Profinite groups with abelian Sylow subgroups

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Abstract

We extend the definition of a finite $\mathcal{A}$-group to profinite groups and give a description of profinite $\mathcal{A}$-groups as a triple semidirect product of two prosoluble groups with a semisimple group, extending an old result of A. M. Broshi (see [2]) to the profinite case. We also prove that a profinite $\mathcal{A}$-group with finitely generated non-trivial Fitting subgroup is metabelian-by-(finite exponent). If, in addition, G is finitely generated then it is virtually metabelian polycyclic.

Keywords: profinite groups, Sylow subgroups, abelian, finite rank.

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1 Introduction

Finite soluble groups have been extensively studied in the past century. The works of P. Hall are an important source of information about these groups; in 1940, P. Hall published a paper entitled "The construction of soluble groups" (see [7]) where he mentioned the term $\mathcal{A}$-group, i.e., a finite group whose Sylow subgroups are abelian. In the next years some other mathematicians worked on properties of finite $\mathcal{A}$-groups (see [19] and [2] for example). Interest in $\mathcal{A}$-groups also broadened due to an important

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relationship to varieties of groups proved by Ol'shanskiĭ ([13]): all the groups of a variety are residually finite if and only if it is generated by a finite $\mathcal{A}$-group. Venkataraman ([20]) gave a bound on the number of isomorphism classes of soluble $\mathcal{A}$-groups of order n.

The objective of this paper is the study of profinite $\mathcal{A}$-groups, i.e., profinite groups with abelian Sylow subgroups. The first result is a profinite version of the main result in [2].

Theorem 1.1. *Let G be a profinite $\mathcal{A}$-group. There are closed subgroups H, S and K of G satisfying:*

(i) $G = HSK$,

(ii) $H \trianglelefteq G$, $K \leqslant N_G(S)$,

(iii) *H, K are prosoluble and S is a direct product of finite simple groups isomorphic to either* $\mathrm{PSL}_2(q)$ *with* $q > 3$, $q \equiv 0, 3, 5 \pmod 8$ *or* J_{11}.

In Theorem 1.1, $\mathrm{PSL}_2(q)$ is the projective special linear group and J_{11} is the Janko group (see [10]).

Profinite groups have features of finite groups (like Sylow theory) and at the same time features of infinite groups (like free objects, combinatorial theory, etc). In particular, a profinite group can be torsion-free.

Corollary 1.2. *A profinite torsion-free $\mathcal{A}$-group is prosoluble.*

Next we describe profinite $\mathcal{A}$-groups with finitely generated Fitting subgroup.

Theorem 1.3. *Let G be a profinite $\mathcal{A}$-group whose Fitting subgroup is non-trivial and finitely generated. Then G is metabelian-by-(finite exponent)*[1]*. If, in addition, G is finitely generated, then G is polycyclic.*

Corollary 1.4. *Let G be a profinite $\mathcal{A}$-group of finite (Prüfer) rank. Then G is virtually polycyclic metabelian.*

The strategy to prove Theorem 1.1 is to use projective limit arguments. We construct some inverse systems to use the result of Broshi for finite $\mathcal{A}$-groups. To prove Theorem 1.3 we use Dixon's ([3]) precise estimate for the bound $\beta(n)$ depending only on n of the index of the Fitting subgroup of a completely

[1]Here we say that a group is $\mathcal{X}$-by-$\mathcal{Y}$ if there is a normal subgroup N of G such that N is $\mathcal{X}$ and G/N is $\mathcal{Y}$. Also, we say that a group G is virtually $\mathcal{X}$ if there is a finite index subgroup N of G such that N is $\mathcal{X}$.

reducible soluble linear group G of degree n.

The structure of the paper is as follows. In section 2 we prove basic properties of profinite $\mathcal{A}$-groups that can be obtained by the projective limit argument using known results for finite $\mathcal{A}$-groups. Among other results, we prove that the Fitting subgroup of a profinite $\mathcal{A}$-group is the direct product of the centers of the derived series and that a commutator of every pair of p-elements is a p'-element.

