A survey of B-V formalism with some investigations on the spinning particle case

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Abstract

This paper is a survey of Batalin-Vilkovisky formalism, especially in supersymmetric(SUSY) systems, with an attempt(failed) to resolve the problems caused by the inversed ghost.

1 Prelude: some definition and physics

All the fundamental interactions that appear in our universe are described by some gauge theory. The quantization of gauge systems is an interesting topic both from a mathematical and from a physical point of view. In this survey article, we will introduce the so-called B-V(Batalin-Vilkovisky) formalism and discuss the BRST(Becchi-Rouet-Stora-Tyutin) cohomology.

1.1 Some history

B-V formalism was first introduced in the quantization of gauge field theory via path integral approach. Here, we will briefly introduce the physical motivation and give some mathematical treatment of gauge field theory.

Definition 1.1 *The pair* (X_0, S_0) *is a* ***gauge theory****.* X_0 *is the corresponding* ***initial(field) configuration space*** *and* S_0 *is the* ***initial action****, which is invariant under the action of the* ***gauge group*** $\mathcal{G}$

After the so-called Wick rotation, the quantization problem of quantum field theory usually lies in the computation of a integral of the following form

$$\langle \mathcal{O} \rangle = \int_{X_0} [D\mu] \mathcal{O} e^{-S_0}, \tag{1.1}$$

where $\mathcal{O}$ is a functional(**observable**) on X_0, $D\mu$ is the measure on the configuration space X_0. This kind of integral over the configuration space of field is called **path integral**.

There are two major obstacles when we want to analyze path integral of some physical systems. The first problem is that the path integral is not mathematically well defined, because the measure in 1.1 in the case of an infinite dimensional configuration space X_0, in general is not well defined.

If we ignore the problem of defining measure on the infinite dimensional configuration space, a new difficulties when we try to quantize gauge field theory. The gauge symmetries would introduce some degeneracies usually lead to some infinities. We usually assume that the gauge group $\mathcal{G}$ acts freely on the space X_0 and that S_0 is $\mathcal{G}$-invariant. Then we should alternatively define the path integral on the quotient space $X_0/\mathcal{G}$ and the previous integral would be proportional to the "volume" of the gauge group $\mathcal{G}$ even though $\mathcal{G}$ could be infinite dimensional or non-compact. To remove the redundant degree of freedom, we have to apply some gauge-fixing procedure but it could blur the physical meaning of the path integral.

The crucial point of BRST construction is to use BRST symmetry to recover the lost gauge symmetry in some sense, which is achieved by introducing **ghost fields**. The initial idea of Faddeev and Popov was to introduce extra fields in the theory in order to cancel the local symmetries and hence to be able to compute the path integral. Subsequently, Becchi, Rouet, Stora and, Tyutin found thtat the introduction of ghost fields leads to some so-called **BRST transformation**. They also discovered that ghost fields are generators of a cohomology complex, know as **BRST cohomology**. Batalin and Vilkovisky, generalized this idea and discovered a quantization procedure known as the antibracket formalism, or also as the Batalin-Vilkovisky or BV formalism.

1.2 B-V formalism

The idea of of B-V formalism is basically to extend to the initial gauge theory (X_0, S_0) to an extended pair $(\tilde{X}, \tilde{S})$. To complete this procedure we have to start from some basic mathematical definitions.

Definition 1.2 *Let B be a commutative unital ring and let $V = \oplus_{i\in\mathbb{Z}} V^i$ be a* ***graded module*** *over B, with free homogeneous components V^i of finite rank and such that $V^0 = 0$.*[1]*.An element $a \in V^i$ is said to be* ***homogeneous*** *of degree i. The* ***symmetric algebra*** *$Sym(V)$ is defined as the following quotient:*

$$Sym(V) = \frac{T(V)}{K} \tag{1.2}$$

where $T(V)$ is the tensor algebra of V and K is the B-module generated by the relation $ab = (-1)^{deg(a)deg(b)}ba$ for homogeneous elements.

