Fictitious quantities

Some irrationals numbers could be considered as measurements, for example sqrt(2) is the length of the side of a right triangle with hypotenuse 2, or 2*pi is the perimeter of a circle of radius 1. These facts were probably known at this time, why did they considered this kind of irrational numbers as "fictions"?

Because a number was defined originally as a multiplicity of units (positive integers). Later this was extended to include multiplicities of units broken into finitely many parts (n/m). Other quantities were still quantities, but not numbers. Extending the notion of number required new definitions and methods for working with quantities, and reconsidering some of the basic rules (e.g. if A>C and B<D, then we're supposed to have A/B>C/D, but this doesn't work if A=D=1 and B=C=-1).

I don't understand the difference between measuring and measuring approximately. I understand that incommensurable ratios cannot be measured exactly because the process to find their exact value is infinite. Why can negative numbers not be measured? Their representation as a directed line segment is exact and finite.

The description that you offer (identifying sign and direction) took a long time to develop and accept as a correct description of negative numbers. Having a debt of 2 francs and taking two steps to the left were not easily reconciled as mathematically identical with each other and with -2.

Ferraro: On page 41, the author states that some mathematicians referred to negative numbers as "false numbers". Did this impact the mathematics they did, or was this simply a disagreement about the terminology?

Some tried to avoid them in their mathematics. Other just called them "false" and used them like "real" numbers.

Why is it that irrational numbers measure a quantity only in approximate way?

Because you don't have a finite precise numerical representation of sqrt(2), other than the artificial construct "sqrt(2)".

When are the square roots of negative numbers useful for studying aspects of quantity?

The first use is in solving polynomial equations. The formula for solving polynomial equations of degree 3 can produce real results by going through roots of negative numbers. Later on, many physical applications emerged.

On Page 41 of paper one it says that sqrt(-4) could be represented in our understanding and take place in our imagination. How? I can't imagine that, I can't imagine a number that multiplied by itself equals -4 except the imaginary numbers and I can't imagine them.

It could not be imagined visually, but can be imagined symbolically, based on formal calculation rules. Today the representation of imaginary numbers in the complex plain allows a sort of visual representation as well.

On page 43 of Ferraro's paper, a "real quantity" is defined. Unfortunately, I'm not able to grasp the definition completely. Could you maybe word it differently?

It's something like a geometrical or physical variable, without any specific value.

On page 44 there is written that the idea that quantity might be reduced to a set of number was not taken into account. What is meant here?
A variable is not identified with the set of its possible values. It is an entity in itself. That's why (p. 36) a statement can be true for a quantity, and still fail for some of its exceptional values.

What exactly are hyperreal numbers?
A modern mathematical model derived from an axiomatization of the real numbers, which includes infinitesimals in a consistent way.

Making the distinction between fictitious and hyper-real numbers seems superfluous. The square root 2 is also approximated using fractions, the same way as the infinitesimal is approximated using 1/x as x tends to infinity. (Later even clarified in the text).
This only works if you deny that zero is the value approximated by 1/x…

In footnote 52 of Ferraro's work he cites Euler, who explains to the reader a syntactical argument, namely that one should write dy/dx = 2x instead of dy = 2x dx, because since dx = 0 and dy = 0, dy = 3x dx would also be a true expression. Why is dy/dx (0/0) only equal to 2x and not also to 3x as he argued with the multiplicative form? Is the formula dx/dy not seen from a purely mathematical but rather a geometrical perspective, and if so, how does 0/0 make sense in the geometrical perspective?
dy and dx do not exist independently, but only in the context of their emergence. If they emerge from the equation y=x^2, then dy/dx can only be 2x.

Did Ferraro mean the statement on p45 by Euler: "an infinitely small quantity is simply an evanescent quantity and therefore actually equal to zero" to be strange because we know otherwise now, or because it hard to understand why he made such a claim? Why does the author state in p 46 that evanescent quantities and limits are not the same?
The strange thing is that infinitesimals appear to be considered at the same time as zero (when used additively) and as non zero (when infinitesimals divide each other). Euler's notion of evanescent quantities is not the same as the modern, formalized epsilon-delta notion of limit – it allows for some more speculative manipulations.

How was Euler's idea of infinitesimal different from that of Newton. Both seem to consider the, evanescent quantities (in the process of vanishing).
Euler thought that the infinitesimal actually equals zero, which was strongly rejected by Newton. But since 0/0 is not determined, the value of quotients of differentials depends on the context.

Maybe it would help if we were going again through all these terms such as "evanescent quantities" and how we distinguish between limits and infinitesimals.
It won't. The whole point is that these terms were unstable and not well formalized. Your confusion about them reflects the confusion of mathematicians at the time. And yet, with all this confusion, they managed to produce remarkable results. So perhaps clarity is not that important in mathematical practice?

