Cutting-Edge Project

Final report - Results & Conclusions

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Group 2

Report composed on the subject of Milliman - Subject 2

In-Depth Analysis of Parameters Impact on Equity-Index Products

Version 1.0

Master in Quantitative Finance (M2QF)

Table of contents

We have used Python as the main numerical tool for all the required computations and to perform model simulations. In order to find our practical approach, please refer to the attached Jupyter Notebook file.

Acknowledgments

We would like to take this opportunity to express our sincere gratitude and appreciation to Sophian MEHALLA, who has played a significant role in the successful completion of our Cutting-Edge project. His expertise, guidance, and support were invaluable in helping us achieve our goals and objectives. We are grateful for his dedication, hard work, and commitment throughout the project.

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1 Introduction

The purpose of this project is to analyze the impact of various parameters on the distribution of the equity index, which is modeled using simulations. Insurers use economic scenarios of financial indicators to calculate risk measures, and the joint law of interest rates and equity indices has a significant impact on the quantile finally calculated. Therefore, this study aims to provide a better understanding of the joint modeling of equity index and interest rates by performing sensitivities and analyzing the impact of parameters on the equity index distribution.

The project focuses on the simulation of the equity index and the analysis of its distribution, including quantiles and expectations. Additionally, the distribution of the couple (interest-rates, S_T) will be analyzed by computing the prices of derivatives whose underlying is the equity index. Simple products like call/put options will be used to evaluate the impact of interest rates on the equity index distribution.

The report will detail the simulation schemes used for interest rates and stock models and discuss the sensitivity analysis conducted on these models to determine the impact of the parameters on the quantile levels. Moreover, it will provide insights into the impact of discretization and other model parameters on the joint law of the interest rate/equity index pair.

Overall, this project will contribute to a better understanding of the joint modeling of interest rates and equity indices, which is essential for insurers to assess and manage the risk in their portfolios accurately.

2 Plan Statement

To achieve our goals, the report will begin with a literature review to establish the background and context of the study and identify the key parameters and variables that affect the distribution of equity indices.

The modeling section will then explain the simulation schemes used for interest rates and stock models and describe the process for analyzing the impact of parameters on the quantile levels. The analysis and results section will present the findings of the sensitivity analysis conducted on the models, including the distribution of the couple (interest-rates, S_T) and the prices of derivatives whose underlying is the equity index.

Finally, the conclusion will summarize the main findings of the study and their significance in improving the understanding of the joint modeling of interest rates and equity indices, as well as highlight any limitations of the study and provide recommendations for future research.

3 Modeling & Methodology

In all what follows, $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$ is a filtered space fulfilling the usual conditions.

Remark 1. The letter B will denote a Brownian motion under $\mathbb{Q}$ for the interest-rates dynamics while W will be reserved to equity-index dynamics.

3.1 Interest Rates Modeling

We have decided to perform the modelling protocol for several *affine rates models*.

3.1.1 Vasicek model

Introduction We choose to implement our first short-rate model with the Vasicek dynamics, defined as a solution of the following EDS :

$$\begin{cases} \mathrm{d}r_t = \kappa(\theta - r_t)\mathrm{d}t + \eta \mathrm{d}B_t \\ r(0) = r_0 \quad \text{deterministic} \end{cases} \tag{1}$$

with : r_0, κ, θ and η positive constants and $(B_t)_t$ a Wiener process under the risk-neutral probability measure $\mathbb{Q}$.

Proposition 1 (Analytical formula of Vasicek model)**.** The solution of (1) can be written as :

$$r(t) := r(s)e^{-\kappa(t-s)} + \theta(1 - e^{-\kappa(t-s)}) + \eta \int_s^t e^{-\kappa(t-u)}\mathrm{d}W_u \tag{2}$$

Proof. We have to apply the Itô formula to the process $(r_t)_t$ with the $\mathcal{C}^2$ function $\phi : t \longmapsto r(t)e^{\kappa t}$ ■

Remark 2. The Vasicek model is an Ornstein–Uhlenbeck stochastic process.

Proposition 2. $(r_t)_t$ is a Gaussian process

Simulation Our goal is to simulate path trajectories issued thanks to the Vasicek dynamics. We have three different strategies to do so :

(1) Compute the path using the analytical formula explained in (2),

(2) Approximate the SDE of (1) using the Euler-Maruyama method

(3) Approximate the SDE of (1) using the Milstein method

We propose to implement these three methods.

Definition 1 (Euler-Maruyama scheme)**.** In Itô calculus, the Euler–Maruyama method (also called the Euler method) is a method for the approximate numerical solution of a stochastic differential equation (SDE).

$$\mathrm{d}X_t := a(t, X_t)\mathrm{d}t + b(t, X_t)\mathrm{d}W_t \tag{3}$$

with $X_0 = x_0 \in \mathbb{R}$ and time horizon in $[0, T]$ with $T > 0$. The general algorithm to perform a Euler method over a general SDE is detailed below :

- ▷ We produce a partition of $N \in \mathbf{N}^*$ sub-divisions of $[0, T]$ with equal width $\Delta t > 0$ defined as :

$$0 =: t_0 < t_1 < \ldots < t_N := T \text{ and } \Delta t := \frac{T}{N} \text{ and } t_k = k\Delta t$$

- ▷ We set $Y_0 = X_0 = x_0$
- ▷ We compute the approximation sequence $(Y_n)_{n\in\mathbb{N}}$ defined recursively as follows :

$$\forall n \in [\![0, N-1]\!],\ Y_{n+1} := Y_n + a(t_n, Y_n)\Delta t + b(t_n, Y_n)\Delta W_n \tag{4}$$

where $\Delta W_n := W_{t_{n+1}} - W_{t_n}$ independent and identically distributed normal random variables with zero mean and variance equals to Δt.

Proposition 3 (Strong and Weak errors of Euleur method)**.** The Euler method has :

- ▷ a strong error of order $\mathcal{O}(\sqrt{\Delta t})$
- ▷ a weak error of order $\mathcal{O}(\Delta t)$

Definition 2 (Milstein scheme)**.** The Milstein method is a technique for the approximate numerical solution of a stochastic differential equation :

$$\mathrm{d}X_t := a(t, X_t)\mathrm{d}t + b(t, X_t)\mathrm{d}W_t \tag{5}$$

with $X_0 = x_0 \in \mathbb{R}$ and time horizon in $[0, T]$ with $T > 0$. The general algorithm to perform a Milstein method over a general SDE is detailed below :

- ▷ We produce a partition of $N \in \mathbf{N}^*$ sub-divisions of $[0, T]$ with equal width $\Delta t > 0$ defined as :

$$0 =: t_0 < t_1 < \ldots < t_N := T \text{ and } \Delta t := \frac{T}{N} \text{ and } t_k = k\Delta t$$

- ▷ We set $Y_0 = X_0 = x_0$
- ▷ We compute the approximation sequence $(Y_n)_{n\in\mathbb{N}}$ defined recursively as follows :

$$\begin{aligned}\forall n \in [\![0, N-1]\!],\ Y_{n+1} := Y_n + a(t_n, Y_n)\Delta t + b(t_n, Y_n)\Delta W_n \\ + \frac{1}{2} b(t_n, Y_n) b'(t_n, Y_n)\Big((\Delta W_n)^2 - \Delta t\Big)\end{aligned} \tag{6}$$

where $\Delta W_n := W_{t_{n+1}} - W_{t_n}$ independent and identically distributed normal random variables with zero mean and variance equals to Δt and we set $b'(.) := \frac{\partial b}{\partial x}(.)$.