In what follows, we define $F^pSym(V)$ be the ideal generated by elements of degree $\geq p$. The sequence of ideals $F^pSym(V)$ forms a descending filtration of $Sym(V)$:

$$Sym(V) \supseteq F^1(Sym(V)) \supseteq F^2(Sym(V)) \supseteq ...... \tag{1.3}$$

Definition 1.3 *The* ***completion*** *$\widehat{Sym}(V)$ of the graded algebra $Sym(V)$ is the* ***inverse limit***[2] *of $Sym(V)/F^p(Sym(V))$ in the category of graded modules.*

$$[\widehat{Sym}(V)]^i = \underset{\leftarrow p}{lim}[Sym(V)]^i/F^pSym(V) \cap [Sym(V)]^i \tag{1.4}$$

$\widehat{Sym}(V)$ still have the structure of graded commutative algebra

Then, we will state the definition of graded space and graded manifolds

Definition 1.4 *M_0 is a topological space. A* ***graded space*** *with support M_0 is a ringed space $M = (M_0, \mathcal{O}_M)$ where $\mathcal{O}_M$ is the* ***structure sheaf*** *of M is a sheaf of $\mathbb{Z}$-graded commutative rings on M_0 such that the stalk $\mathcal{O}_M, x$ at every point x has a unique maximal proper graded ideal (is a graded local ring)*

Definition 1.5 *Given two graded spaces $M = (M_0, \mathcal{O}_M)$ and $N = (N_0, \mathcal{O}_N)$, a* ***morphism*** *$M \to N$ is a pair $f = (f_0, \phi)$ such that:*

- *$f_0 : M_0 \to N_0$ is a homeomorphism;*

[1] The assumption that $V^0 = 0$ is not essential. It can be achieved by replacing B by $Sym_B(V_0)$

[2] some ref

- *$\phi : \mathcal{O}_M \to \mathcal{O}_N$ is a grading-preserving morphism of sheaves of rings such that its fiber ϕ_x also preserve the maximal ideal*

Remark 1.6 *we can specialize the above definition to any subcategory $\mathcal{C}$ of the category of ringed space, such as algebraic varieties, smooth manifolds.*

Definition 1.7 *A graded $\mathcal{C}$-variety with support $M_0 \in \mathcal{C}$ is a graded space $M = (M_0, \mathcal{O}_M)$ such that*

- *M_0 is an object in category $\mathcal{C}$;*
- *$\forall x \in M_0$, $\exists$ neighborhood $U \subseteq M_0$ such that $(U, \mathcal{O}_M|_U) \cong (U, \widehat{Sym}_{\mathcal{O}_M}(\mathcal{E}))$, where $\mathcal{E}$ is some graded $\mathcal{O}_M$-module with homogeneous comoponents of finite rank and $\mathcal{E}^0 = 0$*

Before introducing our crucial notion of B-V variety, we state the a sequence of definitions and will finally lead to **shifted cotangent bundle**. Consider the case where $\mathcal{C}$ is the category of nonsingular algebraic varieties over a field of characteristic zero or smooth manifolds.

Definition 1.8 *A P_0-**algebra** A over a field k is a $\mathbb{Z}-$graded commutative algebra $A = \oplus_{i\in\mathbb{Z}} A^i$ endowed with a degree 1 Poisson bracket $[\_\,,\_] : A \otimes A \to A$, with $deg([a,b]) = deg(a) + deg(b) + 1$*

Definition 1.9 *A **differential** P_0-**algebra** is a P_0-algebra together with a differential operator d such that for any homogeneous elements a, b, c, we have:*

- *$[\_,\_]$ is a bilinear map;*
- *the bracket is graded symmetric:*
$$[a,b] = -(-1)^{(deg(a)-1)(deg(b)-1)}[b,a];$$
- *the bracket is a graded Poisson bracket:*
$$[ab,c] = a[b,c] + (-1)^{deg(a)deg(b)} b[a,c];$$
- *the graded Jacobi identity:*
$$(-1)^{(deg(a)-1)(deg(c)-1)}[a,[b,c]] + (-1)^{(deg(b)-1)(deg(a)-1)}[b,[c,a]]$$
$$+ (-1)^{(deg(c)-1)(deg(b)-1)}[c,[a,b]] = 0;$$

- *d is graded derivative map of degree 1:*

$$d(ab) = d(a)b + (-1)^{deg(a)} a d(b);$$

- *the differential operator d is compatible with the bracket:*

$$d([a,b]) = [d(a), b] + (-1)^{deg(a)-1}[a, d(b)].$$

Definition 1.10 *Choose $V = (V_0, \mathcal{O}_V)$ to be a graded $\mathcal{C}$-variety with $\mathcal{O}_V^i = 0, \forall i < 0$.*

