Fiber Bundles and Connections

L. N. Queiroz Xavier

March 2, 2021

From the standard model of particle physics to the theory of topological phases of matter, gauge theory arises almost everywhere in fundamental physics. Here we will introduce the mathematical basis of gauge theory, which is the theory of *connections on fiber bundles.* This exposition is mostly based on references [1, 2].

1 Basic Differential Geometry

Before we talk about fiber bundles, we need to do a little review of differential geometry. Differential geometry allows us to do calculus on spaces that are not really Euclidean, but are very close to be, in a local sense which we will define later. But why would we need to do this? Can't we just apply the usual rules of calculus to functions over arbitrary spaces and be happy?

The answer is: not really. Recall the definition of limits in real calculus. Given a function $f : \mathbb{R} \to \mathbb{R}$, the limit $\lim_{x\to x_0} f(x)$ of this function approaches L iff for all $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - x_0| < \delta$ implies $|f(x) - L| < \epsilon$. This definition relies on our ability to measure *distances* between points on $\mathbb{R}$. Then, to do calculus on an arbitrary manifold M essentialy we need a way to measure distances between points on M and to be able to tell if they are close or not to each other. Euclidean spaces have a natural definition of distance between points, given by the length of the straight line that joins them. Other spaces might not have such a natural definition of distance between points. Take for instance $M = S^1$, the unity circle. We could define the distance between two points as the length of the arc that joins them, as in figure 1. However, there are two arcs joining them as in figure 2, and the choice of which one to use becomes arbitrary. To solve this problem, we could embed the circle inside $\mathbb{R}^2$ and then use the Euclidean distance to measure how far apart are points in the circle. This is essentialy how one turns S^1 into a manifold.

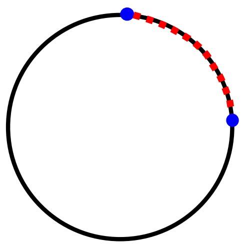

Figure 1: An arc length can be used to measure the distance between two points.

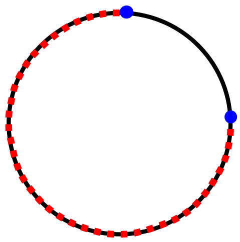

Figure 2: But which arc to use?.

1.1 Manifolds

Let's start by introducing a general, metric-independent way to tell if a function is *continuous*. Let M be a set and let $F = \{U_\alpha\}$ be a family of subsets of M. F is a *topology* iff both M and $\emptyset$ belong to F, the union of arbitraly many $U_\alpha' s$ belongs to F, i.e., $\cup_\alpha U_\alpha \in F$ and the pairwise intersection $U_\alpha \cap U_\beta \in F$. The space M, together with the familiy of open sets F is called a *topological space*. Intuitively, we can see an open set as a collection of nearby points. To be consistent, all points of M seem globally are near, and we demand that the set with no points is also a "local" object. That's why $M, \emptyset \in F$. The union of a collection of nearby points makes a bigger collection of nearby points, that's why we demand $\cup_\alpha U_\alpha \in F$. The intersection between two collections of nearby points can either be empty, meaning that the collections are not near, or not, meaning that they are indeed near. In both cases, the result belongs to F.

A continuous function essentialy sends nearby points to nearby points. This is captured by the following definition: let M, N be two topological spaces and let $f : M \to N$ be a function between them. We say that f is *continuous* iff for all $V \subset N$ open, $f^{-1}(V) \subset M$ is also open. That is, the inverse image of nearby points in N is composed by nearby points.

We can now define a manifold. Let M be a topological space with an *open cover*, that is, a collection of open sets $\{U_\alpha\}$ such that

$$M = \bigcup_\alpha U_\alpha.$$

Define a *chart* as an homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$, i.e., ϕ_α is a continuous invertible function from some open set $U_\alpha \subset M$ to n-dimensional Euclidean space $\mathbb{R}^n$. Suppose we can define a chart ϕ_α for all open sets U_α which cover M. This is basically to say that we can turn each point of M into a point of $\mathbb{R}^n$ in a continuous fashion, constructing an image of M inside

Euclidean space and respecting distances between points, with nearby points in M going to nearby points in $\mathbb{R}^n$. A the family (U_α, ϕ_α) of charts is called an *atlas* of M. Then, we might consider that M is a n-dimensional *manifold* if we can define an atlas on it. In other words, a n-dimensional manifold is a space that can be seem as composed by "small" parts that look like $\mathbb{R}^n$. But there is an ambiguity still. In the intersection $U_\alpha \cap U_\beta$, we can use two different charts $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ and $\phi_\beta : U_\beta \to \mathbb{R}^n$, which might send the same point $p \in U_\alpha \cap U_\beta$ to two diferent points $x = \phi_\alpha(p)$ and $x' = \phi_\beta(p)$ in $\mathbb{R}^n$. To resolve this issue, we demand that the charts are such that there is a *smooth* function relating x and x'. That is, we must have that the *transition function* $\phi_\beta \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. This can be thought as a rule of how to glue together the open sets U_α to form M.

Smooth transition functions also allow us to define uniquely what we mean when we say that a function $f : M \to \mathbb{R}$ is smooth. Such a function is smooth iff $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth for all α. At the intersection $U_\alpha \cap U_\beta$, we can choose two charts ϕ_α and ϕ_β such that $f \circ \phi_\alpha^{-1}$ and $f \circ \phi_\beta^{-1}$ are smooth. But there is no ambiguity because both functions are related through a smooth map:

$$f \circ \phi_\alpha^{-1} = f \circ (\phi_\beta^{-1} \circ \phi_\beta) \circ \phi_\alpha^{-1} = (f \circ \phi_\beta^{-1}) \circ (\phi_\beta \circ \phi_\alpha^{-1}),$$

which is the transition function.

An intuitive way of seeing what is a manifold is to think of $\mathbb{R}^n$ as a "grid", whose points are coordinates in $\mathbb{R}^n$. Then, the charts take pieces of this grid and "plug" them on top of M. On intersections, the grid lines must be joined in a smooth fashion. Hence the name atlas is justified, because it is really in the same spirit as when people put coordinates (latitude and longitude) on the surface of Earth.

Let's consider some simple examples of manifolds. Consider the *unity circle* as the set

$$S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}. \tag{1.1}$$

To make it a manifold, we need 1) an open covering, 2) charts and 3) to make sure that transition functions are smooth. Consider the sets

$$U_1 = S^1 - \{(1, 0)\}$$

and

$$U_2 = S^1 - \{(-1, 0)\}.$$

Both are open, because they are topologically equivalent to open intervals in the real line. Clearly,

$$S^1 = U_1 \cup U_2,$$

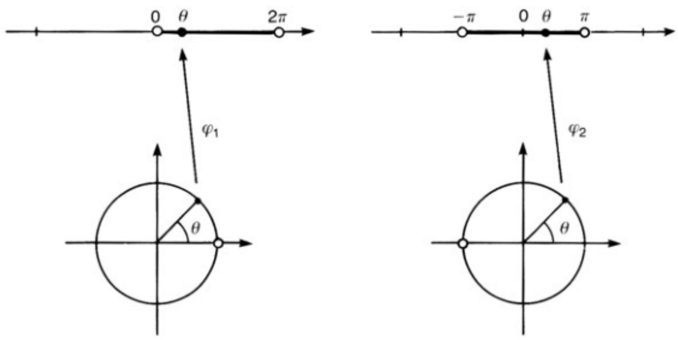

Figure 3: Charts of the circle S^1. Source: [1].

hence we have found our open covering of S^1. Now, recall the change of coordinates

$$(x, y) = (\cos\theta, \sin\theta),$$

where $\theta \in (0, 2\pi]$. This gives us good candidates for charts. For $\theta \in (0, 2\pi)$, the map

$$\phi_1^{-1}(\theta) = (\cos\theta, \sin\theta)$$

sends $\theta \in \mathbb{R}$ to a point in $S^1 - \{(1, 0)\}$. Likewise, for $\theta \in (-\pi, \pi)$, the map

$$\phi_2^{-1}(\theta) = (\cos\theta, \sin\theta)$$

sends $\theta \in \mathbb{R}$ to a point in $S^1 - \{(-1, 0)\}$. Since both maps are clearly continuous and invertible, their inverses being $\phi_1(x, y) = \tan^{-1}(y/x)$, for $(x, y) \in U_1$ and $\phi_2(x, y) = \tan^{-1}(y/x)$ for $(x, y) \in U_2$, we have two good charts for S^1, shown in 3, turning S^1 into a 1-manifold. Since, for $\theta \in (0, \pi)$, $(\phi_1 \circ \phi_2^{-1})(\theta) = \theta$, the transition function is clearly smooth.

As another example, consider the *2-torus* $T^2 = S^1 \times S^1$. Let's use the atlas we built for S^1 to construct an atlas for T^2. Define

$$U = U_1 \times U_1,\ V = U_2 \times U_2.$$

Intuitively, U is a torus with a strip removed from the outer surface, while V is a torus with a strip removed from the inner surface. Thus, $T^2 = U \cup V$. Define also $\phi : U \to \mathbb{R}^2$ such that $\phi^{-1}(\theta, \eta) = (\phi_1^{-1}(\theta), \phi_1^{-1}(\eta))$ and $\psi : V \to \mathbb{R}^2$ such that $\psi^{-1}(\theta, \eta) = (\phi_2^{-1}(\theta), \phi_2^{-1}(\eta))$. It is clear that those are smooth homeomorphisms and that the transition function $\phi \circ \psi^{-1} = \mathrm{id}$ is smooth on $U \cap V$. Hence, T^2 is a 2-manifold.

A last example: the *2-sphere*

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

We may procceed exactly as in the case of S^1. Define the sets

$$U_1 = S^2 - \{(0,0,1)\},\ U_2 = S^2 - \{(0,0,-1)\},$$

which are indeed open because they are topologically equivalent to an open ball. The set U_1 is the sphere with the north pole removed, while U_2 is the sphere with the south pole removed. Thus, $S^2 = U_1 \cup U_2$. Now, for $i = 1, 2$, define the maps $\phi_i : U_i \to \mathbb{R}^2$ such that

$$\phi_i^{-1}(\theta, \eta) = (\sin\theta\cos\eta, \sin\theta\sin\eta, \cos\theta),$$

where for $i = 1$, $\theta \in (0, 2\pi)$ and $\phi \in (0, \pi)$ and, for $i = 2$, $\theta \in (-\pi, \pi)$ and $\phi \in (0, \pi)$. Clearly, those are smooth homeomorphisms and the transition function $\phi_1 \circ \phi_2^{-1} = \text{id}$ is also smooth on $(0, \pi)$. Thus, the 2-sphere is a 2-manifold.

1.2 Tangent vectors

Our motivation in defining manifolds is that we want to do calculus on cool and interesting spaces such as spheres, Klein bottles, torus, Möbius bands and etc. We already know how to check if a scalar function on a n-manifold M, $f : M \to \mathbb{R}$, is smooth. We only have to check the smoothness of $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ for every chart ϕ_α in the atlas of M. The next thing we might want to do is to differentiate the function f along some curve

$$\gamma : [a, b] \to M$$

on the manifold. That is, we want to find how f changes as we move along γ on M. Let's see how that can be done.

