Representation Theory of $\mathrm{GL}(N, \mathbb{C})$

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1 Lie Groups

1.1 GL($N, \mathbb{C}$) as a Lie group

A *Lie group* G is a group which is also a smooth manifold, such that the group product $\mu : G \times G \to G$, $\mu(g, h) = gh$, and inversion $i : G \to G$, $i(g) = g^{-1}$, are smooth maps.

Let $\mathrm{M}(N, \mathbb{C})$ be the vector space of $N \times N$ matrices with complex entries. Clearly, $\mathrm{M}(N, \mathbb{C}) \cong \mathbb{R}^{2N^2}$ and so it is also a smooth manifold. The set

$$\mathrm{GL}(N, \mathbb{C}) = \{M \in \mathrm{M}(N, \mathbb{C}) | \mathrm{det} M \neq 0\} \tag{1.1.1}$$

is a group, because the $N \times N$ identity matrix I is in it, it is closed under the matrix product and any matrix M in $\mathrm{GL}(N, \mathbb{C})$ has an inverse M^{-1} which is also in $\mathrm{GL}(N, \mathbb{C})$. Moreover, it is a smooth manifold. In fact, first note that the determinant

$$\det : \mathrm{M}(N, \mathbb{C}) \to \mathbb{R},\ \mathrm{det} M = \sum_{\sigma \in S_N} \prod_{i=1}^{N} M_{i\sigma_i},$$

where S_N is the group of permutations of N elements, is a polynomial in the entries of M and, therefore, it is a continuous function. This means that, for every open interval $O = (a, b) \subset \mathbb{R}$, $\det^{-1} O$ is an open subset of $\mathrm{M}(N, \mathbb{C})$. Since $\mathrm{GL}(N, \mathbb{C}) = \det^{-1}(\mathbb{R} - \{0\})$ and since $\mathbb{R} - \{0\}$ is open, $\mathrm{GL}(N, \mathbb{C})$ is an open subset of $\mathrm{M}(N, \mathbb{C})$, and so it is homeomorphic to an open subset of $\mathbb{R}^{2N^2}$. Thus, $\mathrm{GL}(N, \mathbb{C})$ is a smooth manifold.

1.2 Invariant integration over GL($N, \mathbb{C}$)

Translation invariance is an important feature of Riemann integrals. In the case of groups, left and right multiplication are equivalent to translation.

Therefore, in Lie group theory, it is important to establish a notion of invariant integration under left and right multiplication. Every locally compact topological group has a so-called *left(right) Haar measure*, i.e., a left(right)-invariant measure [1]. Here we show how to construct such a measure for $\mathrm{GL}(N,\mathbb{C})$.

Consider the volume element $dV(M) = f(M)dM$ in $\mathrm{GL}(N,\mathbb{C})$, where dM is the usual Lebesgue measure $dM = \prod_{i,j=1}^{N} dM_{ij}$ and $f(M) : \mathrm{GL}(N,\mathbb{C}) \to \mathbb{C}$ is some smooth function such that the volume element is left-invariant. A left-translation is a map $L_g : \mathrm{GL}(N,\mathbb{C}) \to \mathrm{GL}(N,\mathbb{C})$, $L_g(M) = gM$. Consider the integral of a smooth function $u : \mathrm{GL}(N,\mathbb{C}) \to \mathbb{C}$,

$$A = \int_{\mathrm{GL}(N,\mathbb{C})} dV(M)u(M) = \int_{\mathrm{GL}(N,\mathbb{C})} dM f(M)u(M).$$

Under a left-translation $M \to M' = gM$, $g \in \mathrm{GL}(N,\mathbb{C})$, the integrand changes as

$$f(M)u(M) \to f(g^{-1}M')u(g^{-1}M'),$$

while the Lebesgue measure changes to

$$dM = |J|dM',$$

where

$$J = \mathrm{det}_{ij,kl}\left(\frac{\partial M_{ij}}{\partial M'_{kl}}\right) = \det\left(g^{-1}\otimes I\right) = (\mathrm{det}g^{-1})^N.$$

Therefore, for the measure to be translation invariant, we must have

$$f(g^{-1}M) = |\mathrm{det}g^{-1}|^{-N} f(M).$$

Thus, we can choose $f(M) = |\mathrm{det}M|^{-N}$, and the left-invariant Haar measure for $\mathrm{GL}(N,\mathbb{C})$ is given by

$$d\mu(M) = |\mathrm{det}M|^{-N} dM. \tag{1.2.1}$$

It is easy to see that, if we imposed that dV is right-invariant, we would obtain the same result, i.e., in this case the left and right Haar measures are equal.

2 Representation Theory

2.1 Representations

Elements of groups are commonly used to represent *symmetry transformations.* A symmetry transformation over some system S is a map that takes

S to itself. For example, a trivial symmetry transformation over some set of numbers $S = \{1, 2, 3, ..., N\}$ is a permutation of the position of the numbers, i.e., $\{1, 2, 3, ..., N\} \to \{2, 4, 8, ..., N\}$. A more sophisticated symmetry transformation is the rotation of a vector in $\mathbb{R}^2$ by some angle θ. Rotation has an inverse (the rotation by $-\theta$) and two consecutive rotations, say, by θ and ϕ, are equivalent to an overral rotation by $\theta + \phi$. Therefore, group elements are suited to describe symmetries. However, to act over some system, for example, over a vector space like $\mathbb{R}^2$, the group elements must be realized as linear transformations over it. Linear transformations $V \to V$, over a finite-dimensional vector space V, form a matrix vector space when expressed in some basis. Since group elements have inverse, this matrix space must contain only invertible matrices, forming, therefore, a matrix group. We are ready, then, to introduce the notion of a representation.

A *representation* of a group G as a group of linear transformations $\mathrm{GL}(V)$ over some finite-dimensional vector space V is a group homomorphism

$$\rho : G \to \mathrm{GL}(V). \tag{2.1.1}$$

Being a group homomorphism is equivalent to say that, for any $g, h \in G$,

$$\rho(gh) = \rho(g)\rho(h).$$

This implies that, if $e \in G$ is the group identity, $\rho(g) = \rho(eg) = \rho(e)\rho(g)$, for any $g \in G$, which means that $\rho(e)$ is the identity in $\mathrm{GL}(V)$, and we write $\rho(e) = I$. Also, if g^{-1} is the inverse of $g \in G$, $\rho(e) = \rho(gg^{-1}) = \rho(g)\rho(g^{-1})$, which is equivalent to say that $\rho(g^{-1})$ is the inverse of $\rho(g) \in \mathrm{GL}(V)$, and thus we write $\rho(g^{-1}) = \rho(g)^{-1}$.

Now, suppose V is a complex vector space with a Hermitian inner product. We say that $\rho : G \to \mathrm{GL}(V)$ is a unitary representation if

$$\langle \rho(g)v, \rho(g)u \rangle = \langle v, u \rangle ,$$

for any $g \in G$, $v, u \in V$. This is equivalent to say that $\rho(g)^\dagger \rho(g) = I$. Every compact Lie group admits a unitary representation, according to the following

Theorem 2.1.1. *Let $\rho : G \to GL(V)$ be a representation of a compact Lie group G into some complex vector space V. Let $(,)$ be some Hermitian inner product on V. Then, there is an inner product in V in which ρ is a unitary representation.*

Proof. For any $u, v \in V$, define the inner product

$$\langle u, v \rangle = \int_G d\mu(g) (\rho(g)u, \rho(g)v),$$

where $d\mu(g)$ is the Haar measure on G. We have that, $\forall h \in G$,

$$\langle \rho(h)u, \rho(h)v \rangle = \int_G d\mu(g)(\rho(g)\rho(h)u, \rho(g)\rho(h)v)$$

$$= \int_G d\mu(g)(\rho(gh)u, \rho(gh)v),$$

and, making the transformation $g \to g' = gh$, since $d\mu(g') = d\mu(gh) = d\mu(g)$,

$$\int_G d\mu(g)(\rho(gh)u, \rho(gh)v) = \int_G d\mu(g')(\rho(g')u, \rho(g')v) = \langle u, v \rangle .$$

Therefore,

$$\langle \rho(h)u, \rho(h)v \rangle = \langle u, v \rangle, \forall h \in G.$$

□

Representations can be seen as linear transformations on vector spaces parametrized by group elements. The vector space V in question can have subspaces which are *invariant* under the action of such linear transformations, meaning that if $v \in V_0$, where $V_0 \subset V$ is an invariant subspace, then for every $g \in G$, $\rho(g)v = w$, where $w \in V_0$ also. Thus, the invariant subspace is closed under the action of such transformations, i.e., no transformation can take a vector outside this subspace. This notion of invariant subspaces is exactly what we want when we consider a representation as a mean to describe symmetry transformations, because, as explained before, symmetries are operations that just sort of "rearrange" the system in some way. Hence, when dealing with representations, we want to find the invariant subspaces under their action, because those are the *fundamental* spaces in some way. Then, we say that a representation $\rho : G \to \mathrm{GL}(V)$ is *irreducible* if the only invariant subspaces are $\{0\}$, i.e, the subspace composed only by the zero vector, and the whole V itself. When the representation is not irreducible, the goal is then to find all invariant subspaces and see if V can be written as a direct sum of invariant subspaces. It turns out that we can always do this with unitary representations.

