Tutorial session 4 - Constraints in continuous optimization

1 Problem

A winner at Euromillions receives a S million prize and decides to live on the investment of her gain. Her welfare at time $n+1$ is supposed to be proportional to $r_n^{1/2}$ where r_n is the money spent between time n and $n+1$. She thus wants to maximize

$$\sum_{n=0}^{N-2} \beta^n \sqrt{r_n}$$

where $N-1 \geq 1$ is her remaining life expectancy and the exponentially decaying term β^n with $\beta \in]0,1[$ accounts for the fact that spending money today is more enjoyable than spending it tomorrow (in other words, the younger, the more fun !). If x_n designates the available amount of money at time $n \in \{0,\dots,N-1\}$, we have

$$(\forall n \in \{0,\dots,N-2\}) \quad x_{n+1} = \gamma x_n - r_n$$

where $\gamma \in [1,+\infty[$ represents the investment rate.

Our purpose is to find the optimal way of spending money.

2 Numerical solution

1. What are the equality and inequality constraints to be handled ?
2. Which kind of optimization problem should be solved ?
3. The provided PPXA Matlab function allows us to tackle the related constrained optimization problem. Use it to solve numerically the problem when $N = 100$ semesters, $S = 130$ millions, $\gamma = 1.03$, and $\beta = 0.96$.
4. Plot $(x_n)_{0\leq n\leq N-1}$ and $(r_n)_{0\leq n\leq N-2}$. Compute the associated cost value.
5. Compare the optimal strategy with the strategy which would consist in spending half of the money available at each time n.
6. The winner might live longer than expected. As a cautious person, she plans to leave a residual amount of money R at duration $N-1$. Solve numerically the modified optimization problem when $R = 10$ millions.

3 Analytic solution

1. Propose a Lagrange formulation of the latter problem.
2. Deduce the closed form expression of the solution.
3. Compare it with the numerical solution.