Proposition 4 (Strong and Weak errors of Milstein method)**.** The Milstein method has :

- ▷ a strong error of order $\mathcal{O}(\Delta t)$
- ▷ a weak error of order $\mathcal{O}(\Delta t)$

Remark 3. Thanks to the last proposition, one can state the Milstein scheme is **more accurate** than Euler's one.

Application of Euler & Milstein schemes In this context, we have :

- ▷ $a(t, X_t) = \kappa(\theta - X_t)$
- ▷ $b(t, X_t) = \eta$

As the functionnal b is deterministic and does not depend on the value of the process $(X_t)_t$, we have : $b'(.) = 0$ and then, for the Vasicek model, the Euler scheme is **equivalent** to the Milstein method.

3.1.2 Cox-Ingersoll-Ross (CIR) model

Introduction We choose to implement our second short-rate model with the Cox-Ingersoll-Ross dynamics, defined as a solution of the following EDS :

$$\begin{cases} \mathrm{d}r_t = \kappa(\theta - r_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}B_t \\ r(0) = r_0 \quad \text{deterministic} \end{cases} \tag{7}$$

with : r_0, κ, θ and σ positive constants and $(B_t)_t$ a Wiener process under the risk-neutral probability measure $\mathbb{Q}$.

Proposition 5 (Feller condition)**.** Under the following condition known as the Feller condition,

$$\boxed{\sigma^2 < 2\kappa\theta} \tag{8}$$

the process $(r_t)_t$ has strictly positive values [2]

Proposition 6. $(r_t)_t$ follows an off centered χ_2 law.

Application of Euler & Milstein schemes In this context, we have :

- ▷ $a(t, X_t) = \kappa(\theta - X_t)$
- ▷ $b(t, X_t) = \sigma\sqrt{X_t} \Longrightarrow b'(t, X_t) = \frac{\partial b}{\partial X_t}(t, X_t) = \frac{\sigma}{2\sqrt{X_t}} \neq 0$

3.1.3 Hull & White (Extended Vasicek) *(Refinement 1)*

Introduction We choose to implement our third short-rate model with the Hull& White dynamics, defined as a solution of the following EDS :

2. This is an alternative to correct the potential negativeness effect on the rates produced by the Vasicek model.

$$
\begin{cases} \mathrm{d}r_t = (\theta(t) - ar_t)\mathrm{d}t + \sigma \mathrm{d}B_t \\ r(0) = r_0 \quad \text{deterministic} \end{cases} \tag{9}
$$

with : r_0, a and σ positive constants, $(B_t)_t$ a Wiener process under the risk-neutral probability measure $\mathbb{Q}$ and $\theta : [0,T] \longrightarrow \mathbb{R}$ a deterministic functional.

Proposition 7 (Analytical formula of Hull & White model)**.** The solution of (9) can be written as :

$$
r(t) := r(s)e^{-a(t-s)} + \int_s^t \theta(u)e^{-a(t-u)}\mathrm{d}u + \sigma \int_s^t e^{-a(t-u)}\mathrm{d}W_u \tag{10}
$$

Proof. We have to apply the Itô formula to the process $(r_t)_t$ with the $\mathcal{C}^2$ function $\phi : t \longmapsto r(t)e^{at}$ ■

Remark 4. The Hull & White model is also denoted as an "extended" Vasicek model, which comes from the similar form of the SDE driving the model dynamics, with an extra property of time-dependency of the mean-reversion factor.

The function θ is parameterized in order to fit the input term structure of interest rates.

3.1.4 Effect of $\theta(t)$

When varying $\theta(t)$, the resulting Hull-White as well as Black Scholes using the interest rates modeled by HW have different properties. Specifically, each $\theta_i(t)$ corresponds to a different function that describes how quickly the interest rates revert back to the long-term mean.

In the provided functions of $\theta(t)$, we have :

- ▷ $\theta_1(t) = 0.1$: This corresponds to a constant mean-reversion speed of 0.1.
- ▷ $\theta_2(t) = 0.2e^{-0.05t}$: This corresponds to a decreasing mean-reversion speed that exponentially approaches 0.2.
- ▷ $\theta_3(t) = 0.1 + 0.02log(t+1)$: This corresponds to an increasing mean-reversion speed that is logarithmically proportional to time.
- ▷ $\theta_4(t) = 0.3\sin(2\frac{\pi t}{365})$: This corresponds to a periodic mean-reversion speed that oscillates with a period of one year.

By simulating Hull-White models with different $\theta(t)$, we can see how changes in the mean-reversion speed affect the behavior of the interest rates. For example, a larger mean-reversion speed tends to make the interest rates revert to the long-term mean more quickly, while a smaller mean-reversion speed makes the interest rates move more slowly.

3.1.5 Ouverture & Further extensions

Other interest rates model can be implemented : Black-Karakinski and Ho-Lee.

3.2 Equity-index modelling

3.2.1 Black & Scholes

We first consider a basic Black & Scholes model with a stochastic interest-rates and constant volatility $\sigma > 0$:

$$\frac{\mathrm{d}S_t}{S_t} := r_t \mathrm{d}t + \sigma \mathrm{d}W_t \tag{11}$$

Remark 5 (Correlation between the two Wiener processes)**.** The classical Brownian motions $(B_t)_t$ *(from the interest rates dynamics)* and $(W_t)_t$ *(from the equity-index dynamics)* are correlated with a parameter $\rho \in [-1, 1]$.

$$\forall t \in \mathbb{R}^+,\ \mathrm{Cor}(B_t, W_t) := \rho t \tag{12}$$

Application of Euler & Milstein schemes In this context, we have :

- ▷ $a(t, X_t) = r_t X_t$
- ▷ $b(t, X_t) = \sigma X_t \Longrightarrow b'(t, X_t) = \frac{\partial b}{\partial X_t}(t, X_t) = \sigma \neq 0$

3.2.2 Heston *(Refinement 2)*

$$\begin{cases} \dfrac{\mathrm{d}S_t}{S_t} = r_t \mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t \\ S_0 \geq 0 \end{cases} \tag{13}$$

$$\begin{cases} \mathrm{d}V_t = \kappa(\theta - V_t)\mathrm{d}t + \eta\sqrt{V_t}\mathrm{d}\widetilde{W}_t \\ V_0 \geq 0 \end{cases} \tag{14}$$

4 Analysis & Results

We have conducted multiple simulations by using different combinations of equity and interest rate models, to look at how our estimation changes, also with respect to the impact of each parameter of the model.
In this section, we report the results of our analysis for each combination of models.

4.1 BS + Vasicek

The Vasicek interest rate model is a one-factor stochastic model used to describe the behavior of interest rates over time.

We started by using the Black-Scholes model for equity and the Vasicek model for interest rate. Our aim is to analyze and understand the impact of each parameter in these models.
First of all, we wanted to have a look at the resulting simulated paths.