Theorem 2.1.2. *If $\rho : G \to GL(V)$ is a unitary representation of G over some finite-dimensional vector space V with a Hermitian inner product $\langle , \rangle$, then V can be written as a direct sum of subspaces that are invariant under the action of ρ.*

Proof. Let $V_0 \subset V$ be an invariant subspace of V. Then, for any $g \in G$, $\rho(g)v \in V_0$ for all $v \in V_0$. Let $V_0^{\perp}$ be the orthogonal complement to V_0, i.e., $V = V_0 \oplus V_0^\perp$ and $\langle V_0, V_0^\perp \rangle = 0$. Then, for any $u \in V_0^\perp$, $v \in V_0$ and any $g \in G$, $0 = \langle v, u \rangle = \langle \rho(g)v, \rho(g)u \rangle = \langle v', \rho(g)u \rangle$, where $v' \in V_0$. Thus, $\rho(g)u \in V_0^\perp$, and $V_0^\perp$ is also invariant. If V_0 and/or $V_0^\perp$ have proper invariant subspaces themselves, just repeate the process, i.e., split the space into the sum of its invariant subspace and its orthogonal complement. Since V is finite dimensional, the process will terminate sometime. $\square$

Lemma 2.1.1 (Schur). *Let $\rho : G \to GL(V)$ and $\pi : G \to GL(W)$ be two irreducible unitary representations of G over finite-dimensional complex vector spaces V and W. Let $T : V \to W$ be a linear transformation such that*

$$T\rho(g) = \pi(g)T, \ \forall g \in G. \tag{2.1.2}$$

Then, either $T = 0$ or T is an isomorphism. In the later case, T must be equal to a escalar multiple of a unitary map $V \to W \cong V$.

Proof. Let $v \in \ker T \subset V$. We have that

$$T\rho(g)v = \pi(g)Tv = 0,$$

that is, $\rho(g)v \in \ker T$ and then $\ker T$ is invariant under ρ. But ρ is irreducible, so either $\ker T = \{0\}$ or $\ker T = V$. If $\ker T = V$, $\mathrm{im} T = \{0\}$ and thus $T = 0$. If $\ker T = \{0\}$, T is injective. Now, for any $u \in V$, $Tu = w \in \mathrm{im} T \subset W$. So, since $\rho(g)u \in V$,

$$T\rho(g)u = \pi(g)w \Leftrightarrow w' = \pi(g)w,$$

where $w' = T\rho(g)u \in \mathrm{im} T$. Hence, $\mathrm{im} T$ is invariant under π, which means that, since π is irreducible, either $\mathrm{im} T = \{0\}$ or $\mathrm{im} T = W$. In ther former case, $T = 0$. In the later case, T is surjective. Therefore, either $T = 0$ or T is injective and surjective, i.e., an isomorphism.

Let then T be an isomorphism $V \to W$. In some basis, $T^\dagger T$ is an Hermitian matrix and hence it is diagonalizable. Moreover, for every $g \in G$, it satisfies

$$T^\dagger T \rho(g) = \rho(g) T^\dagger T.$$

Let $v \in V$ be an eigenvector of $T^\dagger T$, $T^\dagger T v = \lambda v, \lambda \in \mathbb{C}$. Thus,

$$T^\dagger T \rho(g) v = \lambda \rho(g) v,$$

for any $g \in G$, which means that the eigenspace of $T^\dagger T$ corresponding to $\lambda \in \mathbb{C}$ is invariant under ρ. From irreducibility then, either this eigenspace is equal to $\{0\}$ or to V. Since $\{0\}$ cannot be an eigenvector, the eigenspace must be equal to V. Therefore, for every $v \in V$, $T^\dagger T v = \lambda v$, which means that $T^\dagger T = \lambda I$, where I is the identity operator in V. Since T is an isomorphism, $T^\dagger T$ is positive-definite and $\lambda > 0$. Define $T' = \lambda^{-1/2} T$. We have then $T'^\dagger T' = I$, hence $T = \sqrt{\lambda} T'$ is indeed a scalar multiple of an unitary map. $\square$

A consequence of this Lemma is that, given an unitary irreducible representation $\rho : G \to \mathrm{GL}(V)$, where V is a finite-dimensional complex vector space, any linear map $T : V \to V$ such that $T\rho(g) = \rho(g)T$, $\forall g \in G$, is equal to a escalar multiple of the identity I, that is, $T = \alpha I$. To see it, just note that, since T is a multiple of an unitary map, it is diagonalizable. Moreover, ρ leaves its eigenspace invariant, so it must be equal to V because of the irreducibility. Then, $T = \alpha I$ follows.

We say that two representations $\rho : G \to \mathrm{GL}(V)$ and $\pi : G \to \mathrm{GL}(V)$ into finite-dimensional vector spaces V and W are equivalent if and only if there is an isomorphism $T : V \to W$ such that $T\rho(g) = \pi(g)T$, $\forall g \in G$. In this case, we write $\rho \approx \pi$.

Proposition 2.1.1. *Let $\rho : G \to GL(V)$ be a representation of a compact Lie group G into a finite-dimensional vector space V with an inner product. Then, the operator $P : V \to V$ such that*

$$Pv = \int_G \rho(g) v d\mu(g) \tag{2.1.3}$$

projects any $v \in V$ into the subspace of V in which ρ acts trivially as the identity map, i.e.,

$$\rho(g)Pv = Pv, \forall v \in V, g \in G.$$

Proof. Let $v \in V$. First, P is indeed a projector:

$$P^2 v = \int_G \rho(g) \left(\int_G \rho(h) v d\mu(h) \right) d\mu(g) = \int_G \int_G \rho(gh) v d\mu(g) d\mu(h).$$

Making the change of variables $g' = gh$, $\rho(gh) = \rho(g')$, $d\mu(g) = d\mu(gh) = d\mu(g')$. We have then

$$P^2 v = \int_G \rho(g') v d\mu(g') \int_G d\mu(h).$$

Using a normalized Haar measure, $\int_G d\mu(h) = 1$ and thus $P^2 = P$. Now,

$$v^\dagger P^\dagger = \int_G v^\dagger \rho(g)^\dagger d\mu(g) = \int_G v^\dagger \rho(g^{-1}) d\mu(g^{-1} g g^{-1}) = \int_G v^\dagger \rho(g) d\mu(g),$$

where we used the unitarity of ρ, the invariance of the Haar measure and made the change of variables $g^{-1} \to g$. Therefore, $P^\dagger = P$.

Now, the subspace in which P projects any $v \in V$ is such that, for all $g \in G$,

$$\rho(g)Pv = \int_G \rho(g)\rho(h) v d\mu(h) = \int_G \rho(gh) v d\mu(h) = Pv,$$

where we made the change of variables $h \to gh$ and used the invariance of the Haar measure. Thus, ρ acts trivially in this subspace. Moreover, for any $v \in V$ such that $\rho(g)v = v$, we have that $Pv = v$, because $Pv = \int_G \rho(g) v d\mu(g) = v \int_G d\mu(g) = v$. $\square$

Next, we demonstrate some important relations satisfied by irreducible unitary representations.