On page 42 of Ferraro's work he states, that "Fictions, however, were always connected with reality, directly or indirectly". What does Ferraro understand as a direct connection? From the

insight he gives about Euler’s thinking of "fictional numbers" they did not have an existence in the sense of “true numbers” of the 17th-century.
“Directly connected” means related to some observable geometric or physical phenomenon, like sqrt(2) and the diagonal of a square with side 1.

page 43: I don't understand point d) about fictions.
There was no theory that guaranteed that using “fictions” in mathematical arguments would yield correct results concerning “non fictitious” numbers. The only thing that controlled the use of “fictions” was the practice from which they emerged.

To the distinction between „true“ and „fictitious“ numbers: Why should one put effort in such a distinction? Aren’t numbers in general an abstract way to describe reality (and therefore kind of „fictitious“) and „true“ only in the sense that they are useful to describe reality, so why pretending that there are inherently „true“ numbers at all?
This depends on your philosophy of mathematics, which is obviously not that of Euler’s contemporaries…

Both texts: Did Euler see the numbers obtained by summing divergent alternating series as what in Ferraro's text is called a "fictitious number"?
No. The value of a divergent series could be a true number (e.g. 1) or a false one (infinity, or sqrt(2)). But either way, the *value* does not have to be identical to the limit of the partial sums, if this limit does not exist.

Notions of function

If Euler considered functions to be a rule that linked two variables quantities and not a pointwise corresponding between numerical sets, doesn't this sacrifice the continuity of the equation that is differentiable? Could he possibly link two separate variables with a non-continuous function?
In Euler’s time, non continuous functions were considered as several functions joined together at their discontinuities. A single function was supposed to represent a single motion subject to a single “law” or formula that produced a continuous curve.

Could you elaborate the idea behind …1/4, 1/3, 1/2, 1/1, !1/0!, 1/-1, 1/-2,…?
This is simple. Consider the function 1/x. As x tends to zero from above, the function increases indefinitely. As x continues to move beyond zero it suddenly becomes negative, but continues to increase. If you consider the whole process as continuous (as functions given by a unified formula were considered), you derive that when a value increases continuously beyond infinity it becomes negative. But this idea was not taken too seriously and not developed by Euler and other leading mathematicians.

page 45-46: I don't see the connection between Ferraro's point that zero or another number "could be thought of the value of any variable" and the example that follows.
You could assign any value (fictitious or not) to every variable, even if the formula of the functions was not well defined at that value

Page 44, #1: The Euler quote is not very clear..

It expresses the fact that functions and differential formulas were considered as global objects, not as something that is true for some values and false for others. This means that even if the equation is not true in the quantitative sense, it should be true in some formal and abstract sense.

Divergent series

Bernoulli says that 1-1+1-1+1-1+etc.=½. that makes no sense to me. I mean I get that the average of all possible answers is ½ but the actual answer is never going to be ½.
If by "actual" you mean "limit", you are right. But there's noting that forces us to consider only the limit as an actual value of a series.

On the first page of "How Euler did it": Bernoulli considered that 1-1+1-1 etc. should be equal to 1/2. On the other hand, 1+0-1+1+0-1+1 etc. should be equal to 2/3, but these two sums are actually the same. Did he see this as a contradiction?
No, because the sum of the series reflect not just the coefficients, but also how they "arise". If the former arises from 1-x+x^2-x^3… and the latter from 1+0x^2-1x^3+0x^4…, then ignoring the zeros does not make the two series equal.

In the paper Sandifer wrote, I can't follow his explanation of Euler about how negative numbers are the same as the numbers behind infinity. The problem is that Euler wrote: "Of the second sort is the -1 that arises as 1+2+4+8+16+ etc., which is equal to the number one gets by dividing +1 by -1." How does a -1 arise out of this sequence?
1+2x+4x^2+8x^3… = 1/(1-2x) by summing geometric series. Substituting x=1 establishes the result.

Did Euler know the result of the (Leibniz) altering series? And did he know that the harmonic series diverges (this is not presented in the divergent series examples given by Euler on page 3)?
Yes and yes. Euler is also credited for the number gamma, which is the limit of the difference of the harmonic series and the logarithm. But the divergence of the harmonic series was known earlier.

page 1: I read an article once that one can prove that the sum 1+2+3+4+... (to infinity) equals -1/12 if one makes the assumption 1-1+1-1+1-...=1/2. Are there any real-life applications of this formula (or has this been used in the past?)
Yes, this is apparently used in string theory and quantum physics. Generally, methods for summing divergent series can help define analytic continuations of functions, which are sometimes applicable for physical phenomena. These methods also have applications in combinatorics and number theory. The mathematical framework for deriving such results today is, however, different from the one used at the 18th-19th centuries.

