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Analysis of the Mapping of Bilinear Transformation in Digital Filter Design

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***Abstract*—The bilinear transformation is one of the most commonly used methods in IIR digital filter design, which accomplishes the mapping from the s-plane to the z-plane. However, the current understanding of the mapping of the bilinear transformation remains somewhat lacking and the more details need to be revealed. In this paper, we derive some interesting conclusions about the mapping through mathematical analysis: A special line A, parallel to the imaginary axis in the s-plane, is mapped to a line B in the z-plane under the bilinear transformation, all the other lines, parallel to the imaginary axis in the s-plane, are mapped to the circles with different centers and radii in the z-plane. These circles are located on both sides of the line B and are tangent to the line B at the point $z = -1$. The formulas for computing the centers and radii are also derived. Numerical computation further verifies our conclusions.**

***Index Terms*—Bilinear transformation, digital filter design, mapping.**

I. INTRODUCTION

THE bilinear transformation is a widely used method in digital filter design [1], [2]. It transforms the analog filter H(s) to the digital filter H(z) through the mapping from the s-plane to the z-plane. Currently, the mapping characteristics of the bilinear transformation can be summarized as follows [3], [4], [5]:

1） The imaginary axis in the s-plane is mapped to the unit circle in the z-plane.

2) The left half of the s-plane is mapped to the interior of the unit circle in the z-plane.

3) The right half of the s-plane is mapped to the exterior of the unit circle in the z-plane.

However, these conclusions are not comprehensive enough, lacking relevant details. There are still some questions that need to be addressed, such as how on earth the left and right half-planes are mapped to the interior and exterior of the unit circle. Are there any internal characteristics in this mapping? And so on.

In contrast, the research on the mapping of the impulse invariance method [6], which is often compared with the bilinear transformation, is very comprehensive, and we have a thorough understanding of the mapping details. Yet, the research on the mapping of the bilinear transformation remains deficient. Our work aims to fill in the missing piece of the bilinear transformation.

II. ANALYSIS OF THE MAPPING

In this section, we first present the definition of the bilinear transformation, followed by a discussion about the circles in the z-plane as the image of the mapping, and finally, we derive the formulas used to compute the centers and radii of the circles.

A. *The Definition of Bilinear transformation*

In digital filter design, the bilinear transformation is defined as

$$s = c\frac{z-1}{z+1} \tag{1}$$

where $c > 0$ and is often set as $2/T$, where T is the sampling period.

From (1), we can obtain

$$z = \frac{c+s}{c-s} \tag{2}$$

Equation (2) is actually a conformal mapping function in complex analysis theory. According to the properties of the conformal mapping, (2) can map the circles or the straight lines in the s-plane to the circles or the straight lines in the z-plane [7], [8].

The entire s-plane can be seen as composed of countless lines which are parallel to the imaginary axis. Therefore, the mapping from the entire s-plane to the z-plane can be regarded as the mapping from these lines in the s-plane to the z-plane.

Consider any line $\mathrm{Re}(s) = \sigma_0$ which is parallel to the imaginary axis in the s-plane. The points lie on this line can be expressed as $s = \sigma_0 + \mathbf{j}\omega$, where $\omega \in (-\infty, +\infty)$. Substituting s into (2) yields

$$z = \frac{c+\sigma_0+\mathbf{j}\omega}{c-\sigma_0-\mathbf{j}\omega} \tag{3}$$

According to the conformal mapping theory, z obtained from (3) should lie on a circle or a straight line in the z-plane. Since a straight line can be regarded as a circle with an infinite radius, it is reasonable to assume that z should lie on a circle so that subsequently we are able to handle z in a unified manner.

B. Discussion about the Center of the Circle in the z-plane

Suppose that the line L: $\mathbf{Re}(s) = \sigma_0 + \mathbf{j}\omega$ in s-plane is mapped to the circle G in z-plane under the conformal mapping function (3). Let z_0 be any point on the circle G, then there exists $s_0 = \sigma_0 + \mathbf{j}\omega_0$ on the line L, which is mapped to z_0. Assume $s_1=\sigma_0 - \mathbf{j}\omega_0$ and s_1 is mapped to z_1. Note that s_1 is also on the line L, so z_1 also lies on the circle G. According to (3), we have

$$z_0 = \frac{c+\sigma_0+\mathbf{j}\omega_0}{c-\sigma_0-\mathbf{j}\omega_0} \tag{4}$$

and

$$z_1 = \frac{c+\sigma_0-\mathbf{j}\omega_0}{c-\sigma_0+\mathbf{j}\omega_0} \tag{5}$$

It is easy to find that z_0 and z_1 are complex conjugates by comparing (4) and (5). That is to say, if a point lies on the circle G, so does its conjugate.

