Logic Review

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1 Size

Remark 1. *Let A, B be sets. We say $|A| = |B|$ if there is a bijection from A to B. We say $|A| \leq |B|$ if there is an injection from A to B. We say $|A| \geq |B|$ if there is a surjection from A to B.*

Remark 2. $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

Proof. Let $\mathbb{N} \times \mathbb{N}$ be represented by lattice points in the first quadrant. Start at the origin and diagonally snake away outward. □

Definition 1. *A set is* countable *if it has the same size as $\mathbb{N}$.*

Theorem 1. *The union of (at most) countably many (at most) countable sets is (at most) countable.*

Proof. We can find a bijection from this union to $\mathbb{N} \times \mathbb{N}$. □

Definition 2. *(Dedekind) A set is infinite if it can be put in bijection with a strict subset of itself.*

Theorem 2. *(Cantor) Let X be a set. Then $|X| < |\mathcal{P}(X)|$.*

Proof. We get $\leq$ from inclusion. Suppose there was a bijection $f : X \to \mathcal{P}(X)$. Let $A = \{x : x \notin f(x)\}$, and let $a = f^{-1}(A)$. Then $a \in A$ or $a \notin A$ gives us a contradiction, so we get $<$. □

Corollary 1. *We can find arbitrarily large sets by taking power sets.*

Theorem 3. *(Cantor-Schröder-Bernstein) If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$.*

Proof. Let $f : X \to Y$ and $g : Y \to X$ be injections. Divide $X \cup Y$ into orbits, where each orbit is formed by starting at a point, applying f and g in the forward direction, and applying f^{-1} and g^{-1} in the backward direction. The set of orbits partitions $X \cup Y$, and each orbit is either a closed loop, continues infinitely and uniquely in both directions, or continues infinitely in the forward direction. Then we can form our bijection easily using these orbits. □

Definition 3. *We write ${}^{A}B$ to mean the set of functions from A to B. We write B^{A} to mean $|{}^{A}B|$. In general, $|\mathcal{P}(X)| = |{}^{X}2| = 2^{|X|}$.*

Remark 3. $|\mathbb{N}| = |\mathbb{Q}| = |\mathbb{Z}| < |\mathbb{R}| = |(a, b)_{\mathbb{R}}| = |\mathcal{P}(\mathbb{N})| = |{}^{\mathbb{N}}2| = |{}^{\mathbb{N}}\mathbb{N}|$.

Lemma 1. *If $|A| = |B|$ and $|C| = |D|$, then $|{}^{A}C| = |{}^{B}D|$.*

Proof. Define bijections $A \to B$ and $C \to D$. Use them to injectively map functions from $A \to C$ to a function from $B \to D$ (and another injection the other way). □

Lemma 2. $|{}^{B}({}^{C}A)| = |{}^{B \times C}A|$.

Theorem 4. *(Cantor) Transcendental numbers (real numbers which are not the root of a rational polynomial) exist.*

Proof. The set of finite sequences of rationals is countable, so the set of rational polynomials is countable. Any polynomial has finitely many roots, so the set of non-transcendental numbers is countable. But $|\mathbb{N}| < |\mathbb{R}|$. $\square$

Theorem 5. *Suppose A_0, A_1, B_0, B_1 are disjoint nonempty sets. Suppose that for $i = 0, 1$ there exists an injection from $A_i \to B_i$ but there does not exist a surjection. Let $A = A_0 \cup A_1$. Let $B = B_0 \times B_1$. Then there is an injection $A \to B$ but no surjection.*

Proof. Let $f : A_0 \to B_0$, $g : A_1 \to B_1$ be injections, and define $h : A \to B$ where $h(a_0) = (f(a_0), b_1)$ and $h(a_1) = (b_0, g(a_1))$ where $b_0 \in B_0 \setminus f(A_0)$, $b_1 \in B_1 \setminus g(A_1)$.

