gipsa-lab

Convex approaches to K-means clustering and image labeling problems

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Tokyo Tech,

May 2018

The K-means problem

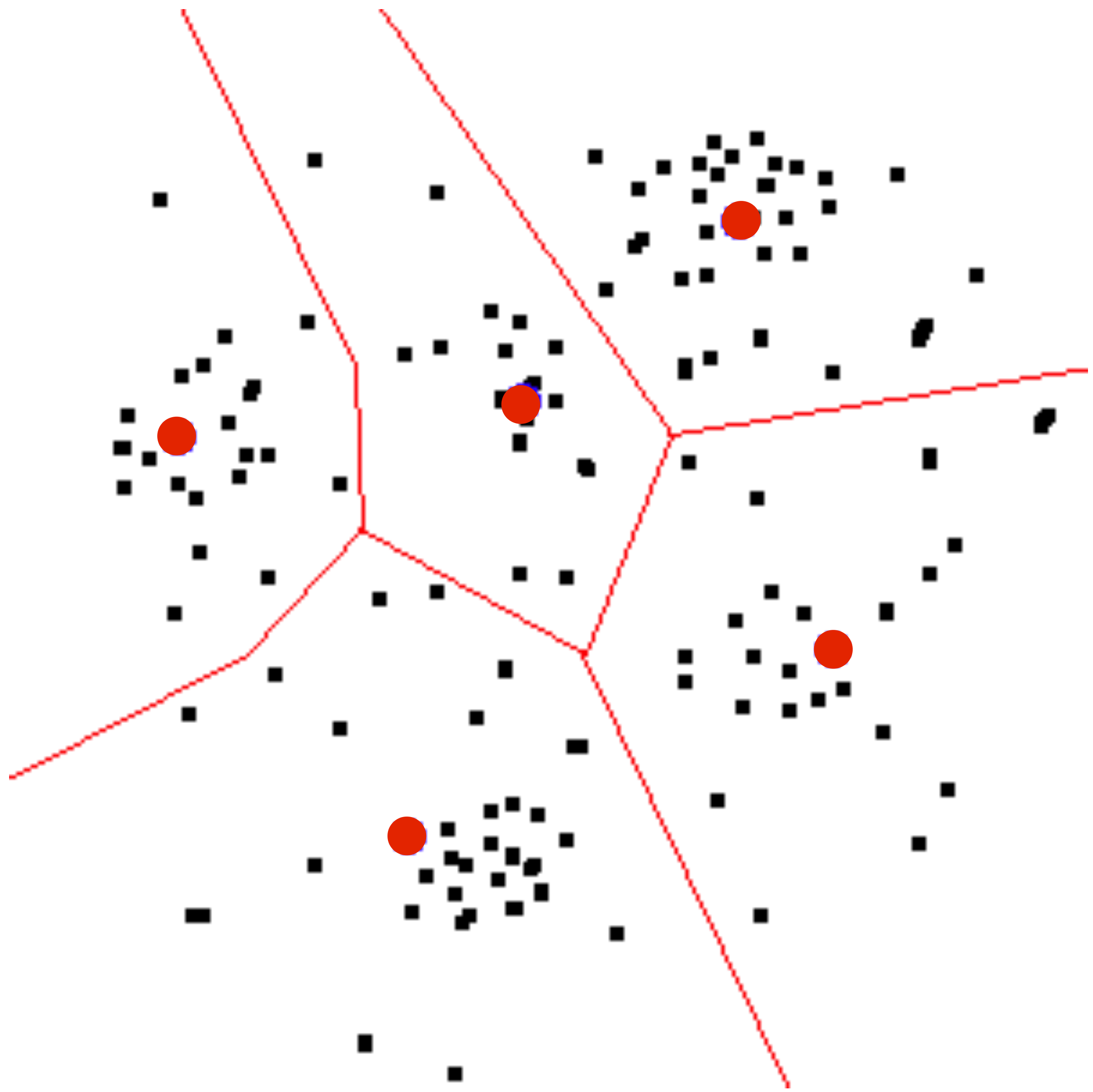

Image quantization

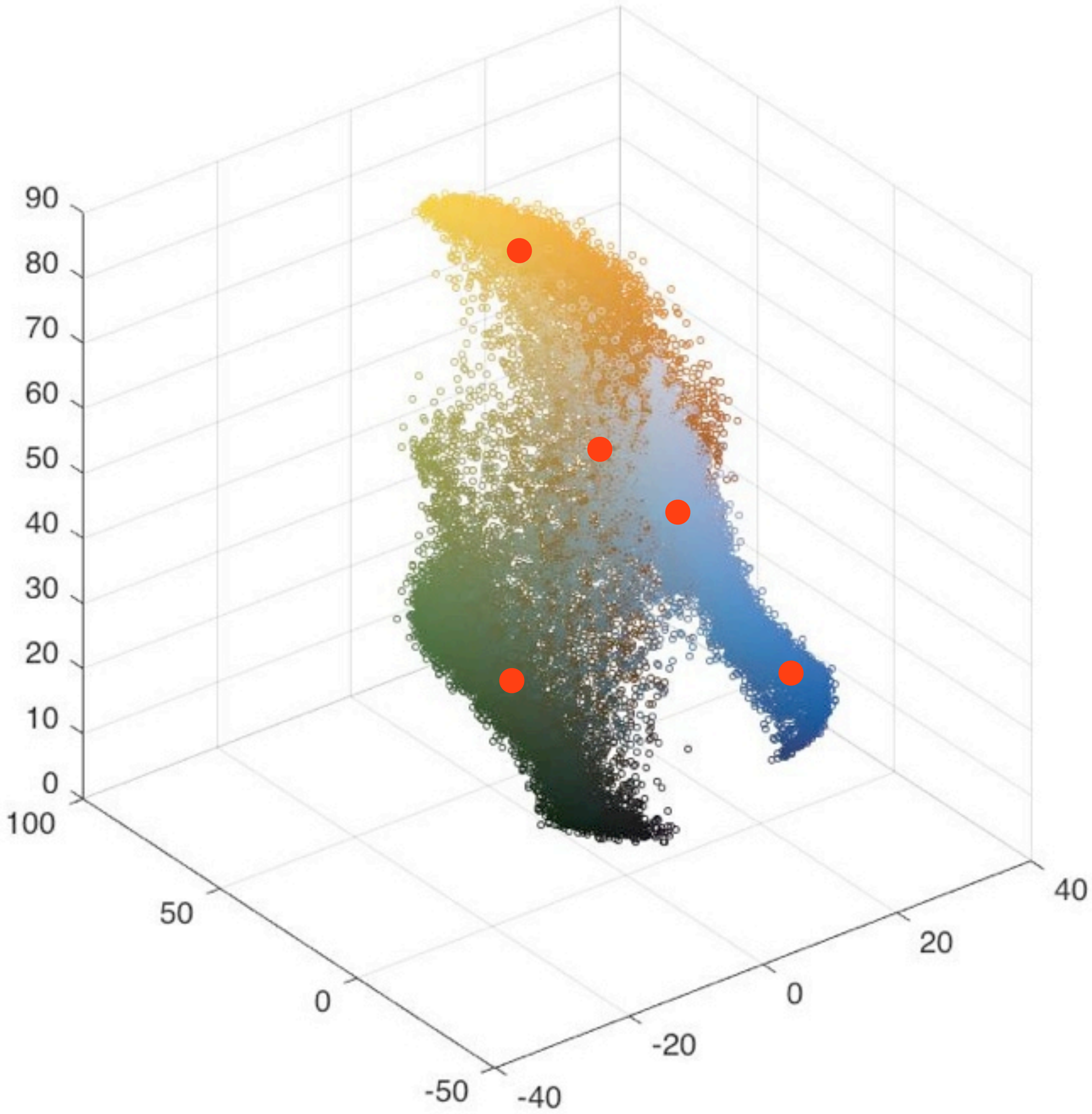

Image quantization

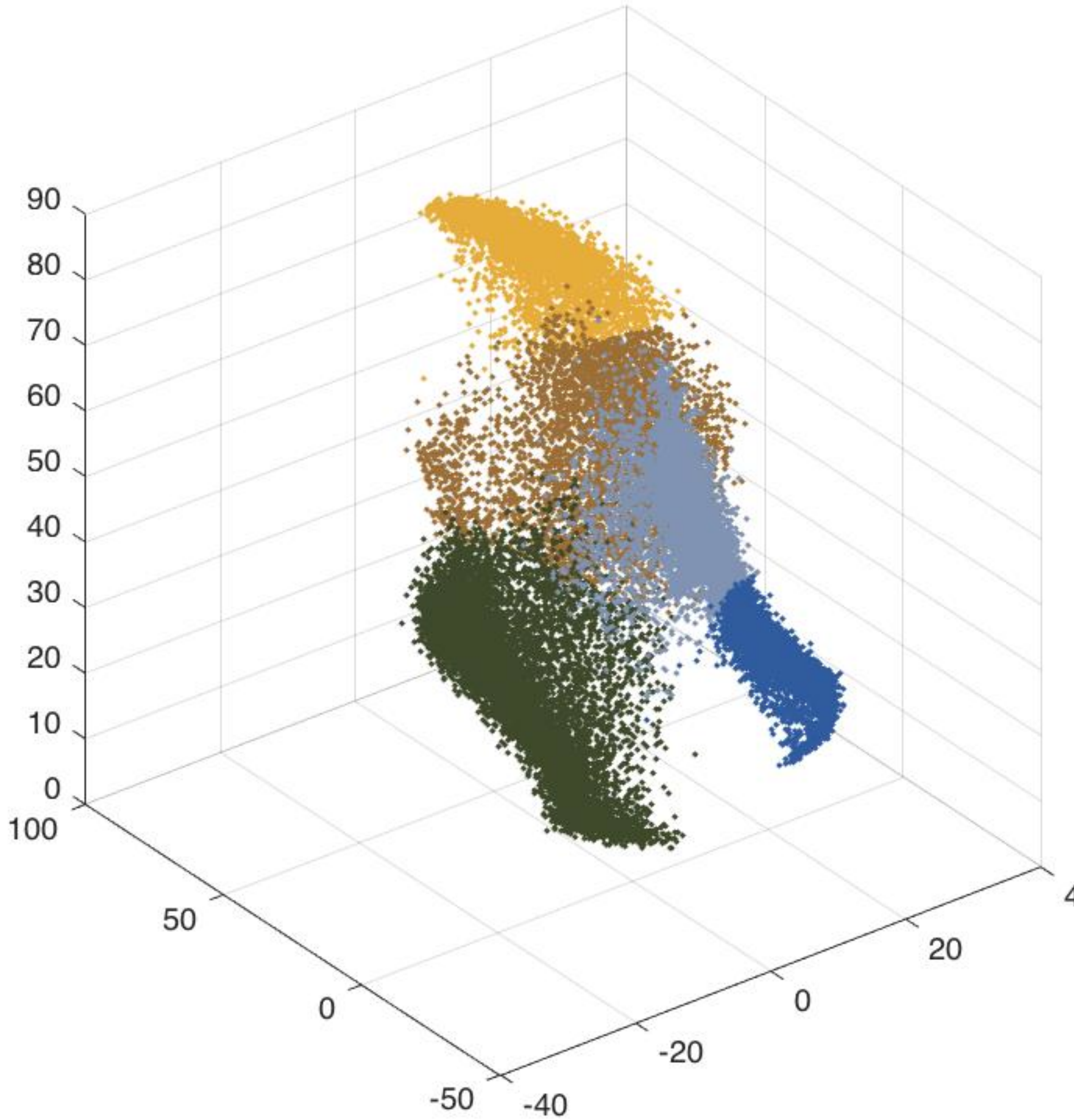

Image segmentation

Result with penalization of the region perimeter

Problem formulation

Data: signal or image $y = (y_n)_{n\in\Omega} \in (\mathbb{R}^d)^\Omega$

Problem: given y, $K \geq 2$, $\lambda \geq 0$,

$$\underset{(\Omega_k)_{k=1}^K,(c_k)_{k=1}^K}{\text{minimize}} \sum_{k=1}^K \sum_{n\in\Omega_k} \|y_n - c_k\|^2 + \lambda \sum_{k=1}^K \text{per}(\Omega_k)$$

with $\bigcup_{k=1}^K \Omega_k = \Omega$ and $\Omega_k \cap \Omega_{k'} = \emptyset$, for all $k \neq k'$.

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Called Potts Problem or piecewise-constant Mumford–Shah problem or multiclass Chan–Vese problem

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NP hard

Problem formulation

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Contribution: a globally convex relaxation

Problem formulation

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Note: centroids = region means: $c_k = \dfrac{1}{|\Omega_k|} \displaystyle\sum_{n\in\Omega_k} y_n$

Problem formulation

Data: signal or image $y = (y_n)_{n\in\Omega} \in (\mathbb{R}^d)^\Omega$

Problem: given y, $K \geq 2$, $\lambda \geq 0$,

$$\underset{(\Omega_k)_{k=1}^K,(c_k)_{k=1}^K}{\text{minimize}} \sum_{k=1}^{K} \sum_{n\in\Omega_k} \|y_n - c_k\|^2 + \lambda \sum_{k=1}^{K} \mathrm{per}(\Omega_k)$$

Note: if $\lambda = 0$, the regions are Voronoi cells:

$$\Omega_k = \left\{ n \in \Omega \;:\; k = \underset{k'}{\arg\min} \|y_n - c_{k'}\| \right\}$$

