Space Debris

Abstract

A satellite has a high probability of being damaged if it comes into contact with a piece of orbital debris due to their high relative speeds. We investigate how easily a spaceship could approach orbiting debris and how long the said debris would otherwise take to burn out. We derive the differential equations describing the orbital trajectory of the debris and spaceship. We use the 4th order Runge Kutta approximation method to solve these equations and obtain the time evolution of the system. We then trial different thrusting methods in order to approach the debris most efficiently. We also approximate the amount of time it would take for debris of various sizes to reach the earth's surface.

1 Introduction

Space debris, which orbits the earth like any other satellite, is a detriment to modern space exploration [1]. The collision of a piece of debris (however tiny) with an active satellite is likely to cause significant damage due to the large relative speeds of the objects, wasting millions in investment and forming more hazardous debris in the process. Our objective therefore is to investigate methods regarding the removal of debris in the most efficient way possible.

We wish to investigate how to guide a cleaning spacecraft to within $1m$ of a piece of debris a small distance away, going almost the same speed and at almost the same orbital height. This requires forming and solving a set of differential equations involving the central gravitational force and the thrust of the spacecraft's engine. This proves very hard to solve analytically, so we turn to Python to approximate a solution with numerical integration methods. Once we have understood the movement of the spacecraft, we investigate the most fuel-efficient way of reaching our target. We make a final study on the time taken for different types of orbital debris to eventually burn out, motivating our need to find a way to capture them. Firstly, we must derive a set of equations to simulate our satellites.

2 Orbital Dynamics

2.1 Equations of Motion

We first need to know the dynamics of a piece of debris. An orbiting satellite has acceleration $a = -\frac{GM_E}{r^2}$ [3], where r is the distance to the origin. Then $a\frac{x}{\sqrt{x^2+y^2}}$ and $a\frac{y}{\sqrt{x^2+y^2}}$ are the x and y components of the acceleration, so using Newton's 2nd law, the trajectory of an object orbiting earth in Cartesian coordinates is satisfied by the equations

$$\begin{aligned} m\ddot{x} &= -\frac{GM_E m}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} + F_x(t), \\ m\ddot{y} &= -\frac{GM_E m}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}} + F_y(t), \end{aligned} \tag{1}$$

where m is the mass of the object, $G = 6.673 \times 10^{-11} Nm^2kg^{-2}$ is the Gravitational constant, $M_E = 5.97 \times 10^{24} kg$ is the Earth's mass, $R_E = 6370km$ is the Earth's radius, and $F_x(t)$ and $F_y(t)$ are forces exerted on the object parallel to the x and y axes respectively.

However, orbital motion is much more suited to polar coordinates. In Cartesian coordinates, modelling two objects moving in circular orbit is rather awkward, whereas in polar coordinates, there is a symmetry in rotating about the origin - it corresponds to a translation in θ. We substitute

$$x = r\sin(\theta), \quad y = r\cos(\theta) \tag{2}$$

into 1 to obtain the following: Q1

$$\ddot{r}\sin(\theta) + 2\dot{r}\dot{\theta}\cos(\theta) + r(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta)) = \frac{-GM_E}{r^2}\sin(\theta) + \frac{F_x(t)}{m}, \tag{3}$$

$$\ddot{r}\cos(\theta) - 2\dot{r}\dot{\theta}\sin(\theta) - r(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)) = \frac{-GM_E}{r^2}\cos(\theta) + \frac{F_y(t)}{m}. \tag{4}$$

Multiplying 3 by $\cos(\theta)$ and 4 by $\sin(\theta)$ and taking the difference gives:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{F_x(t)}{m}\cos(\theta) - \frac{F_y(t)}{m}\sin(\theta). \tag{5}$$

We can then rearrange 5 to make $\ddot{\theta}$ the subject, and substitute into 3 to get

$$\ddot{r}\sin(\theta) + 2\dot{r}\dot{\theta}\cos(\theta) + rG(r,\theta,t) = -\frac{GM_E\sin(\theta)}{r^2} + \frac{F_x(t)}{m}, \tag{6}$$

where

$$G(r,\theta,t) = (\frac{-2\dot{\theta}\dot{r}}{r} + \frac{F_x(t)}{mr}\cos(\theta) - \frac{F_y(t)}{mr}\sin(\theta))\cos(\theta) - \dot{\theta}^2\sin(\theta). \tag{7}$$

We can rearrange and simplify this to finally obtain our system

$$\ddot{r} = \frac{-GM_E}{r^2} + \dot{\theta}^2 r + \frac{F_r}{m}, \tag{8}$$

$$\ddot{\theta} = \frac{-2\dot{\theta}\dot{r}}{r} + \frac{F_\theta}{mr}, \tag{9}$$

where

$$\begin{aligned} F_r &= F_x\sin(\theta) + F_y\cos(\theta), \\ F_\theta &= F_x\cos(\theta) - F_y\sin(\theta) \end{aligned} \tag{10}$$

are the forces exerted on the object along the radial and angular axes.

