Let $\Phi\colon (\mathbb{C}^*)^2 \to \mathbb{C}^3$ be given by $\Phi(s,t) = (s^3, st, t^3)$ and let $Y = \mathbb{V}(xz - y^3)$ be the closure of the image of Φ (as in part of Problem (1.1.6)).

Recall that the action of $T = (\mathbb{C}^*)^2$ on Y is induced by the restriction of Φ to $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^3$ and the action of $(\mathbb{C}^*)^3$ on $\mathbb{C}^3$. Write M for the character lattice of T and R for the coordinate ring $\mathbb{C}[x,y,z]/\langle xz - y^3\rangle$ of Y. Briefly justify all of your answers.

(1) Describe the action of a torus element (s,t) on a point (x,y,z) satisfying $xz - y^3 = 0$ and verify that Y is closed under this action.

(2) Describe the action of T on R. (This action exists because elements of R correspond to functions on Y.)

(3) Which points (x,y,z) in Y are in the torus?

(4) How does R include into the coordinate ring $\mathbb{C}[M]$ of T?

(5) We proved in class that R is spanned by functions f where T acts by characters, meaning that $t \cdot f = \chi^m(t) \cdot f$ for all $t \in T$. Find a specific example of f and m.

(6) Show that the image of R is $\mathbb{C}[\sigma^\vee \cap M]$ for σ the cone spanned by $(3,-2)$ and $(0,1)$ in $\mathbb{R}^2$, which most of you saw in Problem (1.2.13).