In the Section 3 we prove Theorem 1.1 and Section 4 is dedicated to the proof of Theorem 1.3.

All groups in this paper are profinite, the subgroups are closed, the homomorphisms are continuous and the groups are generated in the topological sense. We will restate the theorems in the corresponding sections for the convenience of the reader.

2 The basic definitions and properties

In this section we will extend the definition of $\mathcal{A}$-groups from finite to profinite groups and prove results that can be obtained from known results for finite $\mathcal{A}$-groups using a projective limit argument.

Definition 2.1. We say that a profinite group G is a *profinite $\mathcal{A}$-group* if all Sylow subgroups of G are abelian.

If $G = \varprojlim G_i$ is a profinite $\mathcal{A}$-group decomposed as an inverse limit of finite quotient groups G_i, then each G_i is a finite $\mathcal{A}$-group. Conversely, suppose that each G_i is a finite $\mathcal{A}$-group. Then all Sylow subgroups of G_i are abelian and therefore so are the Sylow subgroups of G (cf. [14, Lemma 2.3.4]). Thus, we can state the following characterization:

Proposition 2.2. *A profinite group G is an $\mathcal{A}$-group if, and only if, it is an inverse limit of finite $\mathcal{A}$-groups.*

Example 2.3. (i) Choose your favorite finite non-abelian $\mathcal{A}$-group H (for instance, S_3 or A_4). Define an infinite direct product $G = \prod H$ of its copies. Then G is a non-abelian infinite profinite $\mathcal{A}$-group.

(ii) Given an infinite sequence of odd primes $p_1, p_2, \ldots$, we denote by $D_{p_i} = C_{p_i} \rtimes C_2$ the dihedral group of order $2p_i$. Then $G = \prod_i D_{p_i}$ is a non-abelian infinite profinite $\mathcal{A}$-group.

The following properties of a profinite $\mathcal{A}$-group can be deduced from Definition 2.1.

Proposition 2.4. *(i) If H is a subgroup of a profinite $\mathcal{A}$-group G, then H is a profinite $\mathcal{A}$-group.*

(ii) If N is a normal subgroup of a profinite $\mathcal{A}$-group G, then G/N is a profinite $\mathcal{A}$-group.

(iii) If $\{G_i\}_{i\in I}$ is a family of profinite $\mathcal{A}$-groups, then $\prod_i G_i$ is a profinite $\mathcal{A}$-group.

Remark. Proposition 2.4 tells us that profinite $\mathcal{A}$-groups form a variety and so one can define a free object in this variety, a free profinite $\mathcal{A}$-group ([14, Chapter 3]).

Suppose that $G = \varprojlim G_i$ is a pronilpotent group. We can write G as the product of its Sylow subgroups (see [14, Proposition 2.3.8]). So, we have the following:

Proposition 2.5. *A pronilpotent group G is a profinite $\mathcal{A}$-group if, and only if, it is abelian.*

The Fitting subgroup $\mathrm{Fit}(G)$ of a profinite group G can be defined as the inverse limit of Fitting subgroups of its finite quotients. It is the unique maximal pronilpotent subgroup of G. Thus we can deduce the following:

Corollary 2.6. *The Fitting subgroup of a profinite $\mathcal{A}$-group is abelian.*

As usual by $Z(G)$ we shall denote the center of G and by $G' = [G, G]$ the (topological) commutator subgroup of it.

