Numerical Fourier method and high-dimensional second-order Taylor scheme for stochastic differential equations

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Outline

Consider the following diffusion process:

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t \mu\left(s, \mathbf{X}_s\right) ds + \int_0^t \sigma\left(s, \mathbf{X}_s\right) dB_s, \quad 0 \leq t \leq T \tag{1.1}$$

with $\mathbf{X}_t \in \mathbb{R}^n$, $\mu : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times q}$ and B is a q-dimensional Brownian motion. We further assume that the drift term μ and diffusion term σ have enough regularity properties. To obtain the second-order scheme, we first define the following differential operators:

$$L^{(0)} := \partial_t + \sum_{i=1}^{n} \mu_i \partial_{x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sigma\sigma^T\right)_{ij} \partial_{x_i}\partial_{x_j}, \tag{1.2}$$

and

$$L^{(i)} := \sum_{j=1}^{n} \sigma_{ji} \partial_{x_j}, \tag{1.3}$$

for $i = 1, 2, \ldots, q$.

Then applying Itô lemma on μ and σ in (1.1), we have, for $i = 1, 2, \cdots, n$,

$$
\begin{aligned}
X_t^i =& X_0^i + \int_0^t \left(\mu_i(0, \mathbf{X}_0) + \int_0^s L^{(0)} \mu_i(u, \mathbf{X}_u)\, du + \sum_{j=1}^q \int_0^s L^{(j)} \mu_i(u, \mathbf{X}_u)\, dB_u^j \right) ds \\
& + \sum_{j=1}^q \int_0^t \left(\sigma_{ij}(0, \mathbf{X}_0) + \int_0^s L^{(0)} \sigma_{ij}(u, \mathbf{X}_u)\, du + \sum_{k=1}^q \int_0^s L^{(k)} \sigma_{ij}(u, \mathbf{X}_u)\, dB_u^k \right) dB_s^j \\
=& X_0^i + \mu_i(0, \mathbf{X}_0)\, t + \sum_{j=1}^q \sigma_{ij}(0, \mathbf{X}_0) \int_0^t dB_s^j + \int_0^t \int_0^s L^{(0)} \mu_i(u, \mathbf{X}_u)\, duds \\
& + \sum_{j=1}^q \int_0^t \int_0^s L^{(j)} \mu_i(u, \mathbf{X}_u)\, dB_u^j ds + \sum_{j=1}^q \int_0^t \int_0^s L^{(0)} \sigma_{ij}(u, \mathbf{X}_u)\, dudB_s^j \\
& + \sum_{j=1}^q \sum_{k=1}^q \int_0^t \int_0^s L^{(k)} \sigma_{ij}(u, \mathbf{X}_u)\, dB_u^k dB_s^j,
\end{aligned}
\tag{1.4}
$$

Next, consider an equidistant time partition $0 =: t_0 < t_1 < t_2 < \cdots < t_M := T$ with the uniform length of each step $\Delta t = \Delta t_m = t_m - t_{m-1} = \frac{T}{M}$, we can develop an approximation scheme at each time step by applying Equation (1.4). For simplicity, we will write X_m as the approximate value of X_t at time $t = t_m$ from now on. Hence, we can obtain (1.5).

$$\begin{aligned} X_{m+1}^i =& X_m^i + \mu_i\left(t_m, \mathbf{X}_m\right)\Delta t + \sum_{j=1}^{q} \sigma_{ij}\left(t_m, \mathbf{X}_m\right)\Delta B_m^j + L^{(0)}\mu_i\left(t_m, \mathbf{X}_m\right)\int_{t_m}^{t_{m+1}}\int_{t_m}^{s} duds \\ &+ \sum_{j=1}^{q} L^{(j)}\mu_i\left(t_m, \mathbf{X}_m\right)\int_{t_m}^{t_{m+1}}\int_{t_m}^{s} dB_u^j ds + \sum_{j=1}^{q} L^{(0)}\sigma_{ij}\left(t_m, \mathbf{X}_m\right)\int_{t_m}^{t_{m+1}}\int_{t_m}^{s} dudB_s^j \\ &+ \sum_{j=1}^{q}\sum_{k=1}^{q} L^{(k)}\sigma_{ij}\left(t_m, \mathbf{X}_m\right)\int_{t_m}^{t_{m+1}}\int_{t_m}^{s} dB_u^k dB_s^j. \end{aligned} \tag{1.5}$$

Outline

Based on the Fourier expansion of the Brownian bridge process, one can obtain the following representations from Kloeden et al. (Stochastic analysis and applications 10: 431-441, 1992) [2]:

$$\int_{t_m}^{t_{m+1}} \int_{t_m}^{s} dB_{s_1}^{j} \, ds = \frac{1}{2} \Delta t \Delta B_m^j - \frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi} \sum_{r=1}^{\infty} \frac{1}{r} G_r^j; \tag{2.1}$$

$$\int_{t_m}^{t_{m+1}} \int_{t_m}^{s} ds_1 \, dB_s^j = \frac{1}{2} \Delta t \Delta B_m^j + \frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi} \sum_{r=1}^{\infty} \frac{1}{r} H_r^j, \tag{2.2}$$

where G^j and H^j are independent and identically distributed (i.i.d.) with $\mathbf{G}_r \sim N(\mathbf{0}, I)$, and $\mathbf{H}_r \sim N(\mathbf{0}, I)$. Here $\Delta t := t_{m+1} - t_m$ and $\Delta B_m^j := B_{m+1}^j - B_m^j$. By saying $N(\mathbf{0}, I)$, we mean that each element in $\mathbf{G}_r$ follows $N(0, 1)$, where $N(0, 1)$ means a random variable which follows the normal distribution with mean equals to 0, and variance equals to 1.

