Here we will construct the Hirzebruch surfaces, whose fans have rays $(1,0),(0,1),(-1,a),(0,-1)$ for $a \in \mathbb{N}$.

(1) Calculate the affine toric variety of each 2-dimensional cone.
(2) Write ring maps for each of the inclusions of coordinate rings implied by the structure of the fan.
(3) For $a = 0$ identify the points of the abstract toric variety X with the points $(a : b, c : d)$ of $\mathbb{P}^1 \times \mathbb{P}^1$.
(4) Draw a fan for $\mathbb{P}^1$.
(5) For general a define a toric map $\varphi\colon X \to \mathbb{P}^1$.
(6) Given a point $(a : b)$ of $\mathbb{P}^1$, show that the preimage of $(a : b)$ under φ is also $\mathbb{P}^1$, at least on closed points. [hint: Start with the case $a = 0$.]