Witten Genus on Symplectic Toric Complete Intersections

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Abstract

This note is a basic summarization of the author's master thesis. We present a general method to calculate elliptic genera on symplectic toric complete intersections and apply it to Witten genus. Finally, we will give a sufficient condition for the witten genus to vanish. For the convenience of reader, the sources of references will be listed in the context.

This short notes is an oversimplified version of my master thesis. It is still not determined whether there will be full version...

1 Toric Variety

[1] Lec N be a lattice, and let $L_{\mathbb{R}} = L \otimes \mathbb{R}$. A subset $C \in L_{\mathbb{R}}$ is a **cone** is C is closed under homotheties and contains no line.

Let $M = Hom_{\mathbb{Z}}(L, \mathbb{Z})$ be the dual lattice, and let $m \cdot n$ be the natural paring. A cone C is **rational polyhedral** if there exist $m_1, m_2, ..., m_s \in M$ such that

$$C = \cap_i \{x \in L_{\mathbb{R}} | m_i \cdot x \geq 0\}.$$

[1] All the contents above canbe found in standard textbooks such as Cox D A, Little J B, Schenck H K. Toric varieties[M]. American Mathematical Soc., 2011. For a brief introduction, one can refer to "Cox, D. (2003, August). What is a toric variety?. In Contemporary Mathematics."

A **face** of C is some subset that can make some of the inequalities equalities. A rational polyhedral cone C is **simplicial** if

$$C = \sum_{i=1}^{k} \mathbb{R}_{\geq 0} n_i, \quad n_i \in L,$$

whrer k is the dimension of the subspace generated by C. A **dual cone** C^* of C is defined as follows

$$C^* = \{v \in M_{\mathbb{R}} | v \cdot u \geq 0, u \in C\}$$

Let Σ be a set of polyhedral cones. Then Σ is called a **fan** is

1. each face of cone in Σ is also a cone in Σ, and
2. the intersection of any two cones in Σ is a face of each.

A fan is **complete** if

$$\cup_{C \in \Sigma} C = L_{\mathbb{R}},$$

and **simplicial** if all the cones in Σ are simplicial.

Let Σ be a fan. We can associate a **toric variety** X_Σ to Σ. If $C \subset L_{\mathbb{R}}$ let $C^* \subset M_{\mathbb{R}}$ be the dual cone. Then we have a $\mathbb{C}$-algebra S_C defined by

$$S_C = \mathbb{C}[\{z^m\}], \ m \in C^* \cap M$$

and we let U_C be the affine variety $Spec S_C$. The variety U_C is the **toric chart** associated to C. And then we can glue up all the affine toric varieties with respect to the combinatorics of Σ to get the **toric variety** X_Σ associated to fan. Let $C_1, C_2, C_1 \cap C_2 \in \Sigma$. Then we can identify $U_{C_1 \cap C_2}$ with a principal opensuvariety of U_{C_1} and U_{C_2}. One can show that X_Σ is separated, and has an open set isomorphic to an algebraic torus $T = (\mathbb{C}^*)^n$. Moreover, X_Σ is complete if and only if Σ is complete, and X_Σ is nonsingular if and only if each cone of Σ is generated by part of a basis of N.

Let $\Sigma(1)$ denote the set of one-dimensional cones in Σ. For each $\rho \in \Sigma(1)$ there is a T-invariant divisor $D_\rho \subset X_\Sigma$. By Poincare duality, there is a cohomology class associated to D_ρ in $H^2(X_\Sigma)$. We still denote this class D_ρ via some abuse of notations. One can show that if X_Σ is nonsingular, then the cohomology class D_ρ generate the cohomology ring $H^*(X_\Sigma, \mathbb{Z})$ as follows. Let X_Σ be a complete simplicial toric variety and fix a numbering $\rho_1, ..., \rho_N$ for the rays in $\Sigma(1)$. Also let n_i be the minimal generator of ρ_i

and introduce a variable D_{ρ_i} for each ρ_i. In the ring $\mathbb{Z}[D_{\rho_1}, ..., D_{\rho_N}]$, let $\mathcal{I}$ be the monomial ideal with square-free generators as follows:

$$\mathcal{I} = \langle D_{i_1}...D_{i_s} | i_j \text{are distinct and} \rho_{i_1} + ... + \rho_{i_s} \text{is not a cone of } \Sigma \rangle.$$

We call $\mathcal{I}$ the Stanley-Reisner ideal. Also let $\mathcal{J}$ be the ideal generated by the linear forms

$$\sum_{i}^{r} \langle m, n_i \rangle D_i$$

where m ranges over M (or equivalently, over some basis for M).

