Stability of discrete shock profiles for systems of conservation laws

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Conservation laws and shocks

We consider a one-dimensional scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R},$$
$$u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R},$$

where the flux $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

Some of the results that will be presented also hold for systems of conservation laws.

Conservation laws and shocks

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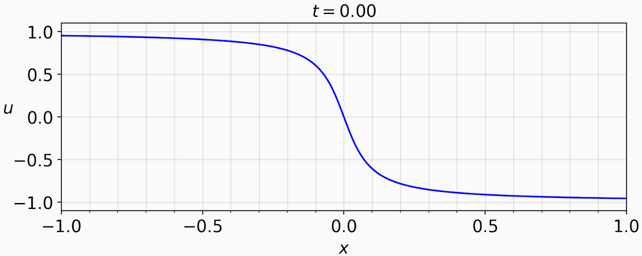
where the flux $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

Observation: This type of PDE tends to have solutions with discontinuities.

Example for Burger's equation: $\quad f(u) := \frac{u^2}{2}.$



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Overarching goal:

When considering a conservative finite difference scheme, we want understand if it will be able to handle/capture discontinuities of solutions.

We consider two distinct states $u^-, u^+ \in \mathbb{R}^2$ and a velocity $s \in \mathbb{R}$ such that:

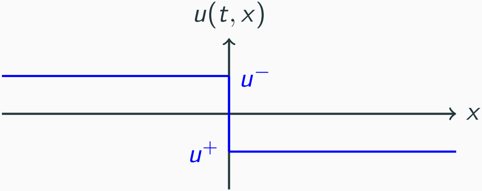
$$f(u^-) - f(u^+) = s(u^- - u^+), \quad \text{(Rankine-Hugoniot condition)}$$
$$f'(u^+) < s < f'(u^-). \quad \text{(Lax shock inequalities)}$$

The function u defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t,x) := \begin{cases} u^- & \text{if } x < st, \\ u^+ & \text{else}, \end{cases}$$

which is a weak solution of the scalar conservation law, is known as a Lax shock.

We focus on steady Lax shock, i.e. $s = 0$.



Conservative finite difference schemes

We introduce a cartesian grid with a space step $\Delta x > 0$ and a time step $\Delta t > 0$.

<u>**Goal:**</u> Compute sequences $u^n := (u_j^n)_{j\in\mathbb{Z}}$ such that u_j^n is close to the solution u on $[n\Delta t, (n+1)\Delta t[\times \left[\left(j-\frac{1}{2}\right)\Delta x, \left(j+\frac{1}{2}\right)\Delta x\right[$.

We consider a conservative explicit finite difference scheme:

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z},$$
$$u_j^{n+1} = u_j^n - \nu\left(F\left(\nu; u_{j-p+1}^n, \dots, u_{j+q}^n\right) - F\left(\nu; u_{j-p}^n, \dots, u_{j+q-1}^n\right)\right)$$

- $u^0 \in \mathbb{R}^{\mathbb{Z}}$: initial condition
- $F :]0, +\infty[\times\mathbb{R}^{p+q} \to \mathbb{R}$: numerical flux
- $p, q \in \mathbb{N}\backslash\{0\}$: integers defining the size of the stencil of the scheme.
- $\nu = \frac{\Delta t}{\Delta x} > 0$: ratio between the time and space steps.

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We consider a conservative explicit finite difference scheme:

$$\boxed{\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n)}$$

where the nonlinear discrete evolution operator $\mathcal{N} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ is defined as:

$$\mathcal{N}(u)_j := u_j - \nu\left(F\left(\nu; u_{j-p+1}, \dots, u_{j+q}\right) - F\left(\nu; u_{j-p}, \dots, u_{j+q-1}\right)\right).$$

"Usual" assumptions on the numerical scheme:

- $\forall u \in \mathbb{R}, \quad F(\nu; u, \dots, u) = f(u)$ (consistency condition)
- For some neighborhood $\mathcal{U}$ of the states $u^{\pm}$

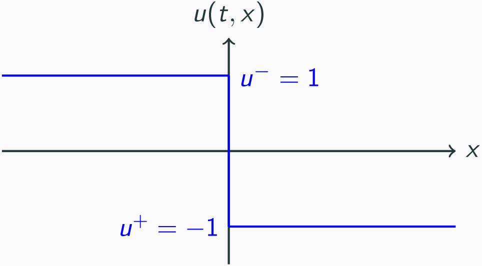
$$\forall u \in \mathcal{U}, \quad -q \leq \nu f'(u) \leq p \quad \text{(CFL condition on } \nu\text{)}$$

- Linear-ℓ^2 stability for constant states $u \in \mathcal{U}$

Diffusion assumption on the numerical scheme:

- The scheme introduces numerical viscosity. This excludes dispersive schemes like for instance the Lax-Wendroff scheme.
  In the present presentation, we consider a first order scheme.

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$.



