

Stability of discrete shock profiles for systems of conservation laws

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Conservation laws and shocks

We consider a one-dimensional scalar conservation law

$$\begin{aligned}\partial_t u + \partial_x f(u) &= 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} &\rightarrow \mathbb{R},\end{aligned}$$

where the flux $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function.

Some of the results that will be presented also hold for systems of conservation laws.

Observation: This type of PDE tends to have solutions with discontinuities.

Overarching goal: When considering a conservative finite difference scheme, understand if it will be able to handle/capture discontinuities of solutions.

We consider two distinct states u^- , $u^+ \in \mathbb{R}^2$ and a velocity $s \in \mathbb{R}$ such that:

$$f(u^-) - f(u^+) = s(u^- - u^+), \quad (\text{Rankine-Hugoniot condition})$$

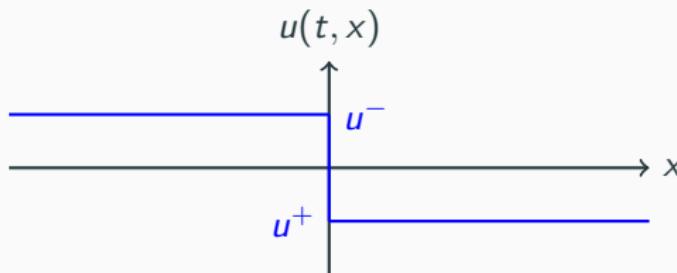
$$f'(u^+) < s < f'(u^-). \quad (\text{Lax shock inequalities})$$

The function u defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t, x) := \begin{cases} u^- & \text{if } x < st, \\ u^+ & \text{else,} \end{cases}$$

which is a weak solution of the scalar conservation law, is known as a [Lax shock](#).

We focus on [steady Lax shock](#), i.e. $s = 0$.



Conservative finite difference schemes

We introduce a cartesian grid with a space step $\Delta x > 0$ and a time step $\Delta t > 0$.

Goal: Compute sequences $u^n := (u_j^n)_{j \in \mathbb{Z}}$ such that u_j^n is close to the solution u on $[n\Delta t, (n+1)\Delta t] \times [(j - \frac{1}{2})\Delta x, (j + \frac{1}{2})\Delta x]$.

We consider a **conservative explicit finite difference scheme**:

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z},$$

$$u_j^{n+1} = u_j^n - \nu (F(\nu; u_{j-p+1}^n, \dots, u_{j+q}^n) - F(\nu; u_{j-p}^n, \dots, u_{j+q-1}^n))$$

- $u^0 \in \mathbb{R}^{\mathbb{Z}}$: initial condition
- $F :]0, +\infty[\times \mathbb{R}^{p+q} \rightarrow \mathbb{R}$: numerical flux
- $p, q \in \mathbb{N} \setminus \{0\}$: integers defining the size of the stencil of the scheme.
- $\nu = \frac{\Delta t}{\Delta x} > 0$: ratio between the time and space steps.

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We consider a **conservative explicit finite difference scheme**:

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n)$$

where the **nonlinear** discrete evolution operator $\mathcal{N} : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}^{\mathbb{Z}}$ is defined as:

$$\mathcal{N}(u)_j := u_j - \nu(F(\nu; u_{j-p+1}, \dots, u_{j+q}) - F(\nu; u_{j-p}, \dots, u_{j+q-1})).$$

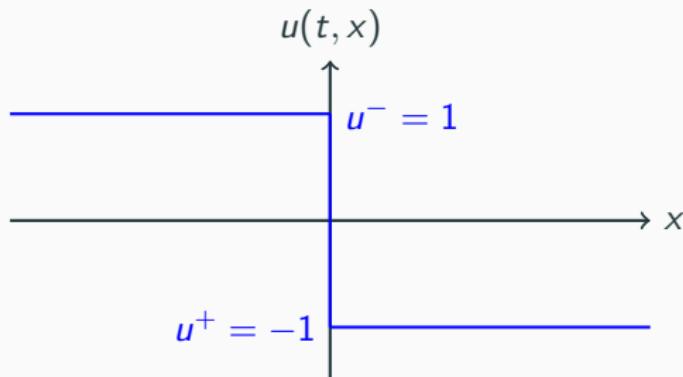
"Usual" assumptions on the numerical scheme:

