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Control theory, stochastic and partial differential equations, as well as their applications to financial mathematics and machine learning

Final year project

MATH 4999

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Abstract

This project contains two major parts including my study objectives and the applications that can be applied to solve those problems. Optimal control is important as a method to create the provably best solution for control problems in engineering and science. My report will develop an application of stochastic equations to solve optimal control problems. I will first investigate optimal control from a mathematical standpoint and gain an understanding of the theory. Then, I will apply these techniques to the problem to be studied.

Contents

1 Introduction

Optimal control theory is a branch of mathematical optimization that studies how to find a control method for the dynamic system in a period to optimize the objective function. It has a wide range of applications in science, engineering, and operations research. For example, the propulsion system could be a spacecraft whose controls correspond to rocket boosters, whose goal might be to get to the moon with minimal fuel expenditure, or the propulsion system could be a national economy whose goal is to minimize unemployment [1]; In this case, control measures can be fiscal and monetary policy within the framework of optimal control theory, and a dynamic system can also be introduced to embed operational research problems [2]. Optimal control is an extension of the variational method, is derived for a control strategy of the mathematical optimization method [3][4]. There are various types of ways to study optimal control problems and my proposed study is I proposed to focus on applying differential equations to study the mathematical model of the system to be controlled.

In mathematics, a differential equation is an equation that relates one or more functions and their derivatives. In application, functions usually represent physical quantities, derivatives represent their rates of change, and differential equations define the relationship between the two. Such relationships are common; As a result, differential equations play a prominent role in many disciplines, including engineering, physics, economics, and biology. The study of differential equations mainly includes the analysis of their solutions (the set of functions satisfying each equation) and the properties of their solutions[1]. Explicit formulas can solve only the simplest differential equations; However, many properties of the solutions of a given differential equation can be determined without exact calculation. Usually, when the closed-form expression of the solution is not available, the solution can be numerically approximated by a computer[5]. Dynamical system theory emphasizes the qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a degree of precision[5].

Among all the types of differential equations, I will mainly focused on (i) Partial Differential Equation (PDE) and (ii) Stochastic Differential Equation (SDE). System of Differential equations are always built to solved the problems in reality. To make sure that the solution is

stable, we need to investigate the well-posedness of the system.

As with deterministic ordinary and partial differential equations, it is important to investigate whether a given partial differential equation has a solution, and whether it is unique. Cauchy-kowalevski's theorem, in a slightly weaker form, essentially states that if all the terms in a partial differential equation are composed of analytic functions, then on certain region, PDE must have solutions which are also analytic functions [6][7].

Specifically, the integral from 0 to any particular time t is a random variable, defined as the limit of a sequence of random variables. Brownian motion paths cannot meet the requirements of applying standard calculus techniques. The main insight is that the integral can be defined as long as the integrand H is adapted, which roughly means that its value at time T can only depend on the information available up to that point (non-anticipating) [1][3]. Roughly speaking, we can choose a sequence of intervals ranging from 0 to T and construct a Riemann sum. Every time we compute a Riemann sum, we use a particular represent point for the integrator. The decision on which point we are using to represent each interval to compute the value of the function is crucial. As the mesh size goes to zero, the limit of the sequences converges to the solution in probability. Ito's integral uses the left end of the interval as a particular instantiation of the integrator while Stratonovic's integral takes the midpoint during calculation [2].

2 Problems concerned and study objectives

2.1 Optimal control problem

The problem of optimal control is finding a control law for a given system to meet a specific optimality criteria. Control problems include an objective functions, which are functions of state and control variables. Optimal control is a set of differential equations describing the path of control variables that minimizes the objective function [8]. Optimal control can be obtained by using Pontryagin's maximum principle, or by solving the Hamilton-Jacobi-Bellman equation.

2.1.1 Problem setting

We want to find a control function u(t) to maximize an integral objective function:

$$J = \varphi\left(\mathbf{x}\left(t_f\right)\right) + \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{1}$$

subject to the state dynamics:

$$\mathbf{dx} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))\mathbf{dt} + \sigma\mathbf{dB}, \quad \mathbf{x}\left(t_0\right) = \mathbf{x}_0, \tag{2}$$

where $[t_0, t_f]$ is the time interval of interest, $\mathbf{x} : [t_0, t_f] \mapsto \mathbb{R}^{n_x}$ is the state vector, $\varphi : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}$ is a terminal cost function, $L : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R} \mapsto \mathbb{R}$ is an intermediate/running cost function, $\mathbf{B} : [t_0, t_f] \mapsto \mathbb{R}^{n_x}$ is a Brownian motion, and $\mathbf{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}$ is a vector field. The equation (2) represents the dynamics of the system and its initial state condition. Problem 1 as defined above is known as the Bolza problem [3]. If $L(\mathbf{x}, \mathbf{u}, t) = 0$, then the problem becomes the Mayer problem, if $\varphi\left(\mathbf{x}\left(t_f\right)\right) = 0$, it is known as the Lagrange problem [8][9].

2.1.2 Hamilton-Jacobi-Bellman equation

Bellman's principle of optimality states that: "An optimal policy (set of decisions) has the property that whatever the initial state and decisions are, the remaining decisions must constitute and optimal policy with regard to the state resulting from the first decision." [9]

Using Bellman's principle of optimality, it is possible to derive the Hamilton-Jacobi-Bellman (HJB) equation:

$$-\frac{\partial J^*}{\partial t} = \min_{\mathbf{u}} \left(L + \frac{\partial J^*}{\partial \mathbf{x}} \mathbf{f} \right). \tag{3}$$

In some situations, the HJB equation can be used to find the analytical solutions to optimal control problems [10].

2.1.3 Solutions of Hamilton-Jacobi-Bellman equation

The HJB equation is usually solved backwards in time, starting with the time $t = T$ and ending with $t = 0$. If $J(x)$ is continuously differentiable and the problem is solved over the whole state, the HJB equation gives a necessary and sufficient condition for the optimal solution when there is no constraint on the terminal state [2]. Once the solution of HJB is

calculated, it can be used to obtain the optimal control by taking the maximizer or minimizer of the Hamiltonian involved in the HJB equation [7][9].

In general, the HJB equation has no classical (smooth) solution [1][11][12]. Several concepts of generalized solutions have been developed to cover this case, including viscosity solutions, minimax solutions, and others.

In addition, it has been shown that Sum-of-squares optimization can produce an approximate polynomial solution to the Hamilton-Jacobi-Bellman equation concerning the L^1 norm arbitrarily well [6][9].

2.1.4 Viscosity solution

In mathematics, the weak solution (also known as the generalized solution) of an ordinary or partial differential equation is a function whose derivatives may not all exist but are still considered to satisfy the equation. We have to consider weak solutions because many differential equations that simulate real-world phenomena do not have sufficiently smooth solutions [8][12].

Viscous solution is a special form of viscous solution and the concept was developed in the early 1980s by Pierre-Louis Lions and Michael G. Crandall to generalize the classical concept of "solutions" to partial differential equations (PDE) [13].

The classical concept was that a PDE

$$F\left(x, u, Du, D^2u\right) = 0 \tag{4}$$

over a domain $x \in \Omega$ has a solution if we can find a function $u(x)$ continuous and differentiable over the entire domain such that x, u, Du, D^2u satisfy the above equation at every point. If a continuous function u is both a supersolution and a subsolution, then it is a viscosity solution of the PDE (4). Under the viscosity solution concept, u does not need to be everywhere differentiable [13].

2.1.5 Forward-Backward sweep method

Sometimes we can't solve the problem explicitly and Forward-Backward sweep method could be used to find the numerical solution.

We need an initial guess for the control vector before we start. Given an approximation of the control function, FBSM first solves the state 'forward' in time and then solves the adjoint 'backward' in time. We update the control vector by entering new values of state and adjoint into the characterization of the optimal control [9][11]. We could solve the above optimality system with FBSM by using Runge-Kutta method.Forward backward sweep method can be executed in MATLAB.

2.2 Differential equations

2.2.1 Ordinary differential equation

An ordinary differential equation (ODE) is a differential equation consisting of one or more functions with one independent variable and its derivatives. The term constant variable is contrasted with the term partial differential equation, which may involve more than one independent variable.

For example,F is a function of x, y, and derivatives of y. An equation of the form

$$F\left(x, y, y', \ldots, y^{(n-1)}\right) = y^{(n)} \tag{5}$$

is called an explicit ordinary differential equation of order n. More generally, an implicit ordinary differential equation of order n takes the form:

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0. \tag{6}$$

If y is a vector whose elements are functions; $y(x) = [y_1(x), y_2(x), ..., y_m(x)]$, and F is a vector-valued function of y and its derivatives, then the ODE becomes an explicit system of ordinary differential equations of order n and dimension m. We can display the ODE in column form

$$\begin{pmatrix} y_1^{(n)} \\ y_2^{(n)} \\ \vdots \\ y_m^{(n)} \end{pmatrix} = \begin{pmatrix} f_1\left(x, \mathbf{y}, \mathbf{y}', \mathbf{y}'', \ldots, \mathbf{y}^{(n-1)}\right) \\ f_2\left(x, \mathbf{y}, \mathbf{y}', \mathbf{y}'', \ldots, \mathbf{y}^{(n-1)}\right) \\ \vdots \\ f_m\left(x, \mathbf{y}, \mathbf{y}', \mathbf{y}'', \ldots, \mathbf{y}^{(n-1)}\right) \end{pmatrix} \tag{7}$$

2.2.2 Partial differential equations

A partial differential equation (PDE) is a kind of equation that imposes the relation between partial derivatives of a multi-variable function. A PDE is called linear if it is linear in the

variables and their derivatives. For example, u is a function of x and y, a second order linear partial differential equation is of the form:

$$a_1(x,y)u_{xx} + a_2(x,y)u_{xy} + a_3(x,y)u_{yx} + a_4(x,y)u_{yy} = f(x,y), \tag{8}$$

where a_i and f are functions of x and y. If f is zero then the linear PDE is homogeneous, otherwise it is inhomogeneous. If the coefficients of a PDE are functions of the independent variables and their lower-order derivatives, then it is called a quasi-linear PDE:

$$a_1\left(u_x, u_y, u, x, y\right) u_{xx} + a_2\left(u_x, u_y, u, x, y\right) u_{yy} = f(u_x, u_y, u, x, y). \tag{9}$$

Partial differential equations can be used to mathematically formulate problems involving functions of several variables, such as the Physics, Astronomy and Economy. In particular, Navier–Stokes equation is a class of partial differential equations which describe the motion of viscous fluid substances in physics [14]. In mathematical finance, the Black–Scholes equation is a partial differential equation (PDE) which is used to price a European call or European put under the Black–Scholes model [11]. Parabolic equations such as heat equation can help to construct Schrödinger equation for a free particle [3][14]. Some elliptic equations like Poisson's equation is of broad use in theoretical physics [2][6][7].Monge–Ampere equation is a non-linear equation which people may encounter in differential geometry and meteorology [5][12].

2.2.3 Stochastic differential equations

The most common form of stochastic differential equations in literature is an ordinary differential equation whose right-hand side is perturbed by a term dependent on the white noise variable [15]. In most cases, SDE is understood as the continuous-time limit of the corresponding stochastic difference equation. For example, this is a general case of linear SDE:

$$dX_t = \left(a(t)X_t + c(t)\right)dt + \left(b(t)X_t + d(t)\right)dW_t \tag{10}$$

$$X_t = \Phi_{t,t_0}\left(X_{t_0} + \int_{t_0}^{t} \Phi_{s,t_0}^{-1}(c(s) - b(s)d(s))ds + \int_{t_0}^{t} \Phi_{s,t_0}^{-1} d(s) dW_s\right) \tag{11}$$

where

$$\Phi_{t,t_0} = \exp\left(\int_{t_0}^{t}\left(a(s) - \frac{b^2(s)}{2}\right)ds + \int_{t_0}^{t} b(s)dW_s\right) \tag{12}$$

The formal mathematical definition of stochastic differential equations was first proposed by Kiyosi Itô in the 1940s and led to what we know today as Itô calculus [16]. Another structure was later presented by the Russian physicist Stratonovich, leading to the so-called Stranovich integral. Itô integrals and Stratonovich integrals are related but different, and the object and choice between them depend on the application being considered. Both Itô's integral and Stratonovich integrals are the stochastic generalizations of Riemann-Stieltjes integral in the analysis. The integrands and the integrators of both integrals are now stochastic processes and the result after integration is another stochastic process [14]. Conversion between Itô and Stratonovich integrals may be performed using the formula

$$\int_0^T f\left(W_t, t\right) \circ \mathrm{d}W_t = \frac{1}{2}\int_0^T \frac{\partial f}{\partial W}\left(W_t, t\right)\mathrm{d}t + \int_0^T f\left(W_t, t\right)\mathrm{d}W_t \tag{13}$$

2.3 Study objectives

2.3.1 Well-posedness and quantitative behavior

Well-posedness refers to a common schematic package of information about a PDE. To say that a PDE is well-posed, one must have: (i) an existence and uniqueness theorem, (ii) the solution's behavior changes continuously with the initial conditions [3]. Common well-posed problems include Dirichlet problem for Laplace's equation, and the heat equation with a given initial boundaries. These problems can be considered "natural" because they are modelled to simulate physical processes. Problems that are not well-posed are called ill-posed problems. The solution of an ill-posed problem is highly sensitive to changes in the initial condition which means that the system may suffer from numerical instability when solved with finite-precision [17]. Problems in complex nonlinear systems provide well-known examples of instability. If the problem is well-posed, it is likely to be solved on a computer using a stable algorithm [5][9].

2.3.2 Forward-Backward stochastic differential equations

Forward-Backward stochastic differential equation (FBDSE) is a new type of stochastic differential equations where the value of the final solution is prescribed at the final point of

each time interval, but the solution is still needed in each time the function a function the underlying Brownian motion [16]. It provides a viable tool to solve certain stochastic control problems. Its applications have been found in many applied sciences, especially in quantitative finance because they can effectively describe the dynamic price behavior of financial derivatives, portfolios, and other financial instruments that are stochastic in nature [17]. We take the following equations as an example:

$$-dY_t = f\left(t, \omega, X_t, Y_t, Z_t\right) dt - Z_t dW_t, \quad Y_T = \xi \tag{14}$$

$$dX_t = g\left(t, \omega, X_t, Y_t, Z_t\right) dt - \sigma\left(t, \omega, X_t, Y_t, Z_t\right) dW_t, \quad X_0 = x_0, \tag{15}$$

where $Y_t, X_t \in \mathbb{R}^d, \xi$ is $\mathcal{F}_{T^-\text{measurable, and}}$ f is $\mathcal{P} \otimes \mathcal{B}^d \otimes \mathcal{B}^{d\times n}$-measurable. $\mathcal{P}$ is the predictable σ-algebra, and $\mathcal{B}^d$ is the Borel σ-algebra on $\mathbb{R}^d$. f is called the generator of the BSDE.

This kind of equation is the nonlinear extension of a linear equation. Scholars also discovered that these FBSDEs provided some probabilistic formulas for solutions of semi-linear PDEs, generalizing the Feynmann-Kac formula for second-order linear PDEs [17]. I want to focus on the existence of an adapted solution and try to apply the FBDSE to solve stochastic control problems.

3 Ito's lemma

What is a financial derivative?

- Financial derivatives are instruments whose value depends on, or derives from, the value of other, more basic, underlying securities.(example:future,option)

Why are they important?

- Options give investors the opportunity to buy or sell a security at a fixed price in the future. All of these examples of derivatives provide some kind of protection against future price movements; they protect the investors from uncertainty and risk. Because of the insurance effect of derivatives, it is essential to price derivatives properly, relative to the level of insurance they provide to the investors, so that people have the incentive to trade derivatives.[8]

Fix a point $x_0 \in R^n$ and consider the following equation:

$$\begin{cases} \dot{x(t)} = b(x(t))(t > 0) \\ x(0) = x_0 \end{cases}$$

In many applications, however, the experimentally measured trajectories of systems modeled by (ODE) are not in fact as smooth as predicted. So the original system may need some modifications.We need to include the random effects disturbing the system:

$$\begin{cases} \dot{x(t)} = b(x(t)) + B(x(t))W(t)(t > 0) \\ x(0) = x_0 \end{cases}$$

This modification gives us more problems:

- How to determine the unknown term $W(t)$
- show that x(t) has a solution and discuss the uniqueness of x(t)

3.1 Stochastic process

Definition 3.1. *A Stochastic Process is a collection of random variables x_t indexed by time.*

Definition 3.2. *A Brownian Motion (Wiener Process) with variance σ^2 is a stochastic process X_t taking values in the real numbers satisfying:*

- $X_0 = 0$*;*
- *For any $a < b \leq c < d, X_b - X_a$ and $X_d - X_c$ are independent;*
- *For any $a < b, X_b - X_a$ follows a Normal Distribution with mean 0 and variance $(b-a)\sigma^2$;*
- *The paths are continuous.*

A random variable X following a Brownian Motion with mean μ and variation σ^2 satisfies $dx = \mu dt + \sigma dW$

Definition 3.3. *An Ito process is a generalized Wiener process in which the parameters μ and σ are functions of the values of the underlying variable x and time t. An Ito process can therefore be written as*

$$dx = \mu(x,t)dt + \sigma(x,t)dW$$

Partition time interval $[0, T]$ into N periods, each of length $\Delta t = \frac{T}{N}$ $t_n = n\Delta t$

$X_{t_{n+1}} = X_{t_n} + \mu_{t_n}\Delta t + \sigma_{t_n}\Delta W_{t_n}$

- drift μ
- volatility σ
- fluctuations $\Delta W_{t_n} = W_{t_{n+1}} - W_{t_n} \sim N(0, \Delta t)$
- Summing the increments,

$$X_T = X_0 + \sum_{n=0}^{N-1} \mu_{t_n}\Delta t + \sum_{n=0}^{N-1} \sigma_{t_n}\Delta W_{t_n}$$

- Continuous-time analogue as $N \to \infty$

$$X_T \overset{?}{\to} X_0 + \int_0^T \mu_t \, \mathrm{d}t + \int_0^T \sigma_t \, \mathrm{d}W_t$$

Then how to calculate $\int_0^T \sigma_t \, \mathrm{d}W_t$?

3.2 Ito's stochastic integral

Definition 3.4.
- *We denote by* $\mathbb{L}^2(0, T)$ *the space of all real-valued, progressively measurable stochastic processes* $G(\cdot)$ *such that*

$$E\left(\int_0^T G^2 dt\right) < \infty$$

- *Likewise,* $\mathbb{L}^1(0, T)$ *is the space of all real-valued, progressively measurable processes* $F(\cdot)$ *such that*

$$E\left(\int_0^T |F| dt\right) < \infty$$

Definition 3.5. *A process* $G \in \mathbb{L}^2(0, T)$ *is called a step process if there exists a partition* $P = \{0 = t_0 < t_1 < \cdots < t_m = T\}$ *such that*

$$G(t) \equiv G_k \quad \textit{for} \quad t_k \leq t < t_{k+1} \quad (k = 0, \ldots, m-1)$$

Definition 3.6. *Let* $G \in \mathbb{L}^2(0, T)$ *be a step process, as above. Then*

$$\int_0^T G dW := \sum_{k=0}^{m-1} G_k \left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)$$

is the Itô stochastic integral of G *on the interval* $(0, T)$*.[18]*

3.3 Stochastic Calculus and Ito's formula

Theorem 3.1. *[12] Suppose f is a C^2 function and B_t is a standard Brownian motion. Then, for every t,*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\, dB_t + \frac{1}{2}\int_0^t f''(B_s)\, ds$$

or,

$$df(B_t) = f'(B_s)\, dB_t + \frac{1}{2} f''(B_s)\, ds$$

Proof. First, we want to show that for $t = 1$

$$f(B_1) = f(B_0) + \int_0^1 f'(B_s)\, dB_t + \frac{1}{2}\int_0^1 f''(B_s)\, ds$$

For any $n \in \mathbb{N}$

$$f(B_1) - f(B_0) = \sum_{j=1}^{n} \left[f\left(B_{j/n}\right) - f\left(B_{(j-1)/n}\right) \right]$$

Using Taylor's Approximation, we get

$$f\left(B_{j/n}\right) - f\left(B_{(j-1)/n}\right) = f'\left(B_{(j-1)/n}\right)\Delta_{j,n} + \frac{1}{2} f'' B_{(j-1)/n} \Bigg) \Delta_{j,n}^2 + o\left(\Delta_{j,n}^2\right)$$

where $\Delta_{j,n} = B_{j/n} - B_{(j-1)/n}$

Therefore,

$$\begin{aligned} f(B_1) - f(B_0) &= \lim_{n\to\infty} \sum_{j=1}^{n} f'\left(B_{(j-1)/n}\right) \left[f\left(B_{j/n}\right) - f\left(B_{(j-1)/n}\right) \right] \\ &+ \frac{1}{2} \lim_{n\to\infty} \sum_{j=1}^{n} f''\left(B_{(j-1)/n}\right) \left[f\left(B_{j/n}\right) - f\left(B_{(j-1)/n}\right) \right]^2 \\ &+ \lim_{n\to\infty} \sum_{j=1}^{n} o\left(\left[f\left(B_{j/n}\right) - f\left(B_{(j-1)/n}\right) \right]^2 \right) \end{aligned}$$

We can see that

$$\lim_{n\to\infty} \sum_{j=1}^{n} f'\left(B_{(j-1)/n}\right) \left[f\left(B_{j/n}\right) - f\left(B_{(j-1)/n}\right) \right] = \int_0^1 f'(b_t)\, dB_t$$

.

