# Structural Dynamics and acoustics:

## Multi-Degrees of Freedom systems

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# Key points of the TP

- Application of Newton's laws to a Multi-Degrees of Freedom system (MDOF)
- Determination of *Normal* modes (modal decomposition)
- Identify the system's *Transfer Function*
- Solve the transient dynamics of a MDOF system
- Design a multistory building

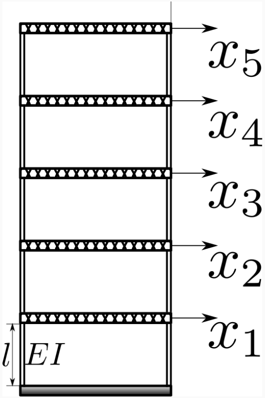
# Outline

The Multi Degree of Freedom system

## Exercice 1

## The Multi Degree of Freedom system

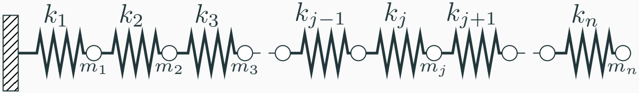
# Problem setting




- $n$ rigid floors of mass $m$
- The floors are connected through walls of stiffness $k$
- No external load.

# 1. Mass and stiffness matrices from the dynamic equation of the floors

## Alternative model



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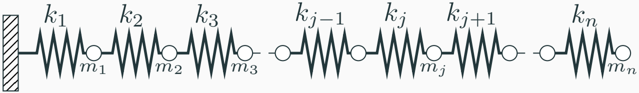


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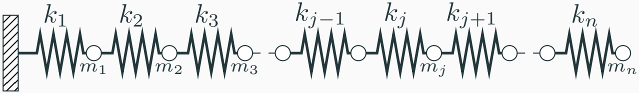


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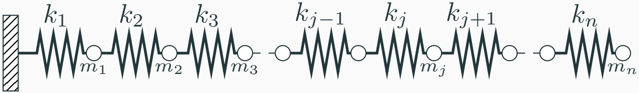


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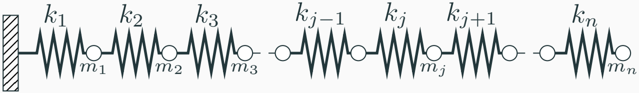


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## Matrix form of the system

$$\begin{bmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & .. & & & \\ & & & m_j & & \\ & & & & .. & \\ & & & & & m_n \end{bmatrix} \underline{\ddot{\boldsymbol{q}}} = - \begin{bmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & & .. & & & \\ & & -k_j & k_j+k_{j+1} & -k_{j+1} & \\ & & & & .. & \\ & & & & -k_n & k_n \end{bmatrix} \underline{\boldsymbol{q}}$$

where $\underline{\boldsymbol{q}}^T = \begin{bmatrix} x_1 & x_2 & .. & x_j & .. & x_m \end{bmatrix}$

# 2. Provide the expression of the boundary conditions

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- clamping at the base: $x_0 = 0$
- free top vertex: $F^{\text{ext}\to n} = -k_n \left(x_n - x_{n-1}\right)$, the only force acting on vertex $n$ is that of spring $k_n$.

# 3. For $x_j = X_j e^{i\omega t}$, what is the displacement of stages $j-1$, $j$ and $j+1$ ?

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$\Rightarrow$ Floor 1: $-\omega^2 m_1 X_1 = -k_1 X_1 - k_2(X_1 - X_2)$

$\Rightarrow$ Floor $j$: $-\omega^2 m_j X_j = -k_j(X_j - X_{j-1}) - k_{j+1}(X_j - X_{j+1})$

$\Rightarrow$ Floor $n$: $-\omega^2 m_n X_n = -k_n(X_n - X_{n-1})$

# 3. For $x_j = X_j e^{i\omega t}$, what is the displacement of stages $j-1$, $j$ and $j+1$ ?

**Harmonic response**

$$\begin{cases} \dfrac{\omega^2}{\omega_1^2} X_1 = X_1 + \dfrac{k_2}{k_1}\left(X_1 - X_2\right) & (1) \\ \dfrac{\omega^2}{\omega_j^2} X_j = \left(X_j - X_{j-1}\right) + \dfrac{k_{j+1}}{k_j}\left(X_j - X_{j+1}\right), \forall j = 2, ..., n-1 & (2) \\ \dfrac{\omega^2}{\omega_n^2} X_n = \left(X_n - X_{n-1}\right) & (3) \end{cases}$$

If $k_i = k$ and $m_i = m$ $\forall i = 1, ..., n$, $\omega_i = \sqrt{\frac{k}{m}} \forall i = 1, ..., n$. Then:

$$\begin{cases} \left(2 - \dfrac{\omega^2}{\omega_0^2}\right) X_1 = X_2 & (4) \\ \left(2 - \dfrac{\omega^2}{\omega_0^2}\right) X_j = X_{j-1} + X_{j+1}, \forall j = 2, ..., n-1 & (5) \\ \dfrac{\omega^2}{\omega_0^2} X_n = \left(X_n - X_{n-1}\right) & (6) \end{cases}$$