- *Then an **endomorphism of degree n** over $\mathcal{O}_V$ is a collection of mutually compatible maps $\phi = \{\phi_U\}$ with U open subsets in V_0, where each ϕ_U is an endomorphism of graded commutative rings with $\phi_U : \mathcal{O}_V^i(U) \to \mathcal{O}_V^{i+n}(U)$, $\forall i \in \mathbb{Z}$. The collection of all endomorphisms of degree n forms a sheaf over V_0 denoted by $\prod_{i=0}^{\infty} \mathcal{H}om(\mathcal{O}_V^i, \mathcal{O}_V^{i+n})$.*

- *A section ξ in $\prod_{i=0}^{\infty} \mathcal{H}om(\mathcal{O}_V^i, \mathcal{O}_V^{i+n})$ is called a **derivation of $\mathcal{O}_V$ of degree n** if locally we have $\xi(ab) = \xi(a)b + (-1)^{n \cdot dega} a\xi(b)$ with a, b homogeneous.*

- *The **tangent sheaf** T_V of V is*

$$T_V = \oplus_{n \geq 0} T_V^n := \oplus_{n \geq 0} \mathcal{H}om^{der}(\mathcal{O}_V^i, \mathcal{O}_V^{i+n}),$$

  with $\mathcal{H}om^{der}(\mathcal{O}_V^i, \mathcal{O}_V^{i+n})$ denoting the set of derivations of degree n.

T_V is a sheaf of graded Lie algebras acting on $\mathcal{O}_V$ with the canonical bracket. This bracket would extends to a Poisson bracket of degree 1 on $\tilde{\mathcal{O}}_M = Sym_{\mathcal{O}_V} T_V[1]$, which is a sheaf of P_0-algebras.[3] We can subsequently define the shifted cotangent bundle

Definition 1.11 *The (-1)**-shifted cotangent bundle** of a graded $\mathcal{C}$-variety V is the graded variety $M = T^*[-1]V = (X, \mathcal{O}_M)$ where $\mathcal{O}_M = \underset{\leftarrow p}{lim} \tilde{\mathcal{O}}_M / F^p \mathcal{O}_M$*

Definition 1.12 *A (-1)**-symplectic variety** is a graded variety $M = (M_0, \mathcal{O}_M)$ that is locally Poisson isomorphic to a (-1)-shifted cotangent bundle: each point $x \in M_0$ has an open neighborhood U such that there exists a morphism $\phi : \mathcal{O}_M|_U \to \mathcal{O}_{T^*[-1]V}|_U$ that respects the Poisson brackets.*

[3]The shifted module $\mathcal{E}[n]$ means $\mathcal{E}[n]^i = \mathcal{E}^{i+n}$

It is important that we can give an explicit description of a graded variety that is globally Poisson isomorphic to the (-1)-shifted cotangent bundle of a non-negatively graded variety, as precisely stated in the following proposition(for the proof, refer to the original paper by Felder and Kazhandan[2]).

Proposition 1.13 *Let $M = (M_0, \mathcal{O}_M)$ be a graded $\mathcal{C}$-variety, . If M is globally Poisson isomorphic to $T^*[-1]V$ for some $\mathbb{Z}_{\geq 0}$-graded variety V , then:*

- *V is isomorphic to a graded variety $(X, Sym_{\mathcal{O}_M}\mathcal{E})$, where $\mathcal{E}$ is a graded $\mathcal{O}_M$-module with homogeneous components*

$$\mathcal{E}^p = \mathcal{O}_M^p / F^{p+1}\mathcal{O}_M^p + I_M \cdot I_M \cap \mathcal{O}_M^p, \quad I_M = F^1\mathcal{O}_M, \quad p \geq 1.$$

- *Moreover, if $\mathcal{E}$ is also a locally free $\mathcal{O}_{M_0}$-module with homogeneous components of finite rank, then the following isomorphism holds:*

$$\mathcal{O}_M \cong \widehat{Sym}_{\mathcal{O}_{M_0}}(T_{M_0}[1] \oplus \mathcal{E} \oplus \mathcal{E}^*[1])$$

Due to the setting of gauge theory the initial configuration space X_0 is a real vector space. It is possible to find a basis for each homogeneous components, and thus we have extended the original configuration space to a $\mathbb{Z}$-graded vector space $\widetilde{X}$.