For the numerical scheme, we consider the modified Lax Friedrichs scheme

$$\forall u \in \mathbb{R}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \quad \mathcal{N}(u)_j := \frac{u_{j+1} + u_j + u_{j-1}}{3} - \nu \frac{f(u_{j+1}) - f(u_{j-1})}{2}.$$

Discrete shock profiles

- Discrete shock profile (DSP): A discrete shock profile is a solution of the numerical scheme of the form

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^n = \overline{u}(j - s\nu n)$$

where the function $\overline{u} : \mathbb{Z} + s\nu\mathbb{Z} \to \mathbb{R}$ verifies that

$$\overline{u}(x) \underset{x \to \pm\infty}{\to} u^{\pm}.$$

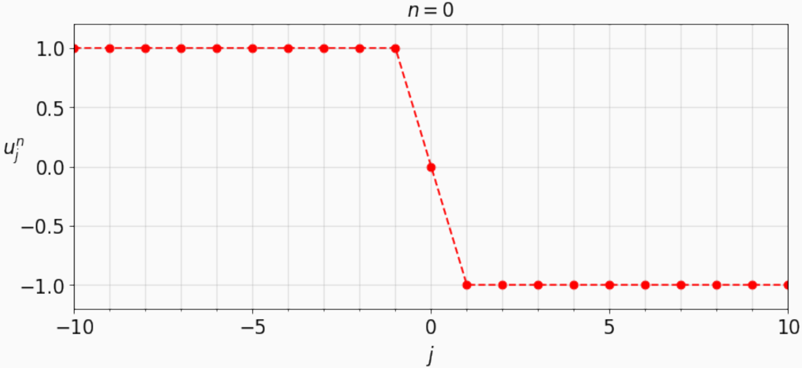
- The particular case of steady Lax shocks, i.e. $s = 0$:

Stationary discrete shock profiles (SDSP) are sequences $\overline{u} = (\overline{u}_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfy

$$\mathcal{N}(\overline{u}) = \overline{u} \quad \text{and} \quad \overline{u}_j \underset{j \to \pm\infty}{\to} u^{\pm}.$$

Example : We consider the following initial condition (mean of the standing shock on each cell):

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$



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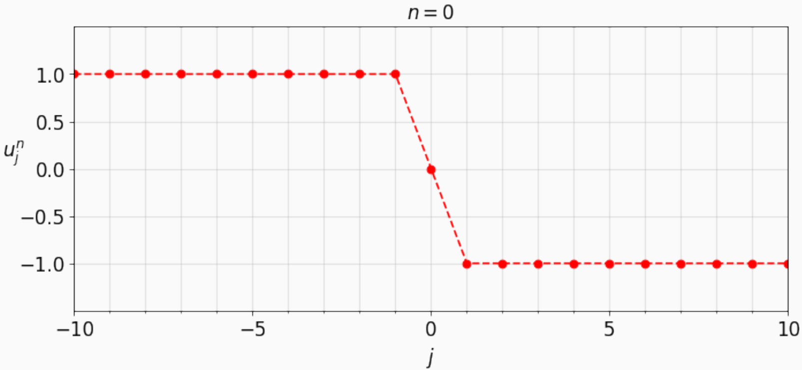
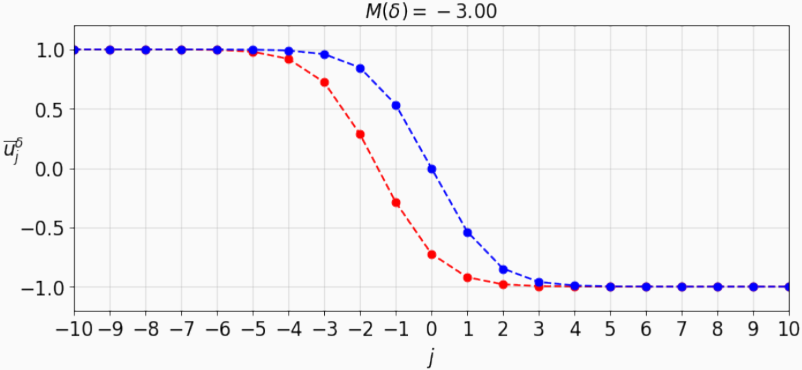


Main goal: Finding conditions on the numerical schemes so that:

stable shock waves for the conservation law	$\Longrightarrow$	stable DSPs for the numerical scheme

This separates the theory surrounding DSPs in two parts:

- Existence of DSPs
- Stability of DSPs

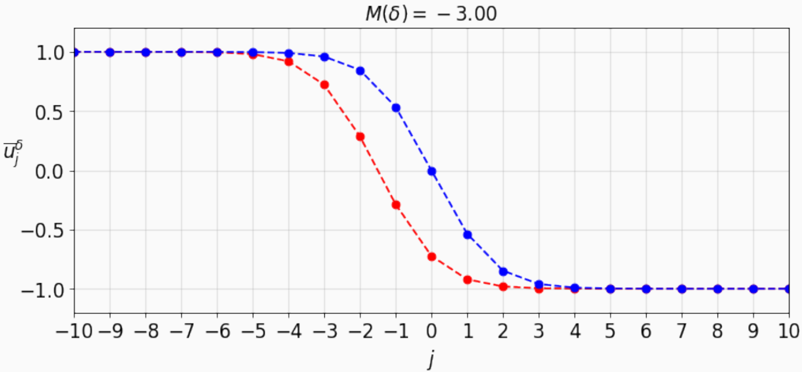


Existence of a continuous one-parameter family $(\overline{u}^\delta)_{\delta \in]-\Delta,\Delta[}$ of SDSPs around [Jennings '74, Majda-Ralston '79, Michelson '84, ...]

From now on, we denote a reference discrete shock profile:

$$\overline{u} := \overline{u}^0.$$

Existence results on SDSPs



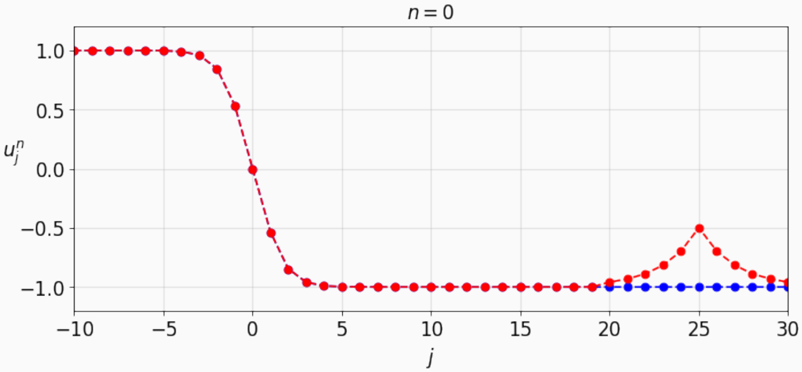
Assumption: The *mass* function M is injective, where:

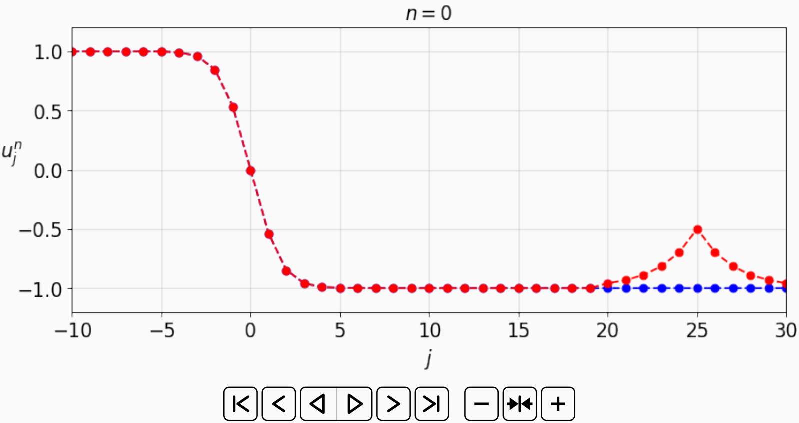
$$\forall \delta \in]-\Delta, \Delta[, \quad M(\delta) := \sum_{j \in \mathbb{Z}} \overline{u}_j^\delta - \overline{u}_j.$$

Nonlinear orbital stability of discrete shock profiles

The end goal would be the nonlinear orbital stability of those DSPs:

For small admissible perturbations $\mathbf{h}$, prove that the solution u^n of the numerical scheme for the initial condition $u^0 = \overline{u} + \mathbf{h}$ converges towards the set of translations of the DSP $\{\overline{u}^\delta, \delta \in]-\Delta, \Delta[\}$.



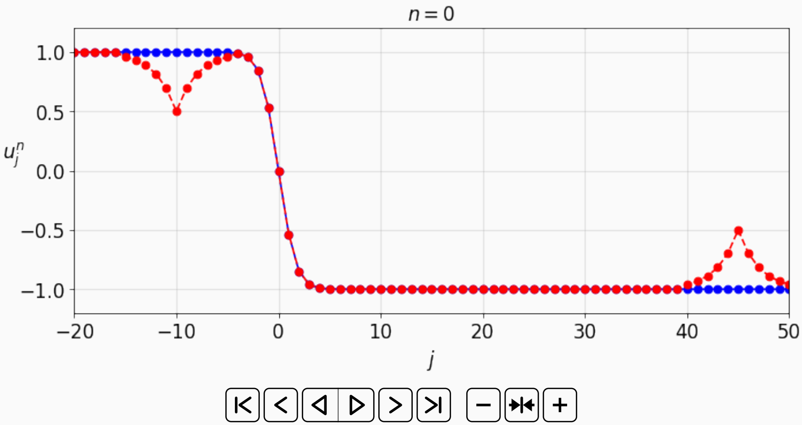


We have conservation of mass:

$$\forall n \in \mathbb{N}, \quad \sum_{j\in\mathbb{Z}} u_j^n - \overline{u}_j = \sum_{j\in\mathbb{Z}} u_j^0 - \overline{u}_j = \sum_{j\in\mathbb{Z}} \mathbf{h}_j.$$

Thus, if $u^n \underset{n\to+\infty}{\to} \overline{u}^\delta$ in ℓ^1, then:

$$\boxed{M(\delta) = \sum_{j\in\mathbb{Z}} \mathbf{h}_j.}$$



The case of zero-mass pertubations **h**.

