Complex Variables

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Properties of the complex field

- $\mathbb{C}$ is a field which cannot be ordered. If $z = a + bi \in \mathbb{C}$, then $\overline{z} = a - bi$ and $|z| = \sqrt{a^2 + b^2}$ so $|z|^2 = z \cdot \overline{z}$.
- If $z, w \in \mathbb{C}$, then $|z \cdot w| = |z| \cdot |w|$, $|z/w| = |z|/|w|$, $\overline{z + w} = \overline{z} + \overline{w}$, and $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$. The triangle inequality holds, and $1/z = \overline{z}/|z|^2$.
- $\mathbb{C}$ is a complete metric space.

Theorem 1. *(Cantor's nested set theorem) In a complete metric space, if $\{F_n\}_{n\in\mathbb{N}}$ is a family of nested closed sets such that $\lim_{n\to\infty} \operatorname{diam} F_n = 0$, there exists a unique point $p \in \bigcap_{n\in\mathbb{N}} F_n$.*

Theorem 2. *If K_1, K_2 are disjoint compact sets in $\mathbb{C}$, there exist $z_1 \in K_1$, $z_2 \in K_2$ such that $|z_1 - z_2|$ is the minimum distance between the two sets.*

Line integrals in $\mathbb{C}$

Remark 1. *We only consider continuous paths $\gamma : [a, b] \to \mathbb{C}$ which are piecewise continuously differentiable, i.e. there exists a partition $a = t_1 < t_2 < \cdots < t_n = b$ such that γ is continuously differentiable on each $[t_i, t_{i+1}]$.*

Definition 1. *An open set $\mathcal{O} \subseteq \mathbb{C}$ is polygonally connected if for every $z_1, z_2 \in \mathcal{O}$, we can produce a finite sequence of alternating horizontal and vertical segments in $\mathcal{O}$ connecting z_1 and z_2.*

Theorem 3. *An open set $\mathcal{O} \subseteq \mathbb{C}$ is polygonally connected if and only if it is connected.*

Proof. Suppose $\mathcal{O}$ is polygonally connected but disconnected by two sets. Draw a polygonal path between a point in each set and disconnect $[0, 1]$, contradiction. Suppose $\mathcal{O}$ is connected, define one set to be points polygonally connected to $z_0 \in \mathcal{O}$, then its complement must be empty otherwise they disconnect $\mathcal{O}$. □

Remark 2. *In the next two results, let $f : [a, b] \to \mathbb{C}$ be continuous, differentiable, and integrable, and let $u(t) = Ref(t)$ and $v(t) = Imf(t)$. Then $f'(t) = u'(t) + iv'(t)$ and $\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$.*

Theorem 4. $\int_a^b if(t)dt = i\int_a^b f(t)dt$.

Proof.

$$\int_a^b if(t)dt = \int_a^b -v(t) + iu(t)dt = -\int_a^b v(t)dt + i\int_a^b u(t)dt = i\int_a^b u(t)dt + iv(t)dt.$$

□

Theorem 5. $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$.

Proof. For any $z \in \mathbb{C} \setminus \{0\}$, take $\alpha = |z|/z$ then $\alpha z = |z|$. Hence,

$$\left| \int_a^b f(t)dt \right| = \alpha \int_a^b f(t)dt = \int_a^b \alpha f(t)dt = \int_a^b Re(\alpha f(t))dt + i \int_a^b Im(\alpha f(t))dt$$
$$\leq \int_a^b |Re(\alpha f(t))|dt \leq \int_a^b |\alpha f(t)|dt = \int_a^b |f(t)|dt.$$

□

Definition 2. *In the next few results, let $\mathcal{O} \subseteq \mathbb{C}$ be open, $f, g : \mathcal{O} \to \mathbb{C}$ be continuous, and $\gamma : [a, b] \to \mathcal{O}$ a path. Then $\int_\gamma f(z)dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt$. Note that as γ is piecewise continuously differentiable, we can break up the integral where γ' is not defined.*

Theorem 6. $\int_\gamma (f + g)(z)dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz$.

