Section 1.0 of the textbook describes the *product* of two affine varieties $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ as a special case of a more general construction. For now we will take $\operatorname{Spec}(R \otimes_{\mathbb{C}} S)$ as the definition of $X \times Y$.

Suppose $R = \mathbb{C}[x_1, \ldots, x_n]/\mathfrak{p}$ where $\mathfrak{p}$ corresponds to the point $(a_1, \ldots, a_n)$ in $\mathbb{C}^n$ and $S = \mathbb{C}[y_1, \ldots, y_m]/\mathfrak{q}$ where $\mathfrak{q}$ corresponds to the point $(b_1, \ldots, b_m)$ in $\mathbb{C}^m$.

(1) Construct a map $X \times Y \to \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m])$.

(2) What is the image of your map from (1) geometrically?

(3) Which ideal in $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ defines the image of $X \times Y$? How does this ideal relate to $\mathfrak{p}$ and $\mathfrak{q}$?

(4) Describe a point in $\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ not of the form you described in (3).

(5) How might you imagine your point from (4) geometrically in $\mathbb{C}^{n+m}$?

(6) Define $f\colon \mathbb{C}^n \to \mathbb{C}^m$ by $a \mapsto (f_1(a), \ldots, f_m(a))$ where $f_1, \ldots, f_m$ are polynomials in n variables. Describe the corresponding map of rings $\mathbb{C}[y_1, \ldots, y_m] \to \mathbb{C}[x_1, \ldots, x_n]$.

(7) What does your map in (6) do to non-maximal ideals ("non-closed points") in $\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$?

(8) Describe the map $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ given by $(a, b) \mapsto a + b$ as a map of rings.