Partial Differential Equations

Exercises

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Lecture I - Ordinary Differential Equations: Theory

NB1: At the beginning of each lab, Section A) contains the list of minimal knowledge and know-hows that are required to pass the course.
NB2: Questions marked with a $*$ or $**$ are optional. They may go beyond the scope of this course and will not be at the exams. However, they will help you have a better understanding of the material. If you choose to skip these questions (which you are allowed to do), you can use the result of this question to answer the next questions of the exercise.

A) Aims of this class

After this class,

- I know what an ODE is.
- I know what a Cauchy problem is.
- I can show that a function is globally Lipschitz continuous.
- I can show the existence and uniqueness of the solution of a Cauchy problem by testing the hypotheses of the adapted Cauchy-Lipschitz theorem.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions I.1 and I.2 must be done before the first lab. The solutions for these exercises are available online. *Do not* look at the solutions before you finish working on these questions.

Question I.1 *Let us consider the initial value problem (IVP) on this Riccati equation with a given $y^0 \in \mathbb{R}$*

$$\begin{cases} y' = y^2, \\ y(0) = y^0. \end{cases}$$

Q. I.1.1 *Prove that this IVP has a unique local solution.*

Q. I.1.2 *Find a closed form for the solution.*

Q. I.1.3 *Give a necessary and sufficient condition on y^0 so the IVP has a global solution.*

Question I.2

Q. I.2.1 *Solve this IVP:*

$$u'(t) = e^{u(t)} \sin t, \quad u(0) = 0.$$

For which t does the solution exist?

Q. I.2.2 *Solve this IVP:*

$$x'(t) = \frac{(x(t))^2}{1+t^2}, \quad x(0) = 1.$$

For which t does the solution exist?

Q. I.2.3 *Solve this IVP:*

$$y' = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

C) Exercises

Exercise I.1 (Shooting Method)
Let f be a continuous function on $[0, 1]$ and q be a non-negative constant.

E. I.1.1 *Find the closed form of the solutions to the Initial Value Problem*

$$\textbf{(IVP)} \qquad \begin{cases} -u''(x) + qu(x) = f(x), & x \in]0,1[, \\ u(0) = 0 \text{ and } u'(0) = k. \end{cases}$$

Deduce the closed form of the solutions to

$$\textbf{(BVP)} \qquad \begin{cases} -u''(x) + qu(x) = f(x), & x \in]0,1[, \\ u(0) = 0 \text{ and } u(1) = 0, \end{cases}$$

which is called a Boundary Value Problem.

E. I.1.2 *What have we achieved? Why is this called a "shooting" method?*

E. I.1.3 *Can we proceed the same way when $q < 0$?*

Exercise I.2
Let us consider these initial value problems:

$$u' = (\sin(u))^3, \quad u(0) = 1, \quad \text{and} \quad v' = (\sin(v))^3, \quad v(0) = \pi.$$

E. I.2.1 *Prove that $f : u \mapsto (\sin(u))^3$ is globally Lipschitz-continuous.*

E. I.2.2 *Prove these two IVPs have a unique solution on $\mathbb{R}$ and that their curves do not intersect one another.*

E. I.2.3 *Give a bound for $|u(t) - v(t)|$ for $t \geq 0$.*

E. I.2.4 *Find v.*

Exercise I.3
The Lotka-Volterra system reads: for $\alpha, \beta, \gamma, \delta > 0$, $x^0, y^0 > 0$,

$$\begin{cases} x' = \alpha x - \beta xy, \\ y' = -\gamma y + \delta xy, \\ x(0) = x^0, \\ y(0) = y^0. \end{cases}$$

E. I.3.1 *Show that there exists a local solution of this system, defined on $[0, T[$, for $T > 0$.*

E. I.3.2 *Show that, for all $T > 0$, $x(t) > 0$ and $y(t) > 0$. Infer from it that the solution is global.*

D) Going further

Exercise I.4 (Proof of the Lyapounov theorem (first part))
The goal of this exercise is to prove the following theorem:

Theorem. *Let v be a singular point of the field $f \in C^1(\mathbb{R}^d)$, that is, $f(v) = 0$. Assume that the Jacobian matrix $Df(v)$ is such that*

$$\mathrm{Sp}(Df(v)) \subset \{\mu \in \mathbb{C} \,:\, \mathrm{Re}(\mu) < 0\}.$$

Then there exists $\mathcal{V}$ a neighborhood of v such that, for all $w \in \mathcal{V}$, the solution of $y' = f(y)$, $y(0) = w$ is global and $y(t) \to w$ as $t \to +\infty$, the convergence being exponential (the point v is said to be exponentially stable).

Let $A = Df(v)$.

E. I.4.1 *Let B be a square complex matrix of size n, $n \geq 1$. Show that the following sum*

$$\sum_{j=0}^{+\infty} \frac{B^j}{j!}.$$

is well defined. It is denoted $\exp(B)$.

E. I.4.2 *Let B be a square real matrix of size n, $n \geq 1$. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of B. Show that there exists a polynomial P such that for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$,*

$$\|e^{tB}x\| \leq P(|t|) \left(\sum_{j=1}^{k} e^{t\mathrm{Re}(\lambda_j)} \right) \|x\|.$$

for any norm $\|\cdot\|$ on $\mathbb{C}^n$.

INDICATION : *Use the Dunford decomposition $B = D + N$, where D is diagonal, N nilpotent and $ND = DN$.*

Let $x \in \mathbb{R}^d$. Let z be the solution of

$$\begin{cases} z' = Az \\ z(0) = x. \end{cases}$$

Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^n$ and $\|\cdot\|$ the associated norm.

E. I.4.3 *Show that there exist two constants $a, C > 0$ such that, for all $t \geq 0$,*

$$\|z(t)\| \leq C\, e^{-at}\, \|x\|$$

Consider the application $b : (\xi, \zeta) \longmapsto \displaystyle\int_0^{+\infty} \langle e^{tA}\xi, e^{tA}\zeta \rangle \mathrm{d}t$.

E. I.4.4 *Show that b is well defined, bilinear, symmetric and continuous.*

Let q be the associated quadratic form, that is, $q : \xi \in \mathbb{R}^d \mapsto b(\xi, \xi)$.

E. I.4.5 *Show that q is positive definite, that is, $q(\xi) = 0$ if and only if $\xi = 0$.*

Let $\xi \in \mathbb{R}^d$.

E. I.4.6 *Compute $(\mathrm{grad}\, q)(\xi)$.*

E. I.4.7 *Show that $\langle (\mathrm{grad}\, q)(\xi), A\xi \rangle = -\|\xi\|^2$.*

We must now show that the solution of the Cauchy problem

$$\begin{cases} y' = f(y) \\ y(0) = x \end{cases}$$

is global.

E. I.4.8 *Justify the existence of $T > 0$ such that y is the solution of the Cauchy problem on $[0, T[$.*

E. I.4.9 *Show that $q(y - v)' = -\|y - v\|^2 + 2b(y - v, f(y) - A(y - v))$.*

E. I.4.10 *Show that for every $\varepsilon > 0$ there exists $\alpha > 0$ such that, if ξ satisfies $q(\xi - v) \leq \alpha$,*

$$\sqrt{q(f(\xi) - A(\xi - v))} \leq \varepsilon \sqrt{q(\xi - v)}.$$

E. I.4.11 *Show that there exists $\beta > 0$ such that, if $y(t)$ satisfies $q(y(t) - v) \leq \alpha$ then $q(y - v)' \leq -\beta q(y - v)$.*

Assume that $q(x - v) < \alpha$.

E. I.4.12 *Show that for all $t \in [0, +\infty[$, $q(y(t) - v) < \alpha$.*

E. I.4.13 *Deduce that y is global.*

To be continued in the next session...