Additionally, let $z_0 = x_0 + \mathbf{j}y_0$ and the center of the circle G be $g = g_0 + \mathbf{j}g_1$, since both z_0 and its conjugate $\bar{z}_0$ lie on the circle G, we have

$$|z_0 - g| = |\bar{z}_0 - g|,$$

That is

$$|x_0 + \mathbf{j}y_0 - g_0 - \mathbf{j}g_1| = |x_0 - \mathbf{j}y_0 - g_0 - \mathbf{j}g_1|$$

$$|(x_0 - g_0) + \mathbf{j}(y_0 - g_1)| = |(x_0 - g_0) - \mathbf{j}(y_0 + g_1)|$$

$$\sqrt{(x_0-g_0)^2 + (y_0-g_1)^2} = \sqrt{(x_0-g_0)^2 + (y_0+g_1)^2}$$

After simplifying the above expression, we obtain that

$$y_0 g_1 = 0$$

Since y_0 can take non-zero values, thus $g_1 = 0$.

So the center g of the circle G lies on the real axis of the z-plane, g actually takes values from the set of real numbers.

C. Formulas for the Center and Radius of the Circle G

Suppose the line L: $\mathbf{Re}(s) = \sigma_0$ in s-plane is mapped to the circle G in the z-plane under (3), according to the discussion earlier, the center and radius of the circle G can be denoted as g and ρ respectively, where g is a real number and ρ is a non-negative real number. The points z on the circle G satisfy

$$|z-g| = \left|\frac{c+\sigma_0+\mathbf{j}\omega}{c-\sigma_0-\mathbf{j}\omega} - g\right| = \rho \tag{6}$$

Equation (6) can be further written as

$$\begin{aligned}|z-g| &= \left|\frac{c+\sigma_0+\mathbf{j}\omega}{c-\sigma_0-\mathbf{j}\omega} - g\right| \\ &= \left|\frac{c+\sigma_0+\mathbf{j}\omega - cg + \sigma_0 g + \mathbf{j}\omega g}{c-\sigma_0-\mathbf{j}\omega}\right| \\ &= \left|\frac{c(1-g)+\sigma_0(1+g)+\mathbf{j}\omega(1+g)}{c-\sigma_0-\mathbf{j}\omega}\right| \\ &= \rho\end{aligned} \tag{7}$$

1) if $g = -1$, then (7) can be written as

$$|z-g| = \left|\frac{2c}{c-\sigma_0-\mathbf{j}\omega}\right| = \rho \tag{8}$$

Since ω varies from $-\infty$ to $+\infty$, (8) holds true only when $\sigma_0 = \pm\infty$ and $\rho = 0$. That is to say, the lines $\mathbf{Re}(s) = \pm\infty$ are mapped to the circle with the center at $g = -1$ and a radius of $\rho = 0$.

2) if $g \neq -1$, then let $c + \sigma_0 = b$, $c - \sigma_0 = d$, the (6) can be further written as

$$\begin{aligned}|z-g| &= \left|\frac{b+\mathbf{j}\omega}{d-\mathbf{j}\omega} - g\right| \\ &= \left|\frac{b+\mathbf{j}\omega - dg + \mathbf{j}\omega g}{d-\mathbf{j}\omega}\right| \\ &= \left|\frac{b-dg+\mathbf{j}\omega(1+g)}{d-\mathbf{j}\omega}\right| \\ &= |1+g|\left|\frac{\frac{b-dg}{1+g}+\mathbf{j}\omega}{d-\mathbf{j}\omega}\right| \\ &= \rho\end{aligned} \tag{9}$$

Since ω varies from $-\infty$ to $+\infty$, (9) holds true only when

$$\frac{b-dg}{1+g} = \pm d \tag{10}$$

Next, we'll consider two cases separately:

(1) when

$$\frac{b-dg}{1+g} = -d$$

It follows that

$$b - dg = -d - dg$$

Thus we obtain

$$b = -d \tag{11}$$

Substitute $c + \sigma_0 = b$, $c - \sigma_0 = d$ into (11) to yield

$$c = 0$$

This result is invalid and should be abandoned.

(2) when

$$\frac{b-dg}{1+g} = d$$

It follows that

$$\begin{aligned}g &= \frac{b-d}{2d} \\ &= \frac{c+\sigma_0-c+\sigma_0}{2d} \\ &= \frac{\sigma_0}{c-\sigma_0}\end{aligned} \tag{12}$$

From (12), we can find that the center g of the circle G is actually a function of σ_0.

Next we will discuss (12) in three different cases, keeping in mind that $c > 0$:

1) if $\sigma_0 > c$, then g is an increasing function of σ_0 in the interval $(c, +\infty)$ according to (12), and g increases from $-\infty$ to -1 as σ_0 varies from c^+ to $+\infty$.