Problem formulation

Data: signal or image $y = (y_n)_{n\in\Omega} \in (\mathbb{R}^d)^\Omega$

Problem: given y, $K \geq 2$, $\lambda \geq 0$,

$$\underset{(\Omega_k)_{k=1}^K,(c_k)_{k=1}^K}{\text{minimize}} \sum_{k=1}^K \sum_{n\in\Omega_k} \|y_n - c_k\|^2 + \lambda \sum_{k=1}^K \text{per}(\Omega_k)$$

☞ for $\lambda = 0$, the **K-means algorithm** alternates between updating $(\Omega_k)_{k=1}^K$ and $(c_k)_{k=1}^K$

Problem formulation

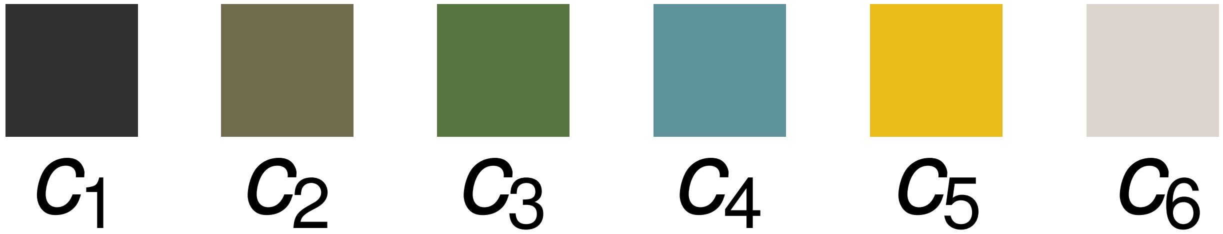
Data: signal or image $y = (y_n)_{n\in\Omega} \in (\mathbb{R}^d)^\Omega$

Problem: given y, $K \geq 2$, $\lambda \geq 0$,

$$\underset{(\Omega_k)_{k=1}^K,(c_k)_{k=1}^K}{\text{minimize}} \sum_{k=1}^{K} \sum_{n\in\Omega_k} \|y_n - c_k\|^2 + \lambda \sum_{k=1}^{K} \text{per}(\Omega_k)$$

If $\lambda > 0$ and the c_k are fixed, there are efficient methods to solve the problem, see e.g. "A convex approach to minimal partitions"

Prior work: example with fixed colors



L. C., "Discrete total variation: New definition and minimization", 2017

Problem formulation

Data: signal or image $y = (y_n)_{n\in\Omega} \in (\mathbb{R}^d)^\Omega$

Problem: given y, $K \geq 2$, $\lambda \geq 0$,

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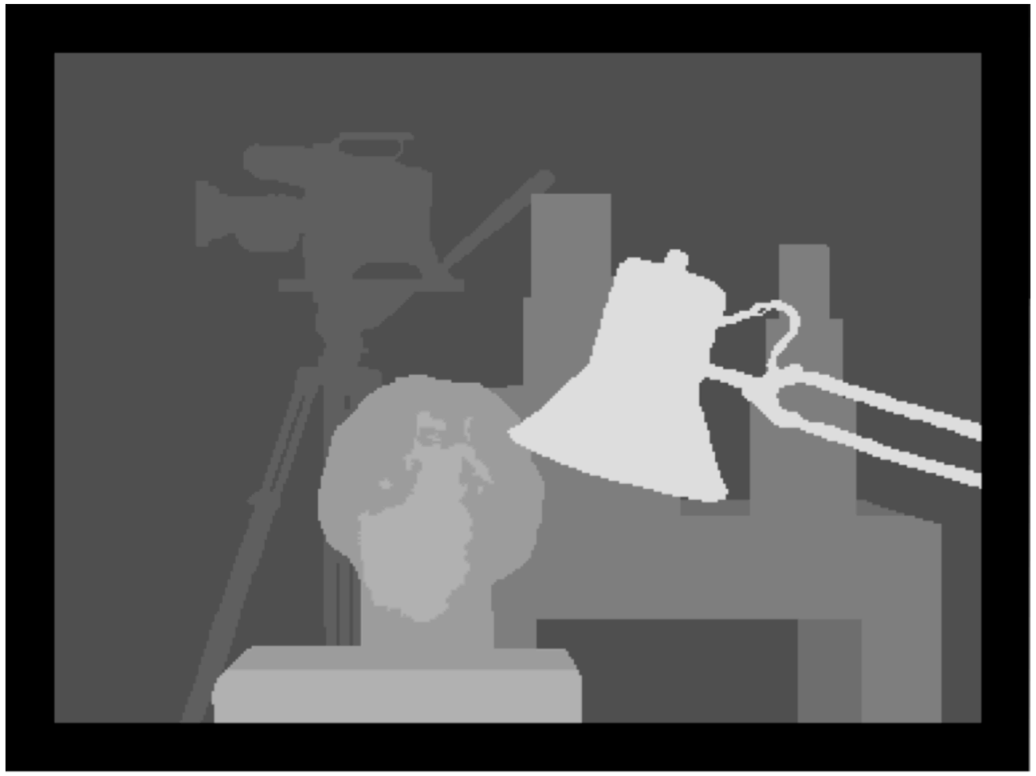
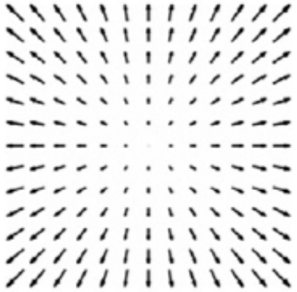
Note: $\|y_n - c_k\|^2$ can be replaced by any *assignment cost* of label c_k to location n

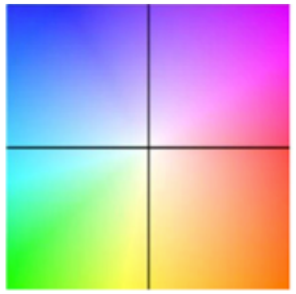
Other labeling problems



http://vision.middlebury.edu/stereo/eval/newEval/tsukuba/

Other labeling problems





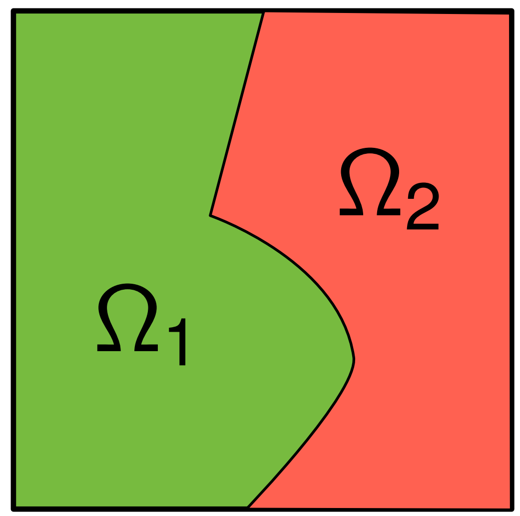
Proposed approach

We discretize the search space of the centroids: they must belong to a finite set $\Gamma = \{a_m\}_{m=1}^{M}$ of M *candidates* of $\mathbb{R}^d$.