As mentioned in Wiktorsson (The Annuals of Applied Probability 11: 470-487, 2001) [3], while numerically simulating iterated Itô integrals, we should carefully deal with the so-called Levy stochastic area [1] defined by

$$A_{ij}(t, t+h) := \frac{I_{ij}(t, t+h) - I_{ji}(t, t+h)}{2},$$

where I_{ij} denotes the following second-order iterated integral:

$$I_{ij}(t, t+h) := \int_t^{t+h} \int_t^s dB_i(u) dB_j(s).$$

Furthermore, we can again incorporate the relationship between iterated integrals and Levy stochastic areas into the representations in Based on the Fourier expansion of the Brownian bridge process, one can obtain the following representations from Kloeden et al. (Stochastic analysis and applications 10: 431-441, 1992) [2] and derive the following equation:

$$\int_{t_m}^{t_{m+1}} \int_{t_m}^{s} dB_{s_1}^k \, dB_s^j = \frac{\Delta B_m^j \Delta B_m^k - \Delta t \delta_{jk}}{2} + \frac{\Delta t}{2\pi} \sum_{r=1}^{\infty} \frac{1}{r} \left[G_r^j \left(H_r^k + \sqrt{\frac{2}{\Delta t}} \Delta B_m^k \right) - G_r^k \left(H_r^j + \sqrt{\frac{2}{\Delta t}} \Delta B_m^j \right) \right] \tag{2.3}$$

where G^j and H^j are i.i.d with $\mathbf{G}_r \sim N(\mathbf{0}, I)$, and $\mathbf{H}_r \sim N(\mathbf{0}, I)$.

Let's consider a path $X_t = \left\{X_t^1, X_t^2\right\}$ in $\mathbb{R}^2$ and $t \in [a, b]$.
For any single index i, let us define the quantity

$$S(X)^i_{a,t} = \int_{a<s<t} dX^i_s = X^i_t - X^i_0,$$

which is the increment of the i-th coordinate of the path at time $t \in [a, b]$. We emphasise that $S(X)^i_{a,\cdot} : [a, b] \mapsto \mathbb{R}$ is itself a real-valued path. Note that a in the subscript of $S(X)^i_{a,t}$ is only used to denote the starting point of the interval $[a, b]$.
Let us define the double-iterated integral

$$S(X)^{i,j}_{a,t} = \int_{a<s<t} S(X)^i_{a,s} dX^j_s = \int_{a<r<s<t} dX^i_r dX^j_s,$$

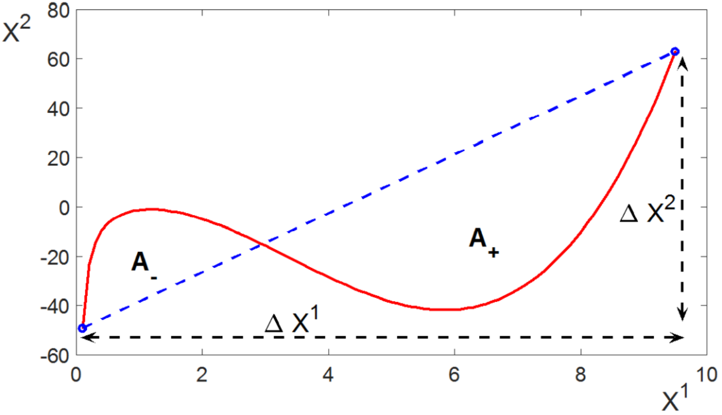
where $S(X)^i_{a,s}$ is given by (6), and the integration limits are simply:

$$a < r < s < t = \begin{cases} a < r < s \\ a < s < t \end{cases}$$

To give geometric meaning to the term $S(X)^{i,j}_{a,b}$ for $i \neq j$, consider the Lévy area, which is a signed area enclosed by the path (solid red line) and the chord (blue straight dashed line) connecting the endpoints. The Lévy area of the two dimensional path $\left\{X^1_t, X^2_t\right\}$ is given by:

$$A = \frac{1}{2}\left(S(X)^{1,2}_{a,b} - S(X)^{2,1}_{a,b}\right).$$

The signed areas denoted by $A_-$and $A_+$are the negative and positive areas, respectively, and ΔX^1 and ΔX^2 represent the increments along each coordinate.



Example of signed Levy area of a curve. Areas above and under the chord connecting two endpoints are negative and positive, respectively.

Here $\Delta t := t_{m+1} - t_m$ and $\Delta B_m^j := B_{m+1}^j - B_m^j$. By saying $N(\mathbf{0}, I)$, we mean that each element in $\mathbf{G}_r$ follows $N(0,1)$, where $N(0,1)$ means a random variable which follows normal distribution with mean equals to 0, and variance equals to 1. As mentioned in Wiktorsson (The Annuals of Applied Probability 11: 470-487, 2001) [3], while numerically simulating iterated Itô integrals, we should carefully deal with the so-called Levy stochastic area defined by

$$A_{ij}(t, t+h) := \frac{I_{ij}(t, t+h) - I_{ji}(t, t+h)}{2},$$

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$$I_{ij}(t, t+h) := \int_t^{t+h} \int_t^s dB_i(u) dB_j(s).$$

Furthermore, we can again incorporate the relationship between iterated integrals and Levy stochastic areas into the representations and derive the following equation:

$$\int_{t_m}^{t_{m+1}} \int_{t_m}^{s} dB_{s_1}^k dB_s^j = \frac{\Delta B_m^j \Delta B_m^k - \Delta t \delta_{jk}}{2} + \frac{\Delta t}{2\pi} \sum_{r=1}^{\infty} \frac{1}{r} \left[G_r^j \left(H_r^k + \sqrt{\frac{2}{\Delta t}} \Delta B_m^k \right) - G_r^k \left(H_r^j + \sqrt{\frac{2}{\Delta t}} \Delta B_m^j \right) \right] \tag{2.4}$$

where G^j and H^j are i.i.d with $\mathbf{G}_r \sim N(\mathbf{0}, I)$, and $\mathbf{H}_r \sim N(\mathbf{0}, I)$. Note that all the introduced representations (2.1), (2.2), and (2.4) above are well-defined in the L^2 sense. We then introduce the discretization scheme for the forward SDE.