(Jurkiewicz-Danilov theorem) When X_Σ is smooth complete toric variety, the cohomology ring $H^*(X_\Sigma, \mathbb{Z})$ is isomorphic to

$$\mathbb{Z}[D_1, ..., D_N]/(\mathcal{I} + \mathcal{J}).$$

We easily derive the corollary that $Picard(X_\Sigma) \cong H^2(X_\Sigma, \mathbb{Z})$ and all $H^{odd}(X_\Sigma, \mathbb{Z}) = 0$. After we quotient the ideal generated by the linear relations, we can equivalently say that the cohomology ring is multiplicative generated by a basis of the Picard group. We denote the basis of Picard group as $p_1, ...p_k$, and express the invariant divisor as

$$D_i = \begin{cases} p_i & 1 \leq i \leq k \\ \displaystyle\sum_{j}^{k} m_{ij} p_j & k < i \leq N \end{cases}$$

2 Multiplicative genus on Toric Varieties

For smooth toric variety we have the following generalized Euler sequence[2]

$$0 \longrightarrow \mathcal{O}(X_\Sigma)^{\oplus k} \longrightarrow \oplus_{\rho \in \Sigma(1)} \mathcal{O}(D_\rho) \longrightarrow TX_\Sigma \longrightarrow 0,$$

from which we can calculate the total Chern class and all the multiplicative genera of TX_Σ. Here the $\mathcal{O}_{D_\rho}$ is a line bundle defined by the divisor D_ρ in the $Picardgroup$, of which the total Chern class is $1 + D_\rho$. k is the dimention of the picard group or equivalently the dimension of $H^2 X_\Sigma, \mathbb{Z}$ we can easily calculate that

$$c(TX_\Sigma) = \prod_{\rho \in \Sigma(1)} (1 + D_\rho).$$

[2]Theorem 8.1.6 in Cox's book Toric variety

We will see that the D_ρs plays the role of Chern roots here. For a general multiplicative genus F, we observe that

$$F(TX_\Sigma) = \frac{F(\oplus_{\rho\in\Sigma(1)}\mathcal{O}(D_\rho))}{F(\mathcal{O}(X_\Sigma)^{\oplus k})} = \frac{\prod_{\rho\in\Sigma(1)} F(\mathcal{O}(D_\rho))}{F(\underline{\mathbb{Q}}^k)},$$

where $\mathbb{Q} = \mathbb{C}$ or $\mathbb{R}$ depending on whether we forget the complex structure. For the case of Elliptic genus, we have

$$Ell(X_\Sigma, y, q) = \frac{\prod_{\rho\in\Sigma(1)} D_\rho \frac{\theta(D_\rho+z,\tau)}{\theta(D_\rho,\tau)}}{G(y,q)^k}$$

For the case of Witten genus,

$$\mathcal{W}(X_\Sigma) = \prod_{\rho\in\Sigma(1)} D_\rho \frac{\theta(D_\rho,\tau)}{\theta'(0,\tau)}.$$

Now let's consider a complete intersection Y as a generic intersection of n hypersurfaces $\{Y_q\}_{1\le q\le n}$. eqch is dual to the cohomology class $\sum_j^k d_{qj}p_j$. Consider the inclusion map $\iota : Y \to X_\Sigma$, we have the adjunction formula:

$$0 \longrightarrow TY \longrightarrow \iota^* TX_\Sigma \longrightarrow \iota^* N \longrightarrow 0,$$

where the normal bundle N is isomorphic to $\oplus_q^n \mathcal{O}(\sum_j^k d_{qj}p_j)$. By the multiplicative properties of Chern class and Witten genus, we have the following data:

$$c(TY) = \iota^*\left(\frac{c(TX)}{c(TN)}\right) = \iota^*\left(\frac{\prod_i^k(1+p_i)\prod_{j=k+1}^N(1+\sum_i^k m_{ji}p_1)}{\prod_q^n(1+\sum_j^k d_{qj}p_j)}\right).$$