To get the rate of change of f along γ at some point $p = \gamma(0)$, we have to take the derivative of $(f \circ \gamma)(t)$ with respect to t at $t = 0$. To properly take derivatives, we must work on Euclidean spaces, i.e., we have to consider that γ is contained on a chart U_α and use the map $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ to map it to $\mathbb{R}^n$. We have then to work with the map

$$\phi_\alpha \circ \gamma : [a, b] \to \mathbb{R}^n.$$

In the same spirit, we have to map the domain of f to Euclidean space and work with the map

$$f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}.$$

The rate of change of the function f along the curve γ at the point $p = \gamma(0)$ is thus given by

$$\frac{d}{dt} \left((f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \gamma) \right)(t) \Big|_{t=0}. \tag{1.2}$$

It is common to write $\phi_\alpha(p) = x^\mu(p)$ to express the fact that $\phi_\alpha(p)$ is a vector in $\mathbb{R}^n$. Then, we have that $(\phi_\alpha \circ \gamma)(t) = x^\mu(\gamma(t))$ and using the chain rule, we can write the directional derivative of f as

$$\frac{\partial}{\partial x^\mu}((f \circ \phi_\alpha^{-1})(x)) \frac{d}{dt} x^\mu(\gamma(t))\Big|_{t=0}, \tag{1.3}$$

where we are using Einstein's summation convention. If we abuse the notation and write

$$\frac{\partial}{\partial x^\mu}((f \circ \phi_\alpha^{-1})(x)) = \frac{\partial f(x)}{\partial x^\mu},$$

we have finally that the directional derivative of f along γ at $p = \gamma(0)$ is given by

$$\frac{\partial}{\partial x^\mu} f(x) \frac{d}{dt} x^\mu(\gamma(t))\Big|_{t=0}. \tag{1.4}$$

Note that this expression is really similar to the usual directional derivative of scalar functions on $\mathbb{R}^n$. In fact, $\partial_\mu f$ is just the μth component of the gradient ∇f. The velocity $\dot{x}^\mu(\gamma(t))$ of the curve is the vector which is tangent to γ at all times.

The expression (1.4) suggests a general way of defining tangent vectors at points along curves. By defining the tangent vector to a curve at some point as the velocity $\dot{x}^\mu(\gamma(t))$, we must specify the chart (or embedding) x^μ. But abstract manifolds have many charts, which can (and will) intersect at various points. To avoid this ambiguity, we define the tangent vector to $\gamma(t)$ at the point $p = \gamma(0)$ as the *linear operator* X, which acts on a scalar function $f : M \to \mathbb{R}$ as

$$X[f] = X^\mu \frac{\partial f}{\partial x^\mu} \tag{1.5}$$

at a given chart $x^\mu = \phi_\alpha(p)$, where

$$X^\mu = \frac{d}{dt} x^\mu(\gamma(t))\Big|_{t=0}.$$

In other words, the tangent vector to γ at p is the diferential operator

$$X = X^\mu \partial_\mu$$

such that, when acting over a scalar function, it returns its directional derivative, the rate of change of the function along γ. Why does this operator represents the tangent vector at p? Note that, if f is a linear function, $X[f]$ is proportional to the velocity $\dot{x}^\mu(\gamma(t))$. So we can expect to now everything about the tangent vector at p through the linear operator X, in such a way that we can consider X as indeed the tangent vector itself.

We said that the tangent vector must not depend on the embedding. Let then $p \in U_\alpha \cap U_\beta$ be a point at the intersection of two charts. If $\phi_\alpha(p) = x^\mu$ and $\phi_\beta(p) = y^\mu$, we have two expressions for the same operator, namely

$$X = X^\mu \frac{\partial}{\partial x^\mu}$$

and

$$X = X'^\mu \frac{\partial}{\partial y^\mu}.$$

For X to be independent of the chart, both expressions must be equal. Thus,

$$X^\mu \frac{\partial}{\partial x^\mu} = X'^\mu \frac{\partial}{\partial y^\mu}.$$

Applying both sides to y^ν, we have that

$$X^\mu \frac{\partial y^\nu}{\partial x^\mu} = X'^\mu \delta^\nu_\mu.$$

Hence, under change of coordinates, the tanget vector coefficients must change as

$$X'^\nu = X^\mu \frac{\partial y^\nu}{\partial x^\mu}, \tag{1.6}$$

rendering the tangent vector independent of coordinates.

It can be shown that the set of tangent vectors at a point $p \in M$ forms a vector space, the *tangent space at* p, T_pM. A basis of this space is given by the set $\{e_\mu = \partial_\mu\}_{\mu=1}^n$, i.e., the partial derivatives at p. Clearly, the tangent space at p, T_pM, is n-dimensional.

We can now introduce the notion of a *vector field*, which is a map that assigns a vector to each point of M. A vector field can be pictured as a bunch of arrows over the manifold (figure 4). Vector fields are natural objects from the physics point of view. For instance, consider the flow of water. Each water molecule has some velocity, which can be pictured as a vector tangent to its motion. The collection of all such velocities forms a vector field.

Vector fields are known to form some kinds of patterns, in such a way that, in the example of water flow, if you follow a pattern of arrows, you are actually tracing the trajectory of some water molecules. Thus, it makes sense talking about the *integral curve* of a vector field (figure 5). Given a vector

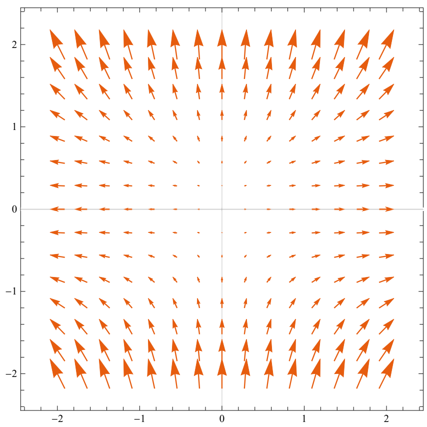

Figure 4: Vector field $V = x\partial_x + y^2\partial_y$.

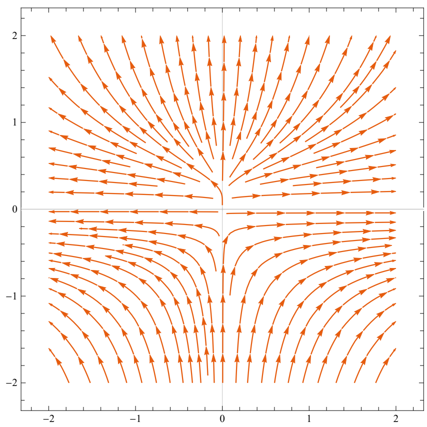

Figure 5: Its integral curves.

field V, its integral curve γ is such that V is tangent to it at each point $\gamma(t)$. That is,

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}. \tag{1.7}$$

For example, consider the vector field $V = x\partial_x + y^2\partial_y$ in $\mathbb{R}^2$ (figure 4). Its integral curve is $\gamma(t) = (x(t), y(t))$ such that

$$\dot{x}(t) = x(t),\ \dot{y}(t) = y(t)^2,$$

i.e.,

$$x(t) = x_0 e^t,\ y(t) = -\frac{y_0}{y_0 t - 1},$$

if the curve passes through (x_0, y_0) at $t = 0$ (figure 5).

Related to integral curves, there is the concept of the *flow* of a vector field. Imagine again the flow of water. Each water molecule has a velocity, and all such velocities form a vector field V. As we saw, we can obtain an integral curve γ from this vector field by "walking" through it and following its direction. It turns out that we can look at this picture in another way. We can build a collection of maps $\{\phi_t\}$, where $t \in [a, b]$ is "time", and each ϕ_t is a map $\phi_t : M \to M$ such that if a water molecule was at p in $t = 0$, it will be at $\phi_t(p)$ in time t. We call $\{\phi_t\}$ the flow generated by V. The equation to find the flow generated by V is just

$$\frac{d}{dt}\phi_t(p) = V|_{\phi_t(p)}.$$

We use the *Lie bracket* to can check if the flow generated by two vector fields commute, i.e., if water going first along the flow of V and then along the flow of W ends up at the same point as if it were first to go along W and then along V. Given two vector fields V, W, the Lie bracket is defined as

$$[V,W]f = V[W[f]] - W[V[f]], \tag{1.8}$$

for any function f. To see that it indeed compares the flows of V and W, consider the directional derivative at p of f along ϕ_t, the flow generated by V,

$$V[f] = \frac{d}{dt}(f \circ \phi_t(p))|_{t=0},$$

and the directional derivative at p of f along ψ_s, the flow generated by W,

$$W[f] = \frac{d}{ds}(f \circ \psi_s(p))|_{s=0},$$

where $\psi_0(p) = \phi_0(p) = p$. The Lie bracket is then given by

$$[V,W]f = \frac{d^2}{dtds}\left[f \circ \psi_s(p) \circ \phi_t(p) - f \circ \phi_t(p) \circ \psi_s(p)\right]\Big|_{s=t=0}.$$

Thus, we first go through ϕ_t, then through ψ_s and apply f, and we compare the result with doing the opposite path. If the flows do not commute, the Lie bracket is zero. The closer the flows, the smaller the Lie bracket.

As an example, in $\mathbb{R}^2$ consider the vectors

$$V = \frac{1}{\sqrt{x^2+y^2}}(x\partial_x + y\partial_y),$$

$$W = \frac{1}{\sqrt{x^2+y^2}}(-y\partial_x + x\partial_y),$$

whose flows are shown in figure 6. The flow of V is given by straight lines starting at the origin, while the flow of W is given by circles around the origin whose radius grow with the distance from the point $(0,0)$. Suppose we start at some point and follow some flow line of V, and then follow some flow line of W. The alternative path is, starting at the same point, we follow an arc of the same length of W and a line of same length of V. It can be seen from the graphics that, by doing this, we end-up at a different point, and thus the flows do not commute. We can intuitively see from the plot that the distance between the two points is given by an arc of a circle. Indeed, we can calculate the Lie bracket

$$[V,W] = VW - WV,$$

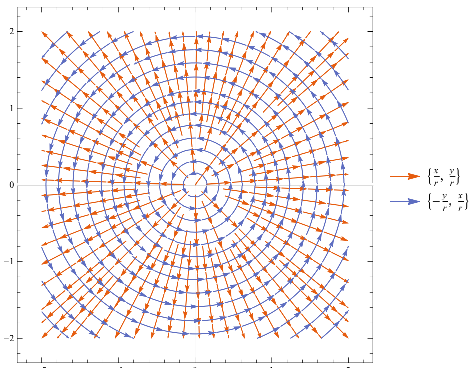

Figure 6: Integral curves of vector fields V and W.

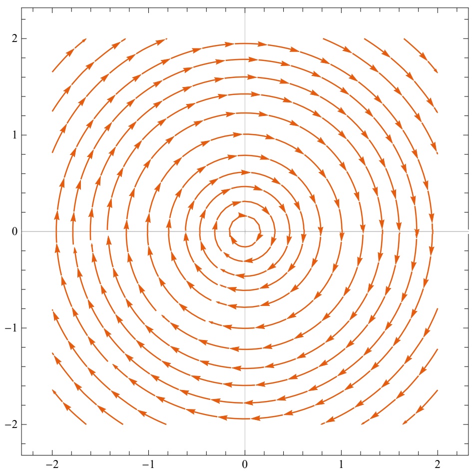

Figure 7: Integral curves of $[V, W]$.

which gives the vector

$$[V, W] = \frac{1}{(x^2 + y^2)^2} \left[(y^3 + x^2 y)\partial_x - (x^3 + xy^2)\partial_y \right] .$$

The flow of this vector is shown in figure 7, and it agrees with our intuition.