Second-order forward discretization scheme

Now, our new idea for this project is to combine representations and the traditional second-order Taylor scheme to develop a new numerical scheme. First of all, one may incorporate (2.1), (2.2) and (2.4) into the above equation and obtain

$$
\begin{aligned}
X_{m+1}^{i} = {} & X_{m}^{i} + \mu_i\left(\mathbf{X}_m\right)\Delta t - \frac{1}{2}\sum_{j=1}^{q} L^{(j)}\sigma_{ij}\left(\mathbf{X}_m\right)\Delta t + \frac{1}{2}L^{(0)}\mu_i\left(\mathbf{X}_m\right)\left(\Delta t\right)^2 \\
& + \sum_{j=1}^{q}\sigma_{ij}\left(\mathbf{X}_m\right)\Delta B_m^j + \frac{1}{2}\sum_{j=1}^{q}\left(L^{(j)}\mu_i\left(\mathbf{X}_m\right) + L^{(0)}\sigma_{ij}\left(\mathbf{X}_m\right)\right)\Delta t\Delta B_m^j \\
& + \frac{1}{2}\sum_{j=1}^{q}\sum_{k=1}^{q} L^{(k)}\sigma_{ij}\left(\mathbf{X}_m\right)\Delta B_m^j\Delta B_m^k \\
& + \frac{\left(\Delta t\right)^{\frac{3}{2}}}{\sqrt{2}\pi}\sum_{j=1}^{q}\left(L^{(0)}\sigma_{ij}\left(\mathbf{X}_m\right) - L^{(j)}\mu_i\left(\mathbf{X}_m\right)\right)\sum_{r=1}^{\infty}\frac{1}{r}G_r^j \\
& + \sum_{j=1}^{q}\sum_{k=1}^{q} L^{(k)}\sigma_{ij}\left(\mathbf{X}_m\right)\frac{\Delta t}{2\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left[G_r^j\left(H_r^k + \sqrt{\frac{2}{\Delta t}}\Delta B_m^k\right) - G_r^k\left(H_r^j + \sqrt{\frac{2}{\Delta t}}\Delta B_m^j\right)\right].
\end{aligned} \tag{2.5}
$$

Also, it is worth noting that we can simplify the last term in (2.5) by

$$
\begin{aligned}
&\sum_{j=1}^{q}\sum_{k=1}^{q} L^{(k)}\sigma_{ij}\left(\mathbf{X}_m\right)\frac{\Delta t}{2\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left[G_r^j\left(H_r^k+\sqrt{\frac{2}{\Delta t}}\Delta B_m^k\right)-G_r^k\left(H_r^j+\sqrt{\frac{2}{\Delta t}}\Delta B_m^j\right)\right]\\
=&\frac{\Delta t}{2\pi}\sum_{\substack{j=k=1\\ j\neq k}}^{q} L^{(k)}\sigma_{ij}\left(\mathbf{X}_m\right)\sum_{r=1}^{\infty}\frac{1}{r}G_r^j\left(H_r^k+\sqrt{\frac{2}{\Delta t}}\Delta B_m^k\right)\\
&-\frac{\Delta t}{2\pi}\sum_{\substack{j=k=1\\ j\neq k}}^{q} L^{(j)}\sigma_{ik}\left(\mathbf{X}_m\right)\sum_{r=1}^{\infty}\frac{1}{r}G_r^j\left(H_r^k+\sqrt{\frac{2}{\Delta t}}\Delta B_m^k\right)\\
=&\frac{\Delta t}{2\pi}\sum_{\substack{j=k=1\\ j\neq k}}^{q}\left(L^{(k)}\sigma_{ij}\left(\mathbf{X}_m\right)-L^{(j)}\sigma_{ik}\left(\mathbf{X}_m\right)\right)\sum_{r=1}^{\infty}\frac{1}{r}G_r^j\left(H_r^k+\sqrt{\frac{2}{\Delta t}}\Delta B_m^k\right).
\end{aligned}
\tag{2.6}
$$

We use $X_{m+1}^{\Delta,m,\mathbf{x},i}$ to denote the approximation value of X at time $t = t_{m+1}$ with the exact information for the value of X at time $t = t_m$.
Consequently, we can step-wise write down the final approximation scheme as follows:

$$\mathbf{X}_m^{\Delta,m,\mathbf{x}} = \mathbf{x} \tag{2.7}$$

$$\begin{aligned}
X_{m+1}^{\Delta,m,\mathbf{x},i} =& X_m^{\Delta,m,\mathbf{x},i} + \zeta_i\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right)\Delta t + \sum_{j=1}^{q} S^{ij}\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right)\Delta B_m^j \\
&+ \frac{1}{2}\sum_{j=1}^{q}\sum_{k=1}^{q} L^{(k)}\sigma_{ij}\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right)\Delta B_m^j \Delta B_m^k \\
&+ \frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi}\sum_{j=1}^{q}\left(L^{(0)}\sigma_{ij}\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right) - L^{(j)}\mu_i\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right)\right)\sum_{r=1}^{\infty}\frac{1}{r}G_r^j \\
&+ \frac{\Delta t}{2\pi}\sum_{\substack{j=k=1 \\ j\neq k}}^{q}\left(L^{(k)}\sigma_{ij}\left(\mathbf{X}_{m'}^{\Delta,m,\mathbf{x}}\right) - L^{(j)}\sigma_{ik}\left(\mathbf{X}_{m'}^{\Delta,m,\mathbf{x}}\right)\right)\sum_{r=1}^{\infty}\frac{1}{r}G_r^j\left(H_r^k + \sqrt{\frac{2}{\Delta t}}\Delta B_m^k\right)
\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&= X_m^{\Delta,m,\mathbf{x},i} + \vartheta_i\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right)\Delta t + \frac{1}{2}\sum_{j=1}^{q} L^{(j)}\sigma_{ij}\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right)\left[\left(\Delta B_m^j\right)^2 - \Delta t\right] \\
&\quad + \sum_{j=1}^{q} S^{ij}\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right)\Delta B_m^j + \frac{1}{2}\sum_{\substack{j=k=1 \\ j\neq k}}^{q} L^{(k)}\sigma_{ij}\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right)\Delta B_m^j \Delta B_m^k \\
&\quad + \frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi}\sum_{j=1}^{q}\left(L^{(0)}\sigma_{ij}\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right) - L^{(j)}\mu_i\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right)\right)\sum_{r=1}^{\infty}\frac{1}{r}G_r^j \\
&\quad + \frac{\Delta t}{2\pi}\sum_{\substack{j=k=1 \\ j\neq k}}^{q}\left(L^{(k)}\sigma_{ij}\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right) - L^{(j)}\sigma_{ik}\left(\mathbf{X}_m^{\Delta,m,\mathbf{x}}\right)\right)\sum_{r=1}^{\infty}\frac{1}{r}G_r^j\left(H_r^k + \sqrt{\frac{2}{\Delta t}}\Delta B_m^k\right)
\end{aligned}
\tag{2.9}
$$

