mathbb{N}$.

Result on the Green's function in the case of systems of conservation laws



Improvement of *Green's function pointwise estimates for the modified Lax-Friedrichs scheme* ('03) by Lafitte solving the following limitations:

- Relaxation of the restriction to the modified Lax-Friedrichs scheme.
- The description of the Green's function was **not uniform** in j_0 .

Methodology (very shortly):

Adaptation of the ideas of Zumbrun and Howard in a fully discrete setting. Similar ideas as in the talk by **Benjamin Melinand** on Tuesday.

From linear to nonlinear stability result

In a continuous setting: Shock tracking technique We consider the solution u for an initial condition $u^0(x) := \bar{u}(x) + h^0(x)$. We want to prove the decay of:

$$v(t, x) := u(t, x + \delta(t)) - \bar{u}(x)$$

for a good choice of δ .

For $h \in \ell^\infty(\mathbb{Z})$ and $n \in \mathbb{N}$, we have:

$$\forall j \in \mathbb{Z}, \quad (\mathcal{L}^n h)_j = \sum_{j_0 \in \mathbb{Z}} h_{j_0} \mathcal{G}(n, j_0, j).$$

Also:

$$\begin{aligned}
& \mathcal{G}(n, j_0, j) \\
&= E \left(\frac{nf'(u^+) \nu + j_0}{\sqrt{n}} \right) V(j) \quad (\text{Excited eigenvector}) \\
&+ \mathbf{1}_{j \in \mathbb{N}} O \left(\frac{1}{\sqrt{n}} \exp \left(-c \left(\frac{|nf'(u^+) \nu - (j - j_0)|^2}{n} \right) \right) \right) \quad (\text{Gaussian wave}) \\
&+ \mathbf{1}_{j \in -\mathbb{N}} O \left(\frac{1}{\sqrt{n}} \exp \left(-c \left(\frac{|nf'(u^+) \nu + j_0|^2}{n} \right) \right) e^{-c|j|} \right) \quad (\text{Exponential residual}) \\
&+ O(e^{-cn - c|j - j_0|})
\end{aligned}$$

From linear to nonlinear stability result

In a continuous setting: Shock tracking technique

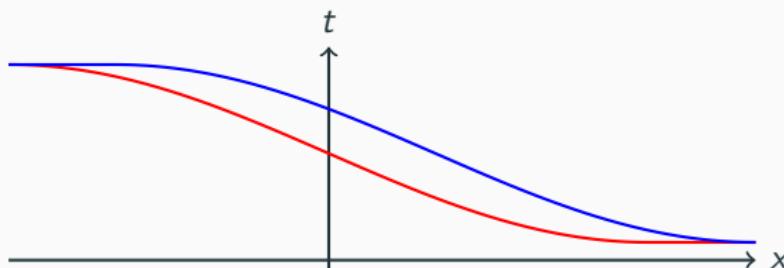
We want to prove the decay of:

$$v(t, x) := u(t, x + \delta(t)) - \bar{u}(x)$$

for a good choice of δ .

Difficulties for the shock tracking technique in the fully discrete setting:

- Restriction of translations on the grid
- "Linearization and translation do not commute in the discrete setting"



Zero-mass assumption

If h^0 is a zero mass perturbation, then if we consider the solution of the numerical scheme $(u^n)_{n \in \mathbb{N}}$ for $u^0 := \bar{u} + h^0$, then we want to prove the decay of $u^n - \bar{u}$.

The polynomially weighted norms

For $r \in [0, +\infty]$ and $\gamma \in [0, +\infty[$, we define the polynomial-weighted spaces ℓ_γ^r :

$$\ell_\gamma^r := \{(h_j)_{j \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad ((1 + |j|)^\gamma h_j)_{j \in \mathbb{Z}} \in \ell^r(\mathbb{Z})\}$$

and

$$\forall h \in \ell_\gamma^r, \quad \|h\|_{\ell_\gamma^r} = \left\| ((1 + |j|)^\gamma h_j)_{j \in \mathbb{Z}} \right\|_{\ell^r}.$$

A nonlinear stability result

Theorem [C., upcoming]

Let us assume that the same assumptions are verified (and thus especially the [spectral stability assumption](#)). We consider a constant $\mathbf{p} \in [0, +\infty[$. There exist two constants $\varepsilon, C \in [0, +\infty[$ such that, if we consider a initial perturbation $h^0 \in \mathbb{R}^{\mathbb{Z}}$ such that:

$$\|h^0\|_{\ell_{\max(\mathbf{1}, \mathbf{p})+\mathbf{p}}^{\mathbf{1}}} < \varepsilon \quad (\text{polynomial weight condition})$$

$$\sum_{j \in \mathbb{Z}} h_j^0 = 0 \quad (\text{zero mass perturbation})$$

then for all $n \in \mathbb{N} \setminus \{0\}$:

$$\|u^n - \bar{u}\|_{\ell_{\max(\mathbf{1}, \mathbf{p})}^{\mathbf{1}}} \leq \frac{C}{n^{\mathbf{p}}} \|h^0\|_{\ell_{\max(\mathbf{1}, \mathbf{p})+\mathbf{p}}^{\mathbf{1}}},$$

$$\|u^n - \bar{u}\|_{\ell_{\max(\mathbf{1}, \mathbf{p})}^{\infty}} \leq \frac{C}{n^{\mathbf{p} + \frac{1}{2\mu}}} \|h^0\|_{\ell_{\max(\mathbf{1}, \mathbf{p})+\mathbf{p}}^{\mathbf{1}}}.$$

Conclusion/ Perspective / Open questions

About the Green's function theorem:

- Bounds on Green's function uniform in j_0
- Proved for a large family of schemes
- The result **is proved** for systems

About the nonlinear stability theorem:

- The result **is not yet proved** for systems
- The "Shock tracking issue" creates limitations

Other Perspectives:

- Proving a more general nonlinear orbital stability result
- Existence of spectrally stable SDSPs?
- What can we say for moving shocks (with rational speed) and/or for under/over-compressive shocks?
- What can we say for dispersive schemes? (Lax-Wendroff for instance)
- Study of the stability for multi-dimensional conservation laws

Thank you for your attention!

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