Personal Notes All the information I need

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1 Affine Group Scheme, Hopf Algebra

Definition 1.1. Let A be a commutative K-algebra. The corresponding affine K-scheme G = Spec(A) is said to be a group scheme if it is endowed with algebraic operations

$$\mu: G \times G \longrightarrow G \ (product),$$

 $e: Spec(K) \longrightarrow G(unit),$
 $\iota: G \longrightarrow G(inverse),$

satisfying the usual axioms of a group, which are expressed by the commutativity of the following three diagrams

(1) Associativity:

$$G \times G \times G \xrightarrow{\mu \times Id} G \times G$$

$$\downarrow^{Id \times \mu} \qquad \qquad \downarrow^{\mu}$$

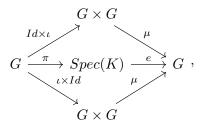
$$G \times G \xrightarrow{\mu} G$$

(2) Unit:

$$G \times Spec(K) \xrightarrow{Id \times e} G \times G \xleftarrow{e \times Id} Spec(K) \times G$$

$$\downarrow \mu \qquad \qquad pr_1 \qquad \downarrow \mu \qquad pr_2$$

(3) Inverse:



where π denotes the structural map of G as a K-scheme. If the algebra A is finitely generated, we say that G is algebraic. We will see below that every affine group scheme is in fact a projective limit of affine group schemes.

Equivalent definition: A group scheme over K is a functor between the categories of commutative K-algebras and abstract groups. Namely, given $G = \operatorname{Spec}(A) A, P, Q$ are K-algebras, one considers the functor:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow_{G} & \downarrow_{G} & \downarrow_{G} \\ Hom_{K-alg}(A,P) = G(P) & \xrightarrow{f \circ g} G(Q) = Hom_{K-alg}(A,Q) \end{array},$$

1.1 Hopf Algebra

The category of affine schemes over K is equivalent to the category of the commutative K-algebras through the contravariant functors

$$A \mapsto \operatorname{Spec}(A)$$

 $G \mapsto \mathcal{O}(G)$.

where $\mathcal{O}(G)$ is the ring of regular functions on G. Thus the defining properties of a group scheme can be transferred to the corresponding algebra, yielding the concept of Hopf algebra.

Definition 1.2. Let H be an associative (not necessarily commutative) K-algebra. Let $\nabla: H \otimes H \longrightarrow H$ be the product of H and $\eta: k \longrightarrow H$ the unit.

(1) We say that H is a **bialgebra** if it is provided with two morphisms of algebras

$$\Delta: H \longrightarrow H \otimes H(coproduct),$$

$$\epsilon: H \longrightarrow k(counit)$$

such that the following diagrams commute:

(a) Coassociativity:

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow^{\Delta} & & \downarrow^{Id \otimes \Delta} \\ H \otimes H & \xrightarrow{\Delta \otimes Id} & H \otimes H \otimes H, \end{array}$$

(b) Counit.

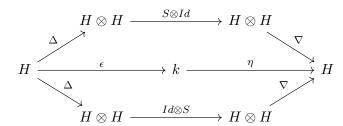
$$H \otimes K \underset{\longrightarrow}{\longleftarrow} H \otimes H \xrightarrow{\epsilon \otimes Id} K \otimes H$$

(2) A bialgebra H is called a **Hopf algebra** if it is further equipped with a morphism of algebras

$$S: H \longrightarrow H \ (antipode)$$

such that the following diagram commutes:

(c) Antipode.



(3) A bialgebra H is called commutative if the product is commutative, and cocommutative if the coproduct satisfies $\Delta = \tau \circ \Delta$, where τ : $H \otimes H \longrightarrow H \otimes H$ is the flip of the factors.

2 Tannakian Categories

K is a field of characteristic 0, \mathcal{A} rigid tensor \mathbb{K} -linear abelian category, L is an extension of K.

Definition 2.1. An L-valued fibre functor is a tensor functor $\omega : \mathcal{A} \longrightarrow Vec_L$ which is faithful and exact.

Definition 2.2. A is

- Neutralized Tannakian if one can endow it with a K-valued fibre functor.
- Neutral Tannakian if $\exists K$ -valued fibre functor.
- **Tannakian** if $\exists L$ -valued fibre functor for some L.

Example 2.3. G is an affine K-group scheme, $A = Rep_K(G), \omega : A \longrightarrow Vec_K$ the forgetful functor.