Space debris is a threat to satellites orbiting earth, which mostly follow circular trajectories, so for simplicities sake we will only study circular orbits. This corresponds to the most simple solution of 8 and 9, where $r = r_0$ is the constant radius, and $\theta = \omega_0 t + \theta_0$ is the initial angle θ_0 and the angular speed ω_0. The orbital period is $T_0 = \frac{2\pi}{\omega_0}$, where $\omega_0 = \sqrt{\frac{GM_E}{r_0^3}}$. Q2

The linear speed of a piece of debris is

$$\begin{aligned} v_0 &= \frac{2\pi r_0}{T_0} \\ &= r_0\omega_0 \\ &= \sqrt{\frac{GM_E}{r_0}}. \end{aligned} \tag{11}$$

The International Space Station has a higher and lower circular orbit, corresponding to heights of 330km and 435km above the surface of the earth. At 330km, the ISS has a circular orbit of period

$$\begin{aligned} T_{330} &= 2\pi\sqrt{\frac{(3.3\times 10^5 + R_E)^3}{GM_E}} \\ &= 5459.393s \\ &= 90.99\text{mins}, \end{aligned} \tag{12}$$

and at 435km, we have

$$\begin{aligned} T_{435} &= 2\pi\sqrt{\frac{(4.35\times 10^5 + R_E)^3}{GM_E}} \\ &= 5588.231s \\ &= 93.14\text{mins}. \end{aligned} \tag{13}$$

Another point to consider when investigating tracking down debris is the scale we are working in. For example, a piece of debris in orbit at 300km above the earth's surface will be traversing the circumference of a circle with radius roughly 6600km. A spaceship beginning a rendezvous from 2km behind will have a difference in angle of 0.0003rad, which will only become smaller as they near each other. Numerical accuracy in computation could become an issue, as well as precisely tracking the relative position between the two objects. To combat this, we use the following trajectories. The debris is given by the reference trajectory

$$r = r_0, \quad \theta = \omega_0 t, \tag{14}$$

and the active spacecraft uses the relative coordinates

$$r = r_0 + z(t), \quad \theta = \omega_0 t + \phi(t), \tag{15}$$

where $z(t)$ and $\phi(t)$ are the radial and angular displacement from the debris. Now instead of working with two moving objects, we can work in a frame of reference where the debris appears to be still, and our two displacement functions give the approach. To do this, we need to reformulate our differential equations in terms of these relative coordinates. Inserting 15 into 8 and 9 gives us

$$\ddot{z} = -\frac{GM_E}{(r_0+z)^2} + (r_0+z)(\omega_0+\dot{\phi})^2 + \frac{F_r}{m}, \tag{16}$$

$$\ddot{\phi} = -2\frac{(\omega_0+\dot{\phi})\dot{z}}{r_0+z} + \frac{F_\theta}{m(r_0+z)}. \tag{17}$$

This is a system of two non-linear second order differential equations, which, rather than trying to find an analytical solution to, we will solve with numerical integration. To do this, we use Python to employ the 4th order Runge Kutta method, which performs the time integration of a system of 4 first order differential equations at discrete time intervals.

To use this method, we need to convert our system into 4 first order ordinary differential equations. Defining $\dot{z} = v_z$ and $\dot{\phi} = v_\phi$, we obtain: **Q3**

$$\begin{aligned} \frac{dz}{dt} &= v_z, \\ \frac{dv_z}{dt} &= -\frac{GM_E}{(r_0+z)^2} + (r_0+z)(\omega_0+v_\phi)^2 + \frac{F_r}{m}, \\ \frac{d\phi}{dt} &= v_\phi, \\ \frac{dv_\phi}{dt} &= -2\frac{(\omega_0+v_\phi)v_z}{r_0+z} + \frac{F_\theta}{m(r_0+z)}, \end{aligned} \tag{18}$$

which we use to simulate the time evolution of our rendezvous.

