Chapter 3

Feynman integrals and algebraic varieties

In this chapter we begin to connect the two topics introduced above, perturbative quantum field theory and motives. We follow first what we referred to as the "bottom-up approach" in the introduction, by showing how to associate to individual Feynman integrals an algebraic variety, in the form of a hypersurface complement, and a (possibly divergent) period integral on a chain with boundary on a normal crossings divisor. The motivic nature of this period, and the question of whether this (after removing divergences) will be a period of a mixed Tate motive can then be formulated as the question of whether a certain relative cohomology of the hypersurface complement relative to the normal crossings divisor is a realization of a mixed Tate motive. We begin by describing the parametric form of Feynman integrals and the main properties of the two Kirchhoff–Symanzik graph polynomials involved in this formulation and the graph hypersurfaces defined by the first polynomial. We show, as in [Aluffi and Marcolli (2008a)], how one can sometimes obtain explicit computations of the classes in the Grothendieck ring of varieties for some families of Feynman graphs, using the classical Cremona transformation and dual graphs. These classes also satisfy a form of deletion/contraction relation, as proved in [Aluffi and Marcolli (2009b)]. We then describe, following the results of [Aluffi and Marcolli (2008b)], a way of formulating Feynman rules directly at the algebro-geometric level of graph hypersurface complements. We give examples of such Feynman rules that factor through the Grothendieck ring and others, defined using characteristic classes of singular varieties, that do not descend to the level of isomorphism classes of varieties in the Grothendieck ring, although they still satisfy an inclusion-exclusion principle. We discuss the difference between working with affine or projective hypersurfaces in describing Feynman integrals. Finally, following [Aluffi and Marcolli (2009a)], we show that one can reformulate the period computation underlying the Feynman amplitudes in terms of a simpler hypersurface complement, using determinant hypersurfaces, whose mixed Tate nature is established, but with a more complicated normal crossings divisor as the second term in the relative cohomology whose motivic nature one wishes to establish. We describe the advantages and the difficulties of this viewpoint.

3.1 The parametric Feynman integrals

Using the Feynman parameters introduced in §1.7 above, we show how to reformulate the Feynman integral

$$U(\Gamma, p_1, \ldots, p_N) = \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1(k_1) \cdots q_n(k_n)} \, d^D k_1 \cdots d^D k_n$$

in the form known as the *Feynman parametric representation.* (We neglect here an overall multiplicative factor in the coupling constants and powers of 2π.)

The first step is to rewrite the denominator $q_1 \cdots q_n$ of (1.37) in the form of an integration over the topological simplex σ_n as in (1.51), in terms of the Feynman parameters $t = (t_1, \ldots, t_n) \in \sigma_n$.

In writing the integral (1.37) we have made a choice of an orientation of the graph Γ, since the matrix $\epsilon_{v,i}$ involved in writing the conservation laws at vertices in (1.37) depends on the orientation given to the edges of the graph. Now we also make a choice of a set of generators for the first homology group $H_1(\Gamma, \mathbb{Z})$, *i.e.* a choice of a maximal set of independent loops in the graph, $\{l_1, \ldots, l_\ell\}$ with $\ell = b_1(\Gamma)$ the first Betti number.

We then define another matrix associated to the graph Γ, the *circuit matrix* $\eta = (\eta_{ik})$, with $i \in E(\Gamma)$ and $k = 1, \ldots, \ell$ ranging over the chosen basis of loops, given by

$$\eta_{ik} = \begin{cases} +1 \text{ if edge } e_i \in \text{ loop } l_k, \text{ same orientation} \\ -1 \text{ if edge } e_i \in \text{ loop } l_k, \text{ reverse orientation} \\ \ \ 0 \text{ if edge } e_i \notin \text{ loop } l_k. \end{cases} \tag{3.1}$$

There is a relation between the circuit matrix and the incidence matrix of the graph, which is given as follows.

Lemma 3.1.1. *The incidence matrix $\epsilon = (\epsilon_{v,i})$ and the circuit matrix $\eta = (\eta_{ik})$ of a graph Γ satisfy the relation $\epsilon\eta = 0$. This holds independently of the choice of the orientation of the graph and the basis of $H_1(\Gamma, \mathbb{Z})$.*

Proof. The result follows from the observation that, independently of the choices of orientations, for two given edges e_i and e_j in a loop l_k, both incident to a vertex v, one has

$$\epsilon_{v,i}\eta_{ik} = -\epsilon_{v,j}\eta_{jk}$$

so that one obtains

$$\sum_i \epsilon_{v,i}\eta_{ik} = 0,$$

since $\epsilon_{v,i}\eta_{ik} \neq 0$ for only two edges in the loop l_k, with different signs, as above; see §2.2 of [Nakanishi (1971)]. $\square$

We then define the Kirchhoff matrix of the graph, also known as the Symanzik matrix.

Definition 3.1.2. The Kirchhoff–Symanzik matrix $M_\Gamma(t)$ of the graph Γ is the $\ell \times \ell$ matrix given by

$$(M_\Gamma(t))_{kr} = \sum_{i=1}^{n} t_i \eta_{ik}\eta_{ir}. \tag{3.2}$$

Equivalently, it can be written as

$$M_\Gamma(t) = \eta^\dagger \Lambda(t)\eta,$$

where $\dagger$ is the transpose and $\Lambda(t)$ is the diagonal matrix with entries $(t_1, \ldots, t_n)$. We think of M_Γ as a function

$$M_\Gamma : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad t = (t_1, \ldots, t_n) \mapsto M_\Gamma(t) = (M_\Gamma(t))_{kr} \tag{3.3}$$

where $\mathbb{A}$ denotes the affine line over a field (here mostly $\mathbb{C}$ or $\mathbb{R}$ or $\mathbb{Q}$).

Definition 3.1.3. The Kirchhoff–Symanzik polynomial $\Psi_\Gamma(t)$ of the graph Γ is defined as

$$\Psi_\Gamma(t) = \det(M_\Gamma(t)). \tag{3.4}$$

Notice that, while the construction of the matrix $M_\Gamma(t)$ depends on the choice of an orientation on the graph Γ and of a basis of $H_1(\Gamma, \mathbb{Z})$, the graph polynomial is independent of these choices.

Lemma 3.1.4. *The Kirchhoff–Symanzik polynomial $\Psi_\Gamma(t)$ is independent of the choice of edge orientation and of the choice of generators for $H_1(\Gamma, \mathbb{Z})$.*

Proof. A change of orientation in a given edge results in a change of sign in one of the columns of $\eta = \eta_{ik}$. The change of sign in the corresponding row of $\eta^\dagger$ leaves the determinant of $M_\Gamma(t) = \eta^\dagger \Lambda(t) \eta$ unaffected. A change of basis for $H_1(\Gamma, \mathbb{Z})$ changes $M_\Gamma(t) \mapsto A M_\Gamma(t) A^{-1}$, where $A \in \mathrm{GL}_\ell(\mathbb{Z})$ is the matrix that gives the change of basis. The determinant is once again unchanged. $\square$

We view it as a function $\Psi_\Gamma : \mathbb{A}^n \to \mathbb{A}$. We define the affine graph hypersurface $\hat{X}_\Gamma$ to be the locus of zeros of the graph polynomial

$$\hat{X}_\Gamma = \{t \in \mathbb{A}^n \mid \Psi_\Gamma(t) = 0\}. \tag{3.5}$$

The polynomial Ψ_Γ is by construction a homogeneous polynomial of degree $\ell = b_1(\Gamma)$, hence we can view it as defining a hypersurface in projective space $\mathbb{P}^{n-1} = (\mathbb{A}^n \smallsetminus \{0\})/\mathbb{G}_m$,

$$X_\Gamma = \{t \in \mathbb{P}^{n-1} \mid \Psi_\Gamma(t) = 0\}, \tag{3.6}$$

of which $\hat{X}_\Gamma$ is the affine cone $\hat{X}_\Gamma = C(X_\Gamma)$.

After rewriting the denominator of the integrand in (1.37) in terms of an integration over σ_n using the Feynman parameters, we want to replace in the Feynman integral $U(\Gamma, p_1, \ldots, p_N)$ the variables k_i associated to the internal edges, and the integration in these variables, by variables x_r associated to the independent loops in the graph and an integration only over these variables, using the linear constraints at the vertices. We set

$$k_i = u_i + \sum_{r=1}^{\ell} \eta_{ir} x_r, \tag{3.7}$$

with the constraint

$$\sum_{i=1}^{n} t_i u_i \eta_{ir} = 0, \quad \forall r = 1, \ldots, \ell, \tag{3.8}$$

that is, we require that the column vector $\Lambda(t)u$ is orthogonal to the rows of the circuit matrix η.

The momentum conservation conditions in the delta function in the numerator of (1.37) give

$$\sum_{i=1}^{n} \epsilon_{v,i} k_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j = 0. \tag{3.9}$$

Lemma 3.1.5. *Using the change of variables* (3.7) *and the constraint* (3.8) *one finds the conservation condition*

$$\sum_{i=1}^{n} \epsilon_{v,i} u_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j = 0. \tag{3.10}$$

Proof. This follows immediately from the orthogonality relation between the incidence matrix and circuit matrix of Lemma 3.1.1. □

The two equations (3.8) and (3.10) constitute the Krichhoff laws of circuits applied to the flow of momentum through the Feynman graph. In particular they determine uniquely the $u_i = u_i(p)$ as functions of the external momenta. We see the explicit form of the solution in Proposition 3.1.7 below. First we give a convenient reformulation of the graph polynomial.

The graph polynomial $\Psi_\Gamma(t)$ has a more explicit combinatorial description in terms of the graph Γ, as follows.

Proposition 3.1.6. *The Kirchhoff–Symanzik polynomial* $\Psi_\Gamma(t)$ *of* (3.4) *is given by*

$$\Psi_\Gamma(t) = \sum_{T \subset \Gamma} \prod_{e \notin E(T)} t_e, \tag{3.11}$$

where the sum is over all the spanning trees T *of the graph* Γ *and for each spanning tree the product is over all edges of* Γ *that are not in that spanning tree.*

Proof. The polynomial $\Psi_\Gamma = \det(M_\Gamma(t))$ can equivalently be described as the polynomial (see [Itzykson and Zuber (2006)], §6-2-3 and [Nakanishi (1971)] §1.3-2)

$$\Psi_\Gamma(t) = \sum_S \prod_{e \in S} t_e, \tag{3.12}$$

where S ranges over all the subsets $S \subset E_{int}(\Gamma)$ of $\ell = b_1(\Gamma)$ internal edges of Γ, such that the removal of all the edges in S leaves a connected graph. This can be seen to be equivalent to the formulation (3.11) in terms of spanning trees of the graph Γ (see [Nakanishi (1971)] §1.3). In fact, each spanning tree has $\#V(\Gamma) - 1$ edges and is the complement of a set S as above. □

In the case of graphs with several connected components, one defines the Kirchhoff–Symanzik polynomial as in (3.11), with the sum over *spanning forests*, by which we mean here collections of a spanning tree in each connected component. This polynomial is therefore multiplicative over connected components.

We then have the following description of the resulting term $\sum_i t_i u_i^2$ after the change of variables (3.7). This will be useful in Theorem 3.1.9 below.

Proposition 3.1.7. *The term* $\sum_i t_i u_i^2$ *is of the form* $\sum_i t_i u_i^2 = p^\dagger R_\Gamma(t) p$, *where* $R_\Gamma(t)$ *is an* $N \times N$ *matrix, with* $N = \#E_{ext}(\Gamma)$ *with*

$$p^\dagger R_\Gamma(t) p = \sum_{v,v' \in V(\Gamma)} P_v (D_\Gamma(t)^{-1})_{v,v'} P_{v'},$$

with

$$(D_\Gamma(t))_{v,v'} = \sum_{i=1}^{n} \epsilon_{v,i} \epsilon_{v',i} t_i^{-1} \tag{3.13}$$

and

$$P_v = \sum_{e \in E_{ext}(\Gamma), t(e)=v} p_e. \tag{3.14}$$

Proof. We give a quick sketch of the argument and we refer the reader to [Nakanishi (1971)] and [Itzykson and Zuber (2006)], §6-2-3 for more details. The result is a consequence of the fact that the u_i, as functions of the external momenta $p = (p_j)$, are determined by the Kirchhoff law (3.10). Thus, there is a matrix $A_{i,v}$ such that $-\sum_v A_{i,v} P_v = t_i u_i$, hence $\sum_i t_i u_i^2 = \sum_i \sum_{v,v'} P_v P_{v'} A_{i,v} A_{i,v'} t_i^{-1}$. The constraints on the u_i given by the Kirchhoff laws also show that $A_{i,v} = \epsilon_{i,v}$ so that one obtains the matrix $D_\Gamma(t)$ as in (3.13). ☐

We set

$$V_\Gamma(t,p) = p^\dagger R_\Gamma(t) p + m^2. \tag{3.15}$$

In the massless case ($m = 0$), we will see below that this is a ratio of two homogeneous polynomials in t,

$$V_\Gamma(t,p)|_{m=0} = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t,p)}, \tag{3.16}$$

of which the denominator is the graph polynomial (3.4) and $P_\Gamma(t,p)$ is a homogeneous polynomial of degree $b_1(\Gamma)+1$. We have the following result; see [Itzykson and Zuber (2006)], §6-2-3.

Proposition 3.1.8. *The function* $V_\Gamma(t,p)$ *satisfies* (3.16) *in the massless case, with the polynomial* $P_\Gamma(t,p)$ *of the form*

$$P_\Gamma(t,p) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e, \tag{3.17}$$

where the sum is over the cut-sets $C \subset \Gamma$, *i.e. the collections of* $b_1(\Gamma)+1$ *edges that divide the graph* Γ *into exactly two connected components* $\Gamma_1 \cup \Gamma_2$.

The coefficient s_C is a function of the external momenta attached to the vertices in either one of the two components,

$$s_C = \left(\sum_{v \in V(\Gamma_1)} P_v \right)^2 = \left(\sum_{v \in V(\Gamma_2)} P_v \right)^2, \tag{3.18}$$

where the P_v are defined as in (3.14)*, as the sum of the incoming external momenta.*

Proof. We give a brief sketch of the argument and refer the reader to the more detailed treatment in [Nakanishi (1971)]. The matrix $D_\Gamma(t)$ of (3.13) has determinant

$$\det(D_\Gamma(t)) = \sum_T \prod_{e \in T} t_e^{-1},$$

where T ranges over the spanning trees of the graph Γ. This is related to the polynomial $\Psi_\Gamma(t)$ by

$$\Psi_\Gamma(t) = (\prod_e t_e) \det(D_\Gamma(t)) = \sum_T \prod_{e \notin T} t_e.$$

Thus, for $m = 0$, the function (3.15) becomes

$$p^\dagger R_\Gamma(t) p = \sum_{v,v' \in V(\Gamma)} P_v (D_\Gamma(t)^{-1})_{v,v'} P_{v'}$$

$$= \frac{1}{\Psi_\Gamma(t)} \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e.$$

□

We can now rewrite the Feynman integral in its parametric form as follows; see [Bjorken and Drell (1964)] §8 and [Bjorken and Drell (1965)] §18.

Theorem 3.1.9. *Up to a multiplicative constant $C_{n,\ell}$,the Feynman integral $U(\Gamma, p_1, \ldots, p_N)$ can be equivalently written in the form*

$$U(\Gamma, p_1, \ldots, p_N) = \frac{\Gamma(n - \frac{D\ell}{2})}{(4\pi)^{D\ell/2}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{n - D\ell/2}}, \tag{3.19}$$

where ω_n is the volume form on the simplex σ_n.

Proof. We first show that we have

$$\int \frac{d^D x_1 \cdots d^D x_\ell}{(\sum_{i=0}^n t_i q_i)^n} = C_{\ell,n} \det(M_\Gamma(t))^{-D/2} (\sum_{i=0}^n t_i (u_i^2 + m^2))^{-n + D\ell/2}, \tag{3.20}$$

where $u_i = u_i(p)$ as above. In fact, after the change of variables (3.7), the left hand side of (3.20) reads

$$\int \frac{d^D x_1 \cdots d^D x_\ell}{(\sum_{i=0}^n t_i(u_i^2 + m^2) + \sum_{k,r}(M_\Gamma)_{kr} x_k x_r)^n} \, .$$

The integral can then be reduced by a further change of variables that diagonalizes the matrix M_Γ to an integral of the form

$$\int \frac{d^D y_1 \cdots d^D y_\ell}{(a + \sum_k \lambda_k y_k^2)^n} = C_{\ell,n}\, a^{-n+D\ell/2} \prod_{k=1}^{\ell} \lambda_k^{-D/2},$$

with

$$C_{\ell,n} = \int \frac{d^D x_1 \cdots d^D x_\ell}{(1 + \sum_k x_k^2)^n}.$$

We then write $\det M_\Gamma(t) = \Psi_\Gamma(t)$ and we use the expression of Proposition 3.1.7 to express the term $(\sum_i t_i(u_i^2 + m^2))^{-n+D\ell/2}$ in terms of

$$\sum_i t_i(u_i^2 + m^2) = \sum_i t_i u_i^2 + m^2 = V_\Gamma(t,p),$$

with $V_\Gamma(t,p)$ as in (3.15). □

In the *massless* case, using the expression (3.16) of $V_\Gamma(t,p)$ in terms of the polynomials $P_\Gamma(t,p)$ and $\Psi_\Gamma(t)$, one writes equivalently the integral $U(\Gamma,p)$ as

$$U(\Gamma, p_1, \ldots, p_N) = \frac{\Gamma(n - \frac{D\ell}{2})}{(4\pi)^{D\ell/2}} \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}. \tag{3.21}$$

Up to a divergent Gamma-factor, one is interested in understanding the motivic nature (*i.e.* the nature as a period) of the remaining integral

$$\mathcal{I}(\Gamma, p_1, \ldots, p_N) = \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}. \tag{3.22}$$

3.2 The graph hypersurfaces

We introduced above the projective hypersurfaces $X_\Gamma \subset \mathbb{P}^{n-1}$, with $n = \# E_{int}(\Gamma)$, and the corresponding affine hypersurface $\hat{X}_\Gamma \subset \mathbb{A}^n$.

We observe here that these hypersurfaces are in general *singular* and with singular locus of positive dimension; see §3.5 below for more details. The fact that we are dealing with singular hypersurfaces in projective spaces implies that, when we consider motives associated to these geometric objects, we will necessarily be dealing with *mixed* motives.

We consider only the case where the spacetime dimension D of the quantum field theory is *sufficiently large*. This means that we look at the parametric Feynman integrals (3.19) in the *stable range* where

$$n \leq D\ell/2, \tag{3.23}$$

for $n = \#E_{int}(\Gamma)$ and $\ell = b_1(\Gamma)$. In this range, the algebraic differential form that is integrated in the parametric representation of the Feynman integral only has singularities along the graph hypersurface X_Γ, while the polynomial $P_\Gamma(t,p)$ only appears in the numerator. This range covers, in particular, the special case of the *log divergent* graphs. These are the graphs with $n = D\ell/2$, for which the integral (3.22) reduces to the simpler form

$$\mathcal{I}(\Gamma, p_1, \ldots, p_N) = \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2}}. \tag{3.24}$$

We will not discuss the *unstable range* of small D, though it is often of significant physical interest, because of additional difficulties in treating the hypersurfaces defined by the second graph polynomial $P_\Gamma(t,p)$ caused by the additional dependence on the external momenta. We only make some general considerations about this case in §3.3 below.

In terms of graph hypersurfaces, the Feynman integral with the condition (3.23) can be regarded (modulo the problem of divergences) as a period obtained by integrating an algebraic differential form defined on the hypersurface complement $\mathbb{P}^{n-1} \smallsetminus X_\Gamma$ over a domain given by the topological simplex σ_n, whose boundary $\partial\sigma_n$ is contained in the normal crossings divisor Σ_n given by the algebraic simplex (the union of the coordinate hyperplanes). Thus, the motivic nature of this period should be detected by the mixed motive

$$\mathfrak{m}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma)), \tag{3.25}$$

whose realization is the relative cohomology

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma)). \tag{3.26}$$

A possible strategy to showing that the Feynman integrals evaluate to numbers that are periods of mixed Tate motives would be to show that the motive (3.25) is mixed Tate. This is one of the main themes that we are going to develop in the rest of the book.

It is important to stress the fact that identifying the motivic nature of the relative cohomology above would only suffice up to the important issue of divergences. Divergences occur where the graph hypersurface X_Γ meets the domain of integration σ_n. Notice that the intersections $X_\Gamma \cap \sigma_n$ can only occur on the boundary $\partial\sigma_n$. In fact, in the interior of σ_n all the coordinates t_e are strictly positive real numbers, and the graph polynomial is then also strictly positive, $\Psi_\Gamma(t) > 0$, as can easily be seen by the expression (3.11). Thus, we have $X_\Gamma \cap \sigma_n \subset X_\Gamma \cap \Sigma_n$. These intersections are usually nonempty, so that some method of regularization needs to be introduced to remove the corresponding divergences in the integral, before one can identify the integration

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}} \tag{3.27}$$

with a period of the motive (3.25). The regularization method proposed in [Bloch, Esnault, Kreimer (2006)] consists of performing a number of blow-ups of (strata of) the locus $X_\Gamma \cap \Sigma_n$, with the result of separating the domain of integration and the hypersurface and therefore regularizing the integral by replacing the original integration with one performed in the blown up variety. This regularization procedure adds a further complication to the problem of identifying the motivic nature of (3.25). Namely, one also needs to know that the motive remains mixed Tate after the blow-ups. We will return to this issue in more detail in §3.14 below, after we discuss the approach of [Aluffi and Marcolli (2009a)].