To prove $(\mathrm{d}B_t)^2 = \mathrm{d}t$, we try to prove

$$\int_0^T (\,\mathrm{d}B_t)^2 = \lim_{N\to\infty} \sum_{n=0}^{N-1} (\Delta B_{t_n})^2 \overset{\mathcal{L}^2}{=} T = \int_0^T \mathrm{d}t$$

$$E\left[\sum_{n=0}^{N-1}\left(\Delta B_{t_n}\right)^2\right]=\sum_{n=0}^{N-1}E\left(\Delta B_{t_n}\right)^2=\sum_{n=0}^{N-1}\Delta t=T$$

$$E\left[\sum_{n=0}^{N-1}\left(\Delta B_{t_n}\right)^2-T\right]^2=\operatorname{Var}\left[\sum_{n=0}^{N-1}\left(\Delta B_{t_n}\right)^2\right]=\sum_{n=0}^{N-1}\operatorname{Var}\left(\Delta B_{t_n}\right)^2$$

$$=\sum_{n=0}^{N-1}\left(E(\Delta B)^4-\left[E(\Delta B)^2\right]^2\right)=\sum_{n=0}^{N-1}\left(3(\Delta t)^2-(\Delta t)^2\right)=\frac{2T^2}{N}\to 0$$

as $N\to\infty$

The transformation is straightforward using the fact that $\left(\mathrm{d}B_t\right)^2=\mathrm{d}t$. We can drop terms $\ll \mathrm{d}t$

$$\begin{aligned}\int_0^T(\,\mathrm{d}t)^p&=\lim_{N\to\infty}\sum_{n=0}^{N-1}(\Delta t)^p\\&=\lim_{N\to\infty}N\left(\frac{T}{N}\right)^p\\&=0\text{ as }N\to\infty\text{ if }p>1\end{aligned}$$

For any other t, because of the Markov Property of Brownian Motion, partitioning the interval $[0,t]$ as we did for $[0,1]$, the proof still holds. □

Example 3.1. *Use Ito's Lemma, write $Z_t=W_t^2$ as a sum of drift and diffusion terms.*

$$Z_t=f\left(W_t\right)\quad\textit{with}\quad\mu_t=0,\sigma_t=1,$$

$$\begin{aligned}\mathrm{d}Z_t&=\mathrm{d}f\left(W_t\right)\\&=f'\left(W_t\right)\mathrm{d}W_t+\frac{1}{2}f''\left(W_t\right)\left(\mathrm{d}W_t\right)^2\\&=2W_t\,\mathrm{d}W_t+\frac{1}{2}2\left(\,\mathrm{d}W_t\right)^2\\&=2W_t\,\mathrm{d}W_t+\mathrm{d}t\end{aligned}$$

Example 3.2. $\int_0^T W_t\,\mathrm{d}W_t$ *with approximating sums* $\sum_{n=0}^{N-1}W_{t_n}\Delta W_{t_n}$

$$\begin{aligned}\sum_{n=0}^{N-1}W_{t_n}\left(W_{t_{n+1}}-W_{t_n}\right)&=\sum_{n=0}^{N-1}\left[\frac{1}{2}\left(W_{t_{n+1}}^2-W_{t_n}^2\right)-\frac{1}{2}\left(W_{t_{n+1}}-W_{t_n}\right)^2\right]\\&=\frac{1}{2}W_T^2-\frac{1}{2}\sum_{n=0}^{N-1}\left(W_{t_{n+1}}-W_{t_n}\right)^2\\&\xrightarrow{\mathcal{L}^2}\frac{1}{2}W_T^2-\frac{1}{2}T\quad\textit{as }N\to\infty\end{aligned}$$

Definition 3.7. *If $(\Omega, \mathcal{F}, P)$ is a given probability space, then a function $X : \Omega \to \mathbb{R}^n$ is called $\mathcal{F}$-measurable if for all open sets $U \in \mathbb{R}^n$ (or, equivalently, for all Borel sets $U \subseteq \mathbb{R}^n$)*

$$X^{-1}(U) := \{\omega \in \Omega : X(\omega) \in U\} \in \mathcal{F}[4]$$

Definition 3.8. *We denote by $\mathbb{L}^2(0,T)$ the space of all real-valued, progressively measurable stochastic processes $G(\cdot)$ such that*

$$E\left(\int_0^T G^2 dt\right) < \infty[11]$$

Definition 3.9. *We denote by $\mathbb{L}^2(0,T)$ the space of all real-valued, progressively measurable stochastic processes $G(\cdot)$ such that*

$$E\left(\int_0^T G^2 dt\right) < \infty[6]$$

Definition 3.10. *Let $G \in \mathbb{L}^2(0,T)$ be a step process, as above. Then*

$$\int_0^T G dW := \sum_{k=0}^{m-1} G_k \left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)$$

is the Itô stochastic integral of G on the interval $(0,T)$. Note carefully that this is a random variable.[19]

Lemma 3.2. *Let $G \in \mathbb{L}^2(0,T)$ be a step process, as above. Then*

$$\int_0^T G dW := \sum_{k=0}^{m-1} G_k \left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)$$

is the Itô stochastic integral of G on the interval $(0,T)$. Note carefully that this is a random variable.[8]

Proof. The first assertion is easy to check. Suppose next $G(t) \equiv G_k$ for $t_k \leq t < t_{k+1}$. Then

$$E\left(\int_0^T G dW\right) = \sum_{k=0}^{m-1} E\left(G_k \left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)\right)$$

As G_k is independent of $W\left(t_{k+1}\right) - W\left(t_k\right)$. Hence

$$E\left(G_k \left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)\right) = E\left(G_k\right) \underbrace{E\left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)}_{=0}$$

Furthermore, $E\left(\left(\int_0^T G dW\right)^2\right) = \sum_{k,j=1}^{m-1} E\left(G_k G_j \left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)\left(W\left(t_{j+1}\right) - W\left(t_j\right)\right)\right)$

Now if $j < k$, then $W(t_{k+1}) - W(t_k)$ is independent of $G_k G_j (W(t_{j+1}) - W(t_j))$. Thus

$$\begin{aligned} &E\left(G_k G_j \left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)\left(W\left(t_{j+1}\right) - W\left(t_j\right)\right)\right) \\ &= \underbrace{E\left(G_k G_j \left(W\left(t_{j+1}\right) - W\left(t_j\right)\right)\right)}_{<\infty} \underbrace{E\left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)}_{=0} \end{aligned}$$

Consequently

$$\begin{aligned} E\left(\left(\int_0^T G dW\right)^2\right) &= \sum_{k=0}^{m-1} E\left(G_k^2 \left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)^2\right) \\ &= \sum_{k=0}^{m-1} E\left(G_k^2\right) \underbrace{E\left(\left(W\left(t_{k+1}\right) - W\left(t_k\right)\right)^2\right)}_{=t_{k+1}-t_k} \\ &= E\left(\int_0^T G^2 dt\right) \end{aligned}$$

□

Lemma 3.3. *If $G \in \mathbb{L}^2(0,T)$, there exists a sequence of bounded step processes $G^n \in \mathbb{L}^2(0,T)$ such that*

$$E\left(\int_0^T \left|G - G^n\right|^2 dt\right) \to 0$$

Proof. We omit the proof, but the idea is this: if $t \mapsto G(t,\omega)$ is continuous for almost every ω, we can set

$$G^n(t) := G\left(\frac{k}{n}\right) \text{ for } \frac{k}{n} \le t < \frac{k+1}{n}, k = 0, \ldots, [nT]$$

For a general $G \in \mathbb{L}^2(0,T)$, define

$$G^m(t) := \int_0^t m e^{m(s-t)} G(s) ds$$

Then $G^m \in \mathbb{L}^2(0,T), t \mapsto G^m(t,\omega)$ is continuous for a.e. ω, and

$$\int_0^T \left|G^m - G\right|^2 dt \to 0 \text{ a.s.}$$

$$E\left(\left(\int_0^T G^n - G^m dW\right)^2\right) = E\left(\int_0^T \left(G^n - G^m\right)^2 dt\right) \to 0 \quad \text{as } n, m \to \infty$$

and so the limit

$$\int_0^T G dW := \lim_{n\to\infty} \int_0^T G^n dW$$

exists in $L^2(\Omega)$ □

3.4 Ito's formula

Definition 3.11. *Suppose that $X(\cdot)$ is a real-valued stochastic process satisfying*

$$X(r) = X(s) + \int_s^r F dt + \int_s^r G dW$$

for some $F \in \mathbb{L}^1(0,T), G \in \mathbb{L}^2(0,T)$ and all times $0 \leq s \leq r \leq T$. We say that $X(\cdot)$ has the stochastic differential

$$dX = Fdt + GdW$$

for $0 \leq t \leq T$.

Theorem 3.4. *Suppose that $X(\cdot)$ has a stochastic differential*

$$dX = Fdt + GdW$$

for $F \in \mathbb{L}^1(0,T), G \in \mathbb{L}^2(0,T)$. Assume $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ is continuous and that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ exist and are continuous. Set

$$Y(t) := u(X(t), t)$$

Then Y has the stochastic differential

$$\begin{aligned} dY &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2 dt \\ &= \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}F + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2\right)dt + \frac{\partial u}{\partial x}GdW \text{[8]} \end{aligned}$$

Lemma 3.5. *We have (i)*

$$d\left(W^2\right) = 2WdW + dt$$

and (ii)

$$d(tW) = Wdt + tdW$$

Theorem 3.6. *Suppose*

$$\begin{cases} dX_1 = F_1 dt + G_1 dW \\ dX_2 = F_2 dt + G_2 dW \end{cases} \quad (0 \leq l \leq T)$$

for $F_i \in \mathbb{L}^1(0,T), G_i \in \mathbb{L}^2(0,T)(i = 1,2)$. Then

$$d\left(X_1 X_2\right) = X_2 dX_1 + X_1 dX_2 + G_1 G_2 dt$$

Proof. We have already established formula (i). To verify (ii), note that

$$\int_0^r tdW = \lim_{n\to\infty} \sum_{k=0}^{m_n-1} t_k^n \left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right)$$

where $P^n = \left\{0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = r\right\}$ is a sequence of partitions of $[0, r]$, with $|P^n| \to 0$.

The limit above is taken in $L^2(\Omega)$ Similarly, since $t \mapsto W(t)$ is continuous a.s.,

$$\int_0^r Wdt = \lim_{n\to\infty} \sum_{k=0}^{m_n-1} W\left(t_{k+1}^n\right)\left(t_{k+1}^n - t_k^n\right)$$

since for almost every ω the sum is an ordinary Riemann sum approximation and for this we can take the right-hand endpoint t_{k+1}^n at which to evaluate the continuous integrand. We add these formulas to obtain

$$\int_0^r tdW + \int_0^r Wdt = rW(r)$$

These integral identities for all $r \geq 0$ are abbreviated $d(tW) = tdW + Wdt$.

1. Choose $0 \leq r \leq T$. First of all, assume for simplicity that $X_1(0) = X_2(0) = 0, F_i(t) \equiv F_i, G_i(t) \equiv G_i$ where F_i, G_i are time-independent, $\mathcal{F}(0)$-measurable random variables $(i = 1, 2)$. Then

$$X_i(t) = F_i t + G_i W(t) \quad (t \geq 0, i = 1, 2)$$

Thus

$$\begin{aligned}
&\int_0^r X_2 dX_1 + X_1 dX_2 + G_1 G_2 dt \\
&= \int_0^r X_1 F_2 + X_2 F_1 dt + \int_0^r X_1 G_2 + X_2 G_1 dW \\
&\quad + \int_0^r G_1 G_2 dt \\
&= \int_0^r \left(F_1 t + G_1 W\right) F_2 + \left(F_2 t + G_2 W\right) F_1 dt \\
&\quad + \int_0^r \left(F_1 t + G_1 W\right) G_2 + \left(F_2 t + G_2 W\right) G_1 dW + G_1 G_2 r \\
&= F_1 F_2 r^2 + \left(G_1 F_2 + G_2 F_1\right)\left[\int_0^r Wdt + \int_0^r tdW\right] \\
&\quad + 2G_1 G_2 \int_0^r WdW + G_1 G_2 r.
\end{aligned}$$

We now use the Lemma above to compute $2\int_0^r WdW = W^2(r) - r$ and $\int_0^r Wdt + \int_0^r tdW = rW(r)$. Employing these identities, we deduce:

$$\int_0^r X_2 dX_1 + X_1 dX_2 + G_1 G_2 dt = F_1 F_2 r^2 + \left(G_1 F_2 + G_2 F_1\right) rW(r) + G_1 G_2 W^2(r) = X_1(r) X_2(r)$$

2. If F_i, G_i are step processes, we apply Step 1 on each subinterval $[t_k, t_{k+1})$ on which F_i and G_i are constant random variables, and add the resulting integral expressions.

3. In the general situation, we select step processes $F_i^n \in \mathbb{L}^1(0,T), G_i^n \in \mathbb{L}^2(0,T)$, with

$$\begin{aligned} &E\left(\int_0^T |F_i^n - F_i|\, dt\right) \to 0 \\ &E\left(\int_0^T (G_i^n - G_i)^2\, dt\right) \to 0 \end{aligned} \qquad \text{as } n \to \infty, i = 1,2$$

Define

$$X_i^n(t) := X_i(0) + \int_0^t F_i^n ds + \int_0^t G_i^n dW \quad (i = 1,2)$$

We apply Step 2 to $X_i^n(\cdot)$ on (s,r) and pass to limits, to obtain the formula

$$X_1(r)X_2(r) = X_1(s)X_2(s) + \int_s^r X_1 dX_2 + X_2 dX_1 + G_1 G_2 dt$$

Suppose $dX = Fdt + GdW$, with $F \in \mathbb{L}^1(0,T), G \in \mathbb{L}^2(0,T)$ 1. We start with the case $u(x) = x^m, m = 0, 1, \ldots$, and first of all claim that

$$d\left(X^m\right) = mX^{m-1}dX + \frac{1}{2}m(m-1)X^{m-2}G^2 dt$$

This is clear for $m = 0, 1$, and the case $m = 2$ follows from the Itô product formula. Now assume the stated formula for $m-1$:

$$\begin{aligned} d\left(X^{m-1}\right) &= (m-1)X^{m-2}dX + \frac{1}{2}(m-1)(m-2)X^{m-3}G^2 dt \\ &= (m-1)X^{m-2}(Fdt + GdW) + \frac{1}{2}(m-1)(m-2)X^{m-3}G^2 dt \end{aligned}$$

and we prove it for m :

$$\begin{aligned} d\left(X^m\right) =& d\left(XX^{m-1}\right) \\ =& Xd\left(X^{m-1}\right) + X^{m-1}dX + (m-1)X^{m-2}G^2 dt \\ & (\text{ by the product rule }) \\ =& X\left((m-1)X^{m-2}dX + \frac{1}{2}(m-1)(m-2)X^{m-3}G^2 dt\right) \\ & + (m-1)X^{m-2}G^2 dt + X^{m-1}dX \\ =& mX^{m-2}dX + \frac{1}{2}m(m-1)X^{m-2}G^2 dt \end{aligned}$$

Since Itô's formula thus holds for the functions $u(x) = x^m, m = 0, 1, \ldots$ and since the operator " d " is linear, Itô's formula is valid for all polynomials u in the variable x.

2. Suppose now $u(x,t) = f(x)g(t)$, where f and g are polynomials. Then

$$\begin{aligned} d(u(X,t)) &= d(f(X)g) \\ &= f(X)dg + gdf(X) \\ &= f(X)g'dt + g\left[f'(X)dX + \frac{1}{2}f''(X)G^2dt\right] \\ &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2dt \end{aligned}$$

This calculation confirms Itô's formula for $u(x,t) = f(x)g(t)$, where f and g are polynomials. Thus it is true for any function u having the form $u(x,t) = \sum_{i=1}^{m} f^i(x)g^i(t)$ where f^i and g^i polynomials. That is, Itô's formula is valid for all polynomial functions u of the variables x, t.

3. Given u as in Itô's formula, there exists a sequence of polynomials u^n such that

$$\begin{gathered} u^n \to u, \frac{\partial u^n}{\partial t} \to \frac{\partial u}{\partial t} \\ \frac{\partial u^n}{\partial x} \to \frac{\partial u}{\partial x}, \frac{\partial^2 u^2}{\partial x^2} \to \frac{\partial^2 u}{\partial x^2} \end{gathered}$$

uniformly on compact subsets of $\mathbb{R} \times [0,T]$. From Step 2 , we know that for all $0 \leq r \leq T$,

$$\begin{aligned} u^n(X(r),r) - u^n(X(0),0) = \int_0^r \tfrac{\partial u^n}{\partial t} + \tfrac{\partial u^n}{\partial x}F + \tfrac{1}{2}\tfrac{\partial^2 u^n}{\partial x^2}G^2dt \\ + \int_0^r \tfrac{\partial u^n}{\partial x}GdW \text{ almost surely} \end{aligned}$$

the argument of the partial derivatives of u^n is $(X(t),t)$ We may pass to limits as $n \to \infty$ in this expression, thereby proving Ito's formula in general. [2] □

4 Stochastic equation

- Remember from the last section that we have introduced the stochastic differential equations of the form

$$\begin{cases} d\mathbf{X} = \mathbf{b}(\mathbf{X},t)dt + \mathbf{B}(\mathbf{X},t)d\mathbf{W} \\ \mathbf{X}(0) = \mathbf{X}_0 \end{cases}$$

which also means

$$\mathbf{X}(t) = \mathbf{X}_0 + \int_0^t \mathbf{b}(\mathbf{X},s)ds + \int_0^t \mathbf{B}(\mathbf{X},s)d\mathbf{W}$$

for all times $t \geq 0$.

- But before we can study and solve such an integral equation, we must first define

$$\int_0^T \mathbf{G} d\mathbf{W}$$

for some wide class of stochastic processes $\mathbf{G}$, so that the right-hand side at least makes sense.

- Observe also that this is not at all obvious. $\int_0^T \mathbf{G} d\mathbf{W}$ simply cannot be understood as an ordinary integral.

Definition 4.1. *Suppose $g : [0,1] \to \mathbb{R}$ is continuously differentiable, with $g(0) = g(1) = 0$. Then let us define*

$$\int_0^1 g dW := -\int_0^1 g' W dt$$

Remark g is an ordinary, deterministic function and not a stochastic process.

Lemma 4.1.
- $E\left(\int_0^1 g dW\right) = 0.$
- $E\left(\left(\int_0^1 g dW\right)^2\right) = \int_0^1 g^2 dt$

Proof.
- $E\left(\int_0^1 g dW\right) = 0.$
- $E\left(\left(\int_0^1 g dW\right)^2\right) = \int_0^1 g^2 dt$
- To confirm (ii), we calculate

$$\begin{aligned}
E\left(\left(\int_0^1 g dW\right)^2\right) &= E\left(\int_0^1 g'(t)W(t)dt \int_0^1 g'(s)W(s)ds\right) \\
&= \int_0^1 \int_0^1 g'(t)g'(s) \underbrace{E(W(t)W(s))}_{=t\wedge s} ds dt \\
&= \int_0^1 g'(t)\left(\int_0^t s g'(s)ds + \int_t^1 t g'(s)ds\right) dt \\
&= \int_0^1 g'(t)\left(tg(t) - \int_0^t g ds - tg(t)\right) dt \\
&= \int_0^1 g'(t)\left(-\int_0^t g ds\right) dt = \int_0^1 g^2 dt
\end{aligned}$$

□

Lemma 4.2. *Suppose $W(\cdot)$ is a one-dimensional Brownian motion. Then*

$$E(W(t)W(s)) = t \wedge s = \min\{s,t\} \quad \textit{for } t \geq 0, s \geq 0 [20]$$

Proof. Assume $t \geq s \geq 0$. Then

$$\begin{aligned} E(W(t)W(s)) &= E((W(s) + W(t) - W(s))W(s)) \\ &= E\left(W^2(s)\right) + E((W(t) - W(s))W(s)) \\ &= s + \underbrace{E(W(t) - W(s))}_{=0}\underbrace{E(W(s))}_{=0} \\ &= s = t \wedge s, \end{aligned}$$

□

Suppose now $g \in L^2(0,1)$. We can take a sequence of C^1 functions g_n, as above, such that $\int_0^1 (g_n - g)^2\, dt \to 0$. In view of property (ii),

$$E\left(\left(\int_0^1 g_m dW - \int_0^1 g_n dW\right)^2\right) = \int_0^1 (g_m - g_n)^2\, dt$$

and therefore $\left\{\int_0^1 g_n dW\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$. Consequently we can define

$$\int_0^1 g dW := \lim_{n\to\infty} \int_0^1 g_n dW$$

But the definition is not for stochastic processes. If we wish to define the integral,

$$\int_0^t \mathbf{B}(\mathbf{X}, s) d\mathbf{W}$$

we must devise a definition for a wider class of integrands.

4.1 Construction by Riemann Sums

To continue our study of stochastic integrals, let us think about what might be an appropriate definition for

$$\int_0^T W dW = ?$$

where $W(\cdot)$ is a 1-dimensional Brownian motion. A reasonable procedure is to construct a Riemann sum approximation.

Definition 4.2. • *If* $[0,T]$ *is an interval, a partition* P *of* $[0,T]$ *is a finite collection of points in* $[0,T]$:

$$P := \{0 = t_0 < t_1 < \cdots < t_m = T\}$$

- *Let the mesh size of* P *be* $|P| := \max_{0\le k\le m-1} |t_{k+1} - t_k|$.
- *For fixed* $0 \le \lambda \le 1$ *and* P *a given partition of* $[0,T]$*, set*

$$\tau_k := (1-\lambda)t_k + \lambda t_{k+1} \quad (k = 0, \ldots, m-1)$$

For such a partition P *and for* $0 \le \lambda \le 1$*, we define*

$$R = R(P,\lambda) := \sum_{k=0}^{m-1} W(\tau_k)\left(W(t_{k+1}) - W(t_k)\right)$$

This is the corresponding Riemann sum approximation of $\int_0^T W dW$. The key question is this: what happens if $|P| \to 0$, with λ fixed?

