Examples--White-Noise_-Random-Walks_-and-Moving-Averages

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Stochastic process (realization) time series

Say a process is a weekly stationary of
Mean Function: $\mu(t)=\mu$
ACF: $\gamma(t_1,t_2)=\gamma(t_2-t_1)=\gamma(\tau)$

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Stationarity: Properties and Examples

- For a Weakly Stationary Process, $-1 \le \rho(t) \le 1$

  This is, of course, perfectly analogous to the property that $-1 \le \rho \le 1$ from elementary statistics. If you have had a linear algebra course, this may feel familiar (that is, if you showed that $|x^T y| \le \|x\|_2\, \|y\|_2$). We know variances are non-negative, so set up a linear combination

$$V[a\,X_1 + b\,X_2] \ge 0$$

  In particular, in the spirit of autocorrelation, set up for lag spacing τ

$$V[a\,X(t) + b\,X(t+\tau)] \ge 0$$

  Your probability teacher probably told you (time and time again) that

$$V[X+Y] = V[X] + V[Y] + 2\,cov(X,Y)$$

  As well as

$$V[aX] = a^2 V[X]$$

  So, immediately,

$$V[a\,X(t) + b\,X(t+\tau)] = a^2 V[X(t)] + b^2 V[X(t+\tau)] + 2ab\,cov\big(X(t), X(t+\tau)\big) \ge 0$$

  We are assuming weak stationarity, so replace variance operators with a notation which suggests constants

$$a^2\sigma^2 + b^2\sigma^2 + 2ab\,cov\big(X(t), X(t+\tau)\big) \ge 0$$

  Two special cases: (1) Let $a = b = 1$

$$2\,\sigma^2 \ge -2cov\big(X(t), X(t+\tau)\big), \quad \sigma^2 \ge -cov\big(X(t), X(t+\tau)\big)$$

$$1 \ge -\frac{cov\big(X(t), X(t+\tau)\big)}{\sigma^2} = -\frac{\gamma(\tau)}{\gamma(0)} = -\rho(\tau)$$

  This gives us

$$\rho(\tau) \ge -1$$

  (2) Let $a = 1, b = -1$

  It's your turn- take a moment to show

$$\rho(\tau) \le 1$$

We have already seen a few simple models: noise, random walks, and moving averages. Can we now show that some of our simple models are, in fact, weakly stationary?

Examples

- White Noise　Stationary

  Is it obvious to you that Gaussian white noise is weakly stationary? Consider a discrete family of independent, identically distributed normal random variables

random variable family
a set – a sequence of iid r.v.

$$X_t \overset{iid}{\sim} N(\mu, \sigma) \qquad X_t \overset{iid}{\sim} (\underline{0}, \sigma^2)$$

  The mean function $\mu(t)$ is obviously constant$^{=0}$, so look at

$$\boxed{\gamma(t_1, t_2) = \begin{cases} 0 & t_1 \ne t_2 \\ \sigma^2 & t_1 = t_2 \end{cases}}$$ 2 diff. r.v.

  And

$$\boxed{\rho(t_1, t_2) = \begin{cases} 0 & t_1 \ne t_2 \\ 1 & t_1 = t_2 \end{cases}}$$

  We are evidently weakly stationary, and could even show strict stationarity if we wanted to.

- Random Walks

  Simple random walks are obviously ***not*** stationary. Think of a walk with N steps built off of IID Z_t where $E[Z_t] = \mu$, $V[Z_t] = \sigma^2$. We would create

$$X_1 = Z_1$$
$$X_2 = X_1 + Z_2$$
$$X_3 = X_2 + Z_3 = X_1 + X_2 + X_3$$
$$\vdots$$
$$X_t = X_{t-1} + Z_t = \sum_{i=1}^{t} Z_i$$

  For the mean, using the idea that "the mean of the sum is the sum of the means":

Index t for position X　　not no. – r.v.

$$E[X_t] = E\left[\sum_{i=1}^{t} Z_i\right] = \sum_{i=1}^{t} E[Z_i] = t\cdot\mu \quad \mu \ne 0 \quad E \uparrow - t$$

  For the variance, using the idea that "the variance of the sum is the sum of the variances *when the random variables are independent*":

$$V[X_t] = V\left[\sum_{i=1}^{t} Z_i\right] = \sum_{i=1}^{t} V[Z_i] = t\cdot\sigma^2 \quad \sigma^2 \ne 0 \quad V\uparrow - t.$$ ≠ constant ≠ stationary

  (Independent random variables have variances which add. All random variables have means which add).

  Even if $\mu = 0$ the variances will still increase along the time series.

