Atomic Norm Minimization for Decomposition into Complex Exponentials

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***Abstract*—This paper is devoted to the study of existence, uniqueness, and computational aspects, of decompositions of discrete signals (vectors) into atoms, which are sampled complex exponentials. In addition, new distances between complex vectors are introduced, within the framework of optimal transport; they are based on the difference between the frequencies of their atoms and they have convex semidefinite formulations.**

***Index Terms*—atomic norm, infinite dictionary, truncated moment problem, trigonometric moments, total variation norm, super-resolution, optimal transport**

I. Introduction

Mathematical analysis aims at understanding complex objects by expressing them in terms of elementary objects, the *atoms*. The discrete Fourier transform is a typical example, which allows us to express a vector of size N as a unique linear combination of N orthogonal atoms, which are the sampled complex exponentials of frequencies the multiples of $1/N$. This framework is restrictive, however, and one may want to remove the constraint that the frequencies lie on this discrete grid. The modern paradigm of signal processing integrates the idea that a signal can be represented in a *dictionary* containing much more than N atoms, by choosing, among the multitude of possible representations offered by the redundancy, the *simplest* one [1]. Here, we study the representation of vectors as linear combinations of sampled complex exponentials, without any constraint on the frequencies. Thus, the dictionary contains the continuous infinity of sampled complex exponentials, with frequencies in $[0,1)$.

We define the unit circle $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ of the complex plane $\mathbb{C}$. For every $z \in \mathbb{C}$, $\arg(z) \in [0, 2\pi)$ is the argument of z, such that $z = |z| \mathrm{e}^{\mathrm{j}\arg(z)}$, where $\mathrm{j} = \sqrt{-1}$. We endow $\mathbb{T}$ with the distance $d : (z_1, z_2) \mapsto \min\big(|\arg(z_1) - \arg(z_2)|/(2\pi), 1 - |\arg(z_1) - \arg(z_2)|/(2\pi)\big)$.

Let $M \geq 1$ be an integer. We define the *real* Hilbert space

$$\mathbb{V} = \big\{v = (v_m)_{m=-M}^{M} \in \mathbb{C}^{2M+1} \ : \ (\forall m) \ v_{-m} = v_m^*\big\}, \quad (1)$$

where $\cdot^*$ indicates complex conjugation, endowed with the usual inner product

$$\langle v, v' \rangle = \sum_{m=-M}^{M} v_m {v'}_m^* = v_0 v_0' + 2\,\mathrm{Re}\Big(\sum_{m=1}^{M} v_m {v'}_m^*\Big) \in \mathbb{R}. \quad (2)$$

We consider the infinite family $\mathcal{A} = \big(\mathrm{a}(z)\big)_{z\in\mathbb{T}}$ of elements of $\mathbb{V}$, defined by

$$\mathrm{a}(z)_m = z^{-m} = \mathrm{e}^{-\mathrm{j}m\arg(z)}, \quad m = -M, \ldots, M. \quad (3)$$

We call $\mathcal{A}$ the *dictionary* and its elements $\mathrm{a}(z)$, which are sampled complex exponentials, the *atoms* [2]–[7]. The frequency of $\mathrm{a}(z)$ is defined as $\arg(z)/(2\pi) \in [0,1)$.

In the following, v is an arbitrary nonzero element of $\mathbb{V}$.

We focus on the decomposition of v into atoms. For this, we need a parameterization of $\mathcal{A}$; that is, a way to express v as a linear combination of atoms. Since $\mathcal{A}$ contains a non-denumerable infinity of atoms, it is natural to work with finite signed Borel measures on $\mathbb{T}$, the set of which is denoted by $\mathcal{M}$. So, the analysis of v consists in determining a measure $\mu \in \mathcal{M}$, such that

$$v_m = \int_{\mathbb{T}} z^{-m} \mathrm{d}\mu(z) = \int_0^1 \mathrm{e}^{-\mathrm{j}2\pi f m} \mathrm{d}\mu(\mathrm{e}^{\mathrm{j}2\pi f}), \quad m = -M, \ldots, M. \quad (4)$$

Thus, the v_m are trigonometric moments, or Fourier coefficients, of μ. Therefore, we write

$$v = \mathcal{F}\mu \quad (5)$$

if the relation in (4) is satisfied, and we say that μ *explains* v.

So, one can view the problem as the identification of a measure from a truncated sequence of its trigonometric moments, the so-called *truncated moment problem* [8]. If one finds μ from $v = (v_m)_{m=-M}^{M}$, one can calculate all its moments $(v_m)_{m\in\mathbb{Z}}$ and so extrapolate v, hence the term *super-resolution* [9]: one recovers high-frequency information of μ, given only low-frequency information. The two "mathematical" and "signal processing" views of the problem consist in reasoning in terms of measures with moments v, or in terms of decompositions of v into atoms (viewing μ as the spectrum of v), respectively; this is just a matter of defining which are the time and frequency domains. We note that the more general setting of the recovery of a measure from linear measurements, by total variation minimization, has been considered by De

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Castro and Gamboa [10] and others; here we focus on the specificities of the measurements being Fourier coefficients.

In this study, we are interested in the representation of v; that is, in the questions of existence and uniqueness of atomic decompositions of v. We do not consider approximation, which consists in estimating a simple approximate atomic decomposition of v, e.g. when v is corrupted by noise; there is a vast literature on approximation, see [11]–[16] and references therein. We review the main properties of atomic decompositions, most of which are folklore results stated as remarks. In addition, our contributions are the following:

1) We establish a precise characterization of the existence and uniqueness of a measure explaining v with minimal total variation norm.
2) We explain how to determine this minimal measure numerically, by a two-step procedure: first solving a convex semidefinite program, for which we propose an iterative algorithm; and then applying Prony's estimation method.
3) We exhibit in Sect. VIII a new construction of an atomic decomposition of v with $2M$ atoms, uniformly located on $\mathbb{T}$. This construction plays an important role in Sect. IX, where the minimal atomic decomposition of a mixture of two atoms with opposite amplitudes is characterized.
4) Finally, three distances are defined on $\mathbb{V}$ in Sect. X, by transferring distances on $\mathbb{T}$, through the formalism of optimal transport.

II. Atomic decompositions and atomic norm

We will see that we can reason with discrete measures only; that is, express v with a finite number of atoms. So, by anticipation, we call an *atomic decomposition* of v, the parameters $K \in \mathbb{N}\backslash\{0\}$, $z_k \in \mathbb{T}$ all distinct, $c_k \in \mathbb{R}\backslash\{0\}$, such that

$$v = \sum_{k=1}^{K} c_k \mathrm{a}(z_k). \tag{6}$$

If v has such an atomic decomposition, the measure $\mu = \sum_{k=1}^{K} c_k \delta_{z_k}$ explains v, where δ_z is the Dirac measure located in $z \in \mathbb{T}$.

Remark 1. The dimension of $\mathbb{V}$ is $2M+1$. The rank of $\mathcal{A}$, which is the largest number of linearly independent atoms, is $2M+1$ as well. Indeed, the invertibility of the Vandermonde matrices

$$\begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{j}\omega_1} & \dots & \mathrm{e}^{-\mathrm{j}(2M+1)\omega_1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \mathrm{e}^{-\mathrm{j}\omega_{2M+1}} & \dots & \mathrm{e}^{-\mathrm{j}(2M+1)\omega_{2M+1}} \end{bmatrix}, \tag{7}$$

for every set $\{\omega_1, \dots, \omega_{2M+1}\}$ of distinct elements of $[0, 2\pi)$, implies the invertibility of the matrices

$$\begin{bmatrix} \mathrm{e}^{\mathrm{j}M\omega_1} & \dots & \mathrm{e}^{\mathrm{j}\omega_1} & 1 & \mathrm{e}^{-\mathrm{j}\omega_1} & \dots & \mathrm{e}^{-\mathrm{j}M\omega_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathrm{e}^{\mathrm{j}M\omega_{2M+1}} & \dots & \mathrm{e}^{\mathrm{j}\omega_{2M+1}} & 1 & \mathrm{e}^{-\mathrm{j}\omega_{2M+1}} & \dots & \mathrm{e}^{-\mathrm{j}M\omega_{2M+1}} \end{bmatrix}, \tag{8}$$

which yields the given value of the rank.

So, every family $\big(\mathrm{a}(z_1), \dots, \mathrm{a}(z_{2M+1})\big)$ of $2M+1$ distinct atoms is a basis of $\mathbb{V}$, i.e. is linearly independent and spans $\mathbb{V}$ by linear combinations with real weights.

Remark 2. The invertibility of Vandermonde matrices also implies that the *spark* of $\mathcal{A}$, which is the smallest number of linearly dependent atoms [4], is $2M+2$; $\mathcal{A}$ has *full spark* [17].

A direct consequence is the following [4, Theorem 3]: if v has two different atomic decompositions, they involve at least $\mathrm{spark}(\mathcal{A}) = 2M+2$ different atoms.

Therefore, if v has an atomic decomposition with $1 \leq K \leq M$ atoms, there is no other atomic decomposition of v with less or as many atoms. Moreover, we will see in Section IV that it is easy to find this decomposition. It is in the case where v does not have a decomposition with $1 \leq K \leq M$ atoms, a property which is easy to test, that it is very difficult to find the decomposition of v with the smallest number of atoms (this number being at most $2M$, see Remark 13).

Remark 3. There is an infinity of atomic decompositions of v with at most $2M+1$ atoms. Indeed, according to Remark 1, given $2M+1$ arbitrary distinct atoms, one can express v as a linear combination of them, with real weights c_k, obtained by solving a linear system. Moreover, if v has a decomposition with more than $2M+1$ atoms, one can re-express all the atoms in terms of a subset of size $2M+1$ of them, to obtain a decomposition of v with at most $2M+1$ atoms. Other said, there is an infinity of discrete measures, made of at most $2M+1$ Dirac measures, explaining v.