For color image quantization and segmentation, we choose the following *palette* of $M = 279$ colors:

Reformulation by lifting

Let x be the segmented image.

Example: $x =$ 

$K = 2$

c_1

c_2

a_1

a_2

a_3

a_4

a_5

$\forall n \in \Omega$, we introduce the assignment vector $z_{:,n} \in \{0, 1\}^M$, with $z_{m,n} = \{1 \text{ if } x_n = a_m,\ 0 \text{ else}\}$.

Reformulation by lifting

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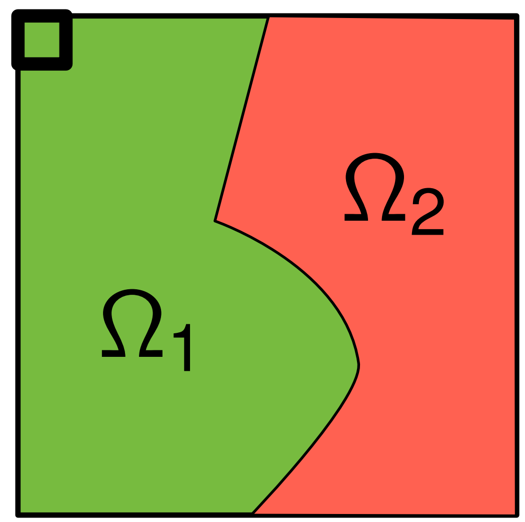
Example: $x =$



$K = 2$

c_1

c_2

a_1

a_2

a_3

a_4

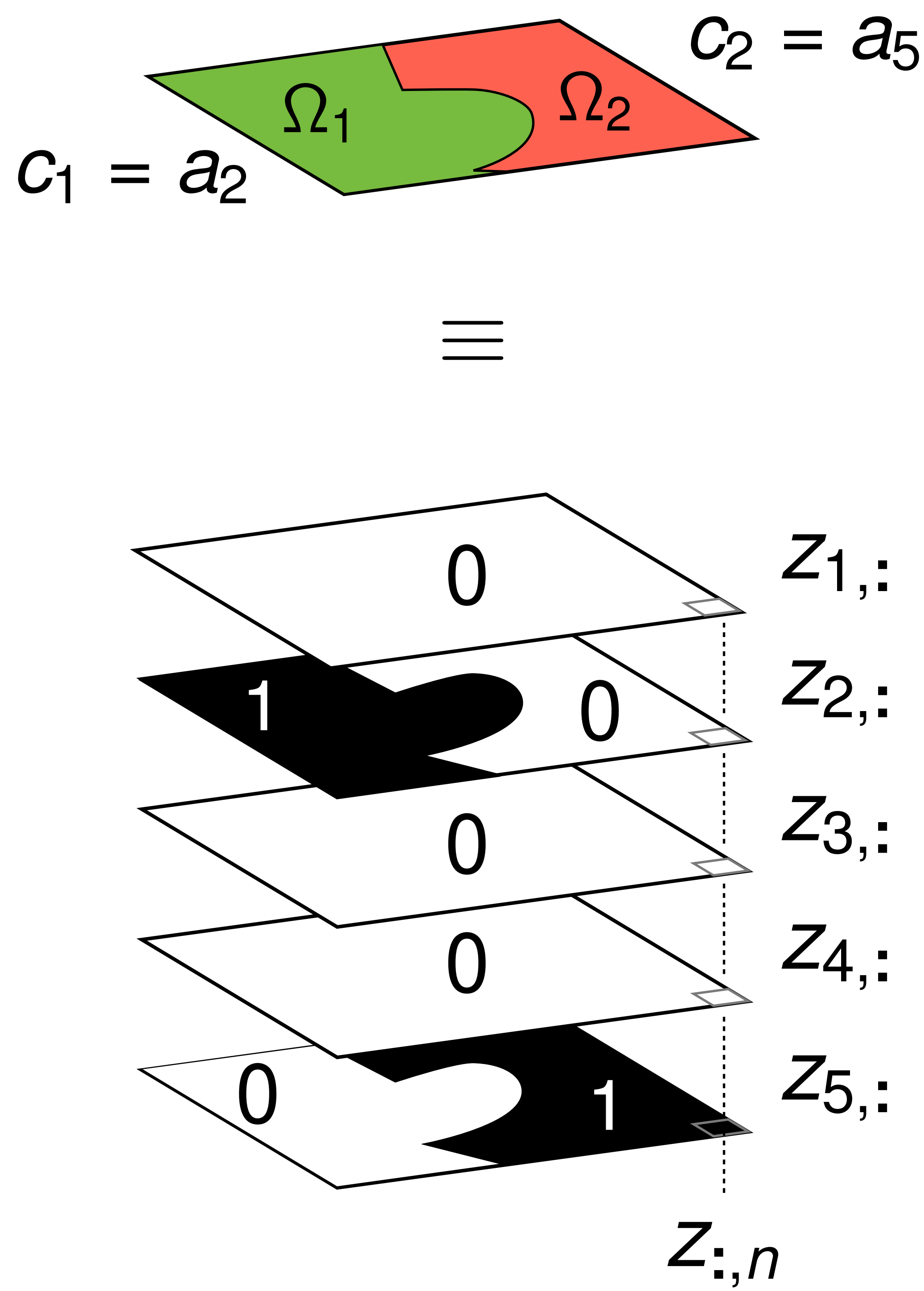
a_5

$\forall n \in \Omega$, we introduce the assignment vector $z_{:,n} \in \{0,1\}^M$, with $z_{m,n} = \{1 \text{ if } x_n = a_m,\ 0 \text{ else}\}$.

Example: $x_n = c_1 = a_2 \ \rightarrow\ z_{:,n} = (0, 1, 0, 0, 0)$

Reformulation by lifting

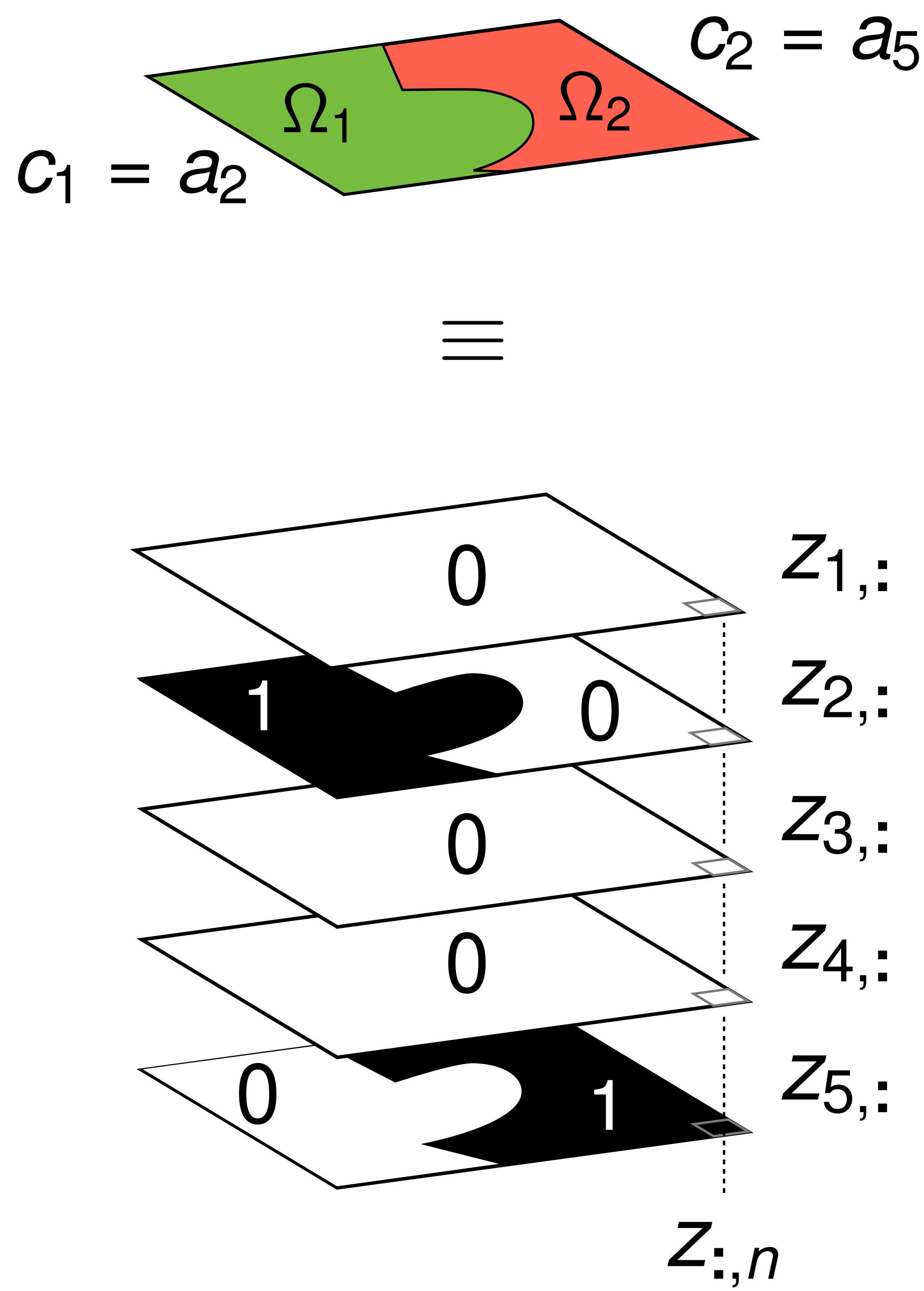
Finding x is equivalent to finding the *assignment array* z



