<< Commutative Algebra >> — CA- ~~Notes~~ Lec 1-1 001

Convention: ① Ring = + Commutative + Identity. ② Ring Homomorphism : $f(1_R) = 1_S$

③ $1=0$ is allowed, which means $R=\{0\}$

By Def, $1 \neq 0$ in a field.

Why Commutative Algebra ?

- DG = Lemma of Level Sets

  AG = $f^{-1}(\{0\})$ Consider $R = k[x_1, \dots, x_n]$

  Example: $n=2$, $R = k[x_1, x_2]$

  $V(\{x_1\}) = x_2$-axis in k^2

  $V(\{x_2 - x_1^2\}) =$ parabola in k^2

  Def: Algebraic Variety Affine. $V(S) = \{x \in k^n \mid \forall f \in S, f(x) = 0\}$

  Fact: $V(\emptyset) = k^n$, $V(\{1\}) = \emptyset$

  $\forall$ nonempty collection of subsets $S \subset R$: $\bigcap_{i \in I} V(S_i) = V(\bigcup_{i \in I} S_i)$

  $V(S) \cup V(S') = V(\{fg \mid f \in S, g \in S'\})$

  Consequence: $(V(S))_{S \subset R}$: Zariski Topology

✓ Def: Conversely, $\forall$ ~~S~~ $X \subset k^n$,

let $I(X) = \{f \in R \mid \forall x \in X, f(x) = 0\} \subset R$

✓ Fact: $S \subset R$, $X \subset R^n$, then $S \subset I(V(S))$, $X \subset V(I(X))$

$S \subset S' \subset R$, $V(S) \supset V(S')$; $X \subset X' \subset R^n$, $I(X) \supset I(X')$

$I(X) \subset R$ is an ideal.

✓ • Radical of an ideal $\mathfrak{a} \subset R$: $r(\mathfrak{a}) = \{a \in R \mid \exists n \geq 1, a^n \in \mathfrak{a}\} \subset R$

An ideal $\mathfrak{a} \subset R$ with $\mathfrak{a} \in r(\mathfrak{a})$ is called radical.

✓ Fact: $\forall$ ideal $\mathfrak{a} \subset R$, then $\mathfrak{a} \subset r(\mathfrak{a})$; $r(\mathfrak{a})$ is an ideal.

$X \subset k^n$, the ideal $I(X)$ is radical.

✓ Theorem: (Hilbert's Nullstellensatz) $\forall$ ideal $\mathfrak{a} \in R$, we have $I(V(\mathfrak{a})) = r(\mathfrak{a})$.

✓ Consequence: maps V, I : $\{$radical maps$\} \underset{I}{\overset{V}{\rightleftarrows}} \{$Affine Algebra Varieties$\}$. (Here V, I is not the same as $V(S), I(X)$)

are usually inverse bijectives (inclusion reversing)

Example: $\forall x = (x_1, \dots, x_n) \in k^n$, the ideal $I(\{x\}) = m_x = (X_1 - x_1, \dots, X_n - \dots)$ is maximal.

Proof: If not, $\exists$ ideal $\mathfrak{a} \subset R$, with $(1) \supsetneq \mathfrak{a} \supsetneq m_x$. but then $\emptyset \subsetneq V(\mathfrak{a}) \subsetneq V(m_x)$ □

Theorem (Weak Nullstellensatz). The ideals m_x are precisely the maximal ideals of $k[x_1, \dots, x_n]$. k: needs to be algebraicly closed.

eg. $k = \mathbb{R}$, $n=1$. x^2+1, irreducible in $\mathbb{R}[x]$. $\mathbb{R}[x]/[x^2+1] \cong \mathbb{C}$, hence $\langle x^2+1 \rangle$ maximal

Consequence: bijection $\{$maximal ideals of $R\} \leftrightarrow k^n$. where $R = k[x_1, \dots, x_n]$.

A: Ring

• Nilpotent : $\exists n \geq 1$, s.t. $a^n = 0$ Zero Divisor: $b \neq 0$, $\exists a$, s.t. $ab = 0$.

✓ Nilpotents are zero divisors but not converse. $(0,1) \times (1,0) = (0,0)$

Def: $N = r((0))$, nilradical. ① N is the set of all Nilpotents in A.

② A/N has no nilpotent

Proof: ① by Def ② $x \in A$, s.t. $\bar{x} \in A/N$, $\bar{x}$ nilpotent. let $\bar{x}^n = (0) = \overline{x^n}$

then $x^n \in N$, s.t. $\exists k > 0$, s.t. $(x^n)^k = 0$. hence $x^{nk} = 0$, $x \in N$, so. $\bar{x} = 0$. contra

Proposition: Nilradical of A is the intersection of all prime ideals.

Proof: • $\forall f \in A$, $f^n = 0$, $f^n \in P$ for every prime ideal P, hence $f \in P$, $f \in N$

• f: not nilpotent, let $I = \{\mathfrak{a} \in A$ ideals $\mid \forall n > 0, f^n \notin \mathfrak{a}\}$.

Apply Zorn's Lemma: $(0) \in \Sigma$, Σ is non-empty, Σ is partially ordered by inclusion. ...
for any chain of ideals $(\mathfrak{a}_i)_{i\in I}$, $\in \Sigma$, the set $\mathfrak{a} = \bigcup_{i\in I} \mathfrak{a}$ is an ideal. (Verify!)
$(\mathfrak{a}_1 \subset \mathfrak{a}_2 \cdots \subset \mathfrak{a}_n \subset \cdots)$
For $\forall n>0$, f^n is not contained in ~~...~~, thus $f^n \notin \mathfrak{a}$, hence $\mathfrak{a} \in \Sigma$
By Zorn's Lemma: then there is a maximal element $p \in \Sigma$. We show that p is a prime ide...
$\forall x,y \notin p$, ideals $\langle x\rangle + p$, $\langle y\rangle + p$, ~~then~~ they're not in Σ.
Let $m,n>0$, $f^n \in p+\langle x\rangle$, $f^m \in p+\langle y\rangle$, then $f^{m+n} \in p+\langle xy\rangle$, so $p+\langle xy\rangle \notin \Sigma$.
hence $xy \notin p$.

✓ $f: A\to B$, ring homomorphism, $p \subset B$, prime ideal. then $f^{-1}(p)$ is a prime ideal of A.
~~Not~~ Notice: not true for maximal ideals in general.
~~By~~ But since maximal ideals $\Rightarrow$ prime ideals ...

✓ $\mathfrak{a} \in A$ ideal, $\pi: A \to A/\mathfrak{a}$, Prop: $\exists$ 1→1 correspondence between ideals of $A/\mathfrak{a}$ and ideals of A containing $\mathfrak{a}$. Via $\mathfrak{b} = \pi^{-1}(\mathfrak{b}')$, $\mathfrak{b}' \in A/\mathfrak{a}$. ($\pi$: homomorphism)

- Corollary: $\mathfrak{a} \in A$, $r(\mathfrak{a})$ is the intersection of all prime ideals contaning $\mathfrak{a}$.
  Proof: Consider $\pi: A \to A/\mathfrak{a}$, $r(\mathfrak{a}) = \pi^{-1}(\mathfrak{N}_{A/\mathfrak{a}})$ (by definition & the proposition before)
  by correspondence, conclude.
- Def: Jacobson Radical $= \mathfrak{R}$ of A. $\mathfrak{R}$: intersection of all maximal ideals.
  Prop: $x \in \mathfrak{R} \Leftrightarrow \forall y \in A$, $1-xy$ is a unit.
  Proof: $\Rightarrow$ $x \in \mathfrak{R}$, $y \in A$, s.t. $1-xy$ is not a unit. (every nonunit is in a maximal ideal)
  $1-xy \in m$ (maximal), $x \in \mathfrak{R} \subset m$. hence $1 = (1-xy)+(xy) \in m$, contra.
  $\Leftarrow$ $x \notin \mathfrak{R}$, then $\exists m$: maximal, $x \notin m$, thus $m + \langle x\rangle = \langle 1\rangle = A$
  then $\exists y \in A$, $u \in m$, s.t. $xy+u=1$, thus $1-xy \in m$, thus $1-xy$ not a uni...

Lecture 2. Lec 1-2 ✓

Def. Local Ring. Residue field.
Example. ① Every field is a local ring $M=\langle 0\rangle$. ② Ring of a formal power series over field $K[[x]]$ with maximal ideal (x).
Prop: (i) A: ring, $m \neq \langle 1\rangle$: ideal, If $\forall x \in A\backslash m$ is unit, then A: local, m: maximal.
(ii) A: ring, $m \subset A$: maxi ideal. If $\forall x = 1+t$, $t \in m$, x: unit, then A: local.
Proof: Like in the book.
Def: Semilocal = finite maximal ideals.
Example: $\mathbb{Z}$ is not semilocal. (maximal ideals are those generated by prime elements)
$\forall m \in \mathbb{Z}$, $\mathbb{Z}/m\mathbb{Z}$ is a semi-local ring, with maximal ideals: $d\mathbb{Z}/m\mathbb{Z}$ for $d|m$ (d prime). But for $p \in \mathbb{Z}$ prime, $\mathbb{Z}/p^n$ $(n>1)$ is a local ring.
Reminder: ~~A~~ $\mathfrak{a}, \mathfrak{b} \subset A$: ideals, $\mathfrak{a}+\mathfrak{b} = \{a+b \mid a \in \mathfrak{a}, b \in \mathfrak{b}\}$, least ideal $\supseteq \mathfrak{a}\cup\mathfrak{b}$
(SUM)
Inf-SUM: $(\mathfrak{a}_i)_{i\in I}$. $\sum_{i\in I} \mathfrak{a}_i = \{\sum_{i\in I} x_i \mid x_i \in \mathfrak{a}_i$ for all $i \in I$, $x_i = 0$ for all but finitely many $i \in I\}$.
$\mathfrak{a}\cdot\mathfrak{b} = \{\sum_{i\in I} x_i y_i \mid x_i \in \mathfrak{a}, y_i \in \mathfrak{b}$, all but finitely many terms $0\}$.
$\prod_{i=1}^{n} \mathfrak{a}_i = \{\sum b_1^i \cdots b_n^i \mid$ finite sums$\}$ = or define by induction.

