Solutions Sheet 10

Krull Dimension

1. We have seen that for a Noetherian ring A we have $\dim(A[X]) = \dim(A) + 1$. Let now A be a ring, which is not necessarily Noetherian. Use Exercise 3 of sheet 9 to prove the inequality

$$\dim(A) + 1 \leqslant \dim(A[X]) \leqslant 2\dim(A) + 1.$$

Solution: Let $n := \dim(A)$. For every prime ideal $\mathfrak{p} \subset A$ the ideal $\mathfrak{p}[X] = \mathfrak{p} \cdot A[X]$ is a prime ideal in $A[X]$, since $A[X]/\mathfrak{p}[X] \cong (A/\mathfrak{p})[X]$ is an integral domain. Its contraction to A is again $\mathfrak{p}$, i.e. $\mathfrak{p}[X] \cap A = \mathfrak{p}$. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a chain of prime ideals in A. The previous arguments show that then $\mathfrak{p}_0[X] \subsetneq \cdots \subsetneq \mathfrak{p}_n[X]$ is a chain of prime ideals in $A[X]$. However, we have the strict inclusions $\mathfrak{p}_n[X] \subsetneq \mathfrak{p}_n[X] + (X) \subsetneq A[X]$, so $\mathfrak{p}_n[X]$ is not a maximal ideal. This proves that $\dim(A) + 1 \leqslant \dim(A[X])$.

Conversely let $s := \dim(A[X])$ and let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_s$ be a chain of prime ideals in $A[X]$. By contraction to A we get a chain of prime ideals $\mathfrak{p}_0 \cap A \subseteq \cdots \subseteq \mathfrak{p}_s \cap A$ in A, with not necessarily strict inclusions. Assume that there are m distinct contractions. By exercise 3 of sheet 9 we know that at least two contractions out of three subsequent prime ideals must differ. This shows $s+1 \leqslant 2m \leqslant 2(\dim(A)+1)$ and hence $s \leqslant 2\dim(A) + 1$.

2. Let A be a Noetherian ring. Prove that $\dim(A[X, X^{-1}]) = \dim(A) + 1$.

Solution: Since A is Noetherian, we know that $\dim(A[X]) = \dim(A) + 1$. Prime ideals in $A[X, X^{-1}]$ correspond to prime ideals in $A[X]$ which do not contain X. Hence $\dim(A[X, X^{-1}]) \leqslant \dim(A[X]) = \dim(A) + 1$. Conversely, a prime ideal $\mathfrak{p}$ in A gives rise to a prime ideal $\mathfrak{p}[X, X^{-1}]$ in $A[X, X^{-1}]$ with contraction $\mathfrak{p}[X, X^{-1}] \cap A = \mathfrak{p}$. However, $\mathfrak{p}[X, X^{-1}]$ is not maximal, because $A[X, X^{-1}]/\mathfrak{p}[X, X^{-1}] \cong (A/\mathfrak{p})[X, X^{-1}]$ is not a field. Hence $\dim(A[X, X^{-1}]) \geqslant \dim(A) + 1$.

3. Let k be a field. Consider the ring $A := k[X, Y, W, Z]/(XW - YZ)$ and its quotient ring $B := A/(X, Y) \cong k[W, Z]$. Define the prime ideal $\mathfrak{p} := (W, Z) \subset A$ and denote by $\mathfrak{q} \subset B$ its image in B, which is again a prime ideal. What is $\dim(A)$? Prove that $\mathrm{ht}(\mathfrak{p}) = 1$, but $\mathrm{ht}(\mathfrak{q}) = 2$.

Solution: By Krull's principal ideal theorem, we conclude that $(XW - YZ)$ has height 1 in $k[X, Y, W, Z]$. Since $\dim(A)$ is its coheight, we have $1 + \dim(A) \leqslant \dim(k[X, Y, W, Z]) = 4$. Hence $\dim(A) \leqslant 3$. On the other hand, there is the chain of prime ideals $(XW-YZ) \subsetneq (X,Y) \subsetneq (X,Y,W) \subsetneq (X,Y,W,Z)$. Thus $\dim(A) \geqslant 3$ and we obtain equality.