Lemma 4.3. *Let* $[a,b]$ *be an interval in* $[0,\infty)$*, and suppose*

$$P^n := \left\{a = t_0^n < t_1^n < \cdots < t_{m_n}^n = b\right\}$$

are partitions of $[a,b]$*, with* $|P^n| \to 0$ *as* $n \to \infty$*. Then*

$$\sum_{k=0}^{m_n-1} \left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right)^2 \to b - a$$

in $L^2(\Omega)$ *as* $n \to \infty$ *[5]*

Proof. Set $Q_n := \sum_{k=0}^{m_n-1} \left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right)^2$. Then

$$Q_n - (b-a) = \sum_{k=0}^{m_n-1} \left(\left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right)^2 - \left(t_{k+1}^n - t_k^n\right)\right)$$

Hence

$$E\left(\left(Q_n - (b-a)\right)^2\right) = \sum_{k=0}^{m_n-1}\sum_{j=0}^{m_n-1} E\left(\left[\left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right)^2 - \left(t_{k+1}^n - t_k^n\right)\right]\right.$$

$$\left.\left[\left(W\left(t_{j+1}^n\right) - W\left(t_j^n\right)\right)^2 - \left(t_{j+1}^n - t_j^n\right)\right]\right)$$

For $k \ne j$, the term in the double sum is

$$E\left(\left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right)^2 - \left(t_{k+1}^n - t_k^n\right)\right) E(\cdots)$$

As $W(t) - W(s)$ is $N(0, t-s)$ for all $t \geq s \geq 0$. Hence

$$E\left(\left(Q_n - (b-a)\right)^2\right) = \sum_{k=0}^{m_n-1} E\left(\left(Y_k^2 - 1\right)^2 \left(t_{k+1}^n - t_k^n\right)^2\right)$$

where

$$Y_k = Y_k^n := \frac{W\left(t_{k+1}^n\right) - W\left(t_k^n\right)}{\sqrt{t_{k+1}^n - t_k^n}} \quad \text{is } N(0,1)$$

Therefore for some constant C we have

$$\begin{aligned} E\left(\left(Q_n - (a-b)\right)^2\right) &\leq C \sum_{k=0}^{m_n-1} \left(t_{k+1}^n - t_k^n\right)^2 \\ &\leq C\left|P^n\right|(b-a) \to 0, \quad \text{as } n \to \infty \end{aligned}$$

□

Lemma 4.4. *If P^n denotes a partition of $[0,T]$ and $0 \leq \lambda \leq 1$ is fixed, define*

$$R_n := \sum_{k=0}^{m_n-1} W\left(\tau_k^n\right)\left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right)$$

Then

$$\lim_{n\to\infty} R_n = \frac{W(T)^2}{2} + \left(\lambda - \frac{1}{2}\right)T$$

the limit taken in $L^2(\Omega)$. That is,

$$E\left(\left(R_n - \frac{W(T)^2}{2} - \left(\lambda - \frac{1}{2}\right)T\right)^2\right) \to 0$$

Proof. We have

$$\begin{aligned} R_n &:= \sum_{k=0}^{m_n-1} W\left(\tau_k^n\right)\left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right) \\ &= \frac{W^2(T)}{2} - \underbrace{\frac{1}{2}\sum_{k=0}^{m_n-1}\left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right)^2}_{=:A} \\ &\quad + \underbrace{\sum_{k=0}^{m_n-1}\left(W\left(\tau_k^n\right) - W\left(t_k^n\right)\right)^2}_{=:B} \\ &\quad + \underbrace{\sum_{k=0}^{m_n-1}\left(W\left(t_{k+1}^n\right) - W\left(\tau_k^n\right)\right)\left(W\left(\tau_k^n\right) - W\left(t_k^n\right)\right)}_{=:C} \end{aligned}$$

We have

$$\begin{aligned} R_n :=& \sum_{k=0}^{m_n-1} W(\tau_k^n)\left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right) \\ =& \frac{W^2(T)}{2} - \underbrace{\frac{1}{2}\sum_{k=0}^{m_n-1}\left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right)^2}_{=:A} \\ &+ \underbrace{\sum_{k=0}^{m_n-1}\left(W(\tau_k^n) - W\left(t_k^n\right)\right)^2}_{=:B} \\ &+ \underbrace{\sum_{k=0}^{m_n-1}\left(W\left(t_{k+1}^n\right) - W(\tau_k^n)\right)\left(W(\tau_k^n) - W\left(t_k^n\right)\right)}_{=:C} \end{aligned}$$

□

It turns out that Itô's definition of $\int_0^T W dW$ corresponds to the choice $\lambda = 0$. That is,

$$\int_0^T W dW = \frac{W^2(T)}{2} - \frac{T}{2}$$

and, more generally,

$$\int_s^r W dW = \frac{W^2(r) - W^2(s)}{2} - \frac{(r-s)}{2} \quad \text{for all } r \geq s \geq 0$$

An alternative definition, takes $\lambda = \frac{1}{2}$; so that

$$\int_0^T W \circ dW = \frac{W^2(T)}{2} \quad \text{(Stratonovich integral)}$$

4.2 Solutions of Stochastic Differential Equations

Let us first suppose $b : \mathbb{R} \to \mathbb{R}$ is C^1, with $|b'| \leq L$ for some constant L, and try to solve the one-dimensional stochastic differential equation

$$\begin{cases} dX = b(X)dt + dW \\ X(0) = x \end{cases}$$

where $x \in \mathbb{R}$

Now the SDE means

$$X(t) = x + \int_0^t b(X)ds + W(t)$$

for all times $t \geq 0$, and this formulation suggests that we try a successive approximation method to construct a solution. So define $X^0(t) \equiv x$, and then

$$X^{n+1}(t) := x + \int_0^t b\left(X^n\right) ds + W(t) \quad (t \geq 0)$$

for $n = 0, 1, \ldots$ Next write

$$D^n(t) := \max_{0 \leq s \leq t} \left| X^{n+1}(s) - X^n(s) \right| \quad (n = 0, \ldots)$$

and notice that for a given continuous sample path of the Brownian motion, we have

$$D^0(t) = \max_{0 \leq s \leq t} \left| \int_0^s b(x)dr + W(s) \right| \leq C$$

for all times $0 \leq t \leq T$, where C depends on ω.

We now claim that

$$D^n(t) \leq C \frac{L^n}{n!} t^n$$

for $n = 0, 1, \ldots, 0 \leq t \leq T$. To see this note that

$$\begin{aligned} D^n(t) &= \max_{0 \leq s \leq t} \left| \int_0^s b\left(X^n(r)\right) - b\left(X^{n-1}(r)\right) dr \right| \\ &\leq L \int_0^t D^{n-1}(s) ds \\ &\leq L \int_0^t C \frac{L^{n-1} s^{n-1}}{(n-1)!} ds \quad \text{by the induction assumption} \\ &= C \frac{L^n t^n}{n!} \end{aligned}$$

In view of the claim, for $m \geq n$ we have

$$\max_{0 \leq t \leq T} |X^m(t) - X^n(t)| \leq C \sum_{k=n}^{\infty} \frac{L^k T^k}{k!} \to 0 \quad \text{as } n \to \infty [7]$$

Remark Thus for almost every ω, $X^n(\cdot)$ converges uniformly for $0 \leq t \leq T$ to a limit process $X(\cdot)$ which solves the system.

4.2.1 Solving SDE by changing variables

Given a general one-dimensional SDE of the form

$$\begin{cases} dX = b(X)dt + \sigma(X)dW \\ X(0) = x \end{cases}$$

let us first solve

$$\begin{cases} dY = f(Y)dt + dW \\ Y(0) = y \end{cases}$$

where f will be selected later, and try to find a function u such that

$$X := u(Y)$$

solves our SDE.

Note that we can in principle at least solve, according to the previous example. Assuming for the moment u and f are known, we compute using Itô's formula that

$$\begin{aligned} dX &= u'(Y)dY + \frac{1}{2}u''(Y)dt \\ &= \left[u'f + \frac{1}{2}u''\right]dt + u'dW \end{aligned}$$

Thus $X(\cdot)$ solves (8) provided

$$\begin{cases} u'(Y) = \sigma(X) = \sigma(u(Y)) \\ u'(Y)f(Y) + \frac{1}{2}u''(Y) = b(X) = b(u(Y)) \end{cases}$$

and

$$u(y) = x$$

So let us first solve the ODE

$$\begin{cases} u'(z) = \sigma(u(z)) \\ u(y) = x \end{cases} \qquad (z \in \mathbb{R})$$

where $' = \frac{d}{dz}$, and then, once u is known, solve for

$$f(z) = \frac{1}{\sigma(u(z))}\left[b(u(z)) - \frac{1}{2}u''(z)\right]$$

4.2.2 A general existence and uniqueness theorem

Lemma 4.5 (Gronwall's lemma)**.** *Let ϕ and f be nonnegative, continuous functions defined for $0 \le t \le T$, and let $C_0 \ge 0$ denote a constant. If*

$$\phi(t) \le C_0 + \int_0^t f\phi ds \quad \textit{for all } 0 \le t \le T$$

then

$$\phi(t) \le C_0 e^{\int_0^t f ds} \quad \textit{for all } 0 \le t \le T [11]$$

Proof. Set $\Phi(t) := C_0 + \int_0^t f\phi ds$. Then $\Phi' = f\phi \leq f\Phi$, and so

$$\left(e^{-\int_0^t f ds}\Phi\right)' = (\Phi' - f\Phi)\, e^{-\int_0^t f ds} \leq (f\phi - f\phi)e^{-\int_0^t f ds} = 0$$

Therefore

$$\Phi(t)e^{-\int_0^t f ds} \leq \Phi(0)e^{-\int_0^0 f ds} = C_0$$

and thus

$$\phi(t) \leq \Phi(t) \leq C_0 e^{\int_0^t f ds}$$

□

Theorem 4.6 (Existence and uniqueness theorem)**.** *Suppose that* $b : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ *and* $\mathbf{B} : \mathbb{R}^n \times [0,T] \to \mathbb{M}^{m\times n}$ *are continuous and satisfy the following conditions:*

$$\begin{aligned} &|\mathbf{b}(x,t) - \mathbf{b}(\hat{x},t)| \leq L|x-\hat{x}| \\ &|\mathbf{B}(x,t) - \mathbf{B}(\hat{x},t)| \leq L|x-\hat{x}| \end{aligned} \quad \textit{for all } 0 \leq t \leq T, x, \hat{x} \in \mathbb{R}^n$$

$$\begin{aligned} &|\mathbf{b}(x,t)| \leq L(1+|x|) \\ &|\mathbf{B}(x,t)| \leq L(1+|x|) \end{aligned} \quad \textit{for all } 0 \leq t \leq T, x \in \mathbb{R}^n$$

for some constant L*. Let* $\mathbf{X}_0$ *be any* $\mathbb{R}^n$*-valued random variable such that*

$$E\left(|\mathbf{X}_0|^2\right) < \infty$$

and $\mathbf{X}_0$ *is independent of* $\mathcal{W}^+(0)$ *where* $\mathbf{W}(\cdot)$ *is a given* m*-dimensional Brownian motion. Then there exists a unique solution for the stochastic differential equation:*

$$\text{(SDE)} \quad \begin{cases} d\mathbf{X} = \mathbf{b}(\mathbf{X},t)dt + \mathbf{B}(\mathbf{X},t)d\mathbf{W} \quad (0 \leq t \leq T) \\ \mathbf{X}(0) = \mathbf{X}_0. \end{cases} \quad \textit{[12]}$$

Let's recall the definition of Martingale before proving the existence and uniqueness theorem.

Definition 4.3 (Martingale)**.** *Let* $X(\cdot)$ *be a stochastic process, such that* $E(|X(t)|) < \infty$ *for all* $t \geq 0$

(i) If

$$X(s) = E(X(t) \mid \mathcal{U}(s)) \text{ a.s.} \quad \textit{for all } t \geq s \geq 0$$

then $X(\cdot)$ is called a martingale.

(ii) If

$$X(s) \leq E(X(t) \mid \mathcal{U}(s)) \text{ a.s.} \quad \text{for all } t \geq s \geq 0$$

$X(\cdot)$ is a submartingale. [14]

Theorem 4.7 (Martingale inequality)**.** *Let $X(\cdot)$ be a stochastic process with continuous sample paths a.s.*

(i) If $X(\cdot)$ is a submartingale, then

$$P\left(\max_{0\leq s\leq t} X(s) \geq \lambda\right) \leq \frac{1}{\lambda} E\left(X(t)^+\right) \quad \text{for all } \lambda > 0, t \geq 0$$

(ii) If $X(\cdot)$ is a martingale and $1 < p < \infty$, then

$$E\left(\max_{0\leq s\leq t} |X(s)|^p\right) \leq \left(\frac{p}{p-1}\right)^p E\left(|X(t)|^p\right)$$

[14]

Proof. 1. Uniqueness. Suppose $\mathbf{X}$ and $\hat{\mathbf{X}}$ are solutions, as above. Then for all $0 \leq t \leq T$,

$$\mathbf{X}(t) - \hat{\mathbf{X}}(t) = \int_0^t \mathbf{b}(\mathbf{X}, s) - \mathbf{b}(\hat{\mathbf{X}}, s) ds + \int_0^t \mathbf{B}(\mathbf{X}, s) - \mathbf{B}(\hat{\mathbf{X}}, s) d\mathbf{W}$$

Since $(a+b)^2 \leq 2a^2 + 2b^2$, we can estimate

$$\begin{aligned} E\left(|\mathbf{X}(t) - \hat{\mathbf{X}}(t)|^2\right) \leq & 2E\left(\left|\int_0^t \mathbf{b}(\mathbf{X}, s) - \mathbf{b}(\hat{\mathbf{X}}, s) ds\right|^2\right) \\ & + 2E\left(\left|\int_0^t \mathbf{B}(\mathbf{X}, s) - \mathbf{B}(\hat{\mathbf{X}}, s) d\mathbf{W}\right|^2\right) \end{aligned}$$

The Cauchy-Schwarz inequality implies that

$$\left|\int_0^t \mathbf{f} ds\right|^2 \leq t \int_0^t |\mathbf{f}|^2 ds$$

for any $t > 0$ and $\mathbf{f} : [0, t] \to \mathbb{R}^n$. We use this to estimate

$$E\left(\left|\int_0^t \mathbf{b}(\mathbf{X}, s) - \mathbf{b}(\hat{\mathbf{X}}, s) ds\right|^2\right) \leq TE\left(\int_0^t |\mathbf{b}(\mathbf{X}, s) - \mathbf{b}(\hat{\mathbf{X}}, s)|^2 ds\right)$$

$$\leq L^2 T \int_0^t E\left(|\mathbf{X} - \hat{\mathbf{X}}|^2\right) ds$$

Furthermore

$$E\left(\left|\int_0^t \mathbf{B}(\mathbf{X},s)-\mathbf{B}(\hat{\mathbf{X}},s)d\mathbf{W}\right|^2\right)=E\left(\int_0^t|\mathbf{B}(\mathbf{X},s)-\mathbf{B}(\hat{\mathbf{X}},s)|^2ds\right)$$
$$\leq L^2\int_0^t E\left(|\mathbf{X}-\hat{\mathbf{X}}|^2\right)ds$$

Therefore for some appropriate constant C we have

$$E\left(|\mathbf{X}(t)-\hat{\mathbf{X}}(t)|^2\right)\leq C\int_0^t E\left(|\mathbf{X}-\hat{\mathbf{X}}|^2\right)ds$$

If we now set $\phi(t):=E\left(|\mathbf{X}(t)-\hat{\mathbf{X}}(t)|^2\right)$, then the foregoing reads

$$\phi(t)\leq C\int_0^t\phi(s)ds \quad \text{for all } 0\leq t\leq T$$

Therefore Gronwall's Lemma, with $C_0=0$, implies $\phi\equiv 0$. Thus $\mathbf{X}(t)=\hat{\mathbf{X}}(t)$ a.s. for all $0\leq t\leq T$. 2. Existence. We will utilize the iterative scheme introduced earlier. Define

$$\begin{cases}\mathbf{X}^0(t):=\mathbf{X}_0\\ \mathbf{X}^{n+1}(t):=\mathbf{X}_0+\int_0^t\mathbf{b}\left(\mathbf{X}^n(s),s\right)ds+\int_0^t\mathbf{B}\left(\mathbf{X}^n(s),s\right)d\mathbf{W}\end{cases}$$

for $n=0,1,\ldots$ and $0\leq t\leq T$. Define also

$$d^n(t):=E\left(\left|\mathbf{X}^{n+1}(t)-\mathbf{X}^n(t)\right|^2\right)$$

We claim that

$$d^n(t)\leq\frac{(Mt)^{n+1}}{(n+1)!} \quad \text{for all } n=0,\ldots,0\leq t\leq T$$

for some constant M, depending on L,T and $\mathbf{X}_0$. Indeed for $n=0$, we have

$$\begin{aligned}d^0(t)&=E\left(\left|\mathbf{X}^1(t)-\mathbf{X}^0(t)\right|^2\right)\\&=E\left(\left|\int_0^t\mathbf{b}\left(\mathbf{X}_0,s\right)ds+\int_0^t\mathbf{B}\left(\mathbf{X}_0,s\right)d\mathbf{W}\right|^2\right)\\&\leq 2E\left(\left|\int_0^t L\left(1+|\mathbf{X}_0|\right)ds\right|^2\right)+2E\left(\int_0^t L^2\left(1+|\mathbf{X}_0|^2\right)ds\right)\\&\leq tM\end{aligned}$$

for some large enough constant M. This confirms the claim for $n=0$.

Next assume the claim is valid for some $n-1$. Then

$$\begin{aligned}
d^n(t) =& E\left(\left|\mathbf{X}^{n+1}(t)-\mathbf{X}^n(t)\right|^2\right)\\
=& E\left(\left|\int_0^t \mathbf{b}\left(\mathbf{X}^n,s\right)-\mathbf{b}\left(\mathbf{X}^{n-1},s\right)ds\right.\right.\\
&\left.\left.+\int_0^t \mathbf{B}\left(\mathbf{X}^n,s\right)-\mathbf{B}\left(\mathbf{X}^{n-1},s\right)d\mathbf{W}\right|^2\right)\\
\leq& 2TL^2E\left(\int_0^t\left|\mathbf{X}^n-\mathbf{X}^{n-1}\right|^2ds\right)\\
&+2L^2E\left(\int_0^t\left|\mathbf{X}^n-\mathbf{X}^{n-1}\right|^2ds\right)\\
\leq& 2L^2(1+T)\int_0^t\frac{M^ns^n}{n!}ds \quad \text{by the induction hypothesis}\\
\leq& \frac{M^{n+1}t^{n+1}}{(n+1)!}
\end{aligned}$$

3. Now note

$$\begin{aligned}
\max_{0\leq t\leq T}\left|\mathbf{X}^{n+1}(t)-\mathbf{X}^n(t)\right|^2 \leq& 2TL^2\int_0^T\left|\mathbf{X}^n-\mathbf{X}^{n-1}\right|^2ds\\
&+2\max_{0\leq t\leq T}\left|\int_0^t\mathbf{B}\left(\mathbf{X}^n,s\right)-\mathbf{B}\left(\mathbf{X}^{n-1},s\right)d\mathbf{W}\right|^2
\end{aligned}$$

Consequently the martingale inequality implies

$$\begin{aligned}
E\left(\max_{0\leq t\leq T}\left|\mathbf{X}^{n+1}(t)-\mathbf{X}^n(t)\right|^2\right) \leq& 2TL^2\int_0^T E\left(\left|\mathbf{X}^n-\mathbf{X}^{n-1}\right|^2\right)ds\\
&+8L^2\int_0^T E\left(\left|\mathbf{X}^n-\mathbf{X}^{n-1}\right|^2\right)ds\\
\leq& C\frac{(MT)^n}{n!} \text{ by the claim above.}
\end{aligned}$$

□

Before continuing our proof, let's introduce the Borel-Cantelli Lemma.

Definition 4.4. *Let $A_1,\ldots,A_n,\ldots$ be events in a probability space. Then the event $\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}A_m=\{\omega\in\Omega\mid \omega$ belongs to infinitely many of the $A_n\}$ is called " A_n infinitely often", abbreviated " A_n i.o.".*

Theorem 4.8 (Borel-Cantelli Lemma)**.** *If $\sum_{n=1}^{\infty}P\left(A_n\right)<\infty$, then $P\left(A_n \text{ i.o.}\right)=0$. [21]*

With this lemma, we can continue our proof.

Proof. 4. The Borel-Cantelli Lemma thus applies, since

$$P\left(\max_{0\le t\le T}\left|\mathbf{X}^{n+1}(t)-\mathbf{X}^n(t)\right|>\frac{1}{2^n}\right)\le 2^{2n}E\left(\max_{0\le t\le T}\left|\mathbf{X}^{n+1}(t)-\mathbf{X}^n(t)\right|^2\right)$$
$$\le 2^{2n}\frac{C(MT)^n}{n!}$$

and

$$\sum_{n=1}^{\infty}2^{2n}\frac{(MT)^n}{n!}<\infty$$

Thus

$$P\left(\max_{0\le t\le T}\left|\mathbf{X}^{n+1}(t)-\mathbf{X}^n(t)\right|>\frac{1}{2^n}\text{ i.o. }\right)=0$$

In light of this, for almost every ω

$$\mathbf{X}^n=\mathbf{X}^0+\sum_{j=0}^{n-1}\left(\mathbf{X}^{j+1}-\mathbf{X}^j\right)$$

converges uniformly on $[0,T]$ to a process $\mathbf{X}(\cdot)$. We pass to limits in the definition of $\mathbf{X}^{n+1}(\cdot)$, to prove

$$\mathbf{X}(t)=X_0+\int_0^t\mathbf{b}(\mathbf{X},s)ds+\int_0^t\mathbf{B}(\mathbf{X},s)d\mathbf{W}\quad\text{ for }0\le t\le T$$

That is, (SDE)

$$\begin{cases}d\mathbf{X}=\mathbf{b}(\mathbf{X},t)dt+\mathbf{B}(\mathbf{X},t)d\mathbf{W}\\ \mathbf{X}(0)=\mathbf{X}_0\end{cases}$$

for times $0\le t\le T$. □

5 The Stratonovic Integral

Let us consider first of all the formal random differential equation

$$\begin{cases}\dot{X}=d(t)X+f(t)X\xi\\ X(0)=X_0\end{cases}$$

where $m=n=1$ and $\xi(\cdot)$ is 1-dimensional "white noise". If we interpret this rigorously as the stochastic differential equation:

$$\begin{cases}dX=d(t)Xdt+f(t)XdW\\ X(0)=X_0\end{cases}$$

we then recall from my previous presentation that the unique solution is

$$X(t) = X_0 e^{\int_0^t d(s) - \frac{1}{2} f^2(s) ds + \int_0^t f(s) dW}$$

More precisely, suppose that $\left\{\xi^k(\cdot)\right\}_{k=1}^{\infty}$ is a sequence of stochastic processes satisfying:

(a) $E\left(\xi^k(t)\right) = 0$

(b) $E\left(\xi^k(t)\xi^k(s)\right) := d^k(t-s)$

(c) $\xi^k(t)$ is Gaussian for all $t \geq 0$,

(d) $t \mapsto \xi^k(t)$ is smooth for all ω,

where we suppose that the functions $d^k(\cdot)$ converge as $k \to \infty$ to δ_0, the Dirac measure at 0.

In light of the formal definition of the white noise $\xi(\cdot)$ as a Gaussian process with $E\xi(t) = 0, E(\xi(t)\xi(s)) = \delta_0(t-s)$, the $\xi^k(\cdot)$ are thus smooth approximations of $\xi(\cdot)$.

Now consider the problem

$$\begin{cases} \dot{X}^k = d(t)X^k + f(t)X^k\xi^k \\ X^k(0) = X_0 \end{cases}$$

For each ω this is just a regular ODE, whose solution is

$$X^k(t) := X_0 e^{\int_0^t d(s) ds + \int_0^t f(s)\xi^k(s) ds}$$

Next look at

$$Z^k(t) := \int_0^t f(s)\xi^k(s) ds$$

For each time $t \geq 0$, this is a Gaussian random variable, with

$$E\left(Z^k(t)\right) = 0$$

Hence as $k \to \infty, Z^k(t)$ converges to a process whose distributions agree with $\int_0^t f(s) dW$. And therefore $X^k(t)$ converges to a process whose distributions agree with

$$\hat{X}(t) := X_0 e^{\int_0^t d(s) ds + \int_0^t f(s) dW}$$

This does not agree with the previous solution.

Thus if we regard the problem as an Itô SDE with $\xi(\cdot)$ a "true" white noise, the first solution should be our solution. But if we approximate $\xi(\cdot)$ by smooth processes $\xi^k(\cdot)$, solve the approximate problems and pass to limits with the approximate solutions $X^k(\cdot)$, we get

a different solution. This means that the SDE is unstable with respect to changes in the random term $\xi(\cdot)$. This conclusion has important consequences in questions of modeling, since it may be unclear experimentally whether we really have $\xi(\cdot)$ or instead $\xi^k(\cdot)$ in similar problems.

In view of all this, it is appropriate to ask if there is some way to redefine the stochastic integral so these difficulties do not come up. One answer is the Stratonovich integral.