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Lec1-2

Lecture 2: $\mathfrak{a}, \mathfrak{b} \subset A$ coprime (ideals) if $\mathfrak{a}+\mathfrak{b}=(1)$.

Fact: If $\mathfrak{a}, \mathfrak{b} \subset A$ coprime, then $\mathfrak{a}\cap\mathfrak{b}$ (always ideal) $= \mathfrak{a}\cdot\mathfrak{b}$.

For general ideals in $\mathfrak{a}, \mathfrak{b}$, $(\mathfrak{a}+\mathfrak{b})\cdot(\mathfrak{a}\cap\mathfrak{b}) \subset \mathfrak{a}\cdot\mathfrak{b} \subset \mathfrak{a}\cap\mathfrak{b}$.

Atiyah P7

Prop: $\mathfrak{a}_1, \dots, \mathfrak{a}_n$: ideals $\subset A$. $\varphi: A \to \prod_{i=1}^{n}(A/\mathfrak{a}_i)$ for the canonical homomorphism.

(i) If $\mathfrak{a}_i, \mathfrak{a}_j$ coprime, $i\neq j$, then $\prod_{i=1}^{n}\mathfrak{a}_i = \bigcap_{i=1}^{n}\mathfrak{a}_i$

(ii) φ: surjective IFF $\mathfrak{a}_i, \mathfrak{a}_j$ coprime $i\neq j$.

(iii) φ: injective IFF $\bigcap_{i=1}^{n}\mathfrak{a}_i = \langle 0 \rangle$.

Proof (iii): $\ker\varphi = \bigcap_{i=1}^{n}\mathfrak{a}_i$

(i) Induction on n, $n=2$ ✓

let $n>2$, let $\mathfrak{b} = \prod_{i=1}^{n-1}\mathfrak{a}_i = \bigcap_{i=1}^{n-1}\mathfrak{a}_i$

$\mathfrak{a}_i + \mathfrak{a}_n = \langle 1 \rangle$, $1\leq i\leq n-1$, $x_i + y_i = 1$, $\exists x_i \in \mathfrak{a}_i$, $y \in \mathfrak{a}_n$.

thus $\prod_{i=1}^{n-1} x_i = \prod_{i=1}^{n-1}(1-y_i) \equiv 1\ (\mathfrak{a}_n)$ thus $\mathfrak{a}_n + \mathfrak{b} = (1)$
$-1+t \in \mathfrak{a}_n$

Thus $\prod_{i=1}^{n}\mathfrak{a}_i = \mathfrak{b}\cdot\mathfrak{a}_n = \mathfrak{b}\cap\mathfrak{a}_n = \bigcap_{i=1}^{n}\mathfrak{a}_i$

(ii) "$\Rightarrow$" φ: surjective. $i\neq j$. $\exists x_i \in A$, s.t. $\varphi(x_i) = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$

thus $x_i \equiv 1(\mathfrak{a}_i)$, $x_i \equiv 0(\mathfrak{a}_j)$, $1 = (1-x_i) + x_i \in \mathfrak{a}_i + \mathfrak{a}_j$

"$\Leftarrow$" Show $\forall k \in \{1, \dots, n\}$, $\exists x_k \in A$, s.t. $\varphi(x_k) = (0, \dots, 0, 1, 0, \dots, 0)$.

$\forall k$, $\forall j \in \{1, \dots, n\}$, $\mathfrak{a}_k + \mathfrak{a}_j = \langle 1 \rangle$, $k\neq j$, thus ~~then~~ $\exists u_j \in \mathfrak{a}_k, v_j \in \mathfrak{a}_j$.

s.t. $u_j + v_j = 1$ let $x = \prod_{\substack{i=1\\ i\neq k}}^{n} v_i$, then $x \equiv 0(\mathfrak{a}_j)$, for $j\neq k$.

and $x = \prod_{i\neq k}^{n}(1-u_i) \equiv 1 \mod(\mathfrak{a}_k)$

Prop: $\mathfrak{a}, \mathfrak{b} \subset A$, ideals. If $r(\mathfrak{a}), r(\mathfrak{b})$ coprime $\Rightarrow \mathfrak{a}, \mathfrak{b}$ = coprime

Proof: $r(\mathfrak{a}+\mathfrak{b}) \overset{Ex}{=} r(r(\mathfrak{a})+r(\mathfrak{b})) \overset{Ex}{=} r((1)) = (1)$

Prop: (i) $P_1, \dots, P_n$: prime ideals in A, let $\mathfrak{a} \subset A$: ideal, $\mathfrak{a} \subset \bigcup_{i=1}^{n} P_i$, then $\mathfrak{a} \subset P_i$ for some i

(ii) $\mathfrak{a}_1, \dots, \mathfrak{a}_n$: ideals, $P \subset A$: prime, s.t. $P \supset (\bigcap_{i=1}^{n}\mathfrak{a}_i)$, then $P \supset \mathfrak{a}_k$ for some k.

if $P = (\bigcap_{i=1}^{n}\mathfrak{a}_i)$, then $P = \mathfrak{a}_i$ for some i.

Proof: (i) Induction on n: $n=1$ ✓. If $n>1$, assume $\mathfrak{a} \not\subset P_i$, for all $1\leq i\leq n$, then ...

$\forall k$, $\mathfrak{a} \not\subset \bigcup_{\substack{i=1\\ i\neq k}} P_k$, $\exists x_k \in \mathfrak{a}$, s.t. $x_k \notin P_i$ for all $i\neq k$.

If $x_k \notin P_k$, for some k, done

If not, then $\forall k$, $x_k \in P_k$, let $y = \sum_{k=1}^{n}\prod_{j\neq k} x_j \in \mathfrak{a}$. $y = \prod_{j\neq k} x_j\ (P_k)$, $\forall k$.

Since $x_j \notin P_k$, $\forall j\neq k$, P_k is a prime ideal, thus $y \notin P_k$, $\forall k$, ~~then~~

Then $\mathfrak{a} \not\subset \bigcup_{i=1}^{n} P_i$

(ii) suppose $\forall i \in \{1, \dots, n\}$, $P \not\supset \mathfrak{a}_i$, then $\exists x_i \in \mathfrak{a}_i$, $x_i \notin P$, $\forall i$. thus $\prod_{i=1}^{n} x_i \in \prod_{i=1}^{n}\mathfrak{a}_i \subset \bigcap_{i=1}^{n}\mathfrak{a}_i$

Since P: prime, $\prod_{i=1}^{n} x_i \notin P$. thus $P \not\supset \bigcap_{i=1}^{n}\mathfrak{a}_i$.

In "=" case, $P = \bigcap_{i=1}^{n}\mathfrak{a}_i \subset \mathfrak{a}_k$, $\forall k$.
$\mathfrak{a}_j \subset$ (for some j)

Def. $\mathfrak{a}, \mathfrak{b} \in A$: ideal. Ideal Quotient $= (\mathfrak{a} : \mathfrak{b}) = \{x \in A \mid x \cdot b \subset \mathfrak{a} \ (\forall b \in \mathfrak{b})\}$. $\Rightarrow$ is an ideal (Fact)

Annihilator of an ideal: $\mathfrak{a} \subset A$, $\mathrm{Ann}(\mathfrak{a}) = ((0) : \mathfrak{a})$

Notation: $x \in A$ $(\mathfrak{a} : x) = (\mathfrak{a} : \langle x \rangle)$

Fact: ~~Set~~ $\{$zero-divisors of $A\} = D \underset{\text{by def}}{=} \bigcup_{x \neq 0} \mathrm{Ann}(x) = \bigcup_{x \neq 0} r(\mathrm{Ann}(x))$

Proof: $D = r(D)$ (Extend the radical def to an non-ideal)

$D = r(D) = r(\bigcup_{x \neq 0} \mathrm{Ann}(x)) = \bigcup_{x \neq 0} r(\mathrm{Ann}(x))$

Properties: let $\mathfrak{a}, \mathfrak{b} \subset A$, ideals. (i) $\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$ (ii) $(\mathfrak{a} : \mathfrak{b}) \cdot \mathfrak{b} \subset \mathfrak{a}$. (iii) $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$

(iv) $(\mathfrak{a}_i)_{i \in I} \subset A$, ideals. $(\bigcap_{i \in I} \mathfrak{a}_i : \mathfrak{b}) = \bigcap_{i \in I} (\mathfrak{a}_i : \mathfrak{b})$

(v) $(\mathfrak{b}_i)_{i \in I} \subset A$, $(\mathfrak{a} : \sum_{i \in I} \mathfrak{b}_i) = \bigcap_{i \in I} (\mathfrak{a} : \mathfrak{b}_i)$

Def: $\mathfrak{a} \subset A$, ideal, $f : A \to B$ ring homomorphism. Def extension of $\mathfrak{a}$ by f be the ideal

$\mathfrak{a}^e = f_*(\mathfrak{a}) = \cancel{B \cdot f(\mathfrak{a})}$ $Bf(\mathfrak{a})$, ideal in B generated by $f(\mathfrak{a})$

$\mathfrak{b} \subset B$, def the contraction of $\mathfrak{b}$ via f to be the ideal $\mathfrak{b}^c = f^*(\mathfrak{b}) = f^{-1}(\mathfrak{b})$

Prop: Let $f : A \to B$, ring hom, $\mathfrak{a} \subset A$, $\mathfrak{b} \subset B$, ideals.

(i) $\mathfrak{a} \subset f^* f_*(\mathfrak{a}) = \mathfrak{a}^{ec}$, $\mathfrak{b} \supset f_* f^*(\mathfrak{b}) = \mathfrak{b}^{ce}$ (ii) $f^*(\mathfrak{b}) = f^* f_* f^*(\mathfrak{b})$, $f_*(\mathfrak{a}) = f_* f^* f_*(\mathfrak{a})$

(iii) C: ~~subset~~ set of contracted ideals in A, E: set of extended ideals in B.

$C = \{\mathfrak{a} \subset A \mid f^* f_*(\mathfrak{a}) = \mathfrak{a}\}$. $E = \{\mathfrak{b} \subset B \mid f_* f^*(\mathfrak{b}) = \mathfrak{b}\}$.

$f_* : C \to E$ bijection with inverse f^*.