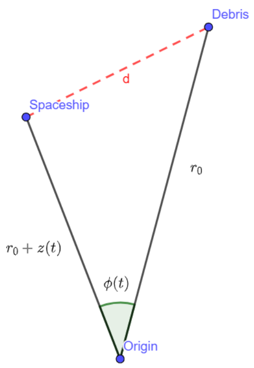


Figure 1: Spacecraft and debris

2.2 Calculating distance

The key to our rendezvous is getting as close as we can to the piece of the debris, so we need to be able to calculate the distance between it and the spaceship. **Q4**

We can simply use the cosine rule to determine the distance d.

$$\begin{aligned} d^2 &= r_0^2 + (r_0 + z(t))^2 - 2r_0(r_0 + z(t))\cos(\phi(t)) \\ &= r_0^2 + r_0^2 + 2r_0 z(t) + z^2(t) - 2r_0(r_0 + z(t))\cos(\phi(t)) \\ &= z^2(t) + 2r_0(1 - \cos(\phi(t)))(r_0 + z(t)), \end{aligned} \tag{19}$$

so we have that $d = \sqrt{z^2(t) + 2r_0(1 - \cos(\phi(t)))(r_0 + z(t))}$. We ensure this is correct with a few examples:

- $z = 0, \phi = 0$ (collision):
  $d = \sqrt{0 + 0} = 0$.
- $z = a, \phi = 0$ (spaceship is directly above/below the debris):
  $d = \sqrt{a^2 + 0} = 0$.
- $z = 0, \phi = a$, where a is assumed to be very small (Figure 2):
  Here, d can be well approximated by the arc length due to how small a is. Our expected value for d is therefore $d = r_0 a$. If we take a Taylor expansion of $\cos(a) = 1 - a^2/2$ to the 2nd term, our formula correctly gives us

  $$\begin{aligned} d &= \sqrt{0 + 2r_0(1 - (1 - a^2/2))r_0} \\ &= r_0 a. \end{aligned}$$

- $z = 0, \phi = \pi/2$ (the debris is a quarter circle out from the spaceship) (Figure 3):
  Since we are dealing with the hypotenuse of a right-angled triangle, we can use Pythagoras and so our expected answer is $\sqrt{2}r_0$. Our formula correctly gives us

  $$\begin{aligned} d &= \sqrt{0 + 2r_0(1 - 0)(r_0)} \\ &= \sqrt{2}r_0. \end{aligned} \tag{20}$$

- $z = a, \phi = \pi$ (the debris is on the other side of the circular trajectory, and the spaceship is above this trajectory):

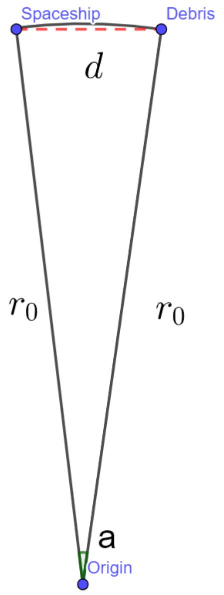


Figure 2: $\phi = a$

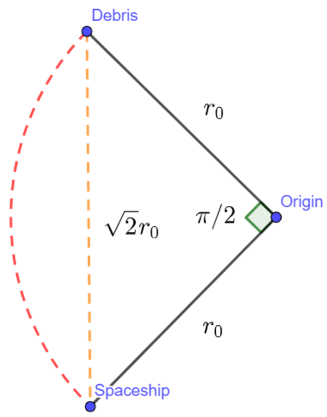


Figure 3: $\phi = \pi/2$

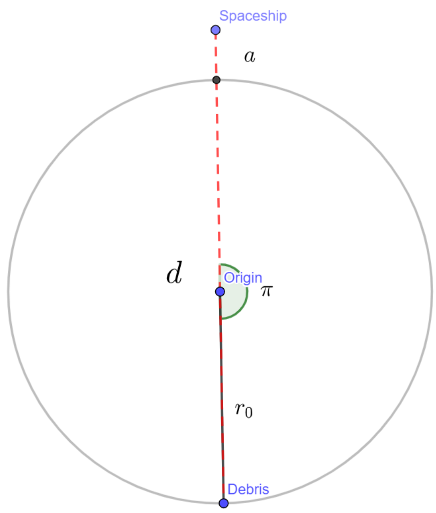


Figure 4: $z = a, \phi = \pi$

Here, we should expect that the distance is $d = 2r_0 + a$, as the increase in height of the spaceship is on the same line as the radius of the debris. Our formula correctly gives us

$$\begin{aligned} d &= \sqrt{a^2 + 2r_0(2)(r_0 + a)} \\ &= \sqrt{a^2 + 4r_0a + 4r_0^2} \\ &= 2r_0 + a. \end{aligned}$$

We now have all the mathematical tools we need to simulate an orbital rendezvous!