Recall that in my previous presentation we defined for 1 -dimensional Brownian motion

$$\int_0^T W dW := \lim_{|P^n|\to 0} \sum_{k=0}^{m_n-1} W\left(t_k^n\right)\left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right) = \frac{W^2(T) - T}{2}$$

where $P^n := \left\{0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = T\right\}$ is a partition of $[0, T]$. This corresponds to a sequence of Riemann sum approximations, where the integrand is evaluated at the left-hand endpoint of each subinterval $\left[t_k^n, t_{k+1}^n\right]$. The Stratonovich integral is instead defined this way:

$$\int_0^T W \circ dW := \lim_{|P^n|\to 0} \sum_{k=0}^{m_n-1} \left(\frac{W\left(t_{k+1}^n\right) + W\left(t_k^n\right)}{2}\right)\left(W\left(t_{k+1}^n\right) - W\left(t_k^n\right)\right) = \frac{W^2(T)}{2}$$

Definition 5.1. *Let $\mathbf{W}(\cdot)$ be an n-dimensional Brownian motion and let $\mathbf{B} : \mathbb{R}^n \times [0, T] \to \mathbb{M}^{n\times n}$ be a C^1 function such that*

$$E\left(\int_0^T |\mathbf{B}(\mathbf{W}, t)|^2 dt\right) < \infty$$

Then we define

$$\int_0^T \mathbf{B}(\mathbf{W}, t) \circ d\mathbf{W} := \lim_{|P^n|\to 0} \sum_{k=0}^{m_n-1} \mathbf{B}\left(\frac{\mathbf{W}\left(t_{k+1}^n\right) + \mathbf{W}\left(t_k^n\right)}{2}, t_k^n\right)\left(\mathbf{W}\left(t_{k+1}^n\right) - \mathbf{W}\left(t_k^n\right)\right)$$

It can be shown that this limit exists in $L^2(\Omega)$.

Remember that Itô's integral can be computed this way:

$$\int_0^T \mathbf{B}(\mathbf{W}, t) d\mathbf{W} = \lim_{|P^n|\to 0} \sum_{k=0}^{m_n-1} \mathbf{B}\left(\mathbf{W}\left(t_k^n\right), t_k^n\right)\left(\mathbf{W}\left(t_{k+1}^n\right) - \mathbf{W}\left(t_k^n\right)\right)$$

This is in general not equal to the Stratonovich integral, but there is a conversion formula

$$\left[\int_0^T \mathbf{B}(\mathbf{W}, t) \circ d\mathbf{W}\right]^i = \left[\int_0^T \mathbf{B}(\mathbf{W}, t) d\mathbf{W}\right]^i + \frac{1}{2}\int_0^T \sum_{j=1}^n \frac{\partial b^{ij}}{\partial x_j}(\mathbf{W}, t) dt$$

for $i = 1, \ldots, n$. Here v^i means the i^{th}-component of the vector function $\mathbf{v}$.

This formula is proved by noting

$$\int_0^T \mathbf{B}(\mathbf{W}, t) \circ d\mathbf{W} - \int_0^T \mathbf{B}(\mathbf{W}, t) d\mathbf{W}$$
$$= \lim_{|P^n| \to 0} \sum_{k=0}^{m_n - 1} \left[\mathbf{B}\left(\frac{\mathbf{W}\left(t_{k+1}^n\right) + \mathbf{W}\left(t_k^n\right)}{2}, t_k^n \right) - \mathbf{B}\left(\mathbf{W}\left(t_k^n\right), t_k^n \right) \right]$$
$$\cdot \left(\mathbf{W}\left(t_{k+1}^n\right) - \mathbf{W}\left(t_k^n\right) \right)$$

and using the Mean Value Theorem.

Definition 5.2. *If $\mathbf{X}(\cdot)$ is a stochastic process with values in $\mathbb{R}^n$, we define*

$$\int_0^T \mathbf{B}(\mathbf{X}, t) \circ d\mathbf{W} := \lim_{|P^n| \to 0} \sum_{k=0}^{m_n - 1} \mathbf{B}\left(\frac{\mathbf{X}\left(t_{k+1}^n\right) + \mathbf{X}\left(t_k^n\right)}{2}, t_k^n \right) \left(\mathbf{W}\left(t_{k+1}^n\right) - \mathbf{W}\left(t_k^n\right) \right)$$

5.1 Stratonovich chain rule

Definition 5.3. *Suppose that the process $\mathbf{X}(\cdot)$ solves the Stratonovich integral equation*

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{b}(\mathbf{X}, s) ds + \int_0^t \mathbf{B}(\mathbf{X}, s) \circ d\mathbf{W} \quad (0 \leq t \leq T)$$

for $\mathbf{b} : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ and $\mathbf{B} : \mathbb{R}^n \times [0, T] \to \mathbb{M}^{n \times m}$. We then write

$$d\mathbf{X} = \mathbf{b}(\mathbf{X}, t) dt + \mathbf{B}(\mathbf{X}, t) \circ d\mathbf{W}$$

the second term on the right being the Stratonovich stochastic differential.

Suppose that $d\mathbf{X} = \mathbf{F}dt + \mathbf{G}d\mathbf{W}, as$ above. Let $u : \mathbb{R}^n \times [0, T]$ be continuous, with continuous partial derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, (i, j = 1, \ldots, n)$. Then

$$d(u(\mathbf{X}(t), t)) = \frac{\partial u}{\partial t} dt + \sum_{i=1}^n \frac{\partial u}{\partial x_i} dX^i$$
$$+ \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \sum_{l=1}^m G^{il} G^{jl} dt$$

Theorem 5.1 (Stratonovich chain rule)**.** *Assume*

$$d\mathbf{X} = \mathbf{b}(\mathbf{X}, t) dt + \mathbf{B}(\mathbf{X}, t) \circ d\mathbf{W}$$

and suppose $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is smooth. Define

$$Y(t) := u(\mathbf{X}(t), t)$$

Then

$$\begin{aligned} dY &= \frac{\partial u}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \circ d\mathbf{X}^i \\ &= \left(\frac{\partial u}{\partial t} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} b^i \right) dt + \sum_{i=1}^{n} \sum_{k=1}^{m} \frac{\partial u}{\partial x_i} b^{ik} \circ d\mathbf{W}^k \end{aligned}$$

[9]

Thus the ordinary chain rule holds for Stratonovich stochastic differentials, and there is no additional term involving $\frac{\partial^2 u}{\partial x_i \partial x_j}$ as there is for Itô's formula.

6 Viscosity solutions

6.1 Uniqueness for $u + H(Du) = f$ in $\mathbf{R}^n$

Theorem 6.1. *Let $u, v, f, g, H \in C(\mathbf{R}^n)$. Assume that u, v are bounded and f, g are uniformly continuous on $\mathbf{R}^n$. Assume that u and v are viscosity solutions of, respectively, $u + H(Du) = f, v + H(Dv) = g$ in $\mathbf{R}^n$. Then we have*

$$\sup_{\mathbf{R}^n} (u - v)^+ \leqslant \sup_{\mathbf{R}^n} (f - g)^+$$

Proof. We begin the proof by assuming the extra conditions

$$\lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0.$$

This will keep the ideas clearer; later the full result is established. We choose a function $\beta \in C^\infty(\mathbf{R}^n)$ with the properties

$$0 \leqslant \beta \leqslant 1, \quad \beta(0) = 1, \quad \beta(x) = 0$$

if $|x| > 1$. Let $M = \max(\|u\|, \|v\|), \varepsilon > 0$, and let $\Phi : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be given by

$$\Phi(x, y) = u(x) - v(y) + 3M\beta_\varepsilon(x - y)$$

, where

$$\beta_\varepsilon(z) = \beta(z/\varepsilon) \text{ for } z \in \mathbf{R}^n.$$

We begin the proof by assuming the extra conditions

$$\lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0.$$

This will keep the ideas clearer; later the full result is established. We choose a function $\beta \in C^\infty(\mathbf{R}^n)$ with the properties

$$0 \leqslant \beta \leqslant 1, \quad \beta(0) = 1, \quad \beta(x) = 0$$

if $|x| > 1$. Let $M = \max(\|u\|, \|v\|), \varepsilon > 0$, and let $\Phi : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be given by

$$\Phi(x, y) = u(x) - v(y) + 3M\beta_\varepsilon(x - y)$$

, where

$$\beta_\varepsilon(z) = \beta(z/\varepsilon) \text{ for } z \in \mathbf{R}^n.$$

Similarly, y_0 is a minimum point of $y \mapsto v(y) - (u(x_0) + 3M\beta_\varepsilon(x_0 - y))$ and so

$$v(y_0) + H(-3M(D\beta_\varepsilon)(x_0 - y_0)) \geqslant g(y_0)$$

Together, the two relations yield

$$u(x_0) - v(y_0) \leqslant f(x_0) - g(y_0).$$

For $x \in \mathbf{R}^n$,

$$u(x) - v(x) + 3M = \Phi(x, x) \leqslant \Phi(x_0, y_0) \leqslant u(x_0) - v(y_0) + 3M;$$

so,

$$\begin{aligned}\sup_{\mathbf{R}^n}(u(x) - v(x))^+ &\leqslant (u(x_0) - v(y_0))^+ \leqslant (f(x_0) - g(y_0))^+ \\ &\leqslant \sup_{\mathbf{R}^n}(f - g)^+ + |g(x_0) - g(y_0)| \\ &\leqslant \sup_{\mathbf{R}^n}(f - g)^+ + \omega_g(\varepsilon)\end{aligned}$$

where $\omega_g(\cdot)$ is the modulus of continuity of g. Now we can prove the theorem upon our letting $\varepsilon \to 0$. The extra assumption was used to guarantee that Φ had a global maximum point (x_0, y_0). To treat the general case, choose $\delta > 0$ and then (x_1, y_1) so that

$$\Phi(x_1, y_1) \geqslant \sup_{\mathbf{R}^n \times \mathbf{R}^n} \Phi(x, y) - \delta.$$

□

Here, we recall the concept of modulus of continuity.

Definition 6.1. *In mathematical analysis, a modulus of continuity is a function $\omega : [0, \infty] \to [0, \infty]$ used to measure quantitatively the uniform continuity of functions. So, a function $f : / \to \mathbf{R}$ admits ω as a modulus of continuity if and only if*

$$|f(x) - f(y)| \leq \omega(|x - y|),$$

for all x and y in the domain of f. [18]

Proof. Now choose $\zeta \in C^{\infty}(\mathbf{R}^n \times \mathbf{R}^n)$ such that

$$\begin{cases} 0 \leqslant \zeta \leqslant 1, \zeta(x_1, y_1) = 1, \zeta(x, y) = 0 \text{ if } |x - x_1|^2 + |y - y_1|^2 > 1, \\ |D\zeta| \leqslant 2 \quad \text{in } \mathbf{R}^n \times \mathbf{R}^n. \end{cases}$$

Finally, set

$$\Psi(x, y) = \Phi(x, y) + 2\delta\zeta(x, y) = u(x) - v(y) + 3M\beta_{\varepsilon}(x - y) + 2\delta\zeta(x, y).$$

We claim that Ψ has a global maximum point $(x_0, y_0) \in \mathbf{R}^n \times \mathbf{R}^n$. Indeed, from these one deduces

$$\Psi(x_1, y_1) = \Phi(x_1, y_1) + 2\delta \geqslant \sup \Phi + \delta$$

whereas lim sup $_{|x|+|y| \to \infty} \Psi(x, y) \leqslant \sup \Phi$.

Moreover, with $\bar{x}, (x_0, y_0)$ as above,

$$\begin{aligned} u(x_0) - v(y_0) + 3M\beta_{\varepsilon}(x_0 - y_0) + 2\delta\zeta(x_0, y_0) \\ \geqslant u(\bar{x}) - v(\bar{x}) + 3M + 2\delta\zeta(\bar{x}, \bar{x}) \end{aligned}$$

hence, $2M + 3M\beta_{\varepsilon}(x_0 - y_0) + 2\delta > 3M$. We conclude that $|x_0 - y_0| \leqslant \varepsilon \quad$ if $M > 2\delta$. Next, using the assumptions on u, v as above, one deduces

$$u(x_0) + H(-3MD\beta_{\varepsilon}(x_0 - y_0) - 2\delta D_x\zeta(x_0, y_0)) \leqslant f(x_0)$$

and

$$v(y_0) + H(-3MD\beta_{\varepsilon}(x_0 - y_0) + 2\delta D_y\zeta(x_0, y_0)) \geqslant g(y_0)$$

These imply

$$u(x_0) - v(y_0) \leqslant f(x_0) - g(y_0) + \omega_{H,r}(8\delta),$$

where $\omega_{H,r}$ denotes the modulus of continuity of H on $\{\xi \in \mathbf{R}^n; |\xi| \leqslant r\}$ and $r = 3M \|D\beta_\varepsilon\| + 8\delta$. Therefore, for $x \in \mathbf{R}^n$, we deduce

$$\begin{aligned} u(x) - v(x) + 3M &\leqslant u(x) - v(x) + 3M + 2\delta\zeta(x,x) = \Psi(x,x) \\ &\leqslant \Psi(x_0, y_0) \leqslant u(x_0) - v(y_0) + 3M + 2\delta \\ &\leqslant \sup_{\mathbf{R}^n}(f-g)^+ + \omega_g(\varepsilon) + \omega_{H,r}(8\delta) + 2\delta + 3M \end{aligned}$$

We conclude upon sending $\delta \to 0$ and then $\varepsilon \to 0$.

□

Theorem 6.2. *Let $0 < T < \infty$ and let $u \in C(\mathbf{R}^n \times (0,T))$ be a viscosity solution of $u_t + H(Du) = 0$. If $\phi \in C^1(\mathbf{R}^n \times (0,T])$, then at each local maximum point (resp. minimum point) of $u - \phi$ on $\mathbf{R}^n \times (0,T]$, we have $\phi_t + H(D\phi) \leqslant 0$ (resp . $\geqslant 0$).*

6.2 Uniqueness for $u_t + H(Du) = 0$

This subsection and the next one concern the Hamilton-Jacobi equation

$$u_t + H(Du) = 0 \text{ in } \mathbf{R}^n \times (0,\infty)$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}^n$$

According to our definitions and to Theorem we recall, $u \in C(\mathbf{R}^n \times (0,T))$ is a viscosity solution of

$$u_t + H(Du) = 0 \quad \text{ on } \mathbf{R}^n \times (0,T)$$

provided that, for every $\phi \in C^1(\mathbf{R}^n \times (0,T))$, we have $\phi_t + H(D\phi) \leqslant 0$ (resp. $\geqslant 0$) at local maxima (resp. minima) of $u - \phi$.

Note that

$$\mathrm{BUC}(\mathbf{R}^n) = \{u \in C_b(\mathbf{R}^n), u \text{ is uniformly continuous on } \mathbf{R}^n\}.$$

Theorem 6.3. *Let $0 < T < \infty$ and let $u, v \in \mathrm{BUC}(\mathbf{R}^n \times [0,T])$ be viscosity solutions of $u_t + H(Du) = 0$. Then*

$$\sup_{\mathbf{R}^n \times [0,T]} (u-v)^+ \leqslant \sup_{\mathbf{R}^n}(u(x,0) - v(x,0))^+.$$

[21]

Proof. Define σ by

$$\sup_{\mathbf{R}^n \times [0,T]} (u - v) = \sup_{\mathbf{R}^n} (u_0(x) - v_0(x)) + \sigma,$$

where here and below $u_0 = u(\cdot, 0), v_0 = v(\cdot, 0)$. If $\sigma = 0$, there is nothing to prove, and so we may assume $\sigma > 0$. Choose $\beta \in C^\infty (\mathbf{R}^n \times \mathbf{R})$ so that

$$\begin{cases} 0 \leqslant \beta \leqslant 1, \quad \beta(0,0) = 1 \\ \beta(x,t) = 0 \quad \text{if } |x|^2 + t^2 > 1. \end{cases}$$

Set $\beta_\varepsilon(x,t) = \beta(x/\varepsilon, t/\varepsilon)$ and $M = \max(\|u\|, \|v\|)$, where $\|h\|$ denotes the norm of h in $C_b (\mathbf{R}^n \times [0,T])$ (note that since $\sigma > 0, M > 0$). Next, let $\lambda > 0$ be fixed and define $\Phi : \mathbf{R}^n \times \mathbf{R}^n \times [0,T] \times [0,T] \to \mathbf{R}$ by

$$\Phi(x,y,t,s) = u(x,t) - v(y,s) - \lambda(t+s) + (5M + 2\lambda T)\beta_\varepsilon(x-y, t-s).$$

If Φ attains its maximum on $\mathbf{R}^{2n} \times [0,T]^2$ at some point, the proof is easily completed, but this need not be so. Therefore we choose $\delta > 0$ and then $(x_0, y_0, t_0, s_0) \in \mathbf{R}^{2n} \times [0,T]^2$ so that

$$\Phi (x_0, y_0, t_0, s_0) > \sup \Phi - \delta.$$

If λ, ε and δ are sufficiently small, then for some $\mu > 0$,

$$t_0, s_0 \geqslant \mu$$

, where μ is independent of $\lambda, \varepsilon, \delta$.

Next select $\zeta \in C^\infty (\mathbf{R}^{2n} \times [0,T]^2)$ satisfying

$$0 \leqslant \zeta \leqslant 1, \zeta (x_0, y_0, t_0, s_0) = 1, \zeta = 0$$

if

$$|x - x_0|^2 + |y - y_0|^2 + |t - t_0|^2 + |s - s_0|^2 \geqslant \mu^2/4.$$

We set $\Psi(x,y,t,s) = \Phi(x,y,t,s) + 2\delta\zeta(x,y,t,s)$. Since

$$\Psi (x_0, y_0, t_0, s_0) = \Phi (x_0, y_0, t_0, s_0) + 2\delta > \sup \Phi + \delta,$$

Ψ attains its maximum at some point (x_1, y_1, t_1, s_1). $(x_1, t_1) \in \mathbf{R}^n \times [0,T]$ is a maximum of

$$u(x,t) - v (y_1, s_1) - \lambda (t + s_1) + (5M + 2\lambda T)\beta_\varepsilon (x - y_1, t - s_1) + 2\delta\zeta (x, y_1, t, s_1)$$

So that Theorem 3 implies

$$\begin{aligned}\lambda - (5M+2\lambda T)\frac{\partial \beta_\varepsilon}{\partial t}(x_1-y_1,t_1-s_1) - 2\delta\zeta_t(x_1,y_1,t_1,s_1)\\ + H\left(-(5M+2\lambda T)D_x\beta_\varepsilon(x_1-y_1,t_1-s_1) - 2\delta D_x\zeta(x_1,y_1,t_1,s_1)\right)\\ \leqslant 0.\end{aligned}$$

Similarly,

$$\begin{aligned}-\lambda + (5M+2\lambda T)\frac{\partial \beta_\varepsilon}{\partial t}(x_1-y_1,t_1-s_1) + 2\delta\zeta_s(x_1,y_1,t_1,s_1)\\ + H\left(-(5M+2\lambda T)D_x\beta_\varepsilon(x_1-y_1,t_1-s_1) + 2\delta D_y\zeta(x_1,y_1,t_1,s_1)\right)\\ \geqslant 0.\end{aligned}$$

Combining these two inequalities and letting $\delta \to 0$, we derive $\lambda = 0$, a contradiction. □

7 Application 1: Pricing of option and portfolio

7.1 Option-pricing using Black-Scholes model

Before we start this subsection, let's introduce the following definition.

Definition 7.1 (derivative)**.** *A derivative is a contract between two or more parties whose value is based on an agreed-upon underlying financial asset (like a security) or set of assets (like an index). Common underlying instruments include bonds, commodities, currencies, interest rates, market indexes, and stocks.*

Definition 7.2 (Call option)**.** *The buyer of the call option has the right, but not the obligation, to buy an agreed quantity of a particular commodity or financial instrument (the underlying) from the seller of the option at a certain time (the expiration date) for a certain price (the strike price). The seller (or "writer") is obliged to sell the commodity or financial instrument to the buyer if the buyer so decides. The buyer pays a fee (called a premium) for this right.*

Definition 7.3 (European call option)**.** *A European call option is a contract with the following conditions: The holder of the option has at a prescribed time in the future, the exercise date , the right to by a prescribed stock for a prescribed amount, the exercise price. The holder of the option is in no way obliged to buy the underlying asset.*

Let us consider a given security whose price at time t is $S(t)$. Suppose that we have the following SDE :

$$\begin{cases} dS = \mu S dt + \sigma S dW \\ S(0) = s_0 \end{cases}$$

where $\mu > 0$ is the drift and $\sigma \neq 0$ the volatility. The initial price s_0 is known. We will investigate a European call option, which is the right to buy one share of the stock S, at the price p at time T. The number p is called the strike price and $T > 0$ the strike (or expiration) time. The basic question is this: What is the "proper price" at time $t = 0$ of this option?

To simplify, we assume hereafter that the prevailing, no-risk interest rate is the constant $r > 0$. This means that \$1 put in a bank at time $t = 0$ becomes $\$e^{rT}$ at time $t = T$. Equivalently, \$1 at time $t = T$ is worth only $\$e^{-rT}$ at time $t = 0$.

As for the problem of pricing our call option, a first guess might be that the proper price should be

$$e^{-rT} E\left((S(T) - p)^+\right)$$

for $x^+ := \max(x, 0)$.

The reasoning behind this guess is that if $S(T) < p$, then the option is worthless. If $S(T) > p$, we can buy a share for the price p, immediately sell at price $S(T)$, and thereby make a profit of $(S(T) - p)^+$. We average this over all sample paths and multiply by the discount factor e^{-rT}.

Other forces are at work in financial markets. Indeed the fundamental factor in options pricing is arbitrage, meaning the possibility of risk-free profits.

Definition 7.4 (Arbitrage)**.** *Arbitrage is the simultaneous purchase and sale of the same asset in different markets in order to profit from tiny differences in the asset's listed price. It exploits short-lived variations in the price of identical or similar financial instruments in different markets or in different forms.*

The financial markets will not allow any buyer to get a risk-free profit. We must price our option so as to create no arbitrage opportunities for others. To convert this principle into mathematics, we introduce also the notion of hedging. This means somehow eliminating our risk as the seller of the call option. The exact details appear below, but the basic idea is that

we can in effect "duplicate" our option by a portfolio consisting of (continually changing) holdings of a risk-free bond and of the stock on which the call is written.

We demonstrate next how use these principles to convert our pricing problem into a PDE. We introduce for $s \geq 0$ and $0 \leq t \leq T$, the unknown price function $u(s,t)$, denoting the proper price of the option at time t, given that $S(t) = s$. Then $u(s_0, 0)$ is the price we are seeking.