Proposition 2.7. *Let G be a profinite $\mathcal{A}$-group. Then*

$$Z(G) \cap G' = 1.$$

Proof. Let $G = \varprojlim_i G_i$ be a decomposition of G as an inverse limit of finite soluble quotient groups. Then

$$Z(G) = \varprojlim Z(G_i)$$

and

$$G' = \varprojlim G_i'.$$

Let $\pi_i : G \longrightarrow G_i$ be the natural epimorphism. Note that $\pi_i(Z(G)) \leqslant Z(G_i)$ and $\pi_i(G') \leqslant G_i'$. By [2, Corollary 4.5], $Z(G_i) \cap G_i' = 1$ for every i, hence, $\pi_i(Z(G)) \cap \pi_i(G') = 1$ for every i. Therefore, $Z(G) \cap G' = 1$. □

Proposition 2.8. *Let G be a prosoluble $\mathcal{A}$-group. Then $\mathrm{Fit}(G)$ is the direct product of the centers of the terms in the derived series of G.*

Proof. Let $G = \varprojlim G_i$ be a decomposition as an inverse limit of finite soluble quotient $\mathcal{A}$-groups of G (see [14, Corollary 4.2.4]) and $\pi_i : G \longrightarrow G_i$ the natural epimorphism. Denoting by $G^{(n)}$ the n-th term of the derived series of G (in the topological sense), we know that

$$G^{(n)} = \varprojlim \pi_i(G^{(n)}) = \varprojlim G_i^{(n)}$$

and

$$Z(G^{(n)}) = \varprojlim \pi_i(Z(G^{(n)})) = \varprojlim Z(G_i^{(n)}).$$

Then

$$\begin{aligned}\prod_{n=0}^{\infty} Z(G^{(n)}) &= \prod_{n=0}^{\infty} \varprojlim Z(G_i^{(n)}) \\ &\overset{(*)}{=} \varprojlim \prod_{n=0}^{\infty} Z(G_i^{(n)}) \\ &\overset{(**)}{=} \varprojlim \mathrm{Fit}(G_i) \\ &= \mathrm{Fit}(G),\end{aligned}$$

where $(*)$ follows by [14, Exercise 1.1.14] and $(**)$ by [19, Theorem 5.4]. □

Definition 2.9. Let G be a profinite group and π a set of primes. A *π-Hall system* is a collection $\{G_p : p \in \pi\}$ of Sylow subgroups of G so that $G_pG_q = G_qG_p$ for any $p, q \in \pi$. A *system of normalizer* is a non-trivial subgroup K of G such that $kG_pk^{-1} = G_p$ for every $k \in K$ and G_p in the π-Hall system.

If G is a prosoluble group then a π-Hall system always exists (see [14, Proposition 2.3.9] and [14, Proposition 2.3.10]) for $\pi = \pi(G)$ (the set of primes with nonzero exponent in the order of G) as well as a system of normalizer. It is shown in [17] that a system of normalizer K of G can be obtained as

$$K = \bigcap_p N_G(G_p)$$

where $N_G(G_p)$ is the normalizer of G_p in G. Moreover, K is pronilpotent (see [15, 9.2.4]). Thus from Proposition 2.5 we can deduce the following:

Proposition 2.10. *Any system normalizer K of a prosoluble $\mathcal{A}$-group G is abelian.*

An element of a Sylow p-subgroup will be called a *p-element* and an element of order coprime to p will be called a *p'-element*. The next proposition shows that if G is a prosoluble $\mathcal{A}$-group, then the commutator of every two p-elements of G is a p'-element and in fact this property characterizes prosoluble profinite $\mathcal{A}$-groups.

Proposition 2.11. *Let G be a profinite group. Then the following conditions are equivalent:*

(i) G is prosoluble with all Sylow subgroups abelian,

(ii) for each prime p the commutator of any two p-elements is a p'-element.

Proof. Suppose that G is a prosoluble group with all Sylow subgroups abelian and write

$$G = \varprojlim G_i$$

in a decomposition of G as an inverse limit of finite soluble groups. By [1], each G_i satisfies the following: each commutator of two p-elements is a p'-element. If $g = (g_i)$ and $h = (h_i)$ are elements of G, then

$$[g, h] = ([g_i, h_i]),$$

then (ii) holds.