Reformulation by lifting

We reformulate the pb. with z as variable:

$\|y - x\|^2 = \langle z, w \rangle$, with
$w_{m,n} = \|y_n - a_m\|^2$



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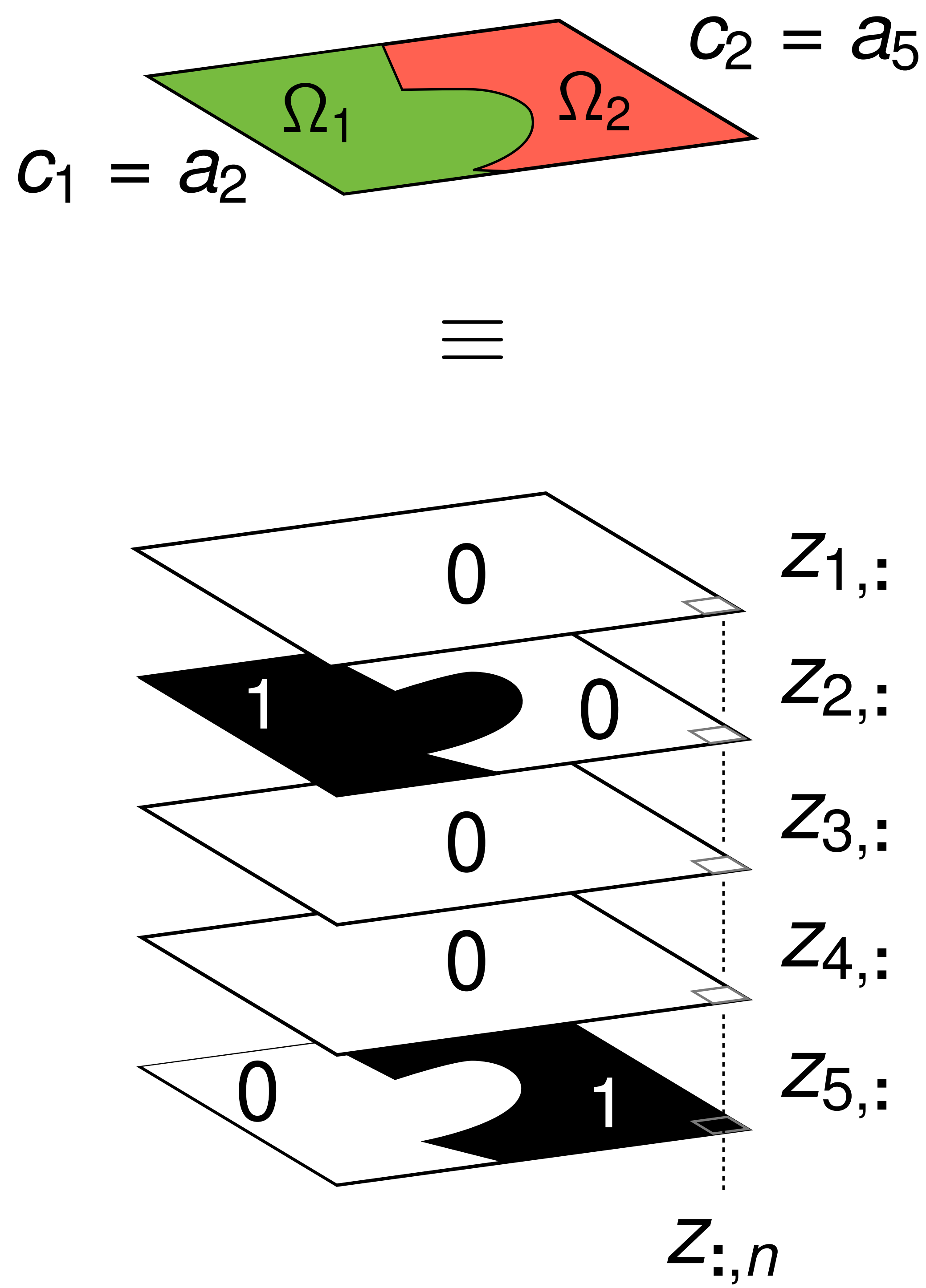
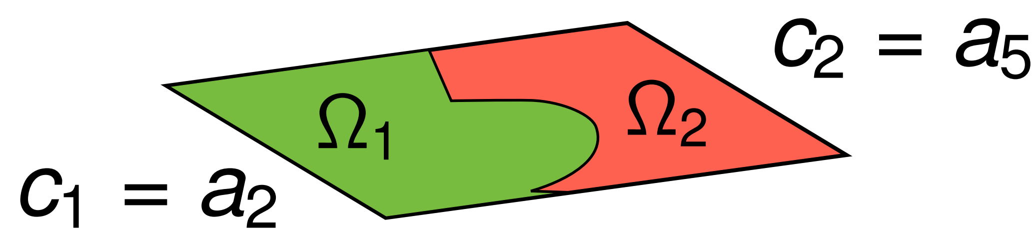
Coarea formula:
$\sum_{k=1}^{K} \mathrm{per}(\Omega_k) = \sum_{m=1}^{M} \mathrm{TV}(z_{m,:})$



proposed in Zach et al. “Fast global labeling for real-time stereo using multiple plane sweeps,” 2008

Reformulation by lifting

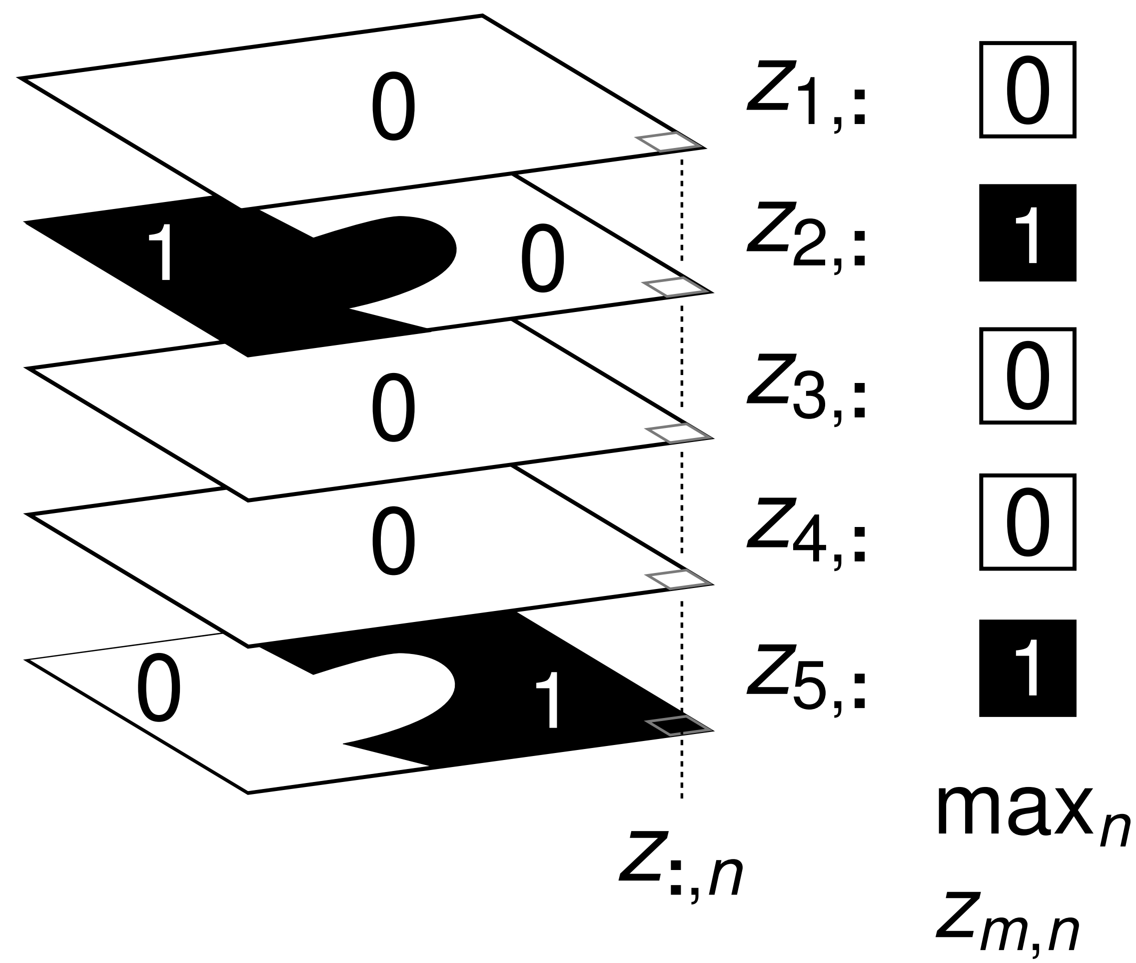
We reformulate the pb.
with z as variable:



$$\equiv$$

Nb. of active candidates is K $\quad\equiv$

$$\sum_{m=1}^{M} \max_{n\in\Omega} z_{m,n} = K$$



Reformulation by lifting

$$\underset{(\Omega_k)_{k=1}^K,(c_k)_{k=1}^K}{\text{minimize}} \sum_{k=1}^{K} \sum_{n\in\Omega_k} \|y_n - c_k\|^2 + \lambda \sum_{k=1}^{K} \text{per}(\Omega_k)$$

$$\equiv$$

$$\underset{z\in\{0,1\}^{M\times\Omega}}{\text{minimize}} \ \langle z, w\rangle + \lambda \sum_{m=1}^{M} \text{TV}(z_{m,:})$$

s.t. $\sum_{m=1}^{M} z_{m,n} = 1,\ \forall n \in \Omega$, and $\sum_{m=1}^{M} \max_{n\in\Omega}\ z_{m,n} = K$

Convex relaxation

$$\underset{z\in[0,1]^{M\times\Omega}}{\text{minimize}}\ \langle z, w\rangle + \lambda \sum_{m=1}^{M} \mathrm{TV}(z_{m,:})$$

s.t. $\sum_{m=1}^{M} z_{m,n} = 1,\ \forall n \in \Omega$, and

$\sum_{m=1}^{M} \max_{n\in\Omega}\ z_{m,n} \leq K$

Convex relaxation

$$\underset{z \in (\Delta_M)^\Omega}{\text{minimize}} \; \langle z, w \rangle + \lambda \sum_{m=1}^{M} \mathrm{TV}(z_{m,:})$$

$$\text{s.t.} \;\; \sum_{m=1}^{M} \max_{n \in \Omega} \; z_{m,n} \leq K$$

Proposed convex problem

$$\underset{z \in (\Delta_M)^\Omega}{\text{minimize}} \ \langle z, w \rangle + \lambda \sum_{m=1}^{M} \mathrm{TV}(z_{m,:})$$

$$\text{s.t.} \ \ \textstyle\sum_{m=1}^{M} \max_{n \in \Omega} \ z_{m,n} \leq K$$

Proposed convex problem

$$\underset{z\in(\Delta_M)^\Omega}{\text{minimize}}\ \langle z, w\rangle + \lambda \sum_{m=1}^{M} \mathrm{TV}(z_{m,:})$$

$$\text{s.t.}\ \ \textstyle\sum_{m=1}^{M} \max_{n\in\Omega}\ z_{m,n} \leq K$$

A vector on the simplex is a vector of *proportions*:

$z_{m,n}$ is the proportion of label a_m at n.

Proposed convex problem

$$\underset{z\in(\Delta_M)^\Omega}{\text{minimize}}\ \langle z, w\rangle + \lambda \sum_{m=1}^{M} \mathrm{TV}(z_{m,:})$$

$$\text{s.t.}\ \ \textstyle\sum_{m=1}^{M} \max_{n\in\Omega}\ z_{m,n} \leq K$$

We can recover x from z by:

$$x_n = \textstyle\sum_{m=1}^{M} z_{m,n} a_m$$

Proposed convex problem

$$\underset{z\in(\Delta_M)^\Omega}{\text{minimize}}\ \langle z, w\rangle + \lambda \sum_{m=1}^{M} \mathrm{TV}(z_{m,:})$$

$$\text{s.t.}\ \ \textstyle\sum_{m=1}^{M} \max_{n\in\Omega}\ z_{m,n} \leq K$$

When $\lambda = 0$ $\rightarrow$ LP relaxation of facility location problems known in op. research (with simplex split and different cost)

Proposed convex problem

$$\underset{z\in(\Delta_M)^\Omega}{\text{minimize}} \; \langle z, w\rangle + \lambda \sum_{m=1}^{M} \mathrm{TV}(z_{m,:})$$

$$\text{s.t.} \;\; \textstyle\sum_{m=1}^{M} \max_{n\in\Omega} \; z_{m,n} \leq K$$

Projection onto the $\ell_{1,\infty}$ ball:
code on my webpage,
average complexity $O(MN\log(N))$.

Proposed convex problem

$$\underset{z\in(\Delta_M)^\Omega}{\text{minimize}}\ \langle z, w\rangle + \lambda \sum_{m=1}^{M} \mathrm{TV}(z_{m,:})$$

$$\text{s.t.}\ \ \textstyle\sum_{m=1}^{M} \max_{n\in\Omega}\ z_{m,n} \leq K$$

☞ Chambolle–Pock algorithm

Splitting the $\ell_{1,\infty}$ constraint

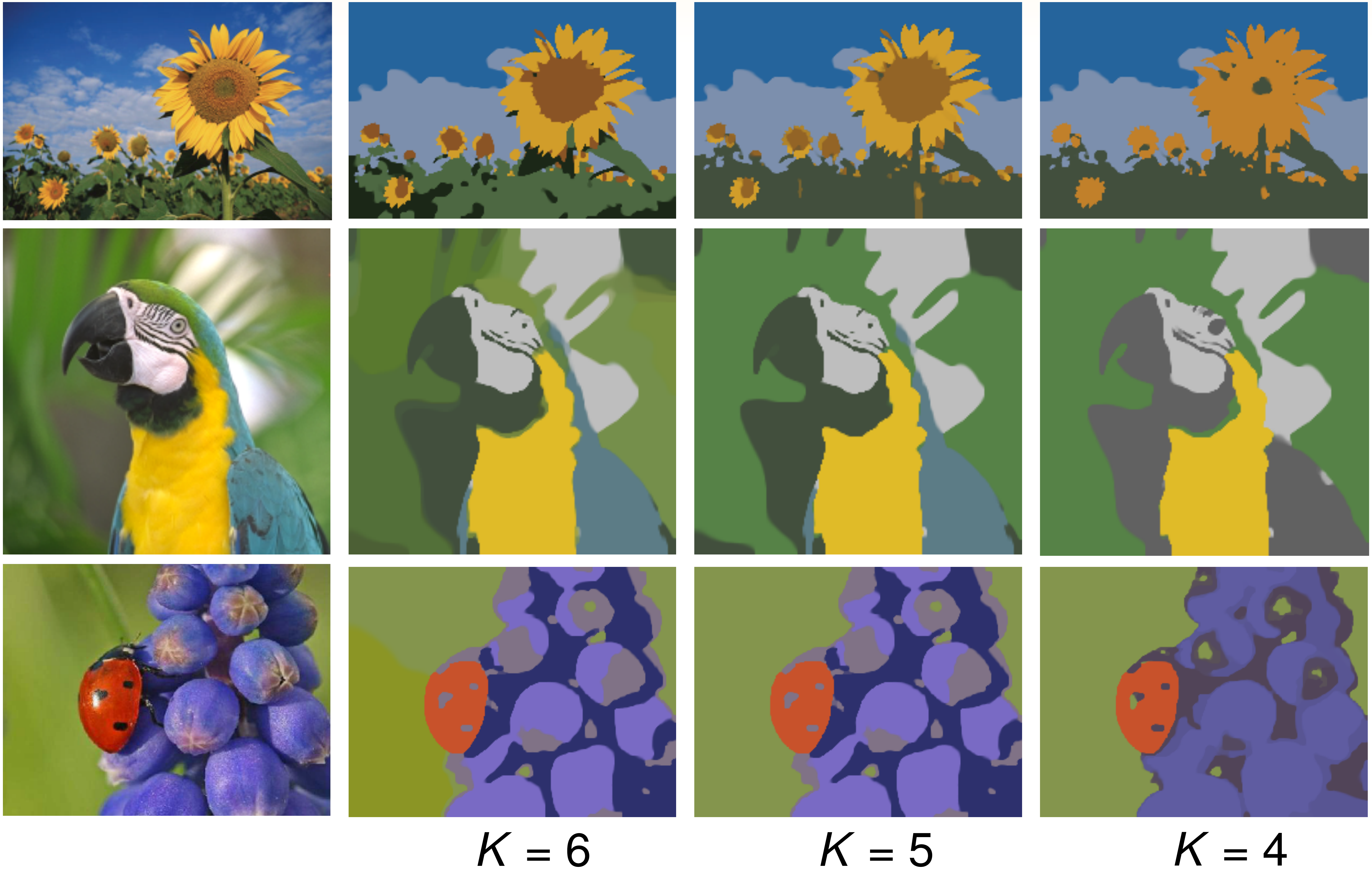
$$\underset{z\in\mathbb{R}^{M\times\Omega},q\in\mathbb{R}^M}{\text{minimize}} \ \langle z,w\rangle + \imath_{\Delta^\Omega}(z) + \lambda\sum_{m=1}^{M}\mathrm{TV}(z_{m,:}) +$$

$$\imath_{\sum_m \cdot\leq K}(q) + \imath_{\leq 0}(z - q\otimes \mathbf{1}_\Omega)$$

