

Cosmic Galois Group: a reading project under supervision of P. Jossen

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1 Preliminary knowledge about Algebraic Group and Tannakian Category

2 Statement of main results

Marcolli and Connes defined it

Theorem 2.1. *For any Feynman graph G with generic kinematics q, m there is a canonical way to associate to a **convergent integral***

- an object mot_G in $\mathcal{H}(S)$, where S is a Zariski open in a space of kinematics
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3 Feynman graph and graph polynomials

A Feynman graph is a graph G defined by $(V_G, E_G, E_G^{\text{ext}})$, where V_G is the set of vertices of G , E_G is the set of internal edges of G , and E_G^{ext} is a set of external legs. Their endpoints are encoded by the maps $\partial : E_G \rightarrow \text{Sym}^2 V_G$ and $\partial : E_G^{\text{ext}} \rightarrow V_G$. We shall assume that the vertices with external legs lie in a single connected component of G . A Feynman graph additionally comes with kinematic data:

- a particle mass $m_e \in \mathbb{R}$ for every internal edge $e \in E_G$.
- a momentum $q_i \in \mathbb{R}^d$ for every external leg $i \in E_G^{\text{ext}}$,

where $d \geq 0$ is the dimension of space-time. All the external legs will be oriented inwards, so all momenta are incoming and are subject to momentum conservation.

In this paper, a subgraph H of G will be graph defined by a triple $(V_H, E_H, E_H^{\text{ext}})$ where $V_H \subset V_G$, $E_H \subset E_G$ and either $E_H^{\text{ext}} = E_G^{\text{ext}}$ or $E_H^{\text{ext}} = \emptyset$.

A tadpole is a subgraph of the the form $\{\{v\}, \{v, v\}, \emptyset\}$. We shall use the following notation for the basic combinatorial invariants of G :

- $h_G = \dim(H^1(G))$ the loop number of G

- $\kappa_G = \dim(H^0(G))$ the number of connected components of G
- $N_G = |E_G|$ the number of connected components of G .

They do not depend on the external legs of G . Euler's formula states that

$$N_G - V_G = h_G - \kappa_G.$$

We define that If a vertex $v \in V_G$ has several incoming momenta q_1, \dots, q_n we can replace it with a single incoming momentum $q_1 + \dots + q_n$. Our notion of Feynman subgraph respects this equivalence relation. Then the graph polynomial defined latter would only depend on the equivalence classes.

We say that a Feynman graph is **of type** (Q, M) if it is equivalent to a graph with at most Q external kinematic parameters and at most M nonzero particle mass. We shall call a graph one-particle irreducible, or 1PI, if every connected component is 2-edge connected (i.e. deleting any edge causes the loop number to drop).

3.1 Graph polynomials

Let G be a Feynman graph. Recall that a tree is a connected graph T with $h_T = 0$. A forest is any graph with $h_T = 0$.

Definition 3.1. *Let G be a connected Feynman graph. The **Kirchhoff polynomial** (or first Symanzik polynomial) is the polynomial in $\mathbb{Z}[\alpha_e, e \in E_G]$ defined by*

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin T} \alpha_e, \quad (1)$$

where the sum is over all spanning trees T of G . If G has several connected components G_1, \dots, G_n , we shall define

$$\Psi_G = \prod_1^n \Psi_{G_i}$$

The second Symanzik polynomial is defined for connected G by

$$\Phi_G(q) = \sum_{T_1 \cup T_2 \subset G} (q^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e, \quad (2)$$

where the sum is over all spanning 2-trees $T = T_1 \cup T_2$ of G , and $q^{T_1} := \sum_{i \in E_{T_1}^{ext}} q_i$ is the total momentum entering T_1 .

Remark 3.2. α_e are just the Schwinger parameters

Definition 3.3. Let G be a Feynman graph. Define

$$\Xi_G(q, m) = \Phi_G(q) + \left(\sum_{e \in E_G} m_e^2 \alpha_e \right) \Psi_G.$$

It is the denominator of Feynman integral, and it is homogeneous in α_e of degree $h_G + 1$

Since the graph polynomials only depend on the total momentum flow, they are well-defined on equivalence classes of graphs.

3.2 Feynman integral in projective space

After omitting certain pre-factors, we define the Feynman integral

$$I_G(q, m) = \int_{\sigma} \omega_G(q, m),$$

where

$$\omega_G(q, m) = \frac{1}{\Psi_G^{d/2}} \left(\frac{\Psi_G}{\Xi_G(q, m)} \right)^{N_G - h_G d/2} \Omega_G$$

and

$$\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \dots \wedge \widehat{d\alpha_i} \wedge \dots \wedge d\alpha_{N_G}$$

Following from the fact that $\deg(\Psi_G) = h_G$ and $\deg(\Xi_G) = h_G + 1$, we know that ω_G is homogeneous of degree 0.

Finally, let $\sigma \subset \mathbb{P}^{N_G-1}(\mathbb{R})$ be the coordinate simplex defined in projective coordinates by

$$\sigma = \{(\alpha_1 : \dots : \alpha_{N_G}) \in \mathbb{P}^{N_G-1}(\mathbb{R}) : \alpha_i \geq 0\}.$$

3.3 Edge subgraphs and their quotients

Let $G = (V_G, E_G, E_G^{ext})$ be a Feynman graph. A set of internal edges $\gamma \subset E_G$ defines a subgraph of G as follows. Write $E_\gamma = \gamma$ and let V_γ be the set of endpoints of elements of E_γ .