Conversely, suppose that each commutator of any two p-elements is a p'-element, for every prime p. Since

$$[g, h] = ([g_i, h_i])$$

for $g = (g_i)$ and $h = (h_i)$, the property (ii) holds for any G_i. By [1], each G_i is a finite soluble $\mathcal{A}$-group so that G is a prosoluble $\mathcal{A}$-group. □

3 A decomposition theorem

In this section we will prove a structure theorem on profinite $\mathcal{A}$-groups. The result is a generalization of a Broshi's theorem for finite $\mathcal{A}$-groups.

We say that a group is *semisimple* if it is a direct product of non-abelian finite simple groups.

Theorem. *Let G be a profinite $\mathcal{A}$-group. There are subgroups H, S and K of G satisfying:*

(i) $G = HSK$,

(ii) $H \trianglelefteq G$, $K \leqslant N_G(S)$,

(iii) *H, K are prosoluble and S is a direct product of finite simple groups of the following type: either* $\mathrm{PSL}_2(q)$ *with* $q > 3$, $q \equiv 0, 3, 5 \pmod 8$ *or* J_{11}.

Proof of Theorem 1.1. Let

$$G = \varprojlim G_i$$

with each G_i a finite group relative to the inverse system (G_i, π_{ij}). Consider $\mathcal{T}_i$ the set of all triples (H_i, S_i, K_i) satisfying

(i) $G_i = H_i S_i K_i$,

(ii) $H_i \trianglelefteq G_i$, $K_i \leqslant N_{G_i}(S_i)$,

(iii) H_i, K_i are soluble and S_i is semisimple.

Since G is a profinite $\mathcal{A}$-group, each G_i is a finite $\mathcal{A}$-group. Thus, by [2], $\mathcal{T}_i \neq \varnothing$ for every i. Define the map $\tilde{\pi}_{ij} : \mathcal{T}_j \to \mathcal{T}_i$ by

$$\tilde{\pi}_{ij}((H_j, S_j, K_j)) = (\pi_{ij}(H_j), \pi_{ij}(S_j), \pi_{ij}(K_j)).$$

To see that it is well-defined note that if $(H_j, S_j, K_j) \in \mathcal{T}_j$, then

(i) $\pi_{ij}(H_j)\pi_{ij}(S_j)\pi_{ij}(K_j) = G_i$,

(ii) $\pi_{ij}(H_j) \trianglelefteq G_i$, $\pi_{ij}(K_j) \leqslant N_{G_i}(\pi_{ij}(S_j))$,

(iii) $\pi_{ij}(H_j), \pi_{ij}(K_j)$ are soluble and $\pi_{ij}(S_j)$ is semisimple.

Hence indeed, a triple of $\mathcal{T}_j$ is mapped to a triple in $\mathcal{T}_i$. Thus, $(\mathcal{T}_i, \tilde{\pi}_{ij})$ is an inverse system of non-empty finite sets. Then

$$\varprojlim \mathcal{T}_i \neq \varnothing.$$

Choosing

$$((H_i, S_i, K_i)) \in \varprojlim \mathcal{T}_i,$$

the structure of the inverse limit give us

$$(\pi_{ij}(H_j), \pi_{ij}(S_j), \pi_{ij}(K_j)) = \tilde{\pi}_{ij}((H_j, S_j, K_j)) = (H_i, S_i, K_i).$$

Therefore, we have $((H_i, S_i, K_i), \tilde{\pi}_{ij})$ and, consequently, inverse systems (H_i, π_{ij}), (S_i, π_{ij}), (K_i, π_{ij}) having the following inverse limits

$$H = \varprojlim H_i,$$

$$K = \varprojlim K_i,$$

$$S = \varprojlim S_i.$$

By construction, H and K are prosoluble. To see that S is semisimple, just observe that the normal subgroups of S_i are products of some factors of S_i and therefore any epimorphic image of S_i is a direct product of finite non-abelian simple groups. It is easy to see that $H \trianglelefteq G$, $K \leqslant N_G(S)$ and $G = HSK$.