for $m = 0, \ldots, M-1$ and $i = 0, \ldots, n$, where

$$
\zeta_i(\mathbf{x}) := \vartheta_i(\mathbf{x}) - \frac{1}{2}\sum_{j=1}^{q} L^{(j)}\sigma_{ij}(\mathbf{x}) := \mu_i(\mathbf{x}) + \frac{1}{2}L^{(0)}\mu_1(\mathbf{x})\Delta t - \frac{1}{2}\sum_{j=1}^{q} L^{(j)}\sigma_{ij}(\mathbf{x})
$$

and

$$
S_{ij}(\mathbf{x}) = \sigma_{ij}(\mathbf{x}) + \frac{1}{2}\left(L^{(j)}\mu_i(\mathbf{x}) + L^{(0)}\sigma_{ij}(\mathbf{x})\right)\Delta t.
$$

Characteristic function

In this subsection, we desire to obtain the characteristic function of $X_{m+1}^{\Delta,m,\mathbf{x},i}$. Recall that in probability theory and statistics, the characteristic function

$$\phi_Y(\omega) = E[e^{i\omega Y}]$$

of a random variable, Y is a function of ω, which determines the behavior and properties of the probability distribution of Y. For the season of simplicity, we write X_m^{Δ} for $X_m^{\Delta,m,\mathbf{x},i}$. We first denote several abbreviations to facilitate the computations:

$$\alpha_j := \sum_{i=1}^{n} \omega_i \left(L^{(0)}\sigma_{ij}\left(\mathbf{X}_m^{\Delta}\right) - L^{(j)}\mu_i\left(\mathbf{X}_m^{\Delta}\right)\right); \tag{2.10}$$

$$\beta_{jk} := \begin{cases} \sum_{i=1}^{n} \omega_i \left(L^{(k)}\sigma_{ij}\left(\mathbf{X}_m^{\Delta}\right) - L^{(j)}\sigma_{ik}\left(\mathbf{X}_m^{\Delta}\right)\right) & \text{if } j \neq k, \\ 0 & \text{if } j = k; \end{cases} \tag{2.11}$$

$$(M)_{jk} := \sum_{i=1}^{n} \omega_i L^{(k)}\sigma_{ij}\left(\mathbf{X}_m^{\Delta}\right); \tag{2.12}$$

$$P := \frac{1}{2}M + \frac{1}{2}M^T. \tag{2.13}$$

where ω_i is the i^{th} entry of the fixed vector $\Omega \in \mathbb{R}^n$. Based on (2.9), we can reach the following theorem, which is a new result that we obtained in the first year of this project.

Theorem 2.1

The characteristic function $\phi_{\mathbf{x}_{m+1}^{\Delta,m,\mathbf{x}\Delta}}$ *of* $\mathbf{X}_{m+1}^{\Delta,m,\mathbf{X}_m^\Delta}$ *can be written as*

$$\phi_{\mathbf{x}_{m+1}^{\Delta,m,\mathbf{x}\Delta}}(\Omega) = \det\left(\prod_{r=1}^{\infty}\left(I + \frac{(\Delta t)^2}{4\pi^2 r^2}\beta^T\beta\right)\right)^{-\frac{1}{2}}$$

$$\det\left(i\Delta t P - 2\sum_{r=1}^{\infty}\left[I - \left(I + \frac{(\Delta t)^2}{4\pi^2 r^2}\beta^T\beta\right)^{-1}\right] - I\right)^{-\frac{1}{2}}$$

$$\exp\left(i\Omega^T \mathbf{X}_m^\Delta + i\Omega^T \zeta\left(\mathbf{X}_m^\Delta\right)\Delta t - \sum_{r=1}^{\infty}\frac{(\Delta t)^3}{4\pi^2 r^2}\alpha^T\alpha + \sum_{r=1}^{\infty}\frac{(\Delta t)^5}{16\pi^4 r^4}\alpha^T\beta\left(I + \frac{(\Delta t)^2}{4\pi^2 r^2}\beta^T\beta\right)^{-1}\beta^T\alpha\right)$$

$$\exp\left(-\frac{\Delta t}{2}\left[\Omega^T S\left(\mathbf{X}_m^\Delta\right) + i\sum_{r=1}^{\infty}\frac{(\Delta t)^2}{2\pi^2 r^2}\alpha^T\beta\left(I + \frac{(\Delta t)^2}{4\pi^2 r^2}\beta^T\beta\right)^{-1}\right]\right.$$

$$\left[I + 2\sum_{r=1}^{\infty}\left[I - \left(I + \frac{(\Delta t)^2}{4\pi^2 r^2}\beta^T\beta\right)^{-1}\right] - i\Delta t P\right]^{-1}$$

$$\left.\left[\Omega^T S\left(\mathbf{X}_m^\Delta\right) + i\sum_{r=1}^{\infty}\frac{(\Delta t)^2}{2\pi^2 r^2}\alpha^T\beta\left(I + \frac{(\Delta t)^2}{4\pi^2 r^2}\right)^{-1}\right]^T\right).$$

Outline

Fourier coefficients

Let h be a function from $\mathbb{R}^{n+1}$ to $\mathbb{R}$, and $[a, b]$ be an interval in $\mathbb{R}$. Recall that the Fourier series expands a periodic (in the spatial variable $\mathbf{x}$) function h into a sum of exponential functions, namely

$$h(t, \mathbf{x}) = \sum_{\mathbf{j}} \mathcal{H}_{\mathbf{j}}(t) \exp(i \sum_{k=1}^{n} j_k (\frac{2\pi}{b-a} x_k - \frac{b+a}{b-a}\pi))$$

where the sum is over all points $\mathbf{j} = (j_1, j_2, \cdots, j_k, \cdots, j_n)$ with integer entries. The Fourier coefficients $\mathcal{H}_{\mathbf{j}}(t)$ are the corresponding coefficients generated by the Fourier series.