3 The Rendezvous

3.1 Gravitational simulation

Our equations of motion have been entered into our simulation program, which should equip us with the time evolution of the system. We need to ensure we are properly simulating a circular orbit. We use a few examples to check, where we have a spacecraft flying at $h = 500km$ above Earth. We set the thrusters to 0 at all times, so we are just simulating an ambient orbit.

- Initial conditions: $z(0) = \dot{z}(0) = \phi(0) = \dot{\phi}(0) = 0$, so we should have a completely stable orbit.

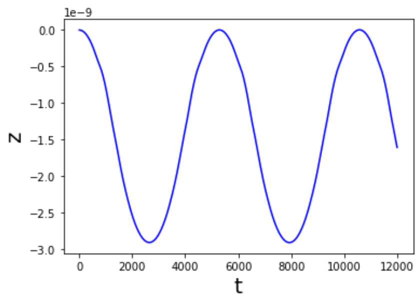


Figure 5: z against t

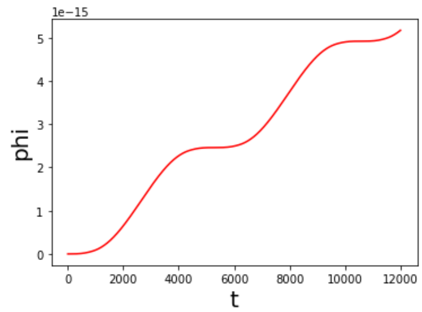


Figure 6: ϕ against t

In the simulation, z and ϕ both stay at 0 (up to numerical accuracy; note the scale on the figures), so this is correct.

- Initial conditions: $z(0) = 1km, \dot{\phi}(0) = \sqrt{\frac{GM_E}{(r_0+z_0)^3}} - \omega_0, \dot{z}(0) = \phi(0) = 0$. Here, the spacecraft is beginning on a circular orbit 1km above the reference orbit.

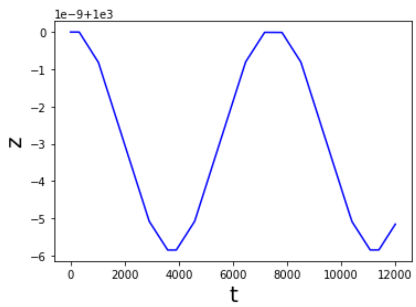


Figure 7: z against t

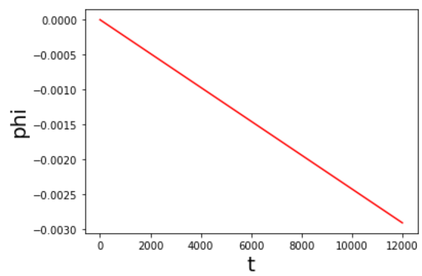


Figure 8: ϕ against t

We have a circular orbit at height 1km higher than the reference orbit, so we should expect z to stay at this height, but then the angular velocity will be smaller than the reference orbit. Here, we see that z stays at 1km within the numerical accuracy of the simulation, and ϕ decreases linearly with time, meaning the angular velocity is at a smaller, but constant value, which is correct.

- Initial conditions: $z(0) = \dot{z}(0) = \phi(0) = 0, \dot{\phi}(0) = 100/(r_0 + z_0)$. Here, the spacecraft is beginning on an elliptical orbit 1km above the reference orbit.
  We have an elliptical orbit, and so z should remain positive, and should increase up to the apogee of the ellipse and then return to 0 after every half orbit. ϕ will vary non-linearly

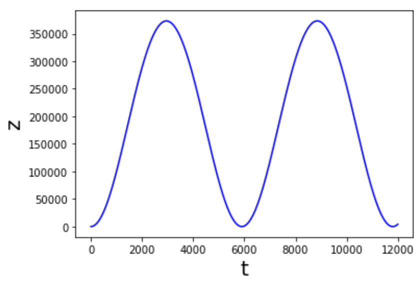


Figure 9: z against t

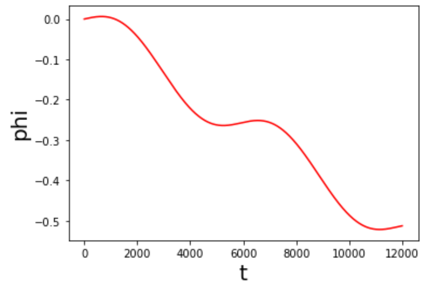


Figure 10: ϕ against t

due to the nature of an ellipse. This is shown in the figure; z increases and decreases in a sinusoidal fashion, and ϕ decreases slowly but non-linearly, as the orbit on an ellipse takes longer.