Remark 4. We can exhibit the "classical" uniform atomic decomposition[1]

$$v = \sum_{k=1}^{2M+1} c_k \mathrm{a}(\mathrm{e}^{\mathrm{j}2\pi k/(2M+1)}), \tag{9}$$

where the sequence $(c_k)_{k=1}^{2M+1}$ is simply the inverse discrete Fourier transform of v:

$$c_k = \frac{1}{2M+1} \sum_{m=-M}^{M} v_m \mathrm{e}^{\mathrm{j}2\pi km/(2M+1)}, \quad k = 1, \dots, 2M+1. \tag{10}$$

However, in most applications, v is a mixture of unknown atoms, which must be identified, with frequencies living in the continuum $[0, 1)$. Enforcing the frequencies to lie on the grid of the multiples of $1/(2M+1)$, or actually on any predetermined finite set of atoms, yields undesirable effects: the reconstruction or approximation of the estimated signal is not parsimonious, with spurious oscillations; this is called spectral leakage, basis mismatch, or gridding error [18]–[20]. This phenomenon is somewhat attenuated by taking a finer and finer grid, while in doing so, with more and more coherent atoms, the estimation problems become increasingly badly conditioned and numerically unstable. This justifies to place ourselves in a continuous framework, as we do in this study, instead of in a discrete framework with a finite, even if redundant, dictionary of predetermined atoms [4], [21], [22].

[1] Strictly speaking, we should remove from this sum the indices k, for which $c_k = 0$.

Thus, we are interested in finding, among the infinite multitude of possible atomic decompositions of v, the "simplest" ones, with respect to some notion of parsimony. Finding the atomic decomposition with the smallest number of atoms is a very difficult nonconvex problem. That is why we focus on its convex relaxation and consider the form of optimality defined in the following.

We define the *atomic norm* of v as

$$\|v\|_{\mathrm{a}} := \inf\big\{\|\mu\|_{\mathrm{TV}} : \ \mu \in \mathcal{M},\, \mu = \mathcal{F}v\big\}, \qquad (11)$$

where $\|\cdot\|_{\mathrm{TV}}$ is the total variation norm of a measure. In that respect, let us recall the following property [23]: every measure $\mu \in \mathcal{M}$ has a unique Jordan decomposition (μ^+, μ^-) of two finite positive measures μ^+ and μ^-, such that $\mu = \mu^+ - \mu^-$, satisfying the following optimality property: for every pair of positive measures (ν^+, ν^-) with $\mu = \nu^+ - \nu^-$, then $\nu^+ \geq \mu^+$ and $\nu^- \geq \mu^-$. Hence, we can define $\|\mu\|_{\mathrm{TV}} = \mu^+(\mathbb{T}) + \mu^-(\mathbb{T})$.

As easily checked, the atomic norm is indeed a norm on $\mathbb{V}$.

We call a *minimal atomic decomposition* of v, an atomic decomposition of v, such that $\sum_{k=1}^K |c_k| = \|v\|_{\mathrm{a}}$.

Remark 5. The atomic norm is defined in several papers [5]–[7] as

$$\|v\|_{\mathrm{a}} = \inf\big\{t \geq 0 \ : \ v \in t\,\mathrm{conv}(\mathcal{A} \cup -\mathcal{A})\big\}, \qquad (12)$$

where conv denotes the convex hull, which, since $\mathbb{V}$ is of dimension $2M+1$, is the set of convex combinations of $2M+1$ signed atoms. Therefore, we can equivalently write (12) as

$$\|v\|_{\mathrm{a}} = \inf\Big\{ \sum_{k=1}^K |c_k| \ : \ v = \sum_{k=1}^K c_k \mathrm{a}(z_k), \text{ for some } K \leq 2M+1, \text{ distinct } z_k \in \mathbb{T},\ c_k \in \mathbb{R}\backslash\{0\}\Big\}. \qquad (13)$$

We will see in Proposition 2 that these two definitions (11) and (12)–(13) coincide, and that the infimum in (11), (12), (13) is attained; that is, it is a minimum. In other words, a minimal atomic decomposition of v exists.

III. Relation to Toeplitz matrices

In order to characterize more precisely the atomic norm and the atomic decompositions of v minimizing it, we introduce the linear operator T, which maps v to the Hermitian Toeplitz matrix

$$\mathrm{T}(v) := \begin{bmatrix} v_0 & v_1 & \cdots & v_M \\ v_{-1} & v_0 & \cdots & v_{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{-M} & v_{-M+1} & \cdots & v_0 \end{bmatrix}. \qquad (14)$$

We can note that T defines a bijection between $\mathbb{V}$ and the set of $(M+1)\times(M+1)$ Hermitian Toeplitz matrices, thereafter denoted by $\mathrm{T}(\mathbb{V})$. So, we will talk equally about an atomic decomposition of v or of $\mathrm{T}(v)$.

$\mathrm{T}(v)$ is said to be indefinite if it is neither positive semidefinite nor negative semidefinite; that is, if it has at least one positive and one negative eigenvalues. $\mathrm{T}(v)$ is said to be singular if it is not invertible; that is, if it has at least one eigenvalue equal to zero.

We remark that, for every $z \in \mathbb{T}$,

$$\begin{aligned} \mathrm{T}(\mathrm{a}(z)) &= \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-M} \\ z & 1 & \cdots & z^{-M+1} \\ \vdots & \vdots & \ddots & \vdots \\ z^M & z^{M-1} & \cdots & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ z \\ \vdots \\ z^M \end{bmatrix} \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-M} \end{bmatrix} \end{aligned} \qquad (15)$$

is positive semidefinite and of rank 1.

Remark 6. If v has an atomic decomposition $v = \sum_{k=1}^K c_k \mathrm{a}(z_k)$ with $1 \leq K \leq M+1$ atoms, the rank of $\mathrm{T}(v)$ is K. Indeed,

$$\begin{aligned} \mathrm{T}(\mathrm{a}(z)) = &\begin{bmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_K \\ \vdots & \vdots & \vdots \\ z_1^M & \cdots & z_K^M \end{bmatrix} \times \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_K \end{bmatrix} \\ &\times \begin{bmatrix} 1 & z_1^{-1} & \cdots & z_1^{-M} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & z_K^{-1} & \cdots & z_K^{-M} \end{bmatrix} \end{aligned} \qquad (16)$$

and the rank of $A^{\mathrm{H}}BA$, where A is a matrix with full column rank and $\cdot^{\mathrm{H}}$ denotes the Hermitian transpose, is equal to the rank of B [24, Lemma 1].

As we see, the rank of $\mathrm{T}(v)$ will play an important role in the analysis. At this stage, one can wonder if the converse of the last property is true; that is, if there always exists an atomic decomposition of v with $K = \mathrm{rank}(\mathrm{T}(v))$ atoms. Unfortunately, this is not the case, as we will see in Section IV.

Remark 7. One can think that if v has an atomic decomposition with $M+1 < K \leq 2M+1$ atoms, the rank of $\mathrm{T}(v)$ is $M+1$. This is not true in general and we can construct examples where the rank is as low as $2M+2-K$. Indeed, if $\sum_{k=1}^{2M+2} c_k \mathrm{a}(z_k) = 0$ (given distinct z_k, such $c_k \neq 0$ exist, since the $\mathrm{a}(z_k)$ are linearly dependent), then for every $M+1 < K \leq 2M+1$, $v = \sum_{k=1}^K c_k \mathrm{a}(z_k) = \sum_{k=K+1}^{2M+2} (-c_k)\mathrm{a}(z_k)$, and there are two different atomic decompositions of v, the second one having less atoms. The rank of $\mathrm{T}(v)$ is then $2M+2-K$.

Remark 8. The trace of $\mathrm{T}(v)$, which is also the sum of its eigenvalues, is equal to $(M+1)v_0$. So, it is equal to $(M+1)\mu(\mathbb{T})$, for every measure μ explaining v, and to $(M+1)\sum_{k=1}^K c_k$, for every atomic decomposition of v.

Remark 9. Let $\mathrm{n}^+(\mathrm{T}(v))$ and $\mathrm{n}^-(\mathrm{T}(v))$ be the number of positive and negative eigenvalues of $\mathrm{T}(v)$, respectively. Then every atomic decomposition $v = \sum_{k=1}^K c_k \mathrm{a}(z_k)$ has at least $\mathrm{n}^+(\mathrm{T}(v))$ positive c_k and at least $\mathrm{n}^-(\mathrm{T}(v))$ negative c_k.

As we see, decomposition into exponentials is intimately related to the spectral properties of Hermitian Toeplitz matrices.

IV. Prony's estimation method

The following estimation method, already known by G. R. de Prony in the 18th century [25], allows us to know, in a simple way, whether v has an atomic decomposition with $K \leq M$ atoms, and if so, to find the corresponding parameters. We outline the method in this section, see [26]–[28] for more details.

First, let us suppose that there exists a decomposition $v = \sum_{k=1}^{K} c_k \mathrm{a}(z_k)$ with $1 \leq K \leq M$. We define the *annihilating polynomial* $H(z) = \sum_{k=0}^{K} h_k z^k := \prod_{k=1}^{K}(z - z_k)$. This polynomial owes its name to the fact that it cancels the sequence v by convolution [29], [30]:

$$\sum_{k=0}^{K} h_k v_{m-k} = \sum_{k=0}^{K} h_k \sum_{l=1}^{K} c_l z_l^{k-m} = \sum_{l=1}^{K} c_l z_l^{-m} H(z_l) = 0, \quad m = -M+K, \ldots, M. \tag{17}$$

Let us form the rectangular Toeplitz matrix of size $(2M+1-K) \times (K+1)$,

$$\mathrm{R}_K(v) := \begin{bmatrix} v_{M-K} & \cdots & v_M \\ v_{M-K-1} & \cdots & v_{M-1} \\ \vdots & \ddots & \vdots \\ v_{-M} & \cdots & v_{-M+K} \end{bmatrix}. \tag{18}$$

Then, the equations (17) can be rewritten as

$$\mathrm{R}_K(v)[h_K \ \cdots \ h_0]^{\mathrm{T}} = [0 \ \cdots \ 0]^{\mathrm{T}}. \tag{19}$$

Thus, the vector $[h_K \ \cdots \ h_0]^{\mathrm{T}}$ is in the kernel of $\mathrm{R}_K(v)$. More precisely, $\mathrm{R}_K(v)$ is of rank K and its kernel, of dimension 1, is spanned by $[h_K \ \cdots \ h_0]^{\mathrm{T}}$.