- $\forall u \in \mathbb{R}, \quad F(\nu; u, \dots, u) = f(u)$ (consistency condition)
- For some neighborhood \mathcal{U} of the states u^\pm
$$\forall u \in \mathcal{U}, \quad -q \leq \nu f'(u) \leq p \quad (\text{CFL condition on } \nu)$$
- Linear- ℓ^2 stability for constant states $u \in \mathcal{U}$

Diffusion assumption on the numerical scheme:

- The scheme introduces numerical viscosity. This excludes dispersive schemes like for instance the Lax-Wendroff scheme.
In the present presentation, we consider a first order scheme.

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$.



For the numerical scheme, we consider the modified Lax Friedrichs scheme

$$\forall u \in \mathbb{R}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \quad \mathcal{N}(u)_j := \frac{u_{j+1} + u_j + u_{j-1}}{3} - \nu \frac{f(u_{j+1}) - f(u_{j-1})}{2}.$$

Discrete shock profiles

- Discrete shock profile (DSP): A discrete shock profile is a solution of the numerical scheme of the form

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^n = \bar{u}(j - s\nu n)$$

where the function $\bar{u} : \mathbb{Z} + s\nu\mathbb{Z} \rightarrow \mathbb{R}$ verifies that

$$\bar{u}(x) \underset{x \rightarrow \pm\infty}{\rightarrow} u^\pm.$$

- The particular case of steady Lax shocks, i.e. $s = 0$:

Stationary discrete shock profiles (SDSP) are sequences

$\bar{u} = (\bar{u}_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfy

$$\mathcal{N}(\bar{u}) = \bar{u} \quad \text{and} \quad \bar{u}_j \underset{j \rightarrow \pm\infty}{\rightarrow} u^\pm.$$

Example : We consider the following initial condition (mean of the standing shock on each cell):

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$

Main goal: Finding conditions on the numerical schemes so that:

stable shock waves for
the conservation law \Rightarrow stable DSPs for the
numerical scheme

This separates the theory surrounding DSPs in two parts:

- Existence of DSPs
- Stability of DSPs

Existence results on SDSPs

Existence of a continuous one-parameter family $(\bar{u}^\delta)_{\delta \in]-\Delta, \Delta[}$ of SDSPs around [Jennings '74, Majda-Ralston '79, Michelson '84, ...]

From now on, we denote a reference discrete shock profile:

$$\bar{u} := \bar{u}^0.$$

Existence results on SDSPs

Assumption: The *mass* function M is injective, where:

$$\forall \delta \in]-\Delta, \Delta[, \quad M(\delta) := \sum_{j \in \mathbb{Z}} \overline{u}_j^\delta - \overline{u}_j.$$

Nonlinear orbital stability of discrete shock profiles

The end goal would be the **nonlinear orbital stability** of those DSPs:

For **small admissible perturbations** \mathbf{h} , prove that the solution u^n of the numerical scheme for the initial condition $u^0 = \bar{u} + \mathbf{h}$ **converges** towards the set of translations of the DSP $\{\bar{u}^\delta, \delta \in]-\Delta, \Delta[\}$.

We have **conservation of mass**:

$$\forall n \in \mathbb{N}, \quad \sum_{j \in \mathbb{Z}} u_j^n - \bar{u}_j = \sum_{j \in \mathbb{Z}} u_j^0 - \bar{u}_j = \sum_{j \in \mathbb{Z}} \mathbf{h}_j.$$

Thus, if $u^n \xrightarrow{n \rightarrow +\infty} \bar{u}^\delta$ in ℓ^1 , then:

$$M(\delta) = \sum_{j \in \mathbb{Z}} \mathbf{h}_j.$$

The case of zero-mass perturbations \mathbf{h} .

[Jennings, '74]

- scalar case
- conservative monotone scheme
- nonlinear orbital stability for ℓ^1 perturbations

[Liu-Xin, '93]

- system case
- Lax-Friedrichs scheme
- weak Lax shocks
- zero mass (dropped in [Ying, 97']) and polynomial weight on the initial perturbation

[Smyrlis, '90]

- scalar case
- stationnary Lax shocks
- Lax-Wendroff scheme
- exponential weight on the initial perturbation

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Goal: We want to prove a result of nonlinear orbital stability that works:

- for **systems** of conservation laws,
- for a **fairly large class** of numerical schemes,
- with **the fewest restrictions** possible on the initial perturbations **h** ,
- replacing the **smallness assumption** on the amplitude of the shock with a **spectral stability assumption** on the discrete shock profile.