The following proposition could be derived using the results:

Proposition 3.1

The Fourier coefficients could be approximated with a trapezoidal rule approximation:

$$\begin{aligned}
\mathcal{H}_{\mathbf{j}}(t) &= \frac{1}{(b-a)^n}\int_{[a,b]^n} h(t,\mathbf{x}) \exp\left(-2\pi i \sum_{k=1}^{n} j_k \left(\frac{x_k}{b-a} - \frac{b+a}{2(b-a)}\right)\right) d\mathbf{x} \\
&= \frac{1}{(b-a)^n}\int_{[a,b]^n} h(t,\mathbf{x}) \exp\left(-i \sum_{k=1}^{n} j_k \left(\frac{2\pi}{b-a}x_k - \frac{b+a}{b-a}\pi\right)\right) d\mathbf{x} \qquad (3.1) \\
&\approx \sum_{g_1=\cdots=g_n=0}^{G-1} h(t,\mathbf{x}) \exp\left(-i \sum_{k=1}^{n} j_k \left(\frac{2g_k+1}{G} - 1\right)\pi\right) \frac{1}{G^n}.
\end{aligned}$$

where G is the number of partitions for the interval $[a,b]$. The last approximation comes from replacing the integration with a trapezoidal rule approximation with the following uniform grid point $x_{k,g} := a + \left(g_k + \frac{1}{2}\right)\frac{b-a}{G}$ and $\Delta x_k = \frac{b-a}{G}$.

With the application of Fourier series approximation, we can estimate an expectation with the following series.
We define $f_{\mathbf{x}_{m+1}^{\Delta,m,\mathbf{x}_m^{\Delta}}}(\mathbf{x})$ to be the conditional probability density function of $\mathbf{x}_{m+1}$ given the value of $\mathbf{x}_m$. A direct consequence of Proposition 3.1 is as follows.

Proposition 3.2

$$
\begin{aligned}
&\mathbb{E}_m\left[h\left(t_{m+1},\mathbf{X}_{m+1}^{\Delta}\right) \mid \mathbf{X}_m^{\Delta}\right] \\
\approx& \int_{[a,b]^n} h\left(t_{m+1},\mathbf{x}\right) f_{\mathbf{x}_{m+1}^{\Delta,m,\mathbf{x}_m^{\Delta}}}(\mathbf{x})d\mathbf{x} \\
\approx& \sum_{\mathbf{j}} \mathcal{H}_{\mathbf{j}}\left(t_{m+1}\right)\mathbb{E}_m\left[\exp\left(\sum_{k=1}^{n}\left(i\frac{2j_k\pi}{b-a}X_{m+1}^{\Delta,k} - i\frac{b+a}{b-a}j_k\pi\right)\right) \mid \mathbf{X}_m^{\Delta}\right] \\
=& \sum_{\mathbf{j}} \mathcal{H}_{\mathbf{j}}\left(t_{m+1}\right)\exp\left(-i\pi\sum_{k=1}^{n} j_k\frac{b+a}{b-a}\right)\phi_{\mathbf{x}_{m+1}^{\Delta,m,\mathbf{x}}}\left(\frac{2\mathbf{j}\pi}{b-a}\right).
\end{aligned}
\tag{3.2}
$$

Outline

Assume that we aim to approximate the quantity

$$I := \mathbb{E}_m^{\mathbf{x}}\left[u\left(t_{m+1}, \mathbf{X}_{m+1}^{\Delta}\right)\right] = \int_{\mathbb{R}^n} u\left(t_{m+1}, \zeta\right) p(\zeta \mid \mathbf{x}) d\zeta \approx \int_{[a,b]^n} u\left(t_{m+1}, \zeta\right) p(\zeta \mid \mathbf{x}) d\zeta \tag{4.1}$$

by using Fourier method, where $p(\zeta \mid \mathbf{x}) = \mathbb{P}\left(\mathbf{X}_{m+1}^{\Delta} = \zeta \mid \mathbf{X}_m^{\Delta} = \mathbf{x}\right)$ denotes the transitional density function. In order to estimate the value of I, we can chop off the integration range $\mathbb{R}^n$ to $[a, b]^n$ so that we only have to estimate $\int_{[a,b]^n} u\left(t_{m+1}, \zeta\right) p(\zeta \mid \mathbf{x}) d\zeta$; and this approach usually works well, provide that the function u (or more precisely, the integrand) has some sort of integrability. This, in turn, left us with an integration truncation error $\int_{\mathbb{R}^n \setminus [a,b]^n} u\left(t_{m+1}, \zeta\right) p(\zeta \mid \mathbf{x}) d\zeta$, which will be negligible.

We denote x_k to be the k^{th} entry of $\mathbf{x}$ and g_k to be the k^{th} entry of G.

Consider the Fourier expansion for u on $[a,b]^n$, which can be written as

$$u\left(t_{m+1},\mathbf{x}\right)=\sum_{\mathbf{j}}\mathcal{U}\left(t_{m+1},\mathbf{j}\right)\exp\left(i\sum_{k=1}^{n}j_k\left(\frac{2\pi}{b-a}x_k-\frac{b+a}{b-a}\pi\right)\right) \tag{4.2}$$

where the Fourier coefficients are defined by

$$\begin{aligned}\mathcal{U}(t,\mathbf{j})&=\frac{1}{(b-a)^n}\int_{[a,b]^n}u(t,\mathbf{x})\exp\left(-i\sum_{k=1}^{n}j_k\left(\frac{2\pi}{b-a}x_k-\frac{b+a}{b-a}\pi\right)\right)d\mathbf{x}\\&\approx\sum_{m_1=\cdots=m_n=0}^{M-1}u(t,\mathbf{x})\exp\left(-i\sum_{k=1}^{n}j_k\left(\frac{2m_k+1}{M}-1\right)\pi\right)\frac{1}{M^n}.\end{aligned} \tag{4.3}$$

with the result from (3.1).