Our gravitational simulation works well, and so now we are equipped to try to complete a rendezvous!

3.2 Approaching a target

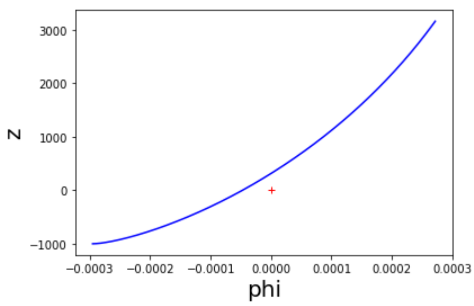
3.2.1 From 2km behind, 1km below

The situation we are considering is a 4000kg cleaning spacecraft attempting to approach a piece of debris on a circular orbit at an altitude of $h = 400km$ above the Earth. The spacecraft begins the rendezvous roughly 2km behind and 1km below the debris. The spacecraft has 4 thrusters, meaning it can can thrust forwards, backwards, upwards and downwards (parallel to the orbit). Each thruster is preset to a certain force (with the limit being $100N$), and the thrusters both start (at $t = 0$) and stop (at $t = t_{\text{thrust}}$) at the same time. We need to determine how strong to make the thrusters, and how long to thrust for, in order to capture the piece of debris - that is, getting within 1 meter of it. The fuel needed to thrust this forcefully for however long is given by Fuel $= (|F_r| + |F_\theta|)t_{\text{thrust}}$, and if possible, we want to minimise this.

Q5

A first attempt is choosing the thrust force to have the same gradient as the displacement to the debris, so $F_r = 50, F_\theta = 100$. The trajectory is shown in Figure 11.

Figure 11: A spacecraft with $F_r = 50, F_\theta = 100$.



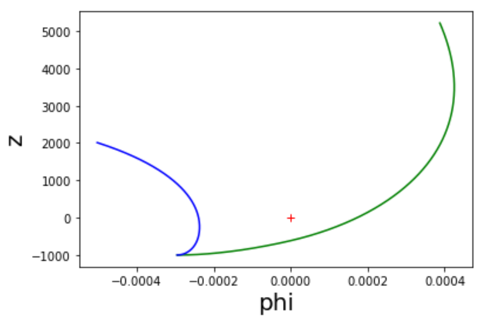


Figure 12: Blue: $F_r = 50, F_\theta = 0$, green: $F_r = 0, F_\theta = 100$, $t_{\text{thrust}} = 340$.

We achieve a minimum distance to the debris (the red cross at $(0, 0)$) of roughly $d = 220m$ using $t_{\text{thrust}} = 340s$, which occurs at about $t = 350s$. Increasing the thrust time simply elongates the trajectory, and does not reduce the minimum distance. But why does this not work? In Cartesian coordinates, if an object was 2 units ahead of us parallel to the x-axis, and 1 unit above us parallel to the y-axis, accelerating in the direction of the vector $(2, 1)$ (i.e. aiming straight at the target) works. Haven't we just done this, but in polar coordinates?

The answer lies in how angular speed is affected by the orbit height. Let's consider the formula for angular speed: $\omega = \frac{v}{r}$, where v is the linear speed perpendicular to the circular orbit and r is the current radius. For a given v, our angular speed decreases the higher we go. So whenever we accelerate into a higher orbit, there is an automatic reduction in ω which must be considered.

Above on 12 we can examine the trajectory of an object being accelerated exclusively along the θ or r axes. The blue (F_r) curve enjoys a short increase in angular velocity, but then the effects of orbital height cause it to veer off to the left. The green (F_θ) curve, despite being exclusively accelerated parallel to the orbit, experiences a large increase in z; this occurs because the acceleration vector is tangent to the circle and hence points outwards. A leftwards curve follows this, again due to the effects of orbital height. The point is that there is an overall smaller increase in angular velocity than previously expected, causing an undershoot to the debris.

Let's find a thrust solution that does work by accommodating for this angular deceleration. If we take $F_r = 32, F_\theta = 100, t_{\text{thrust}} = 266s$, we reach the target in roughly 377s, using $35112kgm/s$ of fuel. Note that we have reduced our radial acceleration as our accommodation.