Lemma 2.1.2 (Weyl Orthogonality Relations). *Let $\rho : G \to GL(V)$ and $\pi : G \to GL(W)$ be two irreducible unitary representations of a compact Lie group G into finite-dimensional vector spaces V and W, with dimensions $dimV$ and $dimW$, respectively. Then, the matrices of the transformations ρ and π in some basis of V and W satisfy*

$$\int_G \rho(g)_{ij} \rho(g)^{\dagger}_{kl} d\mu(g) = \frac{1}{dimV} \delta_{il} \delta_{jk}, \tag{2.1.4}$$

$$\int_G \pi(g)_{ij} \rho(g)^{\dagger}_{kl} d\mu(g) = 0, \ \ \textit{if } \rho \textit{ not} \approx \pi. \tag{2.1.5}$$

Proof. Let $T \in \mathrm{Hom}(V, W) = \{T : V \to W\}$ and define $P : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ as

$$P(T) = \int_G \pi(g) T \rho(g)^{\dagger} d\mu(g).$$

Then,

$$\begin{aligned} \pi(g) P(T) \rho(g)^{\dagger} &= \int_G \pi(g) \pi(h) T \rho(h)^{\dagger} \rho(g)^{\dagger} d\mu(h) \\ &= \int_G \pi(gh) T \rho(gh)^{\dagger} d\mu(gh) \\ &= P(T), \end{aligned}$$

where we used the invariance of the Haar measure. Then, by Schur's Lemma, either $P(T)$ is a scalar multiple of an unitary isomorphism or $P(T) = 0$, $\forall T \in \mathrm{Hom}(V, W)$. If ρ not $\approx \pi$, then there is no isomorphism between V and W and, in particular, $P(T) = 0$ for all T. By representing the linear transformations as matrices in a basis, we then have that

$$\sum_{j,k} \int_G \pi(g)_{ij} T_{jk} \rho(g)^{\dagger}_{kl} d\mu(g) = 0, \ \forall T_{jk},$$

which means that

$$\int_G \pi(g)_{ij} \rho(g)^{\dagger}_{kl} d\mu(g) = 0,$$

for each $i, j = 1, ..., \mathrm{dim}W$ and each $k, l = 1, ..., \mathrm{dim}V$.

Now, if $\rho = \pi$, take any map $T : V \to V$ such that $\rho(g)T\rho(g)^\dagger = T$. The map $A = T^\dagger T$ satisfies $\rho(g)A\rho(g)^\dagger = A$, and $P(A) = A$. Since $A = \lambda I$, where I is the identity in V, we have that, by taking the trace of both sides, $\lambda = \mathrm{Tr}A/\mathrm{dim}V$. Thus, by representing the linear transformations as matrices in a basis, we have that

$$\sum_{jk} \int_G \rho(g)_{ij} A_{jk} \rho^\dagger(g)_{kl} d\mu(g) = \frac{1}{\mathrm{dim}V} \sum_{jk} A_{jk} \delta_{jk} \delta_{il},$$

which means that, since A is arbitrary,

$$\int_G \rho(g)_{ij} \rho(g)^\dagger_{kl} d\mu(g) = \frac{1}{\mathrm{dim}V} \delta_{il} \delta_{jk},$$

for all $i, j, k, l = 1, ..., \mathrm{dim}V$. □

Let G be a compact Lie group and consider the space $L^2(G)$ of square-integrable functions $f : G \to \mathbb{C}$. This is the space in which

$$\int d\mu(g) f(g) \bar{g}(g) < \infty,$$

for $f, g \in L^2(G)$. Consider a maximal set I of mutually inequivalent irreducible representations ρ^α, $\alpha \in I$, of G into vector spaces V_α with dimensions $d_\alpha = \mathrm{dim}V_\alpha$. Weyl's orthogonality relations mean that, if we choose a basis for every V_α, the set of matrix elements $\{d_\alpha^{1/2} \rho^\alpha_{ij} | \alpha \in I,\ i, j = 1, ..., d_\alpha\}$ is orthonormal in $L^2(G)$. In fact, it forms an orthonormal basis of $L^2(G)$, as stated in

Theorem 2.1.3 (Peter-Weyl)**.** *The set* $\{d_\alpha^{1/2} \rho^\alpha_{ij} | \alpha \in I,\ i, j = 1, ..., d_\alpha\}$ *forms an orthonormal basis of* $L^2(G)$.

We will not prove this theorem, but we can say that it follows from the Stone-Weierstrass theorem [2] if the representations are injective, i.e., *faithful*. This is sufficient for this exposition because we will be dealing with representations of $\mathrm{GL}(N, \mathbb{C})$ derived from its regular representation on $\mathbb{C}^N$, which is indeed faithful.

We call a function $f \in L^2(G)$ *central* if $f(h^{-1}gh) = f(g)$, for every $g, h \in G$. A very important example of central function is the *character* of a representation $\rho : G \to \mathrm{GL}(V)$, defined as

$$\chi_\rho(g) = \mathrm{Tr}\rho(g). \tag{2.1.6}$$

The character is indeed central:

$$\chi_\rho(h^{-1}gh) = \mathrm{Tr}\left[\rho(h^{-1})\rho(g)\rho(h)\right] = \mathrm{Tr}\left[\rho(hh^{-1})\rho(g)\right] = \mathrm{Tr}\rho(g) = \chi_\rho(g).$$

We have also that, if ρ is irreducible and unitary,

$$\int d\mu(g)\chi_\rho(g)\bar{\chi}_\rho(g) = \sum_{i,j}\int d\mu(g)\rho(g)_{ii}\rho(g)^\dagger_{jj} = \frac{1}{d_\rho}\sum_{i,j}\delta_{ij}\delta_{ij} = 1, \quad (2.1.7)$$

and that, for some other irreducible and unitary representation π not $\approx \rho$,

$$\int d\mu(g)\chi_\pi(g)\bar{\chi}_\rho(g) = \sum_{i,j}\int d\mu(g)\pi(g)_{ii}\rho(g)^\dagger_{jj} = 0, \quad (2.1.8)$$

where we used Weyl's orthogonality relations. Therefore, the characters of irreducible representations are orthonormal. Moreover, we have the

Theorem 2.1.4. *The characters of irreducible unitary representations form an orthonormal basis for the space of central functions in $L^2(G)$.*

Proof. Let $\{\rho^\alpha, \alpha \in I\}$ be a maximal set of mutually inequivalent irreducible unitary representations. The Peter-Weyl theorem states that $\{d^{1/2}\rho^\alpha_{ij} | \alpha \in I, i, j = 1, ..., d_\alpha\}$ is an orthonormal basis of $L^2(G)$. Let $f \in L^2(G)$. We can write

$$f(g) = \sum_{\alpha\in I}\sum_{i,j=1}^{d_\alpha} d_\alpha^{1/2} f^\alpha_{ij}\rho^\alpha_{ij}(g),$$

where the coefficients

$$f^\alpha_{ij} = d_\alpha^{1/2}\int d\mu(g) f(g)(\rho^\alpha)(g)^\dagger_{ij}.$$

Define

$$P_\alpha f(g) = \sum_{i,j=1}^{d_\alpha} d_\alpha^{1/2} f^\alpha_{ij}\rho^\alpha_{ij}(g).$$

Thus, $f(g) = \sum_{\alpha\in I} P_\alpha f(g)$. $P_\alpha f(g)$ can also be written as

$$P_\alpha f(g) = d_\alpha^{1/2}\mathrm{Tr}\left[(f^\alpha)^T\rho^\alpha(g)\right].$$