Lecture II - Ordinary Differential Equations: Numerics

NB1: At the beginning of each lab, Section A) contains the list of minimal knowledge and know-hows that are required to pass the course.
NB2: Questions marked with a $*$ or $**$ are optional. They may go beyond the scope of this course and will not be at the exams. However, they will help you have a better understanding of the material. If you choose to skip these questions (which you are allowed to do), you can use the result of this question to answer the next questions of the exercise.

A) Aims of this class

After this class,

- I know what an equilibrium point is, and what a stable, asymptotically, exponentially stable equilibrium point is.
- I know how to show that a point of equilibrium is asymptotically stable.
- I know how to define the Forward Euler Scheme.
- I know how to define the Backward Euler Scheme.
- I know how to define the consistency, the consistency error and the stability of a scheme.
- I can show that the Euler schemes are consistent with the ODE.
- I can code the Forward Euler Scheme in Python or Matlab.
- I know how to determine the numerical order of a method.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions II.1 and II.2 must be done before the second lab. The solutions are available online. *Do not* look at the solutions before you finish working on these exercises.

Question II.1 *We consider the ODE*

$$\begin{cases} x' &= -y \\ y' &= x - y + z \\ z' &= x - 2y + 2z \end{cases}$$

where x, y and z are functions defined on $\mathbb{R}$.

Q. II.1.1 *Let $x(0) = x^0$, $y(0) = y^0$ and $z(0) = z^0$ be given. Find a closed form for x, y and z.*

Q. II.1.2 *Give a necessary and sufficient condition on x^0, y^0 and z^0 so the solutions x, y and z are periodical.*

Question II.2 *We consider the IVP*

$$\begin{cases} y'(t) = -3y(t) \\ y(0) = 1 \end{cases}.$$

Q. II.2.1 *Find the closed form of the solution y.*

Q. II.2.2 *Implement the Euler Forward Method in Python to approximate y on $[0, 10]$. Try these steps: $h = 1$, $h = 0.1$, $h = 0.01$ and compute the global error in each case.*

Q. II.2.3 *Implement the Euler Backward Method in Python to approximate y on $[0, 10]$. Try these steps: $h = 1$, $h = 0.1$, $h = 0.01$ and compute the global error in each case.*

Q. II.2.4 *Elaborate.*

C) Exercises

Exercise II.1
In 1963, Edward N. Lorenz developed a model of the atmosphere. While the atmosphere is obviously very complex, the model developed by Lorenz is rather simple. It takes into account three variables:

x the rate of convection,
y the horizontal temperature variation,
z the vertical temperature variation,

and links these variables by these equalities:

$$\begin{cases} x' &= \sigma(y - x), \\ y' &= x(\rho - z) - y, \\ z' &= xy - \beta z, \end{cases}$$

where

- σ is a parameter related to the thermal diffusivity,
- ρ is a parameter related to the free convection,

- β is a parameter related physical dimensions of the layer.

Edward Lorenz chose $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. We will make this choice from now on.

E. II.1.1 *Find a student who took the Transport Phenomena elective class and treat him/her to lunch. Ask this student to explain to you what is thermal diffusivity, free convection and how the Lorenz model is derived.*

E. II.1.2 *Use the Euler method to develop a software in Python approximating the solutions for $t \in [0, 50]$ with the initial condition $x(0) = 1$, $y(0) = 1$ and $z(0) = 1$. You may want to choose a time step of* 0.01.

E. II.1.3 *Change the values of the initial conditions. Elaborate on what you see.*

Exercise II.2
Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be non-negative sequences. Let $q \in \mathbb{R}^+$ and assume

$$\forall n \in \mathbb{N}^*,\ \alpha_{n+1} \leq q\alpha_n + \beta_n.$$

E. II.2.1 *What is the nature of this property?*

E. II.2.2 *Prove*

$$\forall n \in \mathbb{N}^*,\ \alpha_n \leq q^n \alpha_0 + \sum_{k=0}^{n-1} q^{n-1-k}\beta_k.$$

Exercise II.3
For a vector field f, the Crank-Nicolson scheme is defined as

$$\begin{cases} z^0 \text{ given} \\ z^{n+1} = z^n + \dfrac{\Delta t}{2} f(t^n, z^n) + \dfrac{\Delta t}{2} f(t^{n+1}, z^{n+1}). \end{cases}$$

E. II.3.1 *Let $a \in \mathbb{R}$. Assume that $f : (t, z) \mapsto az$. Give a necessary and sufficient condition on a to ensure that the scheme is defined for all $\Delta t > 0$.*

E. II.3.2 *Prove the Crank-Nicolson method is a second-order convergent method.*

D) Going further

End of Exercise E.I.4

Exercise II.4 (Proof of the Lyapounov theorem (second part))
The goal of this exercise is to prove the following theorem:

Theorem. *Let v be a singular point of the field $f \in C^1(\mathbb{R}^d)$. Assume that the Jacobian matrix $Df(v)$ is such that*

$$\mathrm{Sp}(Df(v)) \subset \{\mu \in \mathbb{C} : \mathrm{Re}(\mu) < 0\}.$$

Then v is exponentially stable.

We proved in the first part that the solution is global.

E. II.4.1 *Going through the proof of the fact that the solution is global, show that $q(y-v)$ goes to 0 exponentially fast and conclude.*

From now on, we will be interested in the following problem: let $d \geq 1$, with I an interval of $\mathbb{R}$ that contains 0, $f : (t,y) \in I \times \mathbb{R}^d \mapsto f(t,y) \in \mathbb{R}^d$ a vector field that is of class C^1 and globally Lipschitz continuous with constant L. We consider the following Cauchy problem:

$$\textbf{(P)} \qquad \begin{cases} y' = f(t,y), \\ y(0) = y^0 \end{cases}$$

with $y^0 \in \mathbb{R}^d$. This problem admits one and only one solution in $C^1(I)$. Let $T > 0$ such that $[0,T] \subset I$. For $N \geq 1$, let $\Delta t = T/N$ and $t^n := n\Delta T$, $n \in \{0,\dots,N\}$. Let $z^0 \in \mathbb{R}^d$.

Exercise II.5

A one-step method is defined by

$$\textbf{(S)} \qquad \forall n \in \{0,\dots,N-1\}, \quad z^{n+1} = z^n + \Delta t\, \Phi(t^n, \Delta t, z^n),$$

with Φ is a given function.

E. II.5.1 *Write the Forward Euler Scheme, that was defined in the lecture, under this form. Give a condition that allows to write the Backward Euler Scheme under this form.*

E. II.5.2 *Give the definition of consistency with respect to* **(S)** *with* **(P)**.

E. II.5.3 (**) *Give and show a necessary and sufficient condition of the associated method Φ of* **(S)** *with* **(P)**.

E. II.5.4 *Recall the definition of stability.*

E. II.5.5 *Show that the assumption "Φ globally Lipschitz continuous with respect to z" of constant K ensures that the associated method Φ is stable.*

E. II.5.6 *Prove the Lax theorem:*

Theorem. *A one-step method* **(S)** *that is consistent with the ODE $y' = f(t,y)$ and stable is convergent.*

Lecture III - Modelling with PDEs

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A) Aims of this class

After this class,

- I can write the heat equation.
- I know how to make a problem dimensionless.
- I know how to determine what type of PDE is a given partial differential equation.
- I know the conditions at the boundaries of a problem with partial derivatives.
- I know the fundamental properties of hyperbolic, elliptic and parabolic equations.
- I know the definition of a well-posed problem.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions III.1 and III.2 must be done before the third lab. The solutions are available online. *Do not* look at the solutions before you finish working on these questions.