Proof. (i) $\mathfrak{a} \subset f^{-1} f(\mathfrak{a}) \subset f^{-1} f_*(\mathfrak{a}) = f^* f_*(\mathfrak{a})$ (i) $\mathfrak{b} \supset f f^{-1}(\mathfrak{b})$, $\mathfrak{b}$: ideal, so $\mathfrak{b} \supset f_* f^*(\mathfrak{b})$

(iii) Ex.

CA - ~~Lecture~~ Lec 2-1 - 003

Chapter 2. (A - M.D)

1. Modules. ("vector spaces" for rings)

A: Commutative Ring, with unit

Def. Module M over A, ① Abelian Group (+) ② μ: linear $A \times M \to M$, $(a,x) \to ax$

② μ: $a(x+y) = ax+ay$, $(a+b)x = ax+bx$, $a(bx) = (ab)x$, $1x = x$

$a, b \in A$, $x, y \in M$

Example ① $\{0\}$. ② A: field. any module = A-vector space

③ $I \subset A$, ideal, then I is a module on A, submodule of A

④ $A = \mathbb{Z}$, A-module is an Abelian Group.

$$n \cdot x = \begin{cases} 0, & n=0 \\ x+\cdots+x, & n>0 \\ -(x+\cdots+x), & n<0 \end{cases}$$

Def. M, N, A-module, $f: M \to N$, A-linear, $f(ax+by) = af(x)+bf(y)$

the set of such $f: M \to N$ is denoted $\mathrm{Hom}_A(M,N)$, it is a module on A. with $f+g = f(x)+g(x)$. $(af)(x) = a(f(x))$

If $M \xrightarrow{f} N \xrightarrow{g} P$, then $g \circ f \in \mathrm{Hom}_A(M,P)$

also $g \circ (f \circ h) = (g \circ f) \circ h$. $Id_M \in \mathrm{Hom}_A(M,M)$.

Def. $f: M \to N$ isomorphism. IFF $\exists g: N \to M$, $f \circ g = Id_N$, $g \circ f = Id_M$.
(assume f, g. A-linear)

IFF f: bijective, (A-linear) $g = f^{-1}$ is A-linear.

Rmk: $Q \xrightarrow{h} M \xrightarrow{f} N \xrightarrow{g} P$ (A-linear). $\Rightarrow \forall P$: A-module, we get $\mathrm{Hom}_A(N,P) \xrightarrow{f^*} \mathrm{Hom}_A(M,P)$

i.e. $g \mapsto g \circ f$, and $\mathrm{Hom}_A(Q,M) \longrightarrow \mathrm{Hom}_A(Q,N)$ i.e. $h \to f \circ h$.

⑧ suppose M is an A-module, $N \subset M$ submodule.

Then $\exists$ structure M/N, of A-module. s.t. $M \to M/N$ is A-linear.

Def. $a(x+N) = ax+N$. well-defined since $aN \subset N$.

Def: $f: M \to N$ $\ker(f) = f^{-1}(\{0\}) \subset M$, submodule.

$\mathrm{Im} = f(M) \subset N$, submodule

$\mathrm{Coker} = N/\mathrm{Im}(f)$ A-module.

Note. (1) $\ker(f) = \{0\} \Leftrightarrow f$: injective. $\mathrm{Coker}(f) = \{0\} \Leftrightarrow f$: surjective.

(2) $f: M \xrightarrow{\text{linear}} N$, $M' \subset \ker(f)$ submodule, then $\bar{f}$ induced linear map

$\bar{f}(x+M') = f(x)$ well-defined since $f(M') = \{0\}$.

$\bar{f} \circ p = f$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "p"'] & N \\ M/M' \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$\mathrm{Im}(\bar{f}) = \mathrm{Im}(f)$, $\ker(\bar{f}) = \ker(f)/M'$

In particular, if $M' = \ker(f)$, we get an isomorphism $M/\ker(f) \xrightarrow{\bar{f}} \mathrm{Im}(f)$

M = A-module, $(M_i)_{i \in I}$, submodules $\bigcap_{i \in I} M_i$ also a submodule,

If $X \subset M$, then the intersection of all submodule that contains X is a submodule

containing X, called the submodule generated by X, $\langle X \rangle$.

$\langle X\rangle = \{$ linear combination of elements of $X\}$. $= \{\sum_{i=1}^{k} a_i x_i\}$

We write $\sum_{i\in I} M_i = \langle \bigcup_{i\in I} M_i\rangle$

Def If $M: M = \langle X\rangle$ with X finite, then M is finitely generated.

Warning: not the same in vector spaces, if M finitely generated, $N \subset M$ submodule, N may not be finitely generated.

Example: $A = \mathbb{C}[x_1, \dots, x_n]$, $A = \langle 1\rangle$

$I = (x_1, \dots, x_n, \dots)$ is not finitely generated.
(ideal generated by x_i's)

Lemma (1) $L \supset M \supset N$, $\exists$ isomorphism, $(L/N)/(M/N) \xrightarrow{\sim} L/M$

with $(x+N) + (M/N) \longmapsto x + M$

$L \xrightarrow{p} L/M$, surjective, $L/N \xrightarrow{\bar{p}} L/M$ surjective. $\ker(\bar{p}) = M/N$

thus $(L/N)/(M/N) \xrightarrow{\sim} L/M$

(2) $(M_1+M_2)/M_2 \xrightarrow{\sim} M_1/(M_2 \cap M_1)$

Def $I \subset A$ (ideal), $I\cdot M \overset{\text{Def}}{=} \langle \{ax \mid a\in I, x\in M\}\rangle \subset M$ (M: module) (submodule of M)

M/IM is a A/I-module

Def $(M_i)_{i\in I}$, A-modules. $\prod_{i\in I} M_i$ is a module with $a(x_i) = (ax_i)$

$\bigoplus_{i\in I} M_i \subset \prod_{i\in I} M_i$ submodule of $(x_i)_{i\in I}$, s.t. $x_i = 0$ for all except finitely many i's

If $M_i = M$ $\forall i\in I$, write $\bigoplus_{i\in I} M_i = M^{(I)}$ or M^I if I is finite.

Def. A-module: free iff $\exists$ I: index set, st. $M \xrightarrow{\sim} A^{(I)}$

(1) A, field, then every A-module is free.

(2) $A = \mathbb{Z}$, $A/2\mathbb{Z}$ is not free.

Warning (3) a submodule of a free module is not necessarily free. (eg: ideals in A)

(4) $m, n \geq 0$, $A^m \simeq A^n$, then $m = n$.
$A \neq \{0\}$. Let $I \subset A$, maximal, then we get isomorphism $(A/I)^n \simeq (A/I)^m$

since A/I is a field, then $m = n$.

Prop: (Nakayama's Lemma)

M: finitely generated, $I \subset$ Jacobson radical of $I = \bigcap_{m\subset A, \max} m$

if $IM = M$, then $M = \{0\}$.

eg: A, local ring. $I = m$

Pf: If $M \neq \{0\}$, let $\{x_1, \dots, x_n\}$ be the generating set, $n \geq 1$.

s.t. n: minimal.

Since $IM = M$, $x_n \in IM$, thus $x_n = \sum_{i=1}^{k} a_i y_i$, $y_i \in M$.

thus $x_n = \sum_{i=1}^{k} a_i (\sum_{j=1}^{n} b_{ij} x_j) = \sum_{i=1}^{k}\sum_{j=1}^{n} a_i b_{ij} x_j = \sum_{j=1}^{n} \underbrace{(\sum_{i=1}^{k} a_i b_{ij})}_{c_j \in I} x_j$

so $(1-c_n)x_n = \sum_{j=1}^{n-1} c_j x_j$, $(1-c_n) \equiv 1 \pmod{J}$ $\Rightarrow$

$(1-c_n)$ is invertible in A. so $x_n = (1-c_n)^{-1} \sum_{j=1}^{n-1} c_j x_j$,

$\Rightarrow \{x_1, \dots, x_{n-1}\}$ generates M, contradiction.

Cor. M: finitely generated, $I \subset$ Jacobson radical, $N \subset M$.

If $M = IM + N$, then $M = N$

Pf: $I(M/N) = (IM+N)/N = M/N$, thus $M/N = \{0\}$.

Cor. A: local ring, M: finitely generated, then if $(x_1, \dots, x_n)$ in M, s.t. their classes mod m form a basis of M/mM ($m \subset A$), as M/m-vector space, then they generate M.

Pf: $N = \langle x_1, \dots, x_n \rangle$ use the previous corolly.

Lec 2-2

§2. Exact Sequences.

Def (1) $M' \xrightarrow{f} M'' \xrightarrow{g} M''$ is exact if $\mathrm{Im}(f) = \ker(\cancel{f}\, g)$

(2) $M_1 \xrightarrow{f_1} M_2 \cdots \xrightarrow{f_{n-1}} M_n \to \cdots$ exact, $\forall i$, $\mathrm{Im}(M_i) = \ker(M_{i+1})$

Example. (1) $0 \to M \xrightarrow{g} M''$ is exact $\Leftrightarrow$ $\{0\} = \ker(g) \Leftrightarrow g$: injective.

(2) $M' \xrightarrow{f} M \xrightarrow{g} 0$. is exact $\Leftrightarrow$ $\mathrm{Im}(f) = \ker(0) = M \Leftrightarrow f$: surjective.

(3) short exact sequence. $0 \to M' \xrightarrow{f} \cancel{M} \xrightarrow{g} M'' \to 0$.

f: injective, g: surjective $\mathrm{Im}(f) = \ker(g)$

eg. $0 \to M' \to M' \oplus M'' \to M'' \to 0$ split exact sequence (short exact sequence)

$x \to (x, 0)$ $\quad (x,y) \to y$.

eg. $A = \mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

$x \to 2x \qquad x \to x \bmod 2$

Not split

Theorem (snake lemma). 2 short exact sequences

$$\begin{tikzcd}
& 0 \arrow[r] & \ker(f') \arrow[r] \arrow[d] & \ker(f) \arrow[r] \arrow[d] & \ker(f'') \arrow[d] \arrow[ddll, bend left, "\delta"] & \\
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'"] \arrow[d] & N \arrow[r, "v'"] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
& \mathrm{Coker}(f') \arrow[r] & \mathrm{Coker}(f) \arrow[r] & \mathrm{Coker}(f'') &
\end{tikzcd}$$

$x \mapsto$, y', x''

$\delta(x'') = y' + \mathrm{Im}(f')$

f', f, f'': Linear, The diagram is commutative

s.t. $u' \circ f' = f \circ u$, $f'' \circ v = v' \circ f$.