Theorem 7. *(Path reparameterization) Let $\varphi : [c, d] \to [a, b]$ be continuously differentiable such that $\varphi' > 0$ and $\varphi(c) = a$, $\varphi(d) = b$. Then $\int_{\gamma\circ\varphi} f(z)dz = \int_\gamma f(z)dz$.*

Proof. Let $u = \varphi(t)$.

$$\int_{\gamma\circ\varphi} f(z)dz = \int_c^d f((\gamma \circ \varphi)(t))(\gamma \circ \varphi)'(t)dt = \int_c^d f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)dt$$
$$= \int_a^b f(\gamma(u))\gamma'(u)du = \int_\gamma f(z)dz.$$

□

Theorem 8. *(Reverse path orientation) Same as above, but let $\varphi' < 0$ and $\varphi(d) = a$, $\varphi(c) = b$. Then $\int_{\gamma\circ\varphi} f(z)dz = -\int_\gamma f(z)dz$.*

Theorem 9. *Let F be a primitive of f, i.e. $F'(z) = f(z)$ for $z \in \mathcal{O}$. Then $\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$.*

Theorem 10. *(Integral estimation) $\left|\int_\gamma f(z)dz\right| \leq \max_{t\in[a,b]} |f(\gamma(t))| \cdot \int_a^b |\gamma'(t)|dt$ (the arc length of γ).*

Theorem 11. *(Pass limit through integral) Let $\{f_n\}$ be a family of continuous functions on the image of γ which converges uniformly to f. Then $\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz$.*

Cauchy's theorems

Definition 3. *A function $f : \mathcal{O} \to \mathbb{C}$ is holomorphic, written $f \in H(\mathcal{O})$, if it is complex-differentiable on $\mathcal{O}$.*

Theorem 12. *(Cauchy-Goursat) Let $\mathcal{O} \subseteq \mathbb{C}$ be open, Δ a solid closed triangle in $\mathcal{O}$, and $f \in H(\mathcal{O})$. Then $\int_{\partial\Delta} f(z)dz = 0$.*

Proof. Form four subtriangles using the midpoint of each side. Take the biggest one and repeat the process. But then the integral around the jth triangle must be at least $1/4^j$ times the original integral. If you assume the original integral is nonzero, you get a contradiction as you can find a j such that the integral around the jth triangle is arbitrarily small, using Cantor's nested set theorem to rewrite the integral in terms of a fixed point. □

Theorem 13. *(Cauchy) Let $\mathcal{O} \subseteq \mathbb{C}$ be convex and open, γ a closed path in $\mathcal{O}$, and $f \in H(\mathcal{O})$. Then $\int_\gamma f(z)dz = 0$.*

Proof. Build a primitive for f by taking $F(z)$ to be the straight line integral from some z_0 to z of f. Prove that $F' = f$ using Cauchy-Goursat and continuity. □

Remark 3. *We can "forgive" f and use Cauchy-Goursat or Cauchy's theorem if f is only continuous at some $z_0 \in \mathcal{O}$ but holomorphic on $\mathcal{O} \setminus \{z_0\}$.*

Definition 4. *Let γ be a closed path and let $\mathcal{O}$ be an open set in the complement of γ. Then for $z \in \mathcal{O}$, $\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z} d\zeta$ is the winding number of γ around z, or the number of times that γ travels counterclockwise around z.*

Lemma 1. *Winding numbers are integers, constant for all z in connected open sets in the complement of γ, and 0 for z in unbounded connected open sets in the complement of γ.*

Theorem 14. *(Cauchy's integral formula) Let $\mathcal{O} \subseteq \mathbb{C}$ be open and take $\overline{D(z_0, r)} \subseteq \mathcal{O}$. Let $z \in D(z_0, r)$ and $f \in H(\mathcal{O})$. Then $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$ where γ traces the boundary of $D(z_0, r)$, $\gamma(\theta) = z_0 + r\cos\theta + ir\sin\theta$.*