We now discuss briefly the difference between working with the affine or the projective hypersurfaces. While it is natural to work projectively when one has equations defined by homogeneous polynomials, there are various reasons why the affine context appears to be more natural in this case. In particular, we argue as in [Aluffi and Marcolli (2008b)] that the multiplicative property of the Feynman rules is more naturally reflected by a corresponding multiplicative property of the affine hypersurface complements, which in the projective case is replaced by the more complicated *join* operation.

As we saw in §1.3, the Feynman rules are multiplicative over disjoint unions of graphs, namely

$$U(\Gamma, p) = U(\Gamma_1, p_1) \cdots U(\Gamma_k, p_k), \tag{3.28}$$

for $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k$ a disjoint union of graphs with $N_j = \#E_{ext}(\Gamma_j)$ external edges, with external momenta

$$p = (p_1, \ldots, p_k), \quad \text{with} \quad p_j = (p_{j,1}, \ldots, p_{j,N_j}).$$

The other "multiplicative" property that the Feynman rules satisfy is the one we discussed in the first chapter, which allows one to determine the

Feynman integrals of non-1PI graphs using the integrals for 1PI graphs. In fact, upon representing an arbitrary (finite) graph as a tree T with vertices $v \in V(T)$ replaced by 1PI graphs Γ_v with a number of external edges $\#E_{ext}(\Gamma_v) = \mathrm{val}(v)$ equal to the valence of the vertex, one obtains the expression for the Feynman integral of the resulting graph Γ as

$$V(\Gamma, p) = \prod_{v \in V(T), e \in E_{int}(T), v \in \partial(e)} V(\Gamma_v, p_v) q_e(p_v)^{-1} \delta((p_v)_e - (p_{v'})_e), \quad (3.29)$$

where, as in (1.35) we write

$$V(\Gamma, p_1, \ldots, p_N) = \varepsilon(p_1, \ldots, p_N)\, U(\Gamma, p_1, \ldots, p_N),$$

with $\varepsilon(p_1, \ldots, p_N) = \prod_{e \in E_{ext}(\Gamma)} q_e(p_e)^{-1}$.

In the particular case of a massive theory $m \neq 0$ where one sets all the external momenta equal to zero, this becomes an actual product

$$U(\Gamma, p)|_{p=0} = U(e, p_e)|_{p_e=0}^{\#E_{int}(T)} \prod_{v \in V(T)} U(\Gamma_v, p_v)|_{p_v=0}, \quad (3.30)$$

where in this case the edge propagator $U(e, p_e)|_{p_e=0} = m^{-2}$ is just a constant depending on the mass parameter $m \neq 0$.

One can then, following [Aluffi and Marcolli (2008b)], define an *abstract Feynman rule* in the following way.

Definition 3.2.1. An abstract Feynman rule is an assignment of an element $U(\Gamma)$ in a commutative ring $\mathcal{R}$ for each finite graph Γ, with the property that, for a disjoint union $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k$ this satisfies the multiplicative property

$$U(\Gamma) = U(\Gamma_1) \cdots U(\Gamma_k) \quad (3.31)$$

and for a non-1PI graph obtained by inserting 1PI graphs Γ_v at the vertices of a tree T it satisfies

$$U(\Gamma) = U(L)^{\#E_{int}(T)} \prod_{v \in V(T)} U(\Gamma_v). \quad (3.32)$$

A first difference one encounters between the projective graph hypersurfaces X_Γ and the affine $\hat{X}_\Gamma$, which is directly relevant to the quantum field theory setting, is the fact that the affine hypersurface complements behave multiplicatively as abstract Feynman rules are expected to do, while the projective hypersurface complements do not. More precisely, we have the following result from [Aluffi and Marcolli (2008b)].

Lemma 3.2.2. *Let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k$ be a disjoint union of finite graphs. Then the affine hypersurface complements satisfy*

$$\mathbb{A}^n \smallsetminus \hat{X}_\Gamma = (\mathbb{A}^{n_1} \smallsetminus \hat{X}_{\Gamma_1}) \times \cdots \times (\mathbb{A}^{n_k} \smallsetminus \hat{X}_{\Gamma_k}), \quad (3.33)$$

where $n_i = \#E_{int}(\Gamma_i)$.

Proof. Using the definition of the graph polynomials in terms of the determinant $\Psi_\Gamma(t) = \det \mathcal{M}_\Gamma(t)$ of the Kirchhoff matrix of the graph, one sees easily that if $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k$ is a disjoint union, then one has

$$\Psi_\Gamma(t) = \prod_{i=1}^{k} \Psi_{\Gamma_1}(t_{i,1}, \ldots, t_{i,n_i}),$$

where $t = (t_e) = (t_{i,j})_{i=1,\ldots,k, j=1,\ldots,n_i}$ are the edge variables of the internal edges of Γ. This means that the affine hypersurface $\hat{X}_\Gamma$ is a union

$$\hat{X}_\Gamma = \bigcup_{i=1}^{k} (\mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_{i-1}} \times \hat{X}_{\Gamma_i} \times \mathbb{A}^{n_{i+1}} \times \cdots \times \mathbb{A}^{n_k}),$$

which implies that the hypersurface complement is then given by the product (3.33). $\square$

The situation with the projective hypersurfaces is more complicated. Instead of getting directly a product of the complements one gets a torus bundle. For instance, suppose given a disjoint union of two graphs $\Gamma = \Gamma_1 \cup \Gamma_2$ and assume, moreover, that neither graph is a forest (the assumption is not necessary in the affine case). Then the projective hypersurface complement $\mathbb{P}^{n_1+n_2-1} \smallsetminus X_\Gamma$ is a $\mathbb{G}_m$-bundle over the product $(\mathbb{P}^{n_1-1} \smallsetminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1} \smallsetminus X_{\Gamma_2})$. To see this, observe that in this case the projective hypersurface X_Γ is a union of cones $C^{n_2}(X_{\Gamma_1})$ and $C^{n_1}(X_{\Gamma_2})$ in $\mathbb{P}^{n_1+n_2-1}$, respectively over X_{Γ_1} and X_{Γ_2} with vertices $\mathbb{P}^{n_2-1}$ and $\mathbb{P}^{n_1-1}$, respectively. The hypothesis that neither graph is a forest ensures that there is a regular map

$$\mathbb{P}^{n_1+n_2-1} \smallsetminus X_\Gamma \to (\mathbb{P}^{n_1-1} \smallsetminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1} \smallsetminus X_{\Gamma_2}) \tag{3.34}$$

given by

$$(t_{1,1} : \cdots : t_{1,n_1} : t_{2,1} : \cdots : t_{2,n_2}) \mapsto ((t_{1,1} : \cdots : t_{1,n_1}), (t_{2,1} : \cdots : t_{2,n_2})), \tag{3.35}$$

while if either Γ_i happened to be a forest, the corresponding X_{Γ_i} would be empty, and the map (3.35) would not be defined everywhere. Under the assumption that neither Γ_i is a forest, the map (3.34) is surjective and the fiber over a point $((t_{1,1} : \cdots : t_{1,n_1}), (t_{2,1} : \cdots : t_{2,n_2}))$ is a copy of the multiplicative group $\mathbb{G}_m$ given by all the points of the form

$$(ut_{1,1} : \cdots : ut_{1,n_1} : vt_{2,1} : \cdots : vt_{2,n_2}), \quad \text{with} \quad (u : v) \in \mathbb{P}^1,\ uv \neq 0.$$

Equivalently, notice that if Γ is not a forest, then there is a regular map

$$\mathbb{A}^n \smallsetminus \hat{X}_\Gamma \to \mathbb{P}^{n-1} \smallsetminus X_\Gamma,$$

and (3.34) is then induced by the isomorphism in (3.33). This is no longer the case if Γ is a forest.

3.3 Landau varieties

For simplicity we restrict here everywhere to the case where the differential form in the parametric Feynman integral has singularities only along the graph hypersurface $\hat{X}_\Gamma$ defined by the vanishing of the graph polynomial $\Psi_\Gamma(t)$. This is certainly the case within what we referred to as the "stable range", *i.e.* for D (considered as a variable) sufficiently large so that $n \leq D\ell/2$. This, however, is typically not the range of physical interest for theories where $D = 4$ and the number of loops and internal edges violates the above inequality. Away from this stable range, the second graph polynomial $P_\Gamma(t,p)$ also appears in the denominator, and the differential form of the parametric Feynman integral is then defined on the complement of the hypersurface (family of hypersurfaces)

$$\hat{Y}_\Gamma(p) = \{t \in \mathbb{A}^n \mid P_\Gamma(t,p) = 0\} \tag{3.36}$$

or in projective space, since $P_\Gamma(\cdot, p)$ is homogeneous of degree $b_1(\Gamma)+1$, on the complement of

$$Y_\Gamma(p) = \{t \in \mathbb{P}^{n-1} \mid P_\Gamma(t,p) = 0\}, \tag{3.37}$$

or else on the complement of the union of hypersurfaces $\hat{Y}_\Gamma(p) \cup \hat{X}_\Gamma$, again depending on the values of n, ℓ and D.

We only discuss here the case where $-n + D\ell/2 \geq 0$, though some of the same arguments extend to the other case as in [Marcolli (2008)]. The hypersurfaces $\hat{Y}_\Gamma(p)$ defined by the vanishing of $P_\Gamma(p,t)$, for a fixed value of the external momenta p, are sometimes referred to as the Landau varieties. The general type of arguments described above using the graph hypersurfaces $\hat{X}_\Gamma$ should then be adapted to the case of the Landau varieties. In such cases, however, the situation is more complicated because of the explicit dependence on the additional parameters p. In particular, some of the divergences of the integral, coming from the intersection of the hypersurface $Y_\Gamma(p)$ with the domain of integration σ_n, can occur in the interior of the domain, not only on its boundary as in the case of X_Γ; see [Bjorken and Drell (1965)], §18.

For simplicity, it is also convenient to make the following assumption on the polynomials $P_\Gamma(t,p)$ and Ψ_Γ, so that we avoid cancellations of common factors.

Definition 3.3.1. A 1PI graph Γ satisfies the *generic condition* on the external momenta if, for p in a dense open set in the space of external momenta, the polynomials $P_\Gamma(t,p)$ and $\Psi_\Gamma(t)$ have no common factor.

Recall that $P_\Gamma(t,p)$ is defined in terms of external momenta and cut-sets through the functions P_v and s_C as in Proposition 3.1.8. In terms of spanning trees, one has

$$P_\Gamma(t,p) = \sum_T \sum_{e' \in T} s_{T,e'}\, t_{e'} \prod_{e \in T^c} t_e, \tag{3.38}$$

where $s_{T,e'} = s_C$ for the cut-set $C = T^c \cup \{e'\}$.

The parameterizing space of the external momenta is the hyperplane in the affine space $\mathbb{A}^{D \cdot \# E_{ext}(\Gamma)}$ obtained by imposing the conservation law

$$\sum_{e \in E_{ext}(\Gamma)} p_e = 0. \tag{3.39}$$

Thus, the simplest possible configuration of external momenta is the one where one puts all the external momenta to zero, except for a pair $p_{e_1} = p = -p_{e_2}$ associated to a choice of a pair of external edges $\{e_1, e_2\} \subset E_{ext}(\Gamma)$. Let v_i be the unique vertex attached to the external edge e_i of the chosen pair. We then have, in this case, $P_{v_1} = p = -P_{v_2}$. Upon writing the polynomial $P_\Gamma(t,p)$ in the form (3.38), we obtain in this case

$$P_\Gamma(p,t) = p^2 \sum_T \Big(\sum_{e' \in T_{v_1,v_2}} t_{e'} \Big) \prod_{e \notin T} t_e, \tag{3.40}$$

where $T_{v_1,v_2} \subset T$ is the unique path in T without backtrackings connecting the vertices v_1 and v_2. We use as in (3.18)

$$s_C = \left(\sum_{v \in V(\Gamma_1)} P_v \right)^2 = \left(\sum_{v \in V(\Gamma_2)} P_v \right)^2$$

to get $s_C = p^2$ for all the nonzero terms in this (3.40). These are all the terms that correspond to cut sets C such that the vertices v_1 and v_2 belong to different components. These cut sets consist of the complement of a spanning tree T and an edge of T_{v_1,v_2}. Consider for simplicity the case where the polynomial Ψ_Γ is irreducible. If we denote the linear functions of (3.40) by

$$L_T(t) = p^2 \sum_{e \in T_{v_1,v_2}} t_e, \tag{3.41}$$

we see that, if the polynomial $\Psi_\Gamma(t)$ divides (3.40), one would have

$$P_\Gamma(p,t) = \Psi_\Gamma(t) \cdot L(t),$$

for a degree one polynomial $L(t)$, and this would give

$$\sum_T (L_T(t) - L(t)) \prod_{e \notin T} t_e \equiv 0,$$

for all t. One then sees, for example, that the 1PI condition on the graph Γ is necessary in order to have the condition of Definition 3.3.1. In fact, for a graph that is not 1PI, one may be able to find vertices and momenta as above such that the degree one polynomials $L_T(t)$ are all equal to the same $L(t)$. Generally, the validity of the condition of Definition 3.3.1 can be checked algorithmically to be satisfied for large classes of 1PI graphs, though a complete combinatorial characterization of the graphs satisfying these properties does not appear to be known.

3.4 Integrals in affine and projective spaces

As we have seen above, the homogeneous graph polynomial Ψ_Γ defines either a projective hypersurface $X_\Gamma \subset \mathbb{P}^{n-1}$, with $n = \#E_{int}(\Gamma)$, or an affine hypersurface $\hat{X}_\Gamma \subset \mathbb{A}^n$, the affine cone over X_Γ. Thus, assuming we are in the stable range of sufficiently high D where $-n + D\ell/2 \geq 0$, for $\ell = b_1(\Gamma)$ the number of loops, one can regard the Feynman integral $U(\Gamma, p)$ as computed over the affine hypersurface complement $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$, but one can also reformulate it in terms of the projective hypersurface complement.

In order then to reformulate in projective space $\mathbb{P}^{n-1}$ integrals originally defined in affine space $\mathbb{A}^n$, we first recall some basic facts about differential forms on affine and projective spaces and their relation, following mostly [Dimca (1992)], see also [Gelfand, Gindikin, and Graev (1980)].

The projective analog of the volume form

$$\omega_n = dt_1 \wedge \cdots \wedge dt_n$$

is given by the form

$$\Omega = \sum_{i=1}^{n} (-1)^{i+1} t_i \, dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_n. \tag{3.42}$$

The relation between the volume form $dt_1 \wedge \cdots \wedge dt_n$ and the homogeneous form Ω of degree n of (3.42) is given by

$$\Omega = \Delta(\omega_n), \tag{3.43}$$

where $\Delta : \Omega^k \to \Omega^{k-1}$ is the operator of contraction with the Euler vector field

$$E = \sum_i t_i \frac{\partial}{\partial t_i}, \tag{3.44}$$

$$\Delta(\omega)(v_1, \ldots, v_{k-1}) = \omega(E, v_1, \ldots, v_{k-1}). \tag{3.45}$$

Let $\mathcal{R} = \mathbb{C}[t_1, \ldots, t_n]$ be the ring of polynomials of $\mathbb{A}^n$. Let $\mathcal{R}_m$ denote the subset of homogeneous polynomials of degree m. Similarly, let Ω^k denote the $\mathcal{R}$-module of k-forms on $\mathbb{A}^n$ and let Ω^k_m denote the subset of k-forms that are homogeneous of degree m.

Let f be a homogeneous polynomial function $f : \mathbb{A}^n \to \mathbb{A}$ of degree $\deg(f)$. Let $\pi : \mathbb{A}^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}$ be the standard projection $t = (t_1, \ldots, t_n) \mapsto \pi(t) = (t_1 : \cdots : t_n)$. We denote by $\hat{X}_f = \{t \in \mathbb{A}^n \,|\, f(t) = 0\}$ the affine hypersurface and by $X_f = \{\pi(t) \in \mathbb{P}^{n-1} \,|\, f(t) = 0\}$ the projective hypersurface, and by $\hat{\mathcal{D}}(f) = \mathbb{A}^n \smallsetminus \hat{X}_f$ and $\mathcal{D}(f) = \mathbb{P}^{n-1} \smallsetminus X_f$ the hypersurface complements. With the notation introduced above, we can always write a form $\omega \in \Omega^k(\mathcal{D}(f))$ as

$$\omega = \frac{\eta}{f^m}, \quad \text{with} \quad \eta \in \Omega^k_{m\deg(f)}. \tag{3.46}$$

We then have the following characterization of the pullback along $\pi : \hat{\mathcal{D}}(f) \to \mathcal{D}(f)$ of forms on $\mathcal{D}(f)$.

Proposition 3.4.1. (see [Dimca (1992)], p.180 and [Dolgachev (1982)]) *Given $\omega \in \Omega^k(\mathcal{D}(f))$, the pullback $\pi^*(\omega) \in \Omega^k(\hat{\mathcal{D}}(f))$ is characterized by the properties of being invariant under the $\mathbb{G}_m$ action on $\mathbb{A}^n \smallsetminus \{0\}$ and of satisfying $\Delta(\pi^*(\omega)) = 0$, where Δ is the contraction* (3.45) *with the Euler vector field E of* (3.44).

Thus, since the sequence

$$0 \to \Omega^n \xrightarrow{\Delta} \Omega^{n-1} \xrightarrow{\Delta} \cdots \xrightarrow{\Delta} \Omega^1 \xrightarrow{\Delta} \Omega^0 \to 0$$

is exact at all but the last term, one can write

$$\pi^*(\omega) = \frac{\Delta(\eta)}{f^m}, \quad \text{with} \quad \eta \in \Omega^k_{m\deg(f)}. \tag{3.47}$$

In particular, any $(n-1)$-form on $\mathcal{D}(f) \subset \mathbb{P}^{n-1}$ can be written as

$$\frac{h\Omega}{f^m}, \quad \text{with} \quad h \in \mathcal{R}_{m\deg(f)-n} \tag{3.48}$$

and with $\Omega = \Delta(dt_1 \wedge \cdots \wedge dt_n)$ the $(n-1)$-form (3.42), homogeneous of degree n.

We then obtain the following result formulating integrals in affine spaces in terms of integrals of pullbacks of forms from projective spaces.

Proposition 3.4.2. *Let $\omega \in \Omega^k_{m\deg(f)}$ be a closed k-form, which is homogeneous of degree $m\deg(f)$, and consider the form ω/f^m on $\mathbb{A}^n$. Let*

$\sigma \subset \mathbb{A}^n \smallsetminus \{0\}$ be a k-dimensional domain with boundary $\partial\sigma \neq \emptyset$. Then the integration of ω/f^m over σ satisfies

$$m\deg(f)\int_\sigma \frac{\omega}{f^m} = \int_{\partial\sigma} \frac{\Delta(\omega)}{f^m} + \int_\sigma df \wedge \frac{\Delta(\omega)}{f^{m+1}}. \tag{3.49}$$

Proof. Recall that we have ([Dimca (1992)], [Dolgachev (1982)])

$$d\left(\frac{\Delta(\omega)}{f^m}\right) = -\frac{\Delta(d_f\omega)}{f^{m+1}}, \tag{3.50}$$

where, for a form ω that is homogeneous of degree $m\deg(f)$,

$$d_f\omega = f\,d\omega - m\,df \wedge \omega. \tag{3.51}$$

Thus, we have

$$d\left(\frac{\Delta(\omega)}{f^m}\right) = -\frac{\Delta(d\omega)}{f^m} + m\frac{\Delta(df \wedge \omega)}{f^{m+1}}. \tag{3.52}$$

Since the form ω is closed, $d\omega = 0$, and we have

$$\Delta(df \wedge \omega) = \deg(f)\,f\,\omega - df \wedge \Delta(\omega), \tag{3.53}$$

we obtain from the above

$$d\left(\frac{\Delta(\omega)}{f^m}\right) = m\deg(f)\frac{\omega}{f^m} - \frac{df \wedge \Delta(\omega)}{f^{m+1}}. \tag{3.54}$$

By Stokes' theorem we have

$$\int_{\partial\sigma} \frac{\Delta(\omega)}{f^m} = \int_\sigma d\left(\frac{\Delta(\omega)}{f^m}\right).$$

Using (3.54) this gives

$$\int_{\partial\sigma} \frac{\Delta(\omega)}{f^m} = m\deg(f)\int_\sigma \frac{\omega}{f^m} - \int_\sigma \frac{df \wedge \Delta(\omega)}{f^{m+1}}.$$

which gives (3.49). □

In the range $-n + D(\ell+1)/2 \geq 0$ we consider the Feynman integral $U(\Gamma, p)$ as defined on the affine hypersurface complement $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$ and we use the result above to reformulate the parametric Feynman integrals in terms of integrals of forms that are pullbacks to $\mathbb{A}^n \smallsetminus \{0\}$ of forms on a hypersurface complement in $\mathbb{P}^{n-1}$. For simplicity, we remove here the divergent Γ-factor from the parametric Feynman integral and we concentrate on the residue given by the integration over the simplex σ as in (3.55) below.