The **spectral stability assumption** corresponds to additional information on the point spectrum of the linearization about the SDSP \bar{u} .

Technique: Adaptation of the ideas "à la Zumbrun" which study the stability of traveling waves for parabolic PDEs.

Spectral stability \Rightarrow Linear stability \Rightarrow Nonlinear stability

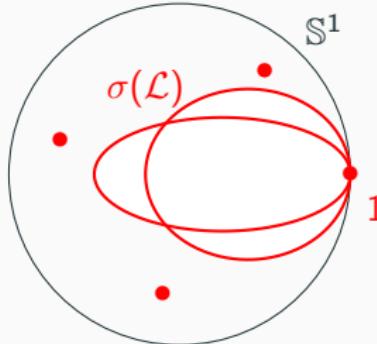
Linearization of the numerical scheme about the SDSP

We define the bounded operator $\mathcal{L} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ obtained by linearizing \mathcal{N} about \bar{u} :

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}h)_j := \sum_{k=-p}^q a_{j,k} h_{j+k},$$

with $a_{j,k} \rightarrow a_k^\pm$ as $j \rightarrow \pm\infty$.

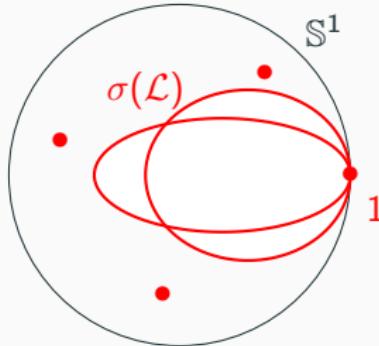
The coefficient $a_{j,k}$ are expressed using the partial derivatives $\partial_k F(\nu; \bar{u}_{j-p}, \dots, \bar{u}_{j+q-1})$.



Observation on the spectrum of \mathcal{L}

- There are curves of essential spectrum corresponding to the spectra of the linearized operators about the constant states u^+ and u^- .
- Outside of those essential spectrum curves, the spectrum is only composed of eigenvalues.
- 1 is an eigenvalue of the operator \mathcal{L} :

$$\text{"} (\forall \delta \in]-\Delta, \Delta[, \quad \mathcal{N}(\bar{u}^\delta) = \bar{u}^\delta) \Rightarrow \left. \frac{\partial \bar{u}^\delta}{\partial \delta} \right|_{\delta=0} \in \ker(Id - \mathcal{L}). \text{"}$$



Spectral stability assumption

- We construct a so-called Evans function. We assume that 1 is a simple zero of the Evans function. As a consequence, 1 is a simple eigenvalue of the operator \mathcal{L} :

$$\text{" } \ker(Id - \mathcal{L}) = \text{Span} \left(\frac{\partial \bar{u}^\delta}{\partial \delta} \Big|_{\delta=0} \right). \text{"}$$

- The operator \mathcal{L} has no other eigenvalue of modulus equal or larger than 1.

We define the **Green's function** associated to \mathcal{L} :

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j \in \mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Green's function associated to the operator \mathcal{L} for $j_0 = 30$

Theorem [C. 23']

