MI Exercise H8.1:
Vapnik-Chervonenkis dimension

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Show that the linear classifier has a Vapnik-Chervonenkis dimension of $d_{VC} = N+1$:

$$\tilde{C}_{(p,N)} = 2\sum_{k=0}^{N}\binom{p-1}{k}, \quad (x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

linear classifier: $y(\bar{x};\bar{w}) = \mathrm{sign}\left(w_0 + \sum_{i=1}^{N} x_i w_i\right)$

recursion property: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

-> for $p = N+1$,

$$\tilde{C}_{(N+1,N)} = 2\sum_{k=0}^{N}\binom{N}{k}$$

where given $x, y = 1 \rightarrow (x+y)^n = 2^n = \sum_{k=0}^{n}\binom{n}{k}$

~~thus~~ $\cancel{2^{N+1} = \sum_{k=0}^{N+1}\binom{N+1}{k} = \sum_{k=0}^{N}\binom{N}{k}}$ ~~and~~ $\cancel{n = N+1}$

and $\tilde{C}_{(N+1,N)} = 2[2^n] = 2^{n+1}$ $\quad n=N$

-> for $p = N+2$,

$$\tilde{C}_{(N+2,N)} = 2\sum_{k=0}^{N}\binom{N+1}{k}$$

and for $n = N+1$, $x, y = 1$ $\rightarrow (x+y)^n = 2^{N+1} = \sum_{k=0}^{N+1}\binom{N+1}{k} = \binom{N+1}{N+1} + \sum_{k=0}^{N}\binom{N+1}{k}$

and together, $\tilde{C}_{(N+2,N)} = 2\left[2^{N+1} - \binom{N+1}{N+1}\right] = 2^{N+2} - 2\binom{N+1}{N+1}$

as $2\binom{N+1}{N+1}$ is positive for all N,

then $\tilde{C}_{(N+2,N)} < 2^{N+2}$

therefore, we have shown that $d_{VC} = N+1$