

1. (30 points) Find the following integrals:

a) $\int \frac{x}{\sqrt{x^4 + 3}} dx$

b) $\int \frac{\cos x dx}{\sqrt{3 - \sin x}}$

c) $\int \frac{dx}{x(1 + \ln^2 x)}$

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nts) Find the following integrals

a) $\int \frac{x^3 + x^2}{x^2 + x - 2} dx$

b) $\int \sin 5x \cos 3x dx$

c) $\int x \tan^{-1} x dx$

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(40 points) i) Find the following improper integrals if they converge, if not explain why they diverge.

a) $\int_0^{+\infty} \frac{x dx}{(1+x^2)^4}$

b) $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

ii) Determine whether the integral is convergent or divergent (**No need to find the exact values of the integral**)

a) $\int_1^{+\infty} \frac{1}{\sqrt{x^5+2}} dx$

b) $\int_1^{\infty} \frac{\cos^2 x}{1+x^2} dx$

4. (20 points) Use the integrating factor to solve the initial value problem:

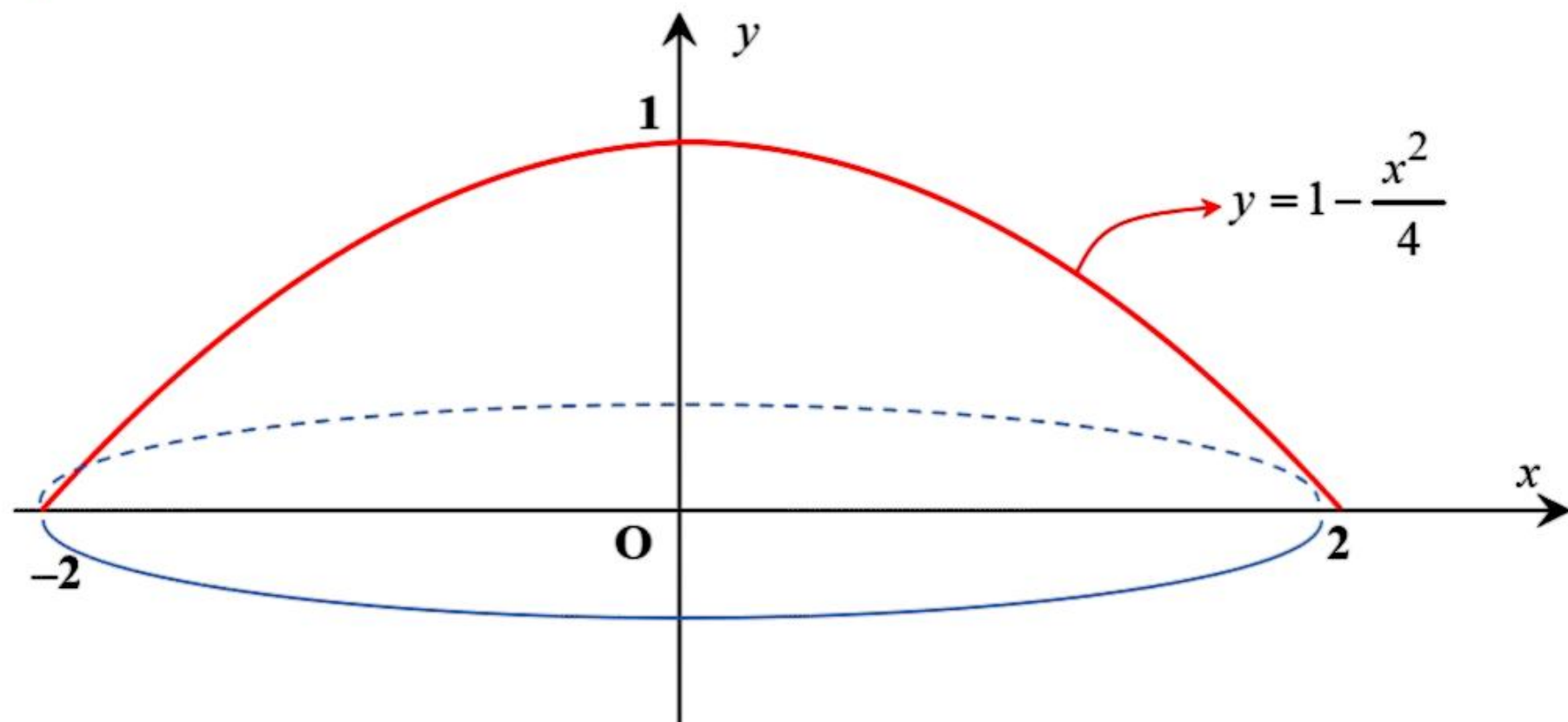
$$\sin(x)y' = \cos(x)y + 1, \quad y\left(\frac{\pi}{4}\right) = 0.$$