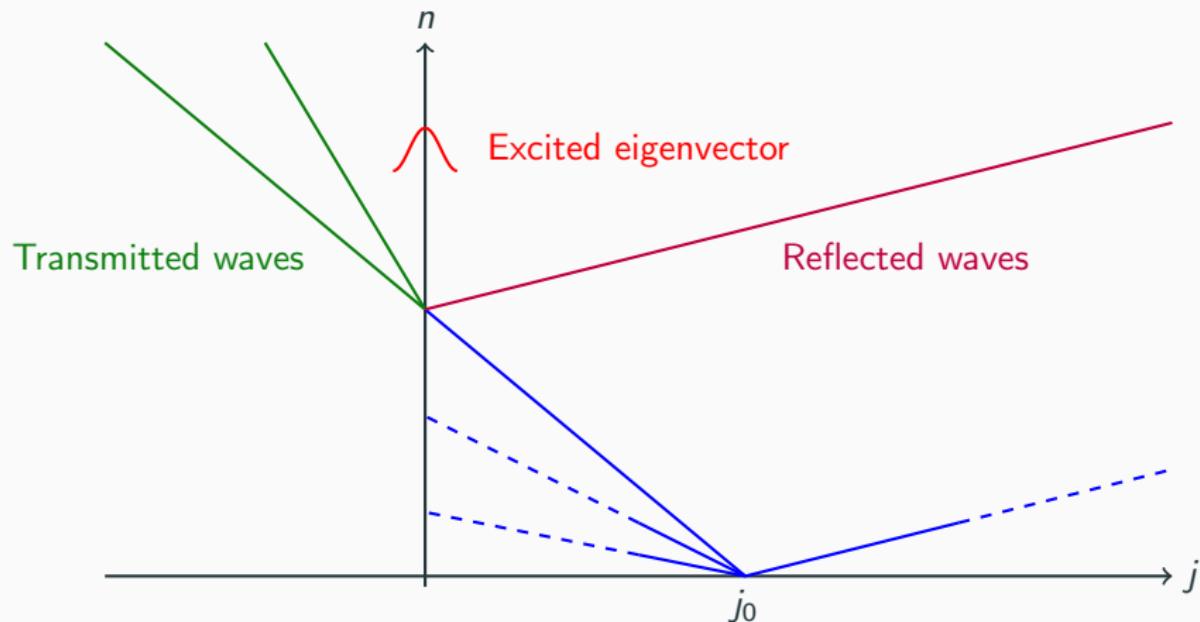
Under some more precise assumptions, there exist a positive constant c , an element V of $\ker(Id - \mathcal{L})$ and an (explicit) function $E : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $n \in \mathbb{N} \setminus \{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$\begin{aligned}\mathcal{G}(n, j_0, j) &= E \left(\frac{nf'(u^+) \nu + j_0}{\sqrt{n}} \right) V(j) \quad (\text{Excited eigenvector}) \\ &+ \mathbf{1}_{j \in \mathbb{N}} O \left(\frac{1}{\sqrt{n}} \exp \left(-c \frac{|nf'(u^+) \nu - (j - j_0)|^2}{n} \right) \right) \quad (\text{Gaussian wave}) \\ &+ \mathbf{1}_{j \in -\mathbb{N}} O \left(\frac{1}{\sqrt{n}} \exp \left(-c \frac{|nf'(u^+) \nu + j_0|^2}{n} \right) e^{-c|j|} \right) \quad (\text{Exponential residual}) \\ &+ O(e^{-cn - c|j - j_0|})\end{aligned}$$

where $E(x) \xrightarrow[x \rightarrow -\infty]{} 1$ and $E(x) \xrightarrow[x \rightarrow +\infty]{} 0$.

There is a similar result for $j_0 \in -\mathbb{N}$.

Result on the Green's function in the case of systems of conservation laws



For $h \in \ell^\infty(\mathbb{Z})$ and $n \in \mathbb{N}$, we have:

$$\forall j \in \mathbb{Z}, \quad (\mathcal{L}^n h)_j = \sum_{j_0 \in \mathbb{Z}} \mathcal{G}(n, j_0, j) h_{j_0}.$$

Also, for $j_0 \in \mathbb{N}$ (a similar description exists for $j_0 \in -\mathbb{N}$):

$$\begin{aligned} & \mathcal{G}(n, j_0, j) \\ &= E \left(\frac{nf'(u^+) \nu + j_0}{\sqrt{n}} \right) V(j) \quad (\text{Excited eigenvector}) \\ &+ \mathbb{1}_{j \in \mathbb{N}} O \left(\frac{1}{\sqrt{n}} \exp \left(-c \left(\frac{|nf'(u^+) \nu - (j - j_0)|^2}{n} \right) \right) \right) \quad (\text{Gaussian wave}) \\ &+ \mathbb{1}_{j \in -\mathbb{N}} O \left(\frac{1}{\sqrt{n}} \exp \left(-c \left(\frac{|nf'(u^+) \nu + j_0|^2}{n} \right) \right) e^{-c|j|} \right) \quad (\text{Exponential residual}) \\ &+ O(e^{-cn - c|j - j_0|}) \end{aligned}$$

The description of the Green's function allows to prove decay estimates on the semigroup $(\mathcal{L}^n)_{n \in \mathbb{N}}$.