Question III.1 *What is the nature of the following problems?*

Q. III.1.1 (Problem 1) *Let $\kappa > 0$ and c be a function defined on $]0,1[$. The problem reads*

$$\begin{cases} -\kappa u''(x) + u'(x) + c(x)u(x) = f(x), & x \in]0,1[, \\ u(0) = 0 \quad \text{and} \quad u'(1) + u(1) = 0, \end{cases}$$

Q. III.1.2 (Problem 2) *Let Ω be the square $]0,1[\times]0,1[$ of $\mathbb{R}^2$, u^0, f 2 functions defined on Ω, A a square matrix of size 2 defined on Ω and g defined on the edges N (north) and S (south) of Ω. The problem reads*

$$\begin{cases} \partial_t u - div_x(A(x)grad\,(u)) = f, \quad t > 0,\, x \in \Omega \\ u(0,\cdot) = u^0 \\ u(t,\cdot)|_{N\cup S} = g \\ \partial_n u(t,\cdot)|_{\partial\Omega\setminus(N\cup S)} = 0 \end{cases}$$

Question III.2 *Let m be a bob with a mass of $500g$, suspended by an inelastic, massless string of length $10cm$. Assume that there is no friction. The gravity of Earth will be approximated by $10.m/s^{-2}$. Denote by θ the angle between the string and the downward vertical axis, measured counter clockwise. Let T be the observation time.*

Q. III.2.1 *Show that the dimensionless equation that is satisfied by the angular movement of the gravity center of the mass is $\theta'' + 100\,T^2\sin(\theta) = 0$. Drawing a picture is strongly recommended.*

Q. III.2.2 *What is the type of the equation?*

C) Exercises

Exercise III.1

We are interested in the spread of an infection on an island, which we model by its open surface Ω assumed to be bounded. We will compute the density v, i.e. the number of individuals infected by square kilometer, using a propagation model involving

- two types of movements: the classical diffusion, representing the average spatial behaviour of the population with constant coefficient $\kappa > 0$, and the difference of the integrated flows, specifically representing the movements of infected individuals at each point x, of positive constant D,
- a modelling of infection or spontaneous healing by a source f depending on time and space.

The equivalent of an energy study conducted in thermal transfer is based on the functional

$$\mathcal{H} : w \mapsto \frac{1}{2}\int_\Omega \left(\kappa\|\nabla w(x,y)\|^2 + c(x,y)w(x,y)^2\right) \mathrm{d}x\,\mathrm{d}y + \frac{D}{2}\left(\int_\Omega w(x,y)\mathrm{d}x\,\mathrm{d}y\right)^2 - \int_\Omega f(x,y)w(x,y)\mathrm{d}x\,\mathrm{d}y,$$

where $\|\cdot\|$ denotes the classical Euclidean norm on $\mathbb{R}^2$, $(\cdot,\cdot)$ the euclidean scalar product on $\mathbb{R}^2$, c and f are functions depending only on space.

E. III.1.1 *Assuming the dimension of c is a daily frequency (d^{-1}), give the dimensions of f, κ, D and $\mathcal{H}$.*

E. III.1.2 *Defining a reference time T, a reference surface S and a reference density W, give the relation between κ, T, S and W so that the dimensionless functional $\mathcal{H}_{adim}$ reads*

$$\mathcal{H}_{adim} : w \longmapsto \frac{1}{2}\int_\Omega \left(\|\nabla w\|^2 + c_{adim} w^2\right) + \frac{D_{adim}}{2}\left(\int_\Omega w\right)^2 - \int_\Omega f_{adim} w.$$

You will give c_{adim}, D_{adim} and f_{adim}.

E. III.1.3 *Explain why the assumptions on c_{adim}, D_{adim} and f_{adim} are the same as the ones on c, D and f.*

E. III.1.4 *Assume the problem can be expressed with a PDE. Give some relevant boundary conditions for this problem.*

Exercise III.2

Let $L > 0$, $c > 0$ and $\nu > 0$. We consider a homogeneous square plate, with side L, whose temperature at the moment $t = 0$ is described by a function u. The edges are maintained at zero temperature and a stationary source f is applied at every point of the plate. The temperature evolution is governed by the following problem

$$\begin{cases} \partial_t u(t,x,y) - \nu \Delta u(t,x,y) = f(t,x,y),\ t > 0,\ x, y \in]0, L[, \\ u(0,x,y) = u^0(x,y),\ x, y \in]0, L[, \\ \text{boundary condition} \end{cases}$$

E. III.2.1 *What is the dimension of ν ?*

E. III.2.2 *Make the problem dimensionless in order to have $\Omega =]0,1[\times]0,1[$.*

E. III.2.3 *What observation time T is it necessary to choose so that the dimensionless coefficient before the Laplacian operator is equal à 1 ?*

E. III.2.4 *What is the type of the dimensionless problem?*

E. III.2.5 *Assuming the solution goes in large time to a limit u_∞, what problem is satisfied by u_∞ ?*

Exercise III.3

Using the characteristics, solve the equation

$$\begin{cases} \partial_t u(t,x) + \partial_x u(t,x) - 3u(t,x) = t, \quad t > 0,\ x \in \mathbb{R} \\ u(0,\cdot) : x \mapsto x^2 \end{cases}$$

D) Going further

Exercise III.4

****E. III.4.1** *Show the following proposition:*

Proposition. *Let I be an interval of $\mathbb{R}$, $V_0 \subset \Omega \subset \mathbb{R}^3$, $\Phi : I \times \Omega \to \Omega$ of class C^1, $V_t := \Phi(t, V_0)$, $\mathbf{u} = \partial_t \Phi$ of class $C^1(I \times \Omega)$. Let $C : (t, \mathbf{x}) \mapsto C(t, \mathbf{x})$ of class $C^1(I \times \Omega)$. Then*

$$\begin{aligned}
\iiint_{V_t} C(t,\mathbf{x})\lambda(d\,\mathbf{x}) &= \iiint_{V_t} \partial_t C(t,\mathbf{x})\lambda(d\,\mathbf{x}) + \iiint_{V_t} div(C\mathbf{u})(t,\mathbf{x})\lambda(d\,\mathbf{x}) \\
&= \iiint_{V_t} \partial_t C(t,\mathbf{x})\lambda(d\,\mathbf{x}) + \iint_{\partial V_t} (C\mathbf{u})(t,\gamma)\cdot \mathbf{n}(t,\gamma)\,\mathrm{d}\gamma.
\end{aligned}$$

E. III.4.1 *Justify the definition of the Robin-Fourier conditions.*

Lecture IV - Distributions

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A) Aims of this class

After this class,

- I know how to define the convergence of a sequence of test functions.
- I can check that a simple linear form is a distribution.
- I know what a regular distribution is.
- I know how to derive in the sense of distributions in dimension 1.
- I can show that a function belongs to the Sobolev space H^1.
- I know the definition of H^1_0.
- I can prove the Poincaré inequality in dimension 1.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions IV.1 and IV.2 must be done before the fourth lab. The solutions are available online.

Question IV.1 *Let $I \subset \mathbb{R}$ be an open interval containing 0. Let us define*

$$T_1 : \varphi \in \mathcal{D}(I) \mapsto \varphi(0)^2$$

$$T_2 : \varphi \in \mathcal{D}(I) \mapsto \int_I \varphi$$

$$T_3 : \varphi \in \mathcal{D}(I) \mapsto \int_I |\varphi|$$

Are T_1, T_2 et T_3 distributions?

Question IV.2 *Define f by:*

$$\begin{cases} f: & [-1,1] \to \mathbb{R}, \\ & x \mapsto \dfrac{|x|+x}{2}. \end{cases}$$

Prove $f \in H^1(-1,1)$ and $f' \notin H^1(-1,1)$.