$\exists \delta$! $\ker(f'') \to \mathrm{Coken}(f')$, s.t. the sequence is a long split sequence, exact

$$0 \to \ker(f') \xrightarrow{u} \ker(f) \xrightarrow{v} \ker(f'') \xrightarrow{\delta} \mathrm{Coker}(f') \xrightarrow{\bar{u}'} \mathrm{Coker}(f) \xrightarrow{\bar{v}'} \mathrm{coker}(f'') \to 0$$

restriction $\qquad$ restriction as well $\qquad$ quotient. $\qquad$ quotient

(check $u(\ker(f')) \subset \ker(f)$

$x \in \ker(f')$, $f(u(x')) = u' \circ f'(x') = u'(0) = 0$)

$\mathrm{Coker}(f') = N'/\mathrm{Im}(f')$

Check that:

well-defined. $\bar{u}'(y' + \mathrm{Im}(f')) = \bar{u}'(y') + \mathrm{Im}(f)$

Check image fall into $\mathrm{Coker}(f)$.

Definition of δ: take $x'' \in \ker(f'')$, $x'' \in M''$

Choose $x \in M$, s.t. $v(x) = x''$, consider $f(x)$

$v'(f(x)) = f''(v(x)) = f''(x'') = 0$, thus $\underline{f(x) \in \ker(v')} = \mathrm{Im}(u')$ ~~thus~~ u' is injective.

$\underline{\exists !\, y' \in N'}$, s.t. $u'(y') = f(x)$, define $\delta(x'') = y' + \mathrm{Im}(f') \in \mathrm{Coker}(f')$

δ: should be well-defined. let $x \to x + \xi$, $\xi \in \ker(v)$, so $\xi = u(\xi')$ for some $\xi' \in M'$.

(exactness, ~~surjective~~) $\mathrm{Im}(\ ) = \ker(\ )$)

$f(x) \to f(x) + f(\xi) = f(x) + f(u(\xi')) = f(x) + u'(f'(\xi'))$

$y' \to y' + f'(\xi') \in y' + \mathrm{Im}(f')$ $\quad$ Same image.

δ: exactness. $\quad \mathrm{Im}(v) = \ker \delta) \quad \mathrm{Im}(\delta) = \ker(\bar{u}')$

$$\begin{cases} \delta \circ v = 0 & (1) \\ \bar{u}' \circ \delta = 0 & (3) \\ \ker(\delta) \subset \mathrm{Im}(v) & (2) \\ \ker(\bar{u}') \subset \mathrm{Im}(\delta) & (4) \end{cases}$$

(1) $x'' = v(x)$, $x \in \ker(f)$.

thus $\delta(x'') \to y' + \mathrm{Im}(f')$

$u'(y') = f(x) = 0$, $y' = 0$, $\delta(x'') = 0$.

(2).

Ex Class _ Week 3.

Vector Spaces VS Modules

V . k-vector space. $\dim_k V = m < \infty$

1) $V \cong K^m$ 2) $\exists \{b_1, \cdots, b_m\}$, $\langle b_1, \cdots, b_m \rangle = V$ 3).

M, A-module.

1) Basis is not necessary $M = \mathbb{Z}/m\mathbb{Z}$. M : $\mathbb{Z}$-module e.g. $\langle [1] \rangle = \mathbb{Z}/m\mathbb{Z}$.

Free Module. bad example ???

$\mathbb{Z}$ free $\mathbb{Z}$-module, $\langle \{2,3\} \rangle = \mathbb{Z}$ $\dim_{\mathbb{Z}} \mathbb{Z} = 1$, ~~by~~ but $3 \notin (2)$, $2 \notin (3)$

K: field, $K^m \cong K^n$ IFF $m=n$ Same Modules. $A^m \cong A^n$ IFF $m=n$

Ex: A: ring, $f \in A$, $f \notin \underset{\text{not a unit.}}{A^\times}$, $f \neq 0$, then. $A[\frac{1}{f}] = \{\sum_{i \in I} a_i f^{-i}$, finite sum$\}$.

A is a free module $\exists k \in \mathbb{N}$, $k > 0$, $\{1, f^{-1}, \cdots, f^{-k}\}$ a set of generators.

$A[x]/(fx-1)$

CA-Lec2-2-005 (CTD from 004)

(3) Let $x'' \in \ker(f'')$ def: $\delta(x'') = \bar{y}'$

to comput $\bar{u}'(\bar{y}')$, $y' \in N'$.

$\bar{u}'(\bar{y}') = u'(y') + \mathrm{Im}(f)$

take x with $v(x) = x''$ (surjen, exact), then y' with $u'(y') = f(x)$. So $\bar{u}'(\bar{y}') = u'(y') + \mathrm{Im}(f) = 0$

(4) Let $\bar{y}' \in \ker \bar{u}')$

i.e. $\exists\, y' \in N'$, $\to \bar{y}'$, s.t. $u'(y') \in \mathrm{Im}(f)$, so

$\exists\, x \in M$, st. $f(x) = u'(y')$, let $x'' = v(x)$. Claim, $\delta(x'') = \bar{y}'$, so $\bar{y}' \in \mathrm{Im}(\delta)$

First, $f''(x'') = f'' \circ v(x) = v' \circ f(x) = v'(u'(y)) = 0$, $v' \circ u' = 0$

So $x'' \in \ker(f'')$

To compute $\delta(x'')$, let $v(x) = x''$ ($x \in \ker f$), $f(x) = u'(y')$. ~~$f(x) \in \ker(v')$~~ So $\delta(x'') = y' + \mathrm{Im}(f') = \bar{y}'$ $\square$

Example: Snake Lemma.

$\{0\} \xrightarrow{v} \ker(f'') \xrightarrow{\delta} \mathrm{Coker}(f') \xrightarrow{\bar{u}'} \{0\}$.

bijective

$\hookrightarrow \delta$: isomorphism.

$f \Big\downarrow$ $M \xrightarrow{v} N$ (30)

Prop: ① $0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ exact. : then $\forall$ A-mod N.

② $0 \to \mathrm{Hom}(M'', N) \xrightarrow{v^*} \mathrm{Hom}(M, N) \xrightarrow{u^*} \mathrm{Hom}(M', N)$ Exact

$f' \mapsto f \circ v$ $\quad g \mapsto g \circ u$

Note: in general, u^* not surjective.

③ $\mathrm{Hom}(N, M') \xrightarrow{u_*} \mathrm{Hom}(N, M) \xrightarrow{v_*} \mathrm{Hom}(N, M'') \to 0$ Exact

u_* not injective in general

Pf: ① without the "left 0", is exact $\iff$ ② is exact, $\forall N$

① without the "right 0", is exact $\iff$ ③ is exact, $\forall N$.

Here prove ① first assume $M' \to M \to M'' \to 0$ exact, $\forall N$: A-mod.

(1) $u^* \circ v^* = 0$ (2) v^*: injective, ③ $\ker(u^*) \subset \mathrm{Im}(v^*)$

pf: (1) $f: M'' \to N$, $u^* \circ v^*(f) = f \circ (v \circ u) = 0$

(2) $\ker(v^*) = 0$, let $v^*(f) = f \circ v = 0$, then $f(\mathrm{Im}(v)) = 0$, to $f = 0$. (v: surjective)

(3) Let $f: M \to N$, s.t. $u^*(f) = f \circ u = 0$.

Then $f(\mathrm{Im}(u)) = 0$, so $f(\ker(v)) = 0$

Thus $\bar{g}: M/\ker(v) \to N$. s.t. $\bar{g} \circ p = f$

v: induces an isomph.

let $g' = \bar{g} \circ \bar{v}^{-1} \in \mathrm{Hom}(M'', N)$

$$\begin{tikzcd} \mathrm{Im}(v) = M'' \arrow[r, shift left, leftarrow, "\bar{v}"] \arrow[r, shift right, "\bar{v}^{-1}"'] & M/\ker(v) \arrow[r, "\bar{g}"] & N \\ & M \arrow[ul, "v"] \arrow[u, "p"] \arrow[ur, "g"'] & \end{tikzcd}$$

Example: $A = \mathbb{Z}$, $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

$f \downarrow$ $f = \mathrm{Id}_{\mathbb{Z}}$

N

~~$\mathbb{Z}/2\mathbb{Z}$~~

u^* ~~v^*~~: $\mathrm{Hom}(\mathbb{Z}, N) \to \mathrm{Hom}(\mathbb{Z}, N)$ is not surjective if $N = \mathbb{Z}$

$f \mapsto f \circ (\times 2)$

BC, if $f = \mathrm{Id}_{\mathbb{Z}}$, then.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "\mathrm{Id}"'] & \mathbb{Z} \arrow[dl, "g"] \\ & \mathbb{Z} & \end{tikzcd}$$

there is no such g. because, linear maps on $\mathbb{Z}$ are $(\times m)$, $m \in \mathbb{Z}$.

Conversely, ($*_N$) ~~...~~ exact

(1) v: surjective. Let $N = \mathrm{Coker}(v)$, show $N = 0$.

$p: M'' \to \mathrm{Coker}(v)$. $\in \mathrm{Hom}(M'', N)$

Let $v^*(p) = p \circ v = 0$, since v^*: injective, $p = 0 \Rightarrow M'' = \ker(p) = \mathrm{Im}(v)$

v: surjective

(2) Take $N = M''$, $f = \mathrm{Id}_{M''}$. $(u^* \circ v^*)(f) = 0$.

$\mathrm{Id}_M \circ v \circ u = 0 \Leftrightarrow v \circ u = 0$.

(3) $N = M/\mathrm{Im}(u)$, $p = M \to N$, projection.

$u^*(p) = p \circ u = 0$. $p \in \mathrm{Ker}(u^*)$, $\exists f \in$ ~~$\mathrm{Im}(v^*)$~~ $\in \mathrm{Hom}(M'', N)$, s.t. $v^*(f) = p$, $f \circ v = p$

$$\begin{tikzcd} M'' \arrow[r, "f"] & N/\mathrm{Im}(u) \\ M \arrow[u, "v"] \arrow[ur, "p"'] & \end{tikzcd}$$

$\mathrm{Ker}(v) \subset \ker(p)$, $\ker(v) \subset \mathrm{Im}(u)$

(2)+(3) $\Rightarrow \ker(v) = \mathrm{Im}(u)$

§3 Tensor Product

Bilinear Map.