To this end, with the application of Fourier series approximations, we can estimate the right-hand side of (4.1) by

$$\begin{aligned}
\int_{[a,b]^n} u\left(t_{m+1},\zeta\right) p(\zeta \mid \mathbf{x}) d\zeta &= \int_{[a,b]^n} \sum_{\mathbf{j}} \mathcal{U}\left(t_{m+1},\mathbf{j}\right) \exp\left(i \sum_{k=1}^{n} j_k \left(\frac{2\pi}{b-a} x_k - \frac{b+a}{b-a}\pi \right)\right) p(\zeta \mid \mathbf{x}) d\zeta \\
&\approx \sum_{\mathbf{j}} \mathcal{U}\left(t_{m+1},\mathbf{j}\right) \mathbb{E}_m \left[\exp\left(\sum_{k=1}^{n} \left(i \frac{2 j_k \pi}{b-a} X_{m+1}^{\Delta,k} - i \frac{b+a}{b-a} j_k \pi \right)\right) \middle| \mathbf{X}_m^{\Delta} \right] \\
&= \sum_{\mathbf{j}} \mathcal{U}\left(t_{m+1},\mathbf{j}\right) \exp\left(-i\pi \sum_{k=1}^{n} j_k \frac{b+a}{b-a} \right) \phi_{\mathbf{x}_{m+1}^{\Delta,m,\mathbf{x}_m^{\Delta}}} \left(\frac{2\mathbf{j}\pi}{b-a} \right) \\
&\approx \sum_{\max|j_k| \le N_1} \mathcal{U}\left(t_{m+1},\mathbf{j}\right) \exp\left(-i\pi \sum_{k=1}^{n} j_k \frac{b+a}{b-a} \right) \phi_{\mathbf{x}_{m+1}^{\Delta,m,\mathbf{x}}} \left(\frac{2\mathbf{j}\pi}{b-a} \right)
\end{aligned} \tag{4.4}$$

where we used (3.2) in the third line. For the final line, we truncated the series by only including vectors whose entries are smaller than N_1.

For the convenience of later use, we denote the operator:

$$\mathcal{C}_m(u) : u \mapsto \sum_{\substack{\mathbf{j} \\ \max|j_k| \leq N_1}} \mathcal{U}\left(t_{m+1}, \mathbf{j}\right) \exp\left(-i\pi \sum_{k=1}^{n} j_k \frac{b+a}{b-a}\right) \phi_{\mathbf{X}_{m+1}^{\Delta,m,}}\left(\frac{2\mathbf{j}\pi}{b-a}\right)$$

for $m = 0, 1, \cdots, M-1$.

A simple SDE of geometric type

Assume that there are two independent Brownian motion $\left(B_t^1\right)_{t\in[0,T]}$ and $\left(B_t^2\right)_{t\in[0,T]}$ defined on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$. We consider the following SDE driven by these two Brownian motions:

$$\begin{cases} dX_t = \mu X_t dt + \sigma_1 X_t dB_t^1 + \sigma_2 X_t dB_t^2, \\ X_0 = x_0, \end{cases} \tag{4.5}$$

where μ, σ_1, σ_2 are constants. One can easily check that

$$\mathbb{E}\left(X_T\right) = x_0 e^{\mu T}. \tag{4.6}$$

We use Matlab to draw a graph between log errors against the log number of partitions in time to generate Figure 1. The plot shows that the convergence rate is approximate in order 2.

A simple SDE of geometric type

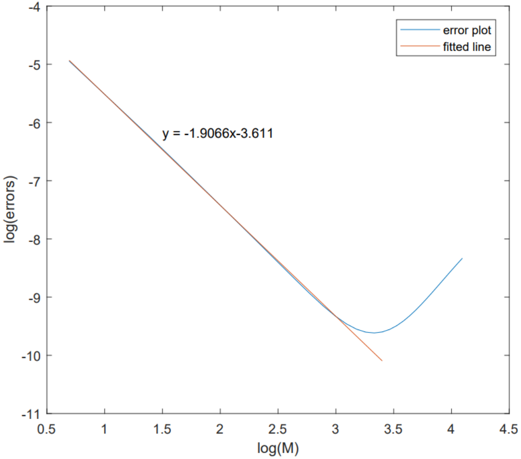

Figure: Log errors versus log number of partitions in time

K. Burrage, P. Burrage, and T. Tian, *Numerical methods for strong solutions of stochastic differential equations: an overview*, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 460 (2004), pp. 373–402.

P. E. Kloeden, E. Platen, and I. Wright, *The approximation of multiple stochastic integrals*, Stochastic analysis and applications, 10 (1992), pp. 431–441.

M. Wiktorsson, *Joint characteristic function and simultaneous simulation of iterated Itô integrals for multiple independent brownian motions*, The Annals of Applied Probability, 11 (2001), pp. 470–487.

The theorem can be similarly derived using techniques applied in Theorem 3.1 of Wiktorsson (2001) [3]. To be specific, we know that

$$
\begin{aligned}
\phi_{\mathbf{X}^{\Delta,m,\mathbf{X}_m^\Delta}_{m+1}}(\Omega) = \mathbb{E}_m \Bigg[& \exp\Bigg(i\Omega^T \mathbf{X}_m^\Delta + i\Omega^T \zeta(\mathbf{X}_m^\Delta)\Delta t + i \sum_{j=1}^{q} \left(\sum_{i=1}^{n} \omega_i S_{ij}(\mathbf{X}_m^\Delta) \right) \Delta B_m^j \\
& + \frac{i}{2} \sum_{j=1}^{q} \sum_{k=1}^{q} \left(\sum_{i=1}^{n} \omega_i L^{(k)} \sigma_{ij}(\mathbf{X}_m^\Delta) \right) \Delta B_m^j \Delta B_m^k \\
& + i \frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi} \sum_{j=1}^{q} \alpha_j \sum_{r=1}^{\infty} \frac{1}{r} G_r^j + i \frac{\Delta t}{2\pi} \sum_{j=1}^{q} \sum_{k=1}^{q} \beta_{jk} \sum_{r=1}^{\infty} \frac{1}{r} G_r^j \left(H_r^k + \sqrt{\frac{2}{\Delta t}} \Delta B_m^k \right) \Bigg) \Bigg] \\
= \exp & \left(i\Omega^T \mathbf{X}_m^\Delta + i\Omega^T \zeta(\mathbf{X}_m^\Delta)\Delta t \right) \times \\
& \mathbb{E}_m \Bigg[\exp \left(i\Omega^T S(\mathbf{X}_m^\Delta) \Delta \mathbf{B}_m + \frac{i}{2} \Delta \mathbf{B}_m^T P \Delta \mathbf{B}_m \right) \\
& \mathbb{E}_m \left[\exp \left(i \sum_{r=1}^{\infty} \frac{1}{r} \left[\frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi} \mathbf{G}_r^T \alpha + \frac{\Delta t}{2\pi} \mathbf{G}_r^T \beta \left(\mathbf{H}_r + \sqrt{\frac{2}{\Delta t}} \Delta \mathbf{B}_m \right) \right] \right) \middle| \Delta \mathbf{B}_m \right] \Bigg].
\end{aligned}
$$