$$\widetilde{X} = \oplus_{p\in\mathbb{Z}}\widetilde{X}^p \tag{1.5}$$

- For $p > 0$, $\widetilde{X}^p = \langle \varphi_1^{(p)} ... \varphi_{q(p)}^{(p)} \rangle$, with $\{\varphi_i^{(p)}\}$ being the generators of homogeneous component $\mathcal{E}^p$
- For $p = 0$, $\widetilde{X}^0 = X_0 = \langle \varphi_1 ... \varphi_q \rangle$, with $\{\varphi_i\}$ being the basis of the initial configuration space X_0.
- For $p = -1$, $\widetilde{X}^{-1} = \langle \varphi_1^* ... \varphi_q^* \rangle$, with $\{\varphi_i^*\}$ being the basis of the shifted tangent space $T_{X_0}[1]$.
- For $p < -1$, $\widetilde{X}^p = \langle \varphi_1^{(-p-1)*} ... \varphi_{q(-p-1)}^{(-p-1)*} \rangle$, with $\{\varphi_i^{(-p-1)*}\}$ being the generators of $\mathcal{E}^*[1]^p$ as the dual of homogeneous component $\mathcal{E}^{-p-1}$.

In the local coordinate, the canonical choice of the Poisson bracket is just

$$[f, g] = \sum_i (-1)^{p(\varphi_i)(p(f)+1)} \left(\frac{\partial f}{\partial \varphi_i} \frac{\partial g}{\partial \varphi_i^*} + (-1)^{p(f)} \frac{\partial f}{\partial \varphi_i^*} \frac{\partial g}{\partial \varphi_i} \right) \tag{1.6}$$

We can easily check that the above equation satisfies the requirements of Def[1.2]. It will also satisfy the canonical relation

$$[\varphi_i, \varphi_j^*] = -[\varphi_j^*, \varphi_i] = \delta_{ij} \tag{1.7}$$

After a long march through a lot of definitions we finally come to the main player B-V variety

Definition 1.14 *Let $S_0 \in \Gamma(X, \mathcal{O}_X^0)$ be a regular function of degree 0 on $X \in \mathcal{C}$. A **B-V variety with support** (X, S_0) is a pair (M, S) consisting of a (-1)-symplectic variety M with support X and a function $S \in \Gamma(X, \mathcal{O}_M^0)$ such that*

- *$S|_X = S_0$;*
- *S is a solution of the classical master equation $[S, S] = 0$;*
- *the cohomology sheaf of the complex $(\mathcal{O}_M/I_M, d_S)$ vanishes in nonzero degree.*

The equation $[S, S] = 0$ is called the **classical master equation**. The classical master equation guarantees that we can define the operator $d_S := [S, \_] : \mathcal{O}_M^n \to \mathcal{O}_M^{n+1}$, which is a differential over the sheaf of P_0-algebras $\mathcal{O}_M$. Because d_S is a derivation of degree 1, it preserves the ideal $I_M = F^1\mathcal{O}_M$, therefore d_S defines also a differential on the reduced sheaf of $\mathbb{Z}_{\leq 0}$-graded algebras $\mathcal{O}_M/I_M$.

After the introduction of B-V variety, we can check the following proposition.

Proposition 1.15 *Let $M = T^*[-1]V$ then $\mathcal{O}_M^{-1}$, is a sheaf of Lie algebra acting on the sheaf $\mathcal{O}_M$ by derivation of degree 0. $\mathfrak{g}_M = \mathcal{O}^{-1} \cap I_M^{(2)}$ is a Lie algebra with $I_M^{(2)}$ the 2-power of ideal I_M*

Definition 1.16 *The adjoint action of the Lie algebra $\mathfrak{g}_M$ exponentiate to a sheaf of groups $G_M = \exp(ad\mathfrak{g}_M)$ Let $\mathfrak{g}(M) = \Gamma(X, \mathfrak{g}_M)$. Then we define the group of Poisson automorphisms $G(M) = exp(ad\mathfrak{g}(M))$ to be the group of **gauge equivalences**.*

Now, we describe how an B-V variety automatically gives rise to a differential graded complex and a notion of cohomology, known as BRST cohomology.

Definition 1.17 *The BRST complex of a B-V variety of a B-V variety (M, S) is the sheaf of differential P_0-algebras $(\mathcal{O}_M, d_S)$, with $d_S = [S, \_]$.*

Lemma 1.18 *For a B-V variety (M, S), the stablizer $G(M, S)$ of S would induce identity operator on the sheaf of BRST cohomology*

2 Felder and Kazhdan's theorems

Then the problem comes. Is it true that for every gauge theory we can find a corresponding B-V variety? In the paper[2] Felder and Kazhdan gave a positive answer.