C) Exercises

Exercise IV.1

Let I be an open interval of $\mathbb{R}$ and $\theta_0 \in \mathcal{D}(I)$ be a given function such that

$$\int_I \theta_0(x)dx = 1.$$

E. IV.1.1 *Let $\varphi \in \mathcal{D}(I)$. Prove there exists a unique $\lambda \in \mathbb{R}$ and a unique $\psi \in \mathcal{D}(I)$ such that*

$$\varphi = \lambda\theta_0 + \psi'.$$

E. IV.1.2 *Let $T \in \mathcal{D}'(I)$ such that $T' = 0$. Prove the T is a constant distribution.*

E. IV.1.3 *Let $T \in \mathcal{D}'(I)$ such that T' is a regular distribution in $C^\infty(I)$. Prove T is, itself, a regular distribution and it belongs to $C^\infty(I)$.*

Exercise IV.2

Let I be an open interval of $\mathbb{R}$ containing 0 and $\theta \in \mathcal{D}(I)$ be a given function satisfying $\theta(0) = 1$.

E. IV.2.1 *Let $\varphi \in \mathcal{D}(I)$. Prove there exists a unique function $\psi \in \mathcal{D}(I)$ such that*

$$\forall x \in I, \quad \varphi(x) = \varphi(0)\theta(x) + x\psi(x).$$

E. IV.2.2 *Consider the equation*

$$xT = 0 \text{ in } \mathcal{D}'(I)$$

where $x \mapsto x$ is obviously in C^∞ and T is the unknown. Solve for T. Hint: use Q.IV.2.1.

E. IV.2.3 *Let $a, b \in I$. Consider the equation*

$$(x-a)(x-b)T = 0 \text{ in } \mathcal{D}'(I).$$

Solve for T. Hint: consider the cases $a \neq b$ and $a = b$ separately.

Exercise IV.3
Let $k \in \mathbb{N}$ and f, g be two functions in $C^k(I)$. We know from previous years that fg is in $C^k(I)$ and its k-th derivative is given by the Leibniz formula:

$$(fg)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}, \quad \text{où } \binom{k}{j} = \frac{k!}{j!(k-j)!}.$$

E. IV.3.1 *For $T \in \mathcal{D}'(\mathbb{R})$ and $\psi \in C^\infty(\mathbb{R})$, prove by induction that*

$$(\psi T)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} \psi^{(k-j)} T^{(j)} \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

E. IV.3.2 (Application) *Let $I = \mathbb{R}$. Compute the first and second distributional derivatives of*

$$x \mapsto H_0(x) \cos x$$

$$x \mapsto H_0(x) \sin x$$

Deduce a solution to the ODE $y'' + y = \delta_0$.

Exercise IV.4 (Estimate)
Prove

$$\forall x \in [0,1], \forall u \in H^1(0,1), \quad |u(x)| \leq \sqrt{2} \|u\|_{H^1}.$$

Hint: you may want to use the Young inequality $\forall a, b \in \mathbb{R}, 2ab \leq a^2 + b^2$.

Exercise IV.5 (A variational problem)
Let $f \in L^2(0,1)$ and

$$a : (u, v) \mapsto \int_{]0,1[} (u'v' + uv) - \frac{u(0)v(0)}{4}.$$

E. IV.5.1 *Prove a is properly defined, bilinear, continuous and coercive on $H^1(0,1) \times H^1(0,1)$.*

E. IV.5.2 *Prove there exists a unique $u \in H^1(0,1)$ such that $\forall v \in H^1(0,1), a(u,v) = \int_{]0,1[} fv$.*

E. IV.5.3 *Prove there exists a unique $u \in H^1(0,1)$ such that $\forall v \in H^1(0,1), a(u,v) = v(0)$.*

To be continued...

D) Going further

Exercise IV.6

E. IV.6.1 *Prove $v \mapsto \|v'\|_{L^2}$ is a norm on $H^1(0,+\infty)$.*

E. IV.6.2 *Is this norm equivalent to $\|v\|_{H^1}$?*

Exercise IV.7
Let $(T_j)_{j\in\mathbb{N}}$ be a sequence of distributions on I an open subset of $\mathbb{R}$ and $(f_j)_{j\in\mathbb{N}}$ a sequence of functions of class $C^\infty(I)$converging uniformly to f, along with all their derivatives, on every compact of I. Show that (f_jT_j) tends to fT.

Lecture V - Theoretical resolution of elliptic problems

NB1: At the beginning of each lab, Section A) contains the list of minimal knowledge and know-hows that are required to pass the course.
NB2: Questions marked with a $*$ or $**$ are optional. They may go beyond the scope of this course and will not be at the exams. However, they will help you have a better understanding of the material. If you choose to skip these questions (which you are allowed to do), you can use the result of this question to answer the next questions of the exercise.

A) Aims of this class

After this class,

- I know how to define a partial derivative in the sense of distributions.
- I know how to define the trace of a H^1 function on the boundary of the domain.
- I know the formulas that extend the integration by parts.
- I can find a variational formulation based on a linear elliptic problem, taking into account correctly the conditions at the boundary.
- I can solve a variational formulation.
- I know how to go back to the initial elliptic problem and to solve it theoretically.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions V.1 and V.2 must be done before the 5th lab. The solutions are available online.

Question V.1 (A variational problem (cont'd)) *Let $f \in L^2(0,1)$ and*

$$a : (u,v) \mapsto \int_{]0,1[} (u'v' + uv) - \frac{u(0)v(0)}{4}.$$

Q. V.1.1 *Prove a is properly defined, bilinear, continuous and coercive on $H^1(0,1) \times H^1(0,1)$.*

Q. V.1.2 *Prove there exists a unique $u \in H^1(0,1)$ such that $\forall v \in H^1(0,1), a(u,v) = \int_{]0,1[} fv$.*

Q. V.1.3 *Prove there exists a unique $u \in H^1(0,1)$ such that $\forall v \in H^1(0,1), a(u,v) = v(0)$.*

Q. V.1.4 *What is the regularity of the solution u in Question V.1.2?*

Q. V.1.5 *What is the regularity of the solution u in question V.1.3? What is the elliptic problem satisfied by u?*

Question V.2 (A convection-diffusion problem) *We solve here, theoretically, the 1D-stationary convection-diffusion problem:*

$$\textbf{(CD)} \qquad \begin{cases} -\kappa u''(x) + bu'(x) + c(x)u(x) = f(x), & x \in]0,1[, \\ u(0) = 0 \quad \text{and} \quad u(1) = 0, \end{cases}$$

with $\kappa \in \mathbb{R}^{+*}$, $b \in \mathbb{R}$, $c \in C^0([0,1], \mathbb{R}^+)$ and $f \in C^0([0,1], \mathbb{R})$.
Variable u (the unknown) represents the temperature if the problem of interest is heat transfer or the concentration if the problem of interest is mass transfer.

Q. V.2.1 *Show that, if $b = 0$, then (CD) has one and only one classical solution, that is, of class $C^2([0,1])$.*

Q. V.2.2 *Deduce that, in the general case, (CD) admits one and only one classical solution.*
Hint: Change unknowns $v : x \mapsto \exp(-\delta x)u(x)$, with δ to be determined.

Q. V.2.3 *We assume that κ, c are constant and $f : x \mapsto \exp(bx/(2\kappa))$. Solve (CD).*

C) Exercises

Exercise V.1 (Lifting)
Let a and $b \in \mathbb{R}$, $c \in C^0([0,1], \mathbb{R}^+)$ and $f \in C^0([0,1])$. Consider the problem

$$\begin{cases} -u'' + cu = f \ \text{ on }]0,1[, \\ u(0) = a \quad \text{and} \quad u(1) = b. \end{cases}.$$

E. V.1.1 *Let u_0 and u_1 be defined from $[0,1]$ to $\mathbb{R}$ by $u_0 : x \mapsto a + (b-a)x$ and $u_1 : x \mapsto a + (b-a)x^2$.*

E.V.1.1.1 *Show that there exists a unique $\tilde{u}$ (resp. $\bar{u}$) in $C^2([0,1])$ such that $u = u_0 + \tilde{u}$ (resp. $v = u_1 + \bar{u}$) is a solution of the problem.*

E.V.1.1.2 *Show that $u = v$. Is the problem well-posed in $C^2([0,1])$?*

E. V.1.2 *Same question for u_0 and u_1 functions of $C^2([0,1])$ such that $u_0(0) = u_1(0) = a$ and $u_0(1) = u_1(1) = b$.*

Exercise V.2
Let the problem:

$$\begin{cases} -\kappa u'' + cu = f \text{ in }]0,1[, \\ u'(0) = \alpha \quad \text{and} \quad u'(1) = 0, \end{cases}$$

where $\alpha \in \mathbb{R}$, $\kappa, c \in C^0([0,1], \mathbb{R}^{+*})$ and $f \in L^2(]0,1[)$.