Now, we consider the converse of these properties. Let K be the rank of $\mathrm{T}(v)$, with $K \leq M$ (if $\mathrm{T}(v)$ is positive definite, see Remark 11, whereas if $\mathrm{T}(v)$ is of full rank and is not positive definite, the method cannot be applied to find an atomic decomposition of v). Let us form the matrix $\mathrm{R}_K(v)$ as in (18). It is of rank K as well. Let $[h_K \ \cdots \ h_0]^{\mathrm{T}}$ be the unique, up to a constant, nonzero vector in the kernel of $\mathrm{R}_K(v)$. For a numerically robust estimation, we compute the singular value decomposition of $\mathrm{R}_K(v)$ and we take $[h_K \ \cdots \ h_0]^{\mathrm{T}}$ as the right singular vector corresponding to the smallest singular value. Then, we compute the roots z_k of the polynomial $H(z) := \sum_{k=0}^{K} h_k z^k$. If we find K distinct roots belonging to $\mathbb{T}$, then v has an atomic decomposition $v = \sum_{k=1}^{K} c_k \mathrm{a}(a_k)$, where the amplitudes c_k are obtained by solving the least-squares linear system

$$Q^{\mathrm{H}} Q [c_1 \ \cdots \ c_K]^{\mathrm{T}} = Q^{\mathrm{H}} [v_{-M} \ \cdots \ v_M]^{\mathrm{T}}, \tag{20}$$

$$\text{where } Q := \begin{bmatrix} z_1^{-M} & \cdots & z_K^{-M} \\ \vdots & \vdots & \vdots \\ z_1^{M} & \cdots & z_K^{M} \end{bmatrix}. \tag{21}$$

In the other cases, v does not have an atomic decomposition with at most M atoms.

Note that the annihilating polynomial can have less than K nonzero roots. For instance, for $v = (2, 1, 1, 1, 1, 1, 2)$,

$$\mathrm{T}(v) = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} \tag{22}$$

is of rank $K = 3$, so $\mathrm{R}_K(v) = \mathrm{T}(v)$ and the annihilating polynomial is $H(z) = z(z-1)$, which has only one nonzero root.

In the case where $H(z)$ has K distinct nonzero roots z_k, but not all in $\mathbb{T}$, we can write $v_m = \sum_{k=1}^{K} c_k z_k^{-m}$, but this amounts to decomposing v into other objects than the atoms of $\mathcal{A}$. This is linked to the fact that the Hermitian Toeplitz matrices of rank 1 are of the form $\mathrm{T}((z^{-M}, \ldots, z^{M}))$, but for an arbitrary $z \in \mathbb{C}\backslash\{0\}$, not necessarily in $\mathbb{T}$.

We will see in the following that if $\mathrm{T}(v)$ is positive semidefinite, Prony's method always yields K distinct roots in $\mathbb{T}$. If $\mathrm{T}(v)$ is indefinite, the key for decomposing v will consist in splitting $\mathrm{T}(v)$ into the sum of two positive semidefinite and negative semidefinite Toeplitz matrices.

V. Case of positive decompositions

For every measure μ explaining v, $\|\mu\|_{\mathrm{TV}} \geq \mu^+(\mathbb{T}) \geq \mu^+(\mathbb{T}) - \mu^-(\mathbb{T}) = v_0$, so we always have $\|v\|_{\mathrm{a}} \geq v_0$. Moreover, if a positive measure μ explains v, we have $\|\mu\|_{\mathrm{TV}} = \mu(\mathbb{T}) = v_0$, so that the infimum in (11) is attained for this measure, with $\|v\|_{\mathrm{a}} = \|\mu\|_{\mathrm{TV}} = v_0$. So, it is interesting to address, in the first place, the case where v is explained by a positive measure.

Remark 10. If a discrete measure $\sum_{k=1}^{K} c_k \delta_{z_k}$ explains v, with $1 \leq K \leq M+1$ and $c_k > 0$, then $\mathrm{T}(v)$ is positive semidefinite and of rank K. Indeed, $\forall z \in \mathbb{T}$, $\mathrm{T}(\mathrm{a}(z))$ is positive semidefinite and the set of positive semidefinite matrices is a closed convex cone. More precisely, for every vector $x \in \mathbb{C}^{M+1}$, $x^{\mathrm{H}}\mathrm{T}(v)x = \int_{\mathbb{T}} x^{\mathrm{H}}\mathrm{T}(\mathrm{a}(z))x \, \mathrm{d}\mu(z) = \int_{\mathbb{T}} |\sum_{m=1}^{M+1} z^{1-m} x_m|^2 \, \mathrm{d}\mu(z) \geq 0$. Moreover, we have seen in Remark 6 that $\mathrm{T}(v)$ is of rank exactly K.

A major result, which implies the converse of the property in Remark 10, dates back to the work of Carathéodory [31], [32]. It can be formulated as follows:

Carathéodory's Theorem. *$\mathrm{T}(v)$ is positive and of rank K, with $1 \leq K \leq M+1$, if and only if there exist distinct elements $z_k \in \mathbb{T}$ and positive reals c_k, such that $v = \sum_{k=1}^{K} c_k \mathrm{a}(z_k)$. Moreover, if $K \leq M$, the discrete measure $\sum_{k=1}^{K} c_k \delta_{z_k}$ is the unique positive measure explaining v.*

Thus, if $\mathrm{T}(v)$ is positive semidefinite, a positive discrete measure $\mu = \sum_{k=1}^{K} c_k \delta_{z_k}$, with $K \leq M+1$, explains v, and we have $\|\mu\|_{\mathrm{TV}} = \sum_{k=1}^{K} c_k = v_0 = \|v\|_{\mathrm{a}}$. In other words, the Theorem states the existence of a minimal atomic decomposition of v, if $\mathrm{T}(v)$ is positive semidefinite. Moreover, according to Remark 2, there exists no decomposition of v with less atoms, so the obtained atomic decomposition is doubly optimal.

Remark 11. If $\mathrm{T}(v)$ is positive definite, with rank $K = M+1$, there exists an infinity of minimal atomic decompositions of v with $M+1$ atoms. We can construct all of them as follows [33, Remark 2.1]: we choose $z_{M+1} \in \mathbb{T}$ and we set

$$c_{M+1} := \big([1\ z_{M+1}^{-1}\ \cdots\ z_{M+1}^{-M}]\mathrm{T}(v)^{-1}[1\ z_{M+1}\ \cdots\ z_{M+1}^{M}]^{\mathrm{T}}\big)^{-1}. \tag{23}$$

Then $c_{M+1} > 0$ and

$$\begin{aligned}&\mathrm{T}\big(v - c_{M+1}\mathrm{a}(z_{M+1})\big)\\ &= \mathrm{T}(v) - [1\ z_{M+1}\ \cdots\ z_{M+1}^{M}]^{\mathrm{T}}[1\ z_{M+1}^{-1}\ \cdots\ z_{M+1}^{-M}]\end{aligned} \tag{24}$$

is positive semidefinite of rank M. Carathéodory's Theorem applied to this residual yields $v - c_{M+1}\mathrm{a}(z_{M+1}) = \sum_{k=1}^{M} c_k \mathrm{a}(z_k)$. All in all, we end up with the positive minimal atomic decomposition $v = \sum_{k=1}^{M+1} c_k \mathrm{a}(z_k)$.

For instance, for $v = (0, \ldots, 0, 1, 0, \ldots, 0)$, i.e. $\mathrm{T}(v) = \mathrm{Id}$, the identity matrix, we get the atomic decompositions, for every $\varphi \in [0, 2\pi)$,

$$v = \frac{1}{M+1} \sum_{k=1}^{M+1} \mathrm{a}\big(\mathrm{e}^{\mathrm{j}(2\pi k+\varphi)/(M+1)}\big). \tag{25}$$

Remark 12. There is a complete proof of Carathéodory's Theorem by Curto and Fialkow [8, section 6]. Given Remarks 10 and 11, there only remains to prove that, if $\mathrm{T}(v)$ is positive semidefinite of rank $1 \leq K \leq M$, then a positive measure explaining v exists, is unique, and is concentrated at K points of $\mathbb{T}$. The difficult part, which is not detailed here, is to show the existence. The rest of the proof is simple: we consider the annihilating polynomial $H(z) = \sum_{k=0}^{K} h_k z^k = \prod_{k=1}^{K}(z - z_k)$ defined in Section IV and we define the vector of size $M+1$, $\tilde{h} = [h_K\ \cdots\ h_0\ 0\ \cdots\ 0]^{\mathrm{T}}$. Let $\mu \in \mathcal{M}$ be a positive measure explaining v. We have

$$\begin{aligned}0 = \tilde{h}^{\mathrm{H}}\mathrm{T}(v)\tilde{h} &= \sum_{0\leq k,l\leq K} h_k^* h_l v_{k-l}\\ &= \int_{\mathbb{T}} \sum_{0\leq k,l\leq K} h_k^* h_l z^{l-k} \mathrm{d}\mu(z)\\ &= \int_{\mathbb{T}} |H(z)|^2 \mathrm{d}\mu(z).\end{aligned} \tag{26}$$

Therefore, μ is a discrete measure concentrated at the roots z_k of $H(z)$ belonging to $\mathbb{T}$. Since μ cannot be made of less than K Diracs measures (else the rank of $\mathrm{T}(v)$ would be less than K, according to Remark 10), all the z_k, $k = 1, \ldots, K$, are in $\mathbb{T}$. Finally, the unique coefficients c_k such that $v = \sum_{k=1}^{K} c_k \mathrm{a}(z_k)$, defined in (20), are positive, according to Remark 9. □

Corollary 1. If a positive measure, which is not a discrete measure concentrated in at most M points, explains v, then $\mathrm{T}(v)$ is of rank $M+1$. This is specific to the positive case, see Remark 7 for the general case.

Corollary 2. An atom cannot be expressed as a linear combination of other atoms with positive coefficients. So, the atoms are the extremal points of the convex hull of $\mathcal{A}$.