Proof. Define $g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$ if $\zeta \neq z$, and $g(z) = f'(z)$. Then g is continuous at z and holomorphic otherwise, so we can apply Cauchy's theorem to get that $\frac{1}{2\pi i}\int_\gamma g(\zeta) d\zeta = 0$. Then we only need to show that $\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z} d\zeta = \operatorname{Ind}_\gamma(z) = 1$. Note that $\frac{\partial}{\partial z}\int_\gamma \frac{1}{\zeta - z} d\zeta = \int_\gamma \frac{\partial}{\partial z}\frac{1}{\zeta - z} d\zeta$. $\square$

Remark 4. *Moreover, $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(\gamma(\theta)) d\theta$ (the mean value property) and $f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$.*

Theorem 15. *(Liouville) If f is entire ($f \in H(\mathbb{C})$) and bounded, then f is constant.*

Proof. By Cauchy's integral formula,

$$|f'(z_0)| = \left| \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta \right| \leq \max_{\zeta \in \gamma} \frac{|f(\zeta)|}{|\zeta - z_0|^2} \cdot 2\pi r = \frac{2\pi r}{r^2} \cdot M$$

which goes to 0 as $r \to \infty$. $\square$

Theorem 16. *(The fundamental theorem of algebra) A non-constant polynomial has at least one root in $\mathbb{C}$.*

Proof. If P has no roots, then $1/P(z)$ is bounded and entire so constant, a contradiction. $\square$

Theorem 17. *(Cauchy-Riemann equations) Let $\mathcal{O} \subseteq \mathbb{C}$ be open and $f : \mathcal{O} \to \mathbb{C}$. If $f'(x_0 + iy_0)$ exists for some $x_0 + iy_0 \in \mathcal{O}$, and $u(x, y) = \Re f(x + iy)$, $v(x, y) = \Im f(x + iy)$, then $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$ and $\frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$.*

Theorem 18. *Let $f \in H(\mathcal{O})$ where $\mathcal{O} \subseteq \mathbb{C}$ is a connected open set. If f has a zero of infinite order, or a limit point in its set of zeros, then f is identically zero.*

Theorem 19. *(Morera) Let $\mathcal{O} \subseteq \mathbb{C}$ be a connected open set and let $f : \mathcal{O} \to \mathbb{C}$ be continuous. If $\int_\gamma f(z) dz = 0$ for every closed path $\gamma \in \mathcal{O}$, then $f \in H(\mathcal{O})$.*

Theorem 20. *(Maximum modulus) Let $f \in H(\mathcal{O})$ where $\mathcal{O} \subseteq \mathbb{C}$ is a connected open set. Suppose there exists $z_0 \in \mathcal{O}$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \mathcal{O}$. Then f is constant.*

Convergence and power series

Remark 5. *The Weierstrass M-test tells us that given a family of functions $g_k : S \to \mathbb{C}$, if each $|g_k|$ is bounded by some M_k and $\sum M_k$ converges, then $\sum g_k$ converges uniformly and absolutely on S.*

Theorem 21. *Let $\mathcal{O} \subseteq \mathbb{C}$ be open, $f \in H(\mathcal{O})$, and $\overline{D(z_0, r)} \subseteq \mathcal{O}$. Then for all $z \in D(z_0, r)$,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \textit{where} \quad a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

and γ traces the boundary of $D(z_0, r)$.

Proof. By Cauchy's integral formula, we can rewrite

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\cdot\frac{1}{1-\frac{z-z_0}{\zeta - z}}d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\zeta - z_0)^n}d\zeta$$

and we can bound each nth term and use the M-test to get the convergent power series. □

Remark 6. *An analytic function is locally equal to a convergent power series. In $\mathbb{C}$, moreover, an analytic function is equal to its Taylor series on any closed disk in the domain.*

Theorem 22. *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with complex coefficients as a function of z. Then there exists ρ, its radius of convergence, such that if $|z-z_0| < \rho$, the series converges absolutely, and if $|z-z_0| > \rho$, the sequence $\{a_n(z-z_0)^n\}_{n\in\mathbb{N}}$ is unbounded.*

Proof. Take $\rho = \sup\{|z| \mid \text{the terms } |a_n||z-z_0|^n \text{ are bounded}\}$. □

Theorem 23. *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a complex power series. Let $f : D(z_0,\rho) \to \mathbb{C}$ be a function such that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. Then $f \in H(D(z_0,\rho))$ and $f'(z) = \sum_{n=0}^{\infty} na_n(z-z_0)^{n-1}$ where f' has radius of convergence ρ as well.*