Now consider the conditional expectation in the last term of the equation above; since G and H are i.i.d multivariate normal distributions, we have

$$
\mathbb{E}_m \left[\exp \left(i \sum_{r=1}^{\infty} \frac{1}{r} \left[\frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi} \mathbf{G}_r^T \alpha + \frac{\Delta t}{2\pi} \mathbf{G}_r^T \beta \left(\mathbf{H}_r + \sqrt{\frac{2}{\Delta t}} \Delta \mathbf{B}_m \right) \right] \right) \middle| \Delta \mathbf{B}_m \right]
$$

Appendix II

$$= \prod_{r=1}^{\infty} \mathbb{E}_m \left[\exp\left(i\mathbf{G}_r^T \left[\frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi r}\alpha + \frac{\Delta t}{2\pi r}\beta \left(\mathbf{H}_r + \sqrt{\frac{2}{\Delta t}}\Delta \mathbf{B}_m \right) \right] \right) \middle| \Delta \mathbf{B}_m \right]$$

$$= \prod_{r=1}^{\infty} \mathbb{E}_m \left[\mathbb{E}_m \left[\exp\left(i\mathbf{G}_r^T \left[\frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi r}\alpha + \frac{\Delta t}{2\pi r}\beta \left(\mathbf{H}_r + \sqrt{\frac{2}{\Delta t}}\Delta \mathbf{B}_m \right) \right] \right) \middle| \Delta \mathbf{B}_m, \mathbf{H} \right] \middle| \Delta \mathbf{B}_m \right]$$

$$= \prod_{r=1}^{\infty} \mathbb{E}_m \left[\exp\left(-\frac{1}{2} \left[\frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi r}\alpha^T + \frac{\Delta t}{2\pi r} \left(\mathbf{H}_r^T + \sqrt{\frac{2}{\Delta t}}\Delta \mathbf{B}_m^T \right) \beta^T \right] \right.\right.$$
$$\left.\left. \left[\frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{2}\pi r}\alpha + \frac{\Delta t}{2\pi r}\beta \left(\mathbf{H}_r + \sqrt{\frac{2}{\Delta t}}\Delta \mathbf{B}_m \right) \right] \right) \middle| \Delta \mathbf{B}_m \right]$$

$$= \prod_{r=1}^{\infty} \mathbb{E}_m \left[\exp\left(-\frac{(\Delta t)^3}{4\pi^2 r^2}\alpha^T\alpha - \frac{(\Delta t)^{\frac{5}{2}}}{\sqrt{8}\pi^2 r^2}\alpha^T\beta \left(\mathbf{H}_r + \sqrt{\frac{2}{\Delta t}}\Delta \mathbf{B}_m \right) \right.\right.$$
$$\left.\left. -\frac{(\Delta t)^2}{8\pi^2 r^2} \left(\mathbf{H}_r^T + \sqrt{\frac{2}{\Delta t}}\Delta \mathbf{B}_m^T \right) \beta^T\beta \left(\mathbf{H}_r + \sqrt{\frac{2}{\Delta t}}\Delta \mathbf{B}_m \right) \right) \middle| \Delta \mathbf{B}_m \right]$$

$$= \prod_{r=1}^{\infty} \det\left(I + \frac{(\Delta t)^2}{4\pi^2 r^2}\beta^T\beta \right)^{-\frac{1}{2}}$$
$$\exp\left(-\frac{1}{2} \left(\frac{2}{\Delta t}\Delta \mathbf{B}_m^T \Delta \mathbf{B}_m + \frac{(\Delta t)^3}{2\pi^2 r^2}\alpha^T\alpha \right) \right.$$

Appendix III

$$+\frac{1}{2}\left(\sqrt{\frac{2}{\Delta t}}\Delta\mathbf{B}_m^T-\frac{(\Delta t)^{\frac{5}{2}}}{\sqrt{8}\pi^2r^2}\alpha^T\beta\right)\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\left(\sqrt{\frac{2}{\Delta t}}\Delta\mathbf{B}_m-\frac{(\Delta t)^{\frac{5}{2}}}{\sqrt{8}\pi^2r^2}\beta^T\alpha\right)\Bigg)$$

$$=\det\left(\prod_{r=1}^{\infty}\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)\right)^{-\frac{1}{2}}\exp\left(-\sum_{r=1}^{\infty}\frac{(\Delta t)^3}{4\pi^2r^2}\alpha^T\alpha+\sum_{r=1}^{\infty}\frac{(\Delta t)^5}{16\pi^4r^4}\alpha^T\beta\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\beta^T\alpha\right)$$

$$\exp\left(-\sum_{r=1}^{\infty}\frac{(\Delta t)^2}{2\pi^2r^2}\alpha^T\beta\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\Delta\mathbf{B}_m-\frac{1}{2}\frac{2}{\Delta t}\Delta\mathbf{B}_m^T\sum_{r=1}^{\infty}\left[I-\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\right]\Delta\mathbf{B}_m\right)$$

and

$$\phi_{\mathbf{X}_{m+1}^{\Delta,m},\mathbf{X}_m^{\Delta}}(\Omega)=\det\left(\prod_{r=1}^{\infty}\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)\right)^{-\frac{1}{2}}$$

$$\exp\left(i\Omega^T\mathbf{X}_m^{\Delta}+i\Omega^T\zeta(\mathbf{X}_m^{\Delta})\Delta t-\sum_{r=1}^{\infty}\frac{(\Delta t)^3}{4\pi^2r^2}\alpha^T\alpha+\sum_{r=1}^{\infty}\frac{(\Delta t)^5}{16\pi^4r^4}\alpha^T\beta\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\beta^T\alpha\right)$$

$$\mathbb{E}_m\left[\exp\left(i\left[\Omega^TS(\mathbf{X}_m^{\Delta})+i\sum_{r=1}^{\infty}\frac{(\Delta t)^2}{2\pi^2r^2}\alpha^T\beta\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\right]\Delta\mathbf{B}_m\right.\right.$$

$$\left.\left.-\frac{i}{2}\Delta\mathbf{B}_m^T\left\{-P-\frac{2i}{\Delta t}\sum_{r=1}^{\infty}\left[I-\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\right]\right\}\Delta\mathbf{B}_m\right)\right]$$