VI. General case

Remark 13. In the previous section, we have seen that if $\mathrm{T}(v)$ is positive semidefinite, v has a decomposition with at most $M+1$ atoms. The analysis is of course the same when $\mathrm{T}(v)$ is negative semidefinite. So, let us direct our attention to the case where $\mathrm{T}(v)$ is indefinite. We have seen in Remark 3 that there are atomic decompositions of v with $2M+1$ atoms. We can now improve this bound and construct decompositions of v with at most $2M$ atoms. We will see in Proposition 3 one such decomposition. Another construction is the following. Let $i^- < 0$ be the smallest eigenvalue of $\mathrm{T}(v)$. $\mathrm{T}(v) - i^-\mathrm{Id}$ is positive semidefinite and singular, so by Carathéodory's Theorem, it can be decomposed into at most M atoms. As in Remark 11, we can decompose $i^-\mathrm{Id}$ into $M+1$ atoms, one of which being an atom of $\mathrm{T}(v) - i^-\mathrm{Id}$. By combining them, $\mathrm{T}(v) = (\mathrm{T}(v) - i^-\mathrm{Id}) + i^-\mathrm{Id}$ can indeed be decomposed into at most $2M$ atoms.

Remark 14. The bound above of $2M$ atoms is optimal: it is possible that v has a decomposition with $2M$ atoms and not less. We can give the example [34, Example 3.4], for $M = 2$, of

$$v = (3, 1, 1, 1, 3) = \frac{3}{2}\mathrm{a}(1) - \frac{1}{2}\mathrm{a}(\mathrm{j}) + \frac{1}{2}\mathrm{a}(-1) - \frac{1}{2}\mathrm{a}(-\mathrm{j}), \tag{27}$$

for which it is shown that no decomposition with less than four atoms exists [34].

We now focus on minimal atomic decompositions of v. For this purpose, let us consider the three equivalent formulations of the same convex optimization problem, where tr denotes the trace, $\succcurlyeq 0$ denotes positive semidefiniteness, and $V := \mathrm{T}(v)$:

$$\begin{aligned}V^+ := \operatorname*{argmin}_X\ &\mathrm{tr}(X) \quad \text{s.t.} \quad X \in \mathrm{T}(\mathbb{V}) \quad \text{and} \quad X \succcurlyeq 0\\ &\text{and} \quad X - V \succcurlyeq 0,\end{aligned} \tag{28}$$

$$\begin{aligned}\equiv\ V^+ := \operatorname*{argmin}_X\ &\tfrac{1}{M+1}\big(\mathrm{tr}(X) + \mathrm{tr}(X - V)\big)\\ &\text{s.t. } X \in \mathrm{T}(\mathbb{V}) \quad \text{and} \quad X \succcurlyeq 0 \quad \text{and} \quad X - V \succcurlyeq 0,\end{aligned} \tag{29}$$

$$\begin{aligned}\equiv\ V^+ := \mathrm{T}(v^+)\,,\ &\text{where}\ \ v^+ := \operatorname*{argmin}_{x\in\mathbb{V}} (2x_0 - v_0)\\ &\text{s.t.} \quad \mathrm{T}(x) \succcurlyeq 0 \quad \text{and} \quad \mathrm{T}(x - v) \succcurlyeq 0.\end{aligned} \tag{30}$$

Proposition 1. *The solution V^+ of the problem* (28) *exists and is unique. Accordingly, let us define $V^- := V^+ - V$. Moreover, if V is positive semidefinite, $V^+ = V$ and $V^- = 0$; if V is negative semidefinite, $V^+ = 0$ and $V^- = -V$; else, V^+ and V^- are singular.*

Proof: the case where V is positive semidefinite or negative semidefinite is obvious, so let us suppose that V is indefinite. We first remark that the feasible set of the problem (28) is not empty: let $i^- < 0$ be the smallest eigenvalue of V; $X = V - i^-\mathrm{Id}$ is positive semidefinite, Hermitian and Toeplitz, as well as $X - V$.

Then the trace defines a norm for the positive semidefinite matrices, and, as a consequence of the Weierstrass theorem [35, Theorem 1.28], a norm attains its minimum on a closed set in finite dimension, so a solution V^+ exists.

Let V^+ be a solution of (28). Set $V^- := V^+ - V$. Suppose that V^+ is positive definite. Then there exists $0 < a < 1$, such that $X := V^+ - aV^-$ is positive semidefinite. Moreover, $X \in \mathrm{T}(\mathbb{V})$ and $X - V = (1-a)V^-$ is positive semidefinite. But $\mathrm{tr}(X) < \mathrm{tr}(V^+)$, which contradicts the optimality of V^+. Hence, V^+ is singular. By the same reasoning, V^- is singular as well.

Let us finally show that the solution of (28) is unique. Let X_1 and X_2 be two solutions. Then $X = (X_1+X_2)/2$ is a solution as well, so X is singular. By Carathéodory's Theorem, X has an atomic decomposition $\sum_{k=1}^K c_k \mathrm{a}(z_k)$, where $1 \leq K \leq M$ is the rank of X. Let us define the annihilating polynomial $H(z) = \sum_{k=0}^K h_k z^k := \prod_{k=1}^K (z - z_k)$ and the vector of size $M+1$, $\tilde{h} = \begin{bmatrix} h_K & \cdots & h_0 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}$. We have $\tilde{h}^{\mathrm{H}} X \tilde{h} = 0$, so $\tilde{h}^{\mathrm{H}} X_1 \tilde{h} + \tilde{h}^{\mathrm{H}} X_2 \tilde{h} = 0$, and since X_1 and X_2 are positive semidefinite, $\tilde{h}^{\mathrm{H}} X_1 \tilde{h} = \tilde{h}^{\mathrm{H}} X_2 \tilde{h} = 0$. As in (26), this implies that the positive atomic decompositions of X_1 and X_2 are of the form $\sum_{k=1}^K c_{1,k} \mathrm{a}(z_k)$ and $\sum_{k=1}^K c_{2,k} \mathrm{a}(z_k)$, respectively, for some $c_{1,k} \geq 0$ and $c_{2,k} \geq 0$. By the same reasoning, $X_1 - V$ and $X_2 - V$ have atomic decompositions of same support, of the form $\sum_{k=1}^{K'} c'_{1,k} \mathrm{a}(z'_k)$ and $\sum_{k=1}^{K'} c'_{2,k} \mathrm{a}(z'_k)$, respectively, with $1 \leq K' \leq M$, some $z'_k \in \mathbb{T}$ all distinct and different from the z_k, and some $c'_{1,k} \geq 0$ and $c'_{2,k} \geq 0$. Therefore, $v = \sum_{k=1}^K c_{1,k} \mathrm{a}(z_k) - \sum_{k=1}^{K'} c'_{1,k} \mathrm{a}(z'_k) = \sum_{k=1}^K c_{2,k} \mathrm{a}(z_k) - \sum_{k=1}^{K'} c'_{2,k} \mathrm{a}(z'_k)$. By subtraction, $\sum_{k=1}^K (c_{1,k} - c_{2,k}) \mathrm{a}(z_k) + \sum_{k=1}^{K'} (c'_{2,k} - c'_{1,k}) \mathrm{a}(z'_k) = 0_{\mathbb{V}}$, the zero sequence of $\mathbb{V}$. Since the atoms are linearly independent, for every k, $c_{1,k} = c_{2,k}$ and $c'_{1,k} = c'_{2,k}$. Hence, $X_1 = X_2$ and the solution, call it V^+, is unique. □

Consequently, to every $V \in \mathrm{T}(\mathbb{V})$, we can assign a unique pair (V^+, V^-) of Hermitian Toeplitz positive semidefinite matrices, with V^+ solution of (28) and $V^- = V^+ - V$. By analogy with the Jordan decomposition of measures, we call (V^+, V^-) the *Jordan decomposition* of V. Equivalently, to every $v \in \mathbb{V}$, we can assign a unique pair $(v^+, v^-) \in \mathbb{V}^2$, with v^+ solution of (30) and $v^- = v^+ - v$. We call (v^+, v^-) the Jordan decomposition of v.

Proposition 2.

(i) *The problem*

$$\underset{\mu \in \mathcal{M}}{\text{minimize}} \ \|\mu\|_{\mathrm{TV}} \quad \textit{s.t.} \quad v_m = \int_{\mathbb{T}} z^{-m} \mathrm{d}\mu(z), \quad m = -M, \ldots, M, \tag{31}$$

has a solution, concentrated at a finite number $K \leq 2M$ of points of $\mathbb{T}$, i.e. of the form $\mu^\star = \sum_{k=1}^K c_k \delta_{z_k}$, for distinct $z_k \in \mathbb{T}$ and nonzero reals c_k. Equivalently, a minimal atomic decomposition $v = \sum_{k=1}^K c_k \mathrm{a}(z_k)$ exists. We obtain the parameters z_k and a_k by combining those of v^+ and v^-; that is $v^+ = \sum_{k=1}^{K^+} c_k \mathrm{a}(z_k)$ and $v^- = \sum_{k=K^++1}^{K^++K^-} (-c_k) \mathrm{a}(z_k)$, where $K = K^+ + K^-$, $K^+ := \mathrm{rank}(V^+)$, $K^- := \mathrm{rank}(V^-)$, $V := \mathrm{T}(v)$. The existence of these two atomic decompositions is guaranteed by Carathéodory's Theorem, since $V^+ \succcurlyeq 0$ and $V^- \succcurlyeq 0$. Thus,

$$\|v\|_{\mathrm{a}} = \|\mu^\star\|_{\mathrm{TV}} = \sum_{k=1}^K |c_k| = \tfrac{1}{M+1}\big(\mathrm{tr}(V^+) + \mathrm{tr}(V^-)\big) = v_0^+ + v_0^- = 2v_0^+ - v_0. \tag{32}$$

(ii) *The solution $\mu^\star$ exhibited in* (i) *is the unique solution of the problem* (31) *if and only if $\mathrm{T}(v)$ is neither positive definite, nor negative definite. More precisely, if $\mathrm{T}(v)$ is neither positive definite, nor negative definite, the measure $\mu^\star = \sum_{k=1}^K c_k \delta_{z_k}$ exhibited in* (i) *is the unique solution of the problem* (31)*. Then we have $0 \leq K^+ \leq M$ and $0 \leq K^- \leq M$. On the contrary, if $\mathrm{T}(v)$ is positive definite (resp. negative definite), there is an infinity of measures, necessarily positive (resp. negative), solution of* (31)*, among which an infinity of discrete measures concentrated at exactly $K = M+1$ points of $\mathbb{T}$; we can construct them explicitly, according to Remark 11.*

Proof: (i) let $\mu \in \mathcal{M}$, with Jordan decomposition (μ^+, μ^-), explaining v. Let $(x^+, x^-) \in \mathbb{V}^2$, such that μ^+ explains x^+ and μ^- explains x^-. We have $x^+ - x^- = v$ and $\|\mu\|_{\mathrm{TV}} = x_0^+ + x_0^- = 2x_0^+ - v_0$. Since x^+ is feasible for the problem (30), the solution v^+ of which exists and is unique, μ is a solution of (31) if and only if $x^+ = v^+$ (and $x^- = v^-$). The rest of the statement follows.