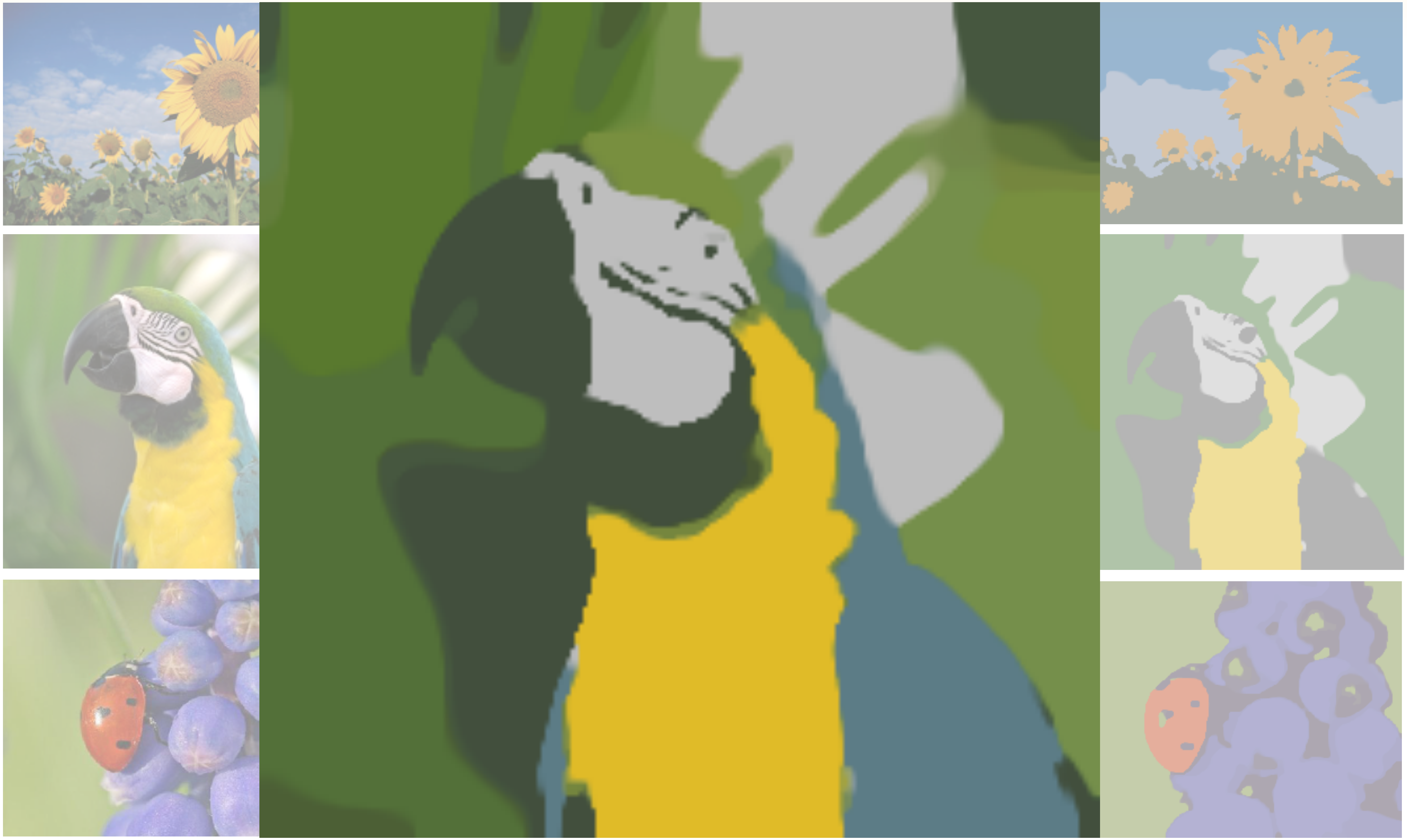
Quantization results

$K = 6$ $\quad K = 5$ $\quad K = 4$

Segmentation results



Segmentation results



$K = 6$

Summary

- Contribution: a globally convex approach to joint estimation of labels and regions
- Rounding strategies in post-processing possible
- More details in the paper "A Convex Approach to K-means Clustering and Image Segmentation", EMMCVPR, 2017.

Link with optimal transport

Let us consider a 1-D nonconvex problem

$$\underset{x \in \mathcal{M}^N}{\text{minimize}} \sum_{n=1}^{N} f_n(x_n) + \sum_{n=1}^{N-1} g(x_n, x_{n+1})$$

Link with optimal transport

Let us consider a 1-D nonconvex problem

$$\underset{x \in \mathcal{M}^N}{\text{minimize}} \sum_{n=1}^{N} f_n(x_n) + \sum_{n=1}^{N-1} g(x_n, x_{n+1})$$

Example: $f_n(x_n) = (x_n - y_n)^2$

Link with optimal transport

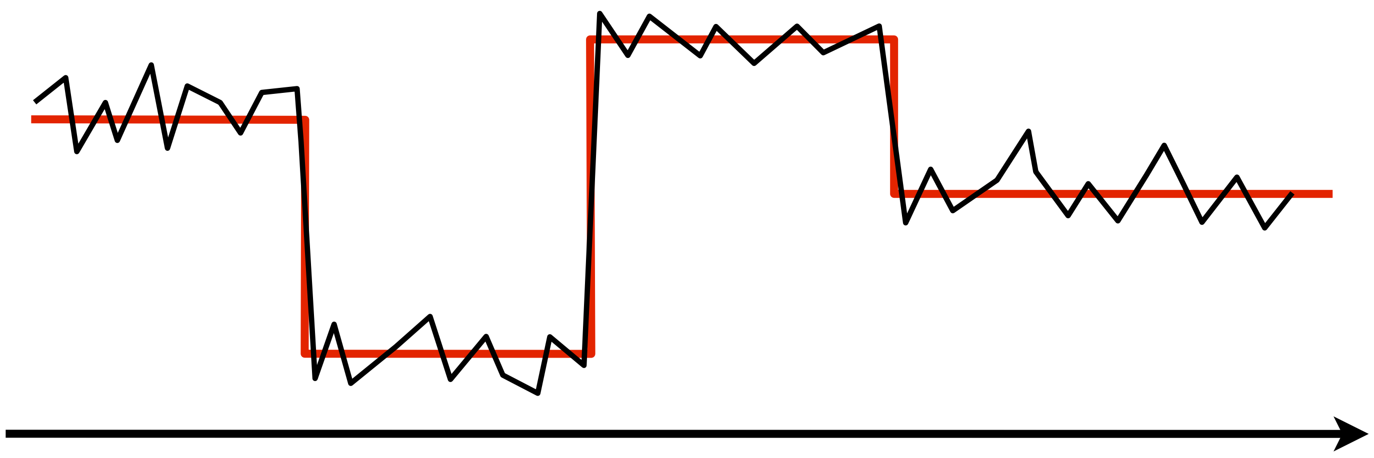
Let us consider a 1-D nonconvex problem

$$\underset{x \in \mathcal{M}^N}{\text{minimize}} \sum_{n=1}^{N} f_n(x_n) + \sum_{n=1}^{N-1} g(x_n, x_{n+1})$$

Example (Potts):

$$g(t, t') = |t - t'|_0$$

$$= \begin{cases} 0 \text{ if } x = x' \\ 1 \text{ else} \end{cases}$$



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$$\underset{x\in\Gamma^N}{\text{minimize}} \sum_{n=1}^{N} f_n(x_n) + \sum_{n=1}^{N-1} g(x_n, x_{n+1})$$

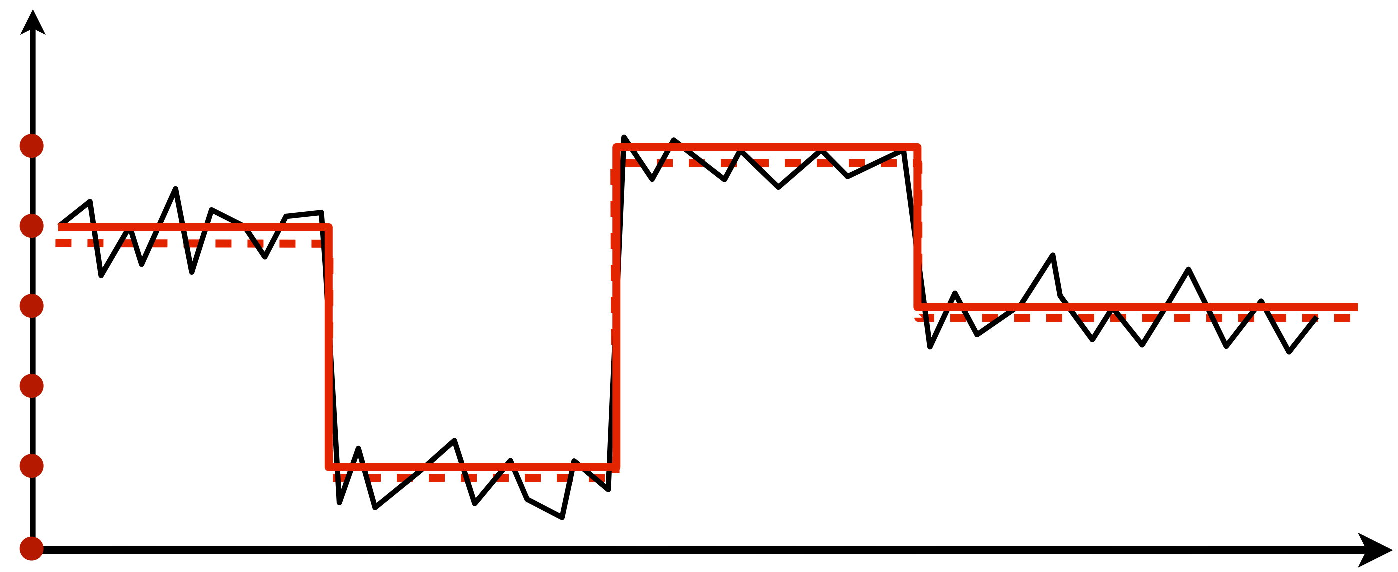
First step: discretize $\mathcal{M}$ into a finite set Γ of candidates $(a_m)_{m=1}^{M}$