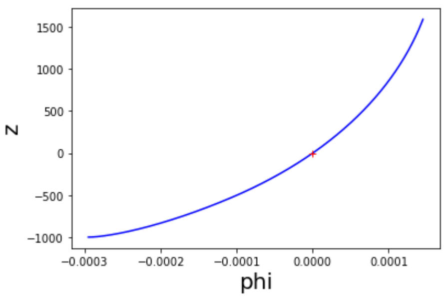


Figure 13: A spacecraft with $F_r = 32, F_\theta = 100$

In this trajectory, the spacecraft reaches the target in just over 6 minutes. Such a short flight period requires the fuel's weight to be drastically larger than the weight of the spacecraft itself (just under 900%.) We now search for more efficient solutions.

Based on initial conditions, here are the most efficient trajectories, shown below. In these three simulations, $F_r = 0$ (allowing the angular acceleration to provide the radial increase.)

- For $F_\theta = 25$, $t_{\text{thrust}} = 49.25s$, Fuel $= 1231kgm/s$, and the time taken was 3301s.
- For $F_\theta = 15$, $t_{\text{thrust}} = 82.55s$, Fuel $= 1238kgm/s$, and the time taken was 3335s.
- For $F_\theta = 10$, $t_{\text{thrust}} = 124.9s$, Fuel $= 1249kgm/s$, and the time taken was 3379s.

The common characteristic in these solutions is that they all go over the reference orbits height, implying a waste of energy. It naturally proceeds that there is a more efficient trajectory to be found, where the spacecraft just reaches the height of the reference orbit, simultaneously capturing the debris. For this, however, the spacecraft must be allowed some freedom in where it can begin to thrust.

3.2.2 From anywhere

Q6

A more energy-friendly solution (shown below) is given by $F_r = 0, F_\theta = 15, t_{\text{thrust}} = 76$, which just brushes the orbital height of the debris' trajectory. The red cross marks the point $z = 0, \phi = 5 \times 10^{-5} rad$.

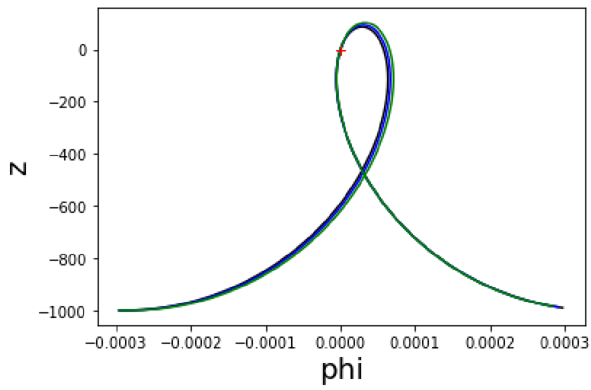


Figure 14: Black: $F_\theta = 25$, Blue: $F_\theta = 15$, Green: $F_\theta = 10$

The fuel cost is $1140kgm/s$, an approximate $90kgm/s$ decrease since our last best solution. This trajectory intercepts the reference height at about $\phi = 5 \times 10^{-5}$, so using the arc length formula, this is approximately $340m$ ahead of the debris.

Hence, our most optimal path is to instead start the rendezvous 1km below and 2.340km behind the debris, and use $F_r = 0, F_\theta = 15, t_{\text{thrust}} = 76$.

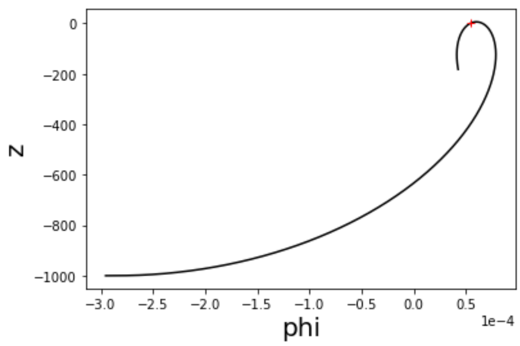


Figure 15: An efficient spacecraft flight.

We now understand that the smallest fuel cost to reach $1km$ above the spacecraft is $1140kgm/s = 1.14kgkm/s$. If the ejection speed of the thruster's gas is $1km/s$, dimensional analysis tells us that the required mass of fuel is $1.14kg$. Now if the spacecraft had $500kg$ of fuel, could we reach several pieces of debris in the range 300km to 500km above the Earth?