Def. M, N, P, A-mod. A map $b: M \times N \to P$ A-bilinear.

$f(ax+by, z) =$ ~~af~~ $af(x,z) + bf(y,z)$. $f(z, ax+by) = af(z,x) + bf(z,y)$.

$\mathrm{Bil}_A(M,N,P)$ set of all bilinear maps: $M \times N \to P$

Tensor product gives A-module: $M \otimes_A N$ s.t. $\mathrm{Bil}_A(M,N,P) = \mathrm{Hom}_A(M \otimes_A N, P)$

Th M, N mod. $\exists (T, \beta)$, T: A-module β: bilinear $M \times N \to T$,

s.t. $\forall$ A-mod P, $\forall$ bilinear map $M \times N \xrightarrow{b} P$.

s.t. $\beta \circ f = b$.

$\exists! f \in \mathrm{Hom}(T, P)$.

$$\begin{tikzcd} M \times N \arrow[r, "b"] \arrow[d, "\beta"'] & P \\ T \arrow[ur, "f"'] & \end{tikzcd}$$

$$\begin{tikzcd} M \times N \arrow[r] \arrow[d] & T \arrow[dl, shift left] \\ T' \arrow[ur, shift left] & \end{tikzcd}$$

Uniqueness: if (T', β') another such pair, $\exists!$ isomorphism $T \to T'$ s.t.

$$\begin{tikzcd} T \arrow[r, "i"] & T' \\ M \times N \arrow[u, "\beta"] \arrow[ur, "\beta'"'] & \end{tikzcd}$$

$i \circ \beta = \beta'$

Prop 2.12. (T, g) $g: M \times N \to T$.

$\forall P$, $\forall f: M \times N \to P$. $\exists! f': T \to P$. s.t. $f = f' \circ g$

(T, g) (T', g') $\exists! j: T \to T'$, $j \circ g = g'$

$\forall$ module P

$\forall$ ~~bijection~~ bilinear.

$$\begin{tikzcd} & T \arrow[d, "\tilde{b}"] \\ M \times N \arrow[ur, "f"] \arrow[r, "b"'] & P \end{tikzcd}$$

Pf: Uniqueness.

M, N: A-modules.

$f_{(b)}: M \times N \to P$. $T, g: M \times N \to T$

? Yoneda Lemma.

$C = A^{(M \times N)}$ free module. $\exists M \times N \to C$

$(x, y) \longrightarrow$ basis

$D \subseteq C$: Def1. Smallest submodule s.t. $M \times N \xrightarrow{\text{induced}} C/D$ is bilinear

$\Updownarrow$ Def2 $\forall a \in A$, $x \in M$, $y \in N$, the books' version

~~<< Commutative Algebra with a view toward Algebraic Geometry >>~~ Lec 3-1 — 006

~~• P31. $f \in k[x_1, \cdots, x_n]$ nonzero. k: infinite, no polynomial other then 0 can vanishes entirely on k^n.~~

~~X ① take $f \in k[x_1, \cdots, x_{n-1}]$, nonzero, only finitely many x_n can make $f \in k[x_1, \cdots, x_n]$ entirely 0~~

~~by induction, only finitely many $[x_2, \cdots, x_n]$ can make $f \in k[x_1]$ entirely 0.~~

~~when $f \in k[x_1] \neq 0$, only finitely x_1 can make $f(x_1) \in k[x_1] = 0$~~

~~Now think from $[x_1] \to [x_1, \cdots, x_n]$ backwards.~~

~~Sketch:~~

Lec 3-1. CA. $T = M \otimes N$. $x \otimes y = f(x, y)$

✓ (1) $M \otimes N \cong N \otimes M$.
$m \otimes n \to n \otimes m$

(2) $(M \otimes N) \otimes P \cong (M \otimes P) \otimes N$
$m \otimes n \otimes p \to m \otimes p \otimes n$.

Note: $S = \{m \otimes n, (m,n) \in M \times N\} \neq M \otimes N$, S generates $M \otimes N$.

(3) $M \otimes (N_1 \oplus N_2) = M \otimes N_1 \oplus M \otimes N_2$
$m \otimes (n_1, n_2) \longmapsto (m \otimes n_1, m \otimes n_2)$

✓ (4) $A \otimes M \cong M$
$a \otimes m \mapsto am$

$b(m, (n_1, n_2)) =$ ~~$m \otimes n_1 + m \otimes n_2 =$~~ $(m \otimes n_1, m \otimes n_2)$

X $b(m+m', (n_1, n_2)) = (m+m') \otimes n_1 + (m+m') \otimes n_2 = m \otimes n_1 + m' \otimes n_2 + m' \otimes n_1 + m' \otimes n_2$

$= m \otimes (n_1 + n_2) + m' \otimes (n_1 + n_2)$

$= b(m, (n_1, n_2)) + b(m', (n_1, n_2))$

✓ (3) Pf:
$\underline{f}(m \otimes (n_1, n_2)) = (m \otimes n_1, m \otimes n_2)$, show f: isomorphism.

Pure tensor: $x \otimes y \in M \otimes N$, ~~for~~ others simply tensors.

$\left(\bigoplus_{i=1}^{m} A_i\right) \otimes A^n = \bigoplus_{i=1}^{m} (A_i \otimes A^n) = \bigoplus_{i=1}^{m} (A^n) \cong A^{mn}$
$A_i \cong A$

$A^m = A \oplus \cdots \oplus A$ (finite free modules)

(4) ⟹ $A \otimes A \cong A$. by (3). (4) $A^m \otimes A^n = A^{mn}$ (Compare $A^m \oplus A^n = A^{m+n}$)

$e_1^{(1)}, \cdots, e_m^{(1)}$ standard basis for A^m, $e_1^{(2)}, \cdots, e_n^{(2)}, \cdots$ for A^n.

✓ then $\{e_i^{(1)} \otimes e_j^{(2)}\}_{i=1,\cdots,m,\ j=1,\cdots,n}$. is a basis of $A^m \otimes A^n$

hence induces the isomorphism $\cong A^{mn}$

$f: A^m \times A^n \to P$, $A^m \ni x = x_1 e_1^{(1)} + \cdots + x_m e_m^{(1)}$

$A^n \ni y = y_1 e_1^{(1)} + \cdots + y_n e_n^{(1)}$

$f(x,y) = \sum x_i y_j f(e_i^{(1)} \otimes e_j^{(2)}) = \sum x_i y_j \cdot \underset{\in P}{a_{ij}}$

Conversely, given an mn-tuple $(a_{ij}) \in P$, define $f: A^m \times A^n \to P$ by the above form.

$$\begin{tikzcd} A^m \times A^n \arrow[r] \arrow[dr, "f"'] & \bigoplus_{i \le m,\ j \le n} A \cdot e_i^{(1)} \otimes e_j^{(2)} \arrow[d, "\exists! \hat{f}"] \\ & P \end{tikzcd}$$

$(e_i^{(1)}, e_j^{(2)})$

$\exists! \hat{f}$: $\hat{f}(e_{ij}) = a_{ij}$.

• Multilinear Case.

• $\mathbb{R}^n$: $\{$inner product on $V\} \subseteq (V \otimes V)^{dual}$ $(A = \mathbb{R})$ Linear $V \otimes V \to \mathbb{R}$ (dual

~~V~~ §. Extension of Scalars.

$f: A \to B$ \
A-mod: M. $\Big\} \to$ B-mod. $\swarrow$ as an A-mod via f.

$M_B = M \otimes_A B$

$a \cdot b = f(a) b$.

$b \cdot (m \otimes z) := m \otimes bz$ extend to a bilinear $B \times M_B \to M_B$

ex. $M = A^m \Rightarrow M_B \cong B^m$ ex. $A = \mathbb{R}$, $B = \mathbb{C}$ $(\mathbb{R}^n)_{\mathbb{C}} = \mathbb{C}^n$.

CA - Lec 3-2.007.

P, M, N modules on A

$M \otimes N := M \otimes_A N$.

- bilinear $f: M \times N \to P$ linear $g: M \otimes N \to P$ $g(x \otimes y) = f(x, y)$
- $\{x \otimes y : x \in M, y \in N\}$ generates but need not equal $M \otimes N$ $R^{10} \otimes R^{10} = R^{100}$. $e_i^1 \otimes e_j^2$
- f: not bilinear then $\nexists$ g: linear : $M \otimes N \to P$, s.t. $g(x \otimes y) = f(x, y)$
  $\Rightarrow$ If f: not bilinear, build $M \otimes N$, g the same way, then g is not linear?

  e.g. $\nexists$ linear $\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$, s.t. $x \otimes y \mapsto x+y$
- $x \otimes y$ depends on modules to which we regard x, y belonging.
- In fact, one can have $x \in M' \subseteq M$, $y \in N' \subseteq N$. s.t. $x \otimes y$ in $M' \otimes N'$. and $x \otimes y$ in $M \otimes N$ not the same

  eg. $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \ni 2 \otimes 1 (\bmod 2)$

  $2\mathbb{Z}/4\mathbb{Z} \ni 2(4) \neq 0$.

  $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$

  $2 \otimes 1(2) \leftrightarrow 2(2) = 0$

Summary: ? $M' \subseteq M, N' \subseteq N \not\Rightarrow M' \otimes N' \subseteq M \otimes N$ but $\exists\, M' \otimes N' \to M \otimes N$ as below

$\otimes$ is a functor? given module morphisms, $f: M' \to M$, $g: N' \to N$ ~~M: category~~

$\exists !\, f \otimes g : M' \otimes N' \to M \otimes N$

$x \otimes y \to f(x) \otimes g(y)$

?

- $(f \circ f') \otimes (g \circ g') = \cancel{(f \otimes g) \otimes f'}\ (f \otimes g) \circ (f' \otimes g')$

  eg: $g = 1$: $M' \xrightarrow{f} M$, N.