Note that $A/\mathfrak{p} \cong k[X,Y]$. Hence $\mathrm{ht}(\mathfrak{p})+2 \leqslant \dim(A) = 3$, which implies $\mathrm{ht}(\mathfrak{p}) \leqslant 1$. Since A is an integral domain and $\mathfrak{p}$ is non-zero in A we conclude that $\mathrm{ht}(\mathfrak{p}) \geqslant 1$ and thus we obtain equality.

Note that $\mathfrak{q}$ is the maximal ideal (W,Z) in B and $\dim(B) = 2$. Since $(0) \subsetneq (W) \subsetneq \mathfrak{q}$ is a chain of prime ideals in B we conclude that $2 \leqslant \mathrm{ht}(\mathfrak{q}) \leqslant \dim(B) = 2$.

4. Give an example of a zero-dimensional ring, which is not Noetherian.

   Solution: Let k be a field and let $A := \prod_{i\in\mathbb{N}} k$ be an infinite product of k. The ideals $I_n := k^n \times \prod_{i>n}\{0\} \subset A$ are a strictly increasing chain of ideals, so A is not Noetherian. On the other hand, the proof of exercise 2 of sheet 5 shows that every localisation of A at a prime ideal is a field. Thus every prime ideal has height 0, which proves that $\dim(A) = 0$.

5. Let k be a field and $A := k[X_1, X_2, \ldots]$ be the polynomial ring with countably many variables. For $i \geqslant 0$ define the prime ideals $\mathfrak{p}_i := (X_{2^i}, \ldots, X_{2^{i+1}-1})$ in A. Define the multiplicative set $S := A \setminus \bigcup_{i=0}^{\infty} \mathfrak{p}_i$ and the ring $B := S^{-1}A$. Show that B has infinite dimension. Prove that B is Noetherian using the lemma below.

   Lemma: *Let A be a ring such that every localisation $A_{\mathfrak{m}}$ at a maximal ideal $\mathfrak{m} \subset A$ is Noetherian, and every non-zero element $x \in A$ is contained in only finitely many maximal ideals. Then A is Noetherian.*

   Solution: Note that $k(X_k, k \notin [2^i, 2^{i+1}-1])[X_{2^i}, \ldots, X_{2^{i+1}-1}]_{\mathfrak{p}_i} \cong A_{\mathfrak{p}_i} \cong B_{S^{-1}\mathfrak{p}_i}$ and so we have $\mathrm{ht}(S^{-1}\mathfrak{p}_i) = 2^i$ and hence $\dim(B) = \infty$.

   We prove that all $S^{-1}\mathfrak{p}_i$ are maximal ideals in B. Let $\alpha \in B \setminus S^{-1}\mathfrak{p}_i$. After clearing denominators we assume that $\alpha \in A \setminus \mathfrak{p}_i$. After removing all monomials of α which contain X_{2^i} we see that $\alpha + X_{2^i} \in S$ and hence is a unit in B. This proves that $S^{-1}\mathfrak{p}_i$ is a maximal ideal in B.

   Next we prove that the $S^{-1}\mathfrak{p}_i$ are the only maximal ideals. Let $\mathfrak{p} \subset B$ be a prime ideal. Thus its contraction to A is contained in $\bigcup_{i=0}^{\infty} \mathfrak{p}_i$. Choose a non-constant element $a \in \mathfrak{p} \cap A$. Consider the set $M := (\mathfrak{p} \cap A) \setminus \bigcup_{i=0}^{n} \mathfrak{p}_i$, where n is large enough, such that a does not contain any variable with index greater or equal to 2^{n+1}. If M is empty, then $\mathfrak{p} \cap A$ is contained in $\bigcup_{i=0}^{n} \mathfrak{p}_i$ and hence contained in one of the $\mathfrak{p}_i$. If M is non-empty then choose $b \in M$. Then b must contain a variable of index greater than 2^{n+1} whose monomial does not contain a variable of index smaller than 2^{n+1}. We conclude that $a+b$ is not contained in any of the $\mathfrak{p}_i$ and is thus a unit in B, which is a contradiction. Hence $\mathfrak{p}$ is contained in one of the prime ideals $S^{-1}\mathfrak{p}_i$.

   According to the first line of the solutions, the localisation of B at any of the maximal ideals $S^{-1}\mathfrak{p}_i$ is Noetherian. Furthermore, every element $\alpha \in B$ has only finitely many monomials, hence is contained in only finitely many maximal ideals $S^{-1}\mathfrak{p}_i$. By using the lemma we conclude that B is Noetherian.