From the basis expansion, if f is central, $f(h^{-1}gh) = f(g)$ implies that

$$\sum_{i,j} f^\alpha_{ij}(\rho^\alpha(h))^\dagger_{ik}\rho^\alpha(g)_{kl}\rho^\alpha(h)_{lj} = f^\alpha_{kl}\rho^\alpha(g)_{kl},$$

for every $\alpha \in I, k, l = 1, ..., d_\alpha$ and every $g, h \in G$, and thus

$$\rho^\alpha(h)(f^\alpha)^T(\rho^\alpha(h))^\dagger = (f^\alpha)^T.$$

Then, Schur's lemma implies that $(f^\alpha)^T = \lambda I$, where I is the identitity matrix. To find the constant λ, we take the trace from both sides:

$$\lambda d_\alpha = \sum_i f_{ii}^\alpha = d_\alpha^{1/2} \int d\mu(g) f(g) \sum_i (\rho^\alpha)(g)_{ii}^\dagger = d_\alpha^{1/2} \int d\mu(g) f(g) \bar{\chi}_\alpha(g),$$

i.e.,

$$\lambda = d_\alpha^{-1/2}(f, \chi_\alpha) = d_\alpha^{-1/2} \int d\mu(g) f(g) \bar{\chi}_\alpha(g),$$

where $(,)$ is the norm of $L^2(G)$. So, since $(f^\alpha)^T = \lambda I$,

$$P_\alpha f(g) = d_\alpha^{1/2} \lambda \mathrm{Tr} \rho^\alpha(g) = (f, \chi_\alpha)\chi_\alpha(g),$$

and then

$$f(g) = \sum_{\alpha \in I} (f, \chi_\alpha)\chi_\alpha(g).$$

□

In theorem 2.1.2, we saw that for any unitary representation $\rho : G \to \mathrm{GL}(V)$ of G on some finite-dimensional vector space V, V can be written as a direct sum of subspaces which are invariant under ρ. It would be useful to have a way of separating those subspaces by means of some projector operator. Suppose that

$$V = V_1 \oplus V_2 \oplus ... \oplus V_n,$$

where each V_α, $\alpha = 1, ..., n$, is invariant under ρ. Define the restriction of ρ to V_α as $\rho^\alpha(g) = \rho(g)|_{V_\alpha}$, $\forall g \in G$ and $\forall \alpha = 1, ..., n$. Each ρ^α is irreducible, i.e., $\rho^\alpha(g)v^{(\alpha)} \in V_\alpha$, $\forall g \in G$ and $\forall \alpha = 1, ..., n$. For each $\alpha = 1, ..., n$, define the operator $P_\alpha : V \to V$ as

$$P_\alpha = d_\alpha \int d\mu(g) \bar{\chi}_\alpha(g) \rho(g), \tag{2.1.9}$$

where $d_\alpha = \mathrm{dim} V_\alpha$ and $\chi_\alpha(g) = \mathrm{Tr} \rho^\alpha(g)$. Take $v_i^{(\alpha)} \in V_\alpha$ some basis element of V_α. Then,

$$P_\alpha v_i^{(\alpha)} = d_\alpha \int d\mu(g) \bar{\chi}_\alpha(g) \rho(g) v_i^{(\alpha)}.$$

Since, $\rho(g)v_i^{(\alpha)} \in V_\alpha$, without loss of generality we can consider here only the restriction of ρ to V_α. Acting on the basis element, we have $\rho^\alpha(g)v_i^{(\alpha)} = \sum_j \rho^\alpha(g)_{ji}v_j^{(\alpha)}$ Then,

$$\int d\mu(g)\bar{\chi}_\alpha(g)\rho^\alpha(g)v_i^{(\alpha)} = \sum_{j,k} v_j^{(\alpha)} \int d\mu(g)(\rho^\alpha(g))^\dagger_{kk}\rho^\alpha(g)_{ji}$$

$$= \sum_{j,k} v_j^{(\alpha)} \frac{1}{d_\alpha}\delta_{ik}\delta_{jk} = \frac{1}{d_\alpha}\sum_j v_j^{(\alpha)}\delta_{ij} = \frac{1}{d_\alpha}v_i^{(\alpha)},$$

where we used Weyl's orthogonality relations, and thus,

$$P_\alpha v_i^{(\alpha)} = v_i^{(\alpha)}.$$

Now, take $v_i^{(\beta)} \in V_\beta$ a basis element, with $\beta \neq \alpha$, i.e., the representation spaces are orthogonal. Then,

$$P_\alpha v_i^{(\beta)} = d_\alpha \sum_k \int d\mu(g)(\rho^\alpha(g))^\dagger_{kk}\rho(g)v_i^{(\beta)}.$$

Again, since $\rho(g)v_i^{(\beta)} \in V_\beta$, we consider only the restriction of ρ to V_β. With $\rho^\beta(g)v_i^{(\beta)} = \sum_j \rho^\beta(g)_{ji}v_j^{(\beta)}$, we have

$$\sum_{k,j} v_j^{(\beta)} \int d\mu(g)(\rho^\alpha(g))^\dagger_{kk}\rho^\beta(g)_{ji} = 0,$$

where we used Weyl's orthogonality relations, and thus

$$P_\alpha v_i^{(\beta)} = 0.$$

Therefore, the operator P_α projects any vector $v \in V$, $v = \sum_\alpha v^{(\alpha)}$, into the subspace in which ρ^α acts irreducibly. However, among the V_α, there might be isomorphic spaces. The number of isomorphic copies of a particular $V_{\alpha'}$ that occurs in V is called the multiplicity of the irreducible representation $\rho^{\alpha'}$ in the representation ρ, and denoted $\mu(\rho^\alpha, \rho)$. It is given by

$$\mu(\rho^\alpha, \rho) = \frac{1}{d_\alpha}\mathrm{Tr}P_\alpha = \int d\mu(g)\bar{\chi}_\alpha(g)\chi_\rho(g). \tag{2.1.10}$$

Two equivalent representations must have the same irreducible components, with the same multiplicities. In fact, the characters can be used to identify different representations. We saw that two representations are equivalent if and only if there is a linear map that intertwines then. Say we have two equivalent irreducible unitary representations $\rho : G \to \mathrm{GL}(V)$ and $\pi : G \to \mathrm{GL}(W)$ of a compact Lie group G into finite-dimensional complex vector spaces V and W. A linear map $T : V \to W$ that intertwines ρ and π, i.e., such that $T\rho(g) = \pi(g)T$ for every $g \in G$, must be an isomorphism multiple of an unitary map, by Schur's lemma. Then, $\rho(g) = T^\dagger \pi(g) T$. The character of ρ is then given by

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)) = \mathrm{Tr}\left[T^\dagger \pi(g) T\right] = \mathrm{Tr}(\pi(g)) = \chi_\pi(g).$$

Therefore, two irreducible representations are equivalent if and only if their characters coincide.

2.2 Representations of S_n

Here we will completely determine all irreducible representations of the symmetric group of permutations of n elements, S_n. But first, we need a few results of the general representation theory of finite groups.

Let G be a finite group, i.e., a group with a finite number $|G|$ of elements. The number $|G|$ is called the order of G. All results from Lie group representation theory are applicable to finite groups by replacing the integral with a sum:

$$\int_G d\mu(g) f(g) \to \frac{1}{|G|} \sum_{g \in G} f(g).$$

Since G is finite, functions $f : G \to \mathbb{C}$ can assign $|G|$ elements to the generator $1 \in \mathbb{C}$. Therefore, $L^2(G)$ is a finite-dimensional vector space, with dimension $\mathrm{dim} L^2(G) = |G|$. Thus, we can define a representation $\rho : G \to \mathrm{GL}(L^2(G))$, given by

$$\rho(g) f(h) = f(g^{-1}h),$$

$\forall g, h \in G$. This representation is unitary, because of theorem 2.1.1.

If $\{\rho^\alpha, \alpha \in I\}$ is a maximal set of mutually inequivalent unitary irreducible representations of G on finite-dimensional vector spaces V_α, the Peter-Weyl theorem implies that $\{\rho^\alpha_{ij} | \alpha \in I, i, j = 1, ..., d_\alpha\}$, where $d_\alpha = \mathrm{dim} V_\alpha$, forms an orthonormal basis for $L^2(G)$. For $\mathcal{V}_\alpha = \mathrm{span}\{\rho^\alpha_{ij} | i, j = 1, ..., d_\alpha\}$, we have the decomposition

$$L^2(G) = \bigoplus_{\alpha \in I} \mathcal{V}_\alpha,$$

which implies that

$$|G| = \sum_{\alpha \in I} d_\alpha^2.$$

Let $C(g) = \{hgh^{-1}|h \in G\}$ be the conjugacy class of $g \in G$. The set of conjugacy classes of G is the union $C = \cup_{g\in G} C(g)$, whose order we denote $|C|$. It is clear then that the space of class functions $L^2_C(G)$ has dimension $|C|$. Since the set of irreducible characters $\{\chi_\alpha | \alpha \in I\}$ forms a basis of $L^2_C(G)$, then

$$|I| = |C|,$$

i.e., the number of nonequivalent irreducible unitary representations is equal to the number of conjugacy classes of G.