[Jennings, '74]

- scalar case
- conservative monotone scheme
- nonlinear orbital stability for ℓ^1 perturbations

[Smyrlis, '90]

- scalar case
- stationnary Lax shocks
- Lax-Wendroff scheme
- exponential weight on the initial perturbation

[Liu-Xin, '93]

- system case
- Lax-Friedrichs scheme
- weak Lax shocks
- zero mass assumption (dropped in [Ying, 97']) and polynomial weight on the initial perturbation

[Michelson, '02]

- system case
- weak stationary Lax shocks
- First and third order schemes

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Goal: We want to prove a result of nonlinear orbital stability that works:

- for systems of conservation laws,
- for a fairly large class of numerical schemes,
- with the fewest restrictions possible on the initial perturbations $\mathbf{h}$,
- replacing the smallness assumption on the amplitude of the shock with a spectral stability assumption on the discrete shock profile.

The spectral stability assumption corresponds to additional information on the point spectrum of the linearization about the SDSP $\overline{u}$.

Technique: Adaptation of the ideas "*à la Zumbrun*" which study the stability of traveling waves for parabolic PDEs.

Spectral stability $\Longrightarrow$ Linear stability $\Longrightarrow$ Nonlinear stability

Spectral stability implies linear stability

The content of this chapter is contained in [C. '25] and is an extension of the result of [Godillon '03] and [Serre '07].

Quick overview of the content:

- Based on [Zumbrun and Howard, '98] and [Mascia and Zumbrun,'02,'03].
- Translate the spectral information on the linearization about the wave into estimates on the semi-group associated with it
- Main difficulty: No spectral gap $\oplus$ Presence of an eigenvalue of modulus 1 in the essential spectrum
- Solution: Pointwise description of the Green's function of the linearized operator using spatial dynamics

Linearization of the numerical scheme about the SDSP

- The linear operators $\mathcal{L}^{\pm} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}^{\pm} h)_j := \sum_{k=-p}^{q} a_k^{\pm} h_{j+k}$$

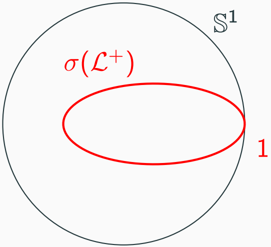
correspond to the linearization of the numerical scheme $\mathcal{N}$ about the constant states $u^{\pm}$.

- We define the linear operator $\mathcal{L} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ obtained by linearizing $\mathcal{N}$ about $\overline{u}$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L} h)_j := \sum_{k=-p}^{q} a_{j,k} h_{j+k},$$

with $a_{j,k} \to a_k^{\pm}$ as $j \to \pm\infty$.

Observation on the spectrum of $\mathcal{L}^{\pm}$



- The spectrum of $\mathcal{L}^{\pm}$ is given by:

$$\sigma(\mathcal{L}^{\pm}) := \left\{ \sum_{k=-p}^{q} a_k^{\pm} e^{i\xi k}, \xi \in \mathbb{R} \right\}.$$

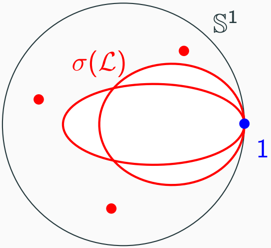
- The consistency condition implies that $1 \in \sigma(\mathcal{L}^{\pm})$.
- The linear ℓ^2-stability implies:

$$\forall \kappa \in \mathbb{S}^1 \backslash \{1\}, \quad \left| \sum_{k=-p}^{q} a_k^{\pm} \kappa^k \right| < 1. \quad (\textit{Dissipativity condition})$$

- The diffusivity condition states there exist two complex constants $\beta^{\pm}$ with positive real parts such that:

$$\sum_{k=-p}^{q} a_k^{\pm} e^{i\xi k} \underset{\xi \to 0}{=} \exp\left(-i\xi\nu f'(u^{\pm}) - \beta^{\pm}\xi^2 + o(\xi^2)\right) \quad (\textit{Diffusion})$$

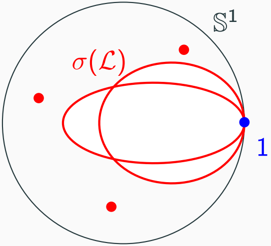
for a first order scheme.