E. V.2.1 *Of what type is this problem?*

E. V.2.2 *Write the variational formulation of this problem.*

E. V.2.3 *Show that the variational problem has a unique solution.*

E. V.2.4 *Give a functional space that makes the initial problem well-posed.*

E. V.2.5 *Rewrite the initial problem as a minimization problem.*

Exercise V.3 (Mixed boundary conditions)
We now turn to the following problem:

$$\textbf{(P)} \qquad \begin{cases} -u''(x) + c(x)u(x) = f(x), \ \ x \in]0,1[, \\ u(0) = 0 \text{ and } u'(1) = 0, \end{cases}$$

with $f \in \mathcal{C}^0([0,1], \mathbb{R})$ and $c \in \mathcal{C}^0([0,1], \mathbb{R}^+)$.

E. V.3.1 *What is the type of the boundary condition?*

E. V.3.2 *Prove the existence and uniqueness of the classical solution.*

Exercise V.4
Let $\Omega =]a,b[\times]c,d[$, with $a,b,c,d \in \mathbb{R}$, $a < b$ and $c < d$. Consider the problem

$$\begin{cases} -\Delta u = 1 \text{ in } \Omega, \\ u(a,y) = 0, \quad \partial_x u(b,y) = 0, \ c < y < d, \\ \partial_y u(x,c) = 1, \quad \partial_y u(x,d) = x, \ a < x < b. \end{cases}$$

E. V.4.1 *Write the associated variational formulation.*

E. V.4.2 *Study this variational formulation.*

D) Going further

Exercise V.5

Consider the problem (F) $\begin{cases} -u'' + qu = f \text{ in }]0,1[, \\ u(0) = 0 \quad \text{and} \quad u(1) = 0, \end{cases}$

where $f \in L^2(]0,1[)$ and q is a nonnegative constant.

E. V.5.1 *Give a variational formulation (FV) of (F) in a Hilbert space H to be precised. Denote respectively $a(\cdot,\cdot)$ and $\ell(\cdot)$ the bilinear (resp. linear) form associated with this variational problem.*

E. V.5.2 *Check that for all $q \geq 0$, (F) admits a unique solution u in a Sobolev space to be precised.*

Let $m \geq 1$. Introduce the finite dimensional vector space H_m generated by the functions

$$\phi_k : x \mapsto \sin(k\pi x), \quad k = 1, \cdots, m.$$

E. V.5.3 *Show that $H_m \subset H, \forall m \in \mathbb{N}^*$. Give the dimension of H_m.*

We approximate the solution of (FV) by $u_m = \sum_{k=1}^{m} \mathbf{u}_k \phi_k$ solution of

$$(\text{FV'}_m) \quad \text{Find } u_m \in H_m \text{ such that } \forall v_m \in H_m, \quad a(u_m, v_m) = \ell(v_m).$$

E. V.5.4 *Write the associated linear system. What can be said about its matrix ?*

E. V.5.5 *Deduce the expression of the coefficients $\mathbf{u}_k$, $k = 1, \ldots, m$, and those of u_m.*

E. V.5.6 *Justify the existence of a Hilbert basis $L^2(]0,1[)$, denoted by $(w_k)_{k\geq 1}$, such that, $\forall k \geq 1$,*

$$w_k \in H_0^1(]0,1[) \quad \textit{and} \quad \forall v \in H_0^1(]0,1[), \quad \int_0^1 w_k' v' \mathrm{d}x = \lambda_k \int_0^1 w_k v \mathrm{d}x.$$

E. V.5.7 *Establish a link between the families $(w_k)_{k\geq 1}$ and $(\phi_k)_{k\geq 1}$.*

E. V.5.8 *Show that $\forall m \in \mathbb{N}^*$, $\|u - u_m\|^2_{L^2(]0,1[)} \leq \dfrac{1}{(\pi^2(m+1)^2 + q)^2} \displaystyle\sum_{k=m+1}^{+\infty} \left(\int_0^1 f(x) \sin(k\pi x) \mathrm{d}x \right)^2$, then that $\|u - u_m\|^2_{L^2(0,1)} \to 0$ as $m \to +\infty$.*

Lecture VI - Finite Element Methods I

NB1: At the beginning of each lab, Section A) contains the list of minimal knowledge and know-hows that are required to pass the course.
NB2: Questions marked with a $*$ or $**$ are optional. They may go beyond the scope of this course and will not be at the exams. However, they will help you have a better understanding of the material. If you choose to skip these questions (which you are allowed to do), you can use the result of this question to answer the next questions of the exercise.

A) Aims of this class

After this class,

- I know the principle of an internal variational approximation (the Céa lemma, the interpolation operator).
- I can define the finite element method $\mathbb{P}_1$ in dimension 1.
- I can calculate the stiffness matrix.
- I installed and tested the Python platform FEniCS.
- I can program in Python or Matlab or FEniCS the approximate resolution of an elliptic problem in dimension 1.
- I know how to determine the order of the method numerically.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions VI.1, VI.2 and VI.3 must be done before the 6th lab. The solutions are available online.

Warning ! The platform FEniCS must be installed and tested on your computer before the lab. The instructions to install it are on edunao. You must attend the class with your computer.

Question VI.1 *Let $f \in L^2(0,1)$. Apply the $\mathbb{P}_1$ method to the problem*

$$(E) \begin{cases} -u'' = f & \text{in }]0,1[, \\ u(0) = a \quad \text{and} \quad u(1) = b. \end{cases}$$

with $a, b \in \mathbb{R}$. Check that the nonhomogeneous Dirichlet boundary conditions appear in the right-hand side of the linear system that we got.

Question VI.2 *Consider the problem*

$$(E) \begin{cases} -u'' + bu' + cu = f & \text{in }]0,1[, \\ u(0) = 0 \quad \text{and} \quad u(1) = 0. \end{cases}$$

The functions f and (b, c) are given respectively in $L^2(0,1)$ and in $C^0([0,1]) \times C^0([0,1])$. We discretize $[0,1]$ uniformly in $J+1$ intervals, $J \geq 1$.

Q. VI.2.1 *Write (E) as a symmetric variational problem (PV). Give a sufficient condition on $c(\cdot)$ so that there exists a unique solution in a well-chosen Hilbert space H.*

Let $H_h \subset H$, the vector subspace of finite dimension J defined as

$$H_h = \{v \in H \;/\; v_{|[x_j, x_{j+1}]} \in \mathbb{P}_1, j \in \{0, \cdots, J\}, \quad \text{and} \quad v(0) = v(1) = 0\}.$$

Q. VI.2.2 *Give the linear system associated with the approximated variational problem in the basis of "hat-functions" :*

$$(PV_h) \quad \text{Find } u_h \in H_h \text{ such that } \quad \forall\, v_h \in H, \quad a(u_h, v_h) = \ell(v_h),$$

where $a(\cdot,\cdot)$ (resp. $\ell(\cdot)$) is the bilinear (resp. linear) form that appears in (PV).

Question VI.3 *Consider the equation of Problem (E) with Neumann boundary conditions:*

$$u'(0) = 0 \quad \text{and} \quad u'(1) = 0.$$

Q. VI.3.1 *Define H and give a sufficient condition on c to ensure that the new problem admits a unique solution.*

Q. VI.3.2 *Define H_h and the associated linear system.*

C) Exercises

The exercises can be found on edunao as Jupyter notebooks.

