MNS: CP4 Luke Longren

Exercise #1: Analytical solutions to ODEs

a) $\dot{x} = -x$, $x_0 = 1$

$$\frac{dx}{dt} = -x \rightarrow \int \frac{dx}{x} = \int dt$$
$$\ln(x) = -t + C$$
$$x = e^{-t+c} = Ce^{-t}$$

where $\rightarrow x(0) = 1 = Ce^{0}$, $C = 1$

thus, $\underline{x(t) = e^{-t}}$

b) $\dot{x} = \frac{1}{x}$, $x_0 = 1$

$$\frac{dx}{dt} = \frac{1}{x} \rightarrow \int x\,dx = \int dt$$
$$\frac{1}{2}x^2 = t + C$$
$$x = \sqrt{2(t+c)}$$

where $\rightarrow x(0) = 1 = \sqrt{2(0+c)}$, $C = \frac{1}{2}$

thus, $\underline{x(t) = \sqrt{2t+1}}$

c) $\dot{x} = 1 - x$, $x_0 = 0$

$$\frac{dx}{dt} = 1 - x \rightarrow \int \frac{dx}{1-x} = \int dt \qquad \begin{cases} u = 1-x \\ du = -dx \end{cases}$$
$$-\int \frac{du}{u} = \int dt$$
$$\ln(u) = -t + C$$
$$\ln(1-x) = -t + C \rightarrow \quad 1 - x = Ce^{-t}, \quad x = 1 - Ce^{-t}$$

where $\rightarrow x(0) = 0 = 1 - Ce^{-0}$, $C = 1$

thus, $\underline{x(t) = 1 - e^{-t}}$

d) $\dot{x} = x(1-x)$, $x_0 = \frac{1}{2}$

$$\frac{dx}{dt} = x(1-x) \rightarrow \int \frac{dx}{x(1-x)} = \int dt$$
$$\cancel{\int \frac{dx}{x - x^2} = t + C}$$
$$-\int \frac{dx}{x^2(1-\frac{1}{x})} \qquad \begin{cases} u = 1 - \frac{1}{x} \\ du = \frac{1}{x^2}dx \end{cases} \rightarrow -\int \frac{du}{u} = -\ln(u)$$

$$\Rightarrow \ln\left(1 - \frac{1}{x}\right) = -t + C$$
$$1 - \frac{1}{x} = Ce^{-t}$$
$$\frac{1}{x} = 1 - Ce^{-t} \rightarrow \quad x = \frac{1}{1 - Ce^{-t}}$$

where $\rightarrow x(0) = \frac{1}{2} = \frac{1}{1-C}$, $C = -1$

thus, $\underline{x(t) = \frac{1}{1+e^{-t}}}$