Substitute (12) into (9) to yield

$$\begin{aligned}\rho &= |1+g| \\ &= \left|1+\frac{\sigma_0}{c-\sigma_0}\right| \\ &= \frac{c}{\sigma_0 - c}\end{aligned} \tag{13}$$

It is easy to find that ρ is actually a decreasing function of σ_0 in the interval $(c, +\infty)$ from (13), and ρ decreases from $+\infty$ to 0 as σ_0 varies from c^+ to $+\infty$.

From (12) and (13), we can deduce that the circle G intersects the real axis in the z-plane at two points, the left and right points are as follows

$$z_L = g - \rho = \frac{c+\sigma_0}{c-\sigma_0}$$

and

$$z_R = g + \rho = -1 \tag{14}$$

Thus, we can find that the lines on the right side of the line **Re**(s) = c in the s-plane are mapped to the circles on the left side of the line **Re**(z) = –1 in the z-plane, and all the circles pass through the point z = –1.

2) if $\sigma_0 = c$, then from (3) we can obtain

$$z = \frac{2c + \mathbf{j}\omega}{-\mathbf{j}\omega} = -1 + \mathbf{j}\frac{2c}{\omega} \tag{15}$$

It is easy to see that (15) represents the line Re(z) = –1 in the z-plane.

3) if $\sigma_0 < c$, then g is an increasing function of σ_0 in the interval (–∞, c) according to (12), and g increases from –1 to +∞ as σ_0 varies from –∞ to c^-. Substitute (12) into (9) to yield

$$\rho = |1 + g| = \left|1 + \frac{\sigma_0}{c - \sigma_0}\right| = \frac{c}{c - \sigma_0} \tag{16}$$

It is easy to find that ρ is actually an increasing function of σ_0 in the interval (–∞, c) from (16), and ρ increases from 0 to +∞ as σ_0 varies from –∞ to c^-.

From (12) and (16), we can obtain that the circle G intersects the real axis in the z-plane at two points, the left and right points are as follows

$$z_L = g - \rho = -1 \tag{17}$$

and

$$z_R = g + \rho = \frac{c + \sigma_0}{c - \sigma_0}$$

Thus, we can deduce that the lines on the left side of the line **Re**(s) = c in the s-plane are mapped to the circles on the right side of the line **Re**(z) = –1 in the z-plane, and all the circles pass through the point z = –1.

We now focus on the special case of $\sigma_0 = 0$, corresponding to the imaginary axis in the s-plane. From (12), we can obtain that the center of the circle G when $\sigma_0 = 0$ is

$$g = \frac{\sigma_0}{c - \sigma_0} = 0,$$

and from (16), we can obtain that the radius of the circle G when $\sigma_0 = 0$ is

$$\rho = \frac{c}{c - \sigma_0} = 1$$

that is to say, the imaginary axis in the s-plane is mapped to the unit circle in the z-plane.

we can draw a brief conclusion that under the bilinear transformation, the line **Re**(s) = c in the s-plane is mapped to the line **Re**(z) = –1 in the z-plane, the lines on the left and right sides of the line **Re**(s) = c are mapped to the circles on the right and left sides of the line **Re**(z) = –1 respectively.

III. NUMERICAL COMPUTATION

In this section, we will plot and compare the image of the mapping of the bilinear transformation obtained from the theoretical analysis and numerical computation respectively.

Let σ_0 take 8 different values from the set {-4, -2, 0, 2, 4, 6, 9,12}, corresponding to 8 lines in the s-plane. These lines are L_1: **Re**(s)= –4, L_2: **Re**(s) = –2, L_3: **Re**(s) = 0, L_4: **Re**(s) = 2, L_5: **Re**(s) = 4, L_6: **Re**(s) = 6, L_7: **Re**(s) = 9, L_8: **Re**(s) = 12.

Let c=4. According to the previous mathematical analysis, we know that these 8 lines in s-plane will be mapped to seven circles and one line in the z-plane. we can use (12), (13) and (16) to get the circles, and use (15) to get the line in the z-plane directly. The mappings are listed as follows:

1) L_1 maps to the circle G_1: $|z + 1/2| = 1/2$,
2) L_2 maps to the circle G_2: $|z + 1/3| = 2/3$,
3) L_3 maps to the circle G_3: $|z - 0| = 1$, unit circle,
4) L_4 maps to the circle G_4: $|z - 1| = 2$,
5) L_5 maps to the line L_5': **Re**(z) = –1,
6) L_6 maps to the circle G_6: $|z + 3| = 2$,
7) L_7 maps to the circle G_7: $|z + 1.8| = 0.8$,
8) L_8 maps to the circle G_8: $|z + 1.5| = 0.5$.

We plot these circles and the line mentioned above in Fig. 1.

In Fig. 2, we plot the points of z computed directly from (2), the definition of the bilinear transformation, where s is chosen from the points on the 8 lines in the s-plane mentioned above.