Lecture VII - Finite Element Methods II

NB1: At the beginning of each lab, Section A) contains the list of minimal knowledge and know-hows that are required to pass the course.
NB2: Questions marked with a $*$ or $**$ are optional. They may go beyond the scope of this course and will not be at the exams. However, they will help you have a better understanding of the material. If you choose to skip these questions (which you are allowed to do), you can use the result of this question to answer the next questions of the exercise.

A) Aims of this class

After this class,

- I know how to get a mesh of a domain with a software in dimension 2.
- I can define the finite element method $\mathbb{P}_1$ in dimension 2.
- I can assemble the stiffness matrix.
- I can program in FEniCS the solution of an elliptic problem in dimension 1 and 2.

B) To become familiar with this class' concepts (to prepare before the examples class)

Question VII.1 must be done before the 7th lab. The solution is available online.

Warning ! You must attend the class with your computer.

Question VII.1 *Let Ω be an open polyhedral bounded subset of $\mathbb{R}^2$. We want to solve approximately the Dirichlet problem over Ω.*

Q. VII.1.1 *Recall the statement of the Dirichlet problem in dimension 2.*

Q. VII.1.2 *For a given triangular mesh $\mathcal{T}$ of Ω, describe the finite element method $\mathbb{P}_1$: type of problem, polynomials to use, and so on.*

Consider a given triangle K and let a_1, a_2, a_3 be its vertices. For $j \in \{1,2,3\}$, denote λ_j the barycentric coordinate associated with the vertex a_j. We already know that these 3 functions $(\lambda_j)_{j\in\{1,2,3\}}$ is a basis of the polynomial space $\mathbb{P}_1$.

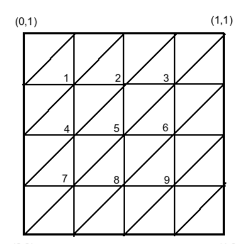
Q. VII.1.3 *Let $M \in K$ with coordinates (x, y). For $j \in \{a, b, c\}$, give the expression of the barycentric coordinate λ_j depending on x and on y.*

Define the elementary stiffness matrix $\mathcal{A}$ associated with the triangle K as the symmetric 9-element matrix: $a_{ij} = \displaystyle\int_K \nabla\lambda_i \cdot \nabla\lambda_j \mathrm{d}x\mathrm{d}y.$

Q. VII.1.4 *Let $h > 0$. Compute $\mathcal{A}$ for the triangle of vertices $(0,0)$, $(0,-h)$, $(h,0)$.*

Q. VII.1.5 *Let $h > 0$. Compute $\mathcal{A}$ for the triangle of vertices $(0,0)$, $(-h,-h)$, $(0,-h)$.*

Q. VII.1.6 *Compute the stiffness matrix for the Dirichlet problem in the square $]0,1[\times]0,1[$ equipped with the following mesh.*



C) Exercises

The exercises can be found on edunao as Jupyter notebooks.

Lecture VIII - Numerical Analysis

NB1: At the beginning of each lab, Section A) contains the list of minimal knowledge and know-hows that are required to pass the course.
NB2: Questions marked with a $*$ or $**$ are optional. They may go beyond the scope of this course and will not be at the exams. However, they will help you have a better understanding of the material. If you choose to skip these questions (which you are allowed to do), you can use the result of this question to answer the next questions of the exercise.

A) Aims of this class

After this class,

- I can characterize a positive definite symmetric matrix.
- I can determine a region of the plane containing all the eigenvalues of a matrix.
- I can code a method that gives me the spectral radius of a matrix.
- I know the difference between a direct method and an iterative method that solve linear systems, and how to use them.
- I know how to use the notion of condition number of a matrix.
- I know how to evaluate the complexity of a numerical method.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions VIII.1 and VIII.2 must be done before the 8th lab. The solutions are available online.

Question VIII.1 (Applications of the Schur theorem) *Let $A \in \mathcal{M}_q(\mathbb{C})$, where $q \geq 1$.*

Q. VIII.1.1 *Recall the decomposition theorem.*

Q. VIII.1.2 *Show that A is normal, that is, $AA^* = A^*A$, if and only if there exists a unitary matrix U and a diagonal matrix D containing the eigenvalues of A such that $A = UDU^*$.*
HINT: *Compute the diagonal elements of TT^* and T^*T where T is the upper triangular matrix such that $A = UTU^*$ and U is unitary.*

Q. VIII.1.3 *Show that if A is Hermitian, it is diagonalizable in an orthogonal basis of eigenvectors and its eigenvalues are real.*

Q. VIII.1.4 *Show that, if A is unitary, it is diagonalizable in an orthogonal basis of eigenvectors and that its eigenvalues are of modulus 1.*

Question VIII.2 (Solving triangular systems) *Let $b \in \mathbb{K}^q$, where $q \geq 1$.*

Q. VIII.2.1 *Let $L \in T_{q,inf}(\mathbb{K})$. Write the resolution algorithm of the linear system $Lx = b$.*

Q. VIII.2.2 *Let $U \in T_{q,sup(\mathbb{K})}$. Write the resolution algorithm of the linear system $Ux = b$.*

Q. VIII.2.3 *Compute the number of operations needed to perform these resolutions.*

C) Exercises

Exercise VIII.1 (An intuitive approach to the resolution of linear systems)
Let $A \in GL_q(\mathbb{R})$. Let $b \in \mathbb{R}^q$. What is the complexity of the resolution of the linear system $A^2x = b$ with a direct method? Propose a less costly method.

Exercise VIII.2 (LU decomposition of the matrix appearing in the Theorem ??)
Let $A \in \mathcal{M}_q(\mathbb{R})$ be a tridiagonal matrix, defined as:

$$A = \begin{pmatrix} 2+c_1 & -1 & & & & \\ -1 & 2+c_2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2+c_{q-1} & -1 \\ & & & & -1 & 2+c_q \end{pmatrix},$$

with $c_i \geq 0, \quad \forall 1 \leq i \leq q$.

E. VIII.2.1 *Let $v \in \mathbb{R}^q$. Show that:*

$$v^T A v = \sum_{i=1}^{q} c_i v_i^2 + \left\{ v_1^2 + v_q^2 + \sum_{i=2}^{q} (v_i - v_{i-1})^2 \right\}.$$

E. VIII.2.2 *Deduce that A admits a Cholesky decomposition.*

E. VIII.2.3 *In the case $c_i = 0$, $(1 \leq i \leq q)$, show that the matrix B such that $A = BB^T$ is bidiagonal and given by:*

$$\begin{cases} B_{i,i} & = \sqrt{\dfrac{i+1}{i}} \quad (1 \leq i \leq q); \\ B_{i+1,i} & = -\sqrt{\dfrac{i}{i+1}} \quad (1 \leq i \leq q-1). \end{cases}$$

Exercise VIII.3 (QR decomposition)
Let $A \in GL_q(\mathbb{R})$.

E. VIII.3.1 *Show that there exists an upper triangular matrix R with positive diagonal such that:*

$$A^T A = R^T R.$$

E. VIII.3.2 *Deduce that there exists an orthogonal matrix $Q \in O_q(\mathbb{R})$ such that: $A = QR$.*

E. VIII.3.3 *Show that this decomposition $A = QR$, R being upper triangular with positive diagonal and Q being orthogonal, is unique.*

Exercise VIII.4
Let $\omega \in \mathbb{R} \setminus \{-1\}$. Let $A \in \mathcal{M}_q(\mathbb{R})$ be a non-singular matrix such that

$$A = (1+\omega)P - (N + \omega P),$$

with P invertible and $P^{-1}N$ having real eigenvalues $1 > \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_q$.

E. VIII.4.1 *Find the values of ω such that the iterative method*

$$\forall n \geq 0, \quad (1+\omega)Px^{(n+1)} = (N + \omega P)x^{(n)} + b$$

converges for all initial vector x_0 to the solution of the system $Ax = b$.