Link with optimal transport

Let us consider a 1-D nonconvex problem

$$\underset{x \in \Gamma^N}{\text{minimize}} \sum_{n=1}^{N} f_n(x_n) + \sum_{n=1}^{N-1} g(x_n, x_{n+1})$$

Example:
$\mathcal{M} = [0, 1]$

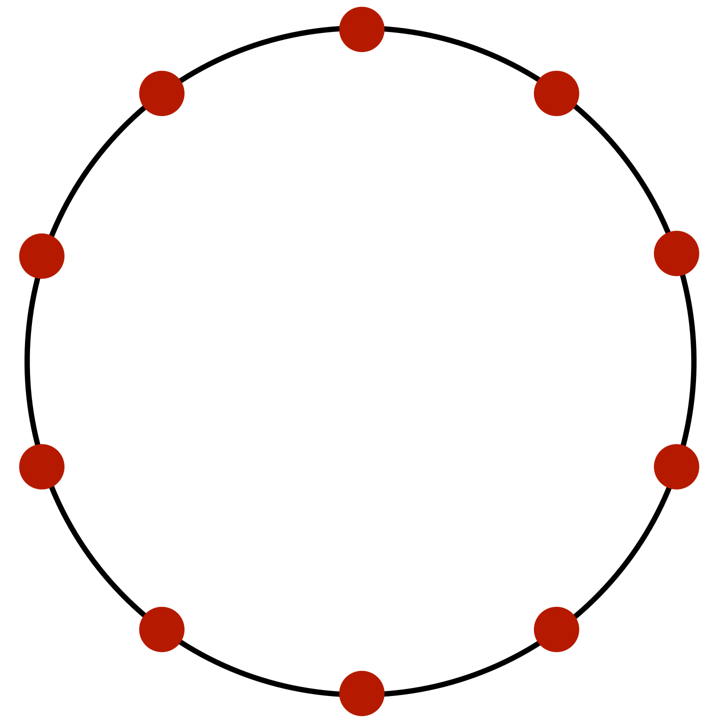


Link with optimal transport

Let us consider a 1-D nonconvex problem

$$\underset{x \in \Gamma^N}{\text{minimize}} \sum_{n=1}^{N} f_n(x_n) + \sum_{n=1}^{N-1} g(x_n, x_{n+1})$$

Example:
$\mathcal{M}$ is the circle



Link with optimal transport

Let us consider a 1-D nonconvex problem

$$\underset{x\in\Gamma^N}{\text{minimize}} \sum_{n=1}^{N} f_n(x_n) + \sum_{n=1}^{N-1} g(x_n, x_{n+1})$$

Second step: lift the problem with assignment vectors $z_{:,n}$ on the simplex, for every n.

Link with optimal transport

Let us consider a 1-D nonconvex problem

$$\underset{z\in\Delta^N}{\text{minimize}}\ \langle z, w\rangle + \sum_{n=1}^{N-1} \mathring{\tilde{g}}(z_{:,n}, z_{:,n+1})$$

with $w_{m,n} = f_n(a_m)$.

Link with optimal transport

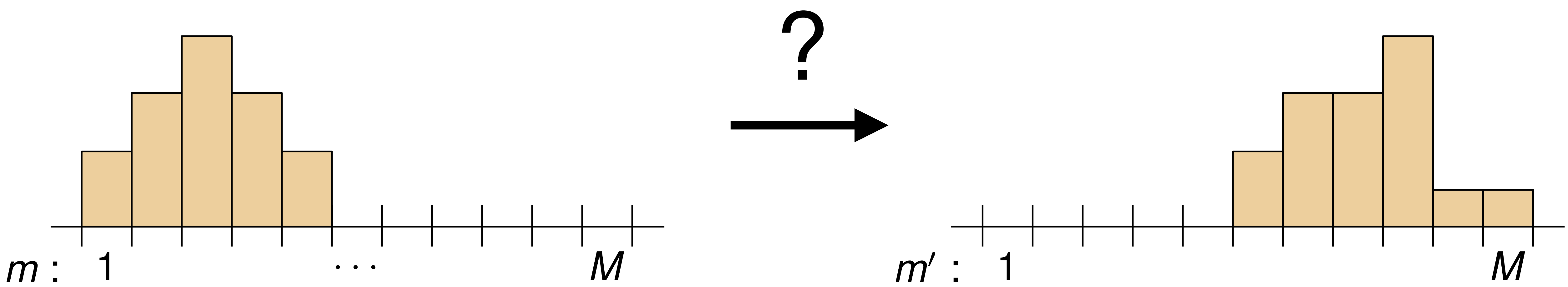
Let us consider a 1-D nonconvex problem

$$\underset{z \in \Delta^N}{\text{minimize}} \; \langle z, w \rangle + \sum_{n=1}^{N-1} \mathring{\tilde{g}}(z_{:,n}, z_{:,n+1})$$

$$\mathring{\tilde{g}} \;:\; (v, v') \in (\mathbb{R}^M)^2 \mapsto \begin{cases} \min_{p \in \Delta_{M \times M}} \langle p, c \rangle \text{ s.t. } Sp = (v, v'), \\ \quad \text{if } v \in \Delta, v' \in \Delta, \\ +\infty \text{ else.} \end{cases}$$

where $c_{m,m'} = g(a_m, a'_m)$,
and S is marginalization.

Link with optimal transport



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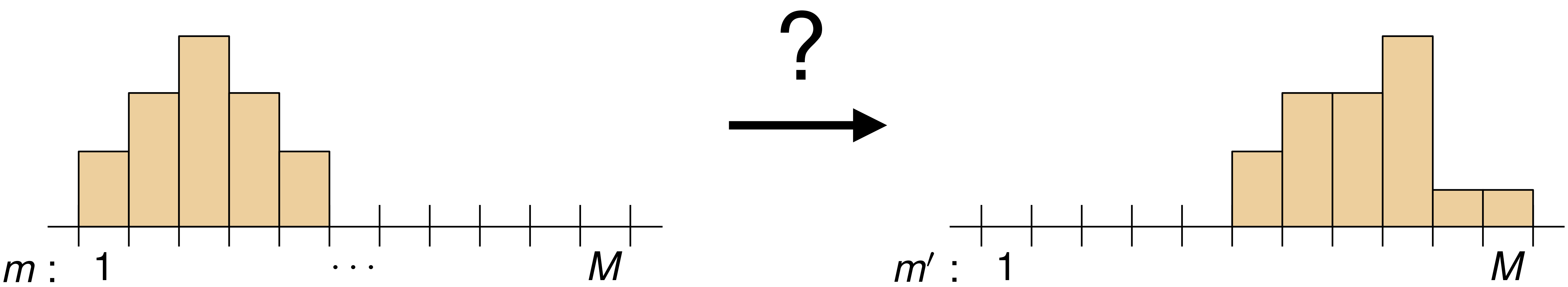
Link with optimal transport



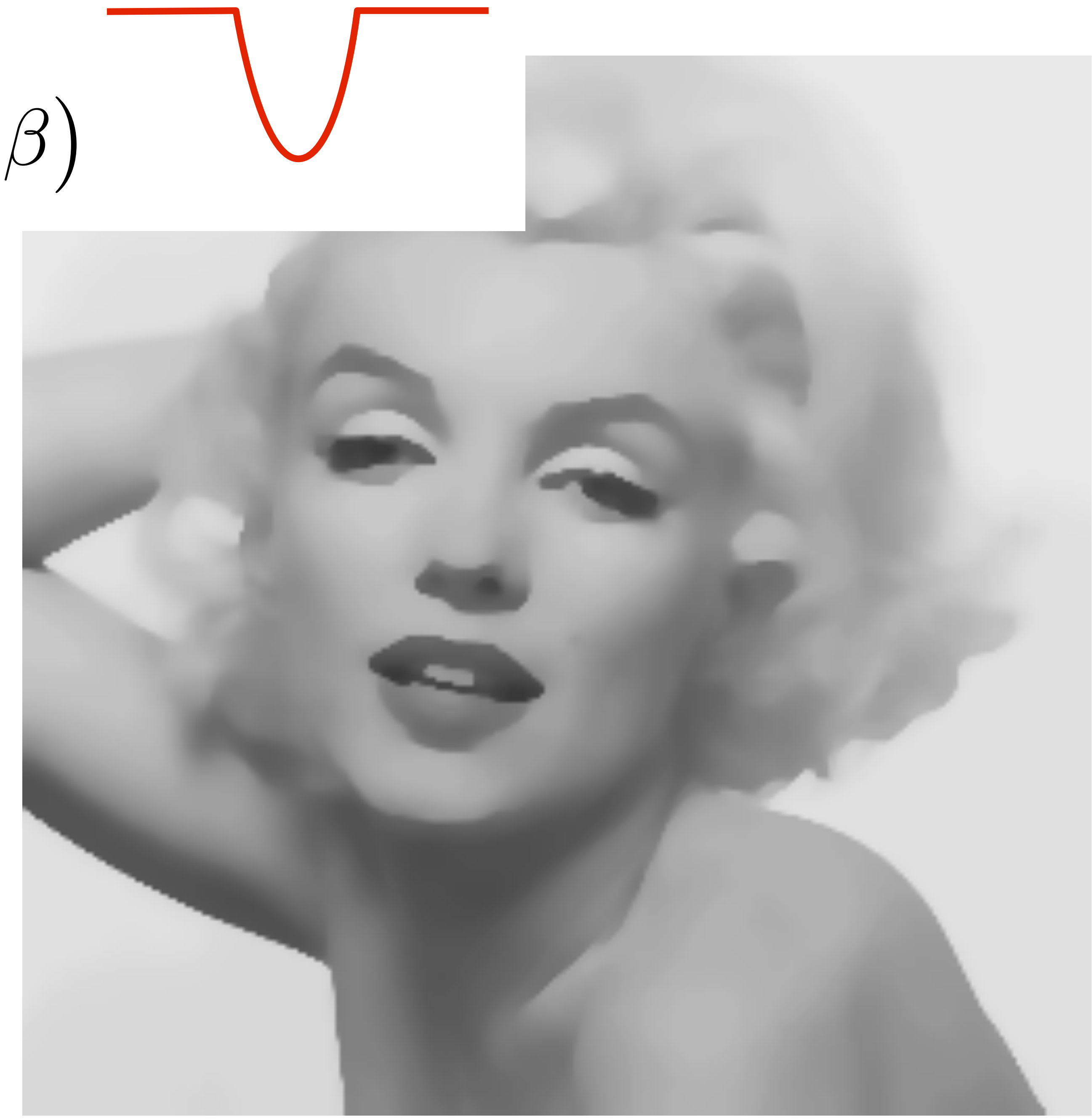
$$\mathring{\tilde{g}} \;:\; (v,v') \in (\mathbb{R}^M)^2 \mapsto \begin{cases} \min_{p\in\Delta_{M\times M}} \langle p, c\rangle \text{ s.t. } Sp = (v,v'), \\ \quad \text{if } v \in \Delta, v' \in \Delta, \\ +\infty \text{ else.} \end{cases}$$