Proof. Show that the derivative of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is as claimed. Note that a geometric series with each term multiplied by n still converges. □

Remark 7. *If $f \in H(\mathcal{O})$, then f is equal to a convergent power series on each disk contained in $\mathcal{O}$. Then by above, f' exists and is a power series on the same disk and so on, so f is infinitely differentiable.*

Theorem 24. *If $f \in H(D(z_0,r)\setminus\{z_0\})$ and f is continuous at z_0, then $f \in H(D(z_0,r)))$.*

Proof. Build a primitive $F(z) = \int_{[z_0,z]} f(\zeta)d\zeta$ such that $F' = f$, then f is locally a power series and therefore holomorphic. □

Remark 8. *If $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, then $a_n = f^{(n)}(z_0)/n!$.*

Theorem 25. *(Cauchy estimates) Let $f \in H(\mathcal{O})$ and let $\overline{D(z_0,r)} \subseteq \mathcal{O}$. Then $|f^{(n)}(z_0)| \leq \frac{n!M(r)}{r^n}$ where $M(r) = \max_{z\in\overline{D(z_0,r)}}|f(z)|$.*

The complex logarithm

- For $z \in \mathbb{C}$, define $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ which is holomorphic on $\mathbb{C}$. Moreover, $e^z \neq 0$ for all $z \in \mathbb{C}$ and $e^z \cdot e^w = e^{z+w}$.
- $\cos z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$ and $\sin z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$.
- (Euler's formula) $e^{iz} = \cos z + i\sin z$. e^z is periodic with period $2\pi i$.
- $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.

Theorem 26. *Let $\mathcal{O} \subseteq \mathbb{C}$ be a convex open set not containing 0. Then there exists $l \in H(\mathcal{O})$ such that $e^{l(z)} = z$ for $z \in \mathcal{O}$.*

Proof. Take $z_0 \in \mathcal{O}$ and let $l(z) = \int_{[z_0,z]}\frac{1}{\zeta}d\zeta$. □

Homotopic deformations

Definition 5. *Let $\gamma_0, \gamma_1 : [0,1] \to \mathcal{O} \subseteq \mathbb{C}$ be two paths from some z_1 to z_2. Then a family $\{\gamma_s\}_{s\in[0,1]}$ of paths in $\mathcal{O}$, each from z_1 to z_2, such that*

1. *the function $\Gamma(s,t) : [0,1]^2 \to \mathcal{O}$ such that $\Gamma(s,t) = \gamma_s(t)$ is continuous*
2. *for each $t \in [0,1]$, $\tau_t : [0,1] \to \mathcal{O}$ where $\tau_t(s) = \gamma_s(t)$ is a path*

is called a homotopic deformation of γ_0 to γ_1.

Definition 6. *A connected open set $\mathcal{O} \subseteq \mathbb{C}$ is simply connected if for all closed paths γ in $\mathcal{O}$ which start and end at some z_0, there exists a homotopic deformation from γ to the trivial path at z_0.*

Theorem 27. *Let $\mathcal{O} \subseteq \mathbb{C}$ be a connected open set and let $f \in H(\mathcal{O})$. Let γ_0 and γ_1 be two paths in $\mathcal{O}$ such that there is a homotopic deformation from γ_0 to γ_1. Then $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$.*

Proof. Divide $[0,1]^2$ into a grid of squares. The line integral around each square is 0 by Cauchy's theorem, so the sum of the integrals is 0. Moreover, the integrals of the segments on the interior of $[0,1]^2$ cancel out. The integrals of the vertical segments of the boundary are 0, and the horizontal segments are γ_0 and γ_1. □

Remark 9. *Hence, Cauchy's theorem holds for simply connected $\mathcal{O}$ by deforming γ to a point.*

Laurent expansions

Theorem 28. *Let $\mathcal{O} \subseteq \mathbb{C}$ be open, $f \in H(\mathcal{O})$, and $A = \{z \mid r < |z - z_0| < R\}$ be an annulus centered at z_0 such that $\overline{A} \subseteq \mathcal{O}$. Then for all $z \in A$,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

and γ traces any concentric circle of radius between (including) r and R centered at z_0. This series is unique, absolutely convergent, and uniformly convergent on every compact subset of A.