Definition 3.4. A set of edges $\gamma \subset E_G$ is **momentum-spanning** if $\partial E_G^{ext} \subset V_\gamma$, and the vertices E_G^{ext} lie in a single connected component of the graph (V_γ, E_γ) .

we define the subgraph associated to $\gamma \subset E_G$ by

$$(V_\gamma, E_\gamma, E_\gamma^{ext}),$$

where $E_\gamma^{ext} = E_G^{ext}$ if γ is momentum-spanning and $E_\gamma^{ext} = \emptyset$ otherwise. We all $(V_\gamma, E_\gamma, E_\gamma^{ext})$ the edge-subgraph associated to γ and denote it also by γ when no confusion arises.

The quotient of G by an edge-subgraph γ is defined by

$$G/\gamma = (V_G/\sim, (E_G \setminus \gamma)/\sim, E_G^{ext}/\sim),$$

where \sim is the equivalence relation on vertices of G where two vertices are equivalent if and only if they are vertices of the same connected component of γ , and the induced equivalence relation on edges (unordered pairs of vertices). It is a Feynman graph. Every connected components of γ corresponds to a unique vertex in G/γ . Note that γ is momentum-spanning if and only if G/γ is equivalent to a graph with no external momenta. (which amounts to compress each component of γ to a single vertex.).

In this way, exactly one of the two Feynman graphs γ and G/γ is equivalent to a Feynman graph with non-zero external momenta: if is momentum spanning it is γ , otherwise it is G/γ .

3.4 Contraction-deletion

Let $G = (V_G, E_G, E_G^{ext})$ be a Feynman graph. The deletion of an edge e in G is the graph G/e defined by deleting the edge e but retaining its endpoints:

$$G/e = (V_G, E_G \setminus \{e\}, E_G^{ext}).$$

In general, it is not a union of Feynman graphs since momentum conservation may not hold on each of its connected components.

One sometimes encounters the following variant of the previous notion of graph-quotient. It will be denoted by a double slash to distinguish it from the ordinary quotient. For an edge-subgraph γ , let $G//\gamma$ be the empty graph if $h_\gamma > 0$ and

$$G//\gamma = G/\gamma$$

if γ is a forest. In the case of a single edge e , G/e is empty whenever e is a tadpole.

It follows from Euler's formula that $h_G = h_\gamma + h_{G/\gamma}$ for any edge-subgraph $\gamma \subset G$ (which is not necessarily connected).

Lemma 3.5. (*Contraction-deletion*) *Let G be connected and $e \in E_G$. Then*

$$\Psi_G = \Psi_{G \setminus e}^0 \alpha_e + \Psi_{G//e},$$

$$\Phi_G(q) = \Phi_{G \setminus e}^0(q) \alpha_e + \Phi_{G//e}(q),$$

where $\Psi_{G \setminus e}^0$ is given by the right hand side of Eq 1: it is $\Psi_{G \setminus e}$ if $G \setminus e$ is connected and 0 otherwise. Likewise $\Phi_{G \setminus e}^0(q)$ is given by the right-hand side of Eq 2: it equals to $\Phi_{G \setminus e}(q)$ if $G \setminus e$ is connected and equals to $\Psi_{G_1} \Psi_{G_2}(q^{G_1})^2 = \Psi_{G_1} \Psi_{G_2}(q^{G_2})^2$ if $G \setminus e$ has two connected components G_1, G_2 .

Proof. Let T be a spanning k -tree of G . The edge e is not an edge of T if and only if T is a spanning k -tree of $G \setminus e$. By the definition of graph polynomials, this gives rise to the first terms in the right-hand side of above equations in the lemma. Note that if e is a tadpole, this is the only case which can occur. Now suppose that e is not a tadpole. If e is an edge of T , the T/e is a spanning k -tree of $G \setminus e$. Conversely, if T' is a spanning k -tree of G/e , then there is a unique component of T' which meets the vertex in G/e . It follows that the inverse image of T' in G with the edge e , is a spanning k -tree in G . This establishes a bijection between the set of spanning k -trees in T which contain e and those G/e . The rest just follows from the definition of graph polynomials. \square

Goal, define and study a glauis group of Feynman amplitude

4 General construction on Tannakian category

5 Motivic Periods over \mathbb{Q}

Definition 5.1. *Category \mathcal{H} of Betti and de Rham realizations:* Consider a category \mathcal{H} whose objects are triples (V_B, V_{dR}, c) consisting of the following data:

1. A finite dimensional \mathbb{Q} -vector space V_B with a finite increasing (Weight) filtration $W_\bullet V_B$.

2. A finite dimensional \mathbb{Q} -vector space V_{dR} with a finite increasing (Weight) filtration $W_{\bullet}V_{dR}$ and a finite decreasing (Hodge) filtration $F^{\bullet}V_{dR}$.
3. An isomorphism

$$c : V_{dR} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} V_B \otimes_{\mathbb{Q}} \mathbb{C}$$

which respect the W_{\bullet} filtration on both sides.

4. A linear involution $F_{\infty} : V_B \xrightarrow{\sim} V_B$ called the real Frobenius.

The category \mathcal{H} has two fibre functors:

$$\omega_{\bullet} : \mathcal{H} \longrightarrow \text{Vec}_{\mathbb{Q}, \bullet = B, dR}$$

Then we can regard \mathcal{H} as a Tannakian category.

Definition 5.2. The ring of \mathcal{H} -periods is $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$. It is equipped with a

- A period homomorphism

$$\text{per} : \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \longrightarrow \mathbb{C}$$

which sends $[(V_B, V_{dR}, c)\sigma, \omega]^{\mathfrak{m}}$