Appendix IV

$$
= \det\left(\prod_{r=1}^{\infty}\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)\right)^{-\frac{1}{2}}
$$

$$
\exp\left(i\Omega^T\mathbf{X}_m^\Delta + i\Omega^T\zeta(\mathbf{X}_m^\Delta)\Delta t - \sum_{r=1}^{\infty}\frac{(\Delta t)^3}{4\pi^2r^2}\alpha^T\alpha + \sum_{r=1}^{\infty}\frac{(\Delta t)^5}{16\pi^4r^4}\alpha^T\beta\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\beta^T\alpha\right)
$$

$$
\left(\frac{1}{2\pi\Delta t}\right)^{\frac{q}{2}}\sqrt{\frac{(2\pi i)^q}{\det\left(-P-\frac{2i}{\Delta t}\sum_{r=1}^{\infty}\left[I-\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\right]-\frac{i}{\Delta t}I\right)}}
$$

$$
\exp\left(-\frac{i}{2}\left[\Omega^T S(\mathbf{X}_m^\Delta) + i\sum_{r=1}^{\infty}\frac{(\Delta t)^2}{2\pi^2r^2}\alpha^T\beta\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\right]\right.
$$

$$
\left[-P-\frac{2i}{\Delta t}\sum_{r=1}^{\infty}\left[I-\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\right]-\frac{i}{\Delta t}I\right]^{-1}
$$

$$
\left.\left[\Omega^T S(\mathbf{X}_m^\Delta) + i\sum_{r=1}^{\infty}\frac{(\Delta t)^2}{2\pi^2r^2}\alpha^T\beta\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\right)^{-1}\right]^T\right)
$$

$$
= \det\left(\prod_{r=1}^{\infty}\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)\right)^{-\frac{1}{2}}\det\left(i\Delta tP - 2\sum_{r=1}^{\infty}\left[I-\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\right]-I\right)^{-\frac{1}{2}}
$$

$$
\exp\left(i\Omega^T\mathbf{X}_m^\Delta + i\Omega^T\zeta(\mathbf{X}_m^\Delta)\Delta t - \sum_{r=1}^{\infty}\frac{(\Delta t)^3}{4\pi^2r^2}\alpha^T\alpha + \sum_{r=1}^{\infty}\frac{(\Delta t)^5}{16\pi^4r^4}\alpha^T\beta\left(I+\frac{(\Delta t)^2}{4\pi^2r^2}\beta^T\beta\right)^{-1}\beta^T\alpha\right)
$$

Appendix V

$$\exp\left(-\frac{\Delta t}{2}\left[\Omega^T S(\mathbf{X}_m^\Delta)+i\sum_{r=1}^{\infty}\frac{(\Delta t)^2}{2\pi^2 r^2}\alpha^T\beta\left(I+\frac{(\Delta t)^2}{4\pi^2 r^2}\beta^T\beta\right)^{-1}\right]\right.$$
$$\left[I+2\sum_{r=1}^{\infty}\left[I-\left(I+\frac{(\Delta t)^2}{4\pi^2 r^2}\beta^T\beta\right)^{-1}\right]-i\Delta tP\right]^{-1}$$
$$\left.\left[\Omega^T S()+i\sum_{r=1}^{\infty}\frac{(\Delta t)^2}{2\pi^2 r^2}\alpha^T\beta\left(I+\frac{(\Delta t)^2}{4\pi^2 r^2}\right)^{-1}\right]^T\right)$$

To this end, we successfully obtain the conditional characteristic function and then focus on constructing the backward scheme for solving the target BSDE.

```
N = 64*8;
G = N*2;
step = 2;
stepmax = 60;
plotrange = step:step:stepmax;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%% Set %%%%%
% mu = 0.5; sigma = [0.3 0.2]; x_0 = 1; T = 1;
mu = 0.5; sigma = [0.3 0.2]; x_0 = 1; T = 1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% b = 32;
b = 32*x_0;

J = -N/2:N/2;
g = 0:G-1;
x = (b/G*(g+1/2));
tic;
E = zeros(1,stepmax/step);
for m=plotrange
    delta_t = T/m;
    FCoeExp = (x*exp(-1i*pi*J'*((2*g+1)/G-1)).'/G).*exp(-1i*pi*J);
    cf_x0 = NumericalBSDE_cf(x_0,2*pi/b*J,mu,sigma,delta_t);
    cf_x = NumericalBSDE_cf(x,2*pi/b*J,mu,sigma,delta_t);
    FCoe_JG = exp(-1i*pi*J'*((2*g+1)/G-1)).'/G;
    Expp = exp(-1i*pi*J);
    for mm = 0:m-1
```

```
        C = FCoeExp*cf_x0;
        FCoeExp = ((FCoeExp*cf_x)...
            *FCoe_JG).*Expp;
    end
    E(m/step) = real(C);
end
logerr = log(abs(E-x_0*exp(mu*T)));
plot(log(plotrange),logerr);
hold on;
% title('');
xlabel('log(M)');
ylabel('log(errors)');
f = polyfit(log(plotrange(1:5)),logerr(1:5),1);
plot(log(plotrange(1:15)),log(plotrange(1:15))*f(1)+f(2));
hold on;
text(1.5,1.5*f(1)+f(2)+0.3,['y = ',num2str(f(1)),'x',num2str(f(2))]);
legend('error plot','fitted line');
set(gcf, 'PaperPosition', [-0.2,0.2,15.8,13.2], 'PaperSize', [15,13]);
saveas(gcf, 'simpleGBM.pdf');
t2 = toc
```