Following these we have

$$\begin{aligned} w_2(T_{\mathbb{R}}Y) &\equiv c_1(TY)(\text{mod } 2) \\ &\equiv \iota^*\left(\sum_i^k p_i + \sum_{j=k+1}^N \sum_i^k m_{ji}p_i + \sum_q^n \sum_i^k d_{qi}p_i\right) \pmod 2 \end{aligned}$$

and

$$p_1(T_{\mathbb{R}}Y) = i^*\left(\sum_i^k p_i^2 + \sum_{j=k+1}^N (\sum_i^k m_{ji}p_i)^2 - \sum_q^n (\sum_i^k d_{qi}p_i)^2\right)$$

Also because of the multiplicative property of Witten genus, we have

$$\mathcal{W}(TY) = \iota^* \left(\frac{\mathcal{W}(TX)}{\mathcal{W}(N)}\right) = \iota^* \left(\frac{\prod_i^k p_i \frac{\theta(p_i,\tau)}{\theta'(0,\tau)} \prod_{j=k+1}^N (\sum_s^k m_{js}p_s) \frac{\theta(\sum_s^k m_{js}p_s,\tau)}{\theta'(0,\tau)}}{\prod_q^n (\sum_j^k d_{qj}p_j) \frac{\theta(\sum_j^k d_{qj}p_j,\tau)}{\theta'(0,\tau)}} \right).$$

Integrate the above formula over the fundamental class of Y, we have

$$\begin{aligned}
\int_Y \mathcal{W}(TY) &= \int_X \iota_! \iota^* \left(\frac{\mathcal{W}(TX)}{\mathcal{W}(N)}\right) \\
&= \int_X \left(\frac{\mathcal{W}(TX)}{\mathcal{W}(N)}\right) \cdot Euler(N) \\
&= \int_X \left(\frac{\prod_i^k p_i \frac{\theta(p_i,\tau)}{\theta'(0,\tau)} \prod_{j=k+1}^N (\sum_s^k m_{js}p_s) \frac{\theta(\sum_s^k m_{js}p_s,\tau)}{\theta'(0,\tau)}}{\prod_q^n (\sum_j^k d_{qj}p_j) \frac{\theta(\sum_j^k d_{qj}p_j,\tau)}{\theta'(0,\tau)}} \right) \cdot \prod_q^n (\sum_j^k d_{qj}p_j) \\
&= \int_X \left(\frac{\prod_i^k p_i \frac{\theta(p_i,\tau)}{\theta'(0,\tau)} \prod_{j=k+1}^N (\sum_s^k m_{js}p_s) \frac{\theta(\sum_s^k m_{js}p_s,\tau)}{\theta'(0,\tau)}}{\prod_q^n \frac{\theta(\sum_j^k d_{qj}p_j,\tau)}{\theta'(0,\tau)}} \right)
\end{aligned}$$

3 Equivariant cohomology and genera as residues

In this section, we will mainly follow the equivariant cohomology chapter of Cox's book"Toric Variety" and Givental's groundbreaking paper "A mirror theorem of Toric complete intersection"

A toric variety is naturally equipped with an effective torus action, where the torus has the dimension $|\Sigma(1)| = N$ and with $|\Sigma(n)|$ isolated fixed points. There is a ring morphism between the ordinary cohomology ring $H^*(X_\Sigma, \mathbb{Q})$ and the equivariant cohomology ring $H_T^*(X_\Sigma, \mathbb{Q})$.

Let's now have a brief recall on the construction of equivariant cohomology over toric varieties. Let G be a compact connected Lie group, classified by the principal G-bundle $EG \to BG$, whose total space is contractible. This bundle is uniquely determined up to homotopy equivalence. Then G acts on the the space$X \times EG$ freely. We consider the famous **Borel's Construction**

$$X \times_G EG = (X \times EG)/G. \tag{1}$$

And the *equivariantcohomology* of X is defined to be

$$H_G^*(X) = H^*(X \times_G EG). \tag{2}$$

A toric variety is naturally equipped with an effective torus action $T = (\mathbb{C}^*)^N$, where the torus has the dimension $|\Sigma(1)|$ and with $|\Sigma(n)|$ isolated fixed points. Then we consider $H_T^*(X_\Sigma)$. Note that

$$H_T^*(point) = H^*(BT) = H^*((\mathbb{CP}^\infty)^N) = \mathbb{Q}[\lambda_1, \lambda_2, ..., \lambda_N],$$

and all λ has degree 2. There is a ring morphism between the ordinary cohomology ring $H^*(X_\Sigma, \mathbb{Q})$ and the equivariant cohomology ring $H_T^*(X_\Sigma, \mathbb{Q})$. $H_T^*(X_\Sigma)$ can be thout of as an $H_T^*(BT)$-module. Note also that the inclusion of a fiber $i_X : X_\Sigma \to X \times_G EG$ induces the "Non-equivariant limit" map $i_X^* : H_T^*(X_\Sigma) \to H^*(X_\Sigma)$, which amounts to mapping all λ_i to 0.