Theorem 2.1 *S_0 being a regular function on a nonsingular affine variety X_0 over a field k of Characteristic 0, there exists*

1. *a B-V variety (M, S) with support (X_0, S_0) such that $M \cong T^*[-1]V$ for some non-negatively graded variety V. It is unique up to a stable equivalence.*
2. *The Poisson automorphisms of (M, S) acts as the identity operator on the cohomology sheaf, which means that the BRST cohomology $\mathcal{H}^\bullet(\mathcal{O}_M, d_S)$ is determine by (X_0, S_0) up to a unique isomorphism.*

Theorem 2.2

1. *The BRST complex forms a cohomology sheaf on the critical locus of S_0 and vanished in the negative degree.*
2. *The BRST cohomology group of degree 0 of this theory describes the gauge invariant observable of initial gauge theory (X_0, S_0), that is:*

$$\mathcal{H}^0(\mathcal{O}_M, d_S) = \{\textit{Observables of the initial gauge theory } (X_0, S_0)\}$$

The second statement of the theorem above is in physical language, the precise mathematical meaning is

$$\mathcal{H}^0(\mathcal{O}_M, d_S) \cong J(S_0)^{L(S_0)} := \{f \in J(S_0) | \xi(f) = 0, \forall f \in L(S_0)\}, \qquad (2.1)$$

where $J(S_0)$ is the Jacobian ring of the Lie algebra the preserve S_0.

The detailed proof requires some knowledge of spectral sequence and Tate resolution, which is still a little out of reach for me and that is why I will omit it here temporarily. These theorems have been verified in pure Yang-Mills theory with arbitrary semisimple gauge group[1] and generalized to many different matrix models[5]

3 Extension to classical field theory

Now, let's introduce the local description of B-V formalism in the case of $0+1$ dimensional field theory. Denote by $\mathcal{A} = \oplus_{j\in\mathbb{Z}}\mathcal{A}^j$ the superspace of all differential expressions in the fields and antifields with $\mathcal{A}^j$ being the homogeneous component of ghost number j. Notice that for example $\mathcal{A}^0$ consists of physical field ϕ and its derivatives $\{\partial^k\phi\}_{k>0}$, which is not finite dimensional which is technically not a sheaf of graded commutative algebra of a B-V variety. Though without a rigorous proof, we can still expect this difference to be non-essential because Felder and Kazhdan's theorems have been check in some higher dimensional case[some ref]. The Poisson bracket defined on this space is

$$\begin{aligned}\llbracket f,g\rrbracket := &\sum_i (-1)^{p(\varphi_i)(p(f)+1)} \\ &\sum_{k,l=0}^{\infty}\left(\partial^l(\partial_{k,\varphi_i}f)\partial^k(\partial_{l,\varphi_i^*}g) + (-1)^{p(f)}\partial^l(\partial_{k,\varphi_i^*}f)\partial^k(\partial_{l,\varphi_i}g)\right)\end{aligned} \tag{3.1}$$

It has been proved that this bracket satisfy the following properties

- $\llbracket f,g\rrbracket = -(-1)^{(p(f)+1)(p(g)+1)}\llbracket g,f\rrbracket$
- $\llbracket f,\llbracket g,h\rrbracket\rrbracket = \llbracket\llbracket f,g\rrbracket,h\rrbracket + (-1)^{(p(f)+1)(p(g)+1)}\llbracket g,\llbracket f,h\rrbracket\rrbracket$
- $\llbracket \partial f,g\rrbracket = \llbracket f,\partial g\rrbracket = \partial\llbracket f,g\rrbracket$

Then, we denote by $\mathcal{F}$ the superspace of funcitonals $\mathcal{A}/\partial\mathcal{A}$, where $\partial\mathcal{A}$ is the subspace of total derivatives. Denote the image of $f\in\mathcal{A}$ in $\mathcal{F}$ by $\int f\ dt$. The above bracket would induce the correct bracket on the superspace $\mathcal{F}$ Notice that in this case the classical master equation could be written as

$$\llbracket S,S\rrbracket = \partial S' \tag{3.2}$$

and the corresponding bracket in $\mathcal{F}$ could be written explicitly via the variational derivatives

$$\left\{\int f, \int g\right\} = \sum_i (-1)^{(p(f)+1)(p(\varphi_i)}\int\left(\frac{\delta f}{\delta\varphi_i}\frac{\delta g}{\delta\varphi_i^*} + (-1)^{p(f)}\frac{\delta f}{\delta\varphi_i^*}\frac{\delta g}{\delta\varphi_i}\right)dt \tag{3.3}$$