Now suppose there were a surjection $h : A \to B$. Let $f : A_0 \to B_0$, $g : A_1 \to B_1$ be the projections. Pick $b_0 \in B_0 \setminus f(A_0), b_1 \in B_1 \setminus g(A_1)$. Then h cannot map to (b_0, b_1), a contradiction. $\square$

2 Order

Definition 4. *An order $(X, <)$ on a set X is* linear *if*

1. *$a \not< a$ for all $a \in X$,*
2. *for all $a, b, c \in X$, if $a < b$ and $b < c$, then $a < c$,*
3. *for all $a, b \in X$, either $a < b$, $b < a$, or $a = b$.*

Definition 5. *A linear ordering $(X, <)$ is a* well-ordering *if every nonempty subset of X has a least element.*

Definition 6. *Let $(X, <)$ be a linear order. An* initial segment *$A \subseteq X$ is a set such that for $a \in A$, if $x \in X$ such that $x < a$, then $x \in A$. A strict initial segment is one which is not all of X.*

Definition 7. *An* order isomorphism *$f : X \to Y$ is a bijection such that $x < y$ if and only if $f(x) < f(y)$.*

Theorem 6. *Suppose $(X, <)$ is a well-ordered set. Suppose $f : X \to X$ satisfies $a < b$ implies $f(a) < f(b)$. Then for every $x \in X$, $x \leq f(x)$. Moreover, $(X, <)$ is not order isomorphic to one of its strict initial segments.*

Proof. Let $S = \{x \in X \mid f(x) < x\}$. If S is nonempty, we can find a least element x_0. But then $f(x_0) < x_0$ so $f(f(x_0)) < f(x_0)$, so $f(x_0) \in S$. Hence, S is empty. Now let f be an order isomorphism from X to a strict initial segment, and extend the codomain to be all of X. Take a point a not in the strict initial segment. But then $a \leq f(a)$, and $f(a)$ is in the initial segment, a contradiction. $\square$

3 Ordinals

Definition 8. *If X is a set of sets, we write $\bigcup X$ to mean the union of sets in X.*

Definition 9. *(von Neumann) An ordinal is a set which is transitive and strictly well-ordered by $\in$.*

Definition 10. *A transitive set α is one such that, equivalently:*

1. *if $x \in y \in \alpha$, then $x \in \alpha$,*

2. *if $x \in \alpha$, then $x \subseteq \alpha$,*
3. *$\bigcup \alpha \subseteq \alpha$.*

Theorem 7. *Every element of an ordinal is an ordinal.*

Proof. Let α be an ordinal, and let $\beta \in \alpha$. Then $\beta \subseteq \alpha$ as α is transitive, so β is well-ordered and we can easily show transitivity. $\square$

Theorem 8. *If α, β are ordinals, then $\alpha \subseteq \beta$ if and only if $\alpha \in \beta$ or $\alpha = \beta$.*

Proof. Suppose $\alpha \subsetneq \beta$ and choose the least element $\gamma \in \beta \setminus \alpha$. Now show $\gamma = \alpha$ by double inclusion, hence $\alpha \in \beta$. $\square$

Remark 4. *If α is an ordinal, we call $\alpha + 1 = \alpha \cup \{\alpha\}$.*

Theorem 9. *If X is a nonempty set of ordinals, then $\bigcup X$ is the least ordinal greater than or equal to all ordinals in X.*

Proof. Let $Y = \bigcup X$. Show that Y is transitive, that $\alpha \notin \alpha$ for all $\alpha \in Y$, that you can compare any two elements, and that $\in$ is transitive. Show that Y is well-ordered by taking a nonempty subset and finding an ordinal with a nonempty intersection. Then the least element of that ordinal is the least element of the subset. Thus, Y is an ordinal.

Show Y is greater than or equal to all ordinals in X. If α is another ordinal greater than or equal to all ordinals in X, then $Y \subseteq \alpha$. $\square$

Remark 5. *We call the union of all finite ordinals ω or ω_0.*

Remark 6. *The class of all ordinals is called* Ord.