  $M' \otimes N \xrightarrow{f \otimes 1} M \otimes N$

- N: flat iff $\forall M' \xrightarrow{f} M$ as before, then (f: injective $\Rightarrow f \otimes 1_N$: injective)

  In other words: $M' \subseteq M \Rightarrow M' \otimes N$ submodule of $M \otimes N$

  Thus: $\mathbb{Z}/2$ is not a flat $\mathbb{Z}$-module, $2\mathbb{Z} \hookrightarrow \mathbb{Z}$ injective

  but $2\mathbb{Z} \otimes \mathbb{Z}/2 \to \mathbb{Z} \otimes \mathbb{Z}/2$ is the zero map, hence not injective.

  eg: $\{0\}$ is a flat A-module, because $M' \otimes \{0\}$ is also a zero module.

- eg. A: flat, A-module. $0 \to M' \xrightarrow{f} M$, $f \leftrightarrow f \otimes 1$

  $0 \to M' \otimes_A A \to M \otimes_A A$,

- $(N_i)_{i \in I}$ cover A. Then $\bigoplus_{i \in I} N_i$: flat $\Leftrightarrow N_i$: flat, $\forall i \in I$

pf: $N_i, i \in I$, flat, let f: injective $M' \xrightarrow{f} M$.

$M' \otimes \left(\bigoplus_{i \in I} N_i\right) \xrightarrow{f \otimes 1} M \otimes \left(\bigoplus_{i \in I} N_i\right)$, consider if $f \otimes 1$ not injective. $N = \bigoplus_{i \in I} N_i$

i.e. $z \in \ker(f \otimes 1)$, $\pi_i : N \to N_i$

by commute of $\otimes, \oplus$,

$$\begin{array}{ccc} \bigoplus_{i \in I} (M' \otimes N_i) & = & M' \otimes \left(\bigoplus_{i \in I} N_i\right) \\ \downarrow p_i & & \downarrow 1 \otimes \pi_i \\ M' \otimes N_i & & M' \otimes N_i \end{array}$$

$z \neq 0 \in M' \otimes \left(\bigoplus_{i \in I} N_i\right)$

$\exists i$, $p_i(z) \neq 0$, $(f \otimes 1)(p_i(z)) \neq 0 \in M \otimes N_i$

by $(f \otimes 1)(p_i(z))$ is the i^{th} component of $(f \otimes 1)(z) = 0$. Contra.

eg. $S = f_1(x_1,\dots,x_n) = \dots = f_m(x_1,\dots,x_n) = 0$. with coeff in $\mathbb{R}$.

Then S has a solution on $\mathbb{R}$ $\Leftrightarrow$ S has a solution on $\mathbb{C}$ (works for any field extension L/K instead of $\mathbb{C}/\mathbb{R}$)

"$\Rightarrow$" ✓ "$\Leftarrow$" take the real/imaginary part of the complex ~~equation~~ Solution.

Second Pf: $A = M$, $M' = \mathbb{R}^n \xrightarrow{f = f_1 \times \dots \times f_m} M = \mathbb{R}^m$

S has a solution on $\mathbb{R}$ $\Leftrightarrow$ $\ker(f) \neq \{0\}$.

$\Leftrightarrow S$ — — — on $\mathbb{C}$ $\Leftrightarrow$ $\ker(f \otimes 1_{\mathbb{C}}) \neq \{0\}$.

Lemma: free $\Rightarrow$ flat. i.e. $\bigoplus_{i \in I} A$: flat by previous results.

Prop: $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ exact.

$\Rightarrow M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ exact

Recall: N flat, $0 \to M' \to M$ exact $\Rightarrow$ $0 \to M' \otimes N \to M \otimes N$ exact

Pf: $\mathrm{Im}(f \otimes 1) \subseteq \ker(g \otimes 1)$ $\Leftarrow g \circ f = 0$

i.e. $(g \otimes 1) \circ (f \otimes 1) = (g \circ f) \otimes 1 = 0$

Show $\dfrac{M \otimes N}{\mathrm{Im}(f \otimes 1)} \xrightarrow{\psi} M'' \otimes N$ is an isomorphism

Construct an inverse:

$M'' \otimes N \xrightarrow{\varphi} M \otimes N / \mathrm{Im}(f \otimes 1)$

$M'' \otimes N \xrightarrow{\exists \varphi_0} M \otimes N / \mathrm{Im}(f \otimes 1)$, $\quad M \times N \to M'' \otimes N$, $\quad M \times N \xrightarrow{\varphi_1} M \otimes N / \mathrm{Im}(f \otimes 1)$

$M \times N$: $(x_1, y), (x_2, y) \longrightarrow (x'', y)$

$\varphi_1(x_1, y) - \varphi_1(x_2, y) = \underbrace{(\text{class of } x_1 \otimes y - x_2 \otimes y \bmod \mathrm{im}(f \otimes 1))}_{=0}$

$x_1 - x_2 \in \ker(g) = \mathrm{Im}(f)$, $f(z)$, $\exists z \in M'$

$\varphi_0(x'', y) = \varphi_1(x, y)$ well-defined.

Check φ_0: bilinear, $\varphi \circ \psi = \psi \circ \varphi = 1$

Cor: N: flat $\Leftrightarrow$ $\cdot \otimes$ preserves ~~exacte~~ exactness of any sequence of modules.

Pf: any exact sequence can be split up into short exact sequences

Example: $\mathfrak{a} \subseteq A$, M, $M \otimes_A A/\mathfrak{a} \cong M/\mathfrak{a}M$

$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$
inj surj exact

$\otimes$ $\mathfrak{a} \otimes M \xrightarrow{\psi} M \to M \otimes A/\mathfrak{a} \to 0$

$x \otimes m \mapsto xm$

$\mathrm{Im}(\psi) = \mathfrak{a}M = \psi(\{\sum x_i \otimes m_i : x_i \in \mathfrak{a}, m_i \in M\})$

$= \{\sum x_i m_i : x_i \in \mathfrak{a}, m_i \in M\}$

$M \otimes A/\mathfrak{a} \cong M/\mathrm{im}(\psi) = M/\mathfrak{a}M$ ■

V, W: VS on $F = A$. $\dim V, W < \infty$

$V^* \otimes W \xrightarrow{\cong} \mathrm{Hom}(V, W)$

$\ell \otimes w \mapsto [v \mapsto \ell(v) w]$

Check these by choosing basis.

$\mathrm{End}(V) = \mathrm{Hom}(V, V) \xrightarrow{\text{trace}} V$

$\cong$

$V^* \otimes V \to \ell(v)$

$\ell \otimes v$

eg. $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$. eg. $m=2, n=3$. $\cong \{0\}$.

$M = \mathbb{Z}/m\mathbb{Z}$, $\mathfrak{a} = n\mathbb{Z}$. $\mathfrak{a}M \cong (n\mathbb{Z} + m\mathbb{Z})/m\mathbb{Z}$

$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/m\mathbb{Z}) / ((n\mathbb{Z}) \cdot (\mathbb{Z}/m\mathbb{Z})) \cong (\mathbb{Z}/m\mathbb{Z}) / (\gcd(m,n) \mathbb{Z}/m\mathbb{Z})$

eg: $n \in \mathbb{Z}_{\geq 0}$, $\mathbb{Z}/n\mathbb{Z}$ flat $\Leftrightarrow$ $n = \pm 1$, $z = 0$.

$\mathbb{Z}/n\mathbb{Z}$ $\begin{cases} n = \pm 1, & \{0\} \\ z = 0, & \mathbb{Z} \end{cases}$ $\quad \Leftarrow$ ✓ $\quad \Rightarrow$ let $|n| \geq 2$

Any finitely generated $\mathbb{Z}$-mod is of the form $\mathbb{Z}^r \oplus \underbrace{\bigoplus_{i=1 \dots N} (\mathbb{Z}/n_i)}_{M_{tor}}$ $(n_i \geq 1)$ $= M$

For such M, M: flat $\Leftrightarrow$ M_{tor} = zero

Def: A, M. M: torsion free if $\forall a \in A$, a: not zero-divisor, $am = 0 \Rightarrow m = 0$

Thm: (i) M: flat $\Rightarrow$ M: torsion free (ii) A: PID, M: torsion free $\Rightarrow$ M: flat

Complete Proof in Bosch (§4.) Chapter 4

Example. X, set, $U \subseteq X$, subset, $A = \{f : X \to \mathbb{R}\}$

Def: $(f_1+f_2)(x) = f_1(x)+f_2(x)$, $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$.

Show: A is a ring with zero $= f(x) \equiv 0$, unity $= f(x) \equiv 1$.

$S = \{f \in A : \forall x \in U, f(x) \neq 0\}$. $B = \{f : U \to \mathbb{R}\}$.

Show: S is multiplicatively closed, and

$j : S^{-1}A \mapsto B$ by $(j[\frac{a}{s}])(x) = s^{-1}(x) \cdot a(x)$ here restrict $s(x)$, $a(x)$ on U.

Show: j is (well-defined) an isomorphism.

Pf: ① well-defined. $[\frac{a_1}{s_1}] = [\frac{a_2}{s_2}]$, then $\exists t \in S$, $t(a_1s_2 - a_2s_1) = 0$

i.e. $\forall x \in X$, $t(x)(a_1(x)s_2(x) - a_2(x)s_1(x)) = 0$.

Restrict these functions on U hence, $\forall x \in U$, $t(x) \neq 0$, $s_1(x), s_2(x) \neq 0$.

i.e. $\forall x \in U$, $s_1^{-1}(x)a_1(x) = s_2^{-1}(x)a_2(x)$.

~~Thus the restriction maps are the same~~.

② $k : B \mapsto S^{-1}A$ by $(kb)(x) = [\tilde{b}/1]$ where $\tilde{b} = \begin{cases} b(x), & x \in U \\ 0, & x \notin U \end{cases}$

$(j \circ k)(b)(x) = j[\tilde{b}/1](x) = \tilde{b}(x)|_U = b(x)$

$(k \circ j)([\frac{a}{s}])(x) = (k(s^{-1}a))(x) = [\widetilde{s^{-1}a}/1]$ where $\widetilde{s^{-1}a} = \begin{cases} s^{-1}a(x), & x \in U \\ 0, & x \notin U. \end{cases}$

Want: $t \in S$, s.t. $ta = s \cdot \widetilde{s^{-1}a} \cdot t$,

i.e. $t(x)(a(x) - s(x) \cdot \widetilde{s^{-1}a}(x)) = 0$, $\forall x \in X$.

i.e. $\begin{cases} t(x)(a(x) - s(x) \cdot s^{-1}(x) \cdot a(x)) = 0, & \forall x \in U \\ t(x) \cdot a(x) = 0 & \forall x \notin U. \end{cases}$

Take $t(x) = \begin{cases} 1, & x \in U \\ 0, & x \notin U. \end{cases} \in S$.