E. VIII.4.2 *Find the value of ω that guarantees a minimal convergence rate, so that the convergence speed is optimized.*

D) Going further

These exercises can be found on edunao as Jupyter notebooks.

Lecture IX - Finite Difference Method

NB1: At the beginning of each lab, Section A) contains the list of minimal knowledge and know-hows that are required to pass the course.
NB2: Questions marked with a $*$ or $**$ are optional. They may go beyond the scope of this course and will not be at the exams. However, they will help you have a better understanding of the material. If you choose to skip these questions (which you are allowed to do), you can use the result of this question to answer the next questions of the exercise.

A) Aims of this class

After this class,

- I can write a finite difference discretization of a stationary partial differential equation.
- I can write the corresponding linear system.
- I can compute the order of consistency of the method.
- I can prove its stability.
- I can prove the convergence of the method.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions IX.1 and IX.2 must be done before the 9th lab. The solutions are available online.

From now on, $J \geq 1$ is fixed and the interval $[0,1]$ is split into $J+1$ subintervals of length h. For $j \in \{0,\ldots,J+1\}$, let $x_j = jh$ and denote the solution to the approximated problem by $V_h = (v_j)_{j\in\{1,\ldots,J\}}$. As well, let $F = (f(x_j))_{j\in\{1,\ldots,J\}}$ and $C = (c(x_j))_{j\in\{1,\ldots,J\}}$.

Question IX.1 *Let $u \in C^6(\mathbb{R})$, $(x,h) \in \mathbb{R}^2$.*

Q. IX.1.1 *Show the following equality*

$$u^{(4)}(x) = \frac{u(x-2h) - 4u(x-h) + 6u(x) - 4u(x+h) + u(x+2h)}{h^4} + \mathcal{O}(h^2)$$

Q. IX.1.2 *Let us consider the finite difference formula*

$$u'(x) = \frac{u(x-2h) - 8u(x-h) + 8u(x+h) - u(x+2h)}{12h} + \mathcal{O}(h^k)$$

What is the order of approximation of this method?

Question IX.2 *Let us establish some results on the approximate resolution of*

$$\textbf{(CD)} \qquad \begin{cases} -\nu u''(x) + bu'(x) + c(x)u(x) = f(x), & x \in]0,1[, \\ u(0) = 1 \text{ and } u(1) = 0, \end{cases}$$

with $\nu \in \mathbb{R}^{+*}$, $b \in \mathbb{R}^+$, $c \in C^2([0,1],\mathbb{R}^+)$ and $f \in C^2([0,1],\mathbb{R})$.
We introduce the three following discretizations of u' $h \in \mathbb{R}$:

$$\text{downwind: } \frac{u(\cdot+h) - u(\cdot)}{h}, \text{ upwind: } \frac{u(\cdot) - u(\cdot-h)}{h}, \text{ centered: } \frac{u(\cdot+h) - u(\cdot-h)}{2h}$$

Q. IX.2.1 *Give the consistency errors of these discretizations. What are their orders?*

Q. IX.2.2 *Recall the formula of the 3-point centered scheme used to discretize u''.*

Q. IX.2.3 *Give the finite difference schemes of (CD) corresponding to the discretizations of the first-order derivative that follow.*

Q. IX.2.4 *Give the orders of these schemes.*

C) Exercises

Exercise IX.1
Let $J \geq 1$, $(\alpha_j)_{j\in\{1,\ldots,J\}} \in \mathbb{R}^J$, β and $\gamma \in \mathbb{R}_*^+$ such that $\alpha_j - \beta - \gamma \geq 0$ for any $j \in \{1,\ldots,J\}$. Let A the square matrix of size $J \geq 1$ with coefficients, for all $(i,j) \in \{1,\ldots,J\}^2$,

$$a_{ij} = \begin{cases} \alpha_i & \text{if } i = j, \\ -\beta & \text{if } i = j+1,\ j \leq J-1, \\ -\gamma & \text{if } i = j-1,\ j \geq 2, \\ 0 & \text{otherwise.} \end{cases} .$$

E. IX.1.1 *Let $G \in (\mathbb{R}^+)^J$ such that $G \in Im(A)$, that is, there exists $V \in \mathbb{R}^J$ such that $G = AV$. Show that $V \in (\mathbb{R}^+)^J$.*

E. IX.1.2 *Deduce that A is invertible, and that the coefficients of A^{-1} are nonnegative.*

Exercise IX.2
This exercise follows Question Q.IX.2, and uses the results of Exercise E.IX.1.

E. IX.2.1 *Write the schemes in a matrix form. Let A_h^+ (resp. A_h^-, A_h^0) the matrix corresponding to the downwind discretization (resp. to the upwind discretization, to the centered scheme). What is the shape of the right-hand side in the different cases ?*

E. IX.2.2 *We want to define a scheme that satisfies the property of the discrete maximum priciple. Which scheme should be used, with respect to the values of b? Sum up the results in a table.*

E. IX.2.3 *Let $V \in \mathbb{R}^J$. Let $v_0 = v_{J+1} = 0$. Show the Discrete Poincaré Inequality:*

$$\|V\|_2^2 := h\sum_{i=0}^{J} v_i^2 \leq \sum_{i=0}^{J} \frac{(v_{i+1} - v_i)^2}{h}.$$

Denote by B_h the matrix corresponding to $b = 0$, $c = 0$.

E. IX.2.4 *Let $V \in \mathbb{R}^J$. Set $v_0 = v_{J+1} = 0$. Prove that $(B_h V, V) = \dfrac{\nu}{h^2}\displaystyle\sum_{i=0}^{J}(v_{i+1} - v_i)^2$.*

Assume from now on that $b > 2\nu/h$. Moreover, assume $v_0 = v_{J+1} = 0$: note that, according to the previous table, an upwind discretization gives a monotonous matrix.

E. IX.2.5 *Explain why considering homogeneous boundary conditions does not contradict (CD).*

E. IX.2.6 *Show that $(A_h V, V) \geq \left(\dfrac{\nu}{h} + \dfrac{b}{2}\right)\|V\|_2^2$, then that $(A_h V_h = G \Rightarrow \|V_h\|_2 \leq \dfrac{1}{(\nu + bh/2)}\|G\|_2)$.*

E. IX.2.7 *Give a L^2 estimate of the error.*

Exercise IX.3 (Boundary conditions of Dirichlet-Neumann type)
Consider the following problem:

$$\textbf{(P)} \qquad \begin{cases} -u''(x) + c(x)u(x) = f(x), & x \in]0,1[, \\ u(0) = 0 \text{ and } u'(1) = 0, \end{cases}$$

with $f \in C^2([0,1], \mathbb{R})$ and $c \in C^2([0,1], \mathbb{R}^+)$. At first, discretize u'' with the classical centered scheme and assume that at the fictive point x_{J+2}, $v_{J+2} = v_{J+1}$.

E. IX.3.1 *Give the discretization matrix A_h.*

E. IX.3.2 *Show that the matrix A_h satisfies the maximum principle.*

E. IX.3.3 A priori, *is the scheme consistent with (P)?*

E. IX.3.4 *Let $Y \in \mathbb{R}^{J+1}$ such that $y_{J+1} = 0$. Show that (V_h solution of $A_h V_h = Y \Rightarrow \|V_h\|_\infty \leq \dfrac{1}{2}\|Y\|_\infty$).*

E. IX.3.5 *Let $Y \in \mathbb{R}^{J+1}$ such that $y_j = 0$ if $j \in \{1, \ldots, J\}$. Show that*
(V_h such that $A_h V_h = Y \Rightarrow \|V_h\|_\infty \leq h\|Y\|_\infty = h|y_J|$).

E. IX.3.6 *Deduce that the scheme is convergent. What is its order ?*

Now set $v_{J+2} = v_J$.

E. IX.3.7 *What discretisation does this correspond to? Compute the order of consistency.*

D) Going further

These exercises can be found on edunao as Jupyter notebooks.