1.3 One-forms

In good old one-variable real calculus, the integral of a function $f : \mathbb{R} \to \mathbb{R}$ in some interval $[a, b]$ is perfomed, intuitively, as follows: we take the area under the curve f describes, approximate it with tiny rectangles and sum the area of these rectangles. The area of a rectangle between x and $x + \Delta x$, for small Δx, is approximately $f(x)\Delta x$. It does depend on we finding the distance between x and $x + \Delta x$. Thus, we have to think a little before we generalize the concept of integration to arbitrary manifolds M.

There is no obvious way to measure distances in a manifold M. But we can at least measure rates of change of functions at points along some curve γ. The tangent vectors give us a sense of direction and "length" along a curve, which we can, roughly speaking, use as a measure of distance between points. Thus, to define integration on arbitrary manifolds, we can introduce some objects which "eats" tangent vectors and returns a real number proportional to its "length". These objects are called *one-forms*.

A one-form at a point $p \in M$ is a linear map $\omega_p : T_pM \to \mathbb{R}$, i.e., it takes tangent vectors at p and returns a real number. More rigorously, let T_p^*M be the *dual space* to T_pM, i.e., it is a space whose elements are linear maps from T_pM to $\mathbb{R}$. A 1-form ω_p at p is an element of T_p^*M.

To gain more intuition on what a 1-form is, let's go back to usual calculus in $\mathbb{R}$ and consider the integral

$$\int_a^b f(x)dx,$$

over the interval $[a, b]$. We introduced one-forms to be able to define integration over manifolds, i.e., one-forms are things that we can integrate. Then, the integrand $f(x)dx$ must be an example of a 1-form. More generally, in $\mathbb{R}^N$, consider the integral over a curve γ

$$\int_\gamma f_\mu dx^\mu.$$

The integrand must also be a 1-form. Locally thus, 1-forms must be given by linear combinations of the differentials $\{dx^\mu\}_{\mu=1}^N$. Since these are intuitively linearly independent, they form a basis of the *cotangent space* T_p^*M at p. Hence, in this basis, 1-forms $\omega_p \in T_p^*M$ are given by

$$\omega_p = \omega_\mu dx^\mu.$$

But T_p^*M and T_pM are dual to each other, which means that there is an inner product $\langle , \rangle : T_p^*M \times T_pM \to \mathbb{R}$ such that

$$\langle dx^\mu, \partial_\nu \rangle = \delta^\mu_\nu.$$

The components ω_μ are then given by

$$\langle \omega_p, \partial_\nu \rangle = \omega_\mu \langle dx^\mu, \partial_\nu \rangle = \omega_\mu \delta^\mu_\nu = \omega_\nu.$$

The inner product between a 1-form ω and a vector V at p is given by

$$\langle \omega, V \rangle = \omega_\mu V^\nu \langle dx^\mu, \partial_\nu \rangle = \omega_\mu V^\mu$$

in local coordinates. The geometrical intuition behind one-forms is the following: vectors are arrows pointing somewhere with some magnitude. Being dual to arrows, 1-forms can only be *surfaces* pierced by the vectors (fig. 8). The inner product between a 1-form and a vector measures how many parallel surfaces the vector traverses, which, in a sense, gives the "length" of the vector. That is why we can use forms to define a sort of distance in a manifold, and thus we are able to integrate functions on it.

An important example of a 1-form is the differential of a function, df. From usual calculus, a plot of $df = \partial_\mu f dx^\mu$ gives equally spaced contour lines, with each contour representing a constant "height" $f(x^\mu)$. To integrate

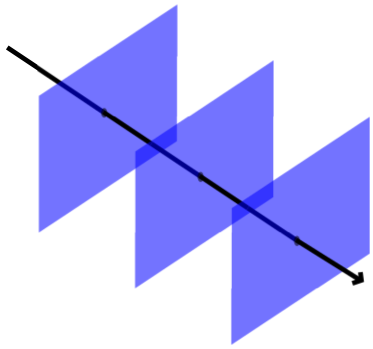

Figure 8: A vector is an arrow while a 1-form is a bunch of planes punctured by the arrow. The inner product between a vector and a one-form counts how many planes the arrow pierces.

df along a curve $\gamma : [0,1] \to M$ is to count how many contour lines of f the curve intercepts, which is a measure of how the "height" varied along the curve. This is nothing but the fundamental theorem of calculus:

$$\int_\gamma df = f(\gamma(1)) - f(\gamma(0)).$$

For $p \in U_\alpha \cap U_\beta$, where we have two charts $x^\mu(p)$ and $y^\nu(p)$, the 1-form ω_p can be expressed in two ways:

$$\omega_\mu dx^\mu = \omega'_\nu dy^\nu.$$

Writing $dy^\nu = (\partial y^\nu / \partial x^\mu) dx^\mu$ and taking the inner product of both sides with $\partial / \partial x^\mu$, we obtain that

$$\omega_\mu = \omega'_\nu \frac{\partial y^\nu}{\partial x^\mu}.$$

1.4 p-forms

As we saw, 1-forms gives us a way to integrate functions along curves. But we also would like to perform surface and volume integrals. Let's then build on our intuition from $\mathbb{R}^N$, specifically $\mathbb{R}^2$. Simple 1-forms dx and dy are a bunch of vertical and horizontal lines in $\mathbb{R}^2$, respectively. But they are also (cotangent) vectors, so we can, being really careful, picture then as arrows.

Recall that with usual vectors a, b we are able to define a *cross product* $a \times b$, whose geometrical interpretation being that it represents the *parallelogram* whose sides are a and b. Thus, it gives us a sense of *area* in two dimensions. Let's define a kind of cross product for 1-forms, which we call the *wedge* *product* $dx \wedge dy$. If 1-forms were vectors on real space, it would define areas enclosed by dx and dy. But they are dual to vectors in real space, and thus the object $dx \wedge dy$ must define dual parallelograms, which are nothing but points on $\mathbb{R}^2$. Thus, we can use the object $dx \wedge dy$ to integrate over an area A: we just count the number of points given by $dx \wedge dy$ which are enclosed by A.

The object $dx \wedge dy$ is an example of a *2-form*. To generalize this notion to arbitrary dimensions, let's define properly the wedge product between two 1-forms ω and μ as

$$\omega \wedge \mu = \frac{1}{2}(\omega \otimes \mu - \mu \otimes \omega).$$

The wedge product is clearly antisymmetric, i.e., $\omega \wedge \mu = -\mu \wedge \omega$. We can then define a *p-form* Ω as the object obtained by taking the wedge product between p 1-forms ω_i:

$$\Omega = \omega_1 \wedge ... \wedge \omega_p.$$

For example, on $\mathbb{R}^3$, we can have a 3-form

$$\begin{aligned}\Omega &= \frac{1}{3!}\omega_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \\ &= \omega_{xyz} dx \wedge dy \wedge dz,\end{aligned}$$

which is the volume form, i.e., its intuitive meaning is as a density of points (like the 2-form in $\mathbb{R}^2$ above). The usual 1-form $\omega_\mu dx^\mu$ can be obtained from a 0-form f, which is a function, through the *exterior derivative* $df = \partial_\mu f dx^\mu$. More on that later.

To sum-up, since 1-forms are vectors, we can take wedge products of an arbitrary number of them to construct higher dimensional objects, as 2-forms, 3-forms and so on. These have also a geometric meaning depending on the dimensionality of the manifold. For example, in 2-dimensions, a 2-form is the volume form and is represented by a bunch of points, while in 3-dimensions the volume form is a 3-form. We can then use these to integrate over arbitrary subspaces of a manifold M.

1.5 Exterior Algebra

It turns out that there is a rich algebraic structure underlying differential forms. Let's consider an arbitrary real vector space V. We can define the wedge product between two vectors $v, w \in V$, but to what space does $v \wedge w$ belongs to?

To answer this question, we only have to notice that elements of the kind $v \wedge w$ exhibit some vector space qualities (because they are defined through

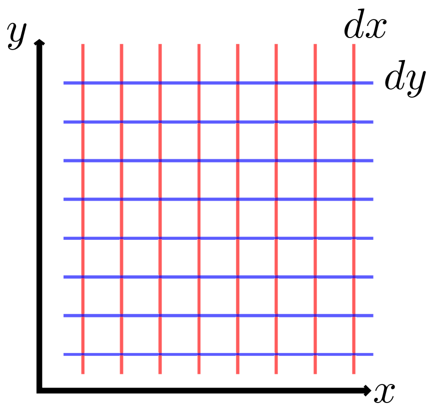

Figure 9: Forms dx and dy in 2D are lines.

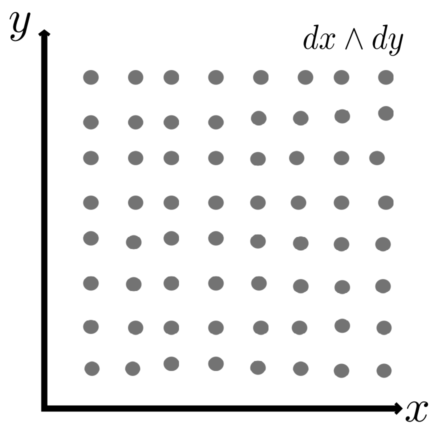

Figure 10: The 2-form $dx \wedge dy$ is a density of points in 2D.

a tensor product operation), and thus we can talk about the vector space generated by linear combinations of wedge products between elements of V. For example, let V be 3-dimensional, and let its basis vectors be given by dx, dy, dx (you can already see where this will end, right?). We define the vector space ΛV as the one generated by elements of the kind

$$a_1 1 + a_2 dx + a_3 dy + a_4 dz + a_5 dx \wedge dy + ... + a_8 dx \wedge dy \wedge dz,$$

i.e., all possible linear combinations of wedge products of elements of V. It turns out that ΛV has the structure of an *algebra* over the reals. It is called the *exterior algebra* over the vector space V.

To give us a feel of what the exterior algebra is, consider two elements of V given by

$$v = v_x dx + v_y dy + v_z dz, \; w = w_x dx + w_y dy + w_z dz.$$

The exterior product between such elements is $v \wedge w \in \Lambda V$, given by

$$\begin{aligned} v \wedge w &= (v_x dx + v_y dy + v_z dz) \wedge (w_x dx + w_y dy + w_z dz) \\ &= (v_x w_y - v_y w_x) dx \wedge dy + (v_x w_z - v_z w_x) dx \wedge dz + (v_y w_z - v_z w_y) dy \wedge dz, \end{aligned}$$

which is nothing but the cross product between the two vectors. Thus, the cross product is an element of the exterior algebra ΛV, and not really of the vector space V itself, although a isomorphism between the two is possible. We will come back to this latter.

If we take the wedge product between three vectors, $v, w, u \in V$, we end up with

$$v \wedge w \wedge u = [(v_x w_y - v_y w_x) dx \wedge dy + (v_x w_z - v_z w_x) dx \wedge dz +$$
$$(v_y w_z - v_z w_y) dy \wedge dz][u_x dx + u_y dy + u_z dz]$$
$$= [u_x(v_y w_z - v_z w_y) - u_y(v_x w_z - v_z w_x) + u_z(v_x w_y - v_y w_x)] dx \wedge dy \wedge dz,$$

which is nothing but the triple product $u \cdot (v \times w)$. Thus, elements of the exterior algebra are some familiar products of vectors we encountered in our linear algebra courses.