Theorem 10. *If α, β are ordinals, then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

Proof. If $\beta \subseteq \alpha$, we are done. Otherwise, let $\gamma \in \beta \setminus \alpha$ be the least element. By choice of least, an element of γ is an element of α, so $\gamma \subseteq \alpha$. Then $\gamma \in \alpha$ (impossible) or $\gamma = \alpha$, so $\alpha \in \beta$ hence $\alpha \subseteq \beta$. $\square$

Theorem 11. *Any set X of ordinals is well-ordered by $\in$.*

Proof. We get that X has a linear order from the lemmas above. Let $Y \subseteq X$ be nonempty. Choose $\alpha \in Y$. If $\alpha \cap Y = \emptyset$, find another $\beta \in Y$. Then $\alpha \in \beta$ or $\beta \in \alpha$ (impossible as then $\beta \in \alpha \cap Y$). Hence, α is the least element of Y. If $\alpha \cap Y \neq \emptyset$, α is well-ordered so repeat above with least element. $\square$

Remark 7. *Elements of an ordinal α are strict initial segments of α, and strict initial segments are elements.*

Theorem 12. *Every well-ordered set is order isomorphic to a unique ordinal.*

Proof. Let A be well-ordered, and let B be the collection of ordinals that are order isomorphic to a strict initial segment of A. Then B is well-ordered as a set of ordinals and transitive. In fact, B is order isomorphic to an initial segment of A, which must be all of A. $\square$

4 Cardinals

Remark 8. *We assume the well-ordering principle, so every set can be well-ordered. Moreover, every set is isomorphic to an ordinal.*

Definition 11. *A* cardinal α *is an ordinal which cannot be put in bijection with any smaller ordinal, i.e. if* $\beta < \alpha$*, then* $|\beta| < |\alpha|$*. Given an ordinal* β*, its* cardinality $|\beta|$ *is the greatest cardinal* $\leq \beta$*.*

Definition 12. *Given any set* X*, its cardinality* $|X|$ *is the cardinal* $|\alpha|$ *where* $(X, <)$ *is isomorphic to* α *under some well-ordering of* X*. No matter what the well-ordering is, we get the same cardinality.*

Remark 9. *We index the infinite cardinals by* $\aleph_\alpha = \omega_\alpha$ *where* α *is an ordinal. We view* $\aleph_\alpha$ *(*ω_α*) as the smallest cardinal (ordinal) greater than (not in bijection with)* $\aleph_\beta$ *(*ω_β*) for* $\beta < \alpha$*.*

Definition 13. *A* successor ordinal α *can be written as* $\beta + 1 = \beta \cup \{\beta\}$ *where* β *is an ordinal. A* limit ordinal *cannot; i.e.* $\alpha = \sup\{\beta : \beta < \alpha\}$*. An infinite cardinal is a successor or limit cardinal if its index is a successor or limit ordinal.*

Lemma 3. *Infinite cardinals are limit ordinals.*

Theorem 13. *If* $\aleph_\alpha$ *is a limit cardinal, then* $\aleph_\alpha = \bigcup_{\beta<\alpha} \aleph_\beta$*. In particular,* $\aleph_\omega = \bigcup_{n<\omega} \aleph_n$*.*

Proof. The union is an ordinal. Suppose there were a smaller ordinal in bijection with the union. Then that ordinal belongs to some $\aleph_\beta$ and so its cardinality is strictly less than that of the union. Finally, use double inclusion to get the equality. $\square$

Remark 10. ω_0 *is the set of finite ordinals.* ω_1 *is the set of countable ordinals.*

Definition 14. *If* κ, λ *are cardinals, then* $\kappa + \lambda = |A \cup B|$ *where* $|A| = \kappa$*,* $|B| = \lambda$*, and* $A \cap B = \emptyset$*.*

Definition 15. *If* κ, λ *are cardinals, then* $\kappa \cdot \lambda = |A \times B|$ *where* $|A| = \kappa$*,* $|B| = \lambda$*.*

Definition 16. *If* κ, λ *are cardinals, then* $\kappa^\lambda = |{}^A B|$ *where* $|A| = \kappa$*,* $|B| = \lambda$*.*