The initial answer is yes, many times over. $500kg$ of fuel is enough for a single rendezvous repeated approximately 438 times. However, there are some preparatory steps required in between captures. 15 shows that the optimal trajectory curls round at its peak, and ends pointing down towards the origin. To rectify this, the thrusters would have to be activated to some capacity to return the spacecraft to a circular orbit, ready to recapture. This adds fuel cost to each rendezvous. After enough trips, the spacecraft's orbital height will approach $h = 500km$, which we have stated as our limit. Some fuel would be required again to drop down to a lower orbit.

Another assumption to consider is the density of debris available for capture. If this density is high enough, after the corrective propulsion, there will always be debris within $1km$ ready to be captured. A high enough density could imply that there are debris nearer, requiring less fuel.

For example, we could assume that each corrective step uses $1kg$ of fuel. Assuming also that all $500kg$ is to be used for capturing, we can make a possible $\frac{500}{2.14} \approx 230$ attempts at capture. However, the spacecraft must leave Earth before getting to space, so this assumption may be unrealistic.

4 Atmospheric Capture of Debris

We have just identified the most efficient procedure to recapture space debris. This naturally leads to the question - what happens to the debris that are not captured? The answer is the Earth's atmosphere does the job for us. The orbital speed of debris is several kilometres per second, which induces a large amount of friction. Collisions with higher-atmospheric particles cause the debris to gradually lose energy, until they reach a denser part of the atmosphere, where the frictional force is so large that it causes the debris to burn up.

We will model a piece of debris on a circular orbit (the frictional forces will initially be so low that the decrease in radius will also be very slow), calculate its energy, and its rate of dissipation.

4.1 Re-entry calculation

We wish to find a height-based formula to find the time of reentry to earth for an arbitrary piece of debris. To do this, we will focus on the loss of energy. The total energy of the debris, E_h, is the sum of its gravitational (P_h) and kinetic (K_h) energies. We define $r_h = R_E + h$. Then:

$$\begin{aligned} K_h &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m(\sqrt{\frac{GM_E}{r_h}})^2 \\ &= \frac{GmM_E}{2r_h}, \end{aligned} \tag{21}$$

using 11. Combined with the gravitational potential energy, $P_h = -\frac{GmM_E}{r_h}$, we get

$$E_h = -\frac{GmM_E}{2r_h}. \tag{22}$$

A piece of debris with surface area A collides with $v_h A\rho_{At}$kg of molecules (if ρ_{At} is the density of atmospheric molecules). The molecules gain a moment which is lost by the debris, meaning the resulting force on the debris is

$$F_{fr} = 2A\rho_{At}v_h^2. \tag{23}$$

We can also obtain the power dissipation of the debris by using $P = Fv$, giving

$$\frac{dE_h}{dt} = -2A\rho_{At}v_h^3. \tag{24}$$

However, we still need to derive the formula for $\rho_{At}(h)$. A set of experimental data allows us to fit it with the function

$$\rho_{At}(h) = ae^{-hl} + B(\frac{h}{h_0})^{-\sigma}, \tag{25}$$

where

$$a = 1.946kg/m^3; \quad l = 0.15/km; \quad B = 8.82 \times 10^7 kg/m^3; \quad \sigma = 7.57; \quad h_0 = 1km. \tag{26}$$

We want to use the equations above to find the height of a piece of debris as a function of time. **Q7**

Using 22, we have

$$\frac{dE_h}{dt} = \frac{GmM_E}{2r_h^2}\frac{dr_h}{dt}. \tag{27}$$

Then rearranging 27 and substituting 24 and 11, and using the fact that $\frac{dh}{dt} = \frac{dr_h}{dt}$, we have:

$$\begin{aligned}\frac{dh}{dt} = \frac{dr_h}{dt} &= \frac{2r_h^2}{GmM_E}\frac{dE_h}{dt} \\ &= -\frac{2r_h^2}{GmM_E}2A\rho_{At}(\frac{GM_E}{r_h})^{\frac{3}{2}} \\ &= -\frac{4A\rho_{At}\sqrt{r_h}\sqrt{GM_E}}{m}. \end{aligned} \tag{28}$$

To make this first order differential equation solvable, we use $r_h \approx R_E$ and $\rho_{At} \approx Bh^{-\sigma}$. This gives us

$$\frac{dh}{dt} = -\frac{4A\sqrt{R_E}\sqrt{GM_E}}{m}Bh^{-\sigma}, \tag{29}$$

which we can solve by separating variables:

$$\int_{h=h}^{h=0} h^{-\sigma}dh = \int_{t=0}^{t=t_{\text{reentry}}} -\frac{4AB\sqrt{R_E GM_E}}{m}dt, \tag{30}$$

which finally gives us

$$t_{\text{reentry}} = \frac{1}{4B(\sigma+1)\sqrt{R_E GM_E}}\frac{m}{A}h^{\sigma+1}, \tag{31}$$

a function of the height of the piece of debris and its mass to surface area ratio.