- Moving Average Processes, $MA(q)$

  A moving average process will create a new set of random variables from an old set, just like the random walk does, but now we build them as, for IID Z_t with $E[Z_t] = 0$ and $V[Z_t] = \sigma_Z^2$

$$MA(q)\ process: \qquad X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}$$

white Noise – no structure
Q=3, MA　no. of components = 3+1=4
lose higher frequencies, gain low frequencies – see structure between neighbors
Q=9 longer scale correlations – neighbors smooth wave

  The parameter q tells us how far back to look along the white noise sequence for our average. Since the Z_t are independent, we immediately have (using the usual linear operator results)

$$E[X_t] = \beta_0 E[Z_t] + \beta_1 E[Z_{t-1}] + \cdots + \beta_q E[Z_{t-q}] = 0$$

$$V[X_t] = \beta_0^2 V[Z_t] + \beta_1^2 V[Z_{t-1}] + \cdots + \beta_q^2 V[Z_{t-q}] = \sigma_Z^2 \sum_{i=0}^{q} \beta_i^2$$

$t > q$　r.v. X_t, X_{t+k} – built – diff. independent r.v. diff. set of noise variables

  The autocovariance isn't all that hard to find either. Consider random variables k steps apart and set up their covariance.

$$cov[X_t, X_{t+k}] = cov[\,\beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}\,,$$
$$\beta_0 Z_{t+k} + \beta_1 Z_{t+k-1} + \cdots + \beta_q Z_{t+k-q}]$$

  This is a little tricky, but please stay focused. There are two numbers to keep track of, the lag spacing k and the support of the MA process, q.

  Now

$$cov[X_t, X_{t+k}] = E[X_t \cdot X_{t+k}] - E[X_t]E[X_{t+k}] = E[X_t \cdot X_{t+k}]$$

$E[X_t] = E[X_{t+k}] = 0$

  Since $E[X_t] = 0$ we really just need

$$E[X_t \cdot X_{t+k}] = E\big[\,(\beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}) \cdot (\beta_0 Z_{t+k} + \beta_1 Z_{t+k-1} + \cdots + \beta_q Z_{t+k-q})\big]$$

  We can rely on matrix results concerning linear combinations of random variables or just work directly. The patient among us will write out

$$\begin{aligned} E[X_t \cdot X_{t+k}] &= \beta_0\beta_0 E[Z_t\, Z_{t+k}] + \beta_0\beta_1 E[Z_t\, Z_{t+k-1}] + \cdots + \beta_0\beta_q E[Z_t\, Z_{t+k-q}] \\ &+ \beta_1\beta_0 E[Z_{t-1}\, Z_{t+k}] + \beta_1\beta_1 E[Z_{t-1}\, Z_{t+k-1}] + \cdots + \beta_1\beta_q E[Z_{t-1}\, Z_{t+k-q}] \\ &+ \cdots + \\ &\beta_q\beta_0 E[Z_{t-q}\, Z_{t+k}] + \beta_q\beta_1 E[Z_{t-q}\, Z_{t+k-1}] + \cdots \beta_q\beta_q E[Z_{t-q}\, Z_{t+k-q}] \end{aligned}$$

  The key to simplifying this is to notice that, since the Z_t are independent, we can say that the expectation of the product is the product of the expectations and so we have

$$E[Z_i \cdot Z_j] = E[Z_i]E[Z_j] = \begin{cases} 0 & i \ne j \\ \sigma_Z^2 & i = j \end{cases}$$

  When the lag spacing k is greater than the order of the process q then the subscripts can never be the same (there is no overlap on the underlying $Z_t's$) and we have $cov[X_t, X_{t+k}] = 0$. When the lag spacing is small enough to have contributions, that is if $q - k \ge 0$, you can visualize the sum like this (we just need to keep track of the $\beta's$):

Z_{t-q}	Z_{t-q+1}	…	$Z_{t-(q-k)}$	…	Z_{t-1}	Z_t	…				
β_q	β_{q-1}		β_{q-k}		β_1	β_0		0	0		
			Z_{t+k-q}	…	$Z_{t+k-k+1}$	Z_{t+k-k}	…	Z_{t+k-1}	Z_{t+k}		
0	0	0	β_q		β_{k+1}	β_k		β_1	β_0		

  This should make clear that, when $k \le q$

$$E[X_t, X_{t+k}] = \sigma_Z^2 \cdot \sum_{i=0}^{q-k} \beta_i\, \beta_{i+k} \quad (no\ t\ dependence)$$

× dependent on location along process
✓ separation of r.v.

  Summing up, then, we have found that

$$\boxed{\gamma(t_1, t_2) = \gamma(k) = \begin{cases} 0 & k > q \\ \sigma_Z^2 \cdot \sum_{i=0}^{q-k} \beta_i\, \beta_{i+k} & k \le q \end{cases}}$$

  We know that the mean function is constant, in fact $\mu(t) = 0$ and the autocovariance function has no t dependence, so we conclude that the *MA(q)* process is (weakly) stationary.

  Let's finish this lecture by finding the autocorrelation function. In general

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$$

  Obviously, then $\rho(0) = 1$. It is easy to see that

$$\boxed{\gamma(0) = \sigma_Z^2 \cdot \sum_{i=0}^{q} \beta_i\, \beta_i = \sigma_Z^2 \cdot \sum_{i=0}^{q} \beta_i^2}$$

  Finally

$$\boxed{\rho(k) = \frac{\sum_{i=0}^{q-k} \beta_i\, \beta_{i+k}}{\sum_{i=0}^{q} \beta_i^2}}$$

  In the next lecture we will simulate an MA(q) process and validate these results numerically.