Let's consider now the symmetric group S_n. It is the group of bijections from the set $\{1, 2, ..., n\}$ to itself. The elements of S_n are permutations of the numbers $\{1, 2, ..., n\}$. As an example, take $\pi \in S_3$ given by

$$\pi(1) = 1, \pi(2) = 3, \pi(3) = 2.$$

Here we express a permutation in terms of its decomposition by *cycles*. For $i \in \{1, ..., n\}$, the elements $i, \pi(i), \pi^2(i), ...$, for some $\pi \in S_n$, cannot be all independent. At some point, we will have a permutation $\pi^p(i)$ that takes i back to itself. A *cycle* is a sequence

$$(i, \pi(i), \pi^2(i), ..., \pi^{p-1}(i)).$$

Equivalently, $(i_1, i_2, ..., i_l)$ is a cycle that takes $i_1 \to i_2$, $i_2 \to ...$ and $i_l \to i_1$. Any permutation can be expressed as a product of cycles. In the S_3 example, we can express π as the following product of cycles:

$$\pi = (1)(2, 3).$$

A k-cycle is a cycle with k elements. For example, (1) is a 1-cycle and $(2, 3)$ is a 2-cycle. The cycle type of a permutation π is an expression of the form

$$(1^{m_1}, 2^{m_2}, ..., n^{m_n}),$$

where m_k is the number of k-cycles in the permutation. For example, the type of $(1)(2, 3)$ is $(1^1, 2^1, 3^0)$. Another way to express the cycle type of a permutation is through a *partition* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ of n, where

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l \geq 0$$

and

$$|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_l = n.$$

Thus, k is repeated m_k times in the partition representation of the type of π. For example, the type of $(1)(2, 3)$ is represented as the following partition:

$$\lambda = (2, 1).$$

Let $\pi \in S_n$ be a permutation and, for some $\tau \in S_n$, consider its conjugate permutation $\tau\pi\tau^{-1}$. We have that, if $\pi = (i_1, ..., i_l)$, the conjugate permutation is given by $\tau\pi\tau^{-1} = (\tau(i_1), ..., \tau(i_l))$. In fact this is true, because $\tau\pi\tau^{-1}(\tau(i_k)) = \tau(\pi(i_k)) = \tau(i_{k+1})$, so $\tau\pi\tau^{-1}$ takes $\tau(i_k)$ to $\tau(i_{k+1})$, and thus it takes $\tau(i_l)$ to $\tau(i_1)$. This result can be extended to a more general permutation $\pi = (i_1, ..., i_l)...(i_m, ..., i_n)$, i.e.,

$$\tau\pi\tau^{-1} = (\tau(i_1), ..., \tau(i_l))...(\tau(i_m), ..., \tau(i_n)).$$

Note that conjugation does not change the cycle type. It follows that two permutations are conjugate if and only if they have the same cycle type. This means that conjugacy classes of S_n are in direct correspondence with partitions of n. For example, the permutations $(1, 2, 3)$, $(1, 2)(3)$, $(1)(2)(3)$ in S_3, of type $\lambda = (3)$, $\lambda = (2, 1)$ and $\lambda = (1, 1, 1)$, respectively, are in different conjugacy classes, while $(1, 2)(3)$ and $(1)(2, 3)$ are in the same conjugacy class, because they both have type $\lambda = (2, 1)$.

Since the number of conjugacy classes is equal to the number of irreducible representations of S_n, it follows that the number of irreducible representations is the number of partitions λ of n.

A representation of S_n is a group homomorphism $\rho : S_n \to \mathrm{GL}(V)$, for some vector space V with dimension $d = \mathrm{dim}V$. We saw that representations can be though of as symmetries of the underlying vector space, i.e., operations that doesn't change the vector space's nature (strictly speaking, its dimension). A natural representation of $\sigma \in S_n$ is thus as some operation $\rho(\sigma) : V \to V$ that take a basis $\{v_1, ..., v_d\} \subset V$ and shuffle its elements, i.e., $\rho(\sigma)v_i = v_{\sigma(i)}$, for any basis element $v_i \in V$. This is clearly a symmetry of V. A natural generalization of this representation is to consider as a representation space the tensor product $V \otimes V$. Then, $\sigma \in S_n$ can be represented as a bilinear map $\rho(\sigma) : V \otimes V \to V \otimes V$ that permutes the basis of $V \otimes V$, i.e., given $v_i \otimes v_j$ basis elements of $V \otimes V$, $\rho(\sigma)v_i \otimes v_j = v_{\sigma(i)} \otimes v_{\sigma(j)}$. However, such representations might not be irreducible, that is, there may be invariant proper subspaces of V and $V \otimes V$, in the sense that any permutation of the basis elements that lie in V or $V \otimes V$ remains in V or $V \otimes V$. In what follows, we will see how to construct *irreducible* representations of S_n based on some of the regular ones.

Consider the *group algebra* $\mathbb{C}S_n$ as the finite-dimensional complex vector space spanned by the basis $\{|\sigma\rangle \, | \sigma \in S_n\}$. That is, the basis elements are indexed by elements of the symmetric group. An arbitrary vector $v \in \mathbb{C}S_n$ can be written as the sum

$$v = \sum_{\sigma \in S_n} v_\sigma \, |\sigma\rangle \, , \; v_\sigma \in \mathbb{C}.$$

Clearly, this space has dimension $d = \dim\mathbb{C}S_n = |S_n| = n!$. Addition and multiplication by scalars in $\mathbb{C}S_n$ are natural: for $u, v \in \mathbb{C}S_n$, $\lambda \in \mathbb{C}$,

$$u + v = \sum_{\sigma\in S_n} u_\sigma \left|\sigma\right\rangle + \sum_{\sigma\in S_n} v_\sigma \left|\sigma\right\rangle = \sum_{\sigma\in S_n} (u_\sigma + v_\sigma) \left|\sigma\right\rangle ,$$

$$\lambda u = \sum_{\sigma\in S_n} \lambda u_\sigma \left|\sigma\right\rangle .$$

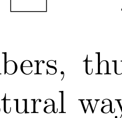
We can also define a multiplication in $\mathbb{C}S_n$. Given $u, v \in \mathbb{C}S_n$,

$$uv = \sum_{\sigma\in S_n} u_\sigma \left|\sigma\right\rangle \sum_{\tau\in S_n} v_\tau \left|\tau\right\rangle = \sum_{\sigma,\tau\in S_n} u_\sigma v_\tau \left|\sigma\tau\right\rangle .$$

This multiplication defines an action $S_n \times \mathbb{C}S_n \to \mathbb{C}S_n$, given on the basis elements of the group algebra by $(\sigma, |\tau\rangle) \to |\sigma\tau\rangle$. This group action, in turn, defines a representation $\rho : S_n \to \mathrm{GL}(\mathbb{C}S_n)$, in which $\rho(\sigma) : \mathbb{C}S_n \to \mathbb{C}S_n$ is such that $\rho(\sigma)\left|\tau\right\rangle = |\sigma\tau\rangle$, for any $\sigma, \tau \in S_n$. That ρ is a group homomorphism follows from the group associativity rule. This is known as the *regular representation* of S_n. Note that, since $\mathbb{C}S_n \cong \mathbb{C}^{n!}$, the regular representation is equivalent to the standard representation on $\mathbb{C}^{n!}$, in which S_n acts by shuffling the basis elements.

Recall that partitions of n are in one to one correspondence with conjugacy classes of permutations in S_n. That is, conjugate permutations have the same type, given by some partition λ of n. Two conjugate permutations, represented as symmetry operators over some vector space, correspond roughly to the same linear transformation, only differing by a change of basis. Thus, different symmetry operators are labelled by partitions of n. Moreover, for a given partition λ of n, we can construct a subspace V_λ of $\mathbb{C}S_n$ which is invariant under the action of the representation $\rho : S_n \to \mathbb{C}S_n$ defined above. In this way, irreducible representations of S_n are labelled by all possible partitions of n. We will now proceed to construct such irreducible representations.

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ be a partition of n, i.e., $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ and $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_k = n$. To such a partition, we can assign a so-called *Young diagram*, made of rows of boxes stacked on top of each other. The ith row has λ_i boxes. For example, the partition of $n = 6$ given by $\lambda = (3, 2, 1)$ is represented by the diagram



.

