Tutorial session 2 - Linear programming

February 20, 2022

1 Introduction

The aim of this tutorial is to deal in an optimization framework with a series of problems related to the multi-agent coverage and the geometrical properties of polyhedral sets in a finite dimensional state space. Such constructions appear in robotics, safe and rescue applications, economic/administrative decisions and other related fields.

All the real-world objectives below need to be translated into an equivalent optimization problems discussed in the Optimization course and solved accordingly.

2 Find the point with the largest distance to the boundaries of a polyhedral set

Consider a mobile robot positioned in a constrained environment. This environment will be represented by a convex set with the boundaries given by the following half-space representation:

$$P = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 3 & -4 \\ 0 & 1 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \tag{1}$$

A mobile robot is positioned at the origin in this set, the wheels are blocked (no possible turning maneuver) and the heading angle allows moves on the line:

$$D = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_1 = x_2 \right\}$$

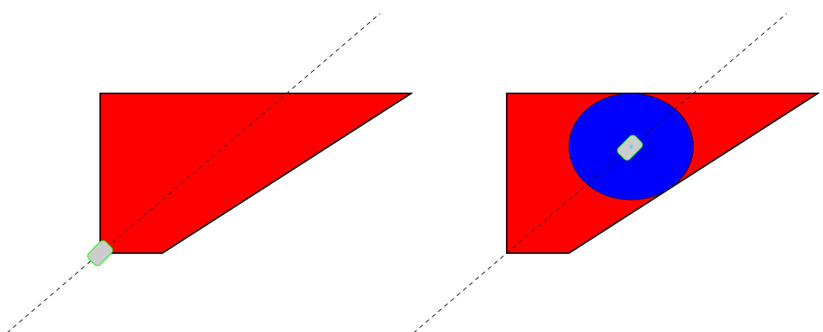
The set P and the restriction D are illustrated in Figure 1a.

The first objective is to find the best possible position of the robot within this constrained set in such a way that the minimal distance to the boundaries is maximized[1]. A graphical illustration of the ideal configuration is presented in Figure 1b.

[1] The distance to the boundaries is denoted as *Tchebychev radius*

Follow the next steps in order to provide a numerical solution:

- describe the ball situated at the position x_c and radius ρ;
- write the objective of the design in terms of an optimization of the radius of such a ball;
- impose the adequate constraints on the x_c and ρ in order to fulfill the limitations on the positioning of the robot and the distance to the boundaries of the given environment;
- show that this is a linear programming problem;
- bring the LP to the standard form;
- solve the problem by completing the the Matlab script that calls the routine `simplex.m` implementing Dantzig's Simplex method (provided);
- plot the results using the provided routine `ellipplot(eye(2),`ρ^2`,'b',`x_c'`)` with the arguments ρ^2, x_c found above.



(a) the initial configuration (b) the intended configuration

Figure 1: Graphical illustration of the mobile robot configurations

3 On the iterations and the structure of the problem

Solve the problem "by hand" with Dantzig's simplex algorithm specifying the intermediate steps all by discussing the uniqueness and boundedness.

Show that ultimately, there are two optimization arguments. Provide a graphical interpretation for the constraints in a two-dimensional representation by:

- drawing the constraints

- identifying the redundant constraints(if any)
- plot the iso-cost profile
- show the iterations of the Dantzig's algorithm on this graphical representation.

4 Two particular cases

Suppose that the limits of the environment are slightly perturbed to:

$$P_1 = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 4 & -4 \\ 0 & 1 \\ -0.5 & 0.5 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0.4 \end{bmatrix} \right\} \tag{2}$$

describe the change in the solution and the uniqueness and boundedness properties.

Same question for the constraints:

$$P_2 = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 3 & -4 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \tag{3}$$

5 From unidirectional to multidirectional robots

We suppose next that the robot or multiagent system can move in any direction of the 2D space (the constraint: $x_1 = x_2$ is relaxed). Describe the new optimization problem which allows the construction of the Tchebychev ball in any given bounded polyhedral set in $\mathbb{R}^2$. Implement and analyze the degrees of freedom and the maximal number of active inequality constraints for the numerical example $x \in P$.

6 Multiple robots and their deployment

We will move next from the deployment of one robot in a stationary configuration to the deployment of fleet of robots that share a common environment. In this framework consider the working space to be described by a box:

$$B = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ 5 \\ 10 \end{bmatrix} \right\} \tag{4}$$

Let 5 agents positioned at the following coordinates at the first time instant:

$$c_1(1) = \begin{bmatrix} 1 \\ 2.5 \end{bmatrix}; c_2(1) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}; c_3(1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}; c_4(1) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}; c_5(1) = \begin{bmatrix} 8 \\ 2.5 \end{bmatrix} \tag{5}$$

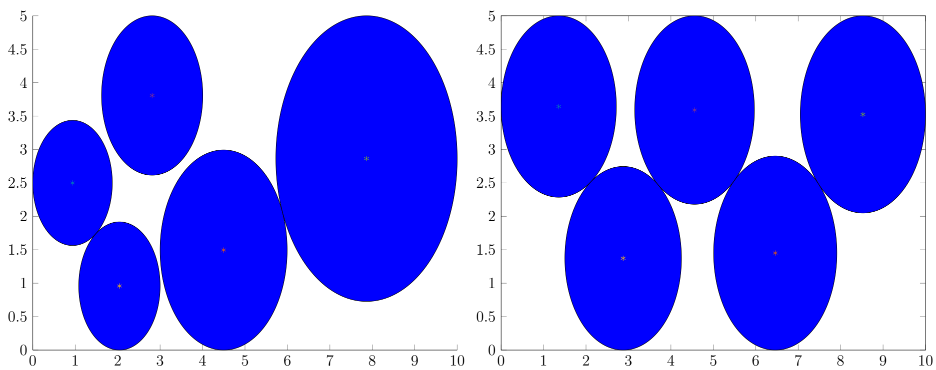
At time instant k at each of such coordinates at $c_i(k), i = 1, \ldots, 5$ corresponds an associated polyhedral cell defined as[2]:

$$V_i(k) = \left\{ x \in B \mid 2(c_j(k) - c_i(k))^T x \leq \|c_j(k)\|^2 - \|c_i(k)\|^2, \forall j \neq i \right\} \tag{6}$$

Compute the Tchebychev center of each of those cells $x_{ci}(k), i = 1, \ldots, 5$ and simulate for 20 iterations the dynamics:

$$c_i(k+1) = x_{ci}(k)$$

and verify the convergence towards the configuration in Figure 2b.



(a) the initial configuration (b) The final configuration

Figure 2: The deployment of fleet of robots

[2]The partition of the given box in according to this disjoint union of cells is also called *Voronoi partition*.