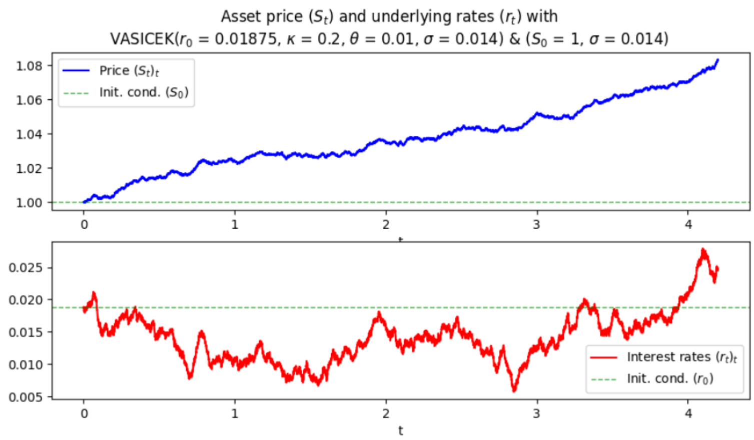

FIGURE 1 – Simulation BS Vascicek

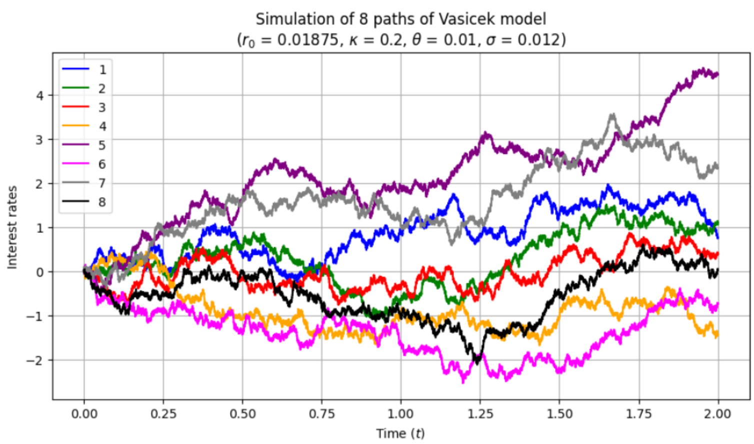

FIGURE 2 – Simulation of 8 paths of Vascicek

Then, we proceeded by trying our different values of the parameters.

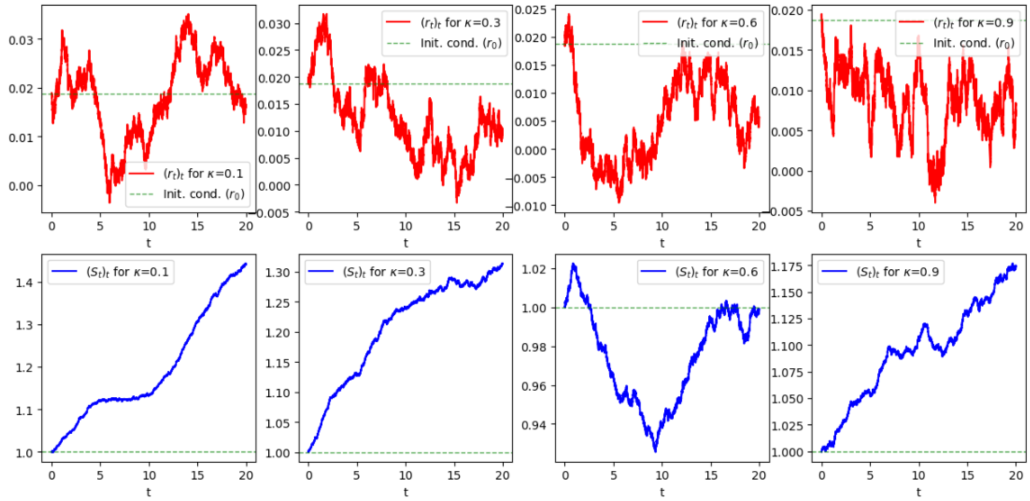

FIGURE 3 – Kappa's effect

A higher kappa value implies a faster mean reversion, meaning that the interest rate will adjust more quickly toward its long-term average.
Increasing kappa would lead to a more volatile interest rate process, with interest rates exhibiting sharper and quicker fluctuations around the mean.

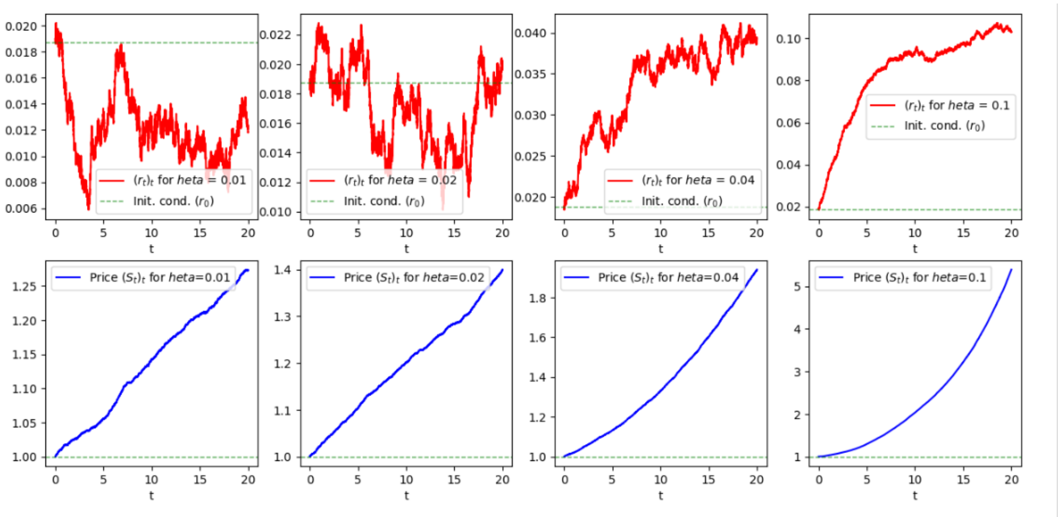

FIGURE 4 – Theta's effect

When theta is increased, it raises the long-term mean of the interest rate process. This means that the interest rate will have a higher average value over time, and the model will reflect an upward shift in the interest rate dynamics.

We also remark that if theta increases, the price plot becomes more and more convex . so, the asset's value is appreciating at an increasing rate.

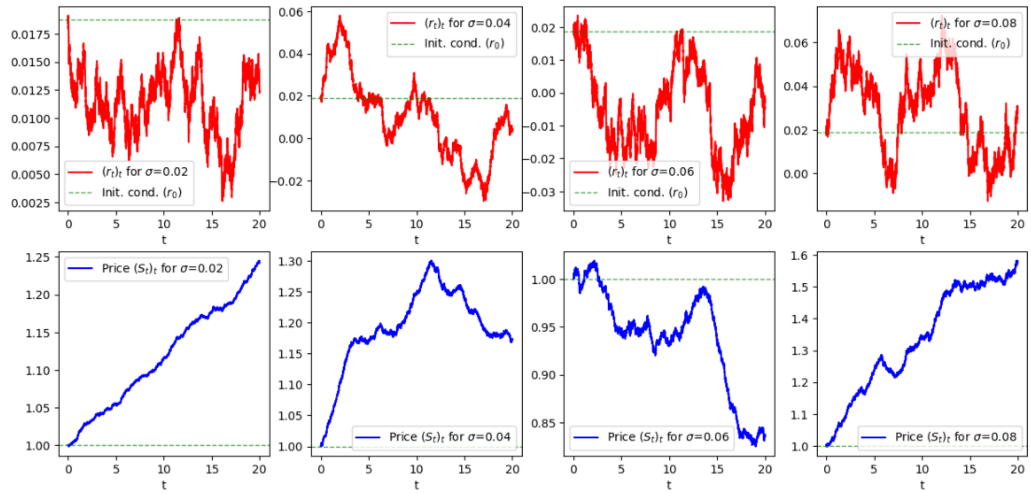

FIGURE 5 – Sigma's effect

Increasing sigma leads to higher volatility in the interest rate process. The interest rate will experience more significant and more frequent fluctuations around its mean.