Observation on the spectrum of $\mathcal{L}$

- There are curves of essential spectrum corresponding to the spectras of the operators $\mathcal{L}^+$ and $\mathcal{L}^-$.
- Outside of those essential spectrum curves, the spectrum is only composed of eigenvalues.
- 1 is an eigenvalue of the operator $\mathcal{L}$:

$$" \left(\forall \delta \in]-\Delta, \Delta[, \quad \mathcal{N}(\overline{u}^\delta) = \overline{u}^\delta\right) \Rightarrow \left.\frac{\partial \overline{u}^\delta}{\partial \delta}\right|_{\delta=0} \in \ker(Id - \mathcal{L})."$$



Spectral stability assumption

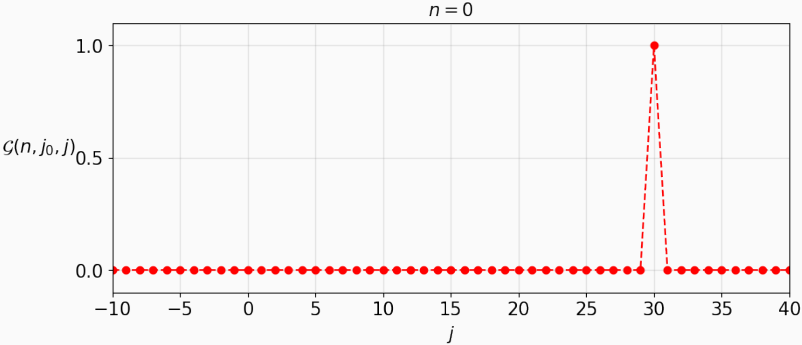
- The operator $\mathcal{L}$ has no other eigenvalue of modulus equal or larger than 1.
- We construct a so-called Evans function. We assume that 1 is a simple zero of the Evans function. As a consequence, 1 is a simple eigenvalue of the operator $\mathcal{L}$:

$$\text{" } \ker(Id - \mathcal{L}) = \mathrm{Span}\left(\left.\frac{\partial \overline{u}^{\delta}}{\partial \delta}\right|_{\delta=0} \right) .\text{"}$$

We define the Green's function associated to $\mathcal{L}$:

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j\in\mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Green's function associated to the operator $\mathcal{L}$ for $j_0 = 30$



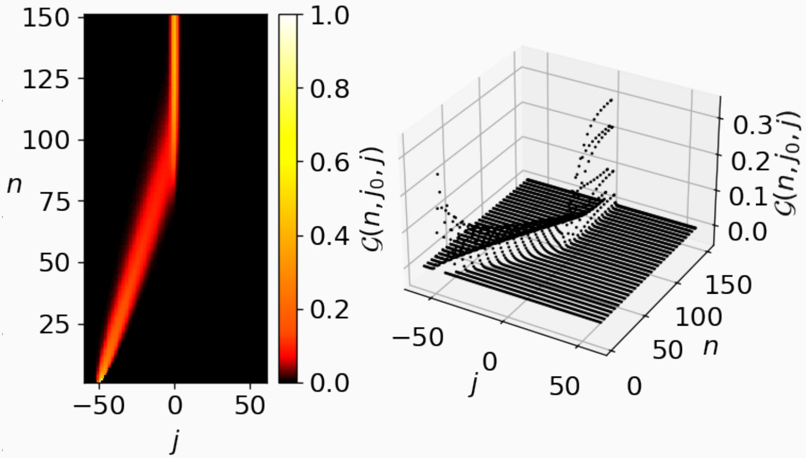
Theorem [C. '25]

Under some more precise assumptions, there exist a positive constant c, an element V of $\ker(Id - \mathcal{L})$ and an (explicit) function $E : \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N}\backslash\{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$
\begin{aligned}
&\mathcal{G}(n, j_0, j) \\
=&E\left(\frac{nf'(u^+)\nu + j_0}{\sqrt{n}}\right) V(j) \quad \text{(Excited eigenvector)} \\
+&\mathbb{1}_{j\in\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\frac{|nf'(u^+)\nu - (j - j_0)|^2}{n}\right)\right) \quad \text{(Gaussian wave)} \\
+&\mathbb{1}_{j\in-\mathbb{N}} O\left(\frac{e^{-c|j|}}{\sqrt{n}} \exp\left(-c\frac{|nf'(u^+)\nu + j_0|^2}{n}\right)\right) \quad \text{(Exponential residual)} \\
+&O(e^{-cn-c|j-j_0|})
\end{aligned}
$$

where $E(x) \underset{x\to-\infty}{\to} 1$ and $E(x) \underset{x\to+\infty}{\to} 0$.
There is a similar result for $j_0 \in -\mathbb{N}$.