Finally, the fact that simple subgroups of S are of the claimed form is the subject of [2, Theorem 3.2]. □

An immediate consequence of this theorem is the Corollary 1.2 stated in the introduction:

Corollary. *If G is torsion-free, then G is prosoluble.*

Corollary 3.1. *If within the hypotheses of Theorem 1.1 P is a Sylow 2-subgroup of S, then $SK = SN_{SK}(P)$ and $N_G(P)$ is prosoluble.*

Proof. If $S = 1$, then its done. Put $L = SK$. Since S is normal in L, by the Frattini Argument (see [14, Exercise 2.3.13]) we have $L = SN_L(P)$. Let Q be a complement of P in $N_S(P)$, which exists by Schur-Zassenhaus theorem (see [14, Theorem 2.3.15]). Note that Q is a $2'$-group and so is prosoluble in view of the Feit-Thompson theorem (see [6]). Also,

$$N_L(P)/N_S(P) = N_L(P)/(N_L(P) \cap S) \simeq K/(K \cap S),$$

so that $N_L(P)/N_S(P)$ is prosoluble, hence, $N_L(P)$ is prosoluble.

Note that $H \cap S = 1$ because, as S is semi-simple, S does not contain any proper normal prosoluble subgroup. Hence $HS = H \rtimes S$ and so $N_{HS}(P) = N_H(P) \rtimes N_S(P)$ is prosoluble. But HS is normal in G and K is prosoluble. Therefore $N_G(P)$ is an extension of $N_{HS}(P)$ by a prosuluble group and so is prosoluble. □

In the next proposition and the rest of the paper we follow the notation

$$G^e = \langle g^e \mid g \in G \rangle$$

where e is a fixed natural number. If a set of numbers is bounded by some natural depending only on e, we will say that this set is *e-bounded*

Proposition 3.2. *Let e be a positive integer and G a profinite $\mathcal{A}$-group such that for every prosoluble subgroup R of G the subgroup R^e is abelian. Then q in the item (iii) of Theorem 1.1 is e-bounded. It follows that the exponent of S is e-bounded.*

Proof. Observe that the order of the group T of upper triangular matrices in $SL_2(q)$ is $q(q-1)$, and T has a maximal abelian normal subgroup U such that T/U is cyclic of order $q-1$. Hence q can not exceed e. □

The following lemma is the prosoluble version of [18, Lemma 3.6] (cf. also [14, Lemma 2.8.15]):

Lemma 3.3. *Let G be a profinite group and N a normal subgroup of G. Let T/N be a prosoluble subgroup of G/N. Then there exists a prosoluble subgroup $U \leqslant G$ such that $UN = T$.*

We finish the section showing that a profinite $\mathcal{A}$-group G is abelian-by-(finite exponent) if this is the case for all prosoluble subgroups of G.

Proposition 3.4. *Let e be a positive integer and G a profinite $\mathcal{A}$-group such that for every prosoluble subgroup $R \leqslant G$ the subgroup R^e is abelian. Then there exists e_1 depending only on e such that G^{e_1} is abelian.*

Proof. By Theorem 1.1, $G = HSK$, where H and K are prosoluble and S is semisimple. Moreover H is normal and K normalizes S. In view of Proposition 3.2, the exponent of S is e-bounded.

Assume first that G has no non-trivial prosoluble normal subgroups. Then $G = SK$ and we can assume $S \neq 1$. Let P be a Sylow 2-subgroup of S. By Corollary 3.1 we can write $G = SN_G(P)$ and $L = N_G(P)$ is prosoluble. Therefore L^e is abelian. It follows that PL^e is abelian since G is an $\mathcal{A}$-group. Write

$$S = \prod_i S_i,$$

where S_i are the simple factors of S and set $P_i = P \cap S_i$. The group G acts on the set $\{S_1, S_2, \dots\}$ by conjugation and so G permutes the simple factors of S. Since L^e centralizes the subgroups P_i, it follows that L^e normalizes each factor S_i. We know from Proposition 3.2 that the order of each S_i is e-bounded and so we deduce that there is a number e_1 depending on e such that L^{e_1} centralizes S. Clearly, $C_G(S)$ is prosoluble and since we assume that G has no non-trivial prosoluble normal subgroups, $L^{e_1} = 1$. Therefore if G has no non-trivial prosoluble normal subgroups, then the exponent of G is e-bounded.