It should be noted that, in 3-dimensions, the wedge product between four vectors $u, v, w, t \in V$ is zero. In fact,

$$(v \wedge w \wedge u) \wedge t = \det \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} dx \wedge dy \wedge dz \wedge [t_x dx + t_y dy + t_z dz],$$

which is indeed zero because $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$.

Given a vector space V, we can define the space $\Lambda^p V$ as the one generated by linear combinations of p-fold wedge products of elements of V:

$$v_1 \wedge ... \wedge v_p \in \Lambda^p V.$$

For example, $\Lambda^1 V = V$, while $\Lambda^0 V = \mathbb{R}$ by definition. Then, it is clear that the exterior algebra ΛV can be written as the direct sum of $\Lambda^p V$:

$$\Lambda V = \bigoplus_{p=0}^{n} \Lambda^p V, \tag{1.9}$$

if V is n-dimensional. Why only n terms? The space $\Lambda^{n+1} V$ would be generated by the product of $n+1$ elements of V:

$$v_1 \wedge ... \wedge v_{n+1}.$$

The wedge product between n vectors can be written as

$$K dx^1 \wedge ... \wedge dx^n,$$

where K is some number. By taking the product of this object with one more element, we would have terms $dx^\mu \wedge dx^\mu = 0$ for $\mu = 1, ..., n$, hence giving a null result. Thus, the space $\Lambda^{n+1} V$ is trivial. The dimension of $\Lambda^p V$ is the number of ways we can set n basis vectors into p positions:

$$\dim \Lambda^p V = \frac{n!}{p!(n-p)!}, \tag{1.10}$$

which means that the dimension of ΛV, as a vector space, is

$$\mathrm{dim}\Lambda V = \sum_{p=0}^{n} \frac{n!}{p!(n-p)!} = 2^n. \tag{1.11}$$

Note that, for V a 3-dimensional space, $\mathrm{dim}\Lambda^2 V = 3$, which means that V and $\Lambda^2 V$ are isomorphic! We saw that the cross product between two vectors in V lies in $\Lambda^2 V$. The isomorphism between the two spaces means that we can map the cross product of two vectors in V back to V itself, that is, we can regard the cross product $v \wedge w$ as a vector in V! However, we must choose an isomorphism (a rule) to do this mapping, which is really the infamous right-hand rule. For example, we can define the isomorphism $* : \Lambda^2 V \to V$ as

$$*dx \wedge dy = dz, \; *dx \wedge dz = -dy, \; *dy \wedge dz = dx,$$

which is the right-hand rule.

A last reamark: for any vector space V, the exterior algebra ΛV is *graded commutative*, which means that, for any $\omega \in \Lambda^p V$ and any $\eta \in \Lambda^q V$,

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega. \tag{1.12}$$

This formula is easy to understand. Note that $\omega \propto \omega_1 \wedge ... \wedge \omega_p$, while $\eta \propto \eta_1 \wedge ... \wedge \eta_q$. To shift the first one of the elements η_1 appearing in η all the way up to the front, we need to anticommute it with all the p vectors appearing in ω. We need to do it for each of the q elements η_i appearing in η, which explains the overral phase $(-1)^{pq}$.

Cool. Now, to relate all of this to differential geometry, let's consider the exterior algebra over the cotangent space $T_p^* M$ of one-forms. We consider this algebra with coefficients in $C^\infty(M)$ (smooth functions on M). The exterior algebra of forms is thus

$$\Omega(M) = \bigoplus_{p=0}^{n} \Omega^p(M), \tag{1.13}$$

where $\Omega^p(M)$ is generated by the wedge product of p 1-forms. That is, the elements of $\Omega^p(M)$ are the p-forms of the previous section. For example, in a n-dimensional manifold M, the 0-forms are functions f, 1-forms are given by

$$\omega_1 = \omega_\mu dx^\mu,$$

2-forms are given by

$$\omega_2 = \omega_{\mu\nu} dx^\mu \wedge dx^\nu,$$

3-forms are given by

$$\omega_3 = \omega_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$$

and so on.

1.6 Exterior Derivative

The 1-forms dx, dy of figure 9 are lines, while the 2-form $dx \wedge dy$ of figure 10 are the points in which dx and dy intersect. There may be thus a relation between the two objects in the form of a map which sends 1-forms to 2-forms. Since the 1-forms are extended objects, this map may be interpreted as taking the *boundary* of those objects.

We define then the map

$$d : \Omega^p(M) \to \Omega^{p+1}(M), \tag{1.14}$$

which takes a p-form and sends it to a $(p+1)$-form. For example, given a 1-form ω, $d\omega$ is a 2-form. More concretely, given a p-form

$$\omega = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p},$$

its exterior derivative is the $(p+1)$-form is given by

$$d\omega = \frac{1}{p!} \partial_\nu \omega_{\mu_1 \ldots \mu_p} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}. \tag{1.15}$$

The intuition behind the exterior derivative is the following: consider the 1-form xdy. We know that dy is represented by horizontal lines. Then, xdy is a bunch of horizontal lines whose density increases as we move from the left to the right. The 2-form $d(xdy) = dx \wedge dy$ is a skewed set of points, which in a way forms the boundary of the figure that xdy describes. From this intuition, we can see that $d(d(xdy)) = 0$, i.e., $d^2 = 0$. This can also be shown from the definition (1.15).

The exterior derivative has the following important property:

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p \omega \wedge d\mu,$$

which can be shown from the definition and from the graded commutativity property of forms.

It is important to notice that the exterior derivative sums up a bunch of known differential operators. For example, in 3-dimensions, a 0-form is a function $f(x, y, z)$. Its exterior derivative is

$$df = \partial_\mu f dx^\mu = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz,$$

which is nothing but the gradient ∇f of $f(x, y, z)$. A 1-form is given by

$$\omega = \omega_\mu dx^\mu = \omega_x dx + \omega_y dy + \omega_z dz,$$

and its exterior derivative is

$$\begin{aligned} d\omega &= \partial_\nu \omega_\mu dx^\nu \wedge dx^\mu \\ &= (\partial_x \omega_y - \partial_y \omega_x) dx \wedge dy - (\partial_x \omega_z - \partial_z \omega_x) dx \wedge dz + (\partial_y \omega_z - \partial_z \omega_y) dy \wedge dz, \end{aligned}$$

which is nothing but the curl $\nabla \times \boldsymbol{\omega}$ of the vector field $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$. A 2-form is given by

$$\omega = \omega_{\mu\nu} dx^\mu \wedge dx^\nu,$$

whose exterior derivative is given by

$$d\omega = \partial_\rho \omega_{\mu\nu} dx^\rho \wedge dx^\mu \wedge dx^\nu,$$

i.e.,

$$\begin{aligned} d\omega = \partial_x \omega_{yz} dx \wedge dy \wedge dz + \partial_y \omega_{zx} dy \wedge dz \wedge dx + \partial_z \omega_{xy} dz \wedge dx \wedge dy \\ = (\partial_x \omega_{yz} + \partial_y \omega_{zx} + \partial_z \omega_{xy}) dx \wedge dy \wedge dz, \end{aligned}$$

which is the divergence $\nabla \cdot \boldsymbol{\omega}$ of the vector field $\boldsymbol{\omega} = (\omega_{yz}, \omega_{zx}, \omega_{xy})$.

From $d^2 = 0$, we can see that the known identities $\nabla \times \nabla f = 0$ and $\nabla \cdot \nabla \times v = 0$ are trivially satisfied.

Thus, we have that p-forms are collections of extended objects and that the exterior derivative measures the boundary of those objects. From this geometrical picture, we can get a nice visualization of the *Stoke's theorem.* For example, in 2-dimensions, Stoke's theorem states that, given a surface S whose boundary is the curve ∂S, and given a 1-form $\omega \in \Omega^1(M)$, we have that

$$\int_S d\omega = \int_{\partial S} \omega. \tag{1.16}$$

We saw that a 1-form in 2D is a bunch of lines. The integral of a 1-form along a curve ∂S counts the (signed) number of times the 1-form lines crosses the curve ∂S. If a line crosses the curve ∂S an even number of times, meaning that it leaves the region S bounded by ∂S, it contributes nothing to the sum, while if the line intersects ∂S an odd number of times, ending inside S, it contributes to the sum. Now, $d\omega$ is a 2-form which is a bunch of points, the boundary of the lines forming the 1-form ω. Its integral in a region S is just the number of points inside S. In this case, these points are the boundaries of the lines of the 1-form that end in S. Thus, the left-hand side counts exactly the same thing as the right-hand side of (1.16), and we have a nice and intuitive identity.

1.7 Pushforwads and Pullbacks

To finish this short review, consider a map $\phi : M \to N$ between two manifolds M and N. It induces a linear map $\phi_* : T_pM \to T_{\phi(p)}N$ on the tangent spaces called the *pushforward*, which acts over a vector $V \in T_pM$ and returns a vector $\phi_* V \in T_{\phi(p)}N$ such that, for any $f : N \to \mathbb{R}$,

$$(\phi_* V)[f] = V[f \circ \phi]. \tag{1.17}$$

The intuition behind this map is straightforward: we take a vector in M and define another vector in N through the mapping ϕ.

Now, we can also define a dual linear map $\phi^* : T^*_{\phi(p)}N \to T^*_pM$, called the *pullback*, acting on 1-forms, such that given a 1-form $\omega \in T^*_{\phi(p)}N$, it returns a 1-form $\phi^*\omega \in T^*_pM$ such that, for any vector $V \in T_pM$,

$$(\phi^*\omega)(V) = \omega(\phi_* V). \tag{1.18}$$

The intuition behind this is also straighforward: we use the mapping ϕ to bring a 1-form we want from N back to M.

An important property of pullbacks: exterior derivatives are natural under them, i.e.,

$$\phi^*(d\omega) = d(\phi^*\omega).$$

Geometricaly, this means that the pullback, in a sense, preserves the boundary of the 1-form: we can take ω back to M and then compute its boundary or we can compute its boundary in N and take the result back to M, the result is the same.

2 Fiber Bundles

Consider a one-dimensional spin chain with periodic boundary conditions, formed by N spin 1/2 particles. At each lattice site, there is a particle whose state can be represented by a vector in a two-dimensional Hilbert space $V = \mathrm{Span}_{\mathbb{C}}\{|{+1}\rangle, |{-1}\rangle\}$. Geometrically, we can picture this as follows: in the thermodynamic limit $N \to \infty$, our spacetime manifold is a cilinder $S^1 \times \mathbb{R}$. A constant-time surface is a circle S^1. At each point of S^1, there lies a 2-dimensional manifold V. Locally, our *total space*, which is our configuration space so-to-speak, is the product space $S^1 \times V$. That is, to say in which state a given particle is, we must communicate its position in space, which is a point p in S^1, and its spin state, which is a vector $|v\rangle \in V$. The question is: what is this space like *globally*, when we glue all points of S^1 together?

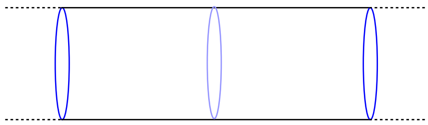

Figure 11: The spacetime is a cilinder of infinite length for the spin chain with periodic boundary conditions.