5. (30 points) The population $P(t)$ of mosquito growing in a garden increases according to the logistic equation with growth constant $k = 0.3 \text{ (day}^{-1}\text{)}$ and carrying capacity $A = 800$.
- (a) Find a formula for the population $P(t)$, assuming an initial population of $P_0 = 80$ larvae.
- (b) After how many days will the mosquito population reach 400?

6. (30 points) a) Write the integral expressing the arc length of the curve $y = \tan^{-1} x, 0 \leq x \leq 2$. **(do not compute the value of the integral)**
- b) Sketch the picture and write the integral expressing the surface area of the solid generated by rotating the following curve about the y-axis: $y = 1 - \frac{x^2}{4}, 0 \leq x \leq 2$.
(do not compute the value of the integral)

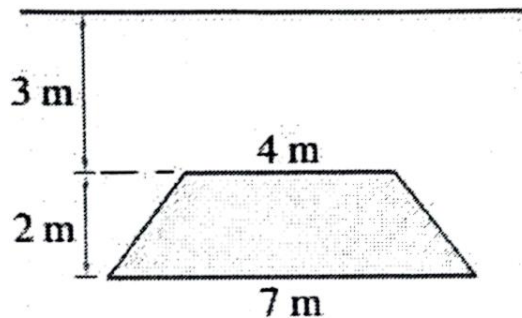
Câu 6.a)
$$L = \int_0^2 \sqrt{1 + (y')^2} \, dx = \int_0^2 \sqrt{1 + \left(\frac{1}{1+x^2} \right)^2} \, dx$$

b)

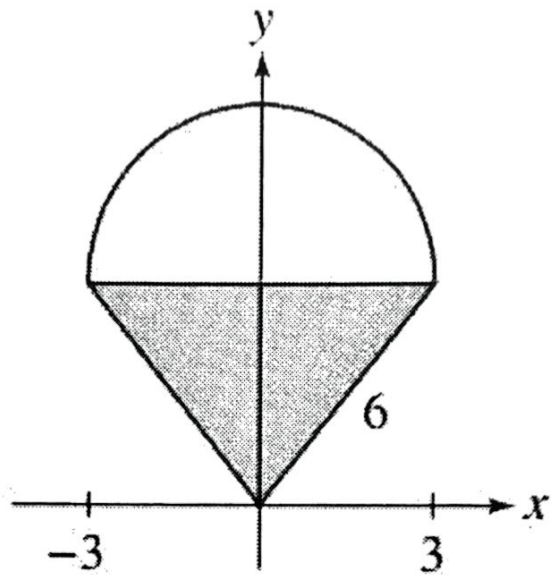


Surface area
$$= 2\pi \int_0^2 x \sqrt{1 + (y')^2} \, dx = 2\pi \int_0^2 x \sqrt{1 + \left(\frac{1}{1+x^2} \right)^2} \, dx$$