Now consider the general case and let R be the largest normal prosoluble subgroup of G. In view of Lemma 3.3 each prosoluble subgroup of G/R is a quotient of some prosoluble subgroup of G an so the hypotheses of the proposition are inherited by G/R. Since G/R has no non-trivial prosoluble normal subgroups, by the preceding paragraph the exponent of G/R is bounded only in terms of e. On the other hand, R^e is abelian by hypothesis. The result follows. $\square$

4 Profinite $\mathcal{A}$-groups with finitely generated Fitting subgroup

In this section we study profinite $\mathcal{A}$-groups with non-trivial finitely generated Fitting subgroup. First we show that a prosoluble subgroup of $G/\operatorname{Fit}(G)$ is abelian-by-(finite exponent), then we prove some lemmas that allow us to prove Theorem 1.3.

Theorem 4.1. *Let G be a profinite $\mathcal{A}$-group having n-generated $\operatorname{Fit}(G) \neq 1$. Then there exists a number $e = e(n)$ depending on n only such that for any prosoluble subgroup H of $G/\operatorname{Fit}(G)$ the subgroup H^e is abelian.*

Proof. Recall that $\operatorname{Fit}(G)$ is the unique maximal abelian normal subgroup of G by Corollary 2.6. Write

$$\operatorname{Fit}(G) = \prod_i P_i$$

where P_i are the Sylow subgroups of $\operatorname{Fit}(G)$ and set $G_i = G/C_G(P_i)$. For a p-Sylow subgroup P_i we can view $V_i = P_i/\Phi(P_i)$ as a vector space over $\mathbb{F}_p$, the finite field of p elements. Since G is an $\mathcal{A}$-group, G_i is a pro-p_i' group. Since the natural action of G_i on P_i is faithful, so is the action of G_i on V_i (see [17, Lemma 2.3]). Thus, the induced representation $\rho : G_i \to \operatorname{GL}(V_i)$ is injective and completely reducible by Maschke's theorem (see [5, Chapter 18, Corollary 2]). Also, since $\mathbb{F}_p$ embeds in its algebraic closure $\overline{\mathbb{F}}_p$, we can consider G_i as embedded in $\operatorname{GL}(V_i \otimes \overline{\mathbb{F}}_p)$.

Recall that $\operatorname{Fit}(G)$ can be generated by n elements. It follows that V_i is a n_i-dimensional vector space with $n_i \leqslant n$.

Now let H_i be the image of H in G_i. By [3, Theorem 1], the index satisfies

$$[H_i : \operatorname{Fit}(H_i)] \leqslant \frac{(2\sqrt[3]{3^2})^{n_i}}{2\sqrt[3]{3}}.$$

Let K_i be the preimage of $\operatorname{Fit}(H_i)$ in H. Consider $K = \bigcap_i K_i$. Then H/K embeds in $\prod_i H/K_i$. Put

$$\beta(n) = \frac{(2\sqrt[3]{3^2})^{n}}{2\sqrt[3]{3}}$$

and $\bar{n} = \lfloor \beta(n) + 1 \rfloor$, the integral part of $\beta(n) + 1$. Observe that

$$\exp(H/K) \leqslant \exp\left(\prod_i H/K_i\right) = \operatorname{lmc}_i(\exp(H/K_i)),$$

hence $\exp(H/K) \leqslant \bar{n}!$. Finally, since $K_i' \leqslant C_G(P_i)$ for all i one has

$$K' \subset \bigcap_i K_i' \subset \bigcap_i C_G(P_i) = \operatorname{Fit}(G),$$

where the last equality follows from the fact that $\mathrm{Fit}(G)$ is maximal normal abelian and so is self-normalized. This means that $K/\mathrm{Fit}(G)$ is abelian and so putting $e(n) = \bar{n}!$ we have the result. □