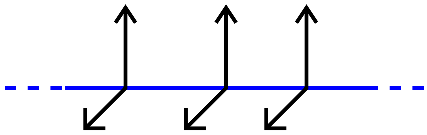

Figure 12: An illustrative representation of what the configuration space for the spin chain might look like. At each point in the circle, there is a two-dimensional Hilbert space.

Let's consider another situation. Let M be a D-dimensional manifold. At each point $p \in M$, there is a tangent space T_pM. Define the space

$$TM = \bigcup_{p\in M} T_pM, \tag{2.1}$$

which is the union of all tangent spaces we can define over M. This space is called the *tangent bundle* of M. An element of the tangent bundle is specified by first picking a point $p \in M$ and then selecting a vector $X|_p \in T_pM$. Locally, the tangent bundle looks like the product $M \times T_pM$. But it is not so globally. To see this, let (U_1, ϕ_1) and (U_2, ϕ_2) be two coordinate neighborhoods of M such that their overlap is not empty, $U_1 \cap U_2 \neq \emptyset$. Let $p \in U_1 \cap U_2$ be a point in the overlap. We can assign different coordinates $x^\mu = \phi_1(p)$ and $y^\mu = \phi_2(p)$ to p. This means that we can represent a vector $X|_p \in T_pM$ in two different coordinate basis:

$$X|_p = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\nu \frac{\partial}{\partial y^\nu}.$$

Then, the components of this tangent vector are related by a change-of-basis matrix:

$$\tilde{X}^\mu = X^\nu \frac{\partial y^\mu}{\partial x^\nu},$$

where $\frac{\partial y^\mu}{\partial x^\nu} \in \mathrm{GL}(D, \mathbb{R})$. Vectors in different coordinate neighborhoods are related in a non-trivial way by a rotation. Therefore, the tangent spaces are glued to form the tangent bundle in such a way that, on passing from one coordinate neighbourhood in M to another, vectors change smoothly by a rotation. This is a constraint we must respect when constructing TM by gluing together all the T_pM's, so the *total space* TM globally does not really looks like the product $M \times T_pM$.

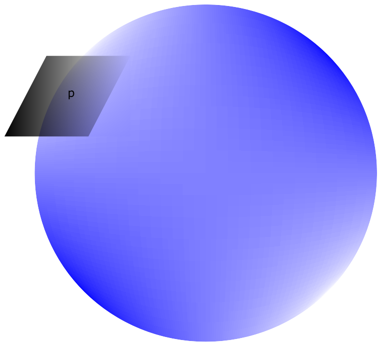

Figure 13: The tangent space to a sphere at point p. The union of all tangent spaces forms the tangent bundle.

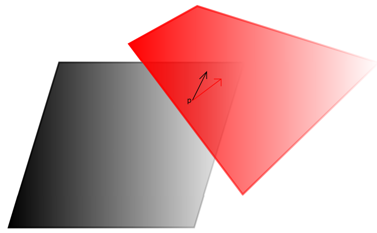

Figure 14: At the intersection of two coordinate charts, the tangent vectors must differ smoothly by a rotation.

In these two examples, we have a manifold in which, at each point of it, we define another manifold. So locally, the *total space* we must consider looks like a product of two manifolds. Globally however, the *total space* may look more complicated. We can define a mathematical object to represent this *total space* and its global complications. This object is called a *fiber bundle*. A fiber bundle is a collection of manifolds and maps. We have a *base space* M, a *total space* E and a *fiber* F. At each point $p \in M$, we define an instance of the fiber $F_p \cong F$. The total space E looks like $M \times F$ locally, but it may be very different globally. For example, in the spin chain case, the base space is S^1, while the fiber is V, and in the tangent bundle case, the base space is the manifold M, the fiber is the tangent space to a point T_pM and the total space is the tangent bundle TM.

To characterize how different from a product manifold the total space is, we need to define a *projection* map $\pi : E \to M$, which is a surjection, and we identify, for each $p \in M$, the inverse image $\pi^{-1}(p) = F_p \cong F$. That is, we identify the inverse image of the projection of a point in M as an instance of the fiber at that point. Then, we take an *open covering* $\{U_i\}$ of M, i.e., a familly of open sets which cover M, and define diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ called *local trivializations*. That is, for each open neighborhood U_i, we make $U_i \times F \cong \pi^{-1}(U_i)$ in a smooth way. This means that every point in an open neighborhood $U_i \subset M$ is mapped to a subset of the total space $\pi^{-1}(U_i) \subset E$ which looks (smoothly!) like the product $U_i \times F$. This is exactly what we want from the examples we saw above. The non-triviality of the *bundle* E arises when we try to sew together the sets of the form $\pi^{-1}(U_i)$ to construct E.

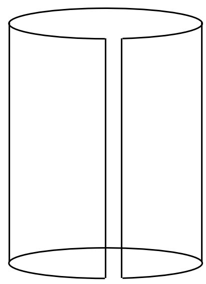

Figure 15: Both spaces $U_1 \times I$ and $U_2 \times I$ look like a cilinder with a strip removed.

Let's look at an example. Take the circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$$

as base space and the unity interval $I = [0, 1]$ as the fiber. The circle is a 1-dimensional manifold whose atlas can be $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$, where

$$U_1 = S^1/\{(1, 0)\},$$

with $\varphi_1(x, y) = \arctan(y/x)$, $\varphi_1 \in (-\pi, \pi)$, and

$$U_2 = S^1/\{(-1, 0)\},$$

with $\varphi_2(x, y) = \arctan(y/x)$, $\varphi_2 \in (0, 2\pi)$. The product of U_1 with the fiber I is like a cilinder without a strip, as in figure 15. Likewise, the product of U_2 with I also looks like a cilinder with a strip removed. Let's see if we can glue these spaces consistently.

We could try to glue a the top of one of the cilinders to the bottom of the other, as in figure 16. However, we would not be able to project this bundle to a circle. There is still a cut in the manifold and some circle points missing. To fix this, we could try to glue the resulting top and bottom circles together, making a torus like the one shown in figure 17. The problem now is that the torus is the trivial bundle $S^1 \times S^1$, which does not look locally like $S^1 \times I$.

Since gluing the circles does not seem to work, let's try gluing the cuts. Note that the cut of $U_1 \times I$ can be seem as the set $\{(1, 0, u) | u \in [0, 1]\}$. Likewise, the cut of $U_2 \times I$ can be seem as the set $\{(-1, 0, u) | u \in [0, 1]\}$. Therefore, the cuts are parametrized by a variable $u \in I$. One obvious way to glue the cuts is to smoothly identify corresponding points. Namely, the point $(1, 0, 0)$ is identified with $(-1, 0, 0)$, the point $(1, 0, 0.1)$ is identified with $(-1, 0, 0.1)$ and so on, until we identify $(1, 0, 1)$ with $(-1, 0, 1)$. This is

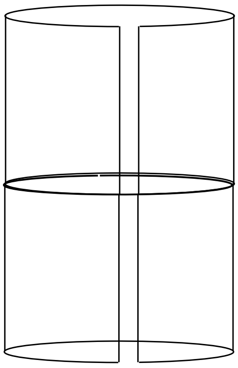

Figure 16: Gluing the top of one of the cilinders with the bootom of the other would result in something like this.

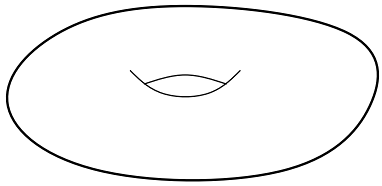

Figure 17: The torus is not a circle bundle.

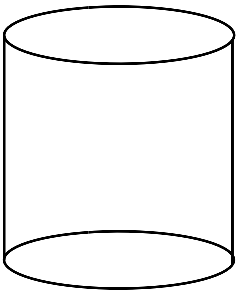

Figure 18: By directly gluing the cuts, we end-up with a cilinder.

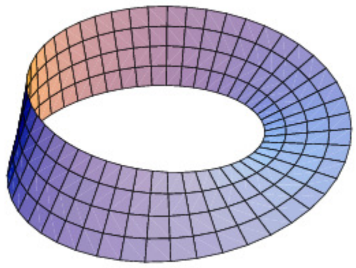

Figure 19: The Möbius strip, a non-trivial circle bundle. Source: Wikipedia

equivalent to define a map $t : I \to I$ that acts as the identity. In doing so, the total space we end-up with is a cilinder $S^1 \times I$, which indeed looks like $S^1 \times I$ locally.

However, nothing is stopping us from gluing the cuts in a different way. For example, we can connect the cuts by identifying the point $(1, 0, 0)$ with the point $(-1, 0, 1)$, the point $(1, 0, 0.1)$ with $(-1, 0, 0.9)$ and so on until we identify $(1, 0, 1)$ with $(-1, 0, 0)$. This is the same as defining a map $t : I \to I$ such that $u \to 1 - u$. By doing so, we construct a *Möbius strip*, shown in figure 19. This is a very non-trivial total space which also locally looks like the product $S^1 \times I$.

Using some maps over the fiber, we can glue the "local" spaces to construct the whole bundle. This leads us to the following last definition we need to do in order to define a fiber bundle properly. Let U_i and U_j be two open neighborhoods of M such that $U_i \cap U_j \neq \emptyset$. For a fixed point $p \in U_i \cap U_j$, the maps $\phi_i(p) : F \to F$ and $\phi_j(p) : F \to F$ are diffeos of the fiber. Define $t_{ij}(p) = \phi_i^{-1}(p) \circ \phi_j(p) : F \to F$. This is the map with which we glue the "local" spaces. We require that this map belong to a group G, called the *structure group*. Then, ϕ_i and ϕ_j are related by a smooth map $t_{ij} : U_i \cap U_j \to G$ as $\phi_j(p, f) = \phi_i(p, t_{ij}(p)f)$. The map t_{ij} is called a *transition function*.

The intuition behind the transition function is simple: if we have a point p which belongs to two open sets, we must have a way to relate the points in the fiber over p in order to build a total space. This relation is given by the transition functions. If all transition functions are identity maps, the bundle is *trivial*, i.e., it is a product $M \times F$ globally.

Transition functions must satisfy the following identities:

$$t_{ii}(p) = \mathrm{id}_F, \ p \in U_i, \tag{2.2}$$

$$t_{ij}(p) = t_{ji}^{-1}(p), \ p \in U_i \cap U_j, \tag{2.3}$$

$$t_{ij}(p)t_{jk}(p) = t_{ik}(p), \ p \in U_i \cap U_j \cap U_k. \tag{2.4}$$

As can be seem from the S^1 example, transition functions are not unique. They depend on the local trivializations we choose to construct the bundle. Suppose we have two sets of local trivializations $\{\phi_i\}$ and $\{\psi_i\}$, defined on the same open covering $\{U_i\}$ and giving rise to the same bundle. For some $p \in U_i \cap U_j$, there are two sets of transition functions:

$$t_{ij}(p) = \phi_i^{-1}(p) \circ \phi_j(p),$$

$$t'_{ij}(p) = \psi_i^{-1}(p) \circ \psi_j(p).$$

We then define

$$g_i(p) = \phi_i^{-1}(p) \circ \psi_i(p)$$

and require that $g_i(p) \in G$. We have a relation between the two transition functions, given by

$$t'_{ij}(p) = g_i(p)^{-1} \circ t_{ij}(p) \circ g_j(p).$$

For trivial bundles, we have that

$$t_{ij}(p) = g_i(p)^{-1} \circ g_j(p).$$

The transition functions are what we will call *gauge transformations*, while the functions g_i are the *gauge degrees of freedom*.