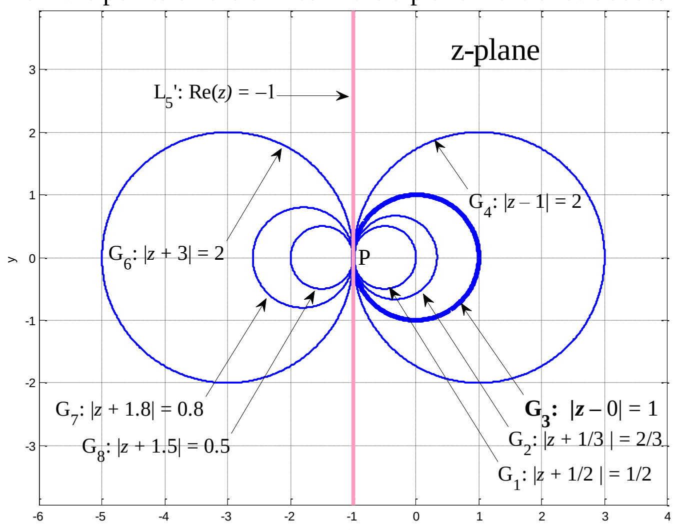

Fig. 1. The image of the mapping of the bilinear transformation obtained from the mathematical analysis results.

It is clear that Fig.1 and Fig.2 are consistent. The numerical computation results further confirm the correctness of the theoretical analysis.

Now let's closely examine these figures and observe how the circles change as σ_0 varies from -∞ to +∞.

We start from the Point P where z=–1 in Fig.1, which is actually the circle |z – (–1) | = 0 mapped by the Line **Re**(s) = σ_0 = –∞ as indicated by (8). As σ_0 increases, the center of the circle moves along the positive direction of the real axis and the radius also increases. For example, when σ_0 = –4, the circle as the image of the mapping is G_1: $|z + 1/2| = 1/2$, when σ_0 = –2, the circle as the image of the mapping is G_2: $|z + 1/3| = 2/3$. Both circles lie inside the unit circle.

When σ_0 increases to 0, the center of the circle moves to 0 and the radius increases to 1, that is exactly the unit circle.

When σ_0 continues to increase from 0, the circles are beginning to lie outside the unit circle. For example, when σ_0 = 2, the circle is G_4: $|z - 1| = 2$.

As σ_0 approaches c from the left side, the center and radius of the circle approach +∞. The line L_5: **Re**(s) = c maps to the

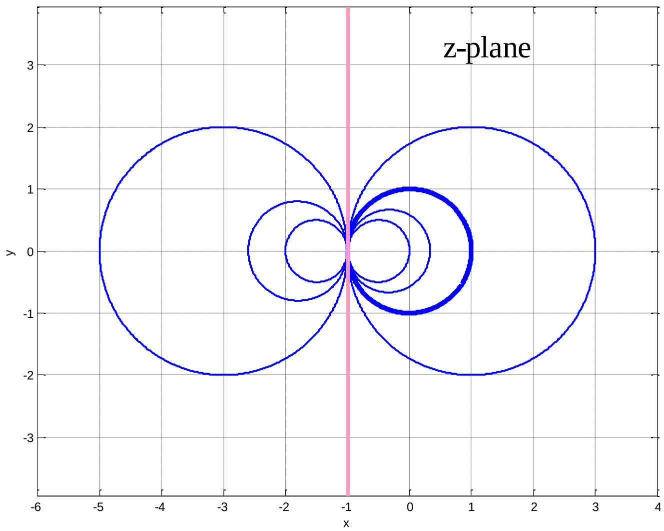

Fig. 2. The image of the mapping of the bilinear transformation obtained from the numerical computation results.

line L_5': $\mathbf{Re}(z) = -1$ in the z-plane which can be considered as a circle with infinite center and radius.

Note that all the circles mentioned above lie on the right side of the line $\mathbf{Re}(z) = -1$.

When σ_0 continues to increase from c to $+\infty$, The center of the circle jumps from $+\infty$ to $-\infty$, then moves along the positive direction of the real axis from $-\infty$ to -1, and the radius decreases from $+\infty$ to 0. The subsequent circles lie on the left side of the line $\mathbf{Re}(z) = -1$. For example, L_6 is mapped to the circle G_6: $|z + 3| = 2$.

When σ_0 approaches $+\infty$, the center of the circle approaches again to the point P but from the left side and the radius of the circle approaches 0.

One more thing to be noted is that all the circles and the line are tangent at the point P where $z = -1$.

IV. Conclusion

In this paper, we delve into the mapping of the bilinear transformation and uncover its internal details. Our work serves as a valuable complement to existing bilinear transformation method. The associated analytical methods and graphical illustrations will contribute to the extensive application of the bilinear transformation in areas such as digital signal processing, control systems, image processing, robotics and so on.

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