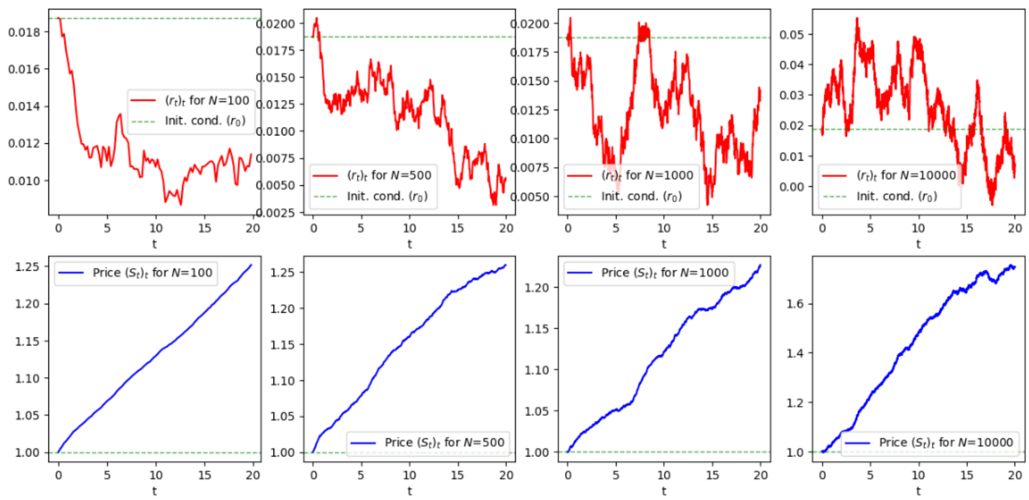

FIGURE 6 – The effect of N

Increasing N results in a finer division of the time horizon, allowing for more accurate modeling of interest rate and equity dynamics.

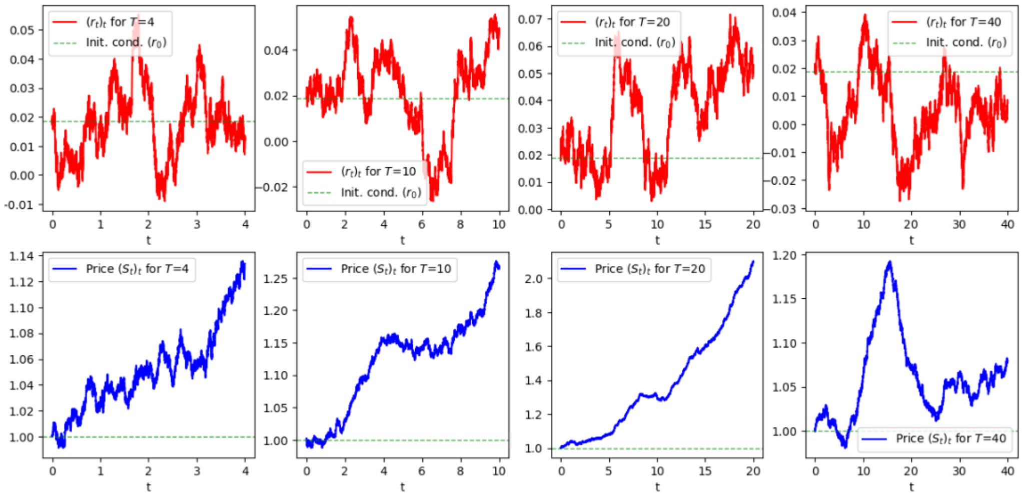

FIGURE 7 – The effect of T

A longer time horizon allows for a greater number of interest rate movements and more time for the mean reversion process to play out.

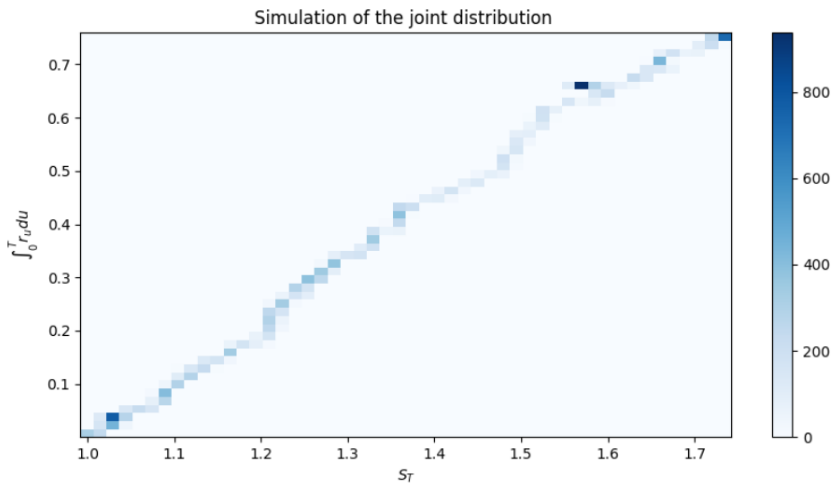

FIGURE 8 – Joint distribution

The joint distribution of S_T and $\int_0^T r_u, du$ has relatively low density throughout most of the plot, with regions of higher density in the upper right and lower left. Additionally, there is a positive correlation between the two variables, indicating that they tend to move together in the same direction

4.2 BS + CIR

In this section, we will have a look at the Black & Scholes model, with interest rates that follow the Cox-Ingersoll-Ross model.

4.2.1 Simulated paths

Before starting the analysis of this combination of models, we wanted to have a look at the simulated path of the CIR model, by setting some plausible values for the model parameters.
We set :

- ▷ r_0, the initial value r_0 of the process $(r_t)_t$ at time $t = 0$, equal to 0.4 ;
- ▷ κ, the mean-reversion speed parameter, equal to 0.2 ;
- ▷ θ, the mean-reversion center parameter, equal to 0.01 ;
- ▷ σ, the (constant) volatility parameter, equal to 0.12.

It should be noted that the following graph is just an example of the results we can get since we have to factor in the random component given by the Brownian motion

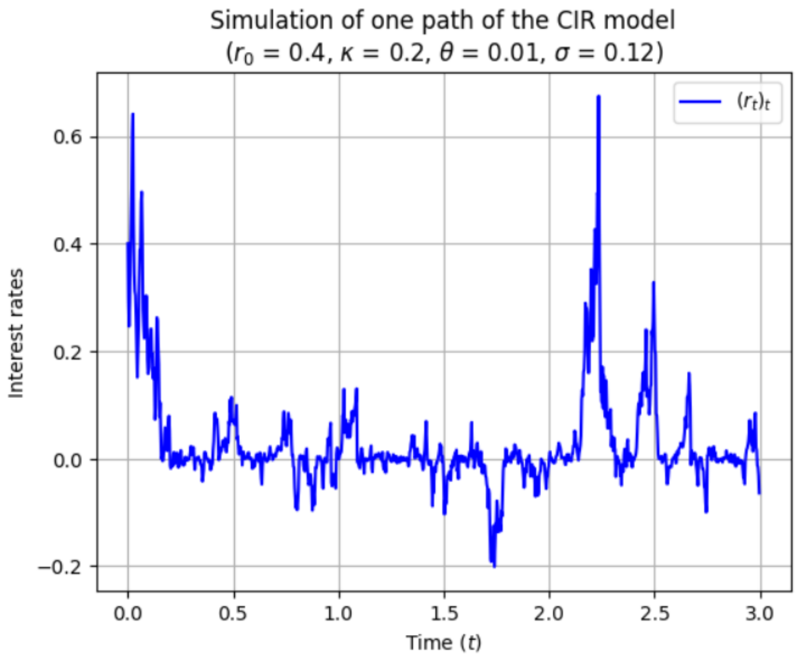

Figure 9 – Simulation of one path of the CIR model

However, we can notice that the evolution of the interest rate is what we can expect for a CIR model. Overall, the path of a simulated CIR model should exhibit mean-reversion behavior, with interest rates fluctuating around the long-term mean over time and the distribution of interest rates becoming more tightly clustered around the mean as the interest rate approaches the mean.
The volatility of the interest rate σ in the model is 0.20, which means that the interest rate is expected to exhibit some fluctuations or jumps in value over time.