$c_{m,m'} = |m - m'|_0$: Potts

$c_{m,m'} = |m - m'|$: 1-Wasserstein distance ...

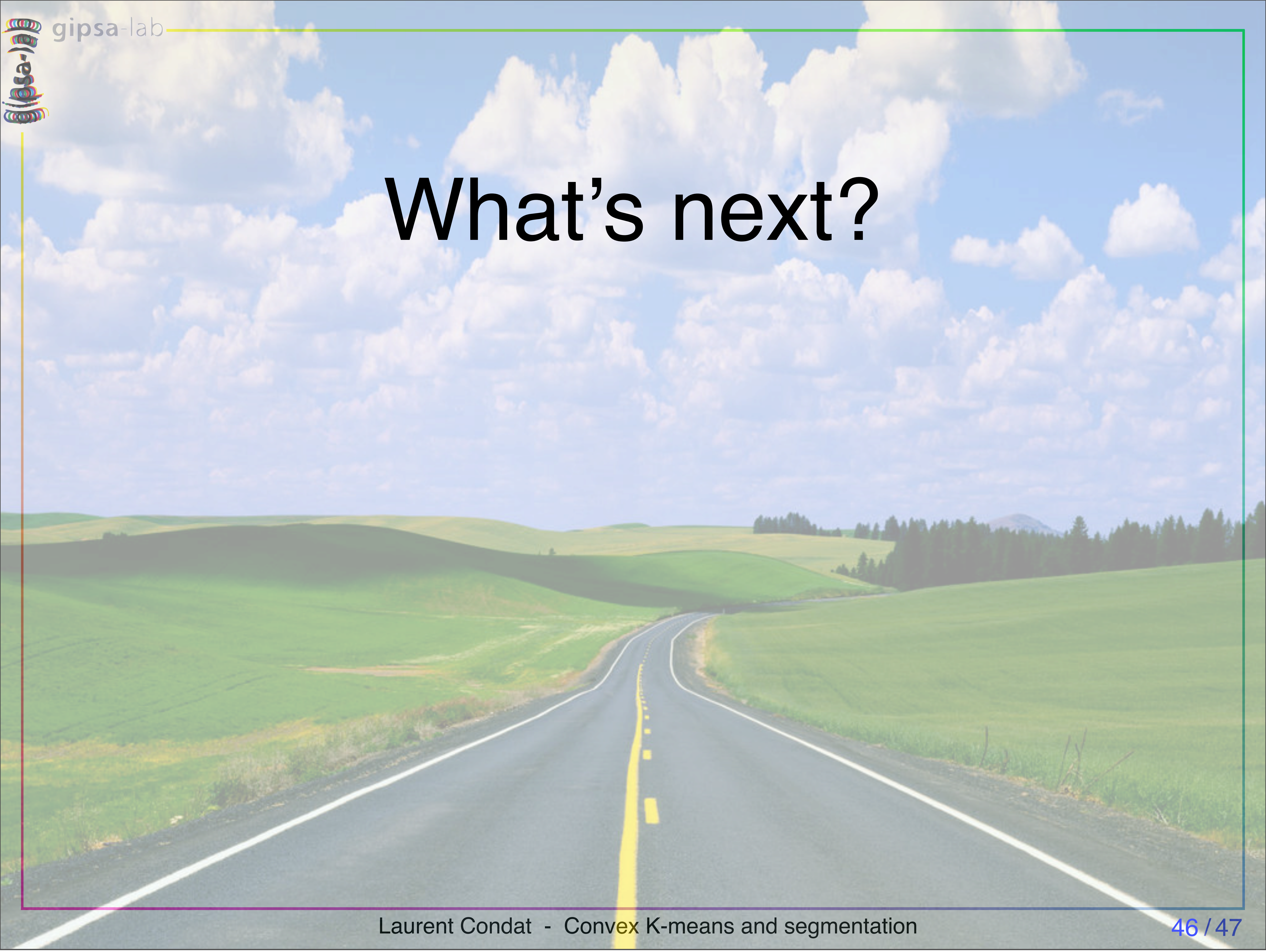
Example of application: Mumford-Shah approximation

$$g(t,t') = \min\big((t-t')^2, \beta\big)$$



From "An Algorithm for Minimizing the Mumford-Shah Functional", Pock et al., 2009

What’s next?



What's next?

Do the same without discretization of the labels...

Thank you!

Remark 1: the TV

Coarea formula:
$\sum_{k=1}^{K} \mathrm{per}(\Omega_k) = \sum_{m=1}^{M} \mathrm{TV}(z_{m,:})$ with classical ‘isotropic’ TV

tighter relaxation in Chambolle et al. "A convex approach to minimal partitions," 2012

Remark 1: the TV

isotropic TV

proposed TV, SIAM IS, 2017

Remark 2: isotonic parametrization

$$r \in \mathbb{R}^{K-1} \quad \text{s.t.} \quad 0 \leq r_1 \leq \cdots \leq r_{K-1} \leq 1$$

differentiate

$$s_k = r_k - r_{k-1}$$

integrate

$$s \in \Delta \quad \equiv \quad s_k \geq 0, \sum_{k=1}^{K} s_k = 1$$

Remark 2: isotonic parametrization

Let $h \in \Gamma_0(\mathbb{R})$. Let $K \geq 2$.

Set $f\ :\ r \in \mathbb{R}^K \mapsto \sum_{k=1}^{K} h(r_k) + \begin{cases} 0 \text{ if } r_1 \leq \cdots \leq r_K \\ +\infty \text{ else} \end{cases}$

Result : $\big(\mathrm{prox}_f(r)\big)_k = \mathrm{prox}_h(z_k)$, where

$z = \arg\min_{v} \|r - v\| \quad \text{s.t.} \quad v_1 \leq \cdots \leq v_K$

very fast pool-adjacent-violators-algorithm

Remark 2: isotonic parametrization

Let $h \in \Gamma_0(\mathbb{R})$. Let $K \geq 2$.

Set $f \ : \ r \in \mathbb{R}^K \mapsto \sum_{k=1}^{K} h(r_k) + \begin{cases} 0 \text{ if } r_1 \leq \cdots \leq r_K \\ +\infty \text{ else} \end{cases}$

Result : $\big(\mathrm{prox}_f(r)\big)_k = \mathrm{prox}_h(z_k)$, where

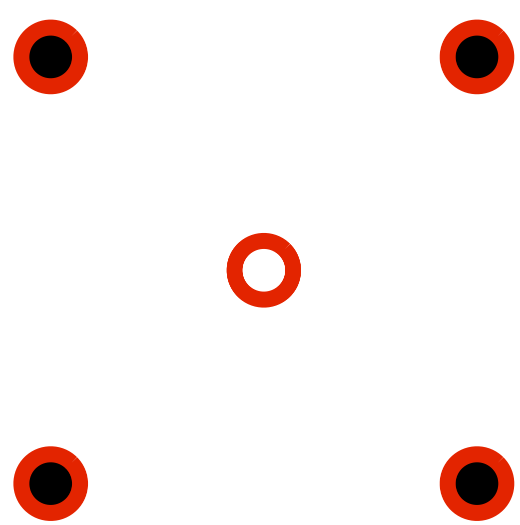
$$z = \arg\min_v \|r - v\| \quad \text{s.t.} \quad v_1 \leq \cdots \leq v_K$$

extended in Pustelnik and Condat, "Proximity Operator of a Sum of Functions; Application to Depth Map Estimation", IEEE SPL 2017

A counter-example

- $N = 4$ data points y_n
- $M = 5$ candidates a_m

$K = 3$



$$z = \begin{bmatrix} 2/3 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$