Proof. Let γ_o be the outer boundary and γ_i be the inner. By a homotopic deformation, we get $\int_{\gamma_o} f(z)dz = \int_{\gamma_i} f(z)dz$. Define $g(\zeta) = \frac{f(\zeta)-f(z)}{\zeta - z}$ if $\zeta \neq z$, and $g(z) = f'(z)$. Hence, $\frac{1}{2\pi i}\int_{\gamma_i} g(\zeta)d\zeta = \frac{1}{2\pi i}\int_{\gamma_o} g(\zeta)d\zeta$, and the result follows. □

Definition 7. *We call the a_{-1} coefficient the residue of f at z_0, denoted* $\operatorname{Res}(f, z_0)$.

Theorem 29. *(Riemann's theorem on removable singularities) Let $f \in H(D(z_0,R) \setminus \{z_0\})$ and let f be bounded. Then we can define $f(z_0)$ such that $f \in H(D(z_0,R))$.*

Proof. For $z \in D(z_0,R) \setminus \{z_0\}$, write $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. We have that

$$|a_n| \leq r \max_{\zeta\in\gamma_r} \frac{|f(\zeta)|}{(\zeta - z_0)^{n+1}} \leq \frac{M}{r^n}$$

where M is the bound of f and $0 < r \leq R$. Then $|a_n| = \lim_{r\to 0} |a_n| = 0$ when $n < 0$, as a_n does not depend on r. Hence, f can be written as a power series on $D(z_0,R) \setminus \{z_0\}$, and so if we define $f(z_0) = a_0$, then $f \in H(D(z_0,R))$. □

The residue theorem

Definition 8. *We say f has an isolated singularity at z_0 if we can find some r such that $f \in H(z_0, r) \setminus \{z_0\}$.*

1. *Removable singularity: f is bounded.*
2. *Pole: There are finitely many terms in the Laurent expansion where $n < 0$, i.e. $f(z) = \sum_{n=-N}^{\infty} a_n(z-z_0)^n$. We define $P(z) = \sum_{n=-N}^{-1} a_n(z-z_0)^n$ to be the principal part of the series.*
3. *Essential singularity: There are infinity many terms in the Laurent expansion where $n < 0$.*

Definition 9. *If f is holomorphic in an open set $\mathcal{O}$ except for isolated poles, then f is meromorphic in $\mathcal{O}$.*

Theorem 30. *(Residue) Let $\mathcal{O} \subseteq \mathbb{C}$ be a simply connected open set and f be meromorphic in $\mathcal{O}$ with finitely many poles at $z_1, \ldots, z_N$. Let γ be a closed path in $\mathcal{O}$ not passing through any of the poles. Then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}(f, z_j) \cdot \operatorname{Ind}_\gamma(z_j).$$

Proof. Let P_j be the principal part of the Laurent series of f at z_j. Then $f(z) - \sum_{j=1}^{N} P_j(z) = g(z) \in H(\mathcal{O})$. Near each z_j, writing f as a Laurent series, we have $f(z) - P_j(z)$ is a power series and thus z_j is a removable discontinuity. Otherwise, for any $z \in \mathcal{O}$ and j such that $z \neq z_j$, we have that P_j is holomorphic near z.

Hence, $\int_\gamma g(z)dz = 0$ so $\int_\gamma f(z)dz = \sum_{j=1}^{N} \int_\gamma P_j(z)dz$. We have that $\int_\gamma P_j(z)dz = \sum_{n=-N}^{-1} \int_\gamma a_n(z-z_j)^n$, so if $n < -1$, the integral is 0 as $a_n(z-z_j)^n$ has a primitive, and if $n = -1$, the integral is $a_{-1} \int_\gamma (z-z_j)^{-1}dz = \operatorname{Res}(f, z_j) 2\pi i \operatorname{Ind}_\gamma(z_j)$. □