However, the speed of mean reversion is relatively slow, so the interest rate is expected to exhibit a high degree of persistence in its movements.
We can find similar results when we look at the simulation of multiple CIR paths with the same parameters :

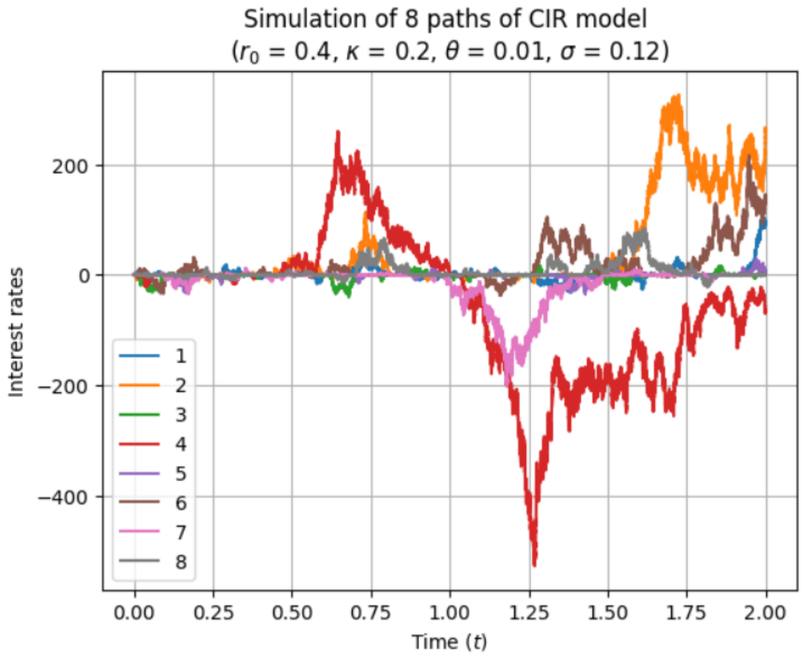

FIGURE 10 – Simulation of 8 paths of the CIR model

We also plotted the evolution of the underlying price and the interest rates over time. From the following plot we can observe that the asset price tends to increase over time, as we expect from the parameters we set, while the interest rate tends to decrease, which might be due to the value we set for r_0.

In the CIR model, the short-term interest rate is assumed to follow a mean-reverting process, where it fluctuates around a long-term mean or equilibrium level. If the initial short-term interest rate is higher than the long-term mean or equilibrium level, then the model predicts that the interest rate will gradually decrease over time as it reverts towards the mean. This could be one reason why we are observing a consistent decrease in the interest rate over time in our plot.

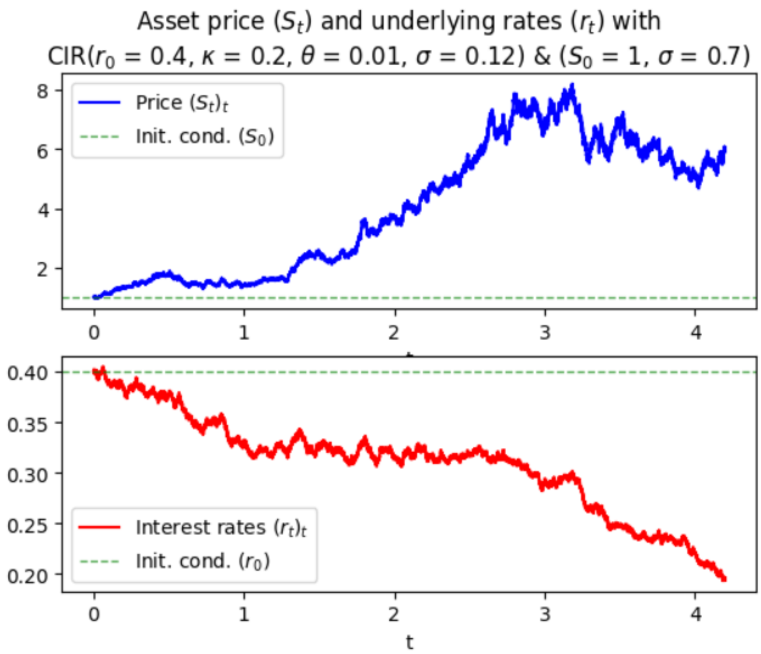

FIGURE 11 – Simulation BS CIR

In the next part, we will have a look at the impact of each parameter on the asset price and the interest rate.

4.2.2 Analysis of the impact of κ

We start our analysis with the mean reversion speed parameter. κ is a parameter that governs the speed of mean reversion of the short-term interest rate towards its long-term mean or equilibrium level. Specifically, κ represents the rate at which deviations of the interest rate from its mean decay over time.

We notice that by increasing κ, we have lower volatility in the interest rates as it causes the rate to revert more quickly to its mean, with fewer and smaller fluctuations around the mean. A higher κ makes the short-term interest rate converge more quickly to its long-term mean or equilibrium level.

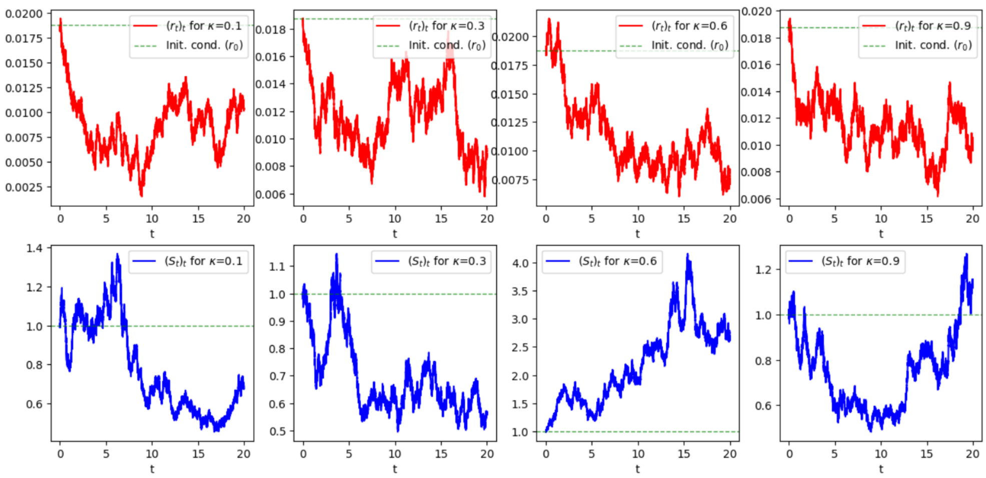

FIGURE 12 – Kappa's effect

4.2.3 Analysis of the impact of θ

Now, we will analyze the impact of θ, which is a parameter that represents the long-term mean or equilibrium level of the short-term interest rate. The model assumes that the short-term interest rate follows a mean-reverting process, where it fluctuates around its long-term mean or equilibrium level. θ represents this long-term mean or equilibrium level towards which the interest rate reverts over time.