Proposition 3.4.3. *Under the generic condition on the external momenta, and assuming that $-n + D\ell/2 \geq 0$, the parametric Feynman integral*

$$U(\Gamma, p) = \int_\sigma \frac{V_\Gamma^{-n+D\ell/2}\,\omega_n}{\Psi_\Gamma^{D/2}}, \tag{3.55}$$

with $V_\Gamma(t,p) = P_\Gamma(t,p)/\Psi_\Gamma(t,p)$, can be computed as

$$U(\Gamma, p) = \frac{1}{C(n, D, \ell)} \left(\int_{\partial\sigma} \pi^*(\eta) + \int_\sigma df \wedge \frac{\pi^*(\eta)}{f} \right), \tag{3.56}$$

where $\pi : \mathbb{A}^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}$ is the projection and η is the form on the hypersurface complement $\mathcal{D}(f)$ in $\mathbb{P}^{n-1}$ with

$$\pi^*(\eta) = \frac{\Delta(\omega)}{f^m}, \tag{3.57}$$

on $\mathbb{A}^n$, where $f = \Psi_\Gamma$ and $m = -n+D(\ell+1)/2$, and $\omega = P_\Gamma(t,p)^{-n+D\ell/2}\omega_n$ with $\omega_n = dt_1 \wedge \cdots \wedge dt_n$ the volume form on $\mathbb{A}^n$. The coefficient $C(n, D, \ell)$ in (3.56) *is given by*

$$C(n, D, \ell) = (-n + D(\ell + 1)/2)\ell. \tag{3.58}$$

Proof. Consider on $\mathbb{A}^n$ the form given by $\Delta(\omega)/f^m$, with f, m, and ω as above. We assume the condition of Definition 3.3.1, *i.e.* for a generic choice of the external momenta the polynomials P_Γ and Ψ_Γ have no common factor. First notice that, since the polynomial Ψ_Γ is homogeneous of degree ℓ and P_Γ is homogeneous of degree $\ell+1$, the form $\Delta(\omega)/f^m$ is $\mathbb{G}_m$-invariant on $\mathbb{A}^n \smallsetminus \{0\}$. Moreover, since it is of the form $\alpha = \Delta(\omega)/f^m$, it also satisfies $\Delta(\alpha) = 0$, hence it is the pullback of a form η on $\mathcal{D}(f) \subset \mathbb{P}^{n-1}$. Also notice that the domain of integration $\sigma \subset \mathbb{A}^n$, given by the simplex $\sigma = \sigma_n = \{\sum_i t_i = 1,\, t_i \geq 0\}$, is contained in a fundamental domain of the action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^n \smallsetminus \{0\}$.

Applying the result of Proposition 3.4.2 above, we obtain

$$\int_\sigma \frac{P_\Gamma(t,p)^{-n+D\ell/2}\, dt_1 \wedge \cdots \wedge dt_n}{\Psi_\Gamma^{-n+D(\ell+1)/2}} = \int_\sigma \frac{\omega}{f^m}$$

$$= \frac{1}{m \deg(f)} \Big(\int_{\partial\sigma} \frac{\Delta(\omega)}{f^m} + \int_\sigma df \wedge \frac{\Delta(\omega)}{f^{m+1}} \Big)$$

$$= C(n, D, \ell)^{-1} \left(\int_{\partial\sigma} \frac{P_\Gamma(t,p)^a \Delta(\omega_n)}{\Psi_\Gamma^m} + \int_\sigma df \wedge \frac{P_\Gamma(t,p)^a \Delta(\omega_n)}{\Psi_\Gamma^{m+1}} \right),$$

with $a = -n + D\ell/2$ and $m = -n + D(\ell + 1)/2$. The coefficient $C(n, D, \ell)$ is given by $C(n, D, \ell) = m \deg(f)$, with m and f as above, hence it is given by (3.58). $\square$

3.5 Non-isolated singularities

The graph hypersurfaces $X_\Gamma \subset \mathbb{P}^{n-1}$ defined by the vanishing of the polynomial $\Psi_\Gamma(t) = \det(M_\Gamma(t))$ usually have non-isolated singularities. This can easily be seen by the following observation.

Lemma 3.5.1. *Let Γ be a graph with $\deg \Psi_\Gamma > 2$. The singular locus of X_Γ is given by the intersection of cones over the hypersurfaces X_{Γ_e}, for $e \in E(\Gamma)$, where Γ_e is the graph obtained by removing the edge e of Γ. The cones $C(X_{\Gamma_e})$ do not intersect transversely.*

Proof. First observe that, since X_Γ is defined by a homogeneous equation $\Psi_\Gamma(t) = 0$, with Ψ_Γ a polynomial of degree m, the Euler formula $m\Psi_\Gamma(t) = \sum_e t_e \frac{\partial}{\partial t_e} \Psi_\Gamma(t)$ implies that $\cap_e Z(\partial_e \Psi_\Gamma) \subset X_\Gamma$, where $Z(\partial_e \Psi_\Gamma)$ is the zero locus of the t_e-derivative. Thus, the singular locus of X_Γ is just given by the equations $\partial_e \Psi_\Gamma = 0$. The variables t_e appear in the polynomial $\Psi_\Gamma(t)$ only with degree zero or one, hence the polynomial $\partial_e \Psi_\Gamma$ consists of only those monomials of Ψ_Γ that contain the variable t_e, where one sets $t_e = 1$. The resulting polynomial is therefore of the form Ψ_{Γ_e}, where Γ_e is the graph obtained from Γ by removing the edge e. In fact, one can see in terms of spanning trees that, if T is a spanning tree containing the edge e, then $T \smallsetminus e$ is no longer a spanning tree of Γ_e, so the corresponding terms disappear in passing from Ψ_Γ to Ψ_{Γ_e}, while if T is a spanning tree of Γ which does not contain e, then T is still a spanning tree of Γ_e and the corresponding monomial m_T of Ψ_{Γ_e} is the same as the monomial m_T in Ψ_Γ without the variable t_e. Thus, the zero locus $Z(\Psi_{\Gamma_e}) \subset \mathbb{P}^{n-1}$ is a cone $C(X_{\Gamma_e})$ over the graph hypersurface $X_{\Gamma_e} \subset \mathbb{P}^{n-2}$ with vertex at the coordinate point $v_e = (0, \ldots, 0, 1, 0, \ldots 0)$ with $t_e = 1$. To see that these cones do not intersect transversely, notice that, in the case where $\deg \Psi_\Gamma > 2$, given any two $C(X_{\Gamma_e})$ and $C(X_{\Gamma_{e'}})$, the vertex of one cone is contained in the graph hypersurface spanning the other cone. $\square$

In fact, the hypersurfaces X_Γ tend to have singularity loci of low codimension. As we observe again later, this has important consequences from the motivic viewpoint. Moreover, it makes it especially useful to adopt methods from singularity theory to study these hypersurfaces. For example, we are going to see in the following sections how characteristic classes of singular varieties can be employed in this context. These provide a way of measuring how singular a hypersurface is, as we discuss more in detail in §3.9 below. Ongoing work [Bergbauer and Rej (2009)] gives an analysis of the singular locus of the graph hypersurfaces using a formula for the Kirchhoff polynomials Ψ_Γ under insertion of subgraphs at vertices.

3.6 Cremona transformation and dual graphs

A useful tool to study the algebraic geometry of the projective graph hypersurfaces X_Γ is the Cremona transformation, which for planar graphs relates the hypersurface of a graph with that of its dual. This was pointed out in [Bloch (2007)] and we illustrate the principle here in the particular completely explicit case of the "banana graphs" computed in [Aluffi and Marcolli (2008a)]. Even for graphs that are non-planar, one can still use the image of the graph hypersurface under the Cremona transformation to derive useful information on its motivic nature, as shown in the recent results of [Bloch (2008)]. We follow [Aluffi and Marcolli (2008a)].

Definition 3.6.1. The standard Cremona transformation of $\mathbb{P}^{n-1}$ is the map

$$\mathcal{C} : (t_1 : \cdots : t_n) \mapsto \left(\frac{1}{t_1} : \cdots : \frac{1}{t_n}\right). \tag{3.59}$$

Let $\mathcal{G}(\mathcal{C})$ denote the closure of the graph of $\mathcal{C}$. Then $\mathcal{G}(\mathcal{C})$ is a subvariety of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ with projections

$$\begin{tikzcd} & \mathcal{G}(\mathcal{C}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^{n-1} \arrow[rr, dashed, "\mathcal{C}"] & & \mathbb{P}^{n-1} \end{tikzcd} \tag{3.60}$$

We introduce the following notation. Let $\mathcal{S}_n \subset \mathbb{P}^{n-1}$ be defined by the ideal

$$\mathcal{I}_{\mathcal{S}_n} = (t_1 \cdots t_{n-1}, t_1 \cdots t_{n-2} t_n, \ldots, t_1 t_3 \cdots t_n, t_2 \cdots t_n). \tag{3.61}$$

Also, let $\mathcal{L}$ be the hyperplane defined by the equation

$$\mathcal{L} = \{(t_1 : \cdots : t_n) \in \mathbb{P}^{n-1} \,|\, t_1 + \cdots + t_n = 0\}. \tag{3.62}$$

The Cremona transformation is a priori defined only away from the algebraic simplex of coordinate axes

$$\Sigma_n := \{(t_1 : \cdots : t_n) \in \mathbb{P}^{n-1} \mid \prod_i t_i = 0\} \subset \mathbb{P}^{n-1}, \tag{3.63}$$

though, as we now show, it is in fact well defined also on the general point of Σ_n, its locus of indeterminacies being only the singularity subscheme $\mathcal{S}_n$ of the algebraic simplex Σ_n.

Lemma 3.6.2. *Let $\mathcal{C}$, $\mathcal{G}(\mathcal{C})$, $\mathcal{S}_n$, and $\mathcal{L}$ be as above.*

(1) The indeterminacy locus of $\mathcal{C}$ is $\mathcal{S}_n$.
(2) $\pi_1 : \mathcal{G}(\mathcal{C}) \to \mathbb{P}^{n-1}$ is the blow-up along $\mathcal{S}_n$.
(3) $\mathcal{L}$ intersects every component of $\mathcal{S}_n$ transversely.
(4) Σ_n cuts out a divisor with simple normal crossings on $\mathcal{L}$.

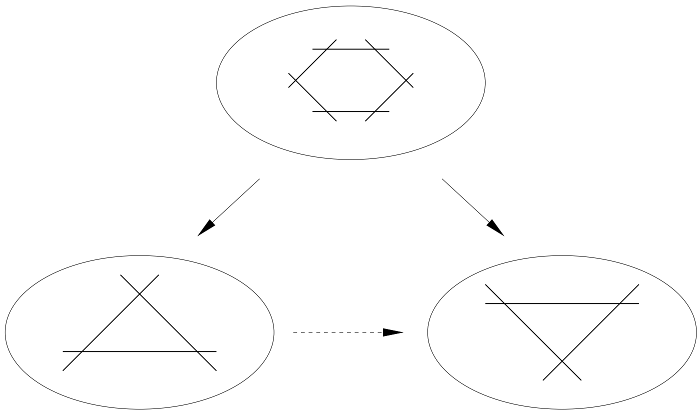
Proof. The components of the map defining $\mathcal{C}$ given in (3.59) can be rewritten as

$$(t_1 : \cdots : t_n) \mapsto (t_2 \cdots t_n : t_1 t_3 \cdots t_n : \cdots : t_1 \cdots t_{n-1}). \tag{3.64}$$

Using (3.64), the map $\pi_1 : \mathcal{G}(\mathcal{C}) \to \mathbb{P}^n$ may be identified with the blow-up of $\mathbb{P}^n$ along the subscheme $\mathcal{S}_n$ defined by the ideal $\mathcal{I}_{\mathcal{S}_n}$ of (3.61) generated by the partial derivatives of the equation of the algebraic simplex (*i.e.* the singular locus of Σ_n). The properties (3) and (4) then follow. □

Properties (3) and (4) of the proposition above only play a role implicitly in the proof of Proposition 3.7.1 below, so we leave here the details to the reader.

An example where it is easier to visualize the effect of the Cremona transformation is the case where $n = 3$. In this case, one can view the algebraic simplex Σ_3 in the projective plane $\mathbb{P}^2$ as a union of three lines, each two intersecting in a point. These three points, the zero-dimensional strata of Σ_3, are blown up in $\mathcal{G}(\mathcal{C})$ and each replaced by a line. The proper transforms of the 1-dimensional components of Σ_3 are then blown down by the other map π_2 of (3.60), so that the resulting image is again a triangle of three lines in $\mathbb{P}^2$. The Cremona transformation (the horizontal rational map in the diagram (3.60)) is an isomorphism of the complements of the two triangles.



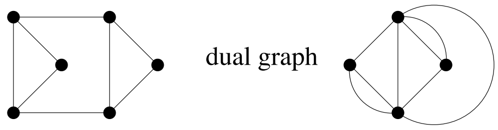
The *dual graph* $\Gamma^\vee$ of a planar graph Γ is defined by choosing an embedding $\iota : \Gamma \hookrightarrow S^2$ in the sphere and then taking as vertices $v \in V(\Gamma^\vee)$ a point in each connected component of $S^2 \smallsetminus \iota(\Gamma)$,

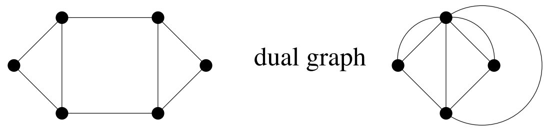
$$\#V(\Gamma^\vee) = b_0(S^2 \smallsetminus \iota(\Gamma)), \tag{3.65}$$

and edges $E(\Gamma^\vee)$ given by adding an edge between two vertices $v, w \in V(\Gamma^\vee)$ for each edge of Γ that is in the boundary between the regions of $S^2 \smallsetminus \iota(\Gamma)$ containing the points v and w. Thus, the dual graph has

$$\#E(\Gamma^\vee) = \#E(\Gamma). \tag{3.66}$$

Notice that it is somewhat misleading to talk about *the* dual graph $\Gamma^\vee$, since in fact this depends not only on Γ itself, but also on the choice of the embedding $\iota : \Gamma \hookrightarrow S^2$. One can give examples of different embeddings ι_1, ι_2 of the same graph Γ, for which the corresponding dual graphs $\Gamma_1^\vee$ and $\Gamma_2^\vee$ are topologically inequivalent, as the following figure shows.





It was shown in [Bloch (2007)] that the Cremona transformation relates the graph hypersurfaces of Γ and its dual $\Gamma^\vee$ in the following way (see also [Aluffi and Marcolli (2008a)]).

Lemma 3.6.3. *Suppose given a planar graph Γ with $\#E(\Gamma) = n$, with dual graph $\Gamma^\vee$. Then the graph polynomials satisfy*

$$\Psi_\Gamma(t_1, \dots, t_n) = (\prod_{e \in E(\Gamma)} t_e) \; \Psi_{\Gamma^\vee}(t_1^{-1}, \dots, t_n^{-1}), \tag{3.67}$$

hence the graph hypersurfaces are related by the Cremona transformation $\mathcal{C}$ of (3.59),

$$\mathcal{C}(X_\Gamma \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n)) = X_{\Gamma^\vee} \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n). \tag{3.68}$$

Proof. This follows from the combinatorial identity

$$\begin{aligned} \Psi_\Gamma(t_1, \dots, t_n) &= \textstyle\sum_{T \subset \Gamma} \prod_{e \notin E(T)} t_e \\ &= \textstyle(\prod_{e \in E(\Gamma)} t_e) \sum_{T \subset \Gamma} \prod_{e \in E(T)} t_e^{-1} \\ &= \textstyle(\prod_{e \in E(\Gamma)} t_e) \sum_{T' \subset \Gamma^\vee} \prod_{e \notin E(T')} t_e^{-1} \\ &= \textstyle(\prod_{e \in E(\Gamma)} t_e) \Psi_{\Gamma^\vee}(t_1^{-1}, \dots, t_n^{-1}). \end{aligned}$$

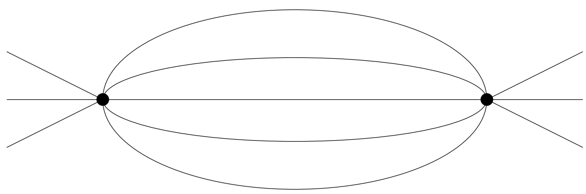
The third equality uses the fact that $\#E(\Gamma) = \#E(\Gamma^\vee)$ and $\#V(\Gamma^\vee) = b_0(S^2 \smallsetminus \Gamma)$, so that $\deg \Psi_\Gamma + \deg \Psi_{\Gamma^\vee} = \#E(\Gamma)$, and the fact that there is a bijection between complements of spanning tree T in Γ and spanning trees T' in $\Gamma^\vee$ obtained by shrinking the edges of T in Γ and taking the dual graph of the resulting connected graph. Written in the coordinates $(s_1 : \cdots : s_n)$ of the target $\mathbb{P}^{n-1}$ of the Cremona transformation, the identity (3.67) gives

$$\Psi_\Gamma(t_1, \dots, t_n) = (\prod_{e \in E(\Gamma^\vee)} s_e^{-1}) \Psi_{\Gamma^\vee}(s_1, \dots, s_n)$$

from which (3.68) follows. $\square$

3.7 Classes in the Grothendieck ring

An example of an infinite family of Feynman graphs for which the classes in the Grothendieck ring $K_0(\mathcal{V})$ can be computed explicitly was considered in [Aluffi and Marcolli (2008a)]. These are known as the *banana graphs*: for $n \geq 2$ the graph Γ_n in this family has two vertices of valence n and n parallel edges between them.



The banana graph Γ_n has graph polynomial

$$\Psi_\Gamma(t) = t_1 \cdots t_n (\frac{1}{t_1} + \cdots + \frac{1}{t_n}).$$

Thus, the parametric integral in this case is

$$\int_{\sigma_n} \frac{(t_1 \cdots t_n)^{(\frac{D}{2}-1)(n-1)-1}\, \omega_n}{\Psi_\Gamma(t)^{(\frac{D}{2}-1)n}},$$

up to an overall multiplicative factor proportional to the sum of external momenta at a vertex.

We gave in [Aluffi and Marcolli (2008a)] an explicit formula for the class in the Grothendieck ring of the graph hypersurfaces for the banana graphs. This is given by the following result.

Proposition 3.7.1. *The class in the Grothendieck ring of the hypersurface X_{Γ_n} of the n-th banana graph is explicitly given by the formula*

$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L}-1)^n - (-1)^n}{\mathbb{L}} - n\,(\mathbb{L}-1)^{n-2}. \tag{3.69}$$

In particular, this formula shows that, in this example, the class $[X_{\Gamma_n}]$ manifestly lies in the mixed Tate part $\mathbb{Z}[\mathbb{L}]$ of the Grothendieck ring. We sketch the proof here and refer the reader to [Aluffi and Marcolli (2008a)] for more details.

Proof. The formula (3.69) is proved using the method of Cremona transformation and dual graphs described above. In fact, one observes easily that, in the case of the banana graphs, the dual graph $\Gamma_n^\vee$ is simply a polygon with n sides, and therefore the associated graph hypersurface $X_{\Gamma_n^\vee} = \mathcal{L}$ is just a hyperplane in $\mathbb{P}^{n-1}$.

One then checks that the complement in this hyperplane of the algebraic simplex has class

$$[\mathcal{L} \smallsetminus (\mathcal{L} \cap \Sigma_n)] = [\mathcal{L}] - [\mathcal{L} \cap \Sigma_n] = \frac{\mathbb{T}^{n-1} - (-1)^{n-1}}{\mathbb{T}+1}$$

where $\mathbb{T} = [\mathbb{G}_m] = [\mathbb{A}^1] - [\mathbb{A}^0]$ is the class of the multiplicative group. Moreover, one finds that $X_{\Gamma_n} \cap \Sigma_n = \mathcal{S}_n$, the scheme of singularities of Σ_n. The latter has class

$$[\mathcal{S}_n] = [\Sigma_n] - n\mathbb{T}^{n-2}.$$

This then gives

$$[X_{\Gamma_n}] = [X_{\Gamma_n} \cap \Sigma_n] + [X_{\Gamma_n} \smallsetminus (X_{\Gamma_n} \cap \Sigma_n)],$$

where one uses the Cremona transformation to identify

$$[X_{\Gamma_n}] = [\mathcal{S}_n] + [\mathcal{L} \smallsetminus (\mathcal{L} \cap \Sigma_n)].$$

$\square$

In particular, since the class in the Grothendieck ring is a universal Euler characteristic, this calculation also yields as a consequence the value for the topological Euler characteristic of the (complex) variety $X_{\Gamma_n}(\mathbb{C})$, which is of the form

$$\chi(X_{\Gamma_n}) = n + (-1)^n,$$

for $n \geq 3$, as one can see from the fact that the Euler characteristic is a ring homomorphism from $K_0(\mathcal{V})$ to $\mathbb{Z}$ and that in (3.69) $\mathbb{L} = [\mathbb{A}^1]$ has Euler characteristic $\chi(\mathbb{A}^1) = 1$. A different computation of the Euler characteristic $\chi(X_{\Gamma_n})$, based on the use of characteristic classes of singular varieties, is also given in [Aluffi and Marcolli (2008a)] along with the more complicated explicit computation of the Chern–Schwartz–MacPherson characteristic class of the hypersurfaces X_{Γ_n}.