Lecture X - Parabolic problems

NB1: At the beginning of each lab, Section A) contains the list of minimal knowledge and know-hows that are required to pass the course.
NB2: Questions marked with a $*$ or $**$ are optional. They may go beyond the scope of this course and will not be at the exams. However, they will help you have a better understanding of the material. If you choose to skip these questions (which you are allowed to do), you can use the result of this question to answer the next questions of the exercise.

A) Aims of this class

After this class,

- I can recognize a parabolic equation.
- I know the fundamental qualitative properties of a parabolic equation (asymptotic behaviour, maximum principle, regularization).
- I know how to discretize a parabolic problem in time with an Euler scheme and in space by a finite element or finite difference method.
- I know the concept of CFL condition.
- I know how to code an iterative algorithm that gives me a numerical solution, evolving over time.

B) To become familiar with this class' concepts (to prepare before the examples class)

Question X.1 must be done before the 10th lab. The solution is available online.

Question X.1 (Diagonalizing the Laplacian matrix)
Let $J \in \mathbb{N}^*$.

Q. X.1.1 *Recall the matrix $A_{\Delta x}$ corresponding to the discretization of the operator $u \mapsto -u''$ with Dirichlet boundary conditions for a space step $\Delta x = 1/(J+1)$.*

Q. X.1.2 *Show that the matrix $A_{\Delta x}$ is diagonalizable with real positive eigenvalues.*

Q. X.1.3 *Using Gershgörin-Hadamard's theorem, show that $\mathrm{Sp}(A_{\Delta x}) \subset]0, 4/(\Delta x)^2]$.*

Q. X.1.4 *Let $\lambda \in \mathrm{Sp}((\Delta x)^2 A_{\Delta x})$. Let $\Lambda = 2 - 2\cos(\theta)$ for some $\theta \in]0, \pi]$. Let v be an eigenvector associated to Λ. Compute v and deduce the possible values of θ.*

Q. X.1.5 *What is the spectrum of $A_{\Delta x}$?*

C) Exercises

We consider a homogeneous rod of length 1 whose temperature at time t and position x is $u(t,x)$. Both ends of the rod (positions $x = 0$ and $x = 1$) are kept at zero temperature. A heat source f is applied to every point of the rod. This heat source is constant over time. The evolution of the temperature in the rod is governed by:

$$(\mathrm{H}) \quad \begin{cases} \partial_t u(t,x) - \alpha \partial^2_{xx} u(t,x) = f(x), \ t > 0, \ x \in]0,1[, \\ u(0,x) = u^0(x), \ x \in]0,1[, \\ u(t,0) = 0, \ u(t,1) = 0, \ t \geq 0. \end{cases}$$

We assume $u^0, f \in L^2(0,1)$. The problem is called *homogeneous* if $u^0 = f = 0$.

Exercise X.1 (Finite Difference Approximation)
Let us assume $f \in C^2([0,1])$.
Let $T > 0$.
Let $(J,N) \in (\mathbb{N}^*)^2$, $\Delta x = 1/(J+1)$ and $\Delta t = T/N$. We discretize the domain $[0,T] \times [0,1]$ with a grid

$$(t^n, x_j) = (n\Delta t, j\Delta x), \ \ n \in \{0, \dots, N\}, \ j \in \{0, \dots, J+1\}$$

Let us consider $y' = a(t,y)$. For $\theta \in [0,1]$., we call *θ-scheme* the method

$$(S_\theta) \quad \begin{cases} z^0 \quad \text{given} \\ z^{n+1} = z^n + \Delta t(1-\theta) a(t^n, z^n) + \Delta t \theta a(t^{n+1}, z^{n+1}). \end{cases}$$

E. X.1.1 *Let $\theta \in [0,1]$. Write the finite difference scheme as a recursive relation in time and space for (H) with the θ-scheme in temps and the three-point stencil in space.*

The scheme for $\theta = 1/2$ is called *Crank-Nicolson*.

E. X.1.2 *Compute the consistency error and the order of the θ-scheme.*

E. X.1.3 *Give a sufficient condition for stability in L^2 of the CFL type for the θ-scheme. Why is Crank-Nicolson so interesting?*

E. X.1.4 *Prove the θ-scheme converges under the CFL-type condition.*

Exercise X.2 (Resolution using separation of variables)
For the sake of simplifying the formulas, we will consider $\alpha = 1$ throughout this exercise. A general α does not introduce any mathematical difficulty, though (provided it is positive).

Consider $\mathcal{H}_0$ and $\mathcal{H}$ defined by

$$\mathcal{H}_0 = C^0([0,+\infty[, L^2(0,1)) \cap C^\infty(]0,+\infty[\times[0,1])$$

$$\mathcal{H} = C^0([0,+\infty[, L^2(0,1)) \cap C^\infty(]0,+\infty[, H^2(0,1))$$

E. X.2.1 *Prove (H) has at most one solution in $\mathcal{H}_0$ by considering the energy associated to the homogeneous problem $E : t \mapsto \int_{]0,1[} \tilde{u}^2(t,x)dx$.*

We now look for a formal solution u to $\partial_t u = \partial^2_{xx} u$ with BC $u(\cdot,0) = u(\cdot,1) = 0$ that **does not** vanish. We shall look for the solution in the form $u(t,x) = T(t)X(x)$.

E. X.2.2 *Prove that X''/X and T'/T are equal to a non-positive constant. We will note $-\lambda^2$ this constant.*

E. X.2.3 *Assuming there exists a solution in $\mathcal{H}_0$, deduce the general expression of the solutions u with separate variables.*

E. X.2.4 *Assuming there exists a solution in $\mathcal{H}$, what is the expression of the solution to (H)?*

E. X.2.5 *Prove that $u \in \mathcal{H}$.*

E. X.2.6 *What is the steady state problem associated to (H)?*

E. X.2.7 *What is the limit function $\bar{u}$ (which is independent of time) toward which converges $u(t,\cdot)$ when t goes to $+\infty$? Prove $\|u(t,\cdot) - \bar{u}(\cdot)\|_{L^2(0,1)} \underset{t\to+\infty}{\longrightarrow} 0$.*

Exercise X.3
Let us consider these schemes, defined for $j \in \{1,\dots,J\}$ and $n \in \{0,\dots,N-1\}$, by:

- The Six-point scheme:

$$\frac{v_{j+1}^{n+1} - v_{j+1}^n}{12\Delta t} + \frac{5(v_j^{n+1} - v_j^n)}{6\Delta t} + \frac{v_{j-1}^{n+1} - v_{j-1}^n}{12\Delta t} - \alpha\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{2\Delta x^2} - \alpha\frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{2\Delta x^2} = f_j,$$

- The DuFort-Frankel scheme:

$$\frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} - \alpha\frac{v_{j+1}^n - v_j^{n+1} - v_j^{n-1} + v_{j-1}^n}{\Delta x^2} = f_j,$$

- The Gear Scheme:
$$\frac{3v_j^{n+1} - 4v_j^n + v_j^{n-1}}{2\Delta t} - \alpha \frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{\Delta x^2} = f_j.$$

E. X.3.1 (Consistency) *Prove the order of the six-point scheme is 2 in time and 4 in space. Prove the order of the Gear scheme is 2 in time and 2 in space. What are the orders of the DuFort-Frankel scheme in space and time?*

E. X.3.2 (Stability in norm L^2) *Prove that all three schemes are inconditionnally stable in the norm L^2.*

E. X.3.3 (Maximum principle) *Prove that, if $2\alpha\Delta t/\Delta x^2 \leq 1$, then the θ-scheme verifies the maximum principle. Prove the same result for the DuFort-Frankel if $\alpha\Delta t/\Delta x^2 \leq 1$. What can we say of the stability in the norm L^∞?*

D) Going further

These exercises can be found on edunao as Jupyter notebooks.