Theta plays a crucial role in the CIR model as it represents the long-term mean or equilibrium level of the interest rate. It determines the level around which the interest rate fluctuates over time, and towards which it reverts when it deviates from this level.

Increasing θ means that the interest rate tends to revert more quickly towards the long-term mean as it fluctuates over time.

If we increase θ while keeping κ fixed, the model should predict that interest rates converge more quickly towards the higher long-term mean as they revert over time. This would result in lower volatility and a flatter term structure of interest rates.

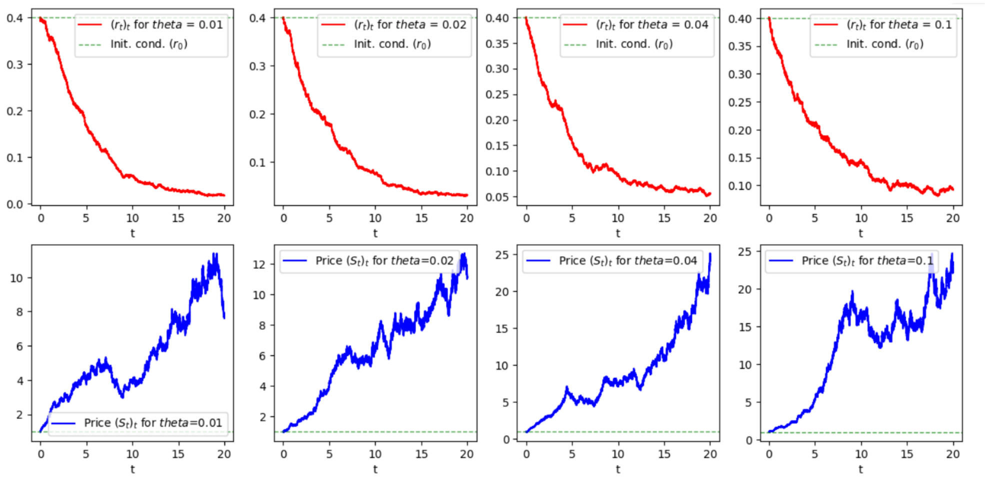

FIGURE 13 – Theta's effect

4.2.4 Analysis of the impact of ρ

In the Cox-Ingersoll-Ross (CIR) model, ρ is a parameter that represents the correlation between the random shock to the interest rate and the random shock to the variance of the interest rate.
The parameter ρ captures the dependence between the two stochastic processes in the CIR model. If ρ is positive, then an increase in the interest rate would tend to be accompanied by an increase in its volatility, and vice versa. This positive correlation between the interest rate and its volatility reflects the market's tendency to demand higher compensation for taking on greater risk.

We can see this in the following plots.

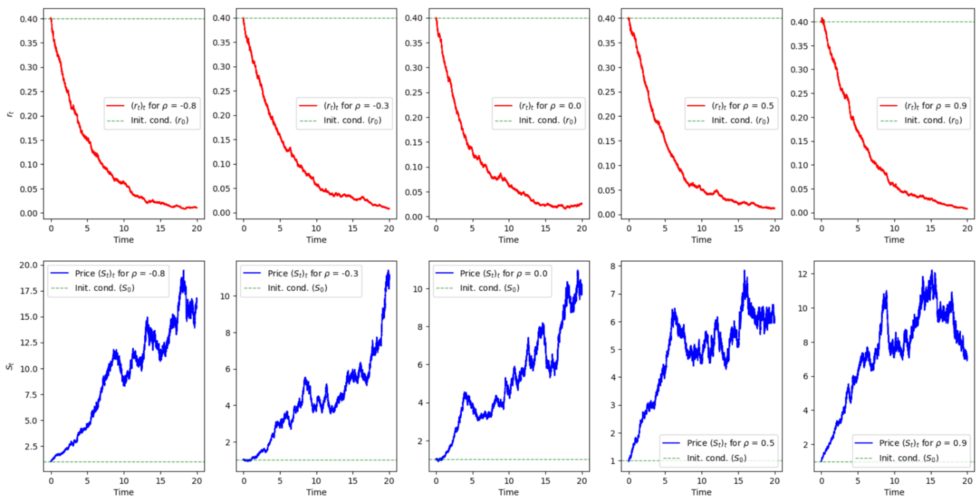

FIGURE 14 – Rho's effect

4.2.5 Simulation of the joint distribution

The joint distribution represents the uncertainty and risk associated with the final stock price and the accumulated interest rate over time.

By simulating the joint distribution, one can gain a better understanding of the potential range of outcomes for the final stock price and interest rate integral and the likelihood of each outcome. This information can be used to price financial derivatives more accurately and make informed investment decisions.

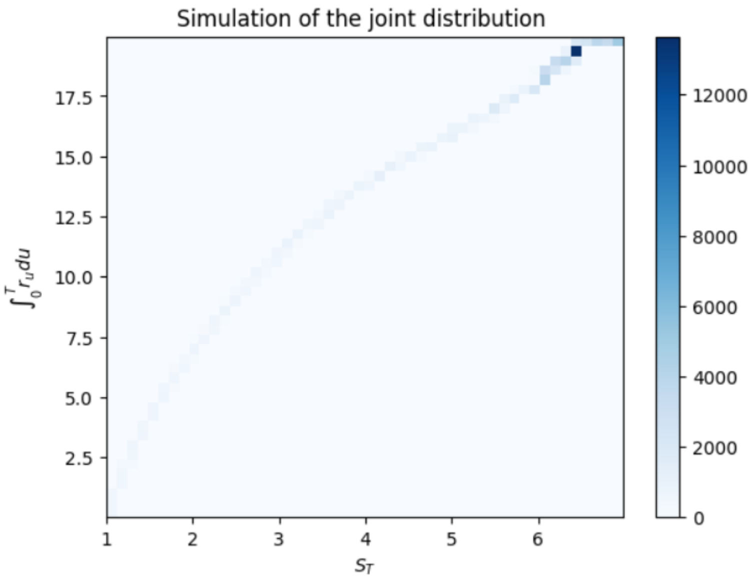

FIGURE 15 – Joint distribution

The diagonal line that we see in the plot represents areas of equal density of the joint distribution of the final stock price and the integral of interest rates. The fact that the line is almost transparent indicates that the density of the joint distribution is relatively low in most areas of the plot.
The darker color in the upper right end of the line indicates that the density of the joint distribution is relatively high in that area. This means that the simulated outcomes for the final stock price and the integral of interest rates are more likely to fall in that area compared to other areas of the plot.

From this plot, we can also say that the two variables have a positive correlation.

4.3 BS + HW

The Hull and White interest rate model is a popular stochastic process used for modeling interest rates. The model includes a deterministic function of time, known as the $\theta(t)$ function, which affects the behavior of interest rates over time.

In this part, we explore the impact of using different functional forms for $\theta(t)$ on the Hull and White model. Specifically, we consider four different functional forms : constant, exponential, logarithmic, and sinusoidal. Each of these functions will be used to model the deterministic component of the Hull and White model, allowing us to compare their impact on the behavior of interest rates over time.