More generally, the use of the Cremona transformation method can lead to interesting results even in the case where the graphs are not necessarily planar. In fact, one can still consider a *dual hypersurface* $X_\Gamma^\vee$ obtained as the image of X_Γ under the Cremona transformation and still have an isomorphism of X_Γ and $X_\Gamma^\vee$ away from the algebraic simplex Σ_n,

$$X_\Gamma \smallsetminus (X_\Gamma \cap \Sigma_n) \overset{\mathcal{C}}{\simeq} X_\Gamma^\vee \smallsetminus (X_\Gamma^\vee \cap \Sigma_n),$$

although in non-planar cases $X_\Gamma^\vee$ is no longer identified with the graph hypersurface of a dual graph $X_{\Gamma^\vee}$. By applying this method to the complete graph Γ_N on N vertices, Bloch proved in [Bloch (2008)] a very interesting result showing that, although the individual hypersurfaces X_Γ of Feynman graphs are not always mixed Tate motives, if one sums the classes over graphs with a fixed number of vertices one obtains a class that is always in the Tate part $\mathbb{Z}[\mathbb{L}]$ of the Grothendieck ring. More precisely, it is shown in [Bloch (2008)] that the class

$$S_N = \sum_{\#V(\Gamma)=N} [X_\Gamma] \frac{N!}{\#\mathrm{Aut}(\Gamma)},$$

where the sum is over all graphs with a fixed number of vertices, is always in $\mathbb{Z}[\mathbb{L}]$. This shows that there will be very interesting cancellations between the classes $[X_\Gamma]$ at least for a sufficiently large number of loops, where one knows by the general result of [Belkale and Brosnan (2003a)] that non-mixed-Tate contributions will eventually appear.

This result fits in well with the general fact that, in quantum field theory, individual Feynman graphs do not represent observable physical processes and only sums over graphs (usually with fixed external edges and external momenta) can be physically meaningful. Although in the more physical setting it would be natural to sum over graphs with a given number of loops and with given external momenta, rather than over a fixed number of vertices, the result of [Bloch (2008)] suggests that a more appropriate formulation of the conjecture on Feynman integrals and motives should perhaps be given directly in terms that involve the full expansion of perturbative quantum field theory, with sums over graphs, rather than in terms of individual graphs. This approach via families of graphs also fits in well with the treatment of gauge theories and Slavonov–Taylor identities within the context of the Connes–Kreimer approach to renormalization via Hopf algebras, as in [van Suijlekom (2007)], [van Suijlekom (2006)] and also [Kreimer and van Suijlekom (2009)] in the setting of Dyson–Schwinger equations.

3.8 Motivic Feynman rules

We have discussed above a definition of "abstract Feynman rules" based on the multiplicative properties (3.31) and (3.32) that express the expectation value of Feynman integrals associated to arbitrary Feynman graphs of a given scalar quantum field theory first in terms of connected graphs and then in terms of 1PI graphs.

This more abstract definition of Feynman rules allows one to make sense of algebro-geometric and motivic $U(\Gamma)$, with the same formal properties as the original Feynman integrals with respect to the combinatorics of graphs. As we discuss later on in the book, this will be closely related to defining Feynman rules in terms of characters of the Connes–Kreimer Hopf algebra of Feynman graphs and to the abstract form of the Birkhoff factorization that gives the renormalization procedure for Feynman integrals.

One can first observe, as in [Aluffi and Marcolli (2008b)], that the hypersurface complement $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$ where the parametric Feynman integral is computed behaves itself like a Feynman integral, in the sense that it satisfies the multiplicative properties (3.31) and (3.32). More precisely, we have seen in Lemma 3.2.2 above that for a disjoint union of graphs $\Gamma = \Gamma_1 \cup \Gamma_2$ the hypersurface complement splits multiplicatively as

$$\mathbb{A}^n \smallsetminus \hat{X}_\Gamma = (\mathbb{A}^{n_1} \smallsetminus \hat{X}_{\Gamma_1}) \times (\mathbb{A}^{n_2} \smallsetminus \hat{X}_{\Gamma_2}).$$

Lemma 3.2.2 shows that the hypersurface complement $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$ satisfies the multiplicative property (3.31), when we think of the Cartesian product $(\mathbb{A}^{n_1} \smallsetminus \hat{X}_{\Gamma_1}) \times (\mathbb{A}^{n_2} \smallsetminus \hat{X}_{\Gamma_2})$ as defining the product operation in a suitable commutative ring. We will specify the ring more precisely below as a refinement of the Grothendieck ring of varieties, namely the *ring of immersed conical varieties* introduced in [Aluffi and Marcolli (2008b)]. One can similarly see that the second property of Feynman rules, the multiplicative property (3.32) reducing connected graphs to 1PI graphs and inverse propagators, is also satisfied by the hypersurface complements.

Lemma 3.8.1. *For a connected graph Γ that is obtained by inserting 1PI graphs Γ_v at the vertices of a finite tree T, the hypersurface complement satisfies*

$$\mathbb{A}^n \smallsetminus \hat{X}_\Gamma = \mathbb{A}^{\#E(T)} \times \prod_{v \in V(T)} (\mathbb{A}^{\#E_{int}(\Gamma_v)} \smallsetminus \hat{X}_{\Gamma_v}). \tag{3.70}$$

Proof. If Γ is a forest with $n = \#E(\Gamma)$, then $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma = \mathbb{A}^n$. Also notice that if Γ_1 and Γ_2 are graphs joined at a single vertex, then the same argument given in Lemma 3.2.2 for the disjoint union still gives the desired result. Thus, in the case of a connected graphs Γ obtained by inserting 1PI graphs at vertices of a tree, these two observations combine to give

$$(\mathbb{A}^{n_1} \smallsetminus \hat{X}_{\Gamma_{v_1}}) \times \cdots \times (\mathbb{A}^{n_{\#V(T)}} \smallsetminus \hat{X}_{\Gamma_{v_{\#V(T)}}}) \times \mathbb{A}^{\#E(T)}. \qquad \square$$

In the above, one can consider the hypersurface complements $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$ modulo certain equivalence relations. For example, if these varieties are considered up to isomorphism, and one imposes the inclusion–exclusion relation $[X] = [X \smallsetminus Y] + [Y]$ for closed subvarieties $Y \subset X$, one can define a Feynman rule with values in the Grothendieck ring of varieties

$$\mathbb{U}(\Gamma) := [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = [\mathbb{A}^n] - [\hat{X}_\Gamma] \in K_0(\mathcal{V}). \tag{3.71}$$

This satisfies (3.31) and (3.32) as an immediate consequence of Lemma 3.2.2 and Lemma 3.8.1.

The relation between the classes in the Grothendieck ring of the affine and projective hypersurface complements is given by the formula

$$[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = (\mathbb{L} - 1)[\mathbb{P}^{n-1} \smallsetminus X_\Gamma],$$

which expresses the fact that $\hat{X}_\Gamma$ is the affine cone over X_Γ.

One can also impose a weaker equivalence relation, by identifying varieties not up to isomorphism, but only up to linear changes of coordinates in an ambient affine space. This leads to the following refinement of the Grothendieck ring of varieties ([Aluffi and Marcolli (2008b)]).

Definition 3.8.2. The *ring of immersed conical varieties* $\mathcal{F}_\mathbb{K}$ is generated by classes $[V]$ of equivalence under linear coordinate changes of varieties $V \subset \mathbb{A}^N$ defined by homogeneous ideals (hence the name "conical") immersed in some arbitrarily large affine space, with the inclusion-exclusion and product relations

$$[V \cup W] = [V] + [W] - [V \cap W]$$

$$[V] \cdot [W] = [V \times W].$$

By imposing equivalence under isomorphisms one falls back on the usual Grothendieck ring $K_0(\mathcal{V})$.

Thus, one can define an $\mathcal{R}$-valued algebro-geometric Feynman rule, for a given commutative ring $\mathcal{R}$, as in [Aluffi and Marcolli (2008b)] in terms of a ring homomorphism $I : \mathcal{F} \to \mathcal{R}$ by setting

$$\mathbb{U}(\Gamma) := I([\mathbb{A}^n]) - I([\hat{X}_\Gamma])$$

and by taking as value of the *inverse propagator*

$$\mathbb{U}(L) = I([\mathbb{A}^1]).$$

This then satisfies both (3.31) and (3.32). The ring $\mathcal{F}$ is then the receptacle of the universal algebro-geometric Feynman rule given by

$$\mathbb{U}(\Gamma) = [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] \in \mathcal{F}.$$

A Feynman rule defined in this way is *motivic* if the homomorphism $I : \mathcal{F} \to \mathcal{R}$ factors through the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$. In this case the inverse propagator is the Lefschetz motive (or the propagator is the Tate motive).

The reason for working with $\mathcal{F}_{\mathbb{K}}$ instead is that it allowed us in [Aluffi and Marcolli (2008b)] to construct invariants of the graph hypersurfaces that behave like algebro-geometric Feynman rules and that measure to some extent how singular these varieties are, and which do not factor through the Grothendieck ring, since they contain specific information on how the $\hat{X}_\Gamma$ are embedded in the ambient affine space $\mathbb{A}^{\#E_{int}(\Gamma)}$.

3.9 Characteristic classes and Feynman rules

In the case of compact smooth varieties, the Chern classes of the tangent bundle can be written as a class $c(V) = c(TV) \cap [V]$ in homology whose degree of the zero dimensional component satisfies the Poincaré–Hopf theorem $\int c(TV) \cap [V] = \chi(V)$, which gives the topological Euler characteristic of the smooth variety.

Chern classes for singular varieties that generalize this property were introduced independently, following two different approaches, in [Schwartz (1965)] and [MacPherson (1974)]. The two definitions were later proved to be equivalent.

The approach followed by Marie Hélène Schwartz generalized the definition of Chern classes as the homology classes of the loci where a family of $k+1$ vector fields become linearly dependent (for the lowest degree case one reads the Poincaré–Hopf theorem as saying that the Euler characteristic measures where a single vector field has zeros). In the case of singular varieties a generalization is obtained, provided that one assigns some radial conditions on the vector fields with respect to a stratification with good properties.

The approach followed by Robert MacPherson is instead based on functoriality. The functor $\mathbb{F}$ of constructible functions assigns to an algebraic variety V the $\mathbb{Z}$-module $\mathbb{F}(V)$ spanned by the characteristic functions 1_W of subvarieties $W \subset V$ and to a proper morphism $f : V \to V'$ the map $f_* : \mathbb{F}(V) \to \mathbb{F}(V')$ defined by $f_*(1_W)(p) = \chi(W \cap f^{-1}(p))$, with χ the Euler characteristic.

A conjecture of Grothendieck–Deligne predicted the existence of a unique natural transformation c_* between the functor $\mathbb{F}$ and the homology (or Chow group) functor, which in the smooth case agrees with $c_*(1_V) = c(TV) \cap [V]$.

MacPherson constructed this natural transformation in terms of data of Mather classes and local Euler obstructions and defined in this way what is usually referred to now as the Chern–Schwartz–MacPherson (CSM) characteristic classes of (singular) algebraic varieties $c_{SM}(X) = c_*(1_X)$.

It is convenient here to work inside an ambient space, for instance an ambient projective space, so that one can regard $c_*(1_X)$ as taking values in the homology (or Chow group) of the ambient $\mathbb{P}^N$, so that the characteristic class $c_{SM}(X) = c_*(1_X)$ can be computed for X an arbitrary locally closed subset of $\mathbb{P}^N$, without requiring a compactness hypothesis. This is important in order to have an inclusion-exclusion relation for these classes, as explained below.

The results of [Aluffi (2006)] show that, in fact, it is possible to compute these classes without having to use Mather classes and Euler obstructions that are usually very difficult to compute. Most notably, these characteristic classes satisfy an inclusion-exclusion formula for a closed $Y \subset X$,

$$c_{SM}(X) = c_{SM}(Y) + c_{SM}(X \smallsetminus Y),$$

but are not invariant under isomorphism, hence they are naturally defined on classes in $\mathcal{F}_{\mathbb{K}}$ but not on $K_0(\mathcal{V}_{\mathbb{K}})$.

For a smooth locally closed X in some ambient projective space $\mathbb{P}^N$, the class $c_{SM}(X)$ can be obtained by a computation using Chern classes of sheaves of differentials with logarithmic poles in a resolution of the closure $\bar{X}$ of X. Then inclusion-exclusion gives a way to compute any embedded c_{SM} without recourse to MacPherson's Euler obstructions or Chern-Mather classes.

The CSM classes give good information on the singularities of a variety: for example, in the case of hypersurfaces with isolated singularities, they can be expressed in terms of Milnor numbers.

To construct a Feynman rule out of these Chern classes, one uses the following procedure, [Aluffi and Marcolli (2008b)]. Given a variety $\hat{X} \subset \mathbb{A}^N$, one can view it as a locally closed locus in $\mathbb{P}^N$, hence one can apply to its characteristic function $1_{\hat{X}}$ the natural transformation c_* that gives an element in the Chow group $A(\mathbb{P}^N)$ or in the homology $H_*(\mathbb{P}^N)$. This gives as a result a class of the form

$$c_*(1_{\hat{X}}) = a_0[\mathbb{P}^0] + a_1[\mathbb{P}^1] + \cdots + a_N[\mathbb{P}^N].$$

One then defines an associated polynomial given by

$$G_{\hat{X}}(T) := a_0 + a_1 T + \cdots + a_N T^N.$$

It is in fact independent of N as it stops in degree equal to $\dim \hat{X}$. It is by construction invariant under linear changes of coordinates. It also satisfies an inclusion-exclusion property coming from the fact that the classes c_{SM} satisfy inclusion-exclusion, namely

$$G_{\hat{X}\cup\hat{Y}}(T) = G_{\hat{X}}(T) + G_{\hat{Y}}(T) - G_{\hat{X}\cap\hat{Y}}(T)$$

It is a more delicate result to show that it is multiplicative, namely that the following holds.

Theorem 3.9.1. *The polynomials $G_{\hat{X}}(T)$ satisfy*

$$G_{\hat{X}\times\hat{Y}}(T) = G_{\hat{X}}(T) \cdot G_{\hat{Y}}(T).$$

Proof. The proof of this fact is obtained in [Aluffi and Marcolli (2008b)] using an explicit formula for the CSM classes of joins in projective spaces, where the join $J(X,Y) \subset \mathbb{P}^{m+n-1}$ of two $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$ is defined as the set of

$$(sx_1 : \cdots : sx_m : ty_1 : \cdots : ty_n), \quad \text{with} \quad (s:t) \in \mathbb{P}^1,$$

and is related to the product in affine spaces by the property that the product $\hat{X} \times \hat{Y}$ of the affine cones over X and Y is the affine cone over $J(X,Y)$. One has

$$c_*(1_{J(X,Y)}) = ((f(H) + H^m)(g(H) + H^n) - H^{m+n}) \cap [\mathbb{P}^{m+n-1}],$$

where $c_*(1_X) = H^n f(H) \cap [\mathbb{P}^{n+m-1}]$ and $c_*(1_Y) = H^m g(H) \cap [\mathbb{P}^{n+m-1}]$, as functions of the hyperplane class H in $\mathbb{P}^{n+m-1}$. We refer the reader to [Aluffi and Marcolli (2008b)] for more details. □

The resulting multiplicative property of the polynomials $G_{\hat{X}}(T)$ shows that one has a ring homomorphism $I_{CSM} : \mathcal{F} \to \mathbb{Z}[T]$ defined by

$$I_{CSM}([\hat{X}]) = G_{\hat{X}}(T)$$

and an associated Feynman rule

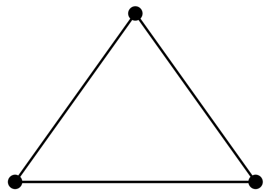
$$\mathbb{U}_{CSM}(\Gamma) = C_\Gamma(T) = I_{CSM}([\mathbb{A}^n]) - I_{CSM}([\hat{X}_\Gamma]).$$

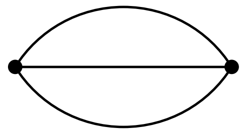
This is not motivic, *i.e.* it does not factor through the Grothendieck ring $K_0(\mathcal{V}_\mathbb{K})$, as can be seen by the example given in [Aluffi and Marcolli (2008b)] of two graphs (see the figure below) that have different $\mathbb{U}_{CSM}(\Gamma)$,

$$C_{\Gamma_1}(T) = T(T+1)^2 \quad C_{\Gamma_2}(T) = T(T^2+T+1)$$

but the same hypersurface complement class in the Grothendieck ring,

$$[\mathbb{A}^n \smallsetminus \hat{X}_{\Gamma_i}] = [\mathbb{A}^3] - [\mathbb{A}^2] \in K_0(\mathcal{V}).$$





There is an interesting *positivity* property that the CSM classes of the graph hypersurfaces X_Γ appear to satisfy, namely the coefficients of all the powers H^k in the expression of the CMS class of X_Γ as a polynomial in H are positive. This was observed in [Aluffi and Marcolli (2008a)] on the basis of numerical computations of these classes, done using the program of [Aluffi (2003)], for sample graphs. As the graphs that are accessible to computer calculations of the corresponding CSM classes are necessarily combinatorially simple (and all planar), it is at present not known whether the observed positivity phenomenon holds more generally. This property may be related to other positivity phenomena observed for CSM classes of similarly combinatorial objects such as Schubert varieties [Aluffi and Mihalcea (2006)]. We see in the last chapter that, in the case of quantum field theories on noncommutative spacetimes and the corresponding modification of the graph polynomials, positivity fails in non-planar cases.

3.10 Deletion-contraction relation

We report here briefly the recent results of [Aluffi and Marcolli (2009b)] on deletion–contraction relations for motivic Feynman rules.

We have seen above that one can construct a polynomial invariant $C_\Gamma(T)$ of graphs that is an algebro-geometric Feynman rule in the sense of [Aluffi and Marcolli (2008b)], which does not factor through the Grothendieck ring of varieties but satisfies an inclusion–exclusion principle. The polynomial $C_\Gamma(T)$ is defined in terms of CSM classes of the graph hypersurface complement.

Several well known examples of polynomial invariants of graphs are examples of Tutte–Grothendieck invariants and are obtained as specializations of the Tutte polynomial of graphs, and they all have the property that they satisfy deletion–contraction relations, which make it possible to compute the invariant indictively from simpler graphs. These relations express the invariant of a given graph in terms of the invariants of the graphs obtained by deleting or contracting edges.

The results of [Aluffi and Marcolli (2009b)] investigate to what extent motivic and algebro-geometric Feynman rules satisfy deletion–contraction relations. Since such relations make it possible to control the invariant for more complicated graphs in terms of combinatorially simpler ones, the validity or failure of such relations can provide useful information that can help to identify where the motivic complexity of the graph hypersurfaces increases beyond the mixed Tate case (see §3.12 below).

In particular, it was shown in [Aluffi and Marcolli (2009b)] that the polynomial $C_\Gamma(T)$ is *not* a specialization of the Tutte polynomial. However, a form of deletion–contraction relation does hold for the motivic Feynman rule $[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] \in K_0(\mathcal{V})$. The latter, unlike the relation satisfied by Tutte–Grothendieck invariants, contains a term expressed as the class of an intersection of hypersurfaces which does not appear to be easily controllable in terms of combinatorial information alone. However, this form of deletion–contraction relations suffices to obtain explicit recursive relation for certain operations on graphs, such as replacing an edge by multiple parallel copies.

We start by recalling the case of the Tutte polynomial of graphs and the form of the deletion–contraction relation in that well known classical case. The Tutte polynomial $\mathcal{T}_\Gamma \in \mathbb{C}[x,y]$ of finite graphs Γ is completely determined by the following properties:

- If $e \in E(\Gamma)$ is neither a looping edge nor a bridge the deletion–contraction relation holds:
$$\mathcal{T}_\Gamma(x,y) = \mathcal{T}_{\Gamma\smallsetminus e}(x,y) + \mathcal{T}_{\Gamma/e}(x,y). \tag{3.72}$$
- If $e \in E(\Gamma)$ is a looping edge then
$$\mathcal{T}_\Gamma(x,y) = y\mathcal{T}_{\Gamma/e}(x,y).$$
- If $e \in E(\Gamma)$ is a bridge then
$$\mathcal{T}_\Gamma(x,y) = x\mathcal{T}_{\Gamma\smallsetminus e}(x,y)$$
- If Γ has no edges then $\mathcal{T}_\Gamma(x,y) = 1$.

Here we call an edge e a "bridge" if the removal of e disconnects the graph Γ, and a "looping edge" if e starts and ends at the same vertex. Tutte–Grothendieck invariants are specializations of the Tutte polynomial. Among them one has well known invariants such as the chromatic polynomial and the Jones polynomial.

A first observation of [Aluffi and Marcolli (2009b)] is that the Tutte polynomial can be regarded as a Feynman rule, in the sense that it has the right multiplicative properties over disjoint unions and over decompositions of connected graphs in terms of trees and 1PI graphs.