Theorem 14. *(Fundamental theorem of cardinal arithmetic) If* κ, λ *are cardinals, neither is 0, and at least 1 is infinite, then* $\kappa \cdot \lambda = \kappa + \lambda = \max\{\kappa, \lambda\}$*.*

Definition 17. *If* X *is a well-ordered set, then* $Y \subseteq X$ *is* cofinal *in* X *if for all* $x \in X$*, there is* $y \in Y$ *such that* $x \leq y$*. The* cofinality *of* X *is the smallest size of a cofinal subset.*

Theorem 15. *Define a relation* $\prec$ *on* $\mathrm{Ord} \times \mathrm{Ord}$ *by* $(\alpha, \beta) \prec (\gamma, \delta)$ *precisely when one of the following holds: (i)* $\max\{\alpha, \beta\} < \max\{\gamma, \delta\}$*, (ii)* $\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$ *and* $\alpha < \gamma$*, (iii)* $\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$ *and* $\alpha = \gamma$ *and* $\beta < \delta$*. Then* $\prec$ *is a well-ordering.*

Theorem 16. *If* κ *is an infinite cardinal, then* $\kappa = \kappa \cdot \kappa$*.*

Proof. Suppose for the sake of contradiction there is a smallest infinite cardinal κ with $\kappa < \kappa \cdot \kappa$. Let f be the order isomorphism from $(\kappa \times \kappa, \preceq)$ to a unique ordinal α. Some element $(\beta, \gamma) \in \kappa \times \kappa$ is sent by f to κ itself, since $\kappa < \alpha$ and f is a surjection. There is an ordinal $\delta < \kappa$ such that $\beta < \delta$, $\gamma < \delta$, since κ is a limit ordinal. Then $|\delta \times \delta| \geq \kappa$, since $\kappa \in f(\delta \times \delta)$ and $|f(\delta \times \delta)| = |\delta \times \delta|$. By hypothesis, $|\delta \times \delta| = |\delta| \cdot |\delta| = |\delta| < \kappa$ (as $\delta < \kappa$ and κ is the least cardinal such that $\kappa < \kappa \cdot \kappa$). Then $\kappa \leq |\delta \times \delta| < \kappa$, a contradiction. $\square$

Lemma 4. *Let* X *be a collection of sets such that* $|X| \leq \mu$ *and for each* $Y \in X$*, we have* $|Y| \leq \lambda$ *for some cardinals* μ, λ*. Then* $|\bigcup X| \leq \mu \cdot \lambda$*.*

Proof. Let $f : X \to \mu$ and, for each $Y \in X$, let $g_Y : Y \to \lambda$ where f, g are injections. We build an injection $h : \bigcup X \to \mu \times \lambda$. For each $y \in \bigcup X$, let $Y \in X$ have the least value $f(Y)$ such that $y \in Y$ (as the collection of such $f(Y)$ is well-ordered). Now, define $h(y) = (f(Y), g_Y(y))$. $\square$

Theorem 17. *Say that an infinite cardinal κ has property (*) if κ cannot be written as the union of fewer than κ sets, each of which has size strictly less than κ. Then for any $n < \omega$, if $\kappa = \aleph_{n+1}$ then κ has property (*). Moreover, $\kappa = \aleph_\omega$ does not have property (*).*

Proof. Suppose that κ does not have property (*). Then we could write $\kappa = \bigcup X$ where $|X| \leq \aleph_n$ and for each $Y \in X$, we have $|Y| \leq \aleph_n$. But $|\bigcup X| \leq \aleph_n \cdot \aleph_n \leq \aleph_n < \kappa$, a contradiction. $\square$

5 Transfinite induction

Definition 18. *(Transfinite induction) Suppose $\mathcal{P}$ is a class of ordinals such that $0 \in \mathcal{P}$, and if $\alpha \in \mathcal{P}$ then $\alpha + 1 \in \mathcal{P}$. Moreover, if α is a limit ordinal and $\beta \in \mathcal{P}$ for all $\beta < \alpha$, then $\alpha \in \mathcal{P}$. Then $\mathcal{P} = \mathrm{Ord}$.*