Sections. Let's go back to the tangent bundle $TM \to M$. A vector field X over M is a smooth map that associates, to each point $p \in M$, a vector in

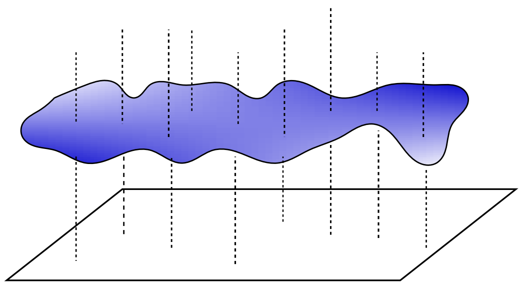

Figure 20: Representation of a section (in blue) of a tangent bundle (vertical lines) over a manifold (rectangle).

T_pM. Thus, it is basically a map $X : M \to TM$ from the base space to the tangent bundle. Given a vector in T_pM, we know that it is defined over the point p. Thus, we must have that $\pi \circ X = \mathrm{id}_M$. That is, the vector field is a map such that if we take a point, map it to a vector through the vector field and then project this vector back to M using the projection map, we have donne nothing.

We say that the vector field is a *section* over the tangent bundle. The picture to have in mind is the one in figure 20. A section is any smooth map $s : M \to TM$ such that

$$\pi \circ s = \mathrm{id}_M. \tag{2.5}$$

A section may be defined only over some open neighborhood U. In such case, we call it a *local section*. The set of all sections is denoted $\Gamma(M, E)$. For example, the set of vector fields $\chi(M)$ is the set of globall sections $\Gamma(M, TM)$ of the tangent bundle. Not all bundles admit global sections.

Sections are important for us because they define, essentially, the *fields* in a quantum field theory. For example, a particle field associates, to each point in spacetime, a particle positioned at that point with some internal degree of freedom (its spin). Thus, it associates a point in some base manifold M to a point in a product-like manifold $M \times F$, where F is a vector space whose vectors are spin states of the particle. A particle field is then nothing but a section of a fiber bundle.

2.1 Fiber Bundles over Contractible Spaces are Trivial

It is a hard problem to tell if a fiber bundle is trivial. Luckily, there are some known cases in which the triviality of the bundle follows from topological properties of the base manifold. Here we will review one such case. First, we need a couple of definitions.

Let $E' \to M'$ and $E \to M$ be two fiber bundles. We say that a smooth map $\tilde{f} : E' \to E$ is a *bundle map* if it induces a smooth map $f : M' \to M$ such that the following diagram commutes:

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f} & M \end{array}$$

that is,

$$\pi \circ \tilde{f} = f \circ \pi'. \tag{2.6}$$

This means that, if $\tilde{f}$ maps a fiber F_p in E' to a fiber F_q in E, then it is a bundle map if it induces a map $f : M' \to M$ such that $f(p) = q$.

Two bundles $E' \to M$ and $E \to M$ are equivalent if there exists a bundle map $\tilde{f} : E' \to E$ which is a diffeomorphism and such that the induced map $f : M \to M$ is the identity id_M.

Let $E \to M$ be a fiber bundle with fiber F. Let N be a smooth manifold and $f : N \to M$ be a smooth map. Define the following subset of $N \times E$:

$$f^*E = \{(p, u) \in N \times E | f(p) = \pi(u)\}.$$

That is, we take points $p \in N$ and $u \in E$ such that the image $f(p)$ is equal to the projection of u over M. This defines the so-called *pullback bundle* f^*E over N. A fiber F_p in f^*E is a copy of a fiber $F_{f(p)}$ in E. The projection is the obvious one: $\pi_1 : f^*E \to N, \pi_1(p, u) = p$.

We will need the following theorem, whose proof can be found in most texts about fiber bundles.

Theorem 1. *Let $E \to M$ be a fiber bundle with fiber F and let $f, g : N \to M$ be two homotopic smooth maps, that is, there is a smooth map $F : N \times I \to M$ such that $F(p, 0) = g(p)$ and $F(p, 1) = f(p)$, for any $p \in N$. Then, f^*E and g^*E are equivalent bundles over N.*

Then, let M be a contractible manifold and let $E \to M$ be a fiber bundle with fiber F over M. Fix a point $p_0 \in M$. Since M is contractible, there is a smooth homotpy $F : M \times I \to M$ such that $F(p, 0) = p$ and $F(p, 1) = p_0$, for any point $p \in M$. Define two smooth maps $h_0, h_1 : M \to M$ such that $h_0(p) = F(p, 0)$ and $h_1(p) = F(p, 1)$. We can construct the pullback bundles h_0^*E and h_1^*E over M. Note that, since $h_1(p) = p_0$ is a constant map, the pullback bundle $h_1^*E = \{(p, u) \in M \times E | \pi(u) = p_0\} = \{p_0\} \times F$, i.e., it is the trivial bundle $h_1^*E = M \times F$. Likewise, since $h_0(p) = p$, $h_0 = \mathrm{id}_M$ and then the pullback bundle $h_0^*E = E$. Since those maps are homotopic, theorem 1 says that the two bundles h_0^*E and h_1^*E are equivalent. Therefore, E is equivalent to the trivial bundle $M \times F$, and we have

Theorem 2. *Let $E \to M$ be a fiber bundle. If M is contractible to a point, then E is trivial.*

2.2 Kinds of Bundles

Bundles are manifolds which are almost like a product $M \times F$. In physics, M tends to represent spacetime, so the interesting stuff that can happen really depends on what kind of manifold F is.

For instance, when F is a vector space, we have a *vector bundle* with some nice properties. Transition functions are given by matrices in $\mathrm{GL}(F)$, which means that the structure group is also $\mathrm{GL}(F)$. Vector bundles arise everywhere in physics. The bundle in the spin chain example and the tangent bundle are vector bundles. A quantum-mechanical wave-function over 3-dimensional Euclidean space is a section of the trivial vector bundle $\mathbb{R}^3 \times \mathbb{C}$.

Now, if F is the structure group G, we have what is called a *principal bundle*. This kind of bundle appears everywhere in physics, as we will see later. As an example of principal bundle, consider the $U(1)$ bundle over S^2, $P \xrightarrow{\pi} S^2$. Take the open covering $\{U_N, U_S\}$ of the sphere such that

$$U_N = \{(\theta, \phi) | 0 \leq \theta \leq \pi/2 + \epsilon, 0 \leq \phi \leq 2\pi\},$$

$$U_S = \{(\theta, \phi) | \pi/2 - \epsilon \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi\}.$$

That is, we take the two hemispheres as an open cover of S^2. The intersection $U_N \cap U_S$ is just the equator, parametrized by the angle ϕ. We know that $\pi^{-1}(U_i) \cong U_i \times U(1)$, for $i = N, S$. This means that, for points $p \in S^2$ far from the equator, the space P made of points $\pi^{-1}(p)$ is like a product $U_i \times U(1)$, depending on which hemisphere p lies. The only problem is in the equator, where we have to find a way of gluing $U_N \times U(1)$ with $U_S \times U(1)$.

Then, fix a point in the equator parametrized by an angle ϕ. The corresponding element of $U_N \times U(1)$ is given by $(\phi, e^{i\alpha_N})$, while the corresponding element of $U_S \times U(1)$ is given by $(\phi, e^{i\alpha_S})$. To relate these points, we introduce the transition map $t_n : U(1) \to U(1)$ such that

$$e^{i\alpha_N} = t_{n,\phi} e^{i\alpha_S} = e^{i(n\phi + \alpha_S)},$$

where $n \in \mathbb{Z}$. That is, if $n = 0$, we just identify corresponding points, and the resulting bundle is trivial, $P = S^2 \times U(1)$. If $n \neq 0$, we have a kind of twisted space. For example, for $n = 1$, at $\phi = 0$, we must identify the charges $e^{i\alpha_N} = e^{i\alpha_S}$, while at $\phi = \pi$, we must identify the charges $e^{i\alpha_N} = e^{i(\pi+\alpha_S)}$ and so on.

Note that, since $U(1) = S^1$, the map is $t_n : S^1 \to U(1)$ and thus n labels classes in the homotopy group $\pi_1(U(1)) = \mathbb{Z}$. Then, the homotopy group $\pi_1(U(1))$ defines how to glue local spaces to build the bundle P.

To any principal bundle, we can construct an *associated* bundle as follows. Let P be a principal bundle and let G act over a manifold F on the left. Define the action of G over $P \times F$ as $(u, f) \to (ug, g^{-1}f)$. The associated bundle is given by the equivalence class $P \times F/G$, where (u, f) and $(ug, g^{-1}f)$ are identified. If F is a vector space, let $\rho : G \to \mathrm{Aut}(F)$ be a representation of G. The associated vector bundle $P \times_\rho F$ is given by the equivalence relation in which we identify (u, f) with $(ug, \rho(g)^{-1}f)$.

An example of associated bundle is given by the *frame bundle*. Consider a vector bundle. At each point, the fiber is a vector space. The frame bundle associated to a vector bundle is such that its fiber is the set of all ordered basis of the vector space. We can see here, at least intuitively, the meaning of the general definition. Points (u, f) and $(ug, \rho(g)^{-1}f)$, differing by a change of basis in the associated (frame) bundle are identified, because they all belong to the set of all ordered basis. In this case, the representation ρ is the standard representation.

Why are principal and vector bundles so important? They are the most common type of bundles that appear in physics. Moreover, they present a rich and tractable mathematical structure. For example, the question of whether a principal or vector bundle is trivial is easily answered. A principal bundle is trivial iff it has a global section [1]. Likewise, a vector bundle is trivial iff its associated bundle has a global section [1].

3 Connections on Fibre Bundles

Since the total space P of a fiber bundle $\pi : P \to M$ is a manifold, we can define tangent spaces over its points. Tangent vectors live on these tangent spaces, and a tangent bundle can be defined whose sections are vector fields over the total space. The total space P looks like a product space $U \times F$ locally, so the tangent space must look locally like a direct sum of tangent spaces T_pM and T_pF. Globally however, P might be the result of a non-trivial gluing of local $U \times F$ pieces. Therefore, the tangent space to P might globally not be just a direct sum of tangent spaces to M and F. The *connection* is a rule to uniquely separate T_uP into a direct sum of vector spaces, as we will see next.

