21/08/25

Generalized Linear Models (GLMs)

Linear Models

1) y_i are independent
2) $y_i \sim N(\mu_i, \sigma^2)$
3) $\mu_i = \theta^T x_i$
$= \theta_1 x_1 + \theta_2 x_2 + \dots$

w.r.t θ, μ_i is linear

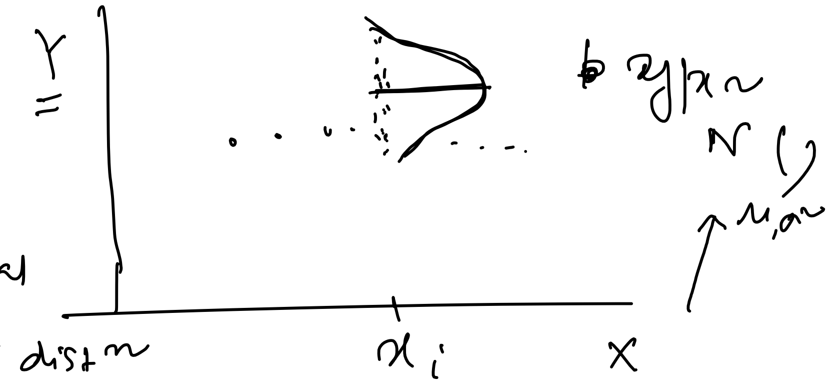
$\mu_i = \theta_1 x_1^2 + \theta_2 \sqrt{x_2} + \theta_3 \log x_3 \dots$

not. $\theta_1^2 x_1 + \cancel{\sqrt{\theta_2} x_2} \dots$

4) θ's can be obtained using LS regression or MLE

GLM

1) ✓
2) $y_i \sim$ Exponential family of distn
G, B, P.



3) $g(\mu_i) = \theta^T x = \eta$

$g(\ ) \leftarrow$ Link fn

$\mu = \theta^T x$

linear Gaussian distn

4) MLE

Exponential family of distn

$$\boxed{f_y(y|\theta, \phi) = \exp\left(\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right)} \quad \text{--- (1)}$$

Link function

(Normal, Poisson, Gamma, binomial, inverse Gaussian)

- Prove that Normal distn has this form.

$$f_Y(y|\theta, \phi) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

$$= \exp\left(\log\left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp(\quad)\right)\right)$$

$$= \exp\left(-\frac{1}{2}\log(2\pi\sigma^2) + \frac{-(y-\mu)^2}{2\sigma^2}\right)$$

$$= \exp\left(-\frac{(y^2 - 2y\mu + \mu^2)}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right)$$

$$= \exp\left(-\frac{y^2}{2\sigma^2} + \frac{\mu y}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right)$$

$$= \exp\left(\frac{y\mu - \frac{1}{2}\mu^2}{\sigma^2} - \frac{1}{2}\left(\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2) \right)\right) \quad (11)$$

$\theta = \mu \quad b(\theta) = \frac{1}{2}\theta^2 \quad a(\phi) = \sigma^2 \quad c(y,\phi) = -\frac{1}{2}\left(\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2)\right)$

Show that binomial distn is GLM

$$f_y(y|\theta,\phi) = \binom{n}{y}\pi^y(1-\pi)^{n-y}$$

$\pi = pr(y=1)$

n = # trials

$$= \exp(\log(\quad))$$

$$= \exp\left(\log\binom{n}{y} + y\log\pi + (n-y)\log(1-\pi)\right)$$

$$= \exp\left(\log\binom{n}{y} + y\log\pi + n\log(1-\pi) - y\log(1-\pi)\right)$$

$$= \exp\left(y(\log\pi - \log(1-\pi)) + n\log(1-\pi) + \log\binom{n}{y}\right)$$

$g()$ canonical link

$$= \exp\left(y \log\frac{\pi}{1-\pi} + n\log(1-\pi) + \log\binom{n}{y}\right)$$

$\theta = \log\frac{\pi}{1-\pi}$

$$= \exp\left(y \underbrace{\log\frac{\pi}{1-\pi}}_{\theta} + \underbrace{-n\log(1+e^\theta)}_{b(\theta)} + \log\binom{n}{y}\right)$$

$\Rightarrow \frac{\pi}{1-\pi} = \exp\theta$

$\Rightarrow \pi = \frac{\exp\theta}{1+\exp\theta}$

$\Rightarrow \log(1-\pi) = \log\left(1 - \frac{e^\theta}{1+e^\theta}\right)$

$= -\log(1+e^\theta)$

$\boxed{b(\theta) = n\log(1+e^\theta)}$

$\boxed{a(\phi) = 1}$

$\boxed{c(y,\phi) = \log\binom{n}{y}}$

Show, posion distn is also GLM

$$f_y(y|\theta,\phi) = \exp(-\mu)\frac{\mu^y}{y!}$$

$$= \exp(-\mu)\exp\left(\log\frac{\mu^y}{y!}\right)$$

$$= \exp(-\mu)\exp\left((\log\mu^y) - \log(y!)\right)$$

$$= \exp\left(-\mu + \log\mu^y - \log(y!)\right)$$

$$= \exp\left(-\mu + y\log\mu - \log(y!)\right)$$

$$= \exp\left(y\theta - \exp(\theta) - \log(y!)\right)$$

$\boxed{\theta = \log\mu}$

$\boxed{a(\phi) = 1}$

$\boxed{b(\theta) = \exp(\theta)}$

$\boxed{c(y,\phi) = -\log(y!)}$

Mean & Variance of Exp family

Mean

$$\boxed{E(y) = b'(\theta)}$$

variance : $\boxed{Var(y) = b''(\theta)\, a(\phi)}$

log likelihood

$$l(\theta, \phi, y) = \frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)$$

$$\boxed{\frac{\partial l}{\partial \theta} = \frac{y - \frac{\partial b(\theta)}{\partial \theta}}{a(\phi)}}$$