Proposition 3.10.1. *The Tutte polynomial invariant defines an abstract Feynman rule with values in the polynomial ring* $\mathbb{C}[x,y]$*, by assigning*

$$U(\Gamma) = \mathcal{T}_\Gamma(x,y), \qquad \textit{with inverse propagator} \qquad U(L) = x. \tag{3.73}$$

Proof. One can first observe that the properties listed above for the Tutte polynomial determine the closed form

$$\mathcal{T}_\Gamma(x,y) = \sum_{\gamma\subset\Gamma} (x-1)^{\#V(\Gamma)-b_0(\Gamma)-(\#V(\gamma)-b_0(\gamma))} (y-1)^{\#E(\gamma)-\#V(\gamma)+b_0(\gamma)}, \tag{3.74}$$

where the sum is over subgraphs $\gamma \subset \Gamma$ with vertex set $V(\gamma) = V(\Gamma)$ and edge set $E(\gamma) \subset E(\Gamma)$. This can be written equivalently as

$$\mathcal{T}_\Gamma(x,y) = \sum_{\gamma\subset\Gamma} (x-1)^{b_0(\gamma)-b_0(\Gamma)} (y-1)^{b_1(\gamma)}.$$

This is sometime referred to as the "sum over states" formula for the Tutte polynomial.

The multiplicative property under disjoint unions of graphs is clear from the closed expression (3.74), since for $\Gamma = \Gamma_1 \cup \Gamma_2$ we can identify subgraphs $\gamma \subset \Gamma$ with $V(\gamma) = V(\Gamma)$ and $E(\gamma) \subset E(\Gamma)$ with all possible pairs of subgraphs (γ_1,γ_2) with $V(\gamma_i) = V(\Gamma_i)$ and $E(\gamma_i) \subset E(\Gamma_i)$, with $b_0(\gamma) = b_0(\gamma_1) + b_0(\gamma_2)$, $\#V(\Gamma) = \#V(\Gamma_1) + \#V(\Gamma_2)$, and $\#E(\gamma) = \#E(\gamma_1) + \#E(\gamma_2)$. Thus, we get

$$\begin{aligned} \mathcal{T}_\Gamma(x,y) &= \textstyle\sum_{\gamma=(\gamma_1,\gamma_2)} (x-1)^{b_0(\gamma_1)+b_0(\gamma_2)-b_0(\Gamma)} (y-1)^{b_1(\gamma_1)+b_1(\gamma_2)} \\ &= \mathcal{T}_{\Gamma_1}(x,y)\, \mathcal{T}_{\Gamma_2}(x,y). \end{aligned}$$

The property for connected and 1PI graphs follows from the fact that, when writing a connected graph in the form $\Gamma = \cup_{v\in V(T)} \Gamma_v$, with Γ_v 1PI graphs inserted at the vertices of the tree T, the internal edges of the tree are all bridges in the resulting graph, hence the property of the Tutte polynomial for the removal of bridges gives

$$\mathcal{T}_\Gamma(x,y) = x^{\#E_{int}(T)}\ \mathcal{T}_{\Gamma\smallsetminus\cup_{e\in E_{int}(T)} e}(x,y).$$

Then one obtains an abstract Feynman rule with values in $\mathcal{R} = \mathbb{C}[x,y]$ of the form (3.73). $\square$

It follows that Tutte–Grothendieck invariants such as the chromatic and Jones polynomials can also be regarded as Feynman rules.

One can then observe, however, that the algebro–geometric Feynman rule

$$C_\Gamma(T) = I_{CSM}([\mathbb{A}^n \smallsetminus \hat{X}_\Gamma])$$

that we discussed in the previous section is not a specialization of the Tutte polynomial.

Proposition 3.10.2. *The polynomial invariant $C_\Gamma(T)$ is not a specialization of the Tutte polynomial.*

Proof. We show that one cannot find functions $x = x(T)$ and $y = y(T)$ such that

$$C_\Gamma(T) = \mathcal{T}_\Gamma(x(T), y(T)).$$

First notice that, if $e \in E(\Gamma)$ is a bridge, the polynomial $C_\Gamma(T)$ satisfies the relation

$$C_\Gamma(T) = (T+1) C_{\Gamma \smallsetminus e}(T). \tag{3.75}$$

In fact, $(T+1)$ is the inverse propagator of the algebro-geometric Feynman rule $U(\Gamma) = C_\Gamma(T)$ and the property of abstract Feynman rules for 1PI graphs connected by a bridge gives (3.75). In the case where $e \in E(\Gamma)$ is a looping edge, we have

$$C_\Gamma(T) = T\ C_{\Gamma/e}(T). \tag{3.76}$$

In fact, adding a looping edge to a graph corresponds, in terms of graph hypersurfaces, to taking a cone on the graph hypersurface and intersecting it with the hyperplane defined by the coordinate of the looping edge. This implies that the universal algebro-geometric Feynman rule with values in the Grothendieck ring $\mathcal{F}$ of immersed conical varieties satisfies

$$\mathbb{U}(\Gamma) = ([\mathbb{A}^1] - 1)\mathbb{U}(\Gamma/e)$$

if e is a looping edge of Γ and $\mathbb{U}(\Gamma) = [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] \in \mathcal{F}$. The property (3.76) then follows since the image of the class $[\mathbb{A}^1]$ is the inverse propagator $(T+1)$. (See §3.9 above and Proposition 2.5 and §2.2 of [Aluffi and Marcolli (2008b)].)

This implies that, if $C_\Gamma(T)$ has to be a specialization of the Tutte polynomial, the relations for bridges and looping edges imply that one has to identify $x(T) = T+1$ and $y(T) = T$. However, this is not compatible with the behavior of the invariant $C_\Gamma(T)$ on more complicated graphs. For example, for the triangle graph one has $C_\Gamma(T) = T(T+1)^2$ while the specialization $\mathcal{T}_\Gamma(x(T), y(T)) = (T+1)^2 + (T+1) + T$. $\square$

One can then investigate what kind of deletion–contraction relations hold in the case of motivic and algebro–geometric Feynman rules. An answer is given in [Aluffi and Marcolli (2009b)] for the motivic case, that is, for the Feynman rule

$$\mathbb{U}(\Gamma) = [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] \in K_0(\mathcal{V}).$$

Theorem 3.10.3. *Let Γ be a graph with $n > 1$ edges.*

(1) If $e \in E(\Gamma)$ is neither a bridge nor a looping edge then the following deletion–contraction relation holds:

$$[\mathbb{A}^n \smallsetminus \widehat{X}_\Gamma] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus (\widehat{X}_{\Gamma \smallsetminus e} \cap \widehat{X}_{\Gamma/e})] - [\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma \smallsetminus e}]. \quad (3.77)$$

(2) If the edge e is a bridge in Γ, then

$$[\mathbb{A}^n \smallsetminus \widehat{X}_\Gamma] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma \smallsetminus e}] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma/e}]. \quad (3.78)$$

(3) If e is a looping edge in Γ, then

$$[\mathbb{A}^n \smallsetminus \widehat{X}_\Gamma] = (\mathbb{L} - 1) \cdot [\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma \smallsetminus e}] = (\mathbb{L} - 1) \cdot [\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma/e}]. \quad (3.79)$$

(4) If Γ contains at least two loops, then the projective version of (3.77) also holds, in the form

$$[\mathbb{P}^{n-1} \smallsetminus X_\Gamma] = \mathbb{L} \cdot [\mathbb{P}^{n-2} \smallsetminus (X_{\Gamma \smallsetminus e} \cap X_{\Gamma/e})] - [\mathbb{P}^{n-2} \smallsetminus X_{\Gamma \smallsetminus e}]. \quad (3.80)$$

(5) Under the same hypotheses, the Euler characteristics satisfy

$$\chi(X_\Gamma) = n + \chi(X_{\Gamma \smallsetminus e} \cap X_{\Gamma/e}) - \chi(X_{\Gamma \smallsetminus e}). \quad (3.81)$$

Proof. (1) and (4): Let Γ be a graph with $n \geq 2$ edges $e_1, \ldots, e_{n-1}, e = e_n$, with $(t_1 : \ldots : t_n)$ the corresponding variables in $\mathbb{P}^{n-1}$. As above we consider the Kirchhoff polynomial Ψ_Γ and the affine and projective graph hypersurfaces $\hat{X}_\Gamma \subset \mathbb{A}^n$ and $X_\Gamma \subset \mathbb{P}^{n-1}$. We assume $\deg \Psi_\Gamma = \ell > 0$.

We consider an edge e that is not a bridge or looping edge. Then the polynomials for the deletion $\Gamma \smallsetminus e$ and contraction Γ/e of the edge $e = e_n$ are both non-trivial and given by

$$F := \frac{\partial \Psi_\Gamma}{\partial t_n} = \Psi_{\Gamma \smallsetminus e} \quad \text{and} \quad G := \Psi_\Gamma|_{t_n=0} = \Psi_{\Gamma/e}. \quad (3.82)$$

The Kirchhoff polynomial Ψ_Γ satisfies

$$\Psi_\Gamma(t_1, \ldots, t_n) = t_n F(t_1, \ldots, t_{n-1}) + G(t_1, \ldots, t_{n-1}). \quad (3.83)$$

Given a projective hypersurface $Y \subset \mathbb{P}^{N-1}$ we denote by $\hat{Y}$ the affine cone in $\mathbb{A}^N$, and we use the notation $\bar{Y}$ for the projective cone in $\mathbb{P}^N$.

It is proved in [Aluffi and Marcolli (2009b)], Theorem 3.3, in a more general context that includes the case under consideration, that the projection from the point $(0 : \cdots : 0 : 1)$ induces an isomorphism

$$X_\Gamma \smallsetminus (X_\Gamma \cap \overline{X}_{\Gamma\smallsetminus e}) \xrightarrow{\sim} \mathbb{P}^{n-2} \smallsetminus X_{\Gamma\smallsetminus e}, \tag{3.84}$$

where $X_{\Gamma\smallsetminus e}$ is the hypersurface in $\mathbb{P}^{n-2}$ defined by the polynomial F. In fact, the projection $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-2}$ from $p = (0 : \cdots : 0 : 1)$ acts as

$$(t_1 : \cdots : t_n) \mapsto (t_1 : \cdots : t_{n-1}).$$

If F is constant (that is, if $\deg \psi = 1$), then $X_{\Gamma\smallsetminus e} = \overline{X}_{\Gamma\smallsetminus e} = \emptyset$ and the statement is trivial. Thus, assume $\deg F > 0$. In this case, $\Psi_\Gamma(p) = F(p) = 0$, hence $p \in X_\Gamma \cap \overline{X}_{\Gamma\smallsetminus e}$, and hence $p \notin X \smallsetminus (X_\Gamma \cap \overline{X}_{\Gamma\smallsetminus e})$. Therefore, the projection restricts to a regular map

$$X_\Gamma \smallsetminus (X_\Gamma \cap \overline{X}_{\Gamma\smallsetminus e}) \to \mathbb{P}^{n-2}.$$

The image is clearly contained in $\mathbb{P}^{n-2} \smallsetminus X_{\Gamma\smallsetminus e}$, and the statement is that this map induces an *isomorphism*

$$X_\Gamma \smallsetminus (X_\Gamma \cap \overline{X}_{\Gamma\smallsetminus e}) \xrightarrow{\sim} \mathbb{P}^{n-2} \smallsetminus X_{\Gamma\smallsetminus e} \quad .$$

Let $q = (q_1 : \cdots : q_{n-1})$. The line from p to q is parametrized by

$$(q_1 : \cdots : q_{n-1} : t).$$

We show that this line meets $X_\Gamma \smallsetminus (X_\Gamma \cap \overline{X}_{\Gamma\smallsetminus e})$ transversely at one point. The intersection with X_Γ is determined by the equation

$$tF(q_1 : \cdots : q_{n-1}) + G(q_1 : \cdots : q_{n-1}) = 0.$$

Since $F(q) \neq 0$, this is a polynomial of degree exactly 1 in t, and determines a reduced point, as needed.

It then follows that, in the Grothendieck ring $K_0(\mathcal{V})$ we have identities

$$[\mathbb{P}^{n-1} \smallsetminus X_\Gamma] = [\mathbb{P}^{n-1} \smallsetminus (X_\Gamma \cap \overline{X}_{\Gamma\smallsetminus e})] - [\mathbb{P}^{n-2} \smallsetminus X_{\Gamma\smallsetminus e}]. \tag{3.85}$$

If Γ has at least two loops, so that $\deg \Psi_\Gamma > 1$, then we also have

$$[\mathbb{P}^{n-1} \smallsetminus X_\Gamma] = \mathbb{L} \cdot [\mathbb{P}^{n-2} \smallsetminus (X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e})] - [\mathbb{P}^{n-2} \smallsetminus X_{\Gamma\smallsetminus e}], \tag{3.86}$$

where $\mathbb{L} = [\mathbb{A}^1]$ is the Lefschetz motive and $X_{\Gamma/e}$ is the hypersurface $G = 0$. Indeed, the ideal of $X_\Gamma \cap \overline{X}_{\Gamma\smallsetminus e}$ is

$$(\psi, F) = (t_n F + G, F) = (F, G).$$

This means that

$$X_\Gamma \cap \overline{X}_{\Gamma\smallsetminus e} = \overline{X}_{\Gamma\smallsetminus e} \cap \overline{X}_{\Gamma/e}. \tag{3.87}$$

If $\deg X_\Gamma > 1$, then F is not constant, hence $\overline{X}_{\Gamma \smallsetminus e} \neq \emptyset$. It then follows that $\overline{X}_{\Gamma \smallsetminus e} \cap \overline{X}_{\Gamma/e}$ contains the point $p = (0 : \cdots : 0 : 1)$. The fibers of the projection

$$\mathbb{P}^{n-1} \smallsetminus (\overline{X}_{\Gamma \smallsetminus e} \cap \overline{X}_{\Gamma/e}) \to \mathbb{P}^{n-2} \smallsetminus X_{\Gamma \smallsetminus e}$$

with center p are then all isomorphic to $\mathbb{A}^1$, and it follows that

$$[\mathbb{P}^{n-1} \smallsetminus (\overline{X}_{\Gamma \smallsetminus e} \cap \overline{X}_{\Gamma/e})] = \mathbb{L} \cdot [\mathbb{P}^{n-2} \smallsetminus (X_{\Gamma \smallsetminus e} \cap X_{\Gamma/e})].$$

The affine version (3.77) follows by observing that, if $\deg X_\Gamma > 1$, then $\deg F > 0$, hence

$$\widehat{X_\Gamma \cap \overline{X}_{\Gamma \smallsetminus e}} \quad \text{and} \quad \widehat{X}_{\Gamma \smallsetminus e} \cap \widehat{X}_{\Gamma/e}$$

contain the origin. In this case, the classes in the Grothendieck ring satisfy

$$[\mathbb{A}^N \smallsetminus \widehat{X_\Gamma \cap \overline{X}_{\Gamma \smallsetminus e}}] = (\mathbb{L} - 1) \cdot [\mathbb{P}^{N-1} \smallsetminus (X_\Gamma \cap \overline{X}_{\Gamma \smallsetminus e})]$$

$$[\mathbb{A}^N \smallsetminus (\widehat{X}_{\Gamma \smallsetminus e} \cap \widehat{X}_{\Gamma/e})] = (\mathbb{L} - 1) \cdot [\mathbb{P}^{N-1} \smallsetminus (X_{\Gamma \smallsetminus e} \cap X_{\Gamma/e})].$$

Then (3.77) follows from the formula for the projective case, while it can be checked directly for the remaning case with $\deg X_\Gamma = 1$.

(2) If e is a bridge, then Ψ_Γ does not depend on the variable t_e and $F \equiv 0$. The equation for $X_{\Gamma \smallsetminus e}$ is $\Psi_\Gamma = 0$ again, but viewed in one fewer variables. The equation for $X_{\Gamma/e}$ is the same.

(3) If e is a looping edge, then Ψ_Γ is divisible by t_e, so that $G \equiv 0$. The equation for $X_{\Gamma/e}$ is obtained by dividing Ψ_Γ through by t_e, and one has $X_{\Gamma \smallsetminus e} = X_{\Gamma/e}$.

(5) Under the hypothesis that $\deg X_\Gamma > 1$, the statement about the Euler characteristics follows from (3.80). One has from (3.84) the identity

$$\chi(\mathbb{P}^{n-1} \smallsetminus (X_\Gamma \cup \overline{X}_{\Gamma \smallsetminus e})) = 0.$$

In the case where $\deg X_\Gamma = 1$ one has $\chi(X_\Gamma) = n - 1$. $\square$

This general result on deletion-contraction relations for motivic Feynman rules was applied in [Aluffi and Marcolli (2009b)] to analyze some operations on graphs, which have the property that the problem of describing the intersection $X_{\Gamma \smallsetminus e} \cap X_{\Gamma/e}$ can be bypassed and the class of more complicated graphs can be computed inductively only in terms of combinatorial data.

One such operation replaces a chosen edge e in a graph Γ with m parallel edges connecting the same two vertices $\partial(e)$. Another operation consists of replacing an edge with a chain of triangles (referred to as a "lemon graph").

The latter operation can easily be generalized to chains of polygons as explained in [Aluffi and Marcolli (2009b)].

Assume that e is an edge of Γ, and denote by Γ_{me} the graph obtained from Γ by replacing e by m parallel edges. (Thus, $\Gamma_{0e} = \Gamma \smallsetminus e$, and $\Gamma_e = \Gamma$.)

Theorem 3.10.4. *Let e be an edge of a graph Γ. Let $\mathbb{T} = \mathbb{L} - 1 = [\mathbb{G}_m] \in K_0(\mathcal{V})$.*

(1) If e is a looping edge, then

$$\sum_{m\geq 0} \mathbb{U}(\Gamma_{me})\, \frac{s^m}{m!} = e^{\mathbb{T}s}\, \mathbb{U}(\Gamma \smallsetminus e). \tag{3.88}$$

(2) If e is a bridge, then

$$\sum_{m\geq 0} \mathbb{U}(\Gamma_{me})\, \frac{s^m}{m!} = \left(\mathbb{T} \cdot \frac{e^{\mathbb{T}s} - e^{-s}}{\mathbb{T}+1} + s\, e^{\mathbb{T}s} + 1\right) \mathbb{U}(\Gamma \smallsetminus e). \tag{3.89}$$

(3) If e is not a bridge nor a looping edge, then

$$\begin{aligned} \sum_{m\geq 0} \mathbb{U}(\Gamma_{me})\, \frac{s^m}{m!} = {} & \frac{e^{\mathbb{T}s} - e^{-s}}{\mathbb{T}+1} \mathbb{U}(\Gamma) \\ & + \frac{e^{\mathbb{T}s} + \mathbb{T}e^{-s}}{\mathbb{T}+1} \mathbb{U}(\Gamma \smallsetminus e) \\ & + \left(s\, e^{\mathbb{T}s} - \frac{e^{\mathbb{T}s} - e^{-s}}{\mathbb{T}+1}\right) \mathbb{U}(\Gamma/e). \end{aligned} \tag{3.90}$$

The result is proved in [Aluffi and Marcolli (2009b)] by first deriving the relation in the case of doubling an edge and then repeating the construction. We do not report here the details of the proof, and we refer the readers to the paper for more details, but it is worth pointing out that the main step that makes it possible to obtain in this case a completely explicit formula in terms of $\mathbb{U}(\Gamma)$, $\mathbb{U}(\Gamma \smallsetminus e)$ and $\mathbb{U}(\Gamma/e)$ is a cancellation that occurs in the calculation of the classes in the operation of doubling an edge. In fact, one first expresses the class $\mathbb{U}(\Gamma_{2e})$ as

$$\mathbb{U}(\Gamma_{2e}) = \mathbb{L} \cdot [\mathbb{A}^n \smallsetminus (\hat{X}_\Gamma \cap \hat{X}_{\Gamma_o})] - \mathbb{U}(\Gamma),$$

where $n = \#E(\Gamma)$ and Γ_o denotes the graph obtained by attaching a looping edge named e to Γ/e. In the notation we used earlier in this section, the equation for Γ_o is given by the polynomial $t_e\, G$. Using inclusion–exclusion for classes in the Grothendieck ring, it is then shown in [Aluffi and Marcolli (2009b)] that one has

$$[\hat{X}_\Gamma \cap \hat{X}_{\Gamma_o}] = [\hat{X}_{\Gamma/e}] + (\mathbb{L} - 1) \cdot [\hat{X}_{\Gamma \smallsetminus e} \cap \hat{X}_{\Gamma/e}].$$

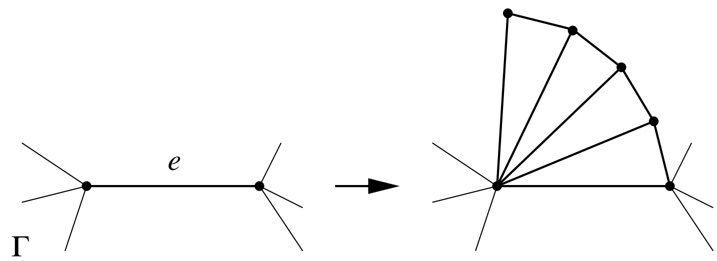
Thus, one obtains

$$\mathbb{U}(\Gamma_{2e}) = (\mathbb{L}-2)\cdot\mathbb{U}(\Gamma) + (\mathbb{L}-1)\cdot\mathbb{U}(\Gamma\smallsetminus e) + \mathbb{L}\cdot\mathbb{U}(\Gamma/e).$$

This expression no longer contains classes of intersections of graph hypersurfaces, and can be iterated to obtain the general formula for $\mathbb{U}(\Gamma_{me})$.