We first define connections on principal bundles. Let $\pi : P \to M$ be a principal bundle whose fiber is the Lie group G. Let $u \in P$ such that $\pi(u) = p \in M$. We define the *vertical space* V_uP as the subspace of T_uP which is tangent to the fiber G. The vector space V_uP is constructed as follows: first, define the right action of G over P as

$$R_g u = ug$$

for any $u \in P$, $g \in G$. Locally, $u = (p, h)$ for $p \in M$ and $h \in G$, so for fixed p, the action is just the right multiplication in the Lie group. As such, it is transitive and free. Based on this intuition, we also have that $\pi(ug) = \pi(u) = p$, i.e., the action acts only over the fiber, therefore not changing the base point. Now, let $\mathfrak{g} = T_eG$ be the Lie algebra associated to G and take any element $A \in \mathfrak{g}$. Through the exponential map, we have that $e^A = g \in G$. Then, the right action

$$R_{e^{tA}} u = ue^{tA}$$

defines a curve parametrized by t passing through u on the total space P. This curve lies only on G, since $\pi(ue^{tA}) = \pi(u) = p$. Define a vector $A^\# \in T_uP$ as

$$A^\# f(u) = \frac{d}{dt} f(ue^{tA})\Big|_{t=0}, \tag{3.1}$$

for any smooth function $f : P \to \mathbb{R}$. Since ue^{tA} is a curve on G, the vector field $A^\#$ belongs to V_uP, which is tangent to G. The map $A \to A^\#$ is, in fact, a vector space isomorphism between $\mathfrak{g}$ and V_uP. Defining such a vector for each $u \in P$, we construct the *fundamental vector field* generated by A. This all means that we can think of the vertical space, which is the tangent space to G, as its Lie algebra $\mathfrak{g}$.

Note that $\pi_* A^\# = 0$. In fact, $(\pi_* A^\#) f(u) = A^\# f(\pi(u)) = \frac{d}{dt} f(u)|_{t=0} = 0$. So, we can identify $V_uP = \text{ker}\pi_*$. This tells us that, if we think of a kind of embedding of T_uP inside the more familiar T_pM, where $p = \pi(u)$, all the part of T_uP which corresponds to the tangent space to the fiber goes to a 0-dimensional subspace of T_pM. This indicates to us that there is indeed a sepration between the tangent space to the fiber and the tangent space to the base manifold, i.e., they do not mix. When we project tangent vectors to the base, we lose all information about the fiber, in a similar way as when we project a point in P to a point in M. We define the *horizontal space* H_uP as the complement of V_uP in T_uP, i.e., as the vector space such that $H_uP \oplus V_uP = T_uP$. It is uniquely specified if a connection is defined on P.

A *connection* is a unique separation of T_uP into the direct sum

$$T_uP = H_uP \oplus V_uP, \tag{3.2}$$

such that any smooth vector field X in P can be written as

$$X = X^H + X^V,$$

where $X^H \in H_uP$ and $X^V \in V_uP$. Also, for any $g \in G$ and $u \in P$, $H_{ug}P = (R_g)_* H_uP$. In this way, a single H_uP generates all horizontal spaces in the same fiber.

3.1 The Connection One-Form

We can give an alternative but equivalent definition of a connection on a principal fiber bundle. Define the *connection one-form* $\omega \in \mathfrak{g} \otimes T^*P$ as a $\mathfrak{g}$-valued 1-form on P such that it is a projection of T_uP into V_uP. It must satisfy the following properties:

$$\omega(A^\#) = A, \tag{3.3}$$

$$R_g^* \omega = Ad_{g^{-1}} \omega, \tag{3.4}$$

for all $A \in \mathfrak{g}$ and all $g \in G$. This last condition means that, for any $X \in T_uP$,

$$R_g^* \omega_{ug}(X) = \omega_{ug}((R_g)_* X) = g^{-1} \omega_u(X) g.$$

The horizontal space H_uP is then defined as the kernel of ω:

$$H_uP = \{X \in T_uP | \omega(X) = 0\}. \tag{3.5}$$

Clearly, this definition satisfies the conditions to define a connection on P. To see this, let's check that it satisfies $(R_g)_* H_uP = H_{ug}P$. Let $X \in H_uP$. So, $\omega(X) = 0$. Let $Y = (R_g)_* X$, for some $g \in G$. We have that $Y \in T_{ug}P$. Then, $\omega(Y) = \omega_{ug}((R_g)_* X) = g^{-1} \omega_u(X) g = 0$, because $\omega(X) = 0$. Thus, $Y = (R_g)_* X \in H_{ug}P$. Since $(R_g)_*$ is a vector space isomorphism, we can write any $Y \in T_{ug}P$ as $Y = (R_g)_* X$, for some $X \in T_uP$. Hence, $(R_g)_* H_uP = H_{ug}P$, and (3.5) defines a connection, known as the *Ehresmann connection*.

Now, let's make things local. Let $\{U_i\}$ be an open covering of the base manifold M. Let $\sigma_i : U_i \to \pi^{-1}(U_i)$ be local sections. These maps induce pullbacks

$$\sigma_i^* : T_u^* P \to T_p^* U_i,$$

where $u = \sigma_i(p)$. Given a connection one-form $\omega \in \mathfrak{g} \otimes T^*P$, we take a local pullback of it into the base manifold and define the $\mathfrak{g}$-valued one form

$$\mathcal{A}_i = \sigma_i^* \omega \in \mathfrak{g} \otimes \Omega^1(U_i).$$

It turns out that, given a local $\mathfrak{g}$-valued one-form $\mathcal{A}_i \in \mathfrak{g} \otimes \Omega^1(U_i)$ and a local section $\sigma_i : U_i \to \pi^{-1}(U_i)$, there exists a connection one-form $\omega \in \mathfrak{g} \otimes T^*P$ such that $\mathcal{A}_i = \sigma_i^* \omega$. So, to define a connection, we can either specify it directly by picking some ω or we can define it through some local data, namely a one-form $\mathcal{A}_i$ over the base space and a bunch of local sections.

To be able to uniquely define a separation of $T_u P$ into horizontal and vertical spaces, the connection one-form ω must be defined uniquely at every open set. That is, at an intersection $U_i \cap U_j \neq \emptyset$, we must have that $\omega_i = \omega_j$. This means that we can define ω at just one open set U_i, and the global ω is given by $\omega = \omega|_{U_i}$. This also means that when we take the pullback of ω into open sets of M by local sections, we must have some kind of compatibility conditions between the local connections one-forms that arise.

Consider two local sections σ_i and σ_j defined on open sets U_i, U_j that intersect, i.e., $U_i \cap U_j \neq \emptyset$. We have that at $p \in U_i \cap U_j$, $\sigma_j(p) = \sigma_i(p) t_{ij}(p)$, where $t_{ij} : U_i \cap U_j \to G$ is the transition function. That is, points in the bundle coming from the same point at the base differ only by a translation in the fiber G. Let $X \in T_p M$. We have that

$$(\sigma_j)_* X = (R_{t_{ij}})_* (\sigma_i)_* X + (t_{ij}^{-1} dt_{ij}(X))^{\#}.$$

In fact, let $\gamma(t) : [0,1] \to M$ be a smooth curve such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X$. We have that

$$\begin{aligned}
(\sigma_j)_* X = \frac{d}{dt} \sigma_j(\gamma(t)) \Big|_{t=0} &= \frac{d}{dt} [\sigma_i(\gamma(t)) t_{ij}(\gamma(t))] \Big|_{t=0} \\
&= \frac{d}{dt} \sigma_i(\gamma(t)) \Big|_{t=0} t_{ij}(p) + \sigma_i(p) \frac{d}{dt} t_{ij}(\gamma(t)) \Big|_{t=0} \\
&= (\sigma_i)_* X t_{ij}(p) + \sigma_j(p) t_{ij}^{-1}(p) \frac{d}{dt} t_{ij}(\gamma(t)) \Big|_{t=0} \\
&= (R_{t_{ij}})_* (\sigma_i)_* X + \sigma_j(p) t_{ij}(p)^{-1} dt_{ij}(X) \\
&= (R_{t_{ij}})_* (\sigma_i)_* X + (t_{ij}^{-1} dt_{ij}(X))^{\#}.
\end{aligned}$$

Consider the local one-forms $\mathcal{A}_i = \sigma_i^* \omega$ and $\mathcal{A}_j = \sigma_j^* \omega$. The are related as follows:

$$\begin{aligned}
\mathcal{A}_j = \sigma_j^* \omega(X) = \omega((\sigma_j)_* X) &= \omega\left((R_{t_{ij}})_* (\sigma_i)_* X + (t_{ij}^{-1} dt_{ij}(X))^{\#}\right) \\
&= \omega\left((R_{t_{ij}})_* (\sigma_i)_* X\right) + \omega\left((t_{ij}^{-1} dt_{ij}(X))^{\#}\right) \\
&= t_{ij}^{-1} \omega((\sigma_i)_* X) t_{ij} + t_{ij}^{-1} dt_{ij}(X) \\
&= t_{ij}^{-1} \sigma_i^* \omega(X) t_{ij} + t_{ij}^{-1} dt_{ij}(X)
\end{aligned}$$

$$= t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}dt_{ij}(X).$$

Hence, the compatibility condition relating local connections on intersecting open sets is nothing but a *gauge transformation*. In a more familiar fashion, since $t_{ij}(p) = g$ is a group element, we can write

$$\mathcal{A}_j = g^{-1}\mathcal{A}_i g + g^{-1}dg.$$

To sum-up a little bit, we learned that, since a principal bundle P is nothing but a bunch of product spaces $M \times G$ sewed smoothly together, we want to be able to define tangent spaces that split into a sum of tangent spaces $TM \oplus TG$. This is convenient because with such a splitting, we can define orthogonal directions on M and on G. We are able to do this by defining a projection ω of TP into $TG \cong \mathfrak{g}$. This projection demands global information about the manifold. Another easier approach is to consider local sections of P, together with local one-forms $\mathcal{A}_i$ over M. Using this data, we can also define a global projection ω. However, we must have some conditions on this local data to be able to uniquely define ω throughout P. These conditions are basically gauge transformations between local one-forms $\mathcal{A}_i$.

3.2 Paralell Transport

Consider a curve $\gamma : [0, 1] \to M$ over M. We can lift this curve to a curve on the principal bundle P through a *horizontal lift*. The lifted curve $\tilde{\gamma} : [0, 1] \to P$ is such that $\pi \circ \tilde{\gamma} = \gamma$ and that all tangent vectors to $\tilde{\gamma}$ belong to the horizontal space $H_{\tilde{\gamma}(t)}P$. Let X be a tangent vector to $\tilde{\gamma}$. Then, $X \in H_{\tilde{\gamma}(t)}P$ and it is such that $\omega(X) = 0$, where ω is the unique connection one-form. Since $\omega(X) = 0$ is a differential equation, theorems of existence and uniqueness of solutions can be employed to convince oneself that $\tilde{\gamma}$ exists and is unique. Let's construct such a curve.