$j_0 = -50$



150
125
100
75
50
25
n
−50
0
50
j
1.0
0.8
0.6
0.4
0.2
0.0
𝒢(n, j₀, j)
0.3
0.2
0.1
0.0
𝒢(n, j₀, j)
150
100
50
0
n
−50
0
50
j

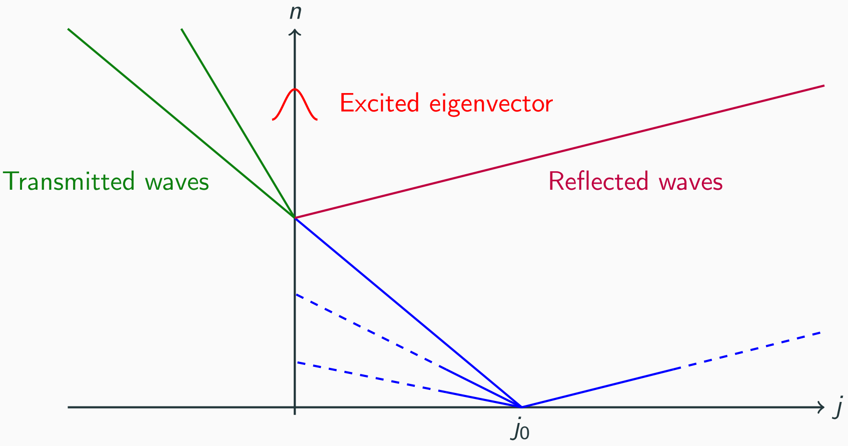
Some precisions on the result:

- We actually extract the leading behavior of the Gaussian wave of the Green's function.
- There is also such a description of the discrete derivative of the Green's function:

$$\mathcal{G}(n, j_0, j) - \mathcal{G}(n, j_0 - 1, j).$$

- The result is proved for a large family of higher order diffusive schemes.
- The result is proved in the case of systems of conservation laws.

Result on the Green's function in the case of systems of conservation laws



Brief idea of the proof

Using the inverse Laplace tranform, we have:

$$\forall n \in \mathbb{N}\setminus\{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi}\int_{\Gamma} z^n G(z, j_0, j)dz$$

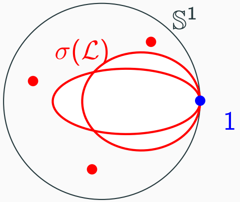
where the spatial Green's function $G(z, j_0, j)$ is defined by:

$$\forall z \notin \sigma(\mathcal{L}), \forall j_0, j \in \mathbb{Z}, \quad G(z, j_0, j) := \left((zId - \mathcal{L})^{-1}\delta_{j_0}\right)_j$$

and the integration path Γ surrounds the spectrum $\sigma(\mathcal{L})$.

<u>Goal:</u>

- Study the spatial Green's function $G(z, j_0, j)$
- Find a suitable choice of path Γ



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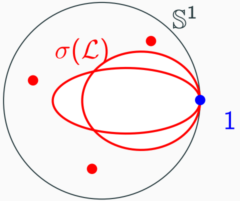
Properties of $G(z, j_0, j)$:

• **Local exponential bounds:** For all z_0 outside of $\sigma(\mathcal{L})$:

$$|G(z, j_0, j)| \leq Ce^{-c|j-j_0|}$$

with z in a neighborhood U of z_0 and two constants C, c.

• **Meromorphical extension** of $G(\cdot, j_0, j)$ near 1 through the essential spectrum with a pole of order 1 at 1



We rewrite the eigenvalue problem:

$$(zId - \mathcal{L})u = 0$$

as a discrete dynamical system:

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z)W_j. \tag{Dyn. Syst.}$$

We are interested in solutions of the dynamical system that tend towards 0 as j tends to $+\infty$ or $-\infty$:

$$E_0^{\pm}(z) := \left\{ W_0, \quad W_j \underset{j\to\pm\infty}{\to} 0 \right\}.$$

We essentially characterize the elements of the vector spaces $E_0^{\pm}(z)$ in two ways:

•**Geometric dichotomy** (projectors) • **Jost solutions** (basis)

Several uses:

- Analysis of the essential spectrum of $\mathcal{L}$
- Characterization of the point spectrum $\mathcal{L}$ and construction of the Evans function.
- Expression of the spatial Green's function $G(z, j_0, j)$.

For $h \in \ell^\infty(\mathbb{Z})$ and $n \in \mathbb{N}$, we have:

$$\forall j \in \mathbb{Z}, \quad (\mathcal{L}^n h)_j = \sum_{j_0 \in \mathbb{Z}} \mathcal{G}(n, j_0, j) h_{j_0}.$$

Also, for $j_0 \in \mathbb{N}$ (a similar description exists for $j_0 \in -\mathbb{N}$):

$$\begin{aligned}
&\mathcal{G}(n, j_0, j) \\
=& E\left(\frac{nf'(u^+)\nu + j_0}{\sqrt{n}}\right) V(j) \quad \text{(Excited eigenvector)} \\
&+ \mathbb{1}_{j \in \mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{\left|nf'(u^+)\nu - (j - j_0)\right|^2}{n}\right)\right)\right) \quad \text{(Gaussian wave)} \\
&+ \mathbb{1}_{j \in -\mathbb{N}} O\left(\frac{e^{-c|j|}}{\sqrt{n}} \exp\left(-c\left(\frac{\left|nf'(u^+)\nu + j_0\right|^2}{n}\right)\right)\right) \quad \text{(Exponential residual)} \\
&+ O(e^{-cn - c|j - j_0|})
\end{aligned}$$

The description of the Green's function allows to prove decay estimates on the semigroup $(\mathcal{L}^n)_{n \in \mathbb{N}}$.