Standards of proof

The article by Sandifer makes it sound as if Euler did not care too much about proving his ideas (Formula 1, convergence of geometric series). Is this because he thought that the proof was trivial, or he only targeted people that would be able to prove it themselves.
Euler's standard of proof or justification was different than ours. There was always some procedural justification (even if it was only inductive – generalizing a formula that is verified only for a few values), but this was not translatable to contemporary standards of proof. Sometimes he just left what he considered simple or standard exercises to the readers.

At page 53 it is written that by calculating the differential of x^n "there is no explicit reference to the limit process". How did Euler then calculate that, for example, the differential of $y = x^3$ was $dy = 3x^2dx$?
Simple algebra shows that dy=(x+dx)^3-x^3 equals 3x^2*dx+3x*dx^2+dx^3. Using the rule that allows to drop higher order differentials yields dy=3x^2*dx.

I find it highly hypocritical how these supposedly fictional entities were perceived with such a distance; naming them irrational and imaginary, nothing and mere tools. At the same time mathematical - or rather algebraic - operations and rules were applied confidently onto them. How could someone justify operating, by inter alia taking roots and logarithms on something merely fictional to produce a result of reality?
A perfect circle, or even a perfectly one dimensional line is also a fiction. Yet this is the basis of geometry… So this is not a very new attitude.

He emphasizes that these fictitious numbers (negative, imaginary, etc.) on mathematical objects did not originate from an arbitrary system of axioms. Although not arbitrary, these fictitious numbers did stem from algebra (and its respective axioms). Especially later he treats dx as a formal variable in Euler's axiomatic system, not as an intuitive thing (unless an infinitesimal as a number is intuitive?)
There was no axiomatization of algebra or fictitious numbers at the time. Their use and the rules governing them were handled intuitively. Even infinitesimals were handled intuitively, based on our intuition of very small quantities and zeros.

If Euler never used sets, when were they introduced/what would Euler have missed by not considering them.
Sets in the mathematical sense emerge in the late 19^{th} century. According to his contemporary standards, Euler missed nothing from not using them. He had different standards and different methods to control his results. Even in contemporary standards there are alternatives to set theory for doing mathematics.

I understand the premise here. Euler clearly deals with his "fictions", meaning that a series value is not originating in the idea of proper quantity, but some kind of attribute of the partial difference progression. But didn't Euler know that by false assumption, one can "prove" anything? By assuming the existence of the series values, he already commits a conceptual error, creating all sorts of paradoxes (like positive series adding up to a negative number).
Even the idea that a convergent series has a value is questionable, and not acceptable by classical Greek standards. Integrating limits into mathematics is very problematic, and took a long time to make consistent.

The Paper contains manipulations that were made by Euler and other mathematicians to prove statements about the convergence of series. However, these proofs can be contradictory and founded on weak theoretical grounds. How is it possible that brilliant minds did not realize the “shakiness” of their productions?
They most certainly did realize it. That’s why it was important for Euler to obtain the same result in many different ways. This was one of his criteria for validity.

Ferraro says on page 42: Fictions were a useful tool for shortening the path of thought and arriving at new results. Does he imply that fictions allowed one to reach "non-fictional" results unavailable without fictions? If so, how did people justify the "reality" of these results if they did consider the tools "fictional"?
In many (but not all) cases the results could be verified, or sometimes even proved by more traditional means. But the fictitious quantities were definitely the vehicle for discovery.

I did not understand how Euler justified the result (-1)^(infinity)=0.
Since (-1)^n with natural n oscillates between -1 and 1, Euler assumed that for infinite n it should have the average value zero.

Another clarification on Euler’s mathematical ethos is needed in Sandifer’s work, where he claims (pp. 4) that negative numbers are greater than infinity. He supposedly shows this by considering the sequence (..., ½, 1/1, 1/0, 1/-1, 1/-2, …) which goes into the negatives after reaching infinity (1/0). However, since positive numbers are surely greater than negative numbers, does this mean that all positive numbers are greater than infinity as well? This clearly is an absurd conclusion that can be drawn from Euler’s writings.
Negative numbers already force you to change your notion of order (because 1/-1 = -1/1, even though the numerator on the left is larger than the one on the right, and the denominator on the left smaller than the one on the right). So the fact that some rules of order do not work when infinities and negative numbers are involved is not surprising. It would turn out (if you accept the above argument) that numbers are arranged in a circle, rather than a line.

Is there any simple example (no monsters, in Lakatos' terminology) where using modern techniques of extending a function's domain by using limits yields different results from Euler's approach?
Analytic functions are not determined by their values on the natural numbers, so there are many possible extensions. Whether you would consider them “simple” or “monsters” is not a question I can answer.