Let $U \subset M$ be an open set and let $\sigma : U \to \pi^{-1}(U)$ be a local section in the principal bundle. Suppose that the curve $\gamma : [0, 1] \to M$ is entirely contained in U. Then, any point in the lifted curve $\tilde{\gamma} : [0, 1] \to \pi^{-1}(U)$ can be written as $\tilde{\gamma}(t) = \sigma(\gamma(t))g(\gamma(t))$, where $g(\gamma(t)) \in G$ belongs to the fiber at $\gamma(t)$. We can define $g(\gamma(0)) = e$, the group identity, in such a way that $\tilde{\gamma}(0) = \sigma(\gamma(0))$. Let X be a tangent vector to the point $\gamma(0)$. We can define a tangent vector to $\tilde{\gamma}(0)$, $\tilde{X}$, as $\tilde{X} = \sigma(\gamma(0))_* X$. Since $\sigma(\gamma(t)) = \sigma(\gamma(0))g(\gamma(t))$, we can define a vector at any $\tilde{\gamma}(t)$ as

$$\tilde{X} = g(\gamma(t))^{-1}\sigma_* X g(\gamma(t)) + [g(\gamma(t))^{-1}dg(X)]^{\#}.$$

Since $\tilde{X}$ is a horizontal vector, $\omega(\tilde{X}) = 0$, which means that

$$0 = g(\gamma(t))^{-1}\omega(\sigma_* X)g(\gamma(t)) + g(\gamma(t))^{-1}dg(X).$$

Multiplying both sides with $g(\gamma(t))$ from the left,

$$\frac{dg(\gamma(t))}{dt} = -\omega(\sigma_* X) g(\gamma(t)).$$

Let $\mathcal{A}(X) = \omega(\sigma_* X) = \sigma^*(\omega(X))$ be the pullback of the connection one-form into a local form on U. We have

$$\frac{dg(\gamma(t))}{dt} = -\mathcal{A}(X) g(\gamma(t)),$$

which can be formally solved. Its solution is

$$\begin{aligned} g(\gamma(t)) &= \mathcal{P}\mathrm{exp}\left(-\int_0^t \mathcal{A}_\mu \frac{dx^\mu}{dt} dt\right) \\ &= \mathcal{P}\mathrm{exp}\left(-\int_{\gamma(0)}^{\gamma(t)} \mathcal{A}(\gamma(t))_\mu dx^\mu\right), \end{aligned} \tag{3.6}$$

where $\mathcal{P}$ is the path-ordering operator. Hence, any point in the lifted curve $\tilde{\gamma}$ can be expressed as $\tilde{\gamma}(t) = \sigma(\gamma(t)) g(\gamma(t))$, where $g(\gamma(t))$ is given by (3.6).

Let $u_0 = \pi^{-1}(\gamma(0))$. There is a unique horizontal lift $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = u_0$. Hence, there is a unique point $\tilde{\gamma}(1) \in \pi^{-1}(\gamma(1))$. We define the *paralell transport* of u_0 to u_1 along the curve $\tilde{\gamma}$ as the map $\Gamma(\tilde{\gamma}) : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$ such that, locally,

$$u_0 \to u_1 = \Gamma(\tilde{\gamma}) u_0 = \sigma(\gamma(1)) \mathcal{P}\mathrm{exp}\left(-\int_{\gamma(0)}^{\gamma(1)} \mathcal{A}(\gamma(t))_\mu dx^\mu\right). \tag{3.7}$$

Let's consider an example. Let $P = M \times \mathbb{R}$ be a trivial bundle, where $M = \mathbb{R}^2 - \{0\}$ and $\mathbb{R}$ is the Abelian group of real numbers. Let

$$\omega = \frac{ydx - xdy}{x^2 + y^2} + df$$

be an one-form over P. In fact, it is a connection one-form on P. To see this, let $A \in \mathbb{R}$ and take the fundamental vector field $A^\# = A\partial/\partial f$. We have that

$$\omega(A^\#) = A \langle df, \partial/\partial f \rangle = A.$$

Also, $R_{g^*}\omega = g^{-1}\omega g = \omega$ because $\mathbb{R}$ is Abelian.

Take a curve $\gamma : [0,1] \to M$ such that $\gamma(t) = (\mathrm{cos} 2\pi t, \mathrm{sin} 2\pi t)$. That is, γ is a circle over M. Let's construct the horizontal lift of γ to P. A tangent vector to $\tilde{\gamma}$ is such that

$$\tilde{X} = \frac{d}{dt} = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{df}{dt}\frac{\partial}{\partial f}.$$

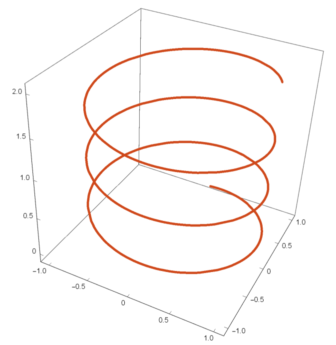

Figure 21: The lifted circle is a helix in P.

For it to be horizontal, it must satisfy

$$\omega(\tilde{X}) = 0,$$

that is,

$$\frac{y}{r^2}\frac{dx}{dt} - \frac{x}{r^2}\frac{dy}{dt} + \frac{df}{dt} = 0,$$

with $r^2 = x^2 + y^2$. Since $x = \cos 2\pi t$ and $y = \sin 2\pi t$, we have that

$$-2\pi \sin^2 2\pi t - 2\pi \cos^2 2\pi t + \frac{df}{dt} = 0,$$

that is,

$$\frac{df}{dt} = 2\pi,$$

whose solution is $f(t) = 2\pi t + \text{const}$. Thus, the lifted curve is given by

$$\tilde{\gamma} = (\cos 2\pi t, \sin 2\pi t, 2\pi t),$$

which is a helix above the unit circle.

Now, if a point travels around the circle, coming back to its initial position, in the lifted curve the point will not come back to where it started. We can see this by making $t \to t + 2\pi$. If we are on the circle, in time t and in time $t + 2\pi$ we are in the same position. However, if we are on the helix, in time $t + 2\pi$ we have climbed the fiber by an amount of 2π. To account for this effect, we will introduce the notion of *holonomy*.

Consider a loop $\gamma : [0, 1] \to M$, i.e., $\gamma(0) = \gamma(1) = p$. The lifted curve in P might be such that $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$, i.e., it might fail do be also a loop, as in the helix case we saw above. When it does, it is because the point $\tilde{\gamma}(1)$ has moved along the fiber to another point such that $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$. Thus, the path γ defines a linear transformation along the fiber $\tau_\gamma : \pi^{-1}(p) \to \pi^{-1}(p)$. This transformation must respect the right action, $\tau_\gamma(ug) = \tau_\gamma(u)g$.

For a point $u \in P$ such that $\pi(u) = p$, take the set of loops $C_p(M)$ at p. The set

$$\Phi_u = \{g \in G | \tau_\gamma(u) = ug, \gamma \in C_p(M)\}$$

is called the *holonomy group* at u.

3.3 Curvature

We now define the curvature of a connection. Consider a vector-valued r-form $\phi \in \Omega^r(P) \otimes V$ over the principal bundle P. For $X_1, ..., X_{r+1} \in T_uP$, we define the *covariant derivative* $D : \Omega^r(P) \to \Omega^{r+1}(P)$ as

$$D\phi(X_1, ..., X_{r+1}) = d_p\phi(X_1^H, ..., X_{r+1}^H), \tag{3.8}$$

where X^H is the horizontal component of the vector X furnished by the connection one-form.

The *curvature 2-form* Ω is defined as

$$\Omega = D\omega \in \Omega^2(P) \otimes \mathfrak{g}, \tag{3.9}$$

i.e., it is the covariant derivative of the connection one-form.

The curvature 2-form is such that

$$\Omega(X, Y) = d_p\omega(X, Y) + [\omega(X), \omega(Y)], \tag{3.10}$$

i.e., it satisfies *Cartan's structure equation.* To see this, first let $X, Y \in H_uP$ be purely horizontal. Then $\omega(X) = \omega(Y) = 0$ and by definition, $\Omega(X, Y) = D\omega(X, Y) = d_p\omega(X, Y)$. Now, if $X \in H_uP$ and $Y \in V_uP$ or vice-versa, we have that $\omega(X) = 0$ and $Y^H = 0$. This last equality means that $\Omega(X, Y) = 0$. Thus, we need that $d_p\omega(X, Y) = 0$. We have that

$$d_p\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Note that $[X, Y] \in H_uP$. In fact, if Y is the flow generated by $g(t)$,

$$[X, Y] = \mathcal{L}_Y X = \lim_{t \to 0} \frac{1}{t} \left[R_{g(t)*}X - X\right].$$

Since horizontal vectors are invariant under right translations, we have a horizontal vector in the right-hand side. Then, $\omega([X, Y]) = 0$. Now, let $Y = A^\#$ for some $A \in \mathfrak{g}$. We have that $\omega(Y) = A$ and thus $XA = 0$.

Lastly, if $X, Y \in V_u P$, $\Omega(X,Y) = 0$ because $X^H = Y^H = 0$, $\omega(X) = X$ and $\omega(Y) = Y$. This means that

$$d_p\omega(X,Y) = -\omega[X,Y],$$

which completes the proof.

But what does the curvature means geometrically? To answer this, first note that, for $X, Y \in H_u P$, $\Omega(X,Y)$ measures how vertical is $[X,Y]$. In fact, since $\omega(X) = \omega(Y) = 0$, we have that

$$\Omega(X,Y) = d_p\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = -\omega([X,Y]),$$

which projects $[X,Y]$ into ist vertical component.

Now, consider a parallelogram γ whose corners are the points $(0,0), (\epsilon,0), (0,\delta), (\epsilon,\delta)$. Let $V = \partial/\partial x^1$ and $W = \partial/\partial x^2$ be tangent vectors to the two directions of γ. Let $\tilde{\gamma}$ be the horizontal lift of γ and let $X, Y \in H_u P$ such that $\pi_* X = V$, $\pi_* Y = W$. We have that

$$\pi_*([X^H, Y^H]) = [V, W] = 0,$$

which means that $[X,Y]$ is vertical. The horizontal lift $\tilde{\gamma}$ does not close, and the failure to close is proportional to $[X,Y]$. Hence, Ω measures the failure of closing the horizontal lift of a loop.

To obtain the local form of the curvature, we only need to consider a local section $\sigma : U \to P$, take its induced pullback $\sigma^* : T^*P \to T^*U$ and define

$$F = \sigma^*\Omega.$$

Since $\sigma^* d_p\omega = d_p\sigma^*\omega = d_p A$ and $\sigma^*[\omega,\omega] = [\sigma^*\omega, \sigma^*\omega] = [A,A] = A \wedge A$, we have that

$$F = dA + A \wedge A, \tag{3.11}$$

or equivalently

$$F(X,Y) = dA(X,Y) + [A(X), A(Y)]. \tag{3.12}$$

In components, it is easy to see that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{3.13}$$

or, using that A and F are Lie-algebra valued, writting $F_{\mu\nu} = F^a_{\mu\nu} t_a$ and $A_\mu = A^a_\mu t_a$,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu, \tag{3.14}$$

where $[t_a, t_b] = f^c_{ab} t_c$.

It can be easily proved that, if F_i and F_j are local curvatures in two open sets U_i, U_j such that $U_i \cap U_j \neq \emptyset$,

$$F_j = t_{ij}^{-1} F_i t_{ij},$$

where t_{ij} is the transition function at $U_i \cap U_j$.

Let's now consider an important identity. Let $\omega = \omega^a t_a$ and $\Omega = \Omega^a t_a$, where t_a are generators of the Lie algebra. We have that

$$\Omega^a = d\omega^a + f^a_{bc}\omega^b \wedge \omega^c.$$

Taking the exterior derivative of both sides, we have

$$d\Omega^a = f^a_{bc} d\omega^b \wedge \omega^c + f^a_{bc}\omega^b \wedge d\omega^c.$$

Since $\omega(X) = 0$ for horizontal X, we have that

$$D\Omega(X, Y) = d\Omega(X^H, Y^H) = 0,$$

which is Bianchi's identity:

$$D\Omega = 0.$$

To find a local form for Bianchi's identity, we just apply the pullback by a local section. The result is

$$dF + [A, F] = 0.$$

References

[1] Mikio Nakahara. *Geometry, Topology and Physics.* CRC Press, 2003.

[2] John C. Baez and Javier P. Muniain. *Gauge Fields, Knots and Gravity*, volume 4. World Scientific Publishing Company, 1994.