By examining the results of each model, we can gain a better understanding of how the choice of $\theta(t)$ function can influence the accuracy and effectiveness of the Hull and White interest rate model.

4.3.1 HW Model using a linear function of $\theta(t)$

In our study, we began by simulating a single path of the Hull and White interest rate model using the Euler scheme. The deterministic function of time, $\theta(t) = 0.1 + 0.2t$. After running the simulation, we found that the interest rate exhibits a linearly increasing trend as time t increases.

This result is in line with our expectation, as the $\theta(t)$ function is a linear function of time. The linearly increasing behavior of the interest rate is an important characteristic of the Hull and White model, and it is used in many applications such as pricing of interest rate derivatives and risk management.

Our finding provides a fundamental understanding of how the Hull and White model behaves under a specific choice of $\theta(t)$ function, and it can serve as a useful reference for future research and applications. The next figure illustrates the linearly increasing behavior of the interest rate under the chosen $\theta(t)$ function, confirming our simulation results.

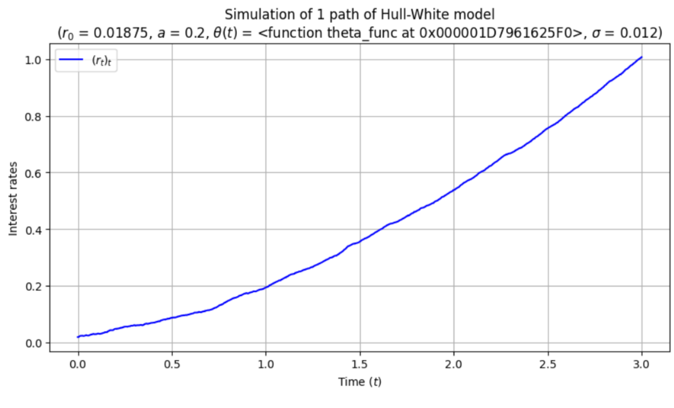

FIGURE 16 – Simulation of 1 path of Hull-White model

4.3.2 Asset price S_t and Interest rates r_t

To begin our analysis of the Hull and White interest rate model, we chose to plot the asset price using a linear function of the mean reversion function. We then examined the evolution of the interest rates and the asset price over time.

The resulting plot revealed that both the underlying rate and the asset price are increasing slowly at the initial values of t and then experience a rapid increase. This increase is particularly noticeable in the asset price, which exhibits a sharp upward trend after the initial slow increase.

The next figure shows the evolution of the underlying rate and the asset price over time, highlighting the observed trends.

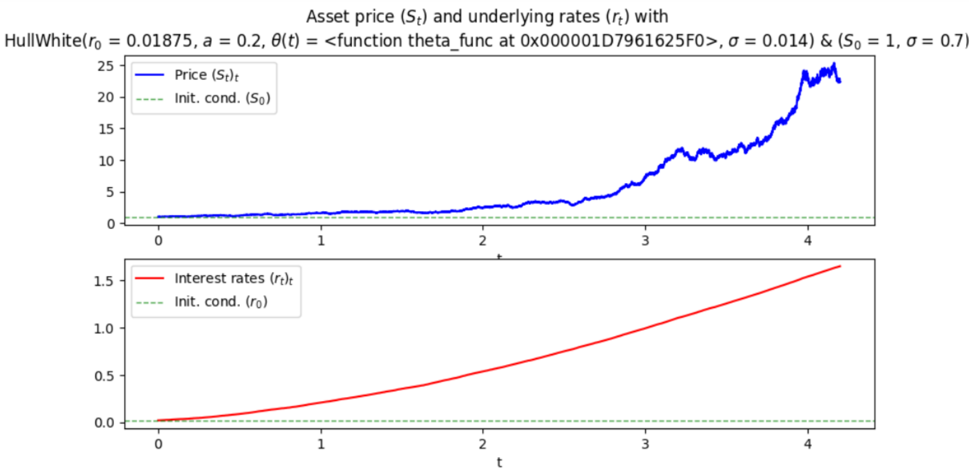

FIGURE 17 – Asset price S_t and interest rates r_t in function of the time t

4.3.3 Analysis of the impact of σ

As part of our analysis of the Hull and White interest rate model, we tested the impact of volatility on the asset price and interest rates. We found that the volatility has a limited impact on the behavior of these variables.

Specifically, as we increased the volatility from small values, we observed that the asset prices and interest rates continue to exhibit a linear increase with some randomness in the asset price graphs as the volatility becomes larger. However, the impact of volatility on the interest rates remained relatively small.

These findings suggest that the Hull and White model is relatively stable under different volatility scenarios, and the choice of volatility should not have a significant impact on the overall behavior of the model.

4.3.4 Analysis of the mean reversion $\theta(t)$

In our study of the Hull and White interest rate model, we also tested the impact of the mean reversion function, $\theta(t)$. We ran simulations using different functional forms of $\theta(t)$ including a constant, exponential, logarithmic, and sinusoidal function.

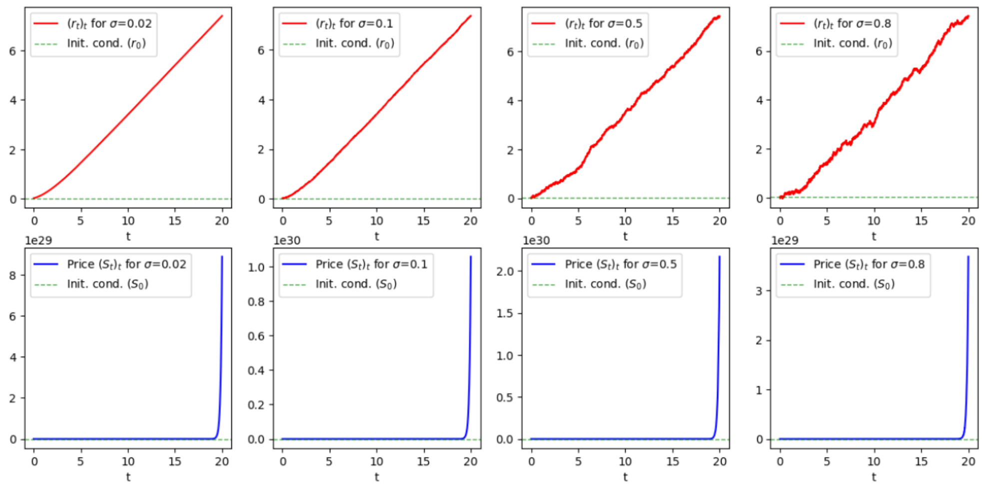

FIGURE 18 – Impact of the volatility for the asset price S_t and interest rates r_t

The resulting plots showed that the choice of the mean reversion can have a significant impact on the behavior of the interest rates and asset prices. Specifically, we observed that the asset prices exhibit a different shape and rate of increase for different functional forms of $\theta(t)$.

Moreover, we observed that the interest rates exhibit different levels of fluctuation and trend depending on the choice of $\theta(t)$. These findings highlight the importance of carefully choosing the functional form of $\theta(t)$ when using the Hull and White model for financial applications.

The different functions of $\theta(t)$:

- ▷ $\theta(t) = 0.1$
- ▷ $\theta(t) = 0.2.e^{-0.05.t}$
- ▷ $\theta(t) = 0.1 + 0.02.log(t + 1)$
- ▷ $\theta(t) = 0.3 \sin(2\pi.\frac{t}{365}) + 0.1$

The above graph presents the results of our simulations using different functional forms of $\theta(t)$, illustrating the impact of this choice on the behavior of the interest rates and asset prices.