It is shown in [Aluffi and Marcolli (2009b)] that the recursive relation obtained in this way for the operation of multiplying edges is very similar in form to the recursive relation that the Tutte polynomial satisfies under the same operation. In fact, they both are solutions to a universal recursive relation with different initial conditions. One also derives from the same universal recursion a conjectural form for the recursion relation, under the operation of multiplying edges $\Gamma \mapsto \Gamma_{me}$, for the polynomial invariant $C_\Gamma(T) = I_{CSM}([\mathbb{A}^n \smallsetminus \hat{X}_\Gamma])$.

The operation of replacing an edge with a chain of triangles, or "lemon graph" depicted in the figure produces, similarly, a recursive formula for the motivic Feynman rule $\mathbb{U}(\Gamma^\Lambda_m)$, where Γ^Λ_m is the graph obtained from Γ by replacing an edge e by a lemon graph Λ_m.



Proposition 3.10.5. *Let e be an edge of a graph Γ, and assume that e is neither a bridge nor a looping edge. Let Γ^Λ_m be the "lemonade graph" obtained by building an m-lemon fanning out from e. Then*

$$\sum_{m\geq 0} \mathbb{U}(\Gamma^\Lambda_m)s^m = \frac{1}{1-\mathbb{T}(\mathbb{T}+1)s-\mathbb{T}(\mathbb{T}+1)^2s^2} \cdot \big((1-(\mathbb{T}+1)s)\,\mathbb{U}(\Gamma) + (\mathbb{T}+1)\mathbb{T}s\,\mathbb{U}(\Gamma\smallsetminus e) + (\mathbb{T}+1)^2 s\,\mathbb{U}(\Gamma/e)\big)\,.$$

This result is proven in [Aluffi and Marcolli (2009b)] by first obtaining a recursive formula for the classes $\mathbb{U}(\Lambda_m)$ of the lemon graphs. This is a succession of operations where one first doubles an edge and then splits the added edge by inserting a vertex. The first operation is covered by the expression derived before for the class $\mathbb{U}(\Gamma_{2e})$ while splitting an edge with a vertex corresponds to taking a cone over the corresponding graph hypersurface. Thus, both operations are well understood at the level of classes in the Grothendieck ring and they give the recursive relation

$$\mathbb{U}(\Lambda_{m+1}) = \mathbb{T}(\mathbb{T}+1)\mathbb{U}(\Lambda_m) + \mathbb{T}(\mathbb{T}+1)^2\mathbb{U}(\Lambda_{m-1}),$$

for $m \geq 1$, which gives then

$$\mathbb{U}(\Lambda_m) = (\mathbb{T}+1)^{m+1} \sum_{i=0}^{m} \binom{m-i}{i} \mathbb{T}^{m-i}.$$

It is interesting to observe that the sequence $a_m = \mathbb{U}(\Lambda_m)$ is a divisibility sequence, in the sense that $\mathbb{U}(\Lambda_{m-1})$ divides $\mathbb{U}(\Lambda_{n-1})$ if m divides n. This is the property satisfied, for instance, by the sequence of Fibonacci numbers. (See [Aluffi and Marcolli (2009b)], Corollary 5.12 for more details.)

Returning to the classes $\mathbb{U}(\Gamma_m^\Lambda)$, one then shows that these also satisfy a relation of the form

$$\mathbb{U}(\Gamma_m^\Lambda) = f_m(\mathbb{T})\mathbb{U}(\Gamma) + g_m(\mathbb{T})\mathbb{U}(\Gamma \smallsetminus e) + h_m(\mathbb{T})\mathbb{U}(\Gamma/e),$$

where the f_m, g_m, h_m satisfy the same recursion that gives $\mathbb{U}(\Lambda_m)$, but with different seeds given by

$$\begin{cases} f_0(\mathbb{T}) = 1 \quad , \quad f_1(\mathbb{T}) = \mathbb{T}^2 - 1 \\ g_0(\mathbb{T}) = 0 \quad , \quad g_1(\mathbb{T}) = \mathbb{T}(\mathbb{T}+1) \\ h_0(\mathbb{T}) = 0 \quad , \quad h_1(\mathbb{T}) = (\mathbb{T}+1)^2. \end{cases}$$

For more details we refer the reader to [Aluffi and Marcolli (2009b)].

3.11 Feynman integrals and periods

Numerical calculations performed in [Broadhurst and Kreimer (1997)] gave very strong evidence for a relation between residues of Feynman integrals and periods of mixed Tate motives. In fact, the numerical evidence indicates that the values computed in the log divergent case are $\mathbb{Q}$-linear combinations of multiple zeta values, which, as we recalled in the previous chapter, can be realized as periods of mixed Tate motives.

Multiple zeta values are real numbers obtained by summing convergent series of the form

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<k_2<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}, \tag{3.91}$$

where the n_i are positive integers with $n_r \geq 2$. There are ways to realize multiple zeta values as periods of mixed Tate motives; see [Goncharov

(2001)], [Goncharov and Manin (2004)], [Terasoma (2002)]. In [Goncharov and Manin (2004)], for instance, as well as in [Brown (2006)], multiple zeta values are explicitly realized as periods on the moduli spaces of curves. It is the occurrence of multiple zeta values in Feynman integral computations, along with the fact that these numbers arise as periods of mixed Tate motives, that gave rise to the idea of a deeper relation between perturbative quantum field theory and motives.

Among the examples of Feynman graphs computed in [Broadhurst and Kreimer (1997)], the wheel with n spokes graphs give simple zeta values $\zeta(2n-3)$. It is also shown in [Broadhurst and Kreimer (1997)] that the non-planar graph given by the complete bipartite graph $K_{3,4}$ evaluates to the double zeta value $\zeta(5,3)$.

Recent results of [Brown (2009a)] for multiloop Feynman diagrams provide a new method for evaluating the case of primitive divergent graphs (*i.e.* those that do not contain any smaller divergent subgraph, and are therefore primitive elements in the Connes–Kreimer Hopf algebra). This shows the occurrence of values at roots of unity of multiple polylogarithms, in addition to the multiple zeta values observed in [Broadhurst and Kreimer (1997)]. The multiple polylogarithm function, introduced in [Goncharov (2001)] is defined as

$$\mathrm{Li}_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = \sum_{0<k_1<k_2<\cdots<k_r} \frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}}. \tag{3.92}$$

It is absolutely convergent for $|x_i|<1$ and it extends analytically to an open domain in $\mathbb{C}^r$ as a multivalued holomorphic function. As shown in [Goncharov (2001)], the multiple polylogarithms are also associated to objects in the category of mixed Tate motives.

3.12 The mixed Tate mystery

The results recalled in the previous section suggest that, in its strongest possible form, one could formulate a conjecture as follows.

Conjecture 3.12.1. *Are all residues of Feynman graphs of perturbative scalar field theories (possibly after a suitable regularization and renormalization procedure is used) periods of mixed Tate motives?*

In this form the conjecture is somewhat vaguely formulated, but it does not matter since it is unlikely to hold true in such generality. One can however refine the question and formulate it more precisely in the following form.

Question 3.12.2. *Under what conditions on the graphs, the scalar field theory, and the renormalization procedure are the residues of Feynman graphs periods of mixed Tate motives?*

As we have seen above, using the Feynman parameters, one can write the Feynman integrals in a form that looks exactly like a period of an algebraic variety obtained as a hypersurface complement, modulo the issue of divergences which needs to be handled separately. In the stable range for D, this hypersurface is the graph variety X_Γ, while in the unstable range it is the Landau variety Y_Γ, or the union of the two.

Kontsevich formulated the conjecture that the graph hypersurfaces X_Γ themselves may always be mixed Tate motives, which would imply Conjecture 3.12.1. Although numerically this conjecture was at first verified up to a large number of loops in [Stembridge (1998)], it was later disproved in [Belkale and Brosnan (2003a)]. They proved that the varieties X_Γ can be arbitrarily complicated as motives: indeed, the X_Γ generate the Grothendieck ring of varieties.

The fact that the classes $[X_\Gamma]$ can be arbitrarily far from the mixed Tate part $\mathbb{Z}[\mathbb{L}]$ of the Grothendieck ring $K_0(\mathcal{V})$ seems to indicate at first that Conjecture 3.12.1 cannot hold. However, that is not necessarily the case. In fact, only a part of the cohomology of the complement of the hypersurface X_Γ is involved in the period computation of the Feynman integral. As we have seen earlier in this chapter, ignoring divergences, one is in fact considering only a certain relative cohomology group, namely

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma)), \tag{3.93}$$

where $n = \#E_{int}(\Gamma)$ and $\Sigma_n = \{t \in \mathbb{P}^{n-1} \mid \prod_i t_i = 0\}$ denotes the normal crossings divisor given by the union of the coordinate hyperplanes. In fact, if one ignores momentarily the issue of divergences, the evaluation of the integral

$$\int_{\sigma_n} \frac{P_\Gamma(p,t)^{-n+D\ell/2}}{\Psi_\Gamma(t)^{-n+(\ell+1)D/2}}\, \omega_n \tag{3.94}$$

can be seen as a pairing of a differential form on the hypersurface complement with a chain σ_n with boundary $\partial\sigma_n \subset \Sigma_n$ contained in the normal crossings divisor. Thus, one is working with a relative cohomology, and we have seen that the question of whether the integral above (*i.e.* the residue of the Feynman graph after dropping the divergent Gamma factor in the parametric Feynman integral) evaluates to a period of a mixed Tate motive can be addressed in terms of the question of whether the relative cohomology (3.93) is a realization of a mixed Tate motive

$$\mathfrak{m}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus X_\Gamma \cap \Sigma_n). \tag{3.95}$$

This could still be the case even though the motive of X_Γ itself need not be mixed Tate. However, this observation makes it clear that, if something like Conjecture 3.12.1 holds, it is due to a very subtle interplay between the geometry of the graph hypersurface X_Γ and the locus of intersection $X_\Gamma \cap \Sigma_n$. Notice that this locus is also the one that is responsible for the divergences of the integral (3.94), so that the situation is further complicated, when one tries to eliminate divergences, by possibly performing blow-ups of this locus. Even without introducing immediately the further difficulties related to eliminating divergences, the main problem remains that as the graphs increase in combinatorial complexity, it becomes extemely difficult to control the motivic nature of (3.95). Thus, at present, not only is the conjecture in its stronger form 3.12.1 still open, but not much is understood on specific conditions that would ensure that (3.95) is in fact mixed Tate, according to the formulation of Question 3.12.2.

Some new work of Francis Brown, which became available in the preprint [Brown (2009b)] while this book was going to print, identifies specific families of graphs for which (3.95) is shown to give a mixed Tate motive and for which the corresponding Feynman integrals evaluate to multiple zeta values.

In fact, a weaker statement than the relative cohomology (3.93) being mixed Tate may still suffice to prove that the residue of the Feynman integral is a period of a mixed Tate motive, since the period only captures a piece of the relative cohomology (3.93), so that the question on the motivic nature of this relative cohomology provides a sufficient condition for a mixed Tate period, but not a necessary one.

Very nice recent results of [Doryn (2008)] compute the middle cohomology of the graph hypersurfaces for a larger class of graphs than those originally considered in the work of [Bloch, Esnault, Kreimer (2006)], consisting of so called "zig-zag graphs". This identifies in a significant class of examples a mixed Tate piece of the cohomology.

As we mentioned in relation to the recent result of [Bloch (2008)] on sums of graphs with fixed number of vertices, one should also take into account that the possible occurrence of non-mixed Tate periods in the Feynman amplitudes for more complicated graphs may cancel out in sums over graphs with fixed external structure or fixed number of vertices and leave only a mixed Tate contribution.

One should also keep in mind that, in the case of divergent Feynman integrals, handling divergences in different ways might affect the nature of the resulting periods, after subtraction of infinities.

3.13 From graph hypersurfaces to determinant hypersurfaces

An attempt to provide specific conditions, as stated in Question 3.12.2 above, that would ensure that certain Feynman integrals evaluate to periods of mixed Tate motives was developed in [Aluffi and Marcolli (2009a)].

One can use the properties of periods, in particular the change of variable formula, to recast the computation of a given integral $\int_\sigma \omega$ computing a period in a different geometric ambient variety, whose motivic nature is easier to control. As long as no information is lost in the period computation, one can hope to obtain in this way some sufficient conditions that will ensure that the periods associated to Feynman integrals are mixed Tate.

A period $\int_\sigma \omega$ associated to the data (X, D, ω, σ) of a variety X, a divisor D, a differential form ω on X, and an integration domain σ with boundary $\partial\sigma \subset D$ can be computed equivalently after a change of variables obtained by mapping it via a morphism f of varieties to another set of data $(X', D', \omega', \sigma')$, with $\omega = f^*(\omega')$ and $\sigma' = f_*(\sigma)$.

In the case of parametric Feynman integrals one can proceed in the following way. The matrix $M_\Gamma(t)$ associated to a Feynman graph Γ determines a linear map of affine spaces

$$\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik} \eta_{ir} \tag{3.96}$$

such that the affine graph hypersurface is obtained as the preimage

$$\hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

under this map of the determinant hypersurface,

$$\hat{\mathcal{D}}_\ell = \{x = (x_{ij}) \in \mathbb{A}^{\ell^2} \mid \det(x_{ij}) = 0\}.$$

One can give explicit combinatorial conditions on the graph that ensure that the map Υ is an embedding. As shown in [Aluffi and Marcolli (2009a)], for any 3-edge-connected graph with at least 3 vertices and no looping edges, which admit a closed 2-cell embedding of face width at least 3, the map Υ

is injective. To see why this is the case, one can start with the following observation on the properties of the matrix $M_\Gamma(t)$ that defines the map Υ of (3.96).

Lemma 3.13.1. *The matrix $M_\Gamma(t) = \eta^\dagger \Lambda \eta$ defining the map Υ has the following properties.*

- *For $i \neq j$, the corresponding entry is the sum of $\pm t_k$, where the t_k correspond to the edges common to the i-th and j-th loop, and the sign is $+1$ if the orientations of the edges both agree or both disagree with the loop orientations, and -1 otherwise.*
- *For $i = j$, the entry is the sum of the variables t_k corresponding to the edges in the i-th loop (all taken with sign $+$).*

Similarly, for a specific edge e, with t_e the corresponding variable, one has the following.

- *The variable t_e appears in $\eta^\dagger \Lambda \eta$ if and only if e is part of at least one loop.*
- *If e belongs to a single loop ℓ_i, then t_e only appears in the diagonal entry (i, i), added to the variables corresponding to the other edges forming the loop ℓ_i.*
- *If there are two loops ℓ_i, ℓ_j containing e, and not having any other edge in common, then the $\pm t_e$ appears by itself at the entries (i, j) and (j, i) in the matrix $\eta^\dagger \Lambda \eta$.*

In the following, we denote by Υ_i the composition of the map Υ with the projection to the i-th row of the matrix $\eta^\dagger \Lambda \eta$, viewed as a map of the variables corresponding only to the edges that belong to the i-th loop in the chosen bases of the first homology of the graph Γ.

Lemma 3.13.2. *If Υ_i is injective for i ranging over a set of loops such that every edge of Γ is part of a loop in that set, then Υ is itself injective.*

Proof. Let $(t_1, \ldots, t_n) = (c_1, \ldots, c_n)$ be in the kernel of τ. Since each (i, j) entry in the target matrix is a combination of edges in the i-th loop, the map τ_i must send to zero the tuple of c_j's corresponding to the edges in the i-th loop. Since we are assuming τ_i to be injective, that tuple is the zero-tuple. Since every edge is in some loop for which τ_i is injective, it follows that every c_j is zero, as needed. □

One can then give conditions based on properties of the graph that ensure that the components Υ_i are injective. This is done in [Aluffi and Marcolli (2009a)] as follows.

Lemma 3.13.3. *The map Υ_i is injective if the following conditions are satisfied:*

- *For every edge e of the i-th loop, there is another loop having only e in common with the i-th loop, and*
- *The i-th loop has at most one edge not in common with any other loop.*

Proof. In this situation, all but at most one edge variable appear by themselves as an entry of the i-th row, and the possible last remaining variable appears summed together with the other variables. More explicitly, if $t_{i_1}, \ldots, t_{i_v}$ are the variables corresponding to the edges of a loop ℓ_i, up to rearranging the entries in the corresponding row of $\eta^\dagger \Lambda \eta$ and neglecting other entries, the map Υ_i is given by

$$(t_{i_1}, \ldots, t_{i_v}) \mapsto (t_{i_1} + \cdots + t_{i_v}, \pm t_{i_1}, \ldots, \pm t_{i_v})$$

if ℓ_i has no edge not in common with any other loop, and

$$(t_{i_1}, \ldots, t_{i_v}) \mapsto (t_{i_1} + \cdots + t_{i_v}, \pm t_{i_1}, \ldots, \pm t_{i_{v-1}})$$

if ℓ_i has a single edge t_v not in common with any other loop. In either case the map Υ_i is injective, as claimed. □

Every (finite) graph Γ may be embedded in a compact orientable surface of finite genus. The minimum genus of an orientable surface in which Γ may be embedded is the *genus* of Γ. Thus, Γ is planar if and only if it may be embedded in a sphere, if and only if its genus is 0.

Definition 3.13.4. An embedding of a graph Γ in an orientable surface S is a *2-cell embedding* if the complement of Γ in S is homeomorphic to a union of open 2-cells (the *faces*, or *regions* determined by the embedding). An embedding of Γ in S is a *closed 2-cell embedding* if the closure of every face is a disk.

It is known that an embedding of a connected graph is minimal genus if and only if it is a 2-cell embedding ([Mohar and Thomassen (2001)], Proposition 3.4.1 and Theorem 3.2.4). We discuss below conditions on the existence of *closed* 2-cell embeddings, *cf.* [Mohar and Thomassen (2001)], §5.5.

For our purposes, the advantage of having a closed 2-cell embedding for a graph Γ is that the faces of such an embedding determine a choice of loops of Γ, by taking the boundaries of the 2-cells of the embedding together with a basis of generators for the homology of the Riemann surface in which the graph is embedded.

Lemma 3.13.5. *A closed 2-cell embedding $\iota : \Gamma \to S$ of a connected graph Γ on a surface of (minimal) genus g, together with the choice of a face of the embedding and a basis for the homology $H_1(S, \mathbb{Z})$ determine a basis of $H_1(\Gamma, \mathbb{Z})$ given by $2g + f - 1$ loops, where f is the number of faces of the embedding.*

Proof. Orient (arbitrarily) the edges of Γ and the faces, and then add the edges on the boundary of each face with sign determined by the orientations. The fact that the closure of each face is a 2-disk guarantees that the boundary is null-homotopic. This produces a number of loops equal to the number f of faces. It is clear that these f loops are *not* independent: the sum of any $f - 1$ of them must equal the remaining one, up to sign. Any $f-1$ loops, however, will be independent in $H_1(\Gamma)$. Indeed, these $f-1$ loops, together with $2g$ generators of the homology of S, generate $H_1(\Gamma)$. The homology group $H_1(\Gamma)$ has rank $2g + f - 1$, as one can see from the Euler characteristic formula

$$b_0(S) - b_1(S) + b_2(S) = 2 - 2g$$

$$= \chi(S) = v - e + f = b_0(\Gamma) - b_1(\Gamma) + f = 1 - \ell + f,$$

so there will be no other relations. $\square$

One refers to the chosen one among the f faces as the "external face" and the remaining $f - 1$ faces as the "internal faces".

Thus, given a closed 2-cell embedding $\iota : \Gamma \to S$, we can use a basis of $H_1(\Gamma, \mathbb{Z})$ constructed as in Lemma 3.13.5 to compute the map Υ and the maps Υ_i of (3.13.1). We then have the following result.

Lemma 3.13.6. *Assume that Γ is closed-2-cell embedded in a surface. With notation as above, assume that*

- *any two of the f faces have at most one edge in common.*

Then the $f-1$ maps Υ_i, defined with respect to a choice of basis for $H_1(\Gamma)$ as in Lemma 3.13.5, are all injective. If further

- *every edge of Γ is in the boundary of two of the f faces,*

then Υ is injective.

Proof. The injectivity of the $f-1$ maps Υ_i follows from Lemma 3.13.3. If ℓ is a loop determined by an internal face, the variables corresponding to edges in common between ℓ and any other internal loop will appear as $(\pm)$ individual entries on the row corresponding to ℓ. Since ℓ has at most one edge in common with the external region, this accounts for all but at most one of the edges in ℓ. By Lemma 3.13.3, the injectivity of Υ_i follows. Finally, as shown in Lemma 3.13.2, the map Υ is injective if every edge is in one of the $f-1$ loops and the $f-1$ maps Υ_i are injective. The stated condition guarantees that the edge appears in the loops corresponding to the faces separated by that edge. At least one of them is internal, so that every edge is accounted for. $\square$

Recall the notions of connectivity of graphs that we discussed in the first chapter in relation to the 1PI condition, Definition 1.5.1.

We also recall the notion of *face width* for an embedding of a graph in a Riemann surface.