We can fill the boxes with numbers, thus making a *Young tableaux*. We choose to fill the boxes in the natural way, that is, starting from the left to the right, from top to botton. For example, the tableaux for $\lambda = (3, 2, 1)$ is

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \; .$$

To a given tableaux λ, we define the sets

$$P_\lambda = \{\sigma \in S_n | \sigma \text{ preserves the rows of } \lambda\}, \tag{2.2.1}$$
$$Q_\lambda = \{\sigma \in S_n | \sigma \text{ preserves the columns of } \lambda\}. \tag{2.2.2}$$

For example, if $n = 3$ and λ is the tableaux

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \; ,$$

$P_\lambda = \{1, (12)\}$ and $Q_\lambda = \{1, (13)\}$. Then, define the elements $a_\lambda, b_\lambda \in \mathbb{C}S_n$ given by

$$a_\lambda = \sum_{\sigma \in P_\lambda} |\sigma\rangle \,, \tag{2.2.3}$$

$$b_\lambda = \sum_{\sigma \in Q_\lambda} \text{sgn}(\sigma) \, |\sigma\rangle \,. \tag{2.2.4}$$

For example, for $n = 3$ and $\lambda = (2, 1)$,

$$a_\lambda = |1\rangle + |(12)\rangle \,,$$

$$b_\lambda = |1\rangle - |(13)\rangle \,.$$

Finally, define the so-called *Young symmetrizer*

$$c_\lambda = a_\lambda b_\lambda. \tag{2.2.5}$$

In our example,

$$c_\lambda = (|1\rangle + |(12)\rangle)(|1\rangle - |(13)\rangle) = |1\rangle - |(13)\rangle + |(12)\rangle - |(132)\rangle \,.$$

Being an element of the group algebra, c_λ acts over the vector space $\mathbb{C}S_n$. The image of this action is a subspace $V_\lambda = \mathbb{C}S_n c_\lambda$, which, as we will show, is irreducible under the representation induced by the group algebra action. For example, take as an ordered basis of $\mathbb{C}S_3$ the set

$$\{1, (12), (13), (23), (123), (132)\}.$$

$c_\lambda = |1\rangle - |(13)\rangle + |(12)\rangle - |(132)\rangle$ acts over this basis by

$$|\sigma\rangle\, c_\lambda = \sum_\tau c_{\tau\sigma} \,|\tau\rangle\,.$$

The matrix of this transformation can be obtained by noting that:

$$\begin{aligned}
|1\rangle\, c_\lambda &= |1\rangle - |(13)\rangle + |(12)\rangle - |(132)\rangle\,, && (2.2.6)\\
|(12)\rangle\, c_\lambda &= |(12)\rangle - |(12)(13)\rangle + |(12)(12)\rangle - |(12)(132)\rangle \\
&= |(12)\rangle - |(132)\rangle + |1\rangle - |(13)\rangle\,, && (2.2.7)\\
|(13)\rangle\, c_\lambda &= |(13)\rangle - |(13)(13)\rangle + |(13)(12)\rangle - |(13)(132)\rangle \\
&= |(13)\rangle - |1\rangle + |(123)\rangle - |(23)\rangle\,, && (2.2.8)\\
|(23)\rangle\, c_\lambda &= |(23)\rangle - |(23)(13)\rangle + |(23)(12)\rangle - |(23)(132)\rangle \\
&= |(23)\rangle - |(123)\rangle + |(132)\rangle - |(12)\rangle\,, && (2.2.9)\\
|(123)\rangle\, c_\lambda &= |(123)\rangle - |(123)(13)\rangle + |(123)(12)\rangle - |(123)(132)\rangle \\
&= |(123)\rangle + |(13)\rangle - |1\rangle - |(23)\rangle\,, && (2.2.10)\\
|(132)\rangle\, c_\lambda &= |(132)\rangle - |(132)(13)\rangle + |(132)(12)\rangle - |(132)(132)\rangle \\
&= |(132)\rangle - |(12)\rangle + |(23)\rangle - |(123)\rangle\,. && (2.2.11)\\
& && (2.2.12)
\end{aligned}$$

Thus, the $c_{\tau\sigma}$ is

$$c_\lambda = \begin{pmatrix} 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}, \qquad (2.2.13)$$

and it is easy to see that only the first and third columns of this matrix are linearly independent. In fact, the second column is identical to the first, the fourth column is equal to minus the sum of the first and third columns, the fifth column is equal to minus the third column and the sixth column is equal to the sum of the first and third columns. Therefore, the image space V_λ of this transformation is generated by the vectors

$$|u\rangle = |1\rangle + |(12)\rangle - |(13)\rangle - |(132)\rangle\,,$$

$$|v\rangle = -\,|1\rangle + |(13)\rangle - |(23)\rangle + |(123)\rangle\,.$$

To show that V_λ is irreducible, we have to show that $\rho(\sigma)\left|\tau\right\rangle = \left|\sigma\right\rangle\left|\tau\right\rangle \in V_\lambda$ for every $\left|\tau\right\rangle \in V_\lambda$, $\sigma \in S_n$. Since any $\left|\tau\right\rangle \in V_\lambda$ can be written as $\left|\tau\right\rangle = a\left|u\right\rangle + b\left|v\right\rangle$, for $a, b \in \mathbb{C}$, it is easy to check that

$$\rho(1)\left|\tau\right\rangle = \left|\tau\right\rangle, \tag{2.2.14}$$

$$\rho((12))\left|\tau\right\rangle = (a-b)\left|u\right\rangle - b\left|v\right\rangle, \tag{2.2.15}$$

$$\rho((13))\left|\tau\right\rangle = a\left|v\right\rangle + b\left|u\right\rangle, \tag{2.2.16}$$

$$\rho((23))\left|\tau\right\rangle = -a\left|u\right\rangle + (b-a)\left|v\right\rangle, \tag{2.2.17}$$

$$\rho((123))\left|\tau\right\rangle = (a-b)\left|v\right\rangle - b\left|u\right\rangle, \tag{2.2.18}$$

$$\rho((132))\left|\tau\right\rangle = (b-a)\left|u\right\rangle - a\left|v\right\rangle. \tag{2.2.19}$$

Hence, V_λ is indeed invariant under ρ, and therefore it is irreducible. The tableaux $\lambda = (2,1)$ gives rise to a two-dimensional representation $V_{(2,1)}$ of S_3. This representation is equivalent to the one in which ρ acts over the subspace of $\mathbb{R}^3$ spanned by the vectors $u_{12} = e_1 - e_2$, $u_{23} = e_2 - e_3$, by permuting indices, where e_1, e_2, e_3 are the canonical basis vectors of $\mathbb{R}^3$. Indeed, we have that

$$\rho((12))u_{12} = -u_{12},\ \rho(12)u_{23} = u_{12} + u_{23}, \tag{2.2.20}$$

$$\rho((13))u_{12} = -u_{23},\ \rho((13))u_{23} = -u_{12}, \tag{2.2.21}$$

$$\rho((23))u_{12} = u_{12} + u_{23},\ \rho((23))u_{23} = -u_{23}, \tag{2.2.22}$$

$$\rho((123))u_{12} = u_{23},\ \rho((123))u_{23} = -(u_{12} + u_{23}), \tag{2.2.23}$$

$$\rho((132))u_{12} = -(u_{12} + u_{23}),\ \rho((132))u_{23} = u_{12}. \tag{2.2.24}$$

As another example, consider the tableaux

$$\lambda = \boxed{1\,|\,2\,|\,3}.$$

The Young symmetrizer is given by $c_\lambda = a_\lambda = \sum_{\sigma \in S_3} \left|\sigma\right\rangle$. As a linear transformation over $\mathbb{C}S_3$, it is straightforward to check that c_λ has as matrix form

$$c_\lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{2.2.25}$$

Thus, its image $V_{(3)}$ is spanned by the vector $\left|u\right\rangle = \sum_{\sigma \in S_3} \left|\sigma\right\rangle$. Since, for any $\tau \in S_3$,

$$\rho(\tau)\left|u\right\rangle = \sum_{\sigma \in S_3} \left|\tau\sigma\right\rangle = \sum_{\sigma'\tau^{-1} \in S_n} \left|\sigma'\right\rangle = \left|u\right\rangle,$$

$V_{(3)}$ is invariant under ρ. Therefore, it furnishes an unidimensional irreducible representation of S_3, called the *trivial representation*, in which the elements of S_3 act as the identity matrix.