Linear stability implies nonlinear orbital stability

We consider an initial perturbation $\mathbf{h} \in \ell^1$ such that there exists $\delta \in]-\Delta, \Delta[$ satisfying:

$$M(\delta) = \sum_{j\in\mathbb{Z}} \mathbf{h}_j.$$

For the solution of the numerical scheme $(u^n)_{n\in\mathbb{N}}$ with $u^0 := \overline{u} + \mathbf{h}$, then the sequence u^n should converge towards $\overline{u}^\delta$.

We thus want to estimate $h^n := u^n - \overline{u}^\delta$ in some suitable norm.

The polynomially weighted norms

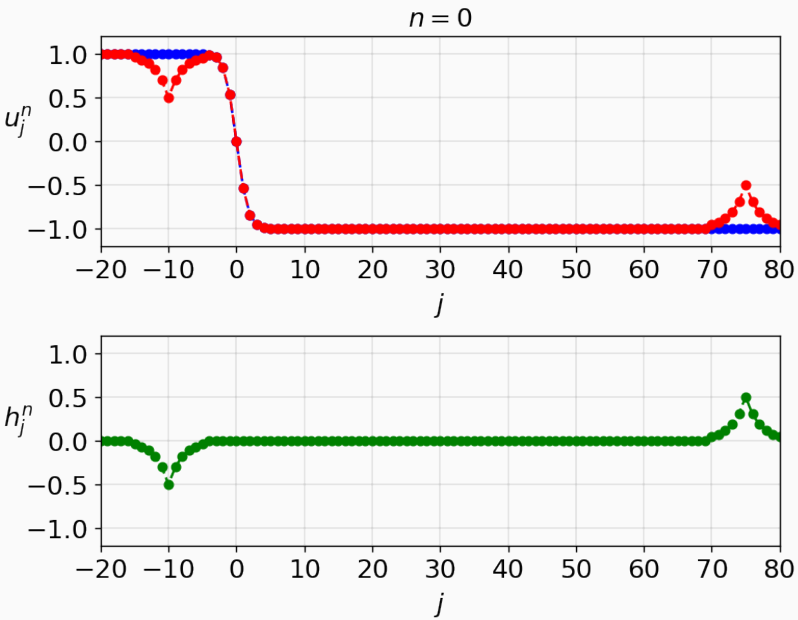
For $r \in [1, +\infty]$ and $\gamma \in [0, +\infty[$, we define the polynomial-weighted spaces ℓ^r_γ:

$$\ell^r_\gamma := \{(h_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad \left((1+|j|)^\gamma h_j\right)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z})\}$$

with the norm:

$$\forall h \in \ell^r_\gamma, \quad \|h\|_{\ell^r_\gamma} = \left\|\left((1+|j|)^\gamma h_j\right)_{j\in\mathbb{Z}}\right\|_{\ell^r}.$$

Observation: We cannot prove decay estimates for the semi-group $(\mathcal{L}^n)_{n\in\mathbb{N}}$ when it acts from $\ell^1(\mathbb{Z})$ into $\ell^r(\mathbb{Z})$ for $r \in [1, +\infty]$.



$n = 0$
1.0
0.5
0.0
−0.5
−1.0
u_j^n
−20
−10
0
10
20
30
40
50
60
70
80
j
h_j^n

Theorem [C. '24]

Let us assume that the same assumptions are verified (and thus especially the spectral stability assumption). We consider a constant $\mathbf{p} > 0$. There exist two constants $\varepsilon, C \in [0, +\infty[$ such that, if we consider a initial perturbation $\mathbf{h} \in \ell^1_{\mathbf{p}+\max\left(\frac{1}{2},\mathbf{p}\right)}$ such that:

$$\|\mathbf{h}\|_{\ell^1_{\mathbf{p}+\max\left(\frac{1}{2},\mathbf{p}\right)}} < \varepsilon \quad \text{(polynomial weight condition)},$$

then there exists $\delta \in]-\Delta, \Delta[$ such that:

$$M(\delta) = \sum_{j\in\mathbb{Z}} \mathbf{h}_j$$

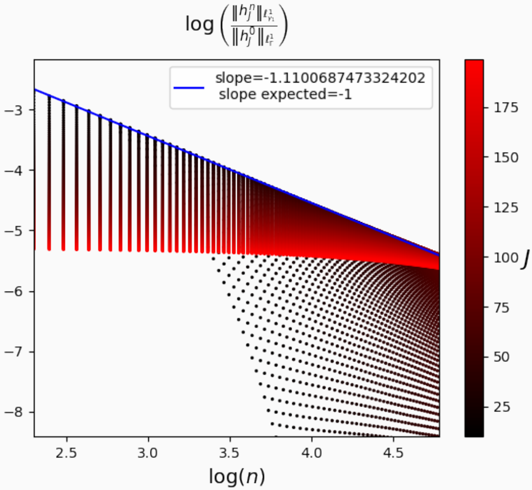
Furthermore, for the solution $(u^n)_n$ of the numerical scheme initialized with $u^0 = \overline{u}$, we have that for all $n \in \mathbb{N}\backslash\{0\}$, the sequence $h^n = u^n - \overline{u}^\delta$ satisfies:

$$\|h^n\|_{\ell^1_{\mathbf{p}}} \leq \frac{C}{n^{\mathbf{p}}} \|\mathbf{h}\|_{\ell^1_{\mathbf{p}+\max\left(\frac{1}{2},\mathbf{p}\right)}} \qquad \text{and} \qquad \|h^n\|_{\ell^\infty_{\mathbf{p}}} \leq \frac{C}{n^{\mathbf{p}+\frac{1}{2}}} \|\mathbf{h}\|_{\ell^1_{\mathbf{p}+\max\left(\frac{1}{2},\mathbf{p}\right)}}.$$

We display the value of:

$$\ln\left(\frac{\|h^n\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})}}}{\|h^0\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}}}\right) \quad \text{and} \quad \ln\left(\frac{\|h^n\|_{\ell^{\infty}_{\max(\mathbf{1},\mathbf{p})}}}{\|h^0\|_{\ell^{\mathbf{1}}_{\max(\mathbf{1},\mathbf{p})+\mathbf{p}}}}\right)$$

for $\mathbf{p} = 1$ and $h^0_J := \delta_J - \delta_0$.



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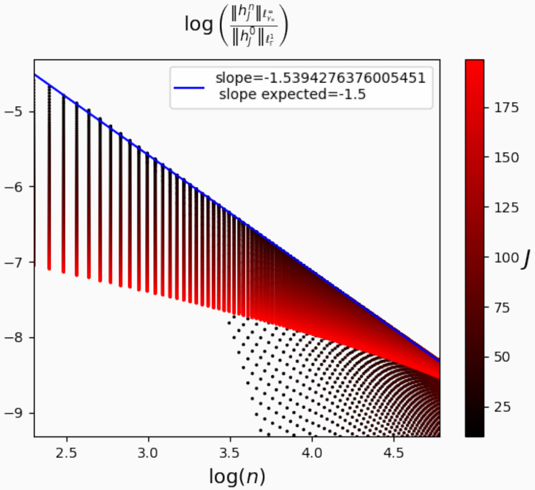

Main calculations: For $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, we have that:

$$u_j^{n+1} = \mathcal{N}(u^n)_j \quad \Rightarrow \quad h_j^{n+1} = (\mathcal{L}^\delta h^n)_j - Q^\delta(h^n)_{j+1} + Q^\delta(h^n)_j$$

where $Q^\delta(h^n)_j$ is a quadratic remainder.

Thus, we have:

$$\forall n \in \mathbb{N}, \quad h^{n+1} = \mathcal{L}^\delta h^n + (Id - \mathcal{T})Q^\delta(h^n)$$

where the shift operator $\mathcal{T}$ is defined by:

$$\mathcal{T} : (h_j)_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} \mapsto (h_{j+1})_{j\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}.$$

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Thus, we have:

$$\forall n \in \mathbb{N}, \quad h^{n+1} = \mathcal{L}h^n + \left(\mathcal{L}^\delta - \mathcal{L}\right) h^n + (Id - \mathcal{T})Q^\delta(h^n)$$

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Finally, using Duhamel's formula:

$$\boxed{\begin{gathered}\forall n \in \mathbb{N}, \\ h^n := \mathcal{L}^n h^0 + \sum_{m=0}^{n-1} \mathcal{L}^{n-1-m} \left(\mathcal{L}^\delta - \mathcal{L}\right) h^m + \sum_{m=0}^{n-1} \mathcal{L}^{n-1-m}(Id - \mathcal{T})Q^\delta(h^m).\end{gathered}}$$

Conclusion/ Perspective / Open questions

Spectral stability implies linear stability:

- Proved for a large family of schemes
- The result is proved for systems

Linear stability implies nonlinear stability:

- The result is not **yet** proved for systems

Other Perspectives:

- What can we say for moving shocks (with rational speed) and/or for under/over-compressive shocks?
- What can we say for dispersive schemes?
  → *Stability of SDSPs for the Lax-Wendroff scheme* [Coulombel and Faye '24]
- Spectral stability of the SDSPs?
- Study of the stability for multi-dimensional conservation laws

Thank you for your attention!