(ii) Suppose that $V = \mathrm{T}(v)$ is neither positive definite nor negative definite. Then V^+ and V^- are singular, according to Proposition 1, so $K^+ = \mathrm{rank}(V^+) \leq M$ and $K^- = \mathrm{rank}(V^-) \leq M$. Let μ be a solution of (30). μ^+ and μ^- explain v^+ and v^-, respectively, as shown above. But according to Carathéodory's Theorem, such positive measures are unique. So, the construction in (i) yields the unique solution. □

So, we have shown that the infimum is attained in (11), (12), (13). Thus, it is licit to replace the infimum by a minimum in these definitions of the atomic norm. In addition, we can strengthen the constraint $K \leq 2M+1$ to $K \leq 2M$ in (13).

Finally, we can write, for every $v \in \mathrm{T}(\mathbb{V})$,

$$\|v\|_{\mathrm{a}} = \min_X \ \big(\tfrac{2}{M+1}\mathrm{tr}(X) - v_0\big) \quad \text{s.t.} \quad X \in \mathrm{T}(\mathbb{V}) \quad \text{and} \quad X \succcurlyeq 0 \quad \text{and} \quad X - \mathrm{T}(v) \succcurlyeq 0. \tag{33}$$

We can note that this semidefinite characterization of the atomic norm was given in [36, eq. 12], without any proof.

If we go back to the problem of finding a decomposition of v with the smallest number of atoms, it amounts to:

$$\underset{X}{\text{minimize}} \ \big(\mathrm{rank}(X) + \mathrm{rank}(X - V)\big) \quad \text{s.t.} \quad X \in \mathrm{T}(\mathbb{V}) \quad \text{and} \quad X \succcurlyeq 0 \quad \text{and} \quad X' - V \succcurlyeq 0. \tag{34}$$

Such nonconvex problems of rank minimization are notoriously difficult, and atomic norm minimization is a convex and efficient alternative.

Given an atomic decomposition $v = \sum_{k=1}^K c_k \mathrm{a}(z_k)$, we define its *separation* $D(\{z_k\}) := \min_{k \neq l} d(z_k, z_l)$, as the minimal distance between its atoms. A remarkable result by Candès and Fernandez-Granda [9] is the following: if v

has an atomic decomposition with large enough separation, then it is both the minimal atomic decomposition of v and the atomic decomposition of v with the fewest atoms; that is, the solutions of (29) and (34) are the same and correspond to this decomposition. A sufficient condition for this is $D(\{z_k\}) \geq 1.87/M$ (and $M \geq 128$) [9]. This condition was further strengthened to $D(\{z_k\}) \geq 1.26/M$ (and $M \geq 1000$) [37]. Some authors conjecture that $D(\{z_k\}) \geq 1/M$ is a sufficient condition [16]. Note that every minimal atomic decomposition has a distance $d(z_k, z_l) \geq 1/(2M)$ between two atoms, if c_k is positive and c_l is negative [16, Corollary 2]. Of course, these sufficient conditions imply that the number of atoms is $K \leq M$, and we have seen that in that case, Prony's method yields this optimal decomposition directly. But we cannot hope for a stable method to recover atomic decompositions with $K \geq M+1$ atoms, characterized by $2K \geq 2M+2$ real parameters, from v, characterized by $2M+1$ real degrees of freedom. More importantly, the fact that separated enough atomic decompositions are represented by solutions of convex optimization problems, opens the door to their robust approximate estimation by convex minimization, from inexact or noisy coefficients $y_m \approx v_m$. For instance, given $y \in \mathbb{V}$, we can

$$\begin{aligned} &\underset{x \in \mathbb{V}}{\text{minimize}} \ \|y - x\|_2^2 + \lambda \|x\|_{\mathrm{a}}, && (35) \\ \equiv \ &\underset{(x^+, x^-) \in \mathbb{V}^2}{\text{minimize}} \ \|y - x^+ + x^-\|_2^2 + \lambda(x_0^+ + x_0^-) \\ &\quad \text{s.t.} \quad \mathrm{T}(x^+) \succcurlyeq 0 \quad \text{and} \quad \mathrm{T}(x^-) \succcurlyeq 0, && (36) \end{aligned}$$

for some regularization parameter $\lambda > 0$. Indeed, there is no straightforward extension of Prony's method to estimate an atomic decomposition with K atoms, in case there is noise in the data, even if K is known [13]. Note that the solution of (35) exists and is unique, since the problem is strongly convex. Approximation goes beyond the scope of this paper, but we can readily give the following stability result, which is a direct consequence of Proposition 2 and [38, Theorem 3.5] (see also [39, Proposition 5]):

Corollary 3. *Suppose that $\mathrm{T}(v)$ is neither positive definite nor negative definite. Let $\mu^\star$ be the solution of* (31)*. For every $n \in \mathbb{N}$, let $e^n \in \mathbb{V}$ be a perturbation with $\|e^n\|_2$ converging monotonically to zero as $n \to \infty$, set $y^n := v + e^n$, and let μ^n be a solution of*

$$\underset{\mu \in \mathcal{M}}{\text{minimize}} \ \sum_{m=-M}^{M} \left(y_m^n - \int_{\mathbb{T}} z^{-m} \mathrm{d}\mu(z) \right)^2 + \lambda^n \|\mu\|_{\mathrm{TV}}, \quad (37)$$

where the sequence $(\lambda^n)_{n \in \mathbb{N}}$ is positive, decreasing, and such that

$$\lambda^n \to 0 \quad \textit{and} \quad \frac{\|e^n\|_2^2}{\lambda^n} \to 0, \quad \textit{as } n \to \infty. \quad (38)$$

Equivalently, μ^n is a measure with minimal total variation explaining $v^n = v^{n,+} - v^{n,-}$, and found easily by Prony's method from $(v^{n,+}, v^{n,-})$, the solution of (36) *with y and λ replaced by y^n and λ^n. Then the sequence $(\mu_n)_{n \in \mathbb{N}}$ converges for the weak-* topology to $\mu^\star$ (and $v^n \to v$), as $n \to \infty$.*

Remark 15. A measure $\mu \in \mathcal{M}$ is said to be *identifiable* if, given its sequence of Fourier coefficients $v = (v_m)_{m=-M}^M$, it is the unique solution of the problem (31). That is, an identifiable measure can be perfectly recovered from $v = \mathcal{F}\mu$. If $\mathrm{T}(v)$ is indefinite, we have seen above that well separated measures are identifiable, but they form a small subset of all the identifiable measures. The picture is different if we restrict ourselves to positive measures; as a consequence of Proposition 2, we have the property: *a positive measure is identifiable if and only if it is concentrated at $K \leq M$ points of $\mathbb{T}$*. It is remarkable that all the information about a positive discrete measure concentrated in at most M points is encoded in its coefficients $(v_m)_{m=1}^M$ (since v_0 can be recovered as the opposite of the smallest eigenvalue of $\mathrm{T}((v_M^*, \ldots, v_1^*, 0, v_1, \ldots, v_M))$), characterized by the critical number $2M$ of real degrees of freedom, without any constraint.

VII. Numerical resolution of (30)

Since the equivalent problems (28) and (30) are semidefinite programs, many efficient algorithmic strategies exist to solve them to an arbitrary precision. It is beyond the scope of this paper to compare them. We present a first order iterative algorithm, which is simple to implement and turns out to be fast. It is an instance of the over-relaxed version [40] of the Chambolle–Pock algorithm [41]. The algorithm is the following, where T^* is the adjoint operator of T and $P_{\preccurlyeq 0}$ denotes the projection onto the convex cone of negative semidefinite matrices, by computing the eigendecomposition and setting the positive eigenvalues to zero.

Algorithm 1. Input: $v \in \mathbb{V}$. Output: estimate $x^{(i)}$ of v^+, the solution of (30). Choose the proximal parameter $\tau > 0$ and the relaxation parameter $\rho \in [1, 2)$. Choose the initial estimate $x^{(0)} \in \mathbb{V}$ of v^+. Set $\sigma := 1/\tau/(2\|\mathrm{T}\|^2) = 1/\tau/(2M+2)$. Set $U_1^{(0)} := U_2^{(0)} := \mathrm{T}(0_{\mathbb{V}})$, the zero matrix. Set $V := \mathrm{T}(v)$. Then iterate, for $i = 0, 1, \ldots$

1. $\tilde{x}^{(i+1)} := x^{(i)} - \tau \mathrm{T}^*(U_1^{(i)} + U_2^{(i)}) - \tau(1, 0, \ldots, 0)$,
2. $\widetilde{U}_1^{(i+1)} := P_{\preccurlyeq 0}\left(U_1^{(i)} + \sigma \mathrm{T}(2\tilde{x}^{(i+1)} - x^{(i)})\right)$,
3. $\widetilde{U}_2^{(i+1)} := P_{\preccurlyeq 0}\left(U_2^{(i)} - \sigma V + \sigma \mathrm{T}(2\tilde{x}^{(i+1)} - x^{(i)})\right)$,
4. $(x^{(i+1)}, U_1^{(i+1)}, U_2^{(i+1)}) := \rho(\tilde{x}^{(i+1)}, \widetilde{U}_1^{(i+1)}, \widetilde{U}_2^{(i+1)}) + (1-\rho)(x^{(i)}, U_1^{(i)}, U_2^{(i)})$.