We can calculate the cohomology of functionals $\mathcal{H}(\mathcal{F},d_S)$ via first calculating the cohomology of functions $\mathcal{H}(\mathcal{A},d_S)$. Notice that because of the existence of the following short exact sequence

$$0 \longrightarrow \tilde{\mathcal{A}} \xrightarrow{\partial} \mathcal{A} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where

$$\tilde{\mathcal{A}}^k = \begin{cases} \mathcal{A}^0/C & (k=0) \\ \mathcal{A}^k & (k \neq 0), \end{cases}$$

there exists a long exact sequence

$$\begin{tikzcd} \ldots \arrow[r] & \mathcal{H}^{-1}(\mathcal{A}, d_S) \arrow[r, "\partial"] & \mathcal{H}^{-1}(\mathcal{A}, d_S) \arrow[r] & \mathcal{H}^{-1}(\mathcal{F}, d_S) \arrow[dll, rounded corners] \\ & \mathcal{H}^{0}(\mathcal{A}/C, d_S) \arrow[r, "\partial"] & \mathcal{H}^{0}(\mathcal{A}, d_S) \arrow[r] & \mathcal{H}^{0}(\mathcal{F}, d_S) \arrow[dll, rounded corners] \\ & \mathcal{H}^{1}(\mathcal{A}, d_S) \arrow[r, "\partial"] & \mathcal{H}^{1}(\mathcal{A}, d_S) \arrow[r] & \mathcal{H}^{1}(\mathcal{F}, d_S) \arrow[r] & \ldots \end{tikzcd}$$

A simple extension of the theorem of Felder and Kazhdan implies that

$$\mathcal{H}^{-k}(\mathcal{A}, d_S) = 0 \quad (k > 0)$$

and the analysis of long exact sequence implies

$$\mathcal{H}^{-k}(\mathcal{F}.d_S) = 0 \quad (k > 1).$$

However, in the next section we will see that the above assertion fails in the case of spinning particle model.

4 Spinning particle

However, unfortunately, we would encounter some difficulties when we try to generalize the the above theorems to supersymmetric case. The difficulty mainly comes from the fact that supersymmetric ghost is invertible.

The spinning relativistic particle is a variant of the plain relativistic particle which has an spin degree of freedom. Coonsider a d-dimensional space with flat metric $\eta_{\mu\nu}$. The spinning particle is a one dimensional field theory with physical fields $x^\mu(t)$ and $\theta^\mu(t)$ along its worldline, where they are of parity 0 and 1 respectively. The corresponding action is

$$S = \frac{1}{2}\eta_{\mu\nu}\left(\partial x^\mu \partial x^\nu - \theta^\mu \partial \theta^\nu\right).$$

Because we want to work with only first order term of ∂x^μ, we use the effective action

$$S = p_\mu \partial x^\mu - \frac{1}{2}\eta_{\mu\nu}\theta^\mu \partial\theta^\nu - \frac{1}{2}\eta^{\mu\nu}p_\mu p_\nu.$$

An additional physical field p_μ with even parity is introduced here. The effective action can be linked to the original action via a Gaussian integral over p_μ.

The above actions defines the free theory of spinning particle, we can easily check that the Felder&Kazhdan's theorem still holds here. In order to get a local supersymmetry, may can couple the spinning particle to supergravity. The resulting action is

$$S_0 = p_\mu \partial x^\mu - \frac{1}{2}\eta_{\mu\nu}\theta^\mu \partial\theta^\nu - \frac{1}{2}e\eta^{\mu\nu}p_\mu p_\nu + \varphi p_\mu \theta^\mu$$

where the the e and *varphi* are physical fields in the supergravity multiplet and $p(e) = 0$, $p(\varphi) = 1$. The differential is now

$$\begin{aligned}
d_{S_0} e^* &= -\frac{1}{2}\eta^{\mu\nu}p_\mu p_\nu \\
d_{S_0} \varphi^* &= -p_\mu \theta^\mu \\
d_{S_0} x^*_\mu &= -\partial p_\mu \\
d_{S_0} \theta^*_\mu &= \eta_{\mu\nu}\partial\theta^\nu + \varphi p_\mu \\
d_{S_0} p^*_\mu &= \partial x^\mu - e\eta^{\mu\nu}p_\nu + \varphi\theta^\mu.
\end{aligned}$$

The local gauge symmetries correspond to the cohomology class of d_{S_0} at ghost number -1.