Hence $[\frac{a}{s}] = [\widetilde{s^{-1}a}/1]$.

Universal Property.

EX Reminder: ① ~~Hom~~ Bilinear $(M \times N, P) \cong \mathrm{Hom}(M \otimes_R N, P)$

↳ The set of bilinear maps ⋯

② $\mathrm{Hom}(\bigoplus_{i \in I} M_i, N) \cong \prod_{i \in I} \mathrm{Hom}(M_i, N)$

Lemma. A, B (rings), $S \subseteq A$, multiplicatively closed, $S \not\ni 0$. (required by $f(S) \subseteq B^\times$).

$\mathrm{Hom}(S^{-1}A, B) \cong T = \{f : A \to B$, morphisms, $f(S) \subseteq B^\times\}$.

Pf: First show T is also a ring. (a subring of $\mathrm{Hom}(A, B)$) ? (This is shown by the lemma itself)

Let $\tau : \mathrm{Hom}(S^{-1}A, B) \to T$, $\tau(\hat{f})(a) = \hat{f}(\frac{a}{1})$.

Show τ: injective, surjective, homomorphism.

① surjective. let $f : A \to B$, then $\hat{f}[\frac{a}{s}] = f(s)^{-1} f(a)$ is $\in \mathrm{Hom}(S^{-1}A, B)$.

$\tau(\hat{f})(a) = \hat{f}(\frac{a}{1}) = f(1)^{-1} f(a) = f(a)$, hence $\tau(\hat{f}) = f$.

② injective. $\tau(\hat{f}_1) = \tau(\hat{f}_2)$, then $\forall a \in A$, $\hat{f}_1([\frac{a}{1}]) = \hat{f}_2([\frac{a}{1}])$

hence $\forall s \in S$, $\hat{f}_1([\frac{s}{1}]) = \hat{f}_2([\frac{s}{1}]) \in B^\times$

$\hat{f}_1, \hat{f}_2$ are morphisms, thus $\hat{f}_1([\frac{1}{s}]) = \hat{f}_2([\frac{1}{s}])$

Thus $\hat{f}_1([\frac{a}{s}]) = \hat{f}_1[\frac{a}{1}] \cdot \hat{f}_1[\frac{1}{s}] \cdots = \hat{f}_2([\frac{a}{s}])$

③ homomorphism. (Omitted.

Def. $\varphi: A \to B$, morphism. $\mathfrak{a} \subseteq A$. $\mathfrak{b} \subseteq B$, ideals.

$\mathfrak{b}^c = \varphi^*(\mathfrak{b}) = \varphi^{-1}(\mathfrak{b})$ also an ideal. $\mathfrak{a}^e = \varphi_*(\mathfrak{a}) = B\varphi(\mathfrak{a})$, ideal generated by $\varphi(\mathfrak{a})$

contraction pullback extension push forward

Note: $\mathfrak{p} \subseteq B$ prime, then $\mathfrak{p}^c = \varphi^*(\mathfrak{p})$ also prime.

φ^*: $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ by $\mathfrak{b} \mapsto \varphi^{-1}(\mathfrak{b})$. ? as set maps / ring morphisms.

Properties: $\tau: A \mapsto S^{-1}A$, $\mathfrak{p} \subseteq A$, prime ideal.

① $\tau^*: \mathrm{Spec}(S^{-1}A) \mapsto \mathrm{Spec}(A)$, $\mathrm{Im}(\tau^*) \overset{=}{\cong} T = \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \cap S = \Phi\}$.

② $\forall \mathfrak{a} \subseteq A$, ideal, $\tau_*(\mathfrak{a}) = \{[\frac{a}{s}], a \in \mathfrak{a}, s \in S\}$.

③ $\tau^*(\mathfrak{a}) = S^{-1}A \iff \mathfrak{a} \cap S \neq \Phi$ ④ $\forall \mathfrak{b} \in S^{-1}A$, ideal, $\varphi_*(\varphi^*(\mathfrak{b})) = \mathfrak{b}$.

Pf: ① (If) As set maps. check $\mathrm{Im}(\tau^*) = T$, τ^* injective. (by def of τ^*)

② $\tau_*(\mathfrak{a}) = B \cdot \tau(\mathfrak{a})$, note $\forall a \in \mathfrak{a}, s \in S$, $[\frac{a}{s}] = \underbrace{[\frac{a}{1}]}_{\in \tau(a)} \cdot \overbrace{[\frac{1}{s}]}^{\in S^{-1}A}$, i.e. $T \subseteq \tau_*(\mathfrak{a})$

Now check $\tau_*(\mathfrak{a}) \supseteq T$, i.e. check T is an ideal. $[\frac{a_1}{s_1}] + [\frac{a_2}{s_2}] = [\frac{a_1 s_2 + a_2 s_1}{s_1 s_2}]$ ($a_1 s_2 + a_2 s_1$ in $\mathfrak{a}$, $s_1 s_2$ in S)

$[\frac{a'}{s'}] \cdot [\frac{a}{s}] = [\frac{a'a}{ss'}]$ ($a'a$ in $\mathfrak{a}$, ss' in S). i.e. T is an ideal.

$a' \in A$, $s, s' \in S$, $a \in \mathfrak{a}$

③ $\tau^*(\mathfrak{a}) = S^{-1}A \iff [\frac{1}{1}] \in \tau^*(\mathfrak{a}) \iff \exists a \in \mathfrak{a}$, s.t. $[\frac{a}{1}] = [\frac{1}{1}]$.

$\iff \exists a \in \mathfrak{a}, s \in S, s = a. \iff S \cap \mathfrak{a} \neq \Phi$.

④ $\varphi_*(\varphi^*(\mathfrak{b})) \subseteq \mathfrak{b}$. (In general), since $\mathfrak{b}$ is the ideal containg $\varphi(\varphi^*(\mathfrak{b}))$, $\mathfrak{b} \supseteq B\varphi(\varphi^*(\mathfrak{b})) = \varphi_*(\varphi^*(\mathfrak{b}))$

Let $[\frac{a}{s}] \in \mathfrak{b}$, then $[\frac{a}{s}] \cdot [\frac{s}{1}] = [\frac{a}{1}] \in \mathfrak{b}$ (ideal), $a \in \varphi^*(\mathfrak{b})$

hence $[\frac{a}{1}] \in \varphi(\varphi^*(\mathfrak{b})) \subset \varphi_*(\varphi^*(\mathfrak{b}))$, now $\varphi_*(\varphi^*(\mathfrak{b}))$ is an ideal,

hence $\underbrace{[\frac{1}{s}]}_{\in S^{-1}A} \cdot \underbrace{[\frac{a}{1}]}_{\in \varphi_*(\varphi^*(\mathfrak{b}))} \in \varphi_*(\varphi^*(\mathfrak{b}))$, i.e. $\mathfrak{b} \subseteq \varphi_*(\varphi^*(\mathfrak{b}))$.

Lemma. $\tau: A \mapsto S^{-1}A$, by $\tau(a) = \frac{a}{1}$ is a ring homomorphism.

with the properties: ① $\tau(S) \subseteq (S^{-1}A)^{\times}$ (units of $S^{-1}A$) (Only for $0 \notin S$)

② $\ker(\tau) = \{a \in A : \exists s \in S, sa = 0\}$

③ $A \neq \{0\}$, then τ: injective IFF S contains no zero divisors of A.

④ $S^{-1}A = \{0\}$ IFF $S \ni 0$ ⑤ τ: isomorphism IFF $S \subseteq A^{\times}$ (Only for $A \neq \{0\}$)

(*1) Pf: ① $\forall s \in S$, $s \mapsto [\frac{s}{1}]$, then $[\frac{1}{s}] \times [\frac{s}{1}] = [\frac{s}{s}] = [\frac{1}{1}]$

Notice: if $0 \in S$, then $\forall [\frac{a}{s}] = [\frac{1}{0}]$, thus $S^{-1}A = \{0\}$, $1_{S^{-1}A} = 0_{S^{-1}A}$

Then $S^{-1}A$ will contain no units.

② $\ker(\tau) = \{a \in A, \tau(a) = [\frac{a}{1}] = [\frac{0}{1}]\} = \{a \in A, \exists s \in S, sa = 0\}$.

③ $A \neq \{0\}$. τ: injective IFF $\ker(\tau) = \{0\}$ IFF S contains no zero divisors.

④ If $0 \in S$, $S^{-1}A = \{0\}$ (Shown in ①)

If $S^{-1}A = \{0\}$, then $[\frac{0}{1}] = [\frac{1}{1}]$, i.e. $\exists s \in S$, $s \cdot 1 = 0 = s$.

~~⑤ If τ: isomorphism, then $A \neq \{0\}$.~~

(*1): First show $\tau: A \to S^{-1}A$ is a ring homomorphism.

⑤ For a general ring ~~homo~~ isomorphism, $\phi: A \to B$, let $b = \phi(a)$, then b: unit in B IFF a: unit in A

Now for this case, $S^{-1}A \cong A \neq \{0\}$, $0 \notin S$, we can use ①

Since $\tau(S) \subseteq (S^{-1}A)^{\times}$, if τ: isomorphism, then $S \subseteq A^{\times}$

Now if $S \subseteq A^{\times}$, $\tau^{-1}: S^{-1}A \to A$ by $\tau^{-1}([\frac{a}{s}]) = s^{-1}a$, show well-defined.

$\tau \circ \tau^{-1}([\frac{a}{s}]) = [\frac{s^{-1}a}{1}] = [\frac{a}{s}]$, $\tau^{-1} \circ \tau(a) = a$.

