Discrete Mathematics - COMP0147

2025 - Term 1
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LIST OF EXERCISES

Recursion and Structural Induction
(Rosen - Chapter 5)

Required reading for this list: *Discrete Mathematics and Its Applications* (Rosen, 7^{th} Edition):

- Chapter 5.3: *Recursive Definitions and Structural Induction*
- Chapter 5.4: *Recursive Algorithms*

Note: The exercises are classified into difficulty levels: easy, medium, and hard. This classification, however, is only indicative. Different people may disagree about the difficulty level of the same exercise. Do not be discouraged if you see a difficult exercise–you may find that it is actually easy, by discovering a simpler way to solve it!

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1. (Rosen 5.3-7) Give a recursive definition for the sequence a_n, $n = 1, 2, 3, ...$ if
   (a) `[Easy]` $a_n = 6n$.
   (b) `[Medium]` $a_n = 2n + 1$.
   (c) `[Medium]` $a_n = 10^n$.
   (d) `[Easy]` $a_n = 5$.

2. (Rosen 5.3-8) Give a recursive definition for the sequence a_n, $n = 1, 2, 3, ...$ if
   (a) `[Easy]` $a_n = 4n - 2$.
   (b) `[Medium]` $a_n = 1 + (-1)^n$.
   (c) `[Medium]` $a_n = n(n + 1)$.
   (d) `[Easy]` $a_n = n^2$.

3. (Rosen 5.3-25) Give a recursive definition of:
   (a) `[Easy]` the set of even integers.
   (b) `[Easy]` the set of positive integers congruent to 2 modulo 3 (that is, the positive integers that have remainder 2 when divided by 3).
   (c) `[Medium]` the set of positive integers not divisible by 5.

4. (Rosen 5.3-27) Let S be a subset of ordered pairs of integers, defined recursively by
   Base step: $(0, 0) \in S$,
   Recursive step: If $(a, b) \in S$, then $(a, b + 1) \in S$, $(a + 1, b + 1) \in S$ and $(a + 2, b + 1) \in S$.
   (a) `[Easy]` List the elements of S produced by the first four applications of the recursive definition.
   (b) `[Medium]` Use structural induction to show that $a \leq 2b$ whenever $(a, b) \in S$.

5. (Rosen 5.3-28) Give a recursive definition for each of the sets of ordered pairs of positive integers. (Hint: Plot the points on the plane and look for lines containing the points of the set.)

(a) [Easy] $S = \{(a,b) \mid a,b \in \mathbb{Z}^+, a+b \text{ is odd}\}$

(b) [Medium] $S = \{(a,b) \mid a,b \in \mathbb{Z}^+, a \mid b\}$

(c) [Medium] $S = \{(a,b) \mid a,b \in \mathbb{Z}^+, 3 \mid a+b\}$

6. (Rosen 5.3-29) Give a recursive definition for each of the sets of ordered pairs of positive integers. (Hint: Plot the points on the plane and look for lines containing the points of the set.)

   (a) [Medium] $S = \{(a,b) \mid a,b \in \mathbb{Z}^+, a+b \text{ is even}\}$

   (b) [Medium] $S = \{(a,b) \mid a,b \in \mathbb{Z}^+,\ a \text{ or } b \text{ is odd}\}$

7. (Rosen 5.3-33)

   (a) [Medium] Give a recursive definition of the function $m(s)$, which returns the smallest digit in a string s of decimal digits. (Ex: $m(3459367) = 3$, $m(12) = 1$, $m(979) = 7$).

   (b) [Hard] Use induction to show that $m(st) = \min(m(s), m(t))$. (Hint: Try induction on the length of the string s, assuming s and t are nonempty decimal-digit strings.)

8. [Hard] (Rosen 5.3-35) Give a recursive definition for the reverse of a string. (Hint: first define the reverse of the empty string λ. Then write a string w of length $n+1$ as xy, where x is a string of length n, and express the reverse of w in terms of x^R and y.)

9. [Hard] (Rosen 5.3-43) Let T be a complete binary tree (that is, a tree in which all internal nodes have exactly two child nodes), let $n(T)$ be the number of nodes in the tree T, and let $h(T)$ be the height (that is, the longest path from the root to a leaf of the tree) of T.

   Use structural induction to show that $n(T) \geq 2h(T) + 1$.

10. (Rosen 5.3-48 to 55) Consider the following inductive definition of a version of Ackermann's function:

$$A(m,n) = \begin{cases} 2n & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \text{ and } n = 0 \\ 2 & \text{if } m \geq 1 \text{ and } n = 1 \\ A(m-1, A(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 2 \end{cases}$$

   4.3-48 [Easy] Find the following values of Ackermann's function: $A(1,0), A(0,1), A(1,1), A(2,2)$.

   4.3-49 [Easy] Show that $A(m,2) = 4$ whenever $m \geq 1$.

   4.3-50 [Easy] Show that $A(1,n) = 2^n$ whenever $n \geq 1$.

   4.3-53 [Hard] Prove that $A(m, n+1) > A(m,n)$ for all $m, n \geq 0$.

   4.3-54 [Medium] Prove that $A(m+1, n) \geq A(m,n)$ whenever m and n are nonnegative integers.

   4.3-55 [Medium] Prove that $A(i,j) \geq j$ whenever i and j are nonnegative integers.

11. [Easy] (Rosen 5.4-7) Give a recursive algorithm for computing $n.x$ whenever n is a positive integer and n is an integer, using just addition.

12. [Easy] (Rosen 5.4-10) Give a recursive algorithm for finding the maximum of a finite set of integers, making use of the fact that the maximum of n integers is the larger of the last integer in the list and the maximum of the first $n-1$ integers in the list.

13. [Easy] (Rosen 5.4-12) Devise a recursive algorithm for finding $x^n \pmod m$ whenever n, x, and m are positive integers based on the fact that $x^n \pmod m = (x^{n-1} \pmod m \cdot x \pmod m) \pmod m$.

14. (Rosen 5.4-51 and 55) The quick sort is an efficient algorithm. To sort $a_1, a_2, \ldots, a_n$, this algorithm begins by taking the first element a_1 and forming two sublists, the first containing those elements that are less than a_1, in the order they arise, and the second containing those elements greater than a_1, in the order they arise. Then a_1 is put at the end of the first sublist. This procedure is repeated recursively for each sublist, until all sublists contain one item. The ordered list of n items is obtained by combining the sublists of one item in the order they occur.

5.4-51 [Medium] Let $a_1, a_2, \ldots, a_n$ be a list of n distinct real numbers. How many comparisons are needed to form two sublists from this list, the first containing elements less than a_1 and the second containing elements greater than a_1?

5.4-55 [Hard] Determine the worst-case complexity of the quick sort algorithm in terms of the number of comparisons used.