$$\begin{aligned}
&d_{S_0}\left(\partial e^* - \eta^{\mu\nu}x^*_\mu p_\nu\right) = 0 \\
&d_{S_0}\left(\partial\varphi^* + \eta^{\mu\nu}\theta^*_\mu p_\nu - x^*_\mu\theta^\mu + 2e^*\varphi\right) = 0.
\end{aligned}$$

Then there is a canonical way to kill these cohomology class by adding to action the term

$$S_1 = \left(\partial e^* - \eta^{\mu\nu}x^*_\mu p_\nu\right)c + \left(\partial\varphi^* + \eta^{\mu\nu}\theta^*_\mu p_\nu - x^*_\mu\theta^\mu + 2e^*\varphi\right)\gamma.$$

c and γ are ghost field with parity $p(c) = 1$ and $p(\gamma) = 0$. We also add to the configuration space the corresponding antighosts. Then we get the table of differentials

$$
\begin{array}{ll}
d_{S_1} c^* = \partial e^* - \eta^{\mu\nu} x^*_\mu p_\nu & d_{S_1} \gamma^* = \partial \varphi^* + \eta^{\mu\nu} \theta^*_\mu p_\nu - x^*_\mu \theta^\mu + 2e^* \varphi \\
d_{S_1} \varphi^* = 2e^* \gamma & d_{S_1} \theta^*_\mu = -x^*_\mu \gamma \\
d_{S_1} p^{*\mu} = -\eta^{\mu\nu} x^*_\nu c + \eta^{\mu\nu} \theta^*_\nu \gamma & d_{S_1} x^\mu = -\eta^{\mu\nu} p_\nu c - \theta^\mu \gamma \\
d_{S_1} \theta^\mu = -\eta^{\mu\nu} p_\nu \gamma & d_{S_1} e = -\partial c + 2\varphi\gamma \\
d_{S_1} \varphi = \partial \gamma &
\end{array}
$$

This choice of S_1 respects the conservation of stress-energy tensor. We now can add the final term

$$
S_2 = -c^* \gamma^2,
$$

and the the corresponding differentials are

$$
\begin{aligned}
d_{S_2} \gamma^* &= -2c^* \gamma \\
d_{S_2} c &= \gamma^2
\end{aligned}
$$

This complete the definition of the theory and the resulting action $S = S_0 + S_1 + S_2$ satisfies the classical master equation

$$
\left\{ \int S, \int S \right\} = 0
$$

A simple calculation in the case of spinning particle in $d = 0$ space can illustrate the deviation form the theorems. In the zero dimensional target space, the matter fields x^μ, θ^μ, p_μ are absent. To show our conclusion, we have to use some trick. Firstly, we consider the following elements of localization $\mathcal{A}_\gamma$ of $\mathcal{A}$ by the inversion of γ:

$$
\begin{aligned}
A_k &= (\varphi^*)^{k+1} c \gamma^{-1} \in \mathcal{A}_\gamma^{-k-1} \quad & k \geq -1, \\
B_k &= \frac{1}{k} (\varphi^*)^k \gamma^{-1} \in \mathcal{A}_\gamma^{-k-1} \quad & k \geq 1.
\end{aligned}
$$

The corresponding differentials are in $\mathcal{A}$:

$$
\begin{aligned}
\alpha_k &= 2(k+1)(\varphi^*)^k e^* c + (\varphi^*)^{k+1} \gamma \in \mathcal{A}^{-k} \quad & k \geq -1, \\
\beta_k &= (\varphi^*)^{k-1} e^* \in \mathcal{A}^{-k} & k \geq 1.
\end{aligned}
$$

Theorem 4.1 *The cohomology group $\mathcal{H}^{-k}(\mathcal{A}, d_S)$ is spanned by the cocycles α_k and β_k and the cocycle 1 in degree 0. All the cohomologies in positive degree vanishes.*

Proof *The new differential operator d_S can be regarded as a quadratic perturbation of $d_{[0]}$, where $d_{[0]}$ is the differential of d_S by retaining only the linear terms. W can calculate the cohomology be spectral sequence argument*

$$\begin{aligned} d_{[0]}c^* &= \partial e^* \\ d_{[0]}\gamma^* &= \partial \varphi^* \\ d_{[0]}e &= -\partial c \\ d_{[0]}\varphi &= \partial \gamma \end{aligned}$$

the cohomology E_1 of the differential $d_{[0]}$ is a graded polynomial ring in generators

$$\mathbf{E}^* = \int e^*, \quad \mathbf{\Phi}^* = \int \varphi^*, \quad \mathbf{C} = \int c, \quad \textit{and} \quad \mathbf{\Gamma} = \int \gamma.$$