As a final example, consider the tableaux

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}\,.$$

The Young symmetrizer associated to it is

$$c_\lambda = b_\lambda = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \left|\sigma\right\rangle .$$

Since

$$\left|\tau\right\rangle c_\lambda = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \left|\tau\sigma\right\rangle = \sum_{\sigma' \in S_3} \operatorname{sgn}(\tau^{-1}\sigma') \left|\sigma'\right\rangle = \operatorname{sgn}(\tau) c_\lambda,$$

it is easy to see that the matrix representation of c_λ, considered as a linear transformation over $\mathbb{C}S_3$, is given by

$$c_\lambda = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}. \tag{2.2.26}$$

Then, the image $\mathbb{C}S_3 c_\lambda = V_{(1,1,1)}$ is spanned by the vector $\left|u\right\rangle = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \left|\sigma\right\rangle$. Since

$$\rho(\tau) \left|u\right\rangle = \operatorname{sgn}(\tau) \left|u\right\rangle ,$$

this space is invariant under ρ, and thus gives another one-dimensional irreducible representation, called the *alternating* (or sign) representation.

We note that the construction of the trivial and alternating representations can be easily extended for arbitrary S_n. This means that, for any $n \in \mathbb{N}$, the partitions (n) and $(1, 1, ..., 1)$ give rise to two one-dimensional irreducible representations of S_n, namely the trivial and the alternating representations, respectively.

The following lemmas will be important to prove that the image of the symmetrizer is irreducible.

Lemma 2.2.1. *Let λ be a partition of n. For any $p \in P_\lambda$, $q \in Q_\lambda$, $\left|p\right\rangle c_\lambda(sgn(q) \left|q\right\rangle) = c_\lambda$. Moreover, any other element that satisfies this identity in $\mathbb{C}S_n$ is a scalar multiple of c_λ.*

Proof. First, we have that, since

$$|p\rangle\, a_\lambda = \sum_{\sigma\in P_\lambda} |p\sigma\rangle = a_\lambda,$$

$$b_\lambda \mathrm{sgn}(q)\,|q\rangle = \sum_{\sigma\in Q_\lambda} \mathrm{sgn}(\sigma q)\,|\sigma q\rangle = b_\lambda,$$

it follows that

$$|p\rangle\, c_\lambda \mathrm{sgn}(q)\,|q\rangle = |p\rangle\, a_\lambda b_\lambda \mathrm{sgn}(q)\,|q\rangle = a_\lambda b_\lambda = c_\lambda.$$

Now, since $c_\lambda = a_\lambda b_\lambda$, it is equal to a linear combination of products of the type $|p\rangle\,|q\rangle = |pq\rangle$, where $p\in P_\lambda$ and $q\in Q_\lambda$. If some other element $\sum_{\sigma\in S_n} v_\sigma\,|\sigma\rangle \in \mathbb{C}S_n$ satisfies

$$|p\rangle \left(\sum_{\sigma\in S_n} v_\sigma\,|\sigma\rangle\right)(\mathrm{sgn}(q)\,|q\rangle) = \sum_{\sigma\in S_n} v_\sigma\,|\sigma\rangle\,,$$

for any $p\in P_\lambda$, $q\in Q_\lambda$, we have that

$$\sum_{\sigma\in S_n} v_\sigma \mathrm{sgn}(q)\,|p\sigma q\rangle = \sum_{\sigma\in S_n} v_\sigma\,|\sigma\rangle\,,$$

which means that

$$v_\sigma \mathrm{sgn}(q) = v_{p\sigma q}.$$

In particular, $v_{pq} = \mathrm{sgn}(q)v_1$. We want to show that $v_\sigma = 0$ if $\sigma \neq pq$ for any $p\in P_\lambda$, $q\in Q_\lambda$. Note that, if p and q were transpositions such that $p\sigma q = \sigma$, we would have $-v_\sigma = v_\sigma$, which means that $v_\sigma = 0$. Thus, we have to show that, if $\sigma\neq pq$ for any $p\in P_\lambda$, $q\in Q_\lambda$, there is a transposition t such that $p = t$ and $q = \sigma^{-1}t\sigma$. Note that, if $T' = \sigma T$ is the tableaux obtained by applying σ to each entry of T, the column stabilizer of T' is $q' = \sigma q\sigma^{-1}$. Therefore, if there exists a transposition such that $p = t$ and $q = \sigma^{-1}t\sigma$, it means that there are two distinct integers in the same row of T which go to the same column of $T' = \sigma T$, and t is their transposition. Suppose then that there are no two such numbers. This means that elements in a row of T goes to different columns in T'. Take q_1' such that the first row of $q_1'T'$ and T have the same elements. Then, take p_1 such that p_1T and $q_1'T'$ have the exact same first row. Repeating this process on the rest of the tableaux, we have that $pT = q'T$ for some $p\in P_\lambda$ and $q'\in Q_\lambda'$. Thus, $pT = \sigma q''\sigma^{-1}\sigma T = \sigma qT$ which means that $p = \sigma q''$, so $\sigma = pq$, where $q = (q'')^{-1}$. □

Corollary 2.2.1. *For any $|v\rangle \in \mathbb{C}S_n$, $c_\lambda\,|v\rangle\,c_\lambda$ is a scalar multiple of c_λ. In particular, $c_\lambda c_\lambda = n_\lambda c_\lambda$.*

We introduce an lexicographic order into the set of partitions: for two partitions λ and μ, $\lambda > \mu$ if for the first i such that $\lambda_i - \mu_i$ is non-vanishing, $\lambda_i - \mu_i > 0$.

Lemma 2.2.2. *If $\lambda > \mu$, for any $|v\rangle \in \mathbb{C}S_n$, $a_\lambda |v\rangle b_\mu = 0$. In particular, $c_\lambda c_\mu = 0$ for $\lambda > \mu$.*

Proof. It is easy to verify that if $\lambda > \mu$, there are two integers in the same row in the tableaux of λ which lie in the same column in the tableaux of μ. Let t be the transposition of these two integers. We have that $(a_\lambda |t\rangle)b_\mu = a_\lambda b_\mu$, because $t \in P_\lambda$, while $a_\lambda(|t\rangle b_\mu) = -a_\lambda b_\mu$, because $t \in Q_\mu$. Thus, $a_\lambda b_\mu = -a_\lambda b_\mu$, which means that $a_\lambda b_\mu = 0$. It follows that $c_\lambda c_\mu = a_\lambda(b_\lambda a_\mu)b_\mu = 0$ if $\lambda > \mu$. □

We are now in position to prove the

Theorem 2.2.1. *Given $n \in \mathbb{N}$, let λ be a partition of n. Let $V_\lambda = \mathbb{C}S_n c_\lambda$ be the space spanned by the Young symmetrizer c_λ. Then, V_λ is an irreducible representation of S_n. Moreover, if λ and μ are distinct partitions of n, V_λ is not isomorphic to V_μ.*

Proof. First, we have that

$$c_\lambda V_\lambda = c_\lambda \mathbb{C}S_n c_\lambda = \mathbb{C}c_\lambda,$$

from corollary (2.2.1). Also, $c_\lambda V_\lambda \neq 0$. In fact, suppose $c_\lambda V_\lambda = 0$. Then $V_\lambda V_\lambda = \mathbb{C}S_n c_\lambda V_\lambda = 0$. But $c_\lambda c_\lambda \in V_\lambda V_\lambda$, and $c_\lambda c_\lambda = n_\lambda c_\lambda$, with $n_\lambda \neq 0$, and we have a contradiction.

Take $W_\lambda \subseteq V_\lambda$ a subrepresentation. $W_\lambda = Ac_\lambda$, where $A \subseteq \mathbb{C}S_n$. By the same argument as before, $c_\lambda W \neq 0$ and $c_\lambda W = \mathbb{C}c_\lambda$. We have then $c_\lambda V_\lambda = c_\lambda W$ and thus $V_\lambda = W_\lambda$. Therefore, V_λ is irreducible.

Next, take $\lambda > \mu$. We have that $c_\lambda V_\mu = c_\lambda \mathbb{C}S_n c_\mu = 0$ from lemma (2.2.2). But $c_\lambda V_\lambda \neq 0$, so V_λ and V_μ are not isomorphic. □

Corollary 2.2.2. *The V_λ account for all irreducible representations of S_n.*

Indeed, since different partitions give rise to non-isomorphic representations, the number of representations V_λ is equal to the number o partitions λ. Since the number of partitions is equal to the number of conjugacy classes of S_n, and the number of conjugacy classes is equal to the number of irreducible representations, we have thus obtained all irreducible representations of S_n.