The algorithm converges to the solution v^+ of (30) [40, Theorem 5.3]. Moreover, the number of positive eigenvalues of $U_1^{(i)} + \sigma \mathrm{T}(2\tilde{x}^{(i+1)} - x^{(i)})$ and $U_2^{(i)} - \sigma V + \sigma \mathrm{T}(2\tilde{x}^{(i+1)} - x^{(i)})$ are robust estimates of $K^+ = \mathrm{rank}(V^+)$ and $K^- = \mathrm{rank}(V^-)$, the number of positive and negative atoms in the minimal atomic decomposition of v, respectively. The complexity of every iteration is dominated by the computation of two eigendecompositions, so the algorithm has complexity $O(M^3)$ per iteration. In practice, we initialize the algorithm with the positive part of the decomposition given in Proposition 3; that is, given the parameters $\{z_k\}$, $\{c_k\}$ of this decomposition, we set $x^{(0)} = \sum_{k=1}^{2M} \max(c_k, 0) \mathrm{a}(z_k)$. So, $x^{(0)}$ is feasible for the problem (30); that is, $\mathrm{T}(x^{(0)}) \succcurlyeq 0$ and $\mathrm{T}(x^{(0)} - v) \succcurlyeq 0$.

We can note that the over-relaxed Chambolle–Pock algorithm can also be used to solve (36).

An alternative way to compute the atomic norm of v is to solve the semidefinite program [6], [7]

$$\begin{array}{ll}\underset{q\in\mathbb{C}^{4M+1}}{\text{minimize}} & \frac{1}{2}(q_0+t)\\ \text{s.t.} & \left[\begin{array}{ccc|c} & \mathrm{T}(q) & & \begin{array}{c}v_{-M}\\ \vdots \\ v_M\end{array}\\ \hline v_M & \cdots & v_{-M} & t\end{array}\right]\succcurlyeq 0,\end{array}\quad \text{s.t. } q_{-m}=q_m^*,\ t\in\mathbb{R} \tag{39}$$

the optimal value of which is equal to $\|v\|_{\mathrm{a}}$. Since the constraint in (39) implies $\mathrm{T}(q)\succcurlyeq 0$, Carathéodory's Theorem is applicable and Prony's method applied to $\mathrm{T}(q)$ allows us to recover the parameters z_k of the minimal atomic decomposition of v. It is easy to show that at optimality, $t=q_0=\|v\|_{\mathrm{a}}$, so that it is better, for faster convergence, to remove the unnecessary variable t and to consider the problem

$$\begin{array}{ll}\underset{q\in\mathbb{C}^{4M+1}}{\text{minimize}} & q_0\\ \text{s.t.} & \left[\begin{array}{ccc|c} & \mathrm{T}(q) & & \begin{array}{c}v_{-M}\\ \vdots \\ v_M\end{array}\\ \hline v_M & \cdots & v_{-M} & q_0\end{array}\right]\succcurlyeq 0.\end{array}\quad \text{s.t. } q_{-m}=q_m^* \tag{40}$$

A typical algorithm to solve (39) or (40) needs one eigendecomposition per iteration, instead of two with the proposed Algorithm 1, but it involves a matrix twice larger. Since the complexity of the eigendecomposition scales with the cube of the matrix size, we expect Algorithm 1 to be faster. We leave for future work a numerical comparison of different formulations and algorithms to compute the atomic norm of v.

VIII. Uniform decomposition with at most $2M$ atoms

In this section, we present a new construction of a *uniform* atomic decomposition of v, with at most $2M$ atoms.

Proposition 3. *There exists an atomic decomposition of v, with at most $2M$ atoms located uniformly on $\mathbb{T}$, with an explicit form:*

$$v=\sum_{k=1}^{2M} c_k \mathrm{a}(z_k), \tag{41}$$

$$\textit{where}\quad z_k:=\mathrm{e}^{\mathrm{j}(2\pi k/(2M)-\phi)}\quad \textit{and}\quad \phi:=\arg(v_M)/M \tag{42}$$

(if $v_M=0$, every ϕ is suitable), and the c_k are obtained by inverse discrete Fourier transform from the sequence $(v_m\mathrm{e}^{-\mathrm{j}m\phi})_{m=-M}^{M-1}$:

$$c_k:=\frac{1}{2M}\sum_{m=-M}^{M-1}(v_m\mathrm{e}^{-\mathrm{j}m\phi})\mathrm{e}^{\mathrm{j}2\pi mk/(2M)},\quad k=1,\dots,2M. \tag{43}$$

Proof: we first remark that the c_k in (43) are real, since $v_{-m}\mathrm{e}^{-\mathrm{j}(-m)\phi}=(v_m\mathrm{e}^{-\mathrm{j}m\phi})^*$ and this value is real for $m=0$ and $m=-M$. Then it is sufficient to notice that

$$v_m\mathrm{e}^{-\mathrm{j}m\phi}=\sum_{k=1}^{2M} c_k\mathrm{e}^{-\mathrm{j}2\pi mk/(2M)},\quad m=-M,\dots,M-1. \tag{44}$$

So (we check that the formula is also valid for $m=M$),

$$v_m=\sum_{k=1}^{2M} c_k\mathrm{e}^{-\mathrm{j}m(2\pi k/(2M)-\phi)},\quad m=-M,\dots,M. \qquad \square \tag{45}$$

Note that the number of atoms in the proposition is at most $2M$, and not exactly $2M$, because some c_k can be zero in (41) and (43).

In terms of extrapolation of $v=(v_m)_{m=-M}^{M}$ to the infinite sequence $(v_m)_{m\in\mathbb{Z}}$ of Fourier coefficients of $\mu=\sum_{k=1}^{2M} c_k\delta_{\mathrm{e}^{\mathrm{j}(2\pi k/(2M)-\phi)}}$, we get

$$v_m=\mathrm{e}^{\mathrm{j}(m-[m])\phi}v_{[m]},\quad m\in\mathbb{Z}, \tag{46}$$

where $[m]=\mathrm{rem}(m+M,2M)-M$ and $\mathrm{rem}(n,N)\in\{0,\dots,N-1\}$ is the remainder of the Euclidean division of $n\in\mathbb{Z}$ by N.

Remark 16. Duval and Peyré showed that there exists a measure with separation $1/(2M)$, which is not identifiable, by exhibiting $\delta_{\mathrm{e}^{-\mathrm{j}2\pi/(2M)}}+\delta_{\mathrm{e}^0}-\delta_{\mathrm{e}^{\mathrm{j}2\pi/(2M)}}$ and proving that it is not the solution of (31) [16, Corollary 1]. According to Proposition 3, measures with separation $1/(2M)$, which are not identifiable, are the rule rather than the exception. Indeed, every $v\in\mathbb{V}$ is explained by the measure with separation $1/(2M)$ constructed in Proposition 3, and there is no reason that this measure has minimal total variation norm, in general.

Now, let us consider the case where v is made of a single atom: $v=c\mathrm{a}(z)$, for some $c>0$ and $z\in\mathbb{T}$ (the case $c<0$ is addressed the same way). $\mathrm{T}(v)$ is positive definite, of rank 1, so the analysis of Section V shows that $v=c\mathrm{a}(z)$ is the unique positive atomic decomposition of v and the unique minimal atomic decomposition of v. Moreover, this single-atom decomposition coincides with the one of Proposition 3: the c_k in (43) are all zero, except $c_{k_0}=c$ for $k_0=\{2M \text{ if } \omega=0,\ 2M(\phi+\omega)/(2\pi) \text{ else}\}$, where $\omega=\arg(z)$. Indeed, if $\omega>0$, $\phi=\arg(\mathrm{e}^{-\mathrm{j}M\omega})/M$, so $\phi=-\omega+2\pi k_0/(2M)$ for some even $k_0\in\{2,4,\dots,2M\}$. Since the sequence $(v_m\mathrm{e}^{-\mathrm{j}m\phi}=c\mathrm{e}^{-\mathrm{j}2\pi mk_0/(2M)})_{m=-M}^{M-1}$ is a sampled complex exponential of frequency multiple of $1/(2M)$, its inverse discrete Fourier transforms gives c_k all equal to zero, except $c_{k_0}=c$.

After this simple case of a single atom, we direct our attention to the case of two atoms with opposite amplitudes, in the next section.

IX. Case of 2 atoms with opposite amplitudes

Let us now consider that v is the sum of two atoms with opposite amplitudes. So, let z_1 and z_2 be two distinct elements of $\mathbb{T}$ and let c be a positive real. We set

$$v=c\mathrm{a}(z_1)-c\mathrm{a}(z_2). \tag{47}$$

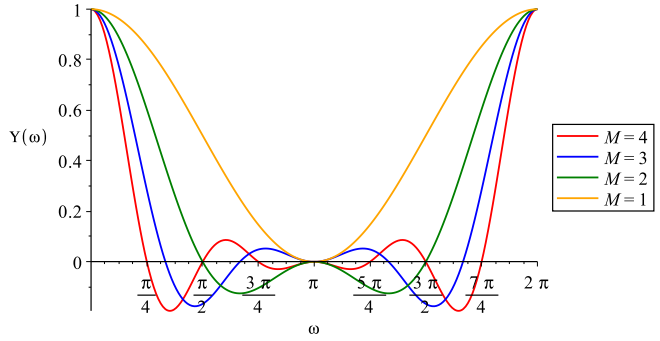

Fig. 1. Plot of the function $\Upsilon(\omega)$, defined in (49), for different values of M.

Set $\omega_1 := \arg(z_1)$, $\omega_2 := \arg(z_2)$. Then we have

$$v_m = c\mathrm{e}^{-\mathrm{j}m\omega_1} - c\mathrm{e}^{-\mathrm{j}m\omega_2}, \quad m = -M, \dots, M. \tag{48}$$

We introduce the 2π-periodic function

$$\Upsilon : \omega \in \mathbb{R} \mapsto \frac{\sin\bigl(\omega(2M+1)/2\bigr) + \sin\bigl(\omega(2M-1)/2\bigr)}{4M\sin(\omega/2)} \tag{49}$$

$$= \frac{1}{4M}\mathrm{e}^{-\mathrm{j}M\omega} + \frac{1}{2M}\sum_{m=-M+1}^{M-1}\mathrm{e}^{\mathrm{j}m\omega} + \frac{1}{4M}\mathrm{e}^{\mathrm{j}M\omega}, \tag{50}$$

extended by continuity to $\Upsilon(0) = 1$. Note that $\Upsilon(2\pi k/(2M)) = 0$ for every $k = 1, \dots, 2M-1$. The function Υ is plotted in Figure 1 for the first values of M.