The differential $d_{[1]}$ on E_1 is given by

$$d_{[1]}\mathbf{\Phi}^* = 2\mathbf{E}^*\mathbf{\Gamma}, \qquad d_{[1]}\mathbf{C}^* = \mathbf{\Gamma}^2$$

The genreal form of a cochain z of degree $-k$ with $k > 0$ is

$$\begin{aligned} z = {} & u_k(\mathbf{\Phi}^*)^k + v_k(\mathbf{\Phi}^*)^{k-1}\mathbf{E}^* \\ & + \sum_{j=1}^{\infty} \Big((\mathbf{\Phi}^*)^{k+j}(u_{k+j}\mathbf{\Gamma}^j + U_{k+j}\mathbf{C}\mathbf{\Gamma}^{j-1}) + (\mathbf{\Phi}^*)^{k+j-1}\mathbf{E}^*(v_{k+j}\mathbf{\Gamma}^j + V_{k+j}\mathbf{C}\mathbf{\Gamma}^{j-1}) \Big) \end{aligned}$$

Requiring $d_{[1]}z = 0$, we find that $U_{k+j} = V_{k+j} - 2(k+j)u_{k+j} = 0$ and a general cocycle should be

$$\begin{aligned} & z = u_{k+1}\alpha_k + v_k\beta_k \\ & d_{[1]}\left(\sum_{j=2}^{\infty} u_{k+j}(\mathbf{\Phi}^*)^{k+j}\mathbf{C}\mathbf{\Gamma}^{j-2} + \sum_{j=1}^{\infty} \frac{v_{k+j}}{2(k+j)}(\mathbf{\Phi}^*)^{k+k}\mathbf{\Gamma}^{j-1} \right). \end{aligned}$$

This verfies that $\mathcal{H}^{-k}(\mathcal{A}, d_S)$ is spanned by α_k and β_k. Similarly, we can prove the degree 0 case.

By analyzing the long exact sequence, we find that

Corollary 4.3 *The cohomology group $\mathcal{H}^{-k}(\mathcal{F}, d_S)$ is spanned by $\int \alpha_k$,$\int \beta_k$,$\int \tilde{\alpha}_k$ and $\int \tilde{\beta}_k$, and $\int 1$ in degree 0, where $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ are defined to be*

$$\begin{aligned} \tilde{\alpha}_k &= \left(c^*\frac{\partial}{\partial e^*} - e\frac{\partial}{\partial c} + \gamma^*\frac{\partial}{\partial \varphi^*} + \varphi\frac{\partial}{\partial \gamma} \right)\alpha_k \quad k \geq 0 \\ \tilde{\beta}_k &= \left(c^*\frac{\partial}{\partial e^*} - e\frac{\partial}{\partial c} + \gamma^*\frac{\partial}{\partial \varphi^*} + \varphi\frac{\partial}{\partial \gamma} \right)\beta_k \quad k \geq 2. \end{aligned}$$

5 A failed attempt and conclusion

Short after the paper[3] was posted on arxiv, under the supervision of Professor Felder, the author followed the calculation and tried to resolve this problem but failed.

The initial idea is quite transparent, we simply generalize to all negative order of γ, which amounts to calculate in the localization $\mathcal{A}_\gamma$ instead of $\mathcal{A}$. This seems to be promising at first, because we can expect the negative degree cohomology to be killed by some complex of 1 degree lower. However after some calculation, we found that the introduction of all negative powers of SUSY ghost γ would indeed kill the cohomology in the $-k$ degree, but, simultaneously and unfortunately, it will also kill all the 0-th cohomology and all the cohomologies in all order.

The author found this unpleasant result via some tedious and cumbersome calculation at first and was quickly pointed out that the complex 1 would be exact in the $\mathcal{H}^0(\mathcal{A}_\gamma, S)$. If we preserve the notion in last section but only calculate in the localization $\mathcal{A}_\gamma$, 1 would be a coboundary:

$$d_{[1]}\mathbf{C}\mathbf{\Gamma}^{-2} = 1$$

For a general cohomology theory, suppose that $d(b) = 1$, then all closed complex would be also exact because $d(ab) = d(a)b + (-1)^{deg(a)}ad(b) = (-1)^{deg(a)}a * 1$, which kills all the cohomologies in all degree, rendering a physically trivial theory.

This effect also appears in non-zero dimensional target space and curved space[4]. The mystery caused by SUSY ghost still remains unsolved and we plan to investigate it in some other SUSY models subsequently.

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