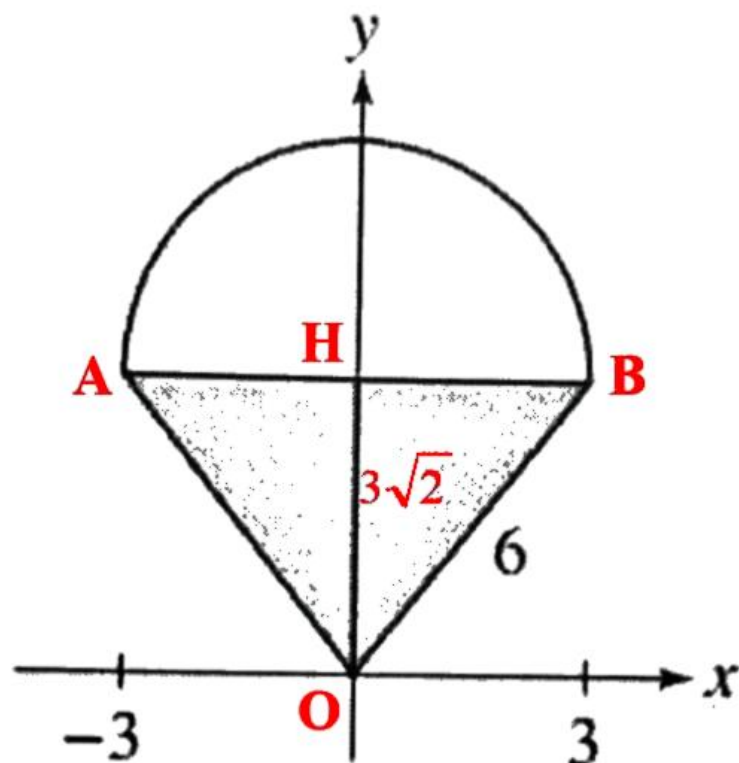
7. (30 points) A vertical gate of a dam has the shape of an isosceles trapezoid with the 4m top, the 7m bottom and 2m height. Given that the top of the gate is 3m below the water surface.
- a) Explain how to approximate the hydrostatic force against one side of the gate by a Riemann sum.
- b) Express the force as an integral and evaluate it.



8. (30 points) Find the centroid of the region which consists of a semicircle on top of an equilateral triangle of side 6 as in the figure below.



Question 8.



Since the triangle is an equilateral triangle of side 6

$$\Rightarrow OH = \frac{6}{\sqrt{2}} = 3\sqrt{2}. \text{ The radius of the semicircle is } R = \frac{6}{2} = 3$$

$H(0, 3\sqrt{2})$ is the center of the circle \Rightarrow the circle equation:

$$(x-0)^2 + (y-3\sqrt{2})^2 = 3^2 \Leftrightarrow x^2 + (y-3\sqrt{2})^2 = 9$$

$$OA: x = \frac{-y}{\sqrt{2}}, \quad OB: x = \frac{y}{\sqrt{2}}, \quad AB: y = 3\sqrt{2}$$

Let D be the given region $\Rightarrow D = D_1 \cup D_2$ where

$$D_1: \begin{cases} 0 \leq y \leq 3\sqrt{2} \\ \frac{-y}{\sqrt{2}} \leq x \leq \frac{y}{\sqrt{2}} \end{cases} \text{ and } D_2: \begin{cases} x^2 + (y-3\sqrt{2})^2 \leq 9 \\ y \geq 3\sqrt{2} \end{cases}$$

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$$m = \iint_D dA = \iint_{D_1} dA + \iint_{D_2} dA = \text{Area}(D_1) + \text{Area}(D_2) = OH \cdot AB + \frac{1}{2}\pi R^2$$

$$= 3\sqrt{2} \cdot 6 + \frac{\pi}{2} \cdot 3^2 = 18\sqrt{2} + \frac{9\pi}{2}$$

$$M_y = \iint_D x dA = 0 \text{ since } f(x, y) = x \text{ is odd with respect to } x, \text{ and the region}$$