By the Proposition 12.4.13 in [Cox's book], every torus-invariant divisor D_ρ has an equivariant counterpart $(D_\rho)_T$, and they generate the equivariant cohomology ring as $H_T^*(X_\Sigma, \mathbb{Q}) = \mathbb{Q}[(D_1)_T, ..., (D_N)_T]/(\mathcal{I} + \mathcal{J})$. And according to [Givental] $(D_\rho)_T = D_\rho - \lambda_\rho$ in $H_T^2(X_\Sigma)$.

This tempts us to define the equivariant version of any polynomial of invariant divisors $P_T(D_\rho) = P((D_\rho)_T)$, for example the equivariant Witten genus:

$$\mathcal{W}_T(X_\Sigma) = \prod_{\rho \in \Sigma(1)} (D_\rho - \lambda_\rho) \frac{\theta(D_\rho - \lambda_\rho, \tau)}{\theta'(0, \tau)}$$

Following Giverntal, we can find some explicit way to proceed calculations via the Atiyah-Bott localization.

$$\int_{X_\Sigma} f = \sum_\alpha \int_{(Z_\alpha)_T} \left(\frac{i_\alpha^*(f)}{Euler(N_\alpha)} \right).$$

Here, the $(X_\Sigma)_T = X \times_T ET$ and Z_α are the fixed points corresponding to the full dimensional cone α in $\Sigma(n)$, and N_α is the normal bundle at the fixed point. combined with the definition of Grothendieck residue symbol, we can get the full explicit formula for a polynomial $f(p, \lambda)$ in $H_T^*(X, \mathbb{Q})$, as is at the bottom of page 22 of [Givental]

$$\int_{X_\Sigma} f(p, \lambda) = \sum_\alpha res_\alpha \frac{f(p, \lambda)}{(D_1)_T \cdot (D_2)_T \cdot ... \cdot (D_N)_T} dp_1 dp_2 ... dp_k$$

If we take the non-equivariant limit we can get an explicit we to calculate the the integration of $f(p, 0)$ over the fundamental class X_Σ.

4 Vanishing result for some string complete intersection

Apply the above equation to Witten genus of the complete intersection and go to non-equivariant limit

$$\int_Y \mathcal{W}(TY) = i_X^* \sum_\alpha res_\alpha \left(\frac{\prod_i^k \frac{\theta(p_i+\lambda_i,\tau)}{\theta'(0,\tau)} \prod_{j=k+1}^N \frac{\theta(\sum_s^k m_{js}p_s+\lambda_j,\tau)}{\theta'(0,\tau)}}{\prod_q^n \frac{\theta(\sum_j^k d_{qj}p_j,\tau)}{\theta'(0,\tau)}} \right) dp_1 ... dp_k$$

One would be expect to get some beautiful explicit result when it comes to elliptic genus and Witten genus. Unfortunately, it is extremely difficult to make sense of the RHS of the above equation because it usually involve some delicate cancellation of infinites when f is a rational function. But we can proceed as in the paper of [Chen and Han]. When f is an elliptic function, the above differential form descends to a differential form on a compact torus and then the sum vanishes due to the global residue theorem. We can derive this via the standard transformation laws of theta functions. Here we list only the result.

When the matrix (m_{ji}) and (d_{ji}) satisfy the following equations

$$\sum_j^n d_{ji} - \sum_{j=k+1}^N m_{ji} - 1 \equiv 0(\mod 2),$$

$$\sum_j^n d_{ji}^2 - \sum_{j=k+1}^N m_{ji}^2 - 1 = 0,$$

and

$$\sum_j^n d_{ji}d_{jl} - \sum_{j=k+1}^N m_{ji}m_{jl} = 0 \text{ for } i \neq l.$$

Then complete intersection is string and the Witten genus vanishes. We can check that this includes the former result by [Chen and Han]