Definition 3.13.7. Let $\iota : \Gamma \hookrightarrow S$ be a given embedding of a graph Γ on a Riemann surface S. The face width $fw(\Gamma, \iota)$ is the largest number $k \in \mathbb{N}$ such that every non-contractible simple closed curve in S intersects Γ at least k times. When S is a sphere, hence $\iota : \Gamma \hookrightarrow S$ is a planar embedding, one sets $fw(\Gamma, \iota) = \infty$.

For a graph Γ with at least 3 vertices and with no looping edges, the condition that an embedding $\iota : \Gamma \hookrightarrow S$ is a *closed* 2-cell embedding is equivalent to the properties that Γ is 2-vertex-connected and that the embedding has face width $fw(\Gamma, \iota) \geq 2$; see Proposition 5.5.11 of [Mohar and Thomassen (2001)]. It is not known whether every 2-vertex-connected graph Γ admits a closed 2-cell embedding. The "strong orientable embedding conjecture" states that this is the case, namely, that every 2-vertex-connected graph Γ admits a closed 2-cell embedding in some orientable surface S, of face width at least two (see [Mohar and Thomassen (2001)], Conjecture 5.5.16).

Theorem 3.13.8. *The following conditions imply the injectivity of* Υ*.*

(1) Let Γ *be a graph with at least* 3 *vertices and with no looping edges, which is closed-*2*-cell embedded in an orientable surface* S*. Then, if any two of the faces have at most one edge in common, the map* Υ *is injective.*

(2) *Let* Γ *be a* 3*-edge-connected graph, with at least* 3 *vertices and no looping edges, admitting a closed-2-cell embedding* $\iota : \Gamma \hookrightarrow S$ *with face width* $fw(\Gamma, \iota) \geq 3$*. Then the maps* Υ_i*,* Υ *are all injective.*

Proof. (1) It suffices to show that, under these conditions on the graph Γ, the second condition of Lemma 3.13.6 is automatically satisfied, so that only the first condition remains to be checked. That is, we show that every edge of Γ is in the boundary of two faces. Assume an edge is not in the boundary of two faces. Then that edge must bound the same face on both of its sides. The closure of the face is a cell, by assumption. Let γ be a path from one side of the edge to the other. Since γ splits the cell into two connected components, it follows that removing the edge splits Γ into two connected components, hence Γ is not 2-edge-connected. However, the fact that Γ has at least 3 vertices and no looping edges and it admits a closed 2-cell embedding implies that Γ is 2-vertex-connected, hence in particular it is 1PI by Lemma 1.5.5, and this gives a contradiction.

(2) The previous result shows that the second condition stated in Lemma 3.13.6 is automatically satisfied, so the only thing left to check is that the first condition stated in Lemma 3.13.6 holds. Assume that two faces F_1, F_2 have more than one edge in common. Since F_1, F_2 are (path)-connected, there are paths γ_i in F_i connecting corresponding sides of the edges. With suitable care, it can be arranged that $\gamma_1 \cup \gamma_2$ is a closed path γ meeting Γ in 2 points. Since the embedding has face width ≥ 3, γ must be null-homotopic in the surface, and in particular it splits it into two connected components. Therefore Γ is split into two connected components by removing the two edges, hence Γ cannot be 3-edge-connected. $\square$

A more geometric description of these combinatorial properties in terms of *wheel neighborhoods* at the vertices of the graph is discussed in [Aluffi and Marcolli (2009a)]. Further, it is shown there that the injectivity property of the map Υ extends from the classes of graphs described in the previous theorem to other graphs obtained from these by simple combinatorial operations such as attaching a looping edge at a vertex of a given graph or subdividing edges by adding intermediate valence two vertices.

The combinatorial conditions of Theorem 3.13.8 are fairly natural from a physical viewpoint. In fact, 2-edge-connected is just the usual 1PI condition, while 3-edge-connected or 2PI is the next strengthening of this condition (the 2PI effective action is often considered in quantum field theory), and the face width condition is also the next strengthening of face width 2, which a well known combinatorial conjecture on graphs [Mohar and Thomassen (2001)] expects should simply follow for graphs that are 2-vertex-connected. (The latter condition is a bit more than 1PI: for graphs with at least two vertices and no looping edges it is equivalent to all the splittings of the graph at vertices also being 1PI, as we showed in Lemma 1.5.5.) The condition that the graph has no looping edges is only a technical device for the proof. In fact, it is then easy to show (see [Aluffi and Marcolli (2009a)]) that adding looping edges does not affect the injectivity of the map Υ.

When the map Υ is injective, it is possible to rephrase the computation of the parametric Feynman integral as a period of the complement of the determinant hypersurface $\hat{\mathcal{D}}_\ell \subset \mathbb{A}^{\ell^2}$. Notice also that if the map Υ is injective then one has a well defined map $\mathbb{P}^{n-1} \to \mathbb{P}^{\ell^2-1}$, which is otherwise not everywhere defined. Motivically, when the map $\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ is injective, the complexity of Feynman integrals of the graph Γ as a period is controlled by the motive $\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_\ell \cap \hat{\Sigma}_\Gamma))$, where $\hat{\Sigma}_\Gamma$ is a normal crossings divisor in $\mathbb{A}^{\ell^2}$ such that $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$. We recall below how to construct the divisor $\hat{\Sigma}_\Gamma$.

One can in fact rewrite the Feynman integral (as usual up to a divergent Γ-factor) in the form

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}},$$

for a polynomial $\mathcal{P}_\Gamma(x,p)$ on $\mathbb{A}^{\ell^2}$ that restricts to $P_\Gamma(t,p)$, and with $\omega_\Gamma(x)$ the image of the volume form. Let then $\hat{\Sigma}_\Gamma$ be a normal crossings divisor in $\mathbb{A}^{\ell^2}$, which contains the boundary of the domain of integration, $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$. The question on the motivic nature of the resulting period can then be reformulated (again modulo divergences) in this case as the question of whether the motive

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)) \tag{3.97}$$

is mixed Tate.

The advantage of moving the period computation via the map $\Upsilon = \Upsilon_\Gamma$ from the hypersurface complement $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$ to the complement of the determinant hypersurface $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ is that, unlike what happens with the graph hypersurfaces, it is well known that the determinant hypersurface $\hat{\mathcal{D}}_\ell$ is a mixed Tate motive, as we have already seen in detail in Theorems 2.6.2 and 2.6.3 above.

One then sees that, in this approach, the difficulty has been moved from understanding the motivic nature of the hypersurface complement to having some control on the other term of the relative cohomology, namely the normal crossings divisor $\hat{\Sigma}_\Gamma$ and the way it intersects the determinant hypersurface.

If the motive of $\hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)$ is mixed Tate, then knowing that $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ is always mixed Tate, the fact that mixed Tate motives form a triangulated subcategory of the triangulated category of mixed motives would show that the motive (3.97) whose realization is the relative cohomology would also be mixed Tate.

Thus, Question 3.12.2 can be reformulated, under the combinatorial conditions given above for the embedding, as a question on when the motive of $\hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)$ is mixed Tate. This requires, first of all, a better description of the divisor $\hat{\Sigma}_\Gamma$ that contains the boundary $\partial(\Upsilon(\sigma_n))$.

A first observation in [Aluffi and Marcolli (2009a)] is that one can use the same normal crossings divisor $\hat{\Sigma}_{\ell,g}$ for all graphs Γ with a fixed number of loops and a fixed genus (that is, the minimal genus of an orientable surface in which the graph can be embedded). This divisor is given by a union of linear spaces.

Proposition 3.13.9. *There exists a normal crossings divisor $\hat{\Sigma}_{\ell,g} \subset \mathbb{A}^{\ell^2}$, which is a union of $N = \binom{f}{2}$ linear spaces,*

$$\hat{\Sigma}_{\ell,g} := L_1 \cup \cdots \cup L_N, \tag{3.98}$$

such that, for all graphs Γ with ℓ loops and genus g closed 2-cell embedding, the preimage under $\Upsilon = \Upsilon_\Gamma$ of the union $\hat{\Sigma}_\Gamma$ of a subset of components of $\hat{\Sigma}_{\ell,g}$ is the algebraic simplex Σ_n in $\mathbb{A}^n$. More explicitly, the components of the divisor $\hat{\Sigma}_{\ell,g}$ can be described by the $N = \binom{f}{2}$ equations

$$\begin{cases} x_{ij} = 0 & 1 \le i < j \le f-1 \\ x_{i1} + \cdots + x_{i,f-1} = 0 & 1 \le i \le f-1, \end{cases} \tag{3.99}$$

where $f = \ell - 2g + 1$ is the number of faces of the embedding of the graph Γ on a surface of genus g.

Proof. The polynomial $\det M_\Gamma(t)$ does not depend on the choice of orientation for the loops of Γ. Thus, we can make the following convenient choice of these orientations. We have chosen a closed 2-cell embedding of Γ into an orientable surface of genus g. Such an embedding has f faces, with $\ell = 2g+f-1$. We can arrange $M_\Gamma(t)$ so that the first $f-1$ rows correspond to the $f-1$ loops determined by the 'internal' faces of the embedding. On each face, we choose the positive orientation (counterclockwise with respect to an outgoing normal vector). Then each edge-variable in common between two faces i, j will appear with a minus sign in the entries (i,j) and (j,i) of $M_\Gamma(t)$. These entries are both in the $(\ell-2g)\times(\ell-2g)$ upper-left minor.

We have in fact seen that the injectivity of an $(\ell-2g)\times(\ell-2g)$ minor of the matrix M_Γ suffices to control the injectivity of the map Υ, and we can arrange so that the minor is the upper-left part of the $\ell\times\ell$ ambient matrix. Then the hyperplanes in $\mathbb{A}^n$ associated to the coordinates t_i can be obtained by pulling back linear spaces along this minor. On the diagonal of the $(f-1)\times(f-1)$ submatrix we find all edges making up each face, with a positive sign. It follows that the pull-backs of the equations (3.99) produce a list of all the edge variables, possibly with redundancies. The components of $\hat{\Sigma}_{\ell,g}$ that form the divisor $\hat{\Sigma}_\Gamma$ are selected by eliminating those components of $\hat{\Sigma}_{\ell,g}$ that contain the image of the graph hypersurface (*i.e.* coming from the zero entries of the matrix $M_\Gamma(t)$). $\square$

A second observation of [Aluffi and Marcolli (2009a)] is then that, using inclusion-exclusion, it suffices to show that arbitrary intersections of the components L_i of $\hat{\Sigma}_{\ell,g}$ have the property that

$$(\cap_{i\in I} L_i)\smallsetminus\hat{\mathcal{D}}_\ell \tag{3.100}$$

is mixed Tate. In fact, this would then imply that also the locus

$$\hat{\Sigma}_\Gamma\smallsetminus(\hat{\mathcal{D}}_\ell\cap\hat{\Sigma}_\Gamma)$$

is mixed Tate, by repeatedly using inclusion–exclusion.

Notice that the intersection $\cap_{i\in I} L_i$ is a linear subspace of codimension $\#I$ in $\mathbb{A}^{\ell^2}$. In general, the intersection of a linear subspace with the determinant is *not* mixed Tate. For example, the intersection of a general $\mathbb{A}^3$ with $\hat{\mathcal{D}}_3$ is a cone over a genus-1 curve. In fact, working projectively, $\mathcal{D}_3$ is a degree-3 hypersurface in $\mathbb{P}^8$, with singularities in codimension >1. Therefore, the intersection with a general $\mathbb{P}^2$ is a nonsingular cubic curve, therefore a curve of genus $=1$. The affine version is a cone over this. Thus, in order to understand under what conditions the locus (3.100) will be mixed Tate, we have to understand in what sense the intersections $\cap_{i\in I} L_i$ are special.

The following characterization, which is given in [Aluffi and Marcolli (2009a)], leads to a reformulation of the problem in terms of certain "manifolds of frames".

Lemma 3.13.10. *Let E be a fixed ℓ-dimensional vector space. Every $I\subseteq\{1,\dots,N\}$ determines a choice of linear subspaces $V_1,\dots,V_\ell$ of E with the property that*

$$\cap_{k\in I} L_k=\{(v_1,\dots,v_\ell)\in\mathbb{A}^{\ell^2}\mid\forall i, v_i\in V_i\}. \tag{3.101}$$

Here, we denote an $\ell \times \ell$ *matrix in* $\mathbb{A}^{\ell^2}$ *by its* ℓ *row-vectors* $v_i \in E$.

Further, $\dim V_i \geq i-1$. *Further still, there exists a basis* $(e_1, \ldots, e_\ell)$ *of* E *such that each space* V_i *is the span of a subset (of cardinality* $\geq i-1$*) of the vectors* e_j.

Proof. Recall (Proposition 3.13.9) that the components L_k of $\hat{\Sigma}_{\ell,g}$ consist of matrices for which either the (i,j) entry x_{ij} equals 0, for $1 \leq i < j \leq \ell - 2g$, or

$$x_{i1} + \cdots + x_{i,\ell-2g} = 0$$

for $1 \leq i \leq \ell - 2g$. Thus, each L_k consists of ℓ-tuples $(v_1, \ldots, v_\ell)$ for which exactly one row v_i belongs to a fixed hyperplane of E, and more precisely to one of the hyperplanes

$$x_1 + \cdots + x_{\ell-2g} = 0, \quad x_2 = 0, \quad \ldots \quad x_{\ell-2g} = 0. \tag{3.102}$$

The statement follows by choosing V_i to be the intersection of the hyperplanes corresponding to the L_k in the i-th row, among those listed in (3.102). Since there are at most $\ell - 2g - i + 1$ hyperplanes L_k in the i-th row,

$$\dim V_i \geq \ell - (\ell - 2g - i + 1) = 2g + i - 1 \geq i - 1 \quad .$$

Finally, to obtain the basis $(e_1, \ldots, e_\ell)$ mentioned in the statement, simply choose the basis dual to the basis $(x_1 + \cdots + x_{\ell-2g}, x_2, \ldots, x_\ell)$ of the dual space to E. $\square$

A sufficient condition that would ensure that the locus $\hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)$ is mixed Tate is then described in terms of *manifolds of frames*.

Definition 3.13.11. For a given ambient space $V = \mathbb{A}^\ell$ and an assigned collection of linear subspaces V_i, $i = 1, \ldots, \ell$, the manifold of frames $\mathbb{F}(V_1, \ldots, V_\ell)$ is defined as the locus

$$\mathbb{F}(V_1, \ldots, V_\ell) := \{(v_1, \ldots, v_\ell) \in \mathbb{A}^{\ell^2} \mid v_k \in V_k\} \cap (\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell). \tag{3.103}$$

More generally, $\mathbb{F}(V_1, \ldots, V_r) \subset V_1 \times \cdots \times V_r$ denotes the locus of r-tuples of linearly independent vectors in a given vector space V, where each v_i is constrained to belong to the given subspace V_i.

The previous characterization of the locus (3.100), together with the use of inclusion–exclusion arguments, shows that if, for a fixed ℓ, the manifold of frames is mixed Tate for all choices of the subspaces V_i, this suffices to guarantee that the motive

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\Sigma}_{\ell,g} \cap \hat{\mathcal{D}}_\ell)) \tag{3.104}$$

is also mixed Tate. When this is the case, the result then implies that, modulo divergences, the residues of Feynman graphs Γ with ℓ loops and genus g, for which the injectivity condition on Υ holds, are all periods of mixed Tate motives.

In the case of two and three loops, one can verify explicitly that the manifolds of frames $\mathbb{F}(V_1, V_2)$ and $\mathbb{F}(V_1, V_2, V_3)$ are mixed Tate. This is done in [Aluffi and Marcolli (2009a)] by exhibiting an explicit stratification, from which, as in the case of the determinant hypersurface discussed above, one can show that the motive defines an object in the category of mixed Tate motives as a subtriangulated category of the category of mixed motives. One can also compute explicitly the corresponding classes in the Grothendieck ring, as a function of the Lefschetz motive $\mathbb{L}$. They are of the following form.

Proposition 3.13.12. *For given subspaces V_1, V_2, the manifold of frames $\mathbb{F}(V_1, V_2)$ is a mixed Tate motive, whose class in the Grothendieck ring is*

$$[\mathbb{F}(V_1, V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L},$$

with $d_i = \dim(V_i)$ and $d_{ij} = \dim(V_i \cap V_j)$.

Proof. We want to parameterize all pairs (v_1, v_2) of vectors such that $v_1 \in V_1$, $v_2 \in V_2$, and $\dim\langle v_1, v_2\rangle = 2$. This locus can be decomposed into the two (possibly empty) pieces

(1) $v_1 \in V_1 \smallsetminus (V_1 \cap V_2)$, and $v_2 \in V_2 \smallsetminus \{0\}$;
(2) $v_1 \in (V_1 \cap V_2) \smallsetminus \{0\}$, and $v_2 \in V_2 \smallsetminus \langle v_1\rangle$.

This exhausts all the possible ways of obtaining linearly independent vectors with the first one in V_1 and the second one in V_2 and the manifold of frames $\mathbb{F}(V_1, V_2)$ is the union of these two loci. The first case describes the locus

$$(V_1 \smallsetminus (V_1 \cap V_2)) \times (V_2 \smallsetminus \{0\})$$

which is clearly mixed Tate, being obtained by taking products and complements of affine spaces. Its class in the Grothendieck ring is of the form

$$(\mathbb{L}^{d_1} - \mathbb{L}^{d_{12}})(\mathbb{L}^{d_2} - 1).$$

The locus defined by the second case can be described equivalently by the following procedure. Consider the projective space $\mathbb{P}(V_1 \cap V_2)$, and the trivial bundles $\mathcal{V}_{12} \subseteq \mathcal{V}_2$ with fiber $V_1 \cap V_2 \subseteq V_2$. The tautological line bundle $\mathcal{O}_{12}(-1)$ over $\mathbb{P}(V_1 \cap V_2)$ sits inside $\mathcal{V}_{12}$, hence inside $\mathcal{V}_2$. The desired pairs of vectors (v_1, v_2) are obtained by choosing a point $p \in \mathbb{P}(V_1 \cap V_2)$, a vector $v_1 \neq 0$ in the fiber of $\mathcal{O}_{12}(-1)$ over p, and a vector v_2 in the fiber of $\mathcal{V}_2 \smallsetminus \mathcal{O}_{12}(-1)$ over p. This again defines a mixed Tate locus, using the same two properties of the existence of distinguished triangles in the category of mixed motives associated to closed embeddings and homotopy invariance, as in Theorems 2.6.2 and 2.6.3. The homotopy invariance property, which is formulated for products, extends in fact to the locally trivial case of vector bundles or projective bundles, see [Voevodsky (2000)]. The class in the Grothendieck ring, in this case, is then given by the expression

$$(\mathbb{L}^{d_{12}} - 1)(\mathbb{L}^{d_2} - \mathbb{L}).$$

Thus, the class of $\mathbb{F}(V_1, V_2)$ is the sum of these two classes,

$$[\mathbb{F}(V_1, V_2)] = (\mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2+d_{12}} + \mathbb{L}^{d_{12}}) + (\mathbb{L}^{d_2+d_{12}} - \mathbb{L}^{d_{12}+1} - \mathbb{L}^{d_2} + \mathbb{L})$$

$$= \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}.$$

□

The case of three subspaces already requires a more delicate analysis of the contribution of the various strata, as in §6.2 of [Aluffi and Marcolli (2009a)], which we reproduce here below.

Theorem 3.13.13. *Let V_1, V_2, V_3 be assigned subspaces of a vector space V. Then the manifold of frames $\mathbb{F}(V_1, V_2, V_3)$ is a mixed Tate motive, with class in the Grothendieck ring given by*

$$[\mathbb{F}(V_1, V_2, V_3)] = (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1)$$

$$-(\mathbb{L} - 1)((\mathbb{L}^{d_1} - \mathbb{L})(\mathbb{L}^{d_{23}} - 1) + (\mathbb{L}^{d_2} - \mathbb{L})(\mathbb{L}^{d_{13}} - 1) + (\mathbb{L}^{d_3} - \mathbb{L})(\mathbb{L}^{d_{12}} - 1)$$

$$+(\mathbb{L} - 1)^2(\mathbb{L}^{d_1+d_2+d_3-D} - \mathbb{L}^{d_{123}+1}) + (\mathbb{L} - 1)^3,$$

where $d_i = \dim(V_1)$, $d_{ij} = \dim(V_i \cap V_j)$, $d_{ijk} = \dim(V_i \cap V_j \cap V_k)$, and $D = D_{ijk} = \dim(V_i + V_j + V_k)$.

Proof. This time one can proceed in the following way to obtain a stratification. One can look for a stratification $\{S_\alpha\}$ of V_3 which is finer than the one induced by the subspace arrangement $V_1 \cap V_3, V_2 \cap V_3$ and such that, for v_3 in S_α, the class $\mathbb{F}_\alpha := [\mathbb{F}(\pi(V_1), \pi(V_2))]$, with $\pi : V \to V' := V/\langle v_3 \rangle$ the projection, depends only on α and not on the chosen vector $v_3 \in S_\alpha$. In other words, we want to construct a stratification of V_3 such that the dimensions of the spaces $\pi(V_1)$, $\pi(V_2)$ and $\pi(V_1 \cap V_2)$ are constant along the strata. The following five loci define such a stratification of $V_3 \smallsetminus \{0\}$:

(1) $S_{123} := (V_1 \cap V_2 \cap V_3) \smallsetminus \{0\}$;

(2) $S_{13} := (V_1 \cap V_3) \smallsetminus (V_1 \cap V_2 \cap V_3)$;
(3) $S_{23} := (V_2 \cap V_3) \smallsetminus (V_1 \cap V_2 \cap V_3)$;
(4) $S_{(12)3} := ((V_1 + V_2) \cap V_3) \smallsetminus ((V_1 \cup V_2) \cap V_3)$;
(5) $S_3 := V_3 \smallsetminus ((V_1 + V_2) \cap V_3)$.