4.2 Examples

Now let us consider some examples of debris:

- A plain aluminium cube of density $\rho = 2700kg/m^3$ and side length $L \approx 1cm$ has $\frac{m}{A} = \frac{2700\times0.01^3}{0.01^2} = 27kg/m^2$. Shown in black below.
- A rectangular rod of length L, square cross-section area l^2 and density ρ has average area $\frac{L\times l}{2}$, $m = l^2L\rho$ and $\frac{m}{A} = 2\rho l$. For an aluminium rod with $L = 10cm, l = 1cm$, $\frac{m}{A} \approx 50kg/m^2$. Shown in green below.
- A square aluminium plate of length L, thickness lmm, mass $m = L^2l\rho$ has an average area $A \approx L^2/2$ and $\frac{m}{A} = 5.4kg/m^2$. Shown in blue below.
- A gemini spacecraft (assumed cubic) of mass $3850kg$ and side length $L = 3m$ has $\frac{m}{A} = 3850/9 = 427.8kg/m^2$. Shown in red below.

NASA [2] states the following: *Debris left in orbits below 600 km normally fall back to Earth within several years. At altitudes of 800 km, the time for orbital decay is often measured in decades. Above 1,000 km, orbital debris will normally continue circling the Earth for a century or more.*

Looking at our diagram, we can see that our prediction lines up well with that statement. Our smaller debris line agree. Moreover, we can see that above 1000km, our Gemini spacecraft is going to stay in orbit for thousands of years due to its enormous mass to area ratio.

5 Conclusion

Throughout the course of this essay, we have motivated the capture of space debris by the need to clear the way for active satellites. We formulated the equations of motion for a spacecraft in polar coordinates, and solved them using computational numerical integration in Python. We then investigated the various methods and limitations in trying to attempt a rendezvous to a

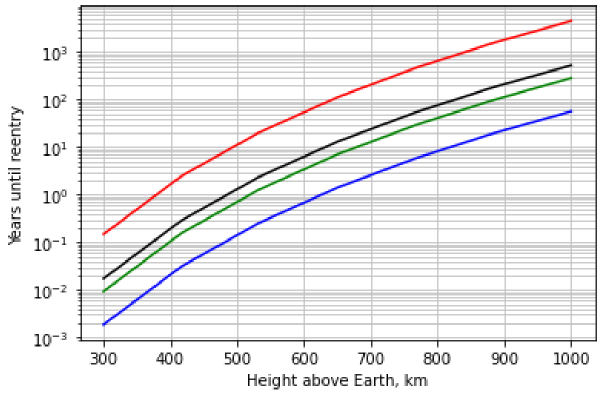


Figure 16: Black: small aluminium cube, green: aluminium rod, blue: aluminium plate, red: gemini spacecraft.

piece of debris and found an efficient way of reaching one in an orbit $1km$ above us. We then considered the fate of space debris we do not recapture, which is gradual burnout.

For a piece of space debris travelling on a circular orbit at around $400km$ above the Earth's surface, a cleaning spacecraft, mass $4000kg$, on an orbit $1km$ below the debris, must expend a minimum of $1.14kg$ of fuel in order to get to within $1m$ of the debris. This comes from using exclusively the forwards thruster, which increases the angular speed as well as the radius of trajectory. This results in the spacecraft needing to be corrected to bring it back to a stable orbital velocity, which then requires more fuel, affecting the number of debris we can afford to capture on one mission.

We then derive the time of reentry of a piece of debris based on its height above Earth and its mass to surface area ratio. For example, we found that a residual 1cm side length aluminium cube would take a year to come down from a height of $700km$, whereas a Gemini spacecraft of mass $3850kg$ would take over a century from the same height.

References

[1] https://www.space.com/kessler-syndrome-space-debris

[2] https://www.nasa.gov/news/debrisfaq.html

[3] https://www.physicsclassroom.com/class/circles/Lesson-4/Mathematics-of-Satellite-Motion