Remark. Keeping the notation of the proof for $\bar{n}$, it follows from the proof that any prosoluble subgroup $H \leqslant G$ possesses a metabelian normal subgroup K such that

$$\exp(H/K) \leqslant \mathrm{lmc}_i\left(\frac{(2\sqrt[3]{3^2})^{n_i}}{2\sqrt[3]{3}}\right) \leqslant \min(3^{\bar{n}}, \bar{n}!),$$

where one used [8, Theorem 1] stating that $\mathrm{lmc}(1, \ldots, k) \leqslant 3^k$.

Corollary 4.2. *Any d-generated prosoluble subgroup L of a profinite $\mathcal{A}$-group G with n-generated $\mathrm{Fit}(G) \neq 1$ is polycyclic and possesses an open (in L) metabelian subgroup K of index bounded in terms of $\beta(n)$ and d.*

Proof. Put $H = L\,\mathrm{Fit}(G)/\mathrm{Fit}(G)$ and

$$\beta(n) = \frac{(2\sqrt[3]{3^2})^n}{2\sqrt[3]{3}}.$$

We use the notation of the proof of Theorem 4.1. Recall that the index $[H : K_i]$ is bounded by $\beta(n)$ and $\bar{n} = \lfloor \beta(n) + 1 \rfloor$ is the integral part of $\beta(n) + 1$. It is shown in [12, Corollary 1.1.2] that the number $s(\bar{n})$ of subgroups with index at most $\bar{n}$ satisfies

$$s(\bar{n}) \leqslant \sum_{k=1}^{\bar{n}} k(k!)^{d-1}.$$

So, in this case we have the explicit bound

$$|H : K| \leqslant \bar{n}^{s(\bar{n})}$$

in terms of n and d only. It remains to note that $H = L\,\mathrm{Fit}(G)/\mathrm{Fit}(G)$ is a prosoluble finitely generated virtually abelian group and, since $\mathrm{Fit}(G)$ is a finitely generated abelian group, then H is polycyclic. □

We are ready to prove Theorem 1.3.

Theorem. *Let G be a profinite $\mathcal{A}$-group having n-generated $\mathrm{Fit}(G) \neq 1$. Then there exists $e = e(n)$ depending on n only such that G^e is metabelian. If, in addition, G is finitely generated then G^e is polycyclic.*

Proof of Theorem 1.3. By Theorem 4.1, there exists number $e = e(n)$ depending on n only such that for any prosoluble subgroup H of $\overline{G} = G/\operatorname{Fit}(G)$ the subgroup H^e is abelian. Then by Proposition 3.4 there exists e_1 depending only on n such that $\overline{G}^{e_1}$ is abelian. Since $\operatorname{Fit}(G)$ is abelian (by Corollary 2.6) we have the first statement.

If G is finitely generated then G/G^{e_1} is finite. Indeed, it follows that a finitely generated profinite $\mathcal{A}$-group of finite exponent is finite (in fact by the celebrated Zelmanov's result [22, Theorem 1] combined with a reduction of the question to pro-p groups due to W. Herfort [9] and J. Wilson [21] any finitely generated compact group of finite exponent is finite. Hence G^{e_1} is prosoluble and finitely generated (see [14, Proposition 2.5.5]). Thus G^{e_1} is polycyclic by Corollary 4.2. □

Recall that the (Prüfer) *rank* r of a profinite group is defined to be

$$r = \sup\{d(H) \mid H \leqslant_c G\}$$

where $d(H)$ denotes the minimal cardinality of a generating set for H.