D is symmetric about Oy

$$M_x = \iint_D y dA = \iint_{D_1} y dA + \iint_{D_2} y dA = I_1 + I_2$$

$$I_1 = \iint_{D_1} y dA = \int_0^{3\sqrt{2}} \int_{-y/\sqrt{2}}^{y/\sqrt{2}} y dx dy = \int_0^{3\sqrt{2}} y \left(\frac{y}{\sqrt{2}} + \frac{y}{\sqrt{2}} \right) dy = \sqrt{2} \int_0^{3\sqrt{2}} y^2 dy$$

$$= \sqrt{2} \frac{y^3}{3} \Big|_0^{3\sqrt{2}} = 36$$

$$I_2 = \iint_{D_2} y dA = \iint_{D_2} (y - 3\sqrt{2}) dA + 3\sqrt{2} \iint_{D_2} dA$$

$$= \int_{-3}^3 \int_{3\sqrt{2}}^{3\sqrt{2} + \sqrt{9-x^2}} (y - 3\sqrt{2}) dy dx + 3\sqrt{2} \cdot \frac{\pi R^2}{2}$$

$$= \int_{-3}^3 \frac{(y - 3\sqrt{2})^2}{2} \Big|_{y=3\sqrt{2}}^{y=3\sqrt{2} + \sqrt{9-x^2}} dx + 3\sqrt{2} \frac{\pi \cdot 3^2}{2}$$

$$= \int_{-3}^3 \frac{9 - x^2}{2} dx + \frac{27\pi}{\sqrt{2}} = \frac{1}{2} \left(9x - \frac{x^3}{3} \right) \Big|_{-3}^3 + \frac{27\pi}{\sqrt{2}} = 18 + \frac{27\pi}{\sqrt{2}}$$

$$\Rightarrow M_x = 36 + 18 + \frac{27\pi}{\sqrt{2}} = 54 + \frac{27\pi}{\sqrt{2}}$$

\Rightarrow the centroid of the given region is (\bar{x}, \bar{y}) where

$$\bar{x} = \frac{M_y}{m} = 0 \text{ and } \bar{y} = \frac{M_x}{m} = \frac{54 + \frac{27\pi}{\sqrt{2}}}{18\sqrt{2} + \frac{9\pi}{2}} = \frac{3\pi\sqrt{2} + 12}{4\sqrt{2} + \pi}$$

9. (30 points) a) Find the general solution of the following differential equation by using separable of variables

$$\sqrt{1-x^2} y' = xy.$$

- b) Find the constant C so that $p(x) = \frac{Ce^{-x}}{1+e^{-2x}}$ is a probability density function on the interval $(-\infty, \infty)$.

Question 9.a)

$$\sqrt{1-x^2} y' = xy \Leftrightarrow \sqrt{1-x^2} \frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = \frac{x dx}{\sqrt{1-x^2}}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{x dx}{\sqrt{1-x^2}} \Leftrightarrow \ln|y| = -\sqrt{1-x^2} + \ln|C|$$

$$\Leftrightarrow |y| = |C|e^{-\sqrt{1-x^2}} \Rightarrow y = \pm Ce^{-\sqrt{1-x^2}}$$

Let $D = \pm C \Rightarrow$ the general solution of the differential equation is $y = De^{-\sqrt{1-x^2}}$

Question 9.b)

$$\begin{aligned}\int_{-\infty}^{+\infty} p(x) dx &= \int_{-\infty}^{+\infty} \frac{C e^{-x}}{1 + e^{-2x}} dx = \int_{-\infty}^{+\infty} \frac{-C d(e^{-x})}{1 + (e^{-x})^2} = -C \arctan(e^{-x}) \Big|_{-\infty}^{+\infty} \\ &= -C \left(0 - \frac{\pi}{2} \right) = \frac{C\pi}{2} \quad \left(\text{since } \lim_{x \rightarrow +\infty} e^{-x} = 0 \text{ and } \lim_{x \rightarrow -\infty} e^{-x} = +\infty \right)\end{aligned}$$