$E\left(\frac{\partial l}{\partial \theta}\right) = 0$ Prove $\Rightarrow$ $E\left(\frac{\partial}{\partial \theta} \log f_y(y|\theta, \phi)\right)$

$$= E\left(\frac{1}{f_y()} \frac{\partial}{\partial \theta} f_y()\right)$$

score fn — gradient of the log of the PDF of prob dist^n

$l = \log f_y()$

$$= \int \cancel{f_y()} \cdot \frac{1}{\cancel{f_y()}} \frac{\partial}{\partial \theta} f_y() \, \partial y$$

$$= \int_y \frac{\partial}{\partial \theta} f_y() \, \partial y$$

$$= \frac{\partial}{\partial \theta} \int_y f_y() \, dy = 0$$

$$E\left(\frac{\partial L}{\partial \theta}\right) = E\left(\frac{y - \frac{\partial b(\theta)}{\partial \theta}}{a(\phi)}\right) = 0$$

$$= E(y) - \frac{\partial b(\theta)}{\partial \theta} = 0$$

$$\Rightarrow E(y) = \frac{\partial b(\theta)}{\partial \theta} = b'(\theta) = \mu$$

Variance $\quad Var(y) = b''(\theta)\, a(\phi)$

$$\boxed{E\left(\frac{\partial^2 l}{\partial \theta^2}\right) = -E\left(\frac{\partial l}{\partial \theta}\right)^2}$$

$\rightarrow$ This is for all types of score fn. regardless of whether it's exponential family or not

$$E\left(\frac{\partial^2 l}{\partial \theta^2}\right) = E\left(\frac{\partial^2}{\partial \theta^2} \log f_y()\right) = E\left(\frac{\partial}{\partial \theta}\left(\frac{1}{f_y()} \frac{\partial}{\partial \theta}(f_y())\right)\right)$$

$$= E\left(-\frac{1}{f_y^2()}\left(\frac{\partial}{\partial \theta}(f_y())\right)^2 + \frac{1}{f_y()} \frac{\partial^2}{\partial \theta^2}(f_y())\right)$$

$$= -E\left[\left(\frac{1}{f_y()} \frac{\partial}{\partial \theta}(f_y())\right)^2\right] + \underbrace{E\left(\quad\right)}_{= 0}$$

$$E\left[\frac{1}{f_y(\cdot)}\frac{\partial^2}{\partial\theta^2}f_y\right] = \int_y \cancel{f_y(\cdot)}\frac{1}{\cancel{f_y(\cdot)}}\frac{\partial^2}{\partial\theta^2}f_y(\cdot)\,dy$$

$$= \int_y \frac{\partial^2}{\partial\theta^2}f_y(\cdot)\,\partial y = \frac{\partial^2}{\partial\theta^2}\int_y f_y(\cdot)\,dy = 0$$

$$= -E\left[\left(\frac{1}{f_y}\frac{\partial}{\partial\theta}(f_y(\cdot))\right)^2\right]$$

$$\frac{\partial}{\partial\theta}\log f_y(\cdot) = \frac{1}{f_\cdot(y)}\frac{\partial}{\partial\theta}f_y$$

$$= -E\left[\left(\frac{\partial}{\partial\theta}\log f_y(\cdot)\right)^2\right]$$

$$= -E\left[\left(\frac{\partial l}{\partial\theta}\right)^2\right]$$

log likelihood

$$l = \frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)$$

$$\frac{\partial l}{\partial\theta} = \frac{y - \frac{\partial}{\partial\theta}b(\theta)}{a(\phi)} \qquad E($$

$$E\left[\frac{\partial^2 l}{\partial\theta^2}\right] = E\left[\frac{-\frac{\partial^2}{\partial\theta^2}b(\theta)}{a(\phi)}\right] = -E\left[\left(\frac{y - \frac{\partial}{\partial\theta}b(\theta)}{a(\phi)}\right)^2\right]$$

$$= \cancel{E}$$

$$= -E\left[\frac{(y-\mu)^2}{(a(\phi))^2}\right]$$

$$\Rightarrow \cancel{-}\frac{\frac{\partial^2}{\partial\theta^2}b(\theta)}{\cancel{a(\phi)}} = \cancel{-}\frac{E[(y-\mu)^2]}{a(\phi)^{\cancel{2}}}$$

$$\Rightarrow \operatorname{Var}(y) = \frac{\partial^2}{\partial\theta^2}b(\theta)\cdot a(\phi)$$

$$= b''(\theta)\cdot a(\phi)$$

$$\theta^T x = \eta$$

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$g(\ )$ $\longrightarrow$ Gaussian : $\eta = \mu = \theta \rightarrow$ identity link

link f^n

Poisson : $\underline{\theta = \log \mu}$; $\eta = \log \mu$ — log link

Binomial : $\theta = \eta = \log\left(\frac{\pi}{1-\pi}\right) \rightarrow$ logit link

Assignment $\boxed{\text{Inverse Gaussian}} \longrightarrow \theta,\ b(\theta),\ a(\phi),\ c(\ \ ),\ \eta,\ g(\ \ )$

$$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi x^3}}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)$$