We need to calculate u. For this, notice first that at the expiration time T, we have

$$u(s,T) = (s-p)^+ \quad (s \geq 0)$$

Furthermore, if $s = 0$, then $S(t) = 0$ for all $0 \leq t \leq T$ and so

$$u(0,t) = 0 \quad (0 \leq t \leq T)$$

7.1.1 Construction of the portfolio

To go further, define the process

$$C(t) := u(S(t), t) \quad (0 \leq t \leq T)$$

Thus $C(t)$ is the current price of the option at time t, and is random since the stock price $S(t)$ is random. According to Itô's formula,

$$\begin{aligned} dC &= u_t dt + u_s dS + \frac{1}{2} u_{ss} (dS)^2 \\ &= \left(u_t + \mu S u_s + \frac{\sigma^2}{2} S^2 u_{ss} \right) dt + \sigma S u_s dW \end{aligned}$$

Now comes the key idea: we propose to "duplicate" C by a portfolio consisting of shares of S and of a bond B. More precisely, assume that B is a risk-free investment, which therefore grows at the prevailing interest rate r :

$$\begin{cases} dB = rBdt \\ B(0) = 1 \end{cases}$$

This just means $B(t) = e^{rt}$, of course. We will try to find processes ϕ and ψ so that

$$C = \phi S + \psi B \quad (0 \leq t \leq T)$$

The point is that if we can construct ϕ, ψ, we can eliminate all risk. To see this more clearly, imagine that your financial firm sells a call option, as above. The firm thereby incurs the risk that at time T, the stock price $S(T)$ will exceed p, and so the buyer will exercise the option. But if in the meantime the firm has constructed the portfolio (9), the profits from it will exactly equal the funds needed to pay the customer. Conversely, if the option is worthless at time T, the portfolio will have no profit.

But to make this work, the financial firm should not have to inject any new money into the hedging scheme, beyond the initial investment to set it up. We ensure this by requiring that the portfolio represented on the right-hand side be self-financing. This means that the changes in the value of the portfolio should depend only upon the changes in S, B. We therefore require that

$$dC = \phi dS + \psi dB \quad (0 \leq t \leq T)$$

Roughly speaking, a portfolio is self-financing if it is financially self contained. To understand this better, let us consider a different model in which time is discrete, and the values of the stock and bond at a time t_i are given by S_i and B_i respectively. Here $\{t_i\}_{i=0}^N$ is an increasing sequence of times and we suppose that each time step $t_{i+1} - t_i$ is small.

A portfolio can now be thought of as a sequence $\{(\phi_i, \psi_i)\}_{i=0}^N$, corresponding to our changing holdings of the stock S and the bond B over each time interval.

Now for a given time interval $(t_i, t_{i+1}), C_i = \phi_i S_i + \psi_i B_i$ is the opening value of the portfolio and $C_{i+1} = \phi_i S_{i+1} + \psi_i B_{i+1}$ represents the closing value. The self-financing condition means that the financing gap $C_{i+1} - C_i$ of cash (that would otherwise have to be injected to pay for our construction strategy) must be zero.This is equivalent to saying that

$$C_{i+1} - C_i = \phi_i (S_{i+1} - S_i) + \psi_i (B_{i+1} - B_i).$$

Combining formulas provides the identity

$$\left(u_t + \mu S u_s + \frac{\sigma^2}{2} S^2 u_{ss}\right) dt + \sigma S u_s dW$$
$$= \phi(\mu S dt + \sigma S dW) + \psi r B dt$$

We observe in particular that the terms multiplying dW on each side will match provided we take

$$\phi(t) := u_s(S(t), t) \quad (0 \leq t \leq T)$$

Then,

$$\left(u_t + \frac{\sigma^2}{2} S^2 u_{ss}\right) dt = r\psi B dt$$

But $\psi B = C - \phi S = u - u_s S$. Consequently,

$$\left(u_t + rSu_s + \frac{\sigma^2}{2} S^2 u_{ss} - ru\right) dt = 0$$

The argument of u and its partial derivatives is $(S(t), t)$. Consequently, we ask that the function $u = u(s,t)$ solve the Black-Scholes-Merton PDE

$$u_t + rsu_s + \frac{\sigma^2}{2} s^2 u_{ss} - ru = 0 \quad (0 \leq t \leq T)$$

The main outcome of all our financial reasoning is the derivation of this partial differential equation. Observe that the parameter μ does not appear.

To price our call option, we solve the boundary-value problem

$$\begin{cases} u_t + rsu_s + \frac{\sigma^2}{2} s^2 u_{ss} - ru = 0 & (s > 0, 0 \leq t \leq T) \\ u = (s-p)^+ & (s > 0, t = T) \\ u = 0 & (s = 0, 0 \leq t \leq T) \end{cases} \quad [22]$$

7.2 Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case

7.2.1 Dynamics of the Model: The Budget Equation

In the usual continuous-time model under certainty, the budget equation is a differential equation. However, when uncertainty is introduced by a random variable, the budget equation must be generalized to become a stochastic differential equation. To see the meaning of such an equation, it is easiest to work out the discrete-time version and then pass to the limit of continuous time.

Define

- $W(t) =$ total wealth at time t
- $X_i(t) =$ price of the i^{th} asset at time $t, (i = 1,, C(t) \quad \ldots, m)$
- $C(t) =$ consumption per unit time at time t

- $w_i(t) =$ proportion of total wealth in the i^{th} asset at time $t, (i = 1, \ldots, m)$
- Note

$$\left(\sum_{i=1}^{m} w_i(t) \equiv 1 \right)$$

The budget equation can be written as

$$W(t) = \left[\sum_{i=1}^{m} w_i \left(t_0\right) \frac{X_i(t)}{X_i \left(t_0\right)} \right] * \left[W \left(t_0\right) - C \left(t_0\right) h \right] \tag{16}$$

where $t \equiv t_0 + h$ and the time interval between periods is h.

By subtracting $W\left(t_0\right)$ from both sides and using $\sum_{i=1}^{m} w_i \left(t_0\right) = 1$, we can rewrite

$$\begin{aligned} & W(t) - W\left(t_0\right) \\ & = \left[\sum_{i=1}^{m} w_i \left(t_0\right) \left(\frac{X_i(t) - X_i \left(t_0\right)}{X_i \left(t_0\right)} \right) \right] \left[W \left(t_0\right) - C \left(t_0\right) h \right] - C \left(t_0\right) h \\ & = \left[\sum_{i=1}^{m} w_i \left(t_0\right) \left(e^{g_i(t_0)\mathrm{h}} - 1 \right) \right] \left[W \left(t_0\right) - C \left(t_0\right) h \right] - C \left(t_0\right) h \end{aligned} \tag{17}$$

where

$$g_i \left(t_0\right) h \equiv \log \left[X_i(t) / X_i \left(t_0\right) \right],$$

the rate of return per unit time on the i^{th} asset.

The $g_i(t)$ are assumed to be generated by a stochastic process. In discrete time, I make the further assumption that $g_i(t)$ is determined as follows,

$$g_i(t)h = \left(a_i - \sigma_i^2 / 2 \right) h + \Delta Y_i \tag{18}$$

where α_i, the "expected" rate of return, is constant; and $Y_i(t)$ is generated by,

$$Y_i(t) - Y_i \left(t_0\right) \equiv \triangle Y_i = \sigma_i Z_i(t) \sqrt{h} \tag{19}$$

where each $Z_i(t)$ is an independent variate with a standard normal distribution for every $t, {\sigma_i}^2$ is the variance for the process Y_i.

Substituting for $g_i(t)$ from (18), we can rewrite (17) as,

$$\begin{aligned} W(t) - W\left(t_0\right) = & \sum_{1}^{m} w_i \left(t_0\right) \left(e^{\left(a_i - \sigma_i^2 / 2(h + \Delta Y) \right.} - 1 \right) \quad \left(W \left(t_0\right) - C \left(t_0\right) h \right) \\ & - C \left(t_0\right) h \end{aligned} \tag{20}$$

Before passing in the limit to continuous time, there are two implications of (20) which will be useful later.

$$E(t_0)[W(t) - W(t_0)] = \left\{ \sum_1^m w_i(t_0) a_i W(t_0) - C(t_0) \right\} h + O(h^2) \tag{21}$$

and

$$E(t_0)\left[(W(t) - W(t_0))^2\right] = \sum_{i=1}^m \sum_{j=1}^m w_i(t_0) w_j(t_0) * E(t_0)(\Delta Y_i \Delta Y_j) * W^2(t_0) + O(h^2) \tag{22}$$

where $E(t_0)$ is the conditional expectation operator (conditional on the knowledge of $W(t_0)$).

The limit of the process described in (19) as $h \to 0$ (continuous time) can be expressed by the formalism of the stochastic differential equation,

$$dY_i = \sigma_i Z_i(t)\sqrt{dt}. \tag{19'}$$

By applying the same limit process to the discrete-time budget equation, we write (20) as

$$dW = \left[\sum_1^m w_i(t) a_i W(t) - C(t)\right] dt + \sum_1^m w_i(t)\sigma_i Z_i(t) W(t)\sqrt{dt}. \tag{20'}$$

From (20), we have

$$E(t_0)\left[\frac{W(t) - W(t_0)}{h}\right] = \sum_1^m w_i(t_0) a_i [W(t_0) - C(t_0) h] - C(t_0) + O(h) \tag{23}$$

Now, take the limit as $h \to 0$, so that (23) becomes the following expression for the defined "mean rate of change of wealth":

$$\overset{\circ}{W}(t_0) \equiv_{\text{def.}} \operatorname{limit}_{h \to 0} E(t_0)\left[\frac{W(t) - W(t_0)}{h}\right] = \sum_1^m w_i(t_0) a_i W(t_0) - C(t_0) \tag{23'}$$

7.2.2 The Two-Asset Model

For simplicity, we first derive the optimal equations and properties for the two-asset model and then display the general equations and results for the m-asset case.

Define

$$w_1(t) = w(t) = \text{proportion invested in the risky asset}$$
$$w_2(t) = 1 - w(t) = \text{proportion invested in the sure asset}$$
$$g_1(t) = g(t) = \text{ return on the risky asset}$$
$$g_2(t) = r = \text{ return on the sure asset}$$

Then, for $g(t)h = (a - \sigma^2/2)\, h + \Delta Y$, equations (20), (21), (22), and $(8')$ can be written as,

$$\begin{aligned} W(t) - W(t_0) = [w(t_0)\left(e^{\left(a-\sigma^2/2\right)h+\Delta Y)} - 1\right) + (1 - w(t_0))\left(e^{rh} - 1\right)] \\ * (W(t_0) - C(t_0)\, h) - C(t_0)\, h \end{aligned} \tag{24}$$

and

$$\begin{aligned} E(t_0)\,[W(t) - W(t_0)] = \{[w(t_0)(a - r) + r]\, W(t_0) - C(t_0)\}\, h \\ + O\left(h^2\right). \end{aligned} \tag{25}$$

and

$$\begin{aligned} E(t_0)\left[(W(t) - W(t_0))^2\right] = w^2(t_0)\, W^2(t_0)\, E(t_0)\left[(\triangle Y)^2\right] + O\left(h^2\right) \\ = w^2(t_0)\, W^2(t_0)\, \sigma^2 h + O\left(h^2\right). \end{aligned} \tag{26}$$

$$\begin{aligned} dW = [(w(t)(a - r) + r)W(t) - C(t)]dt \\ + w(t)\sigma Z(t)W(t)\sqrt{dt} \end{aligned} \tag{27}$$

$$\overset{\circ}{W}(t) = [w(t)(a - r) + r]W(t) - C(t) \tag{28}$$

The problem of choosing optimal portfolio selection and consumption rules is formulated as follows,

$$\text{Max}\, E\left\{\int_0^t e^{-\rho t} U[C(t)]dt + B[W(T), T]\right\} \tag{29}$$

subject to the budget constraint: $C(t) \geqq 0; W(t) > 0; W(0) = W_0 > 0$ and where $U(C)$ is assumed to be a strictly concave utility function (i.e., $U'(C) > 0$; $U''(C) < 0$); where $g(t)$ is a random variable generated by the previously described Wiener process. $B[W(T), T]$ is

to be a specified "bequest valuation function". "E" in (29) is short for $E(0)$, given $W(0) = W_0$ as known.

Define,

$$I[W(t),t] \equiv \underset{\{C(s),w(s)\}}{\text{Max}}\, E(t)[\int_t^T e^{-\rho s}U[C(s)]ds + B[W(T),T]] \tag{30}$$

where (30) is subject to the same constraints as (29).

Therefore,

$$I[W(T),T] = B[W(T),T] \tag{15'}$$

In general, from definition (30),

$$I\left[W\left(t_0\right),t_0\right] = \underset{\{C(s),w(s)\}}{\text{Max}}\, E\left(t_0\right)\left[\int_{t_0}^t e^{-\rho s}U[C(s)]ds \quad + I[W(t),t]\right] \tag{31}$$

and, in particular, (29) can be rewritten as

$$I\left(W_0,0\right) = \underset{\{C(s),w(s)\}}{\text{Max}}\, E\left[\int_0^t e^{-\rho s}U[C(s)]ds + \quad I[W(t),t]\right]. \tag{29'}$$

If $t \equiv t_0 + h$ and the third partial derivatives of $I\left[W\left(t_0\right),t_0\right]$ are bounded, then by Taylor's theorem and the mean value theorem for integrals, (31) can be rewritten as

$$\begin{aligned} I\left[W\left(t_0\right),t_0\right] = \underset{\{C,w\}}{\text{Max}}\, E(t_0)\Big\{ & e^{-\rho\bar{t}}U[C(t)] + I\left[W\left(t_0\right),t_0\right] + \frac{\partial I\left[W\left(t_0\right),t_0\right]}{\partial t} \\ & + \frac{\partial I\left[W\left(t_0\right),t_0\right]}{\partial W}\left[W(t) - W\left(t_0\right)\right] + \frac{1}{2}\frac{\partial^2 I\left[W\left(t_0\right),t_0\right]}{\partial W^2}\cdot \\ & \left[W(t) - W\left(t_0\right)\right]^2 + O\left(h^2\right)\Big\} \end{aligned} \tag{32}$$

where $\bar{t}\,\epsilon\left[t_0,t\right]$.

In (32), take the $E\left(t_0\right)$ operator onto each term and, noting that $I\left[W\left(t_0\right),t_0\right] = E\left(t_0\right)$ $I\left[W\left(t_0\right),t_0\right]$, subtract $I\left[W\left(t_0\right)t_0\right]$ from both sides. Substitute from equations (10) and (11) for $E\left(t_0\right)\left[W(t) - W\left(t_0\right)\right]$ and $E\left(t_0\right)\left[(W(t)\text{ - }W\left(t_0\right))^2\right]$, and then divide the equation by h. Take the limit of the resultant equation as $h \to 0$ and (32) becomes,

$$\begin{aligned} 0 = \underset{\{C(t),w(t)\}}{\text{Max}} \Bigg[& e^{-\rho t}U[C(t)] + \frac{\partial I_t}{\partial t} + \frac{\partial I_t}{\partial W}[(w(t)(a-r)+r)W(t) - C(t)] \\ & +1/2\frac{\partial^2 I_t}{\partial W^2}\sigma^2 w^2(t)W^2(t)\Bigg]\Bigg] \end{aligned} \tag{32'}$$

where I_t is short for $I[W(t),t]$ and the subscript on t_0 has been dropped to reflect that $(17')$ holds for any $t \in [0,T]$. If we define $\phi(w,C;W;t) \equiv \Big\{ \; e^{-\rho t}U(C) \; + \frac{\partial I_t}{\partial t} + \frac{\partial I_t}{\partial W}[(w(t)(a - r) + r)W(t) - C(t)] + 1/2\frac{\partial^2 I_t}{\partial W^2}\sigma^2 w^2(t)W^2(t) \Big\}$, then $(17')$ can be written in the more compact form,

$$\begin{aligned} &\text{Max}\, \phi(w,C;W,t) = 0 \\ &\{C,w\} \end{aligned} \tag{32"}$$

The first-order conditions for a regular interior maximum to (17") are

$$\phi_c\,[w^*,C^*;W,:t] = 0 = e^{-\rho t}U'(C) - \partial I_t/\partial W \tag{33}$$

and

$$\phi_w\,[w^*;C^*;W;t] = 0 = (a-r)\frac{\partial I_t}{\partial W} + \frac{\partial^2 I_t}{\partial W^2}wW\sigma^2. \tag{34}$$

A set of sufficient conditions for a regular interior maximum is $\phi_{ww} < 0; \phi_{cc} < 0; \det \begin{bmatrix} \phi_{ww} & \phi_{wC} \\ \phi_{Cw} & \phi_{CC} \end{bmatrix} > 0$. $\phi_{wC} = \phi_{Cw} = 0$, and if $I[W(t),t]$ were strictly concave in W, then

$$\phi_{CC} = U''(c) < 0, \tag{35}$$

by the strict concavity of U and

$$\phi_{ww} = W(t)\sigma^2\frac{\partial^2 I_t}{\partial W^2} < 0, \tag{36}$$

by the strict concavity of I_t and the sufficient conditions would be satisfied. [20]

The optimality conditions can be re-written as a set of two algebraic and one partial differential equation to be solved for $w^*(t), C^*(t)$, and $I[W(t),t]$.

$$(^*) \left\{ \begin{array}{l} \phi\,[w^*,C^*;W;t] = 0 \\ \phi_C\,[w^*;C^*;W;t] = 0 \\ \phi_w\,[w^*,C^*;W;t] = 0 \\ \text{subject to the boundary condition} \\ I[W(T),T] = B[W(T),T] \text{ and} \\ \text{the solution being a feasible solution to } (14). \end{array} \right\}$$

7.2.3 Constant Relative Risk Aversion

The system (*) of a nonlinear partial differential equation coupled with two algebraic equations is difficult to solve in general. However, if the utility function is assumed to be of the

form yielding constant relative risk-aversion, then (*) can be solved explicitly. Therefore, let $U(C) = C^{\gamma}/\gamma, \gamma < 1$ and $\gamma \neq 0$. Then, system (*) can be written in this particular case as

$$(*') \left\{ \begin{array}{c} 0 = \left[\frac{(1-\gamma)}{\gamma}\frac{\partial I_t}{\partial W}\right]^{\gamma/\gamma-1} e^{-\rho t/1-\gamma} + \frac{\partial I_t}{\partial t} + \frac{\partial I_t}{\partial W} rW - \frac{(a-r)^2}{2\sigma^2}\frac{[\partial I_t/\partial W]^2}{\partial^2 I_t/\partial W^2} \\ C^*(t) = \left[e^{\rho t}\frac{\partial I_t}{\partial W}\right]^{1/\gamma-1} \\ w^*(t) = \frac{-(a-r)\partial I_t/\partial W}{\sigma^2 W \partial^2 I_t/\partial W^2} \\ \text{subject to } I[W(T),T] = \epsilon^{1-\gamma}e^{-\rho T} * [W(T)]^{\gamma}/\gamma, \text{ for } 0 < \epsilon << 1 \end{array} \right\}$$

To solve $(17'')$ of $(*')$, take as a trial solution,

$$\bar{I}_t[W(t),t] = \frac{b(t)}{\gamma} e^{-\rho t}[W(t)]^{\gamma}. \tag{37}$$

By substitution of the trial solution into (32"), a necessary condition that $\bar{I}_t[W(t),t]$ be a solution to (32") is found to be that $b(t)$ must satisfy the following ordinary differential equation,

$$\dot{b}(t) = \mu b(t) - (1-\gamma)[b(t)] - \gamma/1-\gamma \tag{38}$$

where $\mu \equiv \rho - \gamma\left[(a-r)^2/2\sigma^2(1-\gamma) + r\right]$. The resulting decision rules for consumption and portfolio selection, $C^*(t)$ and $w^*(t)$, are from equations (18) and (19) of $(*')$, then

$$C^*(t) = [b(t)]^{1/\gamma-1}W(t) \tag{39}$$

and

$$w^*(t) = \frac{(a-r)}{\sigma^2(1-\gamma)}. \tag{40}$$

The solution to (38) is

$$b(t) = \left\{\left[1 + (v\epsilon - 1)e^{\nu(t-T)}\right]/\nu\right\}^{1-\gamma} \tag{41}$$

where $\nu \equiv \mu/(1-\gamma)$. Then the optimal consumption and portfolio selection rules are,

$$\begin{aligned} C^*(t) = \left[v/\left(1 + (v\epsilon - 1)e^{\nu(t-T)}\right)\right] W(t), &\text{(for none-zero v)} \\ = [1/(T-t+\epsilon)]W(t), &\text{(for v=0)} \end{aligned} \tag{42}$$

and

$$w^*(t) = \frac{(a-r)}{\sigma^2(1-\gamma)}, \tag{43}$$

a constant independent of W or t. [16]

7.2.4 Dynamic Behavior and the Bequest Valuation Function

To examine some of the dynamic properties of $C^*(t)$, let $\epsilon = 0$, and define $V(t) \equiv [C^*(t)/W(t)]$, the instantaneous marginal (in this case, also average) propensity to consume out of wealth. Then, from (42),

$$\dot{V}(t) = [V(t)]^2 e^{\nu(t-T)} \tag{44}$$

and, as observed in figure 1 (for $\epsilon = 0$), $V(t)$ is an increasing function of time. In a generalization of the half-life calculation of radioactive decay, define τ as that $t \in [0, T]$ such that $V(\tau) = nV(0)$ (i.e., τ is the length of time required for $V(t)$ to grow to n times its initial size).

Then, from (42),

$$\tau = \log \left[e^{vT} \left(1 - \frac{1}{n} \right) + \frac{1}{n} \right] / v; \text{for} v \neq 0. \tag{45}$$

$$\tau = \frac{(n-1)}{(n)} T \quad , \text{for} v = 0 \tag{46}$$

To examine the dynamic behavior of $W(t)$ under the optimal decision rules, it only makes sense to discuss the expected or "averaged" behavior because $W(t)$ is a function of a random variable. To do this, we consider equation (28), the averaged budget equation, and evaluate it at the optimal (w^*, C^*) to form

$$\frac{\overset{\circ}{W}(t)}{W(t)} = a_* - V(t) \tag{28'}$$

where

$$a_* = \left[\frac{(a-r)^2}{\sigma^2(1-\gamma)} + r \right]$$

By differentiating $(13')$ and using (45), we get

$$\frac{d}{dt} \left[\frac{\overset{\circ}{W}}{W} \right] = -\dot{V}(t) < 0 \tag{47}$$

which implies that for all finite-horizon optimal paths, the expected rate of growth of wealth is a diminishing function of time. Therefore, if $a_* < V(0)$, the individual will disinvest (i.e., he will plan to consume more than his expected income).

If $a_* > V(0)$, he will plan to increase his wealth for $0 < t < \bar{t}$, and then, dis-invest at an expected rate $a_* < V(t)$ for $\bar{t} < t < T$ where $\bar{t}$ is defined as

$$\bar{t} = T + \frac{1}{v} \log \left[\frac{a_* - v}{a_*} \right]. \tag{48}$$

Further, $\partial \bar{t} / \partial a_* > 0$ which implies that the length of time for which the individual is a net saver increases with increasing expected returns on the portfolio. [7]

7.2.5 Infinite Time Horizon

Although the infinite time horizon case ($T = \infty$) yields essentially the same substantive results as in the finite time horizon case, it is worth examining separately because the optimality equations are easier to solve than for finite time. Therefore, for solving more complicated problems of this type, the infinite time horizon problem should be examined first.