Having all irreducible representations of S_n, one may ask how to compute their associated irreducible characters. To do so, choose a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ of n and let C_i be the conjugacy class of S_n composed of permutations of type $i = (i_1, i_2, ..., i_k)$. The permutations of type i have i_1 1-cycles, i_2 2-cycles and so on. Let $x = (x_1, ..., x_k)$ be a vector of k independent variables, and let

$$P_j(x) = x_1^j + x_2^j + ... + x_k^j.$$

Also, let

$$\Delta(x) = \prod_{i<j}(x_i - x_j)$$

be the Vandermond determinant. For any $f(x) = f(x_1, ..., x_k)$ defined as a formal series in the k variables $x_1, ..., x_k$, and for any tuple $(l_1, ..., l_k)$ of k integers, define

$$[f(x)]_{(l_1,...,l_k)} = \text{coefficient of the term } x_1^{l_1}...x_k^{l_k} \text{ in the series expansion.}$$

Finally, let $l_1 = \lambda_1 + k - 1$, $l_2 = \lambda_2 + k - 1$, ..., $l_k = \lambda_k$. We have that the irreducible character $\chi_\lambda(C_i)$ of the representation V_λ, evaluated at any element of the conjugacy class C_i in S_n is given by the *Forbenius formula*

$$\chi_\lambda(C_i) = \left[\Delta(x)\prod_j P_j(x)^{i_j}\right]_{(l_1,...,l_k)}. \tag{2.2.27}$$

For example, consider $n = 5$, $\lambda = (3, 2)$ and $C_i = [(12)(345)]$. Since $l_1 = 3 + 2 - 1 = 4$ and $l_2 = 2 + 2 - 2 = 2$, we have that

$$\chi_{(3,2)}(C_i) = \left[(x_1 - x_2)(x_1^2 + x_2^2)(x_1^3 + x_2^3)\right]_{(4,2)}$$

$$= \left[x_1^6 + x_1^4x_2^2 + x_1x_2^5 - x_1^5x_2 - x_1^2x_2^4 - x_2^6\right]_{(4,2)} = 1.$$

We will not prove this formula, but we will use it to find the dimension of any representation V_λ. Fix a partition λ of n. Since the type of the permutations in the class of the identity is $i = (n)$ $(i_1 = n,\ i_a = 0\ \forall a \neq n)$, $\dim V_\lambda = \chi_\lambda(C_{(n)})$ and we have that

$$\dim V_\lambda = \left[\Delta(x)\prod_j P_j(x)^{i_j}\right]_{(l_1,...,l_k)} = \left[\Delta(x)(x_1 + ... + x_k)^n\right]_{(l_1,...,l_k)}.$$

Now,

$$\Delta(x) = \det_{ij} x_i^{j-1} = \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma) x_k^{\sigma(1)-1}...x_1^{\sigma(k)-1}$$

and

$$(x_1 + ... + x_k)^n = \sum_{(r_1,...,r_k)} \frac{n!}{r_1!...r_k!} x_1^{r_1}...x_k^{r_k},$$

where the sum is over tuples of integers $(r_1, ..., r_k)$ that sum up to n. Thus,

$$\Delta(x)(x_1 + ... + x_k)^n = \sum_{(r_1,...,r_k)} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \frac{n!}{r_1!...r_k!} x_1^{r_1+\sigma(k)-1} ... x_k^{r_k+\sigma(1)-1}.$$

The coefficient of the term $x_1^{l_1}...x_k^{l_k}$ is obtained by equating $r_1+\sigma(k)-1 = l_1$, ..., $r_k + \sigma(1) - 1 = l_k$, i.e., it is given by

$$\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \frac{n!}{(l_1 - \sigma(k) + 1)!...(l_k - \sigma(1) + 1)!},$$

where the sum is over permutations $\sigma \in S_k$ such that $l_{k-i+1} - \sigma(i) + 1 \geq 0$, $\forall\ 0 \leq i \leq k$. Since

$$\frac{1}{(l_1 - \sigma(k) + 1)!} = \frac{1}{l_1!} l_1(l_1 - 1)(l_1 - 2)...(l_1 - \sigma(k) + 2)$$

and so on, we can write the coefficient as

$$\frac{n!}{l_1!l_2!...l_k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \prod_{j=1}^{k} l_j(l_j - 1)...(l_j - \sigma(k - j + 1) + 2).$$

By doing some explicit examples, we note that the sum is clearly a Vandermon determinant. We have then that

$$\mathrm{dim} V_\lambda = \frac{n!}{l_1!l_2!...l_k!} \prod_{i<j} (l_i - l_j), \tag{2.2.28}$$

where $l_i = \lambda_i + k - i$.

2.3 Representations of GL$(N, \mathbb{C})$

Here we state some facts about how to construct the irreducible representations of GL$(N, \mathbb{C})$. Their proof can be found in [3].

A representation of GL$(N, \mathbb{C})$ is a group homomorphism $\rho : \mathrm{GL}(N, \mathbb{C}) \to \mathrm{GL}(M, \mathbb{C})$. Consider the *standard* representation of GL$(N, \mathbb{C})$ over $\mathbb{C}^N$ given by matrix multiplication,

$$\rho : \mathrm{GL}(N, \mathbb{C}) \to \mathrm{GL}(N, \mathbb{C}),\ \ \rho(g)v = gv,\ \ \forall v \in \mathbb{C}^N, g \in \mathrm{GL}(N, \mathbb{C}).$$

This representation is actually irreducible, because it acts just as a rotation in $\mathbb{C}^N$. From this trivial irreducible representation, we construct many more. All we have to do is to take tensor powers of $V = \mathbb{C}^N$. Define the action of

$\mathrm{GL}(N,\mathbb{C})$ over the nth tensor power of V as $g(v_1 \otimes v_2 \otimes ... \otimes v_n) = gv_1 \otimes gv_2 \otimes ... \otimes gv_n$, for any $g \in \mathrm{GL}(N,\mathbb{C})$ and any basis vector $v_1 \otimes ... \otimes v_n \in V^{\otimes n}$. Then, it is easy to see that $V^{\otimes n}$ is invariant under $\mathrm{GL}(N,\mathbb{C})$. However, now we have a natural action of S_n over $V^{\otimes n}$ by permutation:

$$(v_1 \otimes ... \otimes v_n)\sigma = v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}, \ \forall \sigma \in S_n.$$

Therefore, hidden inside $V^{\otimes n}$ there are many irreducible representations of $\mathrm{GL}(N,\mathbb{C})$, given by the action of S_n over it.

Consider the nth tensor power of the standard representation $V = \mathbb{C}^N$ of $\mathrm{GL}(N,\mathbb{C})$. Let $\lambda = (\lambda_1, ..., \lambda_n)$ be a partition of n. Then, the image of the action of the Young symmetrizer c_λ over V, $\mathbb{S}_\lambda V = V^{\otimes n} c_\lambda$, gives an irreducible representation of $\mathrm{GL}(N,\mathbb{C})$. We have that, if $\lambda_{N+1} \neq 0$, $\mathbb{S}_\lambda V = 0$. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$,

$$\dim \mathbb{S}_\lambda V = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

The tensor power $V^{\otimes n}$ decomposes as a direct sum

$$V^{\otimes n} \cong \bigoplus_\lambda \mathbb{S}_\lambda V^{\otimes m_\lambda},$$

where m_λ is the dimension of the irreducible representation V_λ of S_n. Moreover, the characters of $\mathbb{S}_\lambda V$ are the *Schur* polynomials

$$\chi_{\mathbb{S}_\lambda V}(g) = s_\lambda(g).$$

References

[1] Asim Barut and Ryszard Raczka. *Theory of Group Representations and Applications.* World Scientific Publishing Company, 1986.

[2] Theodor Bröcker and Tammo Tom Dieck. *Representations of Compact Lie Groups*, volume 98. Springer Science & Business Media, 2013.

[3] William Fulton and Joe Harris. *Representation Theory: a First Course*, volume 129. Springer Science & Business Media, 2013.