Upon examining the graphs of the asset price and interest rates under different functional forms of $\theta(t)$, we observed that the behavior of the asset price is largely random. Specifically, the asset price exhibits a Brownian motion behavior, indicating a random walk process.

In contrast, the interest rates exhibit an increasing random behavior that is affected by the choice of $\theta(t)$. In fact, for the first 2 choices of the mean reversion, r_t is random and then for $\theta(t)$ sinusoidal, the interest rate starts to lose its randomness by increasing slowly in time and then drastically increases.

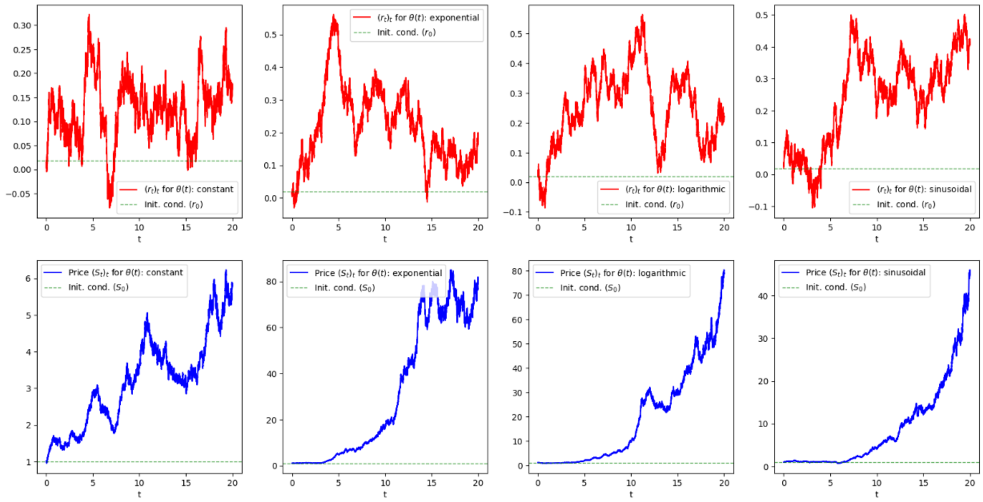

FIGURE 19 – Impact of the mean reversion $\theta(t)$ for S_t and r_t

These findings suggest that the asset price and interest rates exhibit different types of behavior and react differently to changes in the model's parameters. These insights can inform the development of effective pricing and risk management strategies for financial applications that involve the use of the Hull and White interest rate model.

4.4 Vasicek + Heston

In this section we will fix heston's parameters and try to see the effects on the vasicek model. Fixing heston parameters allows us to make sure the respect the feller condition : $2 * \kappa * \theta > \eta^2$ $(0.08 \geq 0.0036)$ and make sure not to have negative variance.

In order to have an idea of the influence of the parameters on the European and Asian option prices , we will just focus on the influence of the parameters on the price and the variance of the underlying and conclude on the effects of the options.

The trajectories being stochastic and non-deterministic, we simulate 10 trajectories for each parameter value and compute the mean and the variance of the trajectories studied, then take the mean of all the calculated means and all the calculated variances.

4.4.1 Study of the correlations

Correlation between the interest rate and the underlying.

4.4.2 Study of the parameters

Kappa's effect with the underlying :

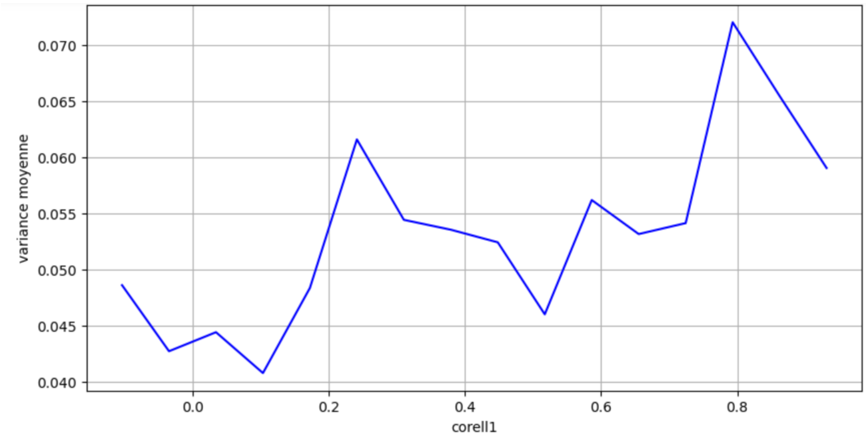

FIGURE 20 – Correlation effect

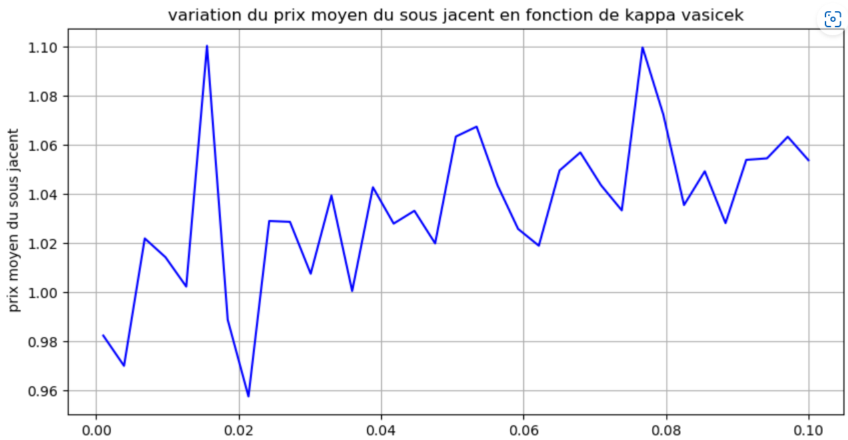

FIGURE 21 – Kappa's effect

Theta's effect with the variance :

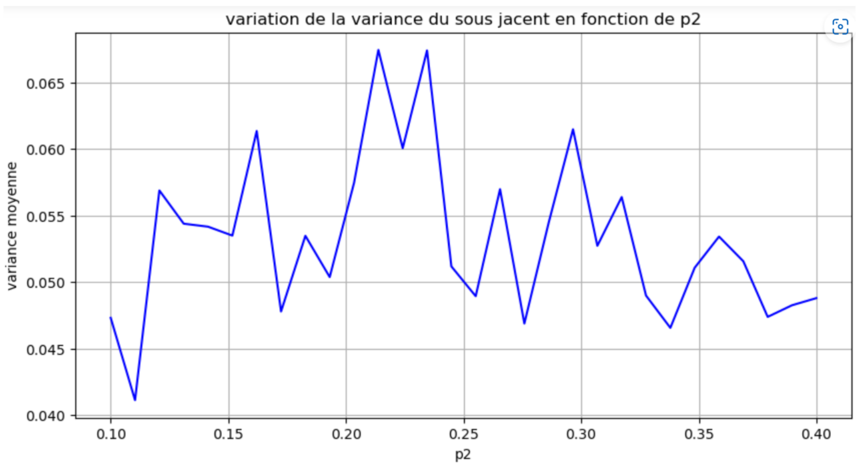

FIGURE 22 – Theta's effect

Conclusion

This academic project allowed us to study and gain knowledge on economic scenario generators implementation. The professional use of software engineering tools and object-oriented programming led us to build a complete simulation Python library used later for an in-depth statistical analysis of parameters impact.