The equation of optimality is, from section III,

$$\begin{aligned} 0 = \max_{\{C,w\}} \Big[& e^{-\rho t} U(C) + \frac{\partial I_t}{\partial t} + \frac{\partial I_t}{\partial W} [(w(t)(a-r)+r) W(t) \\ & + 1/2 \frac{\partial^2 I_t}{\partial W^2} \sigma^2 w^2(t) W^2(t) \Big] \end{aligned} \tag{17'}$$

However $(17')$ can be greatly simplified by eliminating its explicit time-dependence. Define

$$\begin{aligned} J[W(t), t] &\equiv e^{\rho t} I[W(t), t] \\ &= \mathrm{Max}_{C,w}\, E(t) \int_t^\infty e^{-\rho(s-t)} U[C] ds \\ &= \mathrm{Max}_{C,w}\, E \int_0^\infty e^{-\rho v} U[C] dv. \end{aligned} \tag{49}$$

Thus, write $J[W(t), t] = J[W]$ to reflect this independence. Substituting $J[W]$, dividing by $e^{-\rho t}$, and dropping all t subscripts, we can rewrite $(17')$ as,

$$\begin{aligned} 0 = \max \big[& U(C) - \rho J + J'(W). \{(w(t)(a-r)+r) W - C\} \\ & + 1/2 J''(W) \sigma^2 w^2 W^2 \big]. \end{aligned} \tag{50}$$

Note that when (50) is evaluated at the optimum (C^*, w^*), it becomes an ordinary differential equation instead of the usual partial differential equation of $(17')$.

For the iso-elastic case, (50) can be written as

$$0 = \frac{(1-\gamma)}{\gamma}\left[J'(W)\right]^{-\gamma/1-\gamma} - \rho' J(W) - \frac{(a-r)^2}{2\sigma^2}\frac{\left[J'(W)\right]^2}{J''(W)} + rWJ'(W) \tag{51}$$

where the functional equations for C^* and w^* have been substituted in equation (51).

The first-order conditions corresponding to (33) and (34) are

$$0 = U'(C) - J'(W) \tag{52}$$

and

$$0 = (a-r)J'(W) + J''wW^2 \tag{53}$$

and assuming that $\text{limit}_{T\to\infty} B[W(T), T] = 0$, the boundary condition becomes,

$$\text{limit}_{t\to\infty} E[I[W(t), t]] = 0 \tag{54}$$

or

$$\text{limit}_{t\to\infty} E\left[e^{-\rho t} J[W(t)]\right] = 0 \tag{55}$$

Then the rest of the derivation is the same as for the finite horizon case and the optimal decision rules are,

$$C^*_\infty(t) = \left\{\frac{\rho}{1-\gamma} - \gamma\left[\frac{(a-r)^2}{2\sigma^2(1-\gamma)^2} + \frac{r}{1-\gamma}\right]\right\} W(t) \tag{56}$$

and

$$w_\infty * (t) = \frac{(a-r)}{\sigma^2(1-\gamma)}. \tag{57}$$

The ordinary differential equation (50), $J'' = f\,(J, J')$, has "extraneous" solutions other than the one that generates (57) and (58). However, these solutions are ruled out by (54) and conditions A, B, and C of section IV. As was expected, $\text{limit}_{T\to\infty} C^*(t) = C^*_\infty(t)$ and $\text{limit}_{T\to\infty} w^*(t) = w^*_\infty(t)$. [18]

7.2.6 Extension to Many Assets

The model presented in section IV, can be extended to the m-asset case with little difficulty. For simplicity, the solution is derived in the infinite time horizon case, but the result is similar for finite time. Assume the m^{th} asset to be the only certain asset with an instantaneous rate

of return $a_m = \mathrm{r}$ Using the general equations derived in section II, and substituting for $w_m(t) = 1 - \sum_{i=1}^{n} w_i(t)$ where $n \equiv m - 1$, equations (21) and (22) can be written as,

$$\begin{aligned} E(t_0)[W(t) - W(t_0)] &= [w'(t_0)(a - \hat{r}) + r] W(t_0) h \\ -C(t_0) h &+ 0\left(h^2\right) \end{aligned} \tag{58}$$

and

$$E(t_0)\left[(W(t) - W(t_0))^2\right] = ww'(t_0)\Omega w(t_o) W^2(t_0) h + 0\left(h^2\right) \tag{59}$$

where

$w'(t_0) \equiv [w_1(t_0), \ldots, w_n(t_0)]$,

$a \equiv [a_1, \ldots, a_n]$

$\hat{r} \equiv [r, \ldots, r]$ $\Omega \equiv [\sigma_{ij}]$, a $n \times n$ variance-covariance matrix of the risky assets

Ω is symmetric and positive definite.

Then, the general form of (50) for m-assets is, in matrix notation,

$$\begin{aligned} 0 = \mathrm{Max}_{\{C,w\}}[U(C) - \rho J(W) &+ J'(W)\{[w'(a - \hat{r}) + r] W - C\} \\ &+ \frac{1}{2} J''(W) w'\Omega w W^2 \Big] \end{aligned} \tag{60}$$

and instead of two, there will be m first-order conditions corresponding to a maximization of (50) with respect to $w_1, \ldots, w_n$ and C. The optimal decision rules corresponding to (57) and (58) in the two-asset case, are

$$\begin{aligned} C_\infty * (t) = \Bigg\{ \frac{\rho}{1-\gamma} - \gamma \Bigg[&\frac{(a - \hat{r})'\Omega^{-1}(a - \hat{r})}{2(1-\gamma)^2} \\ &+ \frac{r}{1-\gamma} \Bigg] \Bigg\} W(t) \end{aligned} \tag{61}$$

and

$$w_\infty * (t) = \frac{1}{(1-\gamma)} \Omega^{-1}(a - \hat{r}) \tag{62}$$

where

$$w^{*\prime}_\infty(t) = [w^*_1(t), \ldots, w^*_n(t)].$$

7.2.7 Constant Absolute Risk Aversion

System $(*)$ of section III, can be solved explicitly for a second special class of utility functions of the form yielding constant absolute risk-aversion. Let $U(C) = -e^{-\eta C}/\eta, \eta > 0$,

where $-U''(C)/U'(C) = \eta$. For convenience, I return to the two-asset case and infinite-time horizon form of system $(*)$ which can be written in this case as,

$$
\left(
\begin{array}{l}
*'' \left\{
\begin{array}{l}
0 = \frac{-J'(W)}{\eta} - \rho J(W) + J'(W) rW \quad + \frac{J'(W)}{\eta} \log\left[J'(W)\right] \\
\quad - \frac{(a-r)^2}{2\sigma^2} \frac{[J'(W)]^2}{J''(W)} \\
C^*(t) = -\frac{1}{\eta} \log\left[J'(W)\right] \\
w^*(t) = -J'(W)(a-r)/\sigma^2 W J'') \\
\text{subject to } \operatorname{limit}_{t\to\infty} E\left[e^{-\rho t} J(W(t))\right] = 0
\end{array}
\right.
\end{array}
\right.
$$

where $J(W) \equiv e^{\rho t} I[W(t), t]$. [2]

To solve $(17'')$ of $(*'')$, take as a trial solution, $\bar{J}(W) = \frac{-p}{q} e^{-qW}$. By substitution of the trial solution into $(17'')$, a necessary condition that $\bar{J}(W)$ be a solution to $(17'')$ is found to be that p and q must satisfy the following two algebraic equations:

$$q = \eta r \tag{63}$$

and

$$p = e^{\left(\frac{r-\rho-(a-r)^2/2\sigma^2}{r}\right)} \tag{64}$$

The resulting optimal decision rules for portfolio selection and consumption are,

$$C^*(t) = rW(t) + \left[\frac{\rho - r + (a-r)^2/2\sigma^2}{\eta^r}\right] \tag{65}$$

and

$$w^*(t) = \frac{(a-r)}{\eta r^2 \sigma^2 W(t)}. \tag{66}$$

Comparing equations (65) and (66) with their counterparts for the constant relative risk-aversion case, (57) and (58), one finds that consumption is no longer a constant proportion of wealth (i.e., marginal propensity to consume does not equal the average propensity) although it is still linear in wealth. Instead of the proportion of wealth invested in the risky asset being constant (i.e., $w^*(t)$ is a constant), the total dollar value of wealth invested in the risky asset is kept constant (i.e., $w^*(t)W(t)$ is a constant). As one becomes wealthier, the proportion of his wealth invested in the risky asset falls, and asymptotically, as $W \to \infty$, one invests all his wealth in the certain asset and consumes all his (certain) income.

8 Application 2: Dynamics based on analysis of public data for spreading of disease

The coronavirus disease (COVID-19) emerged in China in the end of 2019 year and since then it has spread reaching all countries in the globe. The quick spreading of the disease made the study of spreading of great importance for forecasting, control and prevention. On the other hand, the modelling of natural phenomena up to end of nineteenth century can be viewed as the study of deterministic solutions of differential equations.

At that time thought that whether all initial data could only be collected, one would be able to predict the future with certainly. We now know this is not so since that the existence of chaos has arisen, in which even quite simple differential equation systems have the rather property of giving rise to unpredictable behavior.

Here, we aim to use the stochastic nonlinear differential equation in Itô's calculus for modeling of the dynamic of novel cases daily as well as the cumulative cases number since the beginning of pandemic until today. We use different statistical tests to give a future estimating of novel cases and behavior of the curve of the spreading of disease by solving the stochastic differential equation and using the probability density $P(N,t)$, obtained analytically from solution of the Fokker–Planck equation.

Due to large uncertainly in the official data about the real number of novel cases generated by the low number of tests realized in countries as Brazil, where the effect of randomness in the official data is modelled by the random term in the stochastic differential equation, being the use of this analysis hence, largely adequate to treat the spreading of coronavirus disease.

It is known that the logistic model of population growing can be used as model for some infectious diseases as the smallpox that are spread largely by individuals who can transmit the disease but who exhibit no overt symptoms. The logistic model may be modified so that unbounded growth does not occur. The simplest way to do this is to introduce a factor which will have the effect of making the variation rate of population growth to be $d\wp/dt$ negative when $\wp$ is large. Where the model is given in form $d\wp/dt = -\alpha\wp(1-\nu\wp)(1-\eta\wp)$, where α and η are parameters and $\alpha > 0, 0 < \eta < v$. Moreover, we can allow some randomness in these parameters with aim to obtain a more realistic model of real situation.

8.0.1 Model for dynamics of novel cases daily

The model for spread of novel cases of COVID-19 in each day is given by the modified logistic model of growth with threshold with the addition of randomness term given by the following stochastic differential equation (SDE) of Itô

$$dN(t) = h + [g(t) - \alpha N(t)(1 - vN(t))(1 - \eta N(t))]dt + \beta(t)dW. \tag{67}$$

where α, v and η parameters comes from the logistic model with threshold for spreading of the smallpox with the values $\alpha = v = 0.125$ used by Bernoulli. Thus, the range of η employed in calculations is within the interval $0 < \eta < v$. $g(t)$ is another fit least squares to the set data within the range considered. The deterministic part $g(t)$, can adjust to the reported novel cases in each day t using a n degree $(n \leq 4)$ polynomial.

The constant h is introduced due to the assumption of a constant rate when the novel cases number N is large, being this assumption reasonable in this limit. However, it becomes less reasonable when N is small. Furthermore, $W(t)$ is the Wiener process or Brownian motion. Thus, we choose a multiplicative white noise of the form $\beta_0 t^n$, where β_0 is an arbitrary constant. Particularly, whether $\xi(t)$ presents the properties $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \Gamma\delta(t - t')$ with Γ constant being the amplitude of the white noise.

The dynamics of novel cases, $N(t)$, of infected by COVID-19 registered in the Brazil as a function of time (days) registered in the period from 15th March, 2020 up to 12th August is displayed in Fig. 1. We obtain the time series of the model Eq. (67). From fit of least squares to the set of data supplied of the Brazilian agencies, we obtain the fit $g(t)$ given by $g(t) = 8191.2 - 620.92t + 12.63t^2 - 0.05t^3$. The zigzag behavior in the range of large t values reflects in an increase of the uncertainty in the data due to the low-number of test performed in the population. We plot the time series of the model Eq. (67) for $\alpha = v = 0.125, \eta = 0.0625$ and noise amplitude $\Gamma = 1$ as well as $\beta(t) = 3.0 \times 10^{-6} t^3$.

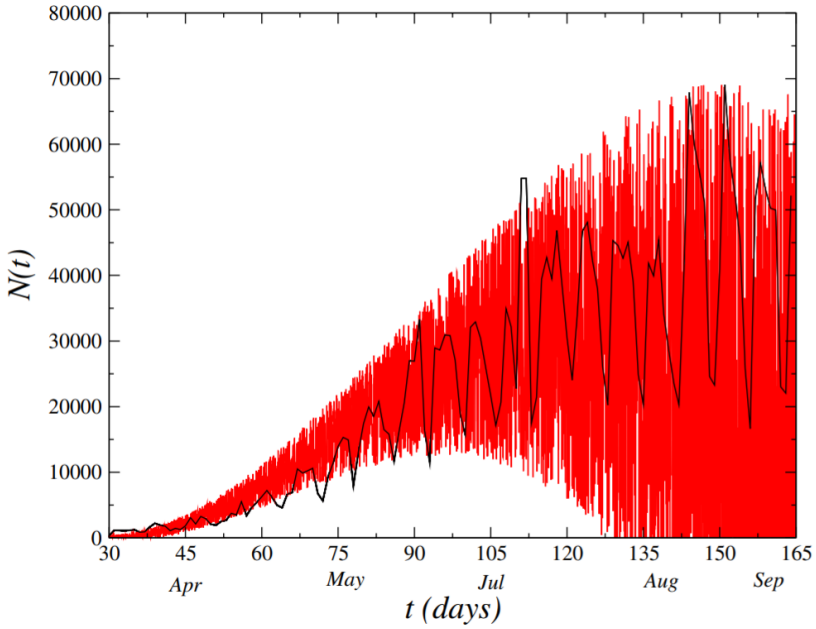

Figure 1: Model for dynamics of novel cases daily in Brazil

Then we plot the time series of the model Eq. (67) for values $\alpha = v = 0.125$, $\eta = 0.0625$, noise amplitude $\Gamma = 1$ and $\beta = 2.0 \times 10^6$.

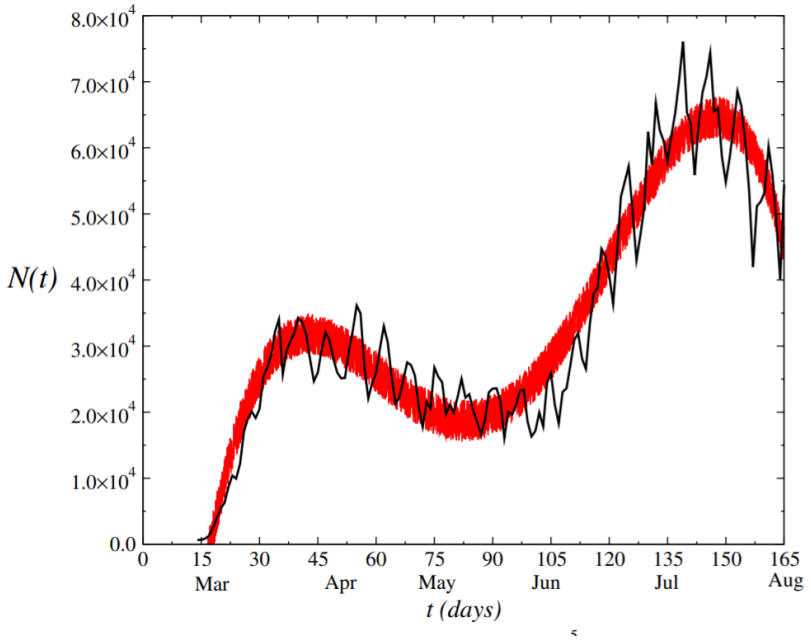

Figure 2: Model for dynamics of novel cases daily in US

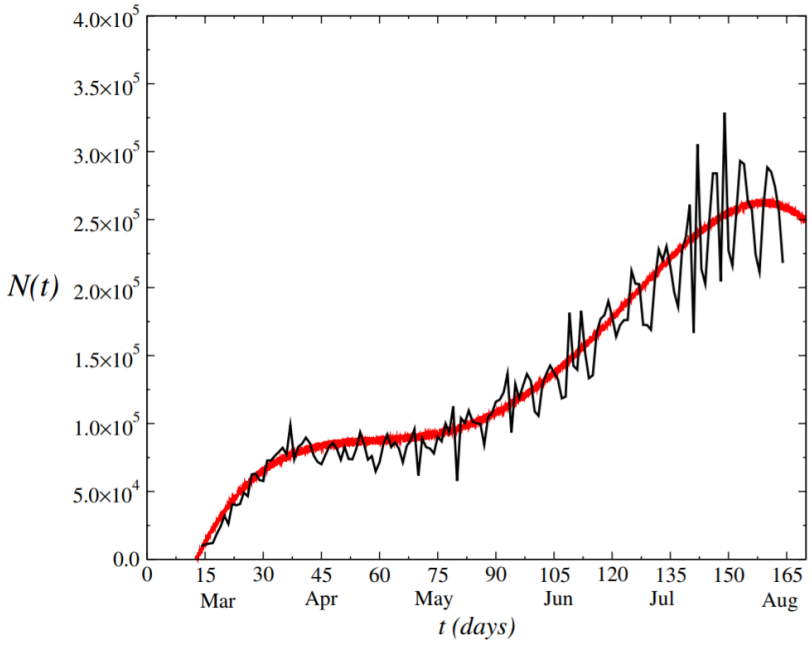

Figure 3: Model for dynamics of novel cases daily in India

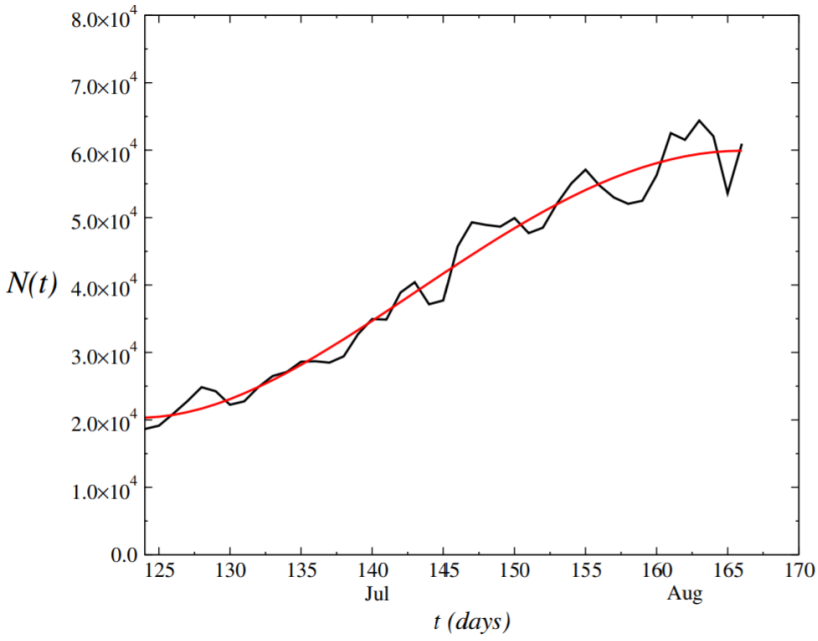

Figure 4: Model for dynamics of novel cases daily over the world

8.1 Analytical results by Fokker–Planck equation

We can perform an analytical analysis solving the correspondent Fokker-Planck equation to the stochastic differential equation Eq. (67). We start from the time development of an

arbitrary function of the stochastic process $N(t), f(N(t))$. Using the Itô formula

$$f[N(t)+dN(t)]-f[N(t)]=$$
$$\partial_N f[N(t)]\left\{\left[g\left(t'\right)-\alpha N\left(t'\right)\left(1-vN\left(t'\right)\right)\left(1-\eta N\left(t'\right)\right)\right]dt+\beta dW\right\}$$
$$+\frac{\beta^2}{2}\partial_N^2 f[N(t)](dW)^2$$

where the higher order terms have been discarded, and $(dW(t))^2 = dt$.

Taking the average of both sides in the equation above, we find

$$\left\langle\frac{\partial f}{\partial t}\right\rangle=\left\langle\left[\frac{\partial f}{\partial x}\{[g(t)-\alpha x(1-vx)(1-\eta x)]dt+\beta dW\}+\frac{\beta^2}{2}\frac{\partial^2 f}{\partial x^2}\right]\right\rangle \tag{68}$$

In following, using

$$\frac{d}{dt}\langle f(N(t))\rangle=\frac{d}{dt}\int_{-\infty}^{\infty}dxf(x)P(x,t)=\int_{-\infty}^{\infty}dxf(x)\frac{\partial}{\partial t}[P(x,t)]$$
$$=\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}\{[g(t)-\alpha x(1-vx)(1-\eta x)]\}P(x,t)dx\quad+\frac{\beta^2}{2}\int_{-\infty}^{\infty}\frac{\partial^2 f}{\partial x^2}P(x,t)dx$$

we integrate by parts and discard surface terms to find

$$\int_{-\infty}^{\infty}dxf(x)\frac{\partial}{\partial t}[P(x,t)]=\frac{\beta^2}{2}\int_{-\infty}^{\infty}f(x)\frac{\partial^2}{\partial x^2}[P(x,t)]dx$$
$$+\int_{-\infty}^{\infty}f(x)\frac{\partial}{\partial x}\{[g(t)-\alpha x(1-vx)(1-\eta x)]P(x,t)\}dx$$

and hence

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}\{[g(t)-\alpha x(1-vx)(1-\eta x)]P(x,t)\}+\frac{\beta^2}{2}\frac{\partial^2}{\partial x^2}P(x,t).$$

Taking the Fourier transform of the above equation, we can guarantee the normalization of the probability density where $P(x)$ is well behaved. We take the boundaries at infinity as $\lim_{x\to\infty}P(x,t)=0$ and $\partial_x P(x)$ being reasonably well behaved. As $\lim_{x\to\infty}\partial_x P(x,t)=0$ so, a nonzero current of probability at infinity will usually require that the terms in the equation above will become infinite there. We use the initial condition $P\left(x_0,0\right)=P_0$. For solving the Fokker-Planck equation time independent we make the power series expansion $P(x,t)=\sum_{n=0}^{\infty}a_n(t)x^n$ to find

$$\frac{\partial P}{\partial t}=\alpha(1-2vx)(1-\eta x)P-[g(t)-\alpha x(1-vx)(1-\eta x)]\frac{\partial P}{\partial x}$$
$$+\frac{\beta^2}{2}\frac{\partial^2 P}{\partial x^2}\sum_{n=0}^{\infty}\left(\frac{da_n}{dt}+g(t)(n+1)a_{n+1}\right)x^n$$

We obtain the following recurrence relations

$$\alpha(1-2vx)(1-\eta x)\sum_{n=0}^{\infty}a_n x^n+\alpha x(1-vx)(1-\eta x)\sum_{n=1}^{\infty}(n+1)a_{n+1}x^n$$
$$+\frac{\beta^2}{2}\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n=k$$

where k is a separation constant. For $k=0$, we obtain other recurrence relations given by

$$a_2=-\frac{\alpha}{\beta^2}a_0,\quad a_3=-\frac{\alpha}{3\beta^2}a_1,\quad a_4=\frac{\alpha^2}{6\beta^4}a_0,\quad a_5=\frac{\alpha^2}{30\beta^4}a_1.$$

Additionally, we have

$$\frac{da_n(t)}{dt}+g(t)(n+1)a_{n+1}(t)=k \tag{69}$$

Therefore, we obtain $P(x)$ in the form

$$P(x)=a_0\left(1-\frac{\alpha}{\beta^2}x^2+\frac{\alpha^2}{\beta^4}x^4+\cdots\right)+a_1\left(x-\frac{\alpha}{3\beta^2}x^3+\frac{\alpha^2}{30\beta^4}x^5\cdots\right)$$

where the constants a_0 and a_1 are determined by the initial conditions $P(0,0)=P_0$ and $\partial_x P(x,0)=0$ in $x=0$.