Linear stability implies nonlinear orbital stability

We consider an initial perturbation $\mathbf{h} \in \ell^1$ such that there exists $\delta \in]-\Delta, \Delta[$ satisfying:

$$M(\delta) = \sum_{j \in \mathbb{Z}} \mathbf{h}_j.$$

For the solution of the numerical scheme $(u^n)_{n \in \mathbb{N}}$ with $u^0 := \bar{u} + \mathbf{h}$, then the sequence u^n should converge towards \bar{u}^δ .

We thus want to estimate $h^n := u^n - \bar{u}^\delta$ in some suitable norm.

The polynomially weighted norms

For $r \in [1, +\infty]$ and $\gamma \in [0, +\infty[$, we define the polynomial-weighted spaces ℓ_γ^r :

$$\ell_\gamma^r := \{(h_j)_{j \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad ((1 + |j|)^\gamma h_j)_{j \in \mathbb{Z}} \in \ell^r(\mathbb{Z})\}$$

with the norm:

$$\forall h \in \ell_\gamma^r, \quad \|h\|_{\ell_\gamma^r} = \left\| ((1 + |j|)^\gamma h_j)_{j \in \mathbb{Z}} \right\|_{\ell^r}.$$

Theorem [C. '24]

Let us assume that the same assumptions are verified (and thus especially the **spectral stability assumption**). We consider a constant $\mathbf{p} > 0$. There exist two constants $\varepsilon, C \in [0, +\infty[$ such that, if we consider a initial perturbation $\mathbf{h} \in \ell_{\mathbf{p}+\max(\frac{1}{2}, \mathbf{p})}^1$ such that:

$$\|\mathbf{h}\|_{\ell_{\mathbf{p}+\max(\frac{1}{2}, \mathbf{p})}^1} < \varepsilon \quad (\text{polynomial weight condition}),$$

then there exists $\delta \in]-\Delta, \Delta[$ such that:

$$M(\delta) = \sum_{j \in \mathbb{Z}} \mathbf{h}_j$$

Furthermore, for the solution $(u^n)_n$ of the numerical scheme initialized with $u^0 = \bar{u}$, we have that for all $n \in \mathbb{N} \setminus \{0\}$, the sequence $h^n = u^n - \bar{u}^\delta$ satisfies:

$$\|h^n\|_{\ell_{\mathbf{p}}^1} \leq \frac{C}{n^{\mathbf{p}}} \|\mathbf{h}\|_{\ell_{\mathbf{p}+\max(\frac{1}{2}, \mathbf{p})}^1} \quad \text{and} \quad \|h^n\|_{\ell_{\mathbf{p}}^\infty} \leq \frac{C}{n^{\mathbf{p}+\frac{1}{2}}} \|\mathbf{h}\|_{\ell_{\mathbf{p}+\max(\frac{1}{2}, \mathbf{p})}^1}.$$

Conclusion/ Perspective / Open questions

About the Green's function theorem:

- Bounds on Green's function uniform in j_0
- Proved for a large family of schemes
- The result **is proved** for systems

About the nonlinear stability theorem:

- The result **is not yet proved** for systems

Other Perspectives:

- What can we say for moving shocks (with rational speed) and/or for under/over-compressive shocks?
- What can we say for dispersive schemes? (Lax-Wendroff for instance)
- Existence of spectrally stable SDSPs?
- Study of the stability for multi-dimensional conservation laws

Thank you for your attention!

Brief idea of the proof

Using the inverse Laplace transform, we have:

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi} \int_{\Gamma} z^n \mathcal{G}(z, j_0, j) dz$$

where the spatial Green's function $\mathcal{G}(z, j_0, j)$ is defined by:

$$\forall z \notin \sigma(\mathcal{L}), \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(z, j_0, j) := ((zId - \mathcal{L})^{-1} \delta_{j_0})_j$$

and the integration path Γ surrounds the spectrum $\sigma(\mathcal{L})$.

Goal:

- Meromorphic extension of the spatial Green's function $\mathcal{G}(\cdot, j_0, j)$ near 1
- Find a suitable choice of path Γ