Proposition 4. *In the present case of two atoms with opposite amplitudes, the decomposition of Proposition 3 can be rewritten as (up to a circular permutation on the indices):*

$$v = \sum_{k=1}^{2M} c_k \mathrm{a}(\mathrm{e}^{\mathrm{j}\omega_k}), \tag{51}$$

$$\textit{where} \quad \omega_k = \frac{2\pi k}{2M} - \frac{2\pi}{4M} + \frac{\omega_1+\omega_2}{2}, \tag{52}$$

$$c_k = c\Upsilon\left(\frac{2\pi k}{2M} - \frac{2\pi}{4M} + \frac{\omega_2-\omega_1}{2}\right) - c\Upsilon\left(\frac{2\pi k}{2M} - \frac{2\pi}{4M} + \frac{\omega_1-\omega_2}{2}\right). \tag{53}$$

Proof: By combining (48), (43) and the fact that $v_{-M}\mathrm{e}^{\mathrm{j}M\phi} = v_M\mathrm{e}^{-\mathrm{j}M\phi} \in \mathbb{R}$, we obtain

$$c_k = c\Upsilon\bigl(\tfrac{2\pi k}{2M} - \phi - \omega_1\bigr) - c\Upsilon\bigl(\tfrac{2\pi k}{2M} - \phi - \omega_2\bigr). \tag{54}$$

Moreover, we have $\phi = \arg(\mathrm{e}^{-\mathrm{j}M\omega_1} + \mathrm{e}^{-\mathrm{j}(M\omega_2+\pi)})/M$, so $\phi = -(\omega_1+\omega_2)/2 + 2\pi/(4M) + 2\pi k_0/(2M)$, for some $k_0 \in \mathbb{Z}$. □

An example is illustrated in Figure 2. In Figure 3, we show the behavior of the atomic norm of v, as a function of $d(z_1, z_2)$. We observe, without proof, that if this distance is larger than or equal to $1/(2M)$, the measure $\delta_{z_1} - \delta_{z_2}$ is identifiable.

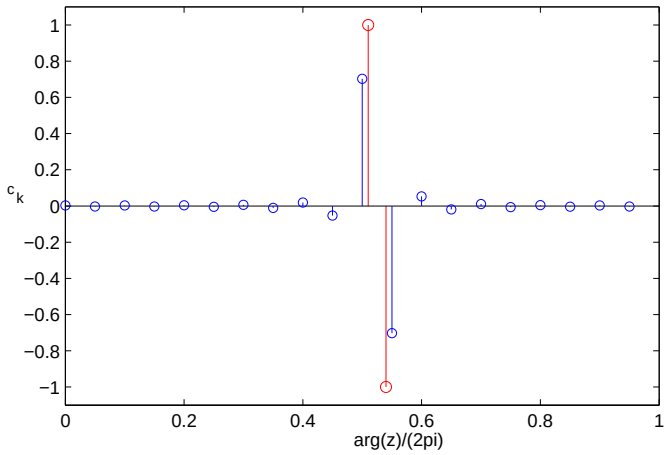

Fig. 2. The two measures $\mu^\sharp = \delta_{\mathrm{e}^{\mathrm{j}2\pi\,0.51}} - \delta_{\mathrm{e}^{\mathrm{j}2\pi\,0.54}}$, in red, and $\mu^\star = \sum_{k=1}^{2M} c_k\delta_{z_k}$ given by Propositions 3 and 4, in blue, for $M = 10$, have the same sequence of Fourier coefficients $v = (v_m)_{m=-M}^{M}$. These two measures have total variation norm $\|\mu^\sharp\|_{\mathrm{TV}} = 2$ and $\|\mu^\star\|_{\mathrm{TV}} = \sum_{k=1}^{2M}|c_k| \approx 1.6$, respectively. Numerical minimization of the atomic norm of v yields the minimal atomic decomposition $v = \sum_{k=1}^{2M} c_k\mathrm{a}(z_k)$, so $\|v\|_{\mathrm{a}} = \|\mu^\star\|_{\mathrm{TV}}$ and $\mu^\star$ is the measure of minimal total variation norm explaining v.

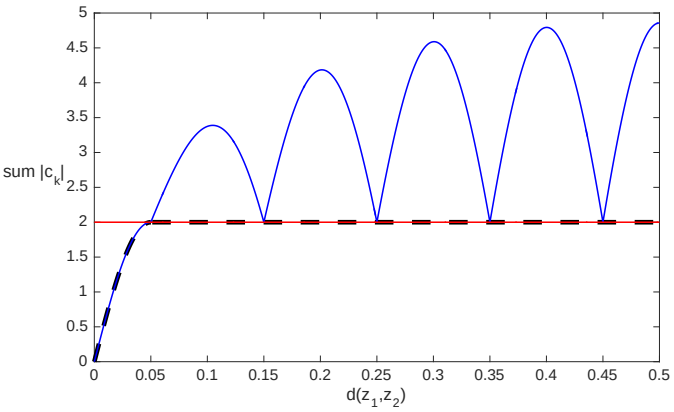

Fig. 3. Value of the total variation norm $\|\mu^\sharp\|_{\mathrm{TV}} = \{0 \text{ if } z_1 = z_2,\ 2 \text{ else}\}$, in red, and $\|\mu^\star\|_{\mathrm{TV}} = \sum_{k=1}^{2M}|c_k|$, in blue, as a function of $d(z_1, z_2)$, for $M = 10$, where the two measures $\mu^\sharp = \delta_{z_1} - \delta_{z_2}$ and $\mu^\star = \sum_{k=1}^{2M} c_k\delta_{z_k}$ given by Proposition 4, have the same sequence of Fourier coefficients $v = (v_m)_{m=-M}^{M}$. In black, the atomic norm of v. We observe that when $d(z_1, z_2) < 1/(2M)$, $\mu^\star$ is the measure of minimal total variation norm explaining v, so $\|v\|_{\mathrm{a}} = \|\mu^\star\|_{\mathrm{TV}} = 2\sin\bigl(\pi M d(z_1, z_2)\bigr)$. Else, when $d(z_1, z_2) \geq 1/(2M)$, $\mu^\sharp$ is the measure of minimal total variation norm explaining v, so $\|v\|_{\mathrm{a}} = \|\mu^\sharp\|_{\mathrm{TV}} = 2$.

X. New distances on $\mathbb{V}$

Building upon the analysis in the previous section, we can define a distance, which we may call the *atomic Radon distance*, denoted by $d_{\mathrm{a}0}$, between two arbitrary elements v and v' of $\mathbb{V}$, with $v_0 = v'_0$:

$$d_{\mathrm{a}0}(v, v') := \tfrac{1}{2}\|v - v'\|_{\mathrm{a}} \tag{55}$$

$$= \min\bigl\{\tfrac{1}{2}\|\mu - \mu'\|_{\mathrm{TV}} : \ (\mu, \mu') \in \mathcal{M}^2, \ \mathcal{F}\mu = v, \ \mathcal{F}\mu' = v'\bigr\}. \tag{56}$$

$$= \min\bigl\{\tfrac{1}{2}\|\eta\|_{\mathrm{TV}} : \ \eta \in \mathcal{M}, \ \mathcal{F}\eta = v - v'\bigr\}. \tag{57}$$

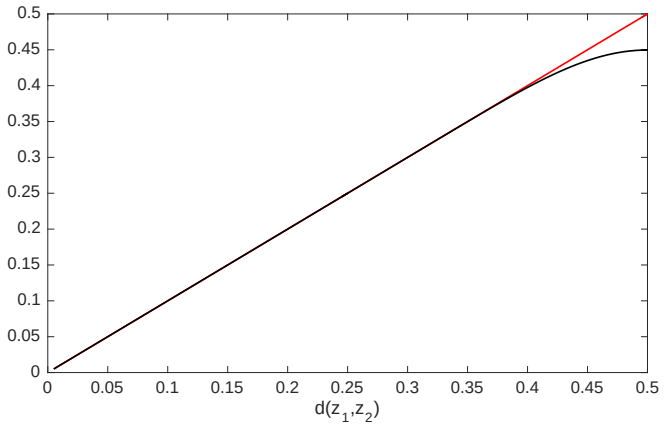

Fig. 4. Value of the atomic 1-Wasserstein distance $d_{\mathrm{a}1}\big(\mathrm{a}(z_1),\mathrm{a}(z_2)\big)$ as a function of $d(z_1,z_2)$, in black, with $M=10$. We observe that the atomic 1-Wasserstein distance is equal to $d(z_1,z_2)$, in red, except for the largest values, where this distance is underestimated.

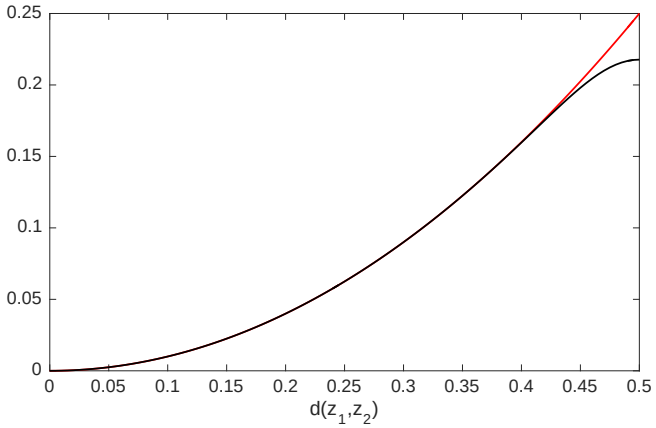

Fig. 5. Value of the squared atomic 2-Wasserstein distance $d_{\mathrm{a}2}\big(\mathrm{a}(z_1),\mathrm{a}(z_2)\big)^2$ as a function of $d(z_1,z_2)$, in black, with $M=10$. We observe that the squared atomic 2-Wasserstein distance is equal to $d(z_1,z_2)^2$, in red, except for the largest values, where this squared distance is underestimated.