We find $a_0=P_0$ and $a_1=0$. From the normalization condition, the second term in the density probability above must be zero and hence, all coefficients a_1 must cancel. Hence, we have

$$P(x,t)=P_0\left(1-\frac{\alpha}{\beta^2}x^2+\frac{\alpha^2}{\beta^4}x^4+\cdots\right)$$

To ensure the normalization of the probability density, P_0 must be non zero only within the interval $-\varepsilon\leq x\leq\varepsilon$ and zero out it. [10]

For $k\neq 0$, we have that $n=0$ and $a_2+\alpha a_0/\beta^2=k$ and all a_n higher are zero. Thus, we find from integration of the Eq. (69)

$$P(t)=\frac{\beta^2}{\alpha}P_0 t\left[1-a_2(t)\right]+a_0(0)-\int_0^t g\left(t'\right)a_1\left(t'\right)dt'.$$

9 Application 3: Supervised learning

Often, machine learning methods are broken into two phases:

- $\boldsymbol{Training}$ **:** A model is learned from a collection of training data.

- ***Application*** **:** A model is learned from a collection of training data.

Some of the main types of machine learning are:

- Supervised Learning, in which the training data is labeled with the correct answers. The two most common types of supervised learning are classification (where the outputs are discrete labels, as in spam filtering) and regression (where the outputs are real-valued).
- Unsupervised learning, in which we are given a collection of unlabeled data, which we wish to analyze and discover patterns within. The two most important examples are dimension reduction and clustering.
- Reinforcement learning, in which an agent (e.g., a robot or controller) seeks to learn the optimal actions to take based the outcomes of past actions.

9.1 Regression

In regression, our goal is to learn a mapping from one real-valued space to another. Linear regression is the simplest form of regression: it is easy to understand, often quite effective, and very efficient to learn and use.
We will start by considering linear regression in just 1 dimension. Here, our goal is to learn a mapping $y = f(x)$, where x and y are both real-valued scalars (i.e., $x \in \mathbb{R}, y \in \mathbb{R}$). We will take f to be an linear function of the form:

$$y = wx + b \tag{70}$$

where w is a weight and b is a bias. These two scalars are the parameters of the model, which we would like to learn from training data. n particular, we wish to estimate w and b from the N training pairs $\{(x_i, y_i)\}_{i=1}^{N}$. Then, once we have values for w and b, we can compute the y for a new x.

Given 2 data points (i.e., $\mathbf{N} = 2$), we can exactly solve for the unknown slope w and offset b. When $N > 2$ we will not be able to find unique parameter values for which $y_i = wx_i + b$ for all i, since we have many more constraints than parameters. The best we can hope for is to find the parameters that minimize the residual errors, i.e., $y_i - (wx_i + b)$.

The most commonly-used way to estimate the parameters is by least-squares regression. We define an energy function (a.k.a. objective function):

$$E(w,b) = \sum_{i=1}^{N} \left(y_i - (wx_i + b)\right)^2 \tag{71}$$

To estimate w and b, we solve for the w and b that minimize this objective function. This can be done by setting the derivatives to zero and solving.

$$\frac{dE}{db} = -2\sum_i \left(y_i - (wx_i + b)\right) = 0 \tag{72}$$

Solving for b gives us the estimate:

$$b^* = \frac{\sum_i y_i}{N} - w\frac{\sum_i x_i}{N} \tag{73}$$

$$= \bar{y} - w\bar{x} \tag{74}$$

where we define $\bar{x}$ and $\bar{y}$ as the averages of the x 's and y 's, respectively. This equation for b^* still depends on w, but we can nevertheless substitute it back into the energy function:

$$E(w,b) = \sum_i \left(\left(y_i - \bar{y}\right) - w\left(x_i - \bar{x}\right)\right)^2 \tag{75}$$

Then:

$$\frac{dE}{dw} = -2\sum_i \left(\left(y_i - \bar{y}\right) - w\left(x_i - \bar{x}\right)\right)\left(x_i - \bar{x}\right) \tag{76}$$

Solving $\frac{dE}{dw} = 0$ then gives:

$$w^* = \frac{\sum_i \left(y_i - \bar{y}\right)\left(x_i - \bar{x}\right)}{\sum_i \left(x_i - \bar{x}\right)^2} \tag{77}$$

Now, suppose we wish to learn a mapping from D-dimensional inputs to scalar outputs: $\mathbf{x} \in \mathbb{R}^D$, $y \in \mathbb{R}$. Now, we will learn a vector of weights $\mathbf{w}$, so that the mapping will be:

$$f(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + b = \sum_{j=1}^{D} w_j x_j + b \tag{78}$$

For convenience, we can fold the bias b into the weights, if we augment the inputs with an additional 1. In other words, if we define

$$\tilde{\mathbf{w}} = \begin{bmatrix} w_1 \\ \vdots \\ w_D \\ b \end{bmatrix}, \quad \tilde{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_D \\ 1 \end{bmatrix} \tag{79}$$

then the mapping can be written:

$$f(\mathbf{x}) = \tilde{\mathbf{w}}^T \tilde{\mathbf{x}}. \tag{80}$$

Given N training input-output pairs, the least-squares objective function is then:

$$E(\tilde{\mathbf{w}}) = \sum_{i=1}^{N} \left(y_i - \tilde{\mathbf{w}}^T \tilde{\mathbf{x}}_i\right)^2 \tag{81}$$

If we stack the outputs in a vector and the inputs in a matrix, then we can also write this as:

$$E(\tilde{\mathbf{w}}) = \|\mathbf{y} - \tilde{\mathbf{X}}\tilde{\mathbf{w}}\|^2 \tag{82}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad \tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{x}_1^T & 1 \\ \vdots & \\ \mathbf{x}_N^T & 1 \end{bmatrix} \tag{83}$$

and $\|\cdot\|$ is the usual Euclidean norm.

Equation 13 is known as a linear least-squares problem, and can be solved by methods from linear algebra. We can rewrite the objective function as:

$$\begin{aligned} E(\mathbf{w}) &= (\mathbf{y} - \tilde{\mathbf{X}}\tilde{\mathbf{w}})^T(\mathbf{y} - \tilde{\mathbf{X}}\tilde{\mathbf{w}}) \\ &= \tilde{\mathbf{w}}^T\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}\tilde{\mathbf{w}} - 2\mathbf{y}^T\tilde{\mathbf{X}}\tilde{\mathbf{w}} + \mathbf{y}^T\mathbf{y} \end{aligned} \tag{84}$$

We can optimize this by setting all values of $dE/dw_i = 0$ and solving the resulting system of equations. The solution is given by:

$$\mathbf{w}^* = \left(\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}\right)^{-1}\tilde{\mathbf{X}}^T\mathbf{y} \tag{85}$$

In the most general case, both the inputs and outputs may be multidimensional. For example, with D-dimensional inputs, and K-dimensional outputs $y \in \mathbb{R}^K$, a linear mapping from input to output can be written as

$$\mathbf{y} = \tilde{\mathbf{W}}^T\tilde{\mathbf{x}} \tag{86}$$

where $\tilde{\mathbf{W}} \in \mathbb{R}^{(D+1)\times K}$.

It is convenient to express $\tilde{\mathbf{W}}$ in terms of its column vectors, i.e.,

$$\tilde{\mathbf{W}} = \begin{bmatrix} \tilde{\mathbf{w}}_1 & \dots & \tilde{\mathbf{w}}_K \end{bmatrix} \equiv \begin{bmatrix} \mathbf{w}_1 & \dots & \mathbf{w}_K \\ b_1 & \dots & b_K \end{bmatrix} \tag{87}$$

In this way we can then express the mapping from the input $\tilde{\mathbf{x}}$ to the j_{th} element of $\mathbf{y}$ as $y_j = \tilde{\mathbf{w}}_j^T \mathbf{x}$. Now, given N training samples, denoted $\{\tilde{\mathbf{x}}_i, \mathbf{y}_i\}_{i=1}^N$ a natural energy function to minimize in order to estimate $\tilde{\mathbf{W}}$ is just the squared residual error over all training samples and all output dimensions, i.e.,

$$E(\tilde{\mathbf{W}}) = \sum_{i=1}^{N} \sum_{j=1}^{K} \left(y_{i,j} - \tilde{\mathbf{w}}_j^T \tilde{\mathbf{x}}_i\right)^2. \tag{88}$$

There are several ways to conveniently vectorize this energy function. One way is to express E solely as a sum over output dimensions.

That is, let $\mathbf{y}'_j$ be the N-dimensional vector comprising the j^{th} component of each output training vector, i.e., $\mathbf{y}'_j = [y_{1,j}, y_{2,j}, \ldots, y_{N,j}]^T$. Then we can write

$$E(\tilde{\mathbf{W}}) = \sum_{j=1}^{K} \left\| \mathbf{y}'_j - \tilde{\mathbf{X}} \tilde{\mathbf{w}}_j \right\|^2 \tag{89}$$

where $\tilde{\mathbf{X}}^T = \begin{bmatrix} \tilde{\mathbf{x}}_1 & \tilde{\mathbf{x}}_2 \ldots \tilde{\mathbf{x}}_N \end{bmatrix}$.
Finally, the solution is again given by:

$$\mathbf{W}^* = \left(\tilde{\mathbf{X}}^T \tilde{\mathbf{X}}\right)^{-1} \tilde{\mathbf{X}}^T \mathbf{Y} \tag{90}$$

9.2 Nonlinear Regression

A common choice for the function $f(\mathbf{x})$ is a basis function representation:

$$y = f(x) = \sum_k w_k b_k(x) \tag{91}$$

for the 1D case. The functions $b_k(x)$ are called basis functions. Often it will be convenient to express this model in vector form, for which we define $\mathrm{b}(x) = [b_1(x), \ldots, b_M(x)]^T$ and $\mathbf{w} = [w_1, \ldots, w_M]^T$ where M is the number of basis functions. We can then rewrite the model as

$$y = f(x) = \mathbf{b}(x)^T \mathbf{w} \tag{92}$$

Two common choices of basis functions are polynomials and Radial Basis Functions (RBF). A simple, common basis for polynomials are the monomials, i.e.,

$$b_0(x) = 1, \quad b_1(x) = x, \quad b_2(x) = x^2, \quad b_3(x) = x^3, \quad \ldots \tag{93}$$

With a monomial basis, the regression model has the form

$$f(x) = \sum w_k x^k, \tag{94}$$

Radial Basis Functions, and the resulting regression model are given by

$$\begin{aligned} b_k(x) &= e^{-\frac{(x-c_k)^2}{2\sigma^2}}, \\ f(x) &= \sum w_k e^{-\frac{(x-c_k)^2}{2\sigma^2}}, \end{aligned} \tag{95}$$

where c_k is the center (i.e., the location) of the basis function and σ^2 determines the width of the basis function. Both of these are parameters of the model that must be determined somehow.

To fit these models, we can again use least-squares regression, by minimizing the sum of squared residual error between model predictions and the training data outputs:

$$E(\mathbf{w}) = \sum_i \left(y_i - f\left(x_i\right)\right)^2 = \sum_i \left(y_i - \sum_k w_k b_k(x)\right)^2 \tag{96}$$

Directly minimizing squared-error can lead to an effect called overfitting, wherein we fit the training data extremely well, yet we obtain a model that produces very poor predictions on future test data whenever the test inputs differ from the training inputs. Overfitting can be understood in many ways, all of which are variations on the same underlying pathology:

1. The problem is insufficiently constrained: for example, if we have ten measurements and ten model parameters, then we can often obtain a perfect fit to the data.

2. Fitting noise: overfitting can occur when the model is so powerful that it can fit the data and also the random noise in the data.

We can add regularization: an extra term to the learning objective function that prefers smooth models. For example, for RBF regression with scalar outputs, and with many other types of basis functions or multi-dimensional outputs, this can be done with an objective function of the form:

$$E(\mathbf{w}) = \underbrace{\|\mathbf{y} - \mathbf{B}\mathbf{w}\|^2}_{\text{data term}} + \underbrace{\lambda\|\mathbf{w}\|^2}_{\text{smoothness term}} \tag{97}$$

This objective function has two terms. The first term, called the data term, measures the model fit to the training data. The second term, often called the smoothness term, penalizes non-smoothness (rapid changes in $f(x)$). This particular smoothness term ($\|\mathbf{w}\|$) is called weight decay, because it tends to make the weights smaller.

Weight decay implicitly leads to smoothness with RBF basis functions because the basis functions themselves are smooth, so rapid changes in the slope of f can only be created in RBFs by adding and subtracting basis functions with large weights.

This regularized least-squares objective function is still quadratic with respect to w and can be optimized in closed-form. To see this, we can rewrite it as follows:

$$\begin{aligned} E(\mathbf{w}) &= (\mathbf{y} - \mathbf{B}\mathbf{w})^T(\mathbf{y} - \mathbf{B}\mathbf{w}) + \lambda \mathbf{w}^T\mathbf{w} \\ &= \mathbf{w}^T\mathbf{B}^T\mathbf{B}\mathbf{w} - 2\mathbf{w}^T\mathbf{B}^T\mathbf{y} + \lambda \mathbf{w}^T\mathbf{w} + \mathbf{y}^T\mathbf{y} \\ &= \mathbf{w}^T\left(\mathbf{B}^T\mathbf{B} + \lambda\mathbf{I}\right)\mathbf{w} - 2\mathbf{w}^T\mathbf{B}^T\mathbf{y} + \mathbf{y}^T\mathbf{y} \end{aligned}$$

To minimize $E(\mathbf{w})$, as above, we solve the normal equations $\nabla E(\mathbf{w}) = \mathbf{0}$ (i.e., $\partial E/\partial w_i = 0$ for all i). This yields the following regularized LS estimate for $\mathbf{w}$:

$$\mathbf{w}^* = \left(\mathbf{B}^T\mathbf{B} + \lambda\mathbf{I}\right)^{-1}\mathbf{B}^T\mathbf{y} \tag{98}$$

Another choice of basis function is the sigmoid function. "Sigmoid" literally means "s-shaped." The most common choice of sigmoid is:

$$g(a) = \frac{1}{1 + e^{-a}}$$

Sigmoids can be combined to create a model called an Artificial Neural Network (ANN). For regression with multi-dimensional inputs $\mathbf{x} \in \mathbb{R}_2^K$, and multidimensional outputs $\mathbf{y} \in \mathbb{R}^{K_1}$:

$$\mathbf{y} = f(\mathbf{x}) = \sum_j \mathbf{w}_j^{(1)} g\left(\sum_k w_{k,j}^{(2)} x_k + b_j^{(2)}\right) + \mathbf{b}^{(1)}$$

This equation describes a process whereby a linear regressor with weights $\mathbf{w}^{(2)}$ is applied to $\mathbf{x}$.

The output of this regressor is then put through the nonlinear Sigmoid function, the outputs of which act as features to another linear regressor. Thus, note that the inner weights $w^{(2)}$ are distinct parameters from the outer weights $\mathbf{w}_j^{(1)}$.

We can again write a regularized squared-error objective function:

$$E(w, b) = \|\mathbf{y} - f(\mathbf{x})\|^2 + \lambda\|\mathbf{w}\|^2$$

Unfortunately, this objective function cannot be optimized in closed-form, and numerical optimization procedures must be used.

Many learning procedures amount to smoothing the training data. RBF fitting is an example of this. However, many of these fitting procedures require making a number of decisions, such as the locations of the basis functions, and can be sensitive to these choices. This raises the question: why not cut out the middleman, and smooth the data directly? This is the idea behind K-Nearest Neighbors regression.

We first select a parameter K, which is the only parameter to the algorithm. Then, for a new input x, we find the K nearest neighbors to x in the training set, based on their Euclidean distance $\|\mathbf{x} - \mathbf{x}_i\|^2$. Then, our new output $\mathbf{y}$ is simply an average of the training outputs for those nearest neigbors. This can be expressed as:

$$\mathbf{y} = \frac{1}{K} \sum_{i \in N_K(\mathbf{x})} \mathbf{y}_i$$

where the set $N_K(\mathbf{x})$ contains the indices of the K training points closest to $\mathbf{x}$.

Alternatively, we might take a weighted average of the K-nearest neighbors to give more influence to training points close to x than to those further away:

$$\mathbf{y} = \frac{\sum_{i \in N_K(\mathbf{x})} w\left(\mathbf{x}_i\right) \mathbf{y}_i}{\sum_{i \in N_K(\mathbf{x})} w\left(\mathbf{x}_i\right)}, \quad w\left(\mathbf{x}_i\right) = e^{-\|\mathbf{x}_i - \mathbf{x}\|^2 / 2\sigma^2}$$

where σ^2 is an additional parameter to the algorithm.

9.3 Quadratics

The objective functions used in linear least-squares and regularized least-squares are multi-dimensional quadratics. We now analyze multidimensional quadratics further.

The general form of a one-dimensional quadratic is given by:

$$f(x) = w_2 x^2 + w_1 x + w_0$$

This can also be written in a slightly different way (called standard form):

$$f(x) = a(x - b)^2 + c$$

where $a = w_2, b = -w_1/(2w_2), c = w_0 - w_1^2/4w_2$.

These two forms are equivalent. The general form for a 2D quadratic function is:

$$f(x_1, x_2) = w_{1,1}x_1^2 + w_{1,2}x_1x_2 + w_{2,2}x_2^2 + w_1x_1 + w_2x_2 + w_0$$

and, for an N-D quadratic, it is:

$$f(x_1, \ldots x_N) = \sum_{1 \le i \le N, 1 \le j \le N} w_{i,j}x_ix_j + \sum_{1 \le i \le N} w_ix_i + w_0$$

Note that there are three sets of terms: the quadratic terms $(\sum w_{i,j}x_ix_j)$, the linear terms $(\sum w_ix_i)$ and the constant term (w_0). Let $\mathbf{x}$ be an N-dimensional column vector, written $\mathbf{x} = [x_1, \ldots x_N]^T$. Then we can write a quadratic as:

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$$

where

$$\mathbf{A} = \begin{bmatrix} w_{1,1} & \ldots & w_{1,N} \\ \vdots & w_{i,j} & \vdots \\ w_{N,1} & \ldots & w_{N,N} \end{bmatrix}$$

$$\mathbf{b} = [w_1, \ldots, w_N]^T$$

$$c = w_0$$

For many manipulations we will want to do later, it is helpful for $\mathbf{A}$ to be symmetric, i.e., to have $w_{i,j} = w_{j,i}$. So, if we are a given a quadratic for which $\mathbf{A}$ is asymmetric, we can symmetrize it as:

$$f(\mathbf{x}) = \mathbf{x}^T\left(\frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)\right)\mathbf{x} + \mathbf{b}^T\mathbf{x} + c = \mathbf{x}^T\tilde{\mathbf{A}}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$$

and use $\tilde{\mathbf{A}} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)$ instead. As before, we can convert the quadratic to a form that leads to clearer interpretation:

$$f(\mathbf{x}) = (\mathbf{x} - \boldsymbol{\mu})^T\mathbf{A}(\mathbf{x} - \boldsymbol{\mu}) + d$$

where $\boldsymbol{\mu} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{b}, d = c - \boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu}$, assuming that $\mathbf{A}^{-1}$ exists.

Suppose we wish to find the stationary points of a quadratic

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c. \tag{99}$$

The stationary points occur where all partial derivatives are zero, i.e., $\partial f/\partial x_i = 0$ for all i. The gradient of a function is the vector comprising the partial derivatives of the function, i.e.,

$$\nabla f \equiv [\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial N]^T$$

At stationary points it must therefore be true that $\nabla f = [0, \ldots, 0]^T$. Let us assume that $\mathbf{A}$ is symmetric.

Due to the linearity of the differentiation operator, we can look at each of the three terms separately. Let us examine the first term.

$$(x_1 \ldots x_N) \begin{pmatrix} a_{11} & \ldots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \ldots & a_{NN} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$$

$$= (x_1 a_{11} + x_2 a_{21} + \ldots + x_N a_{N1} x_1 a_{12} + x_2 a_{22} + \ldots$$

We get:

$$\ldots + x_1 a_{1N} + x_2 a_{2N} + \ldots + x_N a_{NN}) \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$$

$$= x_1^2 a_{11} + x_1 x_2 a_{21} + \ldots + x_1 x_N a_{N1} + x_1 x_2 a_{12} + x_2^2 a_{22} + \ldots + x_N x_2 a_{N2} + \ldots$$

$$\ldots x_1 x_N a_{1N} + x_2 x_N a_{2N} + \ldots + x_N^2 a_{NN}$$

$$= \sum_{ij} a_{ij} x_i x_j$$

The i^{th} element of the gradient corresponds to $\partial f/\partial x_i$. So in the expression above, for the terms in the gradient corresponding to each x_i, we only need to consider the terms involving x_i (others will have derivative zero), namely

$$x_i^2 a_{ii} + \sum_{j \neq i} x_i x_j \left(a_{ij} + a_{ji}\right)$$

The gradient then has a very simple form:

$$\frac{\partial \left(\mathbf{x}^T \mathbf{A} \mathbf{x}\right)}{\partial x_i} = 2 x_i a_{ii} + \sum_{j \neq i} x_j \left(a_{ij} + a_{ji}\right).$$

We can write a single expression for all of the x_i using matrix/vector form:

$$\frac{\partial \mathbf{x}^T \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = \left(\mathbf{A} + \mathbf{A}^T\right) \mathbf{x}.$$

If we assume that $\mathbf{A}$ is symmetric, then we have

$$\frac{\partial \mathbf{x}^T \mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$$

For the next term in the cost function, we have:

$$\mathbf{b}^T\mathbf{x} = b_1x_1 + b_2x_2 + \ldots + b_Nx_N$$

then $\partial f/\partial x_i = b_i$. We can again express this in matrix/vector form:

$$\frac{\partial \left(\mathbf{b}^T\mathbf{x}\right)}{\partial \mathbf{x}} = \mathbf{b}.$$

Combining both of the expressions we have just derived, we get:

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x} + \mathbf{b} = [0, \ldots, 0]^T$$

The optimum is given by the solution to this system of equations:

$$\mathbf{x} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{b}$$

For linear regression with multi-dimensional inputs above: $\mathbf{A} = \mathbf{X}\mathbf{X}^T$ and $\mathbf{b} = -2\mathbf{X}\mathbf{y}^T$.

9.4 Gaussian distribution

The simplest case is a Gaussian PDF over a scalar value x, in which case the PDF is:

$$p\left(x \mid \mu, \sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).$$

The Gaussian has two parameters, the mean μ, and the variance σ^2. The mean specifies the center of the distribution, and the variance tells us how "spread-out" the PDF is.