The dimensions are indeed constant along the strata and given by

	$\dim \pi(V_1)$	$\dim \pi(V_2)$	$\dim(\pi(V_1) \cap \pi(V_2))$
S_{123}	$d_1 - 1$	$d_2 - 1$	$d_{12} - 1$
S_{13}	$d_1 - 1$	d_2	d_{12}
S_{23}	d_1	$d_2 - 1$	d_{12}
$S_{(12)3}$	d_1	d_2	$d_{12} + 1$
S_3	d_1	d_2	d_{12}

This information can then be translated into explicit classes $[\mathbb{F}_\alpha]$ in the Grothendieck ring associated to the strata S_α so that the class of the frame manifold is of the form

$$[\mathbb{F}(V_1, V_2, V_3)] = \sum_\alpha \mathbb{L}^{s_\alpha} [\mathbb{F}_\alpha][S_\alpha], \tag{3.105}$$

where the number s_α is the number of subspaces V_i, $i = 1, 2$ containing the vector $v_3 \in S_\alpha$. This is also independent of the choice of v_3 and only dependent on α by the properties of the stratification. For the table of dimensions obtained above the corresponding classes $[\mathbb{F}_\alpha]$ are as follows.

	$[\mathbb{F}_\alpha]$
S_{123}	$\mathbb{L}^{d_1+d_2-2} - \mathbb{L}^{d_1-1} - \mathbb{L}^{d_2-1} - \mathbb{L}^{d_{12}} + \mathbb{L}^{d_{12}-1} + \mathbb{L}$
S_{13}	$\mathbb{L}^{d_1+d_2-1} - \mathbb{L}^{d_1-1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}$
S_{23}	$\mathbb{L}^{d_1+d_2-1} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2-1} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}$
$S_{(12)3}$	$\mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+2} + \mathbb{L}^{d_{12}+1} + \mathbb{L}$
S_3	$\mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}$

One then obtains the following values for the class of the stratum $[S_\alpha]$ and the number s_α.

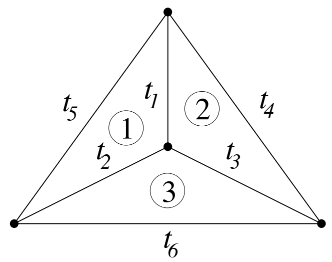
	$[S_\alpha]$	s_α
S_{123}	$\mathbb{L}^{d_{123}} - 1$	2
S_{13}	$\mathbb{L}^{d_{13}} - \mathbb{L}^{d_{123}}$	1
S_{23}	$\mathbb{L}^{d_{23}} - \mathbb{L}^{d_{123}}$	1
$S_{(12)3}$	$\mathbb{L}^{d_1+d_2+d_3-D-d_{12}} - \mathbb{L}^{d_{13}} - \mathbb{L}^{d_{23}} + \mathbb{L}^{d_{123}}$	0
S_3	$\mathbb{L}^{d_3} - \mathbb{L}^{d_1+d_2+d_3-D-d_{12}}$	0

Using the formula (3.105) then gives the stated result. Although the stratification is used there only to compute the class in the Grothendieck group, an argument similar to the one of Theorems 2.6.2 and 2.6.3 can be used to show that the same stratifications used to compute $[\mathbb{F}(V_1, V_2)]$ and $[\mathbb{F}(V_1, V_2, V_3)]$ also show that $\mathbb{F}(V_1, V_2)$ and $\mathbb{F}(V_1, V_2, V_3)$ define objects in the triangulated category of mixed Tate motives. $\square$

One can use these explicit computations of classes of manifolds of frames for two and three subspaces to obtain specific information about the motives

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell))$$

for specific Feynman graphs with up to three loops. For example, we report here briefly the case of the wheel-with-three-spokes graph (the 1-skeleton of a tetrahedron) described in [Aluffi and Marcolli (2009a)].



This graph has matrix $M_\Gamma(t)$ given by

$$\begin{pmatrix} t_1+t_2+t_5 & -t_1 & -t_2 \\ -t_1 & t_1+t_3+t_4 & -t_3 \\ -t_2 & -t_3 & t_2+t_3+t_6 \end{pmatrix}.$$

Labeling entries of the matrix as x_{ij}, we can obtain $t_1, \ldots, t_6$ as pull-backs of the following:

$$\begin{cases} t_1 = -x_{12} \\ t_2 = -x_{13} \\ t_3 = -x_{23} \\ t_4 = x_{21}+x_{22}+x_{23} \\ t_5 = x_{11}+x_{12}+x_{13} \\ t_6 = x_{31}+x_{32}+x_{33} \end{cases}$$

Thus, we can consider as $\hat{\Sigma}_\Gamma$ the normal crossings divisor defined by the equation

$$x_{12}x_{13}x_{23}(x_{11}+x_{12}+x_{13})(x_{21}+x_{22}+x_{23})(x_{31}+x_{32}+x_{33}) = 0.$$

In order to compute the class in the Grothendieck ring of the intersection $\hat{\Sigma}_\Gamma \cap (\mathbb{A}^9 \smallsetminus \hat{\mathcal{D}}_3)$, one can use inclusion-exclusion and compute the classes for all the intersections of subsets of components of the divisor $\hat{\Sigma}_\Gamma$. This divisor has 6 components, hence there are $2^6 = 64$ such intersections. Each of them determines a choice of three subspaces V_1, V_2, V_3 corresponding to the linearly independent vectors given by the rows of the matrix (x_{ij}) in $\mathbb{A}^9$. All of the corresponding classes $[\mathbb{F}(V_1, V_2, V_3)]$, for each of the 64 possibilities, were computed explicitly in [Aluffi and Marcolli (2009a)]. We do not reproduce them all here, but we only give the final result, which shows that the resulting class is then of the form

$$[\hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_3 \cap \hat{\Sigma}_\Gamma)] = \mathbb{L}(6\mathbb{L}^4 - 3\mathbb{L}^3 + 2\mathbb{L}^2 + 2\mathbb{L} - 1)(\mathbb{L} - 1)^3.$$

By arguing as in Theorem 2.6.2, and using the same stratification of the complement $\mathbb{A}^9 \smallsetminus \hat{\mathcal{D}}_3$ induced by the divisor $\hat{\Sigma}_\Gamma$ as in the computation of the class above, one can improve the result from a statement about the class in the Grothendieck ring being a polynomial function of the Lefschetz motive $\mathbb{L}$ to one about the motive $\mathfrak{m}(\hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_3 \cap \hat{\Sigma}_\Gamma))$ itself being an object in the triangulated subcategory $\mathcal{DMT}_{\mathbb{Q}}$ of mixed Tate motives inside the triangulated category $\mathcal{DM}_{\mathbb{Q}}$ of mixed motives. Then this fact, together with the fact that the hypersurface complement itself is a mixed Tate motive, and the distinguished triangle corresponding to the long exact cohomology sequence, suffice to show that the mixed motive

$$\mathfrak{m}(\mathbb{A}^9 \smallsetminus \hat{\mathcal{D}}_3, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_3))$$

is mixed Tate for the case of the wheel with three spokes. The result for other graphs with three loops can be derived from the same analysis used for the wheel with three spokes, by restricting only to certain components of the divisor, as explained in [Aluffi and Marcolli (2009a)].

For the cases of manifolds of frames $\mathbb{F}(V_1, \ldots, V_r)$ with more than three subspaces, it would seem at first that one should be able to establish an inductive argument that would take care of the cases of more subspaces, but the combinatorics of the possible subspace arrangements quickly becomes very difficult to control. In fact, in general it seems unlikely that the frame manifolds $\mathbb{F}(V_1, \ldots, V_\ell)$ will continue to be mixed Tate for large ℓ and for arbitrarily complex subspace arrangements. The situation may in fact be somewhat similar to that of "Murphy's law in algebraic geometry" described in [Vakil (2006)], where a sufficiently general case may be as bad as possible, while specific cases one can explicitly construct are much better behaved. The connection can probably be made more precise, given that both the result of [Belkale and Brosnan (2003a)] on the general motivic properties of the graph hypersurfaces X_Γ and Murphy's law result of [Vakil (2006)] are based on the same universality result for matroid representations of [Mnëv (1988)]. Another observation from the point of view of matroids is that, while the result of [Belkale and Brosnan (2003a)] is non-constructive, in the sense that it shows that the graph hypersurfaces generate the Grothendieck ring of motives but it does not provide an explicit construction of matroids on which to test the mixed Tate property, it may be that the loci $\hat{\Sigma}_\Gamma \cap (\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell)$ considered here may provide a possible way to make that general result more explicit, by constructing explicit matroids, along the lines of [Gelfand and Serganova (1987)]. A reformulation of the problem of understanding when frame manifolds are mixed Tate is given in [Aluffi and Marcolli (2009a)] in terms of intersections of unions of Schubert cells in flag varieties. This version of the problem suggests a possible connection to Kazhdan–Lusztig theory [Kazhdan and Lusztig (1980)].

3.14 Handling divergences

All the considerations above on the motivic nature of the relative cohomology involved in the computation of the parametric Feynman integral, either in the hypersurface complement $\mathbb{P}^{n-1} \smallsetminus X_\Gamma$ or in the complement of the determinant hypersurface $\mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell$, assume that the integral is convergent and therefore directly defines a period. In other words, the issue of divergences is not dealt with in this approach. However, one knows very well that most Feynman integrals are divergent, even when in the parametric form one removes the divergent Gamma factor and only looks at the residue, and that some regularization and renormalization procedure is needed to handle these divergences.

In terms of the geometry, after removing the divergent Gamma factor, the source of other possible divergences in the parametric Feynman integral is the locus of intersection of the graph hypersurface X_Γ with the domain of integration σ_n.

Notice that the poles of the integrand that fall inside the integration domain σ_n occur necessarily along the boundary $\partial\sigma_n$, since in the interior the graph polynomial Ψ_Γ takes only strictly positive real values.

Thus, one needs to modify the integrals suitably in such a way as to eliminate, by a regularization procedure, the intersections $X_\Gamma \cap \partial\sigma_n$, or (to work in algebro-geometric terms) the intersections $X_\Gamma \cap \Sigma_n$ which contains the former.

There are different possible ways to achieve such a regularization procedure. We mention here three possible approaches. We discuss the third one in more detail in the following chapter.

One method was developed in [Belkale and Brosnan (2003b)] in the logarithmically divergent case where $n = D\ell/2$, that is, when the polynomial $P_\Gamma(t, p)$ is not present and only the denominator $\Psi_\Gamma(t)^{D/2}$ appears in the parametric Feynman integral. As we have already mentioned, using dimensional regularization, one can, in this case, rewrite the Feynman integral in the form of a local Igusa L-function

$$I(s) = \int_\sigma f(t)^s \omega,$$

for $f = \Psi_\Gamma$. They prove that this L-function has a Laurent series expansion where all the coefficients are periods. In this setting, the issue of eliminating divergences becomes similar to the techniques used, for instance, in the context of log canonical thresholds. The result was more recently extended algorithmically to the non-log-divergent case [Bogner and Weinzierl (2007a)], [Bogner and Weinzierl (2007b)].

Another method, used in [Bloch, Esnault, Kreimer (2006)], consists of eliminating the divergences by separating Σ_n and X_Γ performing a series of blow-ups. Similarly, working in the setting of the determinant hypersurface complement $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ one can perform blowups to separate the locus of intersection of the determinant hypersurface $\hat{\mathcal{D}}_\ell$ with the divisor $\hat{\Sigma}_{\ell,g}$. To make sure that this operation does not change the motivic properties of the resulting period integrals, one needs to ensure that blow-ups along $\hat{\Sigma}_{\ell,g} \cap \hat{\mathcal{D}}_\ell$ maintain the mixed Tate properties if we know that (3.104) is mixed Tate. For blow-ups performed over a smooth locus one has projective bundles, for which one knows that the mixed Tate property is maintained, since one has an explicit formula for the motive of a projective bundle in the Voevodsky category, which is a sum of Tate twisted copies of the motive of the base, but when the locus is singular, as is typically the case here, then the required analysis is more delicate.

Yet another method was proposed in [Marcolli (2008)], based on deformations instead of resolutions. By considering the graph hypersurface X_Γ as the special fiber X_0 of a family X_s of varieties defined by the level sets $f^{-1}(s)$, for $f = \Psi_\Gamma : \mathbb{A}^n \to \mathbb{A}^1$, one can form a tubular neighborhood

$$D_\epsilon(X) = \cup_{s \in \Delta^*_\epsilon} X_s,$$

for Δ^*_ϵ a punctured disk of radius ϵ, and a circle bundle $\pi_\epsilon : \partial D_\epsilon(X) \to X_\epsilon$. One can then regularize the Feynman integral by integrating "around the singularities" in the fiber $\pi_\epsilon^{-1}(\sigma \cap X_\epsilon)$. The regularized integral has a Laurent series expansion in the parameter ϵ.

Motivically, one should take into account the fact that the presence of singularities in the hypersurface X_Γ increases the likelihood that the part of the cohomology involved in the period computation can be mixed Tate. In fact, if these were smooth hypersurfaces, they would typically not be mixed Tate even for a small number of loops, while it is precisely because of the fact that they are highly singular that the X_Γ continue to be mixed Tate for a fairly large number of loops and even though one knows that eventually one will run into graphs for which X_Γ is no longer mixed Tate, the fact that they continue to be singular in low codimension stil makes it possible to have a significant part of the cohomology that will still be a realization of a mixed Tate motive.

The use of deformations X_ϵ to regularize the integral, which we describe more in detail in the following chapter, is natural from the point of view of singularity theory, where one works with Milnor fibers of singularites. The singularities of the hypersurfaces X_Γ are non-isolated, but many techniques of singularity theory are designed to cover also this more general case.

From this viewpoint, it is not clear though how to perform the regularization and subtraction of divergences in a way that does not alter the motivic properties. In fact, while the special singular fiber may be mixed Tate, the generic fiber in a family used as deformation can be a general hypersurface that will not be motivically mixed Tate any longer. However, there are notions such as a motivic tubular neighborhood [Levine (2005)] that may be useful in this context, as well as the theory of limiting mixed Hodge structures for a degeneration of a family of algebraic varieties. It is not yet clear how to use this type of methods to obtain motivic information on Feynman integrals after subtraction of divergences.

In general, as we discuss more at length in Chapter 4.2, a regularization procedure for Feynman integrals replaces a divergent integral with a function of some regularization parameters (such as the complexified dimension of DimReg, or the deformation parameter ϵ in the example here above) in which the resulting function has a Laurent series expansion around the pole that corresponds to the divergent integral originally considered. One then uses a procedure of extraction of finite values to eliminate the polar parts of these Laurent series in a way that is *consistent over graphs*, that is, a renormalization procedure.

3.15 Motivic zeta functions and motivic Feynman rules

As we mentioned briefly in describing the Grothendieck ring as a universal Euler characteristic, the counting of points of a variety over finite fields is an example of an additive invariant that, as such, factors through the Grothendieck ring. The behavior of the number of points over finite fields is one of the properties of a variety that reveal its motivic nature. For example, if a variety, say defined over $\mathbb{Q}$, is motivically mixed Tate, then the number of points of the reductions mod p is polynomial in p.

The information on the number of points over finite fields is conveniently packaged in the form of a zeta function

$$Z_X(t) = \exp\left(\sum_n \frac{\#X(\mathbb{F}_{q^n})}{n}\, t^n \right) \tag{3.106}$$

Notice that this can be written equivalently in terms of symmetric products as

$$Z_X(t) = \sum_{n\geq 0} \#s^n(X)(\mathbb{F}_q)\, t^n,$$

where $s^n(X)$ denotes the n-th symmetric power of X.

The fact that the counting of the number of points behaves like an Euler characteristic, hence it descends from the Grothendieck ring, suggested that the zeta function itself may be lifted at the motivic level. This was done in [Kapranov (2000)], where the *motivic zeta function* is defined as

$$Z_X(t) = \sum_{n\geq 0} [s^n(X)]\, t^n \tag{3.107}$$

where $[s^n(X)]$ are the classes in the Grothendieck ring and the zeta function can be seen as an element of $K_0(\mathcal{V}_{\mathbb{K}})[[t]]$. For example, one has

$$Z_{\mathbb{P}^1}(t) = (1-t)^{-1}(1-\mathbb{L}t)^{-1}.$$

Kapranov proved that, when X is the motive of a curve, then the zeta function is a rational function, in the sense that, given a motivic measure $\mu : K_0(\mathcal{M}) \to \mathcal{R}$, the zeta function $Z_{X,\mu}(T) \in \mathcal{R}[[T]]$ is a rational function of T. Later, it was proved in [Larsen and Lunts (2003)] that in general this is not true in the case of algebraic surfaces.

One recovers the zeta function of the variety, for a finite field $\mathbb{K} = \mathbb{F}_q$, by applying a "counting of points" homomorphism $K_0(\mathcal{V}_{\mathbb{K}}) \to \mathbb{Z}$ to the motivic zeta function. One obtains other kinds of zeta functions by applying other "motivic measures" $\mu : K_0(\mathcal{V}_{\mathbb{K}}) \to \mathcal{R}$, which give

$$Z_{X,\mathcal{R}}(t) = \sum_{n\geq 0} \mu(s^n(X))\, t^n \in \mathcal{R}[[t]].$$

It is well known that the motivic zeta function contains all the information on the behavior of the number of points of reductions mod p, hence potentially the information on the mixed Tate nature of a variety. A recent survey of these ideas is given in [André (2009)], where it is also observed that the motivic zeta function should play a useful role in the case of motives associated to Feynman graphs.

There are other reasons why one can expect that, indeed, Kapranov's motivic zeta functions may be especially suitable tools to investigate motivic properties of Feynman integrals. One is, for example, the observation made at the end of [Marcolli (2008)] which suggests that the function Ψ_Γ^s, for s a complex variable, that appears in the dimensional regularization of parametric Feynman integrals, regarded as a zeta function

$$Z_\Gamma(T) := \sum_{n\geq 0} \frac{\log^n \Psi_\Gamma}{n!} T^n = \Psi_\Gamma^T,$$

may be related to a motivic zeta function

$$Z_{\mathrm{Log},\Gamma}(T) := \sum_{n\geq 0} s^n(\mathrm{Log}_\Gamma)\, T^n,$$

where the motive Log_Γ is the pullback of the logarithmic motive (2.39), via the map Ψ_Γ.

Another reason is the formulation of parametric Feynman integrals as local Igusa L-functions given in [Belkale and Brosnan (2003b)]. There is a motivic Igusa L-function constructed by [Denef and Loeser (1998)], which may provide the right tool for a motivic formulation of the dimensionally regularized parametric Feynman integrals.

In quantum field theory it is customary to consider the full partition function of the theory, arranged in an asymptotic series by loop number, instead of looking only at the contribution of individual Feynman graphs. Besides the loop number $\ell = b_1(\Gamma)$, other suitable gradings $\delta(\Gamma)$ are given by the number $n = \#E_{int}(\Gamma)$ of internal edges, or by $\#E_{int}(\Gamma) - b_1(\Gamma) = \#V(\Gamma) - b_0(\Gamma)$, the number of vertices minus the number of connected components. As shown in [Connes and Marcolli (2008)] p.77, these all define gradings on the Connes–Kreimer Hopf algebra of Feynman graphs.

When one considers motivic Feynman rules, these partition functions appear to be interesting analogs of the motivic zeta functions considered in [Kapranov (2000)], [Larsen and Lunts (2003)]. For instance, one can consider a partition function given by the formal series

$$Z(t) = \sum_{N \geq 0} \sum_{\delta(\Gamma)=N} \frac{U(\Gamma)}{\#\mathrm{Aut}(\Gamma)} \, t^N, \tag{3.108}$$

where $\delta(\Gamma)$ is any one of the gradings described above and where $U(\Gamma) = [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] \in K_0(\mathcal{V}_\mathbb{K})$. Given a motivic measure $\mu : K_0(\mathcal{V}_\mathbb{K}) \to \mathcal{R}$, this gives a zeta function with values in $\mathcal{R}[[t]]$ of the form

$$Z_\mathcal{R}(t) = \sum_{N \geq 0} \sum_{\delta(\Gamma)=N} \frac{\mu(U(\Gamma))}{\#\mathrm{Aut}(\Gamma)} \, t^N.$$

In the case of the sums over graphs

$$S_N = \sum_{\#V(\Gamma)=N} [X_\Gamma] \frac{N!}{\#\mathrm{Aut}(\Gamma)}$$

considered in [Bloch (2008)], one finds that cancellations between the classes occur in the Grothendieck ring $K_0(\mathcal{V}_\mathbb{K})$ and the resulting class S_N is always in the Tate subring $\mathbb{Z}[\mathbb{L}]$ even if the individual terms $[X_\Gamma]$ may be non-mixed Tate. Similarly, it is possible that interesting cancellations of a similar nature may occur in suitable evaluations of the zeta functions (3.108).