**Interpretations**
In that case, the system reads:

$$
\begin{cases}
\left(2 - \dfrac{\omega^2}{\omega_0^2}\right) = e^z & (7)\\
\left(2 - \dfrac{\omega^2}{\omega_0^2}\right) = e^{-z} + e^z, \forall j = 2, ..., n-1 & (8)\\
\dfrac{\omega^2}{\omega_0^2} = 1 - e^{-z} & (9)
\end{cases}
$$

- If $\Im(z) = 0$, $e^{-z} + e^z = 2\cosh(z)$. In that case $X_j$ exponentially grows with $j$
- If $\boxed{\Re(z) = 0}$, $e^{-z} + e^z = 2\cos(\Im(z))$

# 5. For $z = \pm i\lambda$, what does *dispersion relationship* mean?

**Dispersion relation**
For a solution of the type $x_j = Ae^{i\lambda x + i\omega t}$, the relation $\omega = \omega(\lambda)$ is the dispersion relation.
In this problem, it is given from the $j$-th equation of the system.

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$$\left(2 - \frac{\omega^2}{\omega_0^2}\right) = e^{\pm i\lambda} + e^{\mp i\lambda} = 2\cos(\lambda) \tag{10}$$

$$\frac{\omega^2}{2\omega_0^2} = 1 - \cos(\lambda) \tag{11}$$

with $\cos(\lambda) = \cos(\frac{\lambda}{2})^2 - \sin(\frac{\lambda}{2})^2$, it comes:

$$\frac{\omega^2}{2\omega_0^2} = 1 - \cos(\frac{\lambda}{2})^2 + \sin(\frac{\lambda}{2})^2 \tag{12}$$

$$\frac{\omega^2}{2\omega_0^2} = \sin(\frac{\lambda}{2})^2 + \sin(\frac{\lambda}{2})^2 \tag{13}$$

$$\Rightarrow \sin(\frac{\lambda}{2}) = \frac{\omega}{2\omega_0} \tag{14}$$

# 6. Deduce the general form of the modal deformations, taking into account the boundary conditions

**Form of the solution**

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- $X_j = A\left(\cos\lambda j + i\sin\lambda j\right) + B\left(\cos\lambda j - i\sin\lambda j\right) = (A+B)\cos\lambda j + i\,(A-B)\sin\lambda j$

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**Boundary conditions**

Clamping at $X_0 : \Re(X_0) = 0 \quad \Rightarrow \boxed{A + B = 0}$

Then, the general form of the solution is: $X_j = i\left(A - B\right)\sin\lambda j$

# 7. Calculate the possible values of $\lambda$ from the boundary conditions with this solution

**Free top-storey boundary condition**

$$\frac{\omega^2}{\omega_0^2} X_n = (X_n - X_{n-1}) \Rightarrow \frac{\omega^2}{\omega_0^2} \sin \lambda n = \sin \lambda n - \sin(\lambda(n-1)) \tag{15}$$

Useful trigonometric identities:

$$\sin a - \sin b = 2 \sin \frac{a-b}{2} \cos \frac{a+b}{2} \tag{16}$$

$$2 \sin a \sin b = \cos(a-b) - \cos(a+b) \tag{17}$$

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**Calculation of $\lambda$**

Equation (16) $\Rightarrow \sin \lambda n - \sin(\lambda(n-1)) = 2 \sin \frac{\lambda}{2} \cos(\lambda n - \frac{\lambda}{2})$

With the dispersion relation: $\frac{\omega^2}{\omega_0^2} \sin \lambda n = 4 \sin^2 \frac{\lambda}{2} \sin \lambda n$

# 7. Calculate the possible values of $\lambda$ from the boundary conditions with this solution

**Calculation of $\lambda$**
The free top-storey boundary condition is then:

$$4\sin^2\frac{\lambda}{2}\sin\lambda n = 2\sin\frac{\lambda}{2}\cos(\lambda n - \frac{\lambda}{2}) \tag{18}$$

$$2\sin\frac{\lambda}{2}\left(2\sin\frac{\lambda}{2}\sin\lambda n - \cos(\lambda n - \frac{\lambda}{2}\right) = 0 \tag{19}$$

With equation (17) $2\sin\frac{\lambda}{2}\sin\lambda n = \cos(\frac{\lambda}{2} - \lambda n) - \cos(\frac{\lambda}{2} + \lambda n)$
Therefore:

$$2\sin\frac{\lambda}{2}\cos(\lambda n + \frac{\lambda}{2}) = 0, \quad \text{with } \sin\frac{\lambda}{2} \neq 0 \text{ for } \omega \neq 0 \tag{20}$$

The solution(s) is(are) then $\lambda_r(n+\frac{1}{2}) = \frac{\pi}{2}(2r-1), r = 1, ..., n$

$$\boxed{\lambda_r = \pi\frac{2r-1}{2n-1}, r = 1, ..., n}$$

# 8. Deduce the natural and modal deformed frequencies of the $r$th mode

**Modes and frequencies**

- Mode solution: $X_j = i\,(A - B)\sin\lambda_r j$
- Associated frequency: $\omega_r = 2\omega_0 \sin\frac{\lambda_r}{2}$