6 Ultrafilters

Definition 19. *If X is a set, then $\mathcal{F} \subseteq \mathcal{P}(X)$ is a* filter *if*

1. *$X \in \mathcal{F}$,*
2. *$\emptyset \notin \mathcal{F}$,*
3. *if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ (finite intersection),*
4. *if $A \in \mathcal{F}$ and $A \subseteq B \subseteq \mathcal{F}$, then $B \in \mathcal{F}$ (upward closed).*

Definition 20. *A collection of sets has* finite intersection property (FIP) *if the intersection of finitely many sets in the collection is nonempty.*

Definition 21. *A filter $\mathcal{F}$ over X is an* ultrafilter *if for all $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.*

Remark 11. *We might have said an ultrafilter is a "maximal filter", i.e. there is no filter which strictly contains an ultrafilter.*

Proof. Given an ultrafilter $\mathcal{D}$, if a set is not in $\mathcal{D}$ then its complement is in $\mathcal{D}$ so it is maximal. Given a maximal filter $\mathcal{F}$, if $\mathcal{F}$ is not decisive about a set, we can apply the lemma below to get a larger filter, a contradiction. $\square$

Lemma 5. *If $F \subseteq \mathcal{P}(X)$ has FIP and $A \subseteq X$, then one of $\mathcal{F} \cup \{A\}$ and $\mathcal{F} \cup \{X \setminus A\}$ has FIP.*

Proof. If both $\mathcal{F} \cup \{A\}$ and $\mathcal{F} \cup \{X \setminus A\}$ fail FIP, then we can find sets $X_1 \cap \cdots \cap X_n \cap A = \emptyset$ and $Y_1 \cap \cdots \cap Y_m \cap X \setminus A = \emptyset$. But then $X_1 \cap \cdots \cap X_n \cap Y_1 \cap \cdots \cap Y_m = \emptyset$. $\square$

Remark 12. *Given a collection $\mathcal{F}$ with FIP, we can talk about the filter generated by $\mathcal{F}$, $\langle \mathcal{F} \rangle$, which is closed under finite intersection and upward closed.*

Definition 22. *A* principal *ultrafilter $\mathcal{F}$ is $\{Y \subseteq X : a \in Y\}$ where $a \in X$.*

Lemma 6. *Let $\mathcal{D}$ be an ultrafilter. Then $\mathcal{D}$ is principal if and only if $\mathcal{D}$ contains a finite set. Equivalently, $\mathcal{D}$ is nonprincipal if and only if $\mathcal{D}$ contains all cofinite sets.*

Lemma 7. *If X is finite, the only ultrafilters on X are principal.*

Lemma 8. *If we have $\{\mathcal{F}_\beta : \beta < \alpha\}$ where each $\mathcal{F}_\beta \subseteq \mathcal{P}(X)$ and has FIP, and if $\beta < \beta' < \alpha$ then $\mathcal{F}_\beta \subseteq \mathcal{F}_{\beta'}$, then $\bigcup_{\beta<\alpha} \mathcal{F}_\beta$ has FIP.*

Theorem 18. *Let X be an infinite set. Then there exists a nonprincipal ultrafilter on X.*

Proof. Let $\kappa = |X|$. Denote the subsets of X as A_β where $\beta < 2^\kappa$. We build by induction on $\beta < 2^\kappa$ a sequence $\{\mathcal{F}_\beta : \beta < 2^\kappa\}$ such that each $\mathcal{F}_\beta \subseteq \mathcal{P}(X)$ and has FIP. Let F_0 be the collection of cofinite sets. For each $\beta < 2^\kappa$, let $\mathcal{F}_{\beta+1}$ be $\mathcal{F}_\beta$ but including $A_{\beta+1}$ or $X \setminus A_{\beta+1}$. If β is a limit ordinal, let $\mathcal{F}_\beta = \bigcup_{\gamma<\beta} \mathcal{F}_\gamma$. Then by the lemma above, $\mathcal{F}_{2^\kappa}$ is a nonprincipal ultrafilter on X. $\square$