Eulerian Infinitesimals and their ratios appear to be strikingly similar to modern differential calculus - what is the concrete difference? What is the mental leap to defining this process as a "derivation" instead of simply making observations about the ratios of infinitesimals?
The definition of a function without referring to formulas, and the modern Weierstrassian epsilon-delta definitions together make huge differences (the whole world of non continuous functions and measure theory, rejection of divergent series, etc.).

What does all this tells us? It's interesting that "incorrect" mathematical tools can lead to very interesting (and perhaps even "correct") results!
Or that our criteria of correctness are too narrow?

We would say that Euler's proofs about divergent series are wrong but isn't there the possibility to build an alternative mathematics where all series converge?
In contemporary mathematics, there are extensions of the notion of limit the make every bounded series convergent (see Hahn Banach theorem or ultrafilter limits), and all sorts of non standard models of analysis. None of them captures Euler's reasoning, but they can sometime shed a new light on his results.

On page 39: "A theory was acceptable only if it conformed to the reality." But in the first class we described math as something beyond reality, as something ultimate true. Also, the next sentence: "reality is unique" is striking.
Different mathematical cultures conceived the relations between mathematics and reality differently. The classical Greek and early modern approaches differ on this point.

Miscellanea

On page 37 of Ferraro's paper, Breger is cited saying how it is impossible to consider an open segment. In the next sentence, he says "the point zero is not a part of continuum ...". I don't understand how this follows. In my opinion, from his arguments, it follows directly that zero IS actually a part of the continuum.
Zero (like any other point) is not an element of the segment, but a point that is placed on the line to define the segment. It is therefore inseparable from the segment *and* not part of the line (it's kind of like a door and a door-handle: the handle is not part of the wood plank that is the door, but without it the door is not a door…).

What kind of number is k, when Euler does the manipulation a^omega = 1 + psi => a^omega = 1 + k * omega?
A real number. More specifically, it turns out to be the natural logarithm of a.

In the sentence: "He also observed that, by ignoring infinitely small quantities but not naughts, it was still possible to commit extremely serious errors", does he mean zero with naughts?
Yes

What kind of expansion did Leibniz use to derive 1/(1-x) = 1 + x + x^2 + ..?
This is the formula for geometric series.

Sandifer: I have really difficulties to imagine and understanding an "asymptotically convergent", I mean, I do not understand how a series can converge and then diverge.
The situation is the following: S_x(N) = \sum_{n=1}^N (a(x,n)). For all x, the sequences S_x(N) diverge as N goes to infinity. However, for every x, the sums S_x(N) approach a certain value L up to some N, and then start diverging to infinity. Moreover, for larger values of x, this behavior holds for larger values of N.

Can you explain in more detail what is meant with: "First, since a single number was a specific determination of quantity, a single number expressed a quantum rather than a quantity." (pg.44) How was that actually meant in Euler's time?
This is not an Eulerian terminology, but an explanation introduced by Ferraro. A quantum is fixed, whereas a quantity is something decreasing/increasing (see page 36).

Ferraro writes that all the fictitious numbers were "well founded in nature". Does he mean specifically for Euler or was this generally accepted? For me it does not seem far-fetched to assume that somebody would question this with regards to the missing theoretical foundation of fictitious numbers.
This was the main attitude at the time, although there were obvious objections (e.g. Berkeley). Later on, this approach had more serious objections its dominance.

Ferraro p.43 What is the difference between general quantities and variables?
A variable (in the modern sense) can be restricted to particular quantities, a general quantity cannot.

When were numbers created (as opposed to e.g. Roman numerals)? Were Euler, Leibniz etc. the creators of modern-day numbers?
What do you mean by numbers? Sexagesimal representations are about 5,000 years old (Babylonia). Decimal numbers are probably around 2,000 years old. The idea that any quantity or ration is representable by a number is relatively recent, and starts (in the west) with early modernity.

What are "evanescent quantities" (p.45)?
Quantities that diminish until they disappear. In modern terms: quantities converging to zero.

Page 41: Footnote 19: In what sense is 0 the principle of natural numbers? Or alternatively: in what sense is a point the principle of a line?
Principle here means something like starting point and cause (the line can be considered as the motion of a point – if you believe in infinitesimals this can be extended to zero and numbers).

On page 52 (Ferraro), the author writes: "So it implicitly assumes the point of view of modern mathematics to be the only possible point of view [...]" So the other mathematicians see mathematics as a universal truth, but what is the view of Ferraro? Can this be deducted from this text?
Ferraro believes there is no universal point of view to different historical expressions of mathematics, and that each should be understood on its own terms.

Was Euler's interest in calculus mainly driven by a mathematical motivation or also by some aspects of physics as in the case of Newton?
Physics played an important role for Euler.

What does dx^n denote? What is the meaning behind this?
Differential of order n (differential of differential of differential…)

p42: What is the meaning of ontological in this context?
Ontology here is the study of what exists and what doesn't, and the categorization of existents into kinds.