The PDF for D-dimensional vector x, the elements of which are jointly distributed with a the Gaussian density function, is given by

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^D|\boldsymbol{\Sigma}|}} \exp\left(-(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})/2\right)$$

where $\boldsymbol{\mu}$ is the mean vector, and Σ is the $D \times D$ covariance matrix, and $|A|$ denotes the determinant of matrix A.

An important special case is when the Gaussian is isotropic (rotationally invariant). In this case the covariance matrix can be written as $\mathbf{\Sigma} = \sigma^2\mathbf{I}$ where $\mathbf{I}$ is the identity matrix. This is called a spherical or isotropic covariance matrix. In this case, the PDF reduces to:

$$p\left(\mathbf{x} \mid \boldsymbol{\mu}, \sigma^2\right) = \frac{1}{\sqrt{(2\pi)^D \sigma^{2D}}} \exp\left(-\frac{1}{2\sigma^2}\|\mathbf{x} - \boldsymbol{\mu}\|^2\right)$$

We will define a function G to be the Gaussian density function, i.e.,

$$G(\mathbf{x}; \boldsymbol{\mu}, \mathbf{\Sigma}) \equiv \frac{1}{\sqrt{(2\pi)^D |\mathbf{\Sigma}|}} \exp\left(-(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})/2\right)$$

When we want to specify that a random vector has a Gaussian PDF, it is common to use the notation:

$$\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{\Sigma} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{\Sigma})$$

9.5 Estimation

We now consider the problem of determining unknown parameters of the world based on measurements. The general problem is one of inference, which describes the probabilities of these unknown parameters. Given a model, these probabilities can be derived using Bayes' Rule. The simplest use of these probabilities is to perform estimation, in which we attempt to come up with single "best" estimates of the unknown parameters.

Definition 9.1 (Prior distribution)**.** *In Bayesian statistical inference, a prior probability distribution, often simply called the prior, of an uncertain quantity is the probability distribution that would express one's beliefs about this quantity before some evidence is taken into account.*

Definition 9.2 (Prior probability)**.** *Similarly, the prior probability of a random event or an uncertain proposition is the unconditional probability that is assigned before any relevant evidence is taken into account.*

For a simple example, we return to coin-flipping. We flip a coin N times, with the result of the i-th flip denoted by a variable $\mathbf{c}_i$: "$\mathbf{c}_i$ = heads" means that the i-th flip came up heads. The probability that the coin lands heads on any given trial is given by a parameter θ. We have no prior knowledge as to the value of θ, and so our prior distribution on θ is uniform.

In other words, we describe θ as coming from a uniform distribution from 0 to 1 , so $p(\theta) = 1$. We further assume that the individual coin flips are independent, i.e., $P(\mathbf{c}_{1:N} \mid \theta) = \prod_i p(\mathbf{c}_i \mid \theta)$. (The notation " $\mathbf{c}_{1:N}$ " indicates the set of observations $\{\mathbf{c}_1, \ldots, \mathbf{c}_N\}$.)

Suppose we wish to learn about a coin by flipping it 1000 times and observing the results $\mathbf{c}_{1:1000}$, where the coin landed heads 750 times. We now need to solve for $p(\theta \mid \mathbf{c}_{1:1000})$, i.e., our belief about θ after seeing the 1000 coin flips.

Applying Bayes' rule gives:

$$p(\theta \mid \mathbf{c}_{1:1000}) = \frac{P(\mathbf{c}_{1:1000} \mid \theta)\, p(\theta)}{P(\mathbf{c}_{1:1000})}$$

The numerator may be written using

$$P(\mathbf{c}_{1:1000} \mid \theta)\, p(\theta) = \prod P(\mathbf{c}_i \mid \theta) = \theta^{750}(1-\theta)^{1000-750}$$

The denominator may be solved for by the marginalization rule:

$$P(\mathbf{c}_{1:1000}) = \int_0^1 P(\mathbf{c}_{1:1000}, \theta)\, d\theta = \int_0^1 \theta^{750}(1-\theta)^{1000-750} d\theta = Z$$

where Z is a constant.

Hence, the final probability distribution is:

$$p(\theta \mid \mathbf{c}_{1:1000}) = \theta^{750}(1-\theta)^{1000-750}/Z.$$

This form gives a probability distribution over θ that expresses our belief about θ after we've flipped the coin 1000 times.

Suppose we just take the peak of this distribution, and it can be shown that the peak is at $\theta = .75$. This makes sense: if a coin lands heads 75% of the time, then we would probably estimate that it will land heads 75% of the time of the future.

More generally, suppose the coin lands heads H times out of N flips; we can compute the peak of the distribution as follows:

$$\arg\max p(\theta \mid \mathbf{c}_{1:N}) = H/N$$

Estimation is the most common form of learning — given some data from the world, we wish to "learn" how the world behaves, which we will describe in terms of a set of unknown variables. We will describe some of the main ways to do this, including Maximum A Posterior (MAP), and Maximum Likelihood (ML).

We can now define the MAP learning rule: choose the parameter value θ that maximizes the posterior, i.e.,

$$\begin{aligned}\hat{\theta} &= \arg\max_{\theta} p(\theta \mid \mathcal{D}) \\ &= \arg\max_{\theta} P(\mathcal{D} \mid \theta) p(\theta)\end{aligned}$$

Note that we don't need to be able to evaluate the evidence term $p(\mathcal{D})$ for MAP learning, since there are no θ terms in it.

Very often, we will assume that we have no prior assumptions about the value of θ, which we express as a uniform prior: $p(\theta)$ is a uniform distribution over some suitably large range. In this case, the $p(\theta)$ term can also be ignored from MAP learning, and we are left with only maximizing the likelihood. Hence, the Maximum Likelihood (ML) learning principle (i.e., estimator) is

$$\hat{\theta}_{ML} = \arg\max_{\theta} P(\mathcal{D} \mid \theta)$$

We now consider the problem of learning a Gaussian distribution from N training samples $\mathbf{x}_{1:N}$. Maximum likelihood learning of the parameters μ and Σ entails maximizing the likelihood:

$$p\left(\mathbf{x}_{1:N} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$$

We assume here that the data points come from a Gaussian. We further assume that they are drawn independently. We can therefore write the joint likelihood over the entire set of data as the produce of the likelihoods for each individual datum, i.e.,

$$\begin{aligned}p\left(\mathbf{x}_{1:N} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) &= \prod_{i=1}^{N} p\left(\mathbf{x}_{1:N} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \\ &= \prod_{i=1}^{N} \frac{1}{\sqrt{(2\pi)^M |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}\left(\mathbf{x}_i - \boldsymbol{\mu}\right)^T \boldsymbol{\Sigma}^{-1}\left(\mathbf{x}_i - \boldsymbol{\mu}\right)\right)\end{aligned}$$

where M is the dimension of the data $\mathbf{x}_i$.

It is somewhat more convenient to minimize the negative log-likelihood:

$$\begin{aligned}L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) &\equiv -\ln p\left(\mathbf{x}_{1:N} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \\ &= -\sum_i \ln p\left(\mathbf{x}_i \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \\ &= \sum_i \frac{\left(\mathbf{x}_i - \boldsymbol{\mu}\right)^T \boldsymbol{\Sigma}^{-1}\left(\mathbf{x}_i - \boldsymbol{\mu}\right)}{2} + \frac{N}{2} \ln |\boldsymbol{\Sigma}| + \frac{NM}{2} \ln(2\pi)\end{aligned}$$

Solving for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ by setting $\partial L/\partial \boldsymbol{\mu} = 0$ and $\partial L/\partial \boldsymbol{\Sigma} = 0$ (subject to the constraint that $\boldsymbol{\Sigma}$ is symmetric) gives the maximum likelihood estimates:

$$\boldsymbol{\mu}^* = \frac{1}{N}\sum_i \mathbf{x}_i$$
$$\boldsymbol{\Sigma}^* = \frac{1}{N}\sum_i \left(\mathbf{x}_i - \boldsymbol{\mu}^*\right)\left(\mathbf{x}_i - \boldsymbol{\mu}^*\right)^T$$

Let us revisit the nonlinear regression model, but now assume that there exists noise in measurements and modelling errors. We'll now write the model as

$$y = \mathbf{w}^T\mathbf{b}(\mathbf{x}) + n$$

where n is a Gaussian random variable, i.e.,

$$n \sim \mathcal{N}\left(0, \sigma^2\right).$$

It is straightforward to show that, given $\mathbf{x}$ and $\mathbf{w}$, y is also Gaussian, i.e.,

$$p(y \mid \mathbf{x}, \mathbf{w}) = G\left(y; \mathbf{w}^T\mathbf{b}(\mathbf{x}), \sigma^2\right) \equiv \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(y-\mathbf{w}^T\mathbf{b}(\mathbf{x})\right)^2/2\sigma^2}$$

It follows that, for a collection of N independent training points, $(y_{1:N}, \mathbf{x}_{1:N})$, the likelihood is given by

$$\begin{aligned} p\left(y_{1:N} \mid \mathbf{w}, \mathbf{x}_{1:N}\right) &= \prod_{i=1}^{N} G\left(y_i; \mathbf{w}^T\mathbf{b}\left(\mathbf{x}_i\right), \sigma^2\right) \\ &= \frac{1}{\left(2\pi\sigma^2\right)^{N/2}} \exp\left(-\sum^{N} \frac{\left(y_i - \mathbf{w}^T\mathbf{b}\left(\mathbf{x}_i\right)\right)^2}{2\sigma^2}\right) \end{aligned}$$

Furthermore, let us assume:

$$\mathbf{w} \sim \mathcal{N}(0, \alpha\mathbf{I}).$$

That is, for $\mathbf{w} \in \mathbb{R}^M$,

$$p(\mathbf{w}) = \prod_{k=1}^{M} \frac{1}{\sqrt{2\pi\alpha}} e^{-w_k^2/2\alpha} = \frac{1}{(2\pi\alpha)^{M/2}} e^{-\mathbf{w}^T\mathbf{w}/2\alpha}$$

Now, to estimate the model parameters (i.e., w), let's consider the posterior distribution over $\mathbf{w}$ conditioned on our N training pairs, $(\mathbf{x}_i, y_i)$. Based on the formulation above, assuming independent training samples, it follows that

$$\begin{aligned} p\left(\mathbf{w} \mid y_{1:N}, \mathbf{x}_{1:N}\right) &= \frac{p\left(y_{1:N} \mid \mathbf{w}, \mathbf{x}_{1:N}\right) p\left(\mathbf{w} \mid \mathbf{x}_{1:N}\right)}{p\left(y_{1:N} \mid \mathbf{x}_{1:N}\right)} \\ &= \frac{\left(\prod_i p\left(y_i \mid \mathbf{w}, \mathbf{x}_i\right)\right) p(\mathbf{w})}{p\left(y_{1:N} \mid \mathbf{x}_{1:N}\right)} \end{aligned}$$

Note that $p(\mathbf{w} \mid \mathbf{x}_{1:N}) = p(\mathbf{w})$, since we can assume that $\mathbf{x}$ alone provides no information about $\mathbf{w}$. In MAP estimation, we want to find the parameters w that maximize their posterior probability:

$$\begin{aligned}\mathbf{w}^* &= \arg\max_{\mathbf{w}} p(\mathbf{w} \mid y_{1:N}, \mathbf{x}_{1:N}) \\ &= \arg\min_{\mathbf{w}} -\ln p(\mathbf{w} \mid y_{1:N}, \mathbf{x}_{1:N})\end{aligned}$$

The negative log-posterior is:

$$\begin{aligned}L(\mathbf{w}) &= -\ln p(\mathbf{w} \mid y_{1:N}, \mathbf{x}_{1:N}) \\ &= \left(\sum_i \frac{1}{2\sigma^2}\left(y_i - \mathbf{w}^T\mathbf{b}(\mathbf{x}_i)\right)^2\right) + \frac{N}{2}\ln\left(2\pi\sigma^2\right) \\ &\quad + \frac{1}{2\alpha}\|\mathbf{w}\|^2 + \frac{M}{2}\ln(2\pi\alpha) + \ln p(y_{1:N} \mid \mathbf{x}_{1:N})\end{aligned}$$

Now, we can discard terms that do not depend on w, since they are irrelevant for optimization:

$$L(\mathbf{w}) = \left(\sum_i \frac{1}{2\sigma^2}\left(y_i - \mathbf{w}^T\mathbf{b}(\mathbf{x}_i)\right)^2\right) + \frac{1}{2\alpha}\|\mathbf{w}\|^2 + \text{ constants}$$

Furthermore, we can multiply by a constant, without changing where the optima are, so let us multiply the whole expression by $2\sigma^2$. Then, if we define $\lambda = \sigma^2/\alpha$, we have the exact same objective function as used in nonlinear regression with regularization.

9.6 Classification

The general goal of classification is to learn a decision boundary, often specified as the level set of a function, e.g., $a(\mathbf{x}) = 0$. The purpose of the decision boundary is to identity the regions of the input space that correspond to each class. For binary classification the decision boundary is the surface in the feature space that separates the test inputs into two classes; points $\mathbf{x}$ for which $a(\mathbf{x}) < 0$ are deemed to be in one class, while points for which $a(\mathbf{x}) > 0$ are in the other. The points on the decision boundary, $a(\mathbf{x}) = 0$, are those inputs for which the two classes are equally probable. [8]

Suppose we have two mutually exclusive classes C_1 and C_2. The prior probability of a data vector coming from class C_1 is $P(C_1)$, and $P(C_2) = 1 - P(C_1)$. Each class has a distribution for its data: $p(\mathbf{x} \mid C_1)$, and $p(\mathbf{x} \mid C_2)$. In other words, to sample from this model, we would first randomly choose a class according to $P(C_1)$, and then sample a data vector $\mathbf{x}$ from that class.

Given labeled training data $\{(\mathbf{x}_i, y_i)\}$, we can estimate the distribution for each class by maximum likelihood. Once we have trained the parameters of our generative model, we perform classification by comparing the posterior class probabilities:

$$P(C_1 \mid \mathbf{x}) > P(C_2 \mid \mathbf{x})?$$

Equivalently, we can compare their ratio to 1 :

$$\frac{P(C_1 \mid \mathbf{x})}{P(C_2 \mid \mathbf{x})} > 1?$$

If this ratio is greater than 1 (i.e. $P(C_1 \mid \mathbf{x}) > P(C_2 \mid \mathbf{x})$) then we classify $\mathbf{x}$ as belonging to class 1 , and class 2 otherwise. The quantities $P(C_i \mid \mathbf{x})$ can by computed using Bayes' Rule as:

$$P(C_i \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid C_i) P(C_i)}{p(\mathbf{x})}$$

so that the ratio is:

$$\frac{p(\mathbf{x} \mid C_1) P(C_1)}{p(\mathbf{x} \mid C_2) P(C_2)}$$

Consider a generative model in which the inputs associated with the i^{th} class (for $i = 1, 2$) are modeled with a Gaussian distribution, i.e.,

$$p(\mathbf{x} \mid C_i) = G(\mathbf{x}; \mu_i, \mathbf{\Sigma}_i)$$

Also, let's assume that the prior class probabilities are equal:

$$P(C_i) = \frac{1}{2}$$

The values of μ_i and Σ_i can be estimated by maximum likelihood on the individual classes in the training data. Given this models, you can show that the log of the posterior ratio is given by

$$-\frac{1}{2}(\mathbf{x} - \mu_1)^T \mathbf{\Sigma}_1^{-1}(\mathbf{x} - \mu_1) - \frac{1}{2}\ln|\mathbf{\Sigma}_1| + \frac{1}{2}(\mathbf{x} - \mu_2)^T \mathbf{\Sigma}_2^{-1}(\mathbf{x} - \mu_2) + \frac{1}{2}\ln|\mathbf{\Sigma}_2|$$

The sign of this function determines the class of $\mathbf{x}$.

Noting that $p(\mathbf{x})$ can be written as

$$p(\mathbf{x}) = p(\mathbf{x}, C_1) + p(\mathbf{x}, C_2) = p(\mathbf{x} \mid C_1) P(C_1) + p(\mathbf{x} \mid C_2) P(C_2),$$

we can express the posterior class probability as

$$P\left(C_1 \mid \mathbf{x}\right) = \frac{p\left(\mathbf{x} \mid C_1\right) P\left(C_1\right)}{p\left(\mathbf{x} \mid C_1\right) P\left(C_1\right) + p\left(\mathbf{x} \mid C_2\right) P\left(C_2\right)}.$$

Dividing both the numerator and denominator by $p\left(\mathbf{x} \mid C_1\right) P\left(C_1\right)$ we obtain:

$$\begin{aligned} P\left(C_1 \mid \mathbf{x}\right) &= \frac{1}{1 + e^{-a(\mathbf{x})}} \\ &= g(a(\mathbf{x})) \end{aligned}$$

where $a(\mathbf{x}) = \ln \frac{p(\mathbf{x}|C_1)P(C_1)}{p(\mathbf{x}|C_2)P(C_2)}$ and $g(a)$ is the sigmoid function. Note that $g(a)$ is monotonic, so that the probability of class C_1 grows as a grows and is precisely $\frac{1}{2}$ when $a = 0$.

For the case of Gaussian class conditionals where both Gaussians have the same covariance, a is a linear function of x. In this case the classification probability can be written as

$$P\left(C_1 \mid \mathbf{x}\right) = \frac{1}{1 + e^{-\mathbf{w}^T\mathbf{x} - b}} = g\left(\mathbf{w}^T\mathbf{x} + b\right),$$

or, if we augment the data vector with a 1 and the weight vector with b,

$$P\left(C_1 \mid \mathbf{x}\right) = \frac{1}{1 + e^{-\mathbf{w}^T\mathbf{x}}}.$$

Logistic regression can also be applied to multi-class classification, i.e., where we wish to classify a data point as belonging to one of K classes. In this case, the probability of data vector $\mathbf{x}$ being in class i is:

$$P\left(C_i \mid \mathbf{x}\right) = \frac{e^{-\mathbf{w}_i^T\mathbf{x}}}{\sum_{k=1}^{K} e^{-\mathbf{w}_k^T\mathbf{x}}}$$

You should be able to see that this is equivalent to the method described above in the two-class case.

We can apply the KNN idea to classification as well. For class labels $\{-1, 1\}$, the classifier is:

$$y_{new} = \operatorname{sign}\left(\sum_{i \in N_K(\mathbf{x})} y_i\right)$$

where

$$\operatorname{sign}(z) = \begin{cases} -1 & z \leq 0 \\ 1 & z > 0 \end{cases}$$

Alternatively, we might take a weighted average of the K-nearest neighbors:

$$y = \operatorname{sign}\left(\sum_{i \in N_K(\mathbf{x})} w\left(\mathbf{x}_i\right) y_i\right), \quad w\left(\mathbf{x}_i\right) = e^{-\|\mathbf{x}_i - \mathbf{x}\|^2 / 2\sigma^2}$$

where σ^2 is an additional parameter to the algorithm.

9.7 Naive Bayes

In Gaussian Class Conditional models, with d-dimensional input vectors, we need to estimate the class mean and class covariance matrix for each class. The mean will be a $d-$ dimensional vector, but the number of unknowns in the covariance matrix grows quadratically with d. That is, the covariance is a $d \times d$ matrix.

Naïve Bayes aims to simplify the estimation problem by assuming that the different input features (e.g., the different elements of the input vector), are conditionally independent. That is, they are assumed to be independent when conditioned on the class. Mathematically, for inputs $\mathbf{x} \in \mathbb{R}^d$, we express this as

$$p(\mathbf{x} \mid C) = \prod_{i=1}^{d} p\left(x_i \mid C\right)$$

In discrete Naive Bayes, the inputs are a discrete set of "features". Each data vector is described by a list of discrete features $F_{1:D} = [F_1, \ldots, F_D]$. Each feature F_i has a set of possible values that it can take; to keep things simple, we'll assume that each feature is binary: $F_i \in \{0, 1\}$.

Consider the distribution over 3 inputs, for class $C = 1$, i.e., $P\left(F_{1:3} \mid C = 1\right)$. Using basic rules of probability, we find that

$$\begin{aligned} P\left(F_{1:3} \mid C = 1\right) &= P\left(F_1 \mid C = 1, F_2, F_3\right) P\left(F_2, F_3 \mid C = 1\right) \\ &= P\left(F_1 \mid C = 1, F_2, F_3\right) P\left(F_2 \mid C = 1, F_3\right) P\left(F_3 \mid C = 1\right) \end{aligned}$$

First we want to know the probability $P\left(F_3 = 1 \mid C = 1\right)$. Now consider the second factor $P\left(F_2 \mid C = 1, F_3\right)$.

In this case, because F_2 depends on F_3, and there are two possible states of F_3, there are two distributions we need to model, namely $P\left(F_2 \mid C = 1, F_3 = 0\right)$ and $P\left(F_2 \mid C = 1, F_3 = 1\right)$. Accordingly, we will need two parameters, one for $P\left(F_2 = 1 \mid C = 1, F_3 = 0\right)$ and one for $P\left(F_2 = 1 \mid C = 1, F_3 = 1\right)$. Using the same logic, to model $P\left(F_1 \mid C = 1, F_2, F_3\right)$ will require one model parameter for each possible setting of (F_2, F_3), and of course there are 2^2 such settings. For D-dimensional binary inputs, there

are $O\left(2^{D-1}\right)$ parameters that one needs to learn. The number of parameters required grows prohibitively large as D increases.

The Naive Bayes model, by comparison, only have D parameters to be learned. This means that the likelihood of a feature vector for a particular class j is given by

$$P\left(F_{1:D} \mid C = j\right) = \prod_i P\left(F_i \mid C = j\right)$$

where C denotes a class $C \in \{1, 2, \ldots K\}$. The probabilities $P\left(F_i \mid C\right)$ are parameters of the model:

$$P\left(F_i = 1 \mid C = j\right) = a_{i,j}$$

We must also define class priors $P(C = j) = b_j$. To classify a new feature vector using this model, we choose the class with maximum probability given the features.

By Bayes' Rule this is:

$$\begin{aligned} P\left(C = j \mid F_{1:D}\right) &= \frac{P\left(F_{1:D} \mid C = j\right) P(C = j)}{P\left(F_{1:D}\right)} \\ &= \frac{\left(\prod_i P\left(F_i \mid C = j\right)\right) P(C = j)}{\sum_{\ell=1}^{K} P\left(F_{1:D}, C = \ell\right)} \\ &= \frac{\left(\prod_{i:F_i=1} a_{i,j} \prod_{i:F_i=0} \left(1 - a_{i,j}\right)\right) b_j}{\sum_{\ell=1}^{K} \left(\prod_{i:F_i=1} a_{i,\ell} \prod_{i:F_i=0} \left(1 - a_{i,\ell}\right)\right) b_\ell} \end{aligned}$$

Suppose there are N_k examples of each class, and N examples total. Then the prior estimate is simply:

$$b_k = \frac{N_k}{N}$$

Similarly, if class k has $N_{i,k}$ examples where $F_i = 1$, then

$$a_{i,k} = \frac{N_{i,k}}{N_k}$$

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