Theorem 31. *(Weierstrass-Casorati) Suppose f has an essential isolated singularity at some z_0. Then for any positive real ε, we have $f(D(z_0, \varepsilon) \setminus \{z_0\})$ is dense in $\mathbb{C}$.*

Proof. Suppose the image is not dense. Then we can find some w_0, δ such that $f(z) \notin D(w_0, \delta)$ for all $z \in D(z_0, \varepsilon) \setminus \{z_0\}$. Then $|f(z) - w_0| \geq \delta$ and so $1/(f(z) - w_0)$ is bounded and hence holomorphic by Riemann's theorem. Writing $1/(f(z) - w_0)$ as a power series, we have $f(z) - w_0 = \frac{1}{a_N(z-z_0)^N + \ldots}$ and so $f(z) = w_0 + \frac{1}{(z-z_0)^N} \cdot \frac{1}{a_N + a_{N+1}(z-z_0) + \ldots}$. We have that $\frac{1}{a_N + \ldots}$ is holomorphic around z_0 as its denominator stays close to a_N which is non-zero. But then we can write $f(z) = w_0 + \frac{1}{(z-z_0)^N} \sum_{n=0}^{\infty} b_n(z-z_0)^n$, implying that f has only a pole at z_0. □

The argument principle

Definition 10. *We say f has a pole of order n at z_0 if the Laurent series at z_0 has $a_{-n} \neq 0$. A pole of order 1 is called a simple pole.*

Theorem 32. *(Argument principle) Let $f : \mathcal{O} \to \mathbb{C}$ be meromorphic, $\overline{D(z_0, r)} \subseteq \mathcal{O}$, and f is non-zero and has no poles on the boundary γ of $D(z_0, r)$. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \#\text{ of zeros of } f \text{ in } D(z_0, r) - \#\text{ of poles of } f \text{ in } D(z_0, r) = \operatorname{Ind}_{f \circ \gamma}(0)$$

counted with multiplicity.

Proof. We have that f'/f has simple poles at the zeros and poles of f, and $\operatorname{Res}(f'/f, z_0)$ is the order of z_0 if z_0 is a zero, or the negative of the order of z_0 if z_0 is a pole. The result follows from the residue theorem. Moreover, we can rewrite $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{1}{z} dz$. □

Remark 10. *For $f \in H(\mathcal{O})$, we say that if $f(z) = a$ for some $z \in \mathcal{O}$, then z is an $a-$point. If $\overline{D(z_0, r)} \subseteq \mathcal{O}$ and doesn't have any $a-$points on its boundary γ, then the number of $a-$points inside $D(z_0, r)$ is $\mathrm{Ind}_{f\circ\gamma}(a)$.*

Proof. Let $g(z) = f(z) - a$. Then $\mathrm{Ind}_{\gamma\circ g}(0) = \mathrm{Ind}_{\gamma\circ f}(a)$ is the number of zeros of g. □

Theorem 33. *(Rouché) Let $f, g : \mathcal{O} \to \mathbb{C}$, $f, g \in H(\mathcal{O})$, and let $\overline{D(z_0, r)} \subseteq \mathcal{O}$. Assume $|f(z)| > |g(z)|$ for all z on the boundary of $D(z_0, r)$. Then f and $f + g$ have the same number of zeros in $D(z_0, r)$.*

Theorem 34. *(Open mapping) Suppose $\mathcal{O} \subseteq \mathbb{C}$ is a connected open set and $f \in H(\mathcal{O})$ is nonconstant. Then f is an open mapping.*

Proof. Take $z_0 \in \mathcal{O}$ and $D(z_0, \delta) \subseteq \mathcal{O}$ such that $f(z) \neq f(z_0)$ on the boundary γ. Let $\Gamma = f \circ \gamma$, and we have that $\mathrm{Ind}_\Gamma(f(z_0)) \geq 1$. Now let $w \in D(f(z_0), \varepsilon) \subseteq \mathbb{C} \setminus \Gamma$. But then as $D(f(z_0), \varepsilon)$ is connected, we have that $\mathrm{Ind}_\Gamma(w) = \mathrm{Ind}_\Gamma(f(z_0)) \geq 1$, so f has at least one $w-$point so $w \in f(D(z_0, \delta))$. □