Corollary. *Let G be a profinite $\mathcal{A}$-group of finite (Prüfer) rank. Then G is virtually polycyclic metabelian.*

Proof. By [16, Lemma 4.4] G is virtually prosoluble. Hence by [16, Lemma 4.3] $\operatorname{Fit}(G) \neq 1$ and is finitely generated by hypothesis. Thus the result follows from Theorem 1.3. □

References

[1] Brandl, R. (2010). Groups with Abelian Sylow subgroups. *Journal of the Australian Mathematical Society*, 88(1):43–47. doi:10.1017/S1446788709000184.

[2] Broshi, A. M. (1971). Finite groups whose Sylow subgroups are abelian. *Journal of Algebra*, 17(1):74–82. doi:10.1016/0021-8693(71)90044-5.

[3] Dixon, J. D. (1967). The Fitting subgroup of a linear solvable group. *Journal of the Australian Mathematical Society*, 7(4):417–424. doi:10.1017/S1446788700004353.

[4] Dixon, J. P. (1971). *The structure of linear groups*. London: Van Nostrand Reinhold Company

[5] Dummit, D. S., Foote, R. M. (2004). *Abstract algebra*. Hoboken: Wiley Hoboken.

[6] Feit, W., Thompson, J. G. (1963). *Solvability of groups of odd order*. Pacific Journal of Mathematics. doi:pjm/1103053942.

[7] Hall, P. (1940). The construction of soluble groups. *Journal für die reine und angewandte Mathematik*,1940(182): 206-214. doi:10.1515/crll.1940.182.206.

[8] Hanson, D. (1972). On the product of the primes. *Canadian Mathematical Bulletin*, 15(1):33–37. doi:10.4153/CMB-1972-007-7.

[9] Herfort, W. N. (1979) *Compact torsion groups and finite exponent*. *Archiv der Mathematik*, 33: 404-410. doi:10.1007/BF01222776.

[10] Janko, Z. (1966). A new finite simple group with abelian Sylow 2-subgroups and its characterization. *Journal of Algebra*, 3(2):147–186. doi:10.1016/0021-8693(66)90010-X.

[11] Kargapolov, M. I., Merzljakov, J. I. (1979). *Fundamentals of the Theory of Groups*. New York: Springer-Verlag

[12] Lubotzky, A., Segal, D. (2003). *Subgroup growth*. Basel: Birkhäuser. doi:10.1007/978-3-0348-8965-0.

[13] Ol'shanskii, A. (1969). Varieties of finitely approximable groups. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 33(4):915–927. doi:10.1070/IM1969v003n04ABEH000807.

[14] Ribes, L., Zalesskii, P. (2010); *Profinite Groups*. Berlin: Springer-Verlag. doi:10.1007/978-3-662-04097-3.

[15] Robinson, D. J. S. (1995). *A course in the Theory of Groups*. New York: Springer-Verlag.

[16] Shumyatsky, P. (2022) On profinite groups in which centralizers have bounded rank. *Communications in Contemporary Mathematics*. doi:10.1142/S0219199722500559.

[17] Shumyatsky, P., Zalesskii, P. (2021). Profinite groups in which centralizers are virtually procyclic. *Journal of Algebra*, 586:467–478. doi:10.1016/j.jalgebra.2021.07.008.

[18] Shumyatsky, P., Zalesskii, P., Zapata, T. (2019). Profinite groups in which centralizers are abelian. *Israel Journal of Mathematics*, 230(2):831–854. doi:10.1007/s11856-019-1838-x.

[19] Taunt, D. R. (1949). On A-groups. *Mathematical proceedings of the cambridge philosophical society*, 45(1):24–42. doi:10.1017/S0305004100000414.

[20] Venkataraman, G. (1997). Enumeration of finite soluble groups with abelian sylow subgroups. *Quarterly Journal of Mathematics*, 48(189):107–126. doi:10.1093/qmath/48.1.107.

[21] Wilson, J. S. (1983) On the structure of compact torsion groups. *Monatshefte für Mathematik*, 96: 57–66. doi:10.1007/BF01298934.

[22] Zelmanov, E. I. (1992). On periodic compact groups. *Israel Journal of Mathematics*, 77(1):83–95. doi:10.1007/BF02808012.

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