This distance allows us to quantify whether v and v' are different. $d_{\mathrm{a}0}$ is convex with respect to the pair (v,v'). We could actually show that $d_{\mathrm{a}0}$ is the largest convex function such that, for every $(z_1,z_2)\in\mathbb{T}^2$ and $c\in\mathbb{R}$, $d_{\mathrm{a}0}\big(c\mathrm{a}(z_1),c\mathrm{a}(z_2)\big)\leq\{0$ if $z_1=z_2$, $|c|$ else $\}$. So, $d_{\mathrm{a}0}$ is the best convex function we can hope for, to discriminate between elements of $\mathbb{V}$. Its precision naturally depends on M.

Further on, we define another distance, which we call the *atomic 1-Wasserstein distance*, denoted by $d_{\mathrm{a}1}$, between two elements v and v' of $\mathbb{V}$, with $v_0=v'_0$; it is the minimal first Wasserstein distance between measures explaining them:

$$d_{\mathrm{a}1}(v,v') = \min\big\{d_{\mathcal{W}1}(\mu,\mu') \;:\; (\mu,\mu')\in\mathcal{M}^2, \quad \mathcal{F}\mu=v,\ \mathcal{F}\mu'=v'\big\}. \tag{58}$$

We recall that the first Wasserstein distance $d_{\mathcal{W}1}$ between two positive measures μ and μ' on $\mathbb{T}$ is the minimum, over all positive measures ν on $\mathbb{T}\times\mathbb{T}$ with marginals μ and μ' on the first and second factors, respectively, of the transport cost $\int_{\mathbb{T}^2} d(z_1,z_2)\mathrm{d}\nu(z_1,z_2)$ [42]. It is known that this value depends only on $\mu-\mu'$ and is equal to the minimum over the real α of $1/(2\pi)\int_{\mathbb{T}}|F(z)-F'(z)-\alpha|\mathrm{d}z$, where F and F' are the cumulative distribution functions of μ and μ', respectively [43]. Consequently, $d_{\mathcal{W}1}$ can be extended to a pair of arbitrary measures $(\mu,\mu')\in\mathcal{M}^2$, with $\mu(\mathbb{T})=\mu'(\mathbb{T})$, as $d_{\mathcal{W}1}(\mu,\mu')=d_{\mathcal{W}1}(\mu^+ + \mu'^-,\mu'^+ + \mu^-)$.

Thus, we can equivalently define the atomic 1-Wasserstein distance as

$$d_{\mathrm{a}1}(v,v') := \min\big\{\|\eta\|_{\mathrm{TV}} \;:\; \eta\in\mathcal{M},\ \mathcal{F}\eta=w \quad \text{with} \quad \mathrm{j}2\pi m w_m = v_m - v'_m,\quad m=-M,\dots,M\big\}. \tag{59}$$

$d_{\mathrm{a}1}$ is convex with respect to the pair (v,v'). We can express it as a semidefinite program: let $w=\big((v_m-v'_m)/(\mathrm{j}2\pi m)\big)_{m=-M}^{M}$, with $w_0=0$, and $W=\mathrm{T}(w)$; then

$$\begin{aligned} d_{\mathrm{a}1}(v,v') &= \min_{X,\alpha}\ \big(\tfrac{2}{M+1}\mathrm{tr}(X)+\alpha\big) \quad \text{s.t.} \quad X\in\mathrm{T}(\mathbb{V}) \quad \text{and} \\ &\qquad X\succcurlyeq 0 \quad \text{and} \quad X-W+\alpha\mathrm{Id}\succcurlyeq 0, \qquad (60)\\ &= \min_{X}\ \big(\tfrac{2}{M+1}\mathrm{tr}(X)+\mathrm{i}^+(W-X)\big) \quad \text{s.t.} \\ &\qquad X\in\mathrm{T}(\mathbb{V}) \quad \text{and} \quad X\succcurlyeq 0, \qquad (61)\end{aligned}$$

where i^+ denotes the largest eigenvalue.

In Figure 4, we can see that when v and v' are atoms, $d_{\mathrm{a}1}(v,v')$ closely matches the distance between their frequencies.

Finally, we define the *atomic 2-Wasserstein distance*, denoted by $d_{\mathrm{a}2}$, between two elements v and v' of $\mathbb{V}$, with $v_0=v'_0$; it is the minimal second Wasserstein distance between measures explaining them:

$$d_{\mathrm{a}2}(v,v') := \min\big\{d_{\mathcal{W}2}(\mu,\mu') \;:\; (\mu,\mu')\in\mathcal{M}^2, \quad \mathcal{F}\mu=v,\ \mathcal{F}\mu'=v'\big\}. \tag{62}$$

The second Wasserstein distance $d_{\mathcal{W}2}$ between two positive measures μ and μ' on $\mathbb{T}$ is the square root of the minimum, over all positive measures ν on $\mathbb{T}\times\mathbb{T}$ with marginals μ and μ' on the first and second factors, respectively, of the transport cost $\int_{\mathbb{T}^2} d(z_1,z_2)^2\mathrm{d}\nu(z_1,z_2)$ [42] ($d_{\mathcal{W}2}$ is extended to signed measures like $d_{\mathcal{W}1}$). It seems difficult to transfer this definition to a finite-dimensional convex program for the squared atomic 2-Wasserstein distance between any v and v'. There is, however, a closed form expression when one of the two vectors is an atom: the squared atomic 2-Wasserstein distance between $a\in\mathcal{A}$ and $v\in\mathbb{V}$, with $v_0=1$ and $\mathrm{T}(v)\succcurlyeq 0$, can be first written as

$$d_{\mathrm{a}2}(a,v)^2 = \min\big\{\eta(\mathbb{T}) \;:\; \eta\in\mathcal{M} \text{ is positive},\ \mathcal{F}\eta=w, \text{ with } -4\pi^2m^2w_m = v_m - 2a_m + a_m^2 v_m^*, \ m=-M,\dots,M\big\}. \tag{63}$$

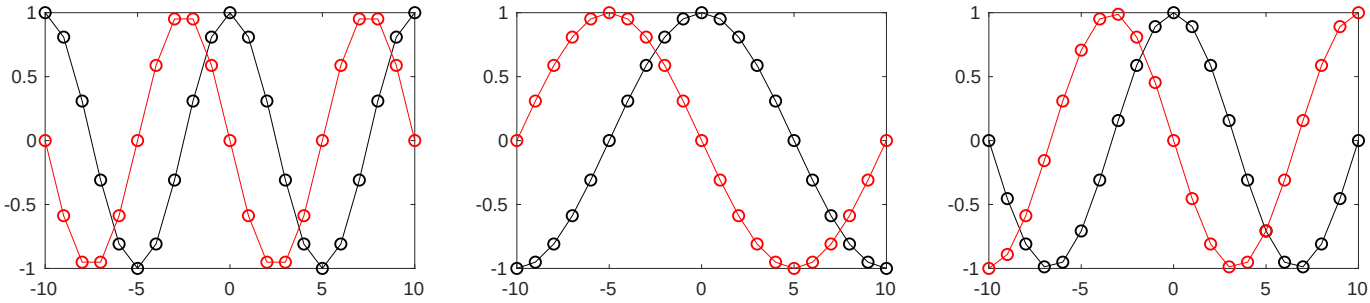

Fig. 6. In (a) and (b): two atoms $\mathrm{a}(z) = \big(\mathrm{a}(z)_m\big)_{m=-M}^{M}$ and $\mathrm{a}(z') = \big(\mathrm{a}(z')_m\big)_{m=-M}^{M}$, respectively, with $z = \mathrm{e}^{\mathrm{j}2\pi/10}$, $z' = \mathrm{e}^{\mathrm{j}2\pi/20}$, $M = 10$; the index m is in abscissa, the real and imaginary parts of the values are in ordinate, with black and red colors, respectively. In (c): the atomic Wasserstein barycenter $v \in \mathbb{V}$, which minimizes $d_{\mathrm{a2}}(\mathrm{a}(z), v)^2 + d_{\mathrm{a2}}(\mathrm{a}(z'), v)^2$; it turns out that v is the atom $\mathrm{a}(\mathrm{e}^{\mathrm{j}3\pi/20})$, whose frequency is the mean of the frequencies of the two given atoms.

This function is convex with respect to v. It has a closed-form expression: let $w = \big((v_m - 2a_m + a_m^2 v_m^*)/(-4\pi^2 m^2)\big)_{m=-M}^{M}$, with $w_0 = 0$, and $W = \mathrm{T}(w)$; then

$$d_{\mathrm{a2}}(a, v)^2 = \mathrm{i}^+(-W). \tag{64}$$

In Figure 5, we can see that when v and v' are atoms, $d_{\mathrm{a2}}(v, v')^2$ closely matches the squared distance between their frequencies.

In Figure 6, we show a toy application of the proposed atomic 2-Wasserstein distance: we compute the Wasserstein barycenter [44] $v \in \mathbb{V}$ of two atoms $\mathrm{a}(z)$ and $\mathrm{a}(z')$, with $z = \mathrm{e}^{\mathrm{j}2\pi/10}$, $z' = \mathrm{e}^{\mathrm{j}2\pi/20}$, $M = 10$. That is, v minimizes $d_{\mathrm{a2}}(\mathrm{a}(z), v)^2 + d_{\mathrm{a2}}(\mathrm{a}(z'), v)^2$. The over-relaxed Chambolle–Pock algorithm is used again to solve this convex optimization problem. The solution v turns out to be the atom $\mathrm{a}(\mathrm{e}^{\mathrm{j}3\pi/20})$, whose frequency is the mean of the frequencies of the two given atoms. So, we can interpolate between two sampled complex exponentials, by solving a convex optimization problem, without having to first identify their frequencies. This opens the door to many signal processing applications.

XI. Conclusion

We studied the properties of decompositions of vectors into complex exponentials and we proposed convex formulations to implement these decompositions. Roughly speaking, the analysis is based on splitting the vectors (and associated Toeplitz matrices) into negative and positive (semidefinite) parts and applying the powerful characterization of Toeplitz matrices by Carathéodory to them. Further theoretical investigations are needed to characterize the values taken by the three defined atomic distances. In future work, the author will use these distances for the approximation and reconstruction of signals and images.

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