$p(x)$ is a probability density function on the interval $(-\infty; +\infty)$

$$\Leftrightarrow \begin{cases} p(x) \geq 0, \forall x \in (-\infty, +\infty) \\ \int_{-\infty}^{+\infty} p(x) dx = 1 \end{cases} \Leftrightarrow \begin{cases} \frac{C e^{-x}}{1 + e^{-2x}} \geq 0, \forall x \in (-\infty, +\infty) \\ \frac{C\pi}{2} = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} C \geq 0 \\ C = \frac{2}{\pi} \end{cases} \Leftrightarrow C = \frac{2}{\pi}$$

Hence $C = \frac{2}{\pi}$

10. (30 points) A 1000-liter (L) tank contains 500 L of water with a salt concentration of 10 g/L. Water with a salt concentration of 50 g/L flows into the tank at a rate of $R_{\text{in}} = 80$ L/minutes (min). The fluid mixes instantaneously and is pumped out at a specified rate $R_{\text{out}} = 40$ L/min.
- (a) Set up and solve the differential equation to find the quantity of salt in the tank at time t .
- (b) What is the salt concentration when the tank overflows?

Question 10.a)

Let $y(t)$ be the amount of salt (in gram) after t minutes.

$$y(0) = 500 \cdot 10 = 5000 \text{ (g)}$$

Note that $\frac{dy}{dt}$ is the rate change of the amount of salt, so

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

where (rate in) is the rate at which salt enters the tank and (rate out) is the rate at which salt leaves the tank. We have

$$\text{rate in} = \left(50 \frac{\text{g}}{\text{L}} \right) \left(80 \frac{\text{L}}{\text{min}} \right) = 4000 \frac{\text{g}}{\text{min}}$$

The volume at time t is:

$$V(t) = 500 + (R_{\text{in}} - R_{\text{out}})t = 500 + (80 - 40)t = 500 + 40t \text{ (L)}$$

$$\Rightarrow \text{rate out} = \left(\frac{y(t)}{500 + 40t} \frac{\text{g}}{\text{L}} \right) \left(40 \frac{\text{L}}{\text{min}} \right) = \frac{2y(t)}{25 + 2t} \frac{\text{g}}{\text{min}}$$

Thus, the differential equation is

$$\frac{dy}{dt} = 4000 - \frac{2y(t)}{25 + 2t}, \quad y(0) = 5000$$

Question 10.b) (EM PHẢI CHÉP 10.a VÀO TRƯỚC)

$$\frac{dy}{dt} = 4000 - \frac{2y(t)}{25+2t} \Leftrightarrow \frac{dy}{dt} + \frac{2}{25+2t}y(t) = 4000 \quad (1)$$

$$P(t) = \frac{2}{25+2t}, \quad Q(t) = 4000. \text{ The integrating factor is}$$

$$I(t) = e^{\int P(t)dt} = e^{\int \frac{2}{25+2t}dt} = e^{\ln(25+2t)} = 25+2t$$

Multiplying both sides of (1) by $(25+2t)$, we get

$$(25+2t)\frac{dy}{dt} + 2y(t) = 4000(25+2t)$$

$$\Leftrightarrow \frac{d}{dt}[(25+2t)y(t)] = 4000(25+2t)$$

$$\Rightarrow (25+2t)y(t) = \int 4000(25+2t)dt = 4000(25t + t^2) + C$$

$$\Rightarrow y(t) = \frac{4000(25t + t^2) + C}{25+2t}$$

$$\text{Since } y(0) = 5000 \Rightarrow \frac{C}{25} = 5000 \Rightarrow C = 125000$$

$$\Rightarrow y(t) = \frac{4000(25t + t^2) + 125000}{25+2t}$$

The tank overflows when $V(t) = 1000 \Leftrightarrow 500 + 40t = 1000 \Leftrightarrow t = 12.5$

$\Rightarrow y(12.5) = 40000$. So the concentration when the tank overflows is

$$\frac{40000 \text{ (g)}}{1000 \text{ (L)}} = 40 \frac{\text{g}}{\text{L}}$$