Examples. ① $A \ni f$, $S_f = \{f^n : n \geq 0\}$ (multiplicatively closed)

$A[\frac{1}{f}] = S_f^{-1}A$, e.g. $\mathbb{Z}[\frac{1}{2}] = \{[\frac{z}{2^n}] : z \in \mathbb{Z}, n \geq 0\}$

② ~~A: integral dom~~ A: integral domain ($A - \{0\}$ is multipliatively closed requires this)

$0 \notin S \subseteq A$, $A[\frac{1}{S}]$ = subring of $(A-\{0\})^{-1}A$ generated by $\{[\frac{a}{1}] : a \in A\} \cup \{[\frac{1}{s}] : s \in S\}$

$A[\frac{1}{S}] = \bar{S}^{-1} \cdot A$

③ $p \subseteq A$, ideal, $S_p = A - p$ then

S_p: multiplicatively closed $\Leftrightarrow$ $\begin{cases} 1 \in S_p & \text{i.e. } p \neq \langle 1 \rangle \\ \forall t_1, t_2 \in S_p, t_1 \cdot t_2 \in S_p & \text{i.e. If } t_1 \cdot t_2 \notin S_p / t_1 \cdot t_2 \in p, \text{ then } t_1 \notin S_p / t_1 \in p, \text{ or } t_2 \notin S_p / t_2 \in p. \end{cases}$

$\Leftrightarrow$ p: prime.

~~Lec~~ CA_ Lec 4-1-(1)

Localization.

Motivation: $\cdot\ \mathbb{Z} \mapsto \mathbb{Q}$; $\mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^n} ; a \in \mathbb{Z}\}, n \geq 0\}$.

$\cdot$ A : integral domain, Frac(A) = quotient field

Equivalance Relation $\sim$ on A^2, $(a_1 b_2 - a_2 b_1) = 0$ IFF $(a_1, b_1) \sim (a_2, b_2)$

$\mathrm{Frac}(A) = \{[\frac{a}{b}] : a \in A, b \neq 0, b \in A\}$.

Then $+$: $[\frac{a_1}{b_1}] + [\frac{a_2}{b_2}] = [\frac{a_1 b_2 + a_2 b_1}{b_1 b_2}]$

$\cdot$: $[\frac{a_1}{b_1}] \cdot [\frac{a_2}{b_2}] = [\frac{a_1 a_2}{b_1 b_2}]$

Show: $(+, \cdot)$ well defined.

Show: (Frac(A), $+, \cdot$) is a quotient field with zero $= [\frac{0}{1}]$, unity $= [\frac{1}{1}]$

Def. Multiplicatively Closed $S \subseteq A$.

A (ring), $S \subseteq A$, S : $\square$ IFF (① $1 \in A$, $1 \in S$ ② $\forall s_1, s_2 \in S$, $s_1 \cdot s_2 \in S$)

IFF ($\forall s_1, \cdots, s_n \in S$, $s_1 \cdot s_2 \cdots s_n \in S$) with empty product defined as 1.

IFF ($\forall s_1, s_2 \in S$, $s_1 \cdot s_2 \in S$) with empty product ...

$\bar{S}$: Multiplicative Closure of $S \subseteq A$.

$\bar{S} = \{ s_1 \cdots s_n ; s_1, \cdots, s_n \in S \}$ with empty ...

$\bar{S}$: least multiplicative (closed) set containing S.

Def $\sim$ on $A \times S$

$S \subseteq A$, multiplicatively closed, $(a_1, s_1), (a_2, s_2) \in A \times S$, $(a_1, s_1) \sim (a_2, s_2)$ IFF

$\exists t \in S$, s.t. $a_1 s_2 \cdot t = a_2 s_1 \cdot t$

Show: $\sim$ is an ~~equaiv~~ equivalance relation. ① reflexive ② symmetry ③ transitivity.

Check ③. let $(a_1, s_1) \sim (a_2, s_2) \sim (a_3, s_3)$, then

$\exists t', t'' \in S$, s.t. $a_1 s_2 t' = a_2 s_1 t'$, $a_2 s_3 t'' = a_3 s_2 t''$

Now $(a_1 s_3) s_2 t' t'' = (a_2 s_1 t')(s_3 t'') = (a_3 s_2 t'')(s_1 t') = (a_3 s_1)(s_2 t'' t')$

$s_2 t' t''$, $s_2 t'' t' \in S$ (This Motivates S to be required multiplicative closed)

Hence $(a_1, s_1) \sim (a_3, s_3)$

Def $S^{-1}A = (A \times S)/\sim$

A (ring), $S \subseteq A$, multiplicatively closed, $\sim$ as defined above.

$S^{-1}A$ is the set of equivalence class under $\sim$, write $[(a,s)] = [\frac{a}{s}]$

Define $[\frac{a_1}{s_1}] + [\frac{a_2}{s_2}] = [\frac{a_1 s_2 + a_2 s_1}{s_1 s_2}]$,

$[\frac{a_1}{s_1}] \cdot [\frac{a_2}{s_2}] = [\frac{a_1 a_2}{s_1 s_2}]$

Show $(+, \cdot)$ well-defined.

Show $S^{-1}A$ is a ring with zero $[\frac{0}{1}]$, unity $[\frac{1}{1}]$

EX A : integral domain, the Quotient field of A = Frac(A) is ~~A-{0}~~ $(A - \{0\})^{-1} \cdot A$

Lemma. $\tau: A \mapsto S^{-1}A$, by $\tau(a) = \frac{a}{1}$ is a ring homomorphism.
with the properties: ① $\overset{0\notin S}{\wedge}$ $\tau(S) \subseteq (S^{-1}A)^{\times}$ (units of $S^{-1}A$) (Only for $0 \notin S$)
② $\ker(\tau) = \{a \in A : \exists s \in S, sa = 0\}$
③ $A \neq \{0\}$, then τ: injective IFF S contains no zero divisors of A.
④ $S^{-1}A = \{0\}$ IFF $S \ni 0$ ⑤ $\overset{A\neq\{0\}}{\wedge}$ τ: isomorphism IFF $S \subseteq A^{\times}$ (Only for $A \neq \{0\}$)

(*1) $\longrightarrow$

Pf: ① $\forall s \in S$, $s \mapsto [\frac{s}{1}]$, then $[\frac{1}{s}] \times [\frac{s}{1}] = [\frac{s}{s}] = [\frac{1}{1}]$

Notice: if $0 \in S$, then $\forall [\frac{a}{s}] = [\frac{1}{0}]$, thus $S^{-1}A = \{0\}$, $1_{S^{-1}A} = 0_{S^{-1}A}$

Then $S^{-1}A$ will contain no units.

② $\ker(\tau) = \{a \in A, \tau(a) = [\frac{a}{1}] = [\frac{0}{1}]\} = \{a \in A, \exists s \in S, sa = 0\}$.

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④ If $0 \in S$, $S^{-1}A = \{0\}$ (Shown in ①)

If $S^{-1}A = \{0\}$, then $[\frac{0}{1}] = [\frac{1}{1}]$, i.e. $\exists s \in S$, $s \cdot 1 = 0 = s$.

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(*1): First show $\tau: A \to S^{-1}A$ is a ring homomorphism.

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Now if $S \subseteq A^{\times}$, $\tau^{-1}: S^{-1}A \to A$ by $\tau^{-1}([\frac{a}{s}]) = s^{-1}a$, show well defined.

$\tau \circ \tau^{-1}([\frac{a}{s}]) = [\frac{s^{-1}a}{1}] = [\frac{a}{s}]$, $\tau^{-1} \circ \tau(a) = a$.

Examples. ① $A \ni f$, $S_f = \{f^n : n \geq 0\}$ (multiplicatively closed)

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② ~~A: integral dom~~ A: integral domain ($A - \{0\}$ is multiplicatively closed requires this)

$0 \notin S \subseteq A$, $A[\frac{1}{S}]$ = subring of $(A-\{0\})^{-1}A$ generated by $\{[\frac{a}{1}] : a \in A\} \cup \{[\frac{1}{s}] : s \in S\}$

$A[\frac{1}{S}] = \langle S \rangle^{-1} \cdot A$

③ $p \subseteq A$, ideal, $S_p = A - p$ then

S_p: multiplicatively closed $\Leftrightarrow$ $\begin{cases} 1 \in S_p \text{ i.e. } p \neq \langle 1 \rangle \\ \forall t_1, t_2 \in S_p,\ t_1 \cdot t_2 \in S_p \text{ i.e. If } t_1 \cdot t_2 \notin S_p / t_1 \cdot t_2 \in p, \text{ then } t_1 \notin S_p / t_1 \in p, \text{ or } t_2 \notin S_p / t_2 \in p. \end{cases}$

$\Leftrightarrow$ p: prime.

Recall: $\tau: A \longmapsto S^{-1}A$.

$\tau_*(\mathfrak{a}) = \{\frac{a}{s}; a \in \mathfrak{a}, s \in S\}$ $\quad \tau_* \tau^*(b) = b, \forall b \in S^{-1}A$.

$\tau_*(\mathfrak{a}) = \langle 1 \rangle \Leftrightarrow \mathfrak{a} \cap S \neq \emptyset$

general ideals (not nece prime)

- $q_1 \subseteq q_2 \Leftrightarrow \tau^* q_1 \subseteq \tau^* q_2$ $\quad$ ($\Leftarrow$ use $\underbrace{q_j = \tau_* q_j^*}$) ?
  $q_i = \tau_*(\tau^*(q_i))$
- $k(p) = \mathrm{Frac}(A/p)$, residue field of the prime ideal p
  p: prime

The bijection $\varphi^*: \mathrm{spec}(S^{-1}A) \longleftrightarrow \mathrm{spec}(A)$, induces isomorphism.

$$k(\tau^* q) \cong k(q)$$

- $p = m$ [maximal ideal, $\Leftrightarrow A/p$ field $\Leftrightarrow k(p) = A/p = \mathrm{Frac}(A/p)$]
  (prime)

EX: $p = (y) \subseteq A = \mathbb{C}[x,y]$, $A/p \cong \mathbb{C}[x]$, $k(p) \cong \mathbb{C}(x)$.

- Pf: $\mathrm{Spec}(S^{-1}A) \longmapsto \{\mathrm{Spec}(A), \underbrace{p \cap S = \emptyset}\}$
  bijection τ^*, τ_* $\quad$ i.e. $S \subset A - p$

Check: p: prime $\Rightarrow \tau_* p$: prime. $(*2)$

$$\underset{(\checkmark)}{\tau_* \tau^*(q) = q}, \quad \tau^* \tau_*(p) = \underset{(*1)\ (\checkmark)}{p}$$

$(*1)$: $\tau^* \tau_*(p) \subseteq p$. let $a \in A$, $\tau(a) = \frac{a}{1} \in \tau_* p$

$\Rightarrow \frac{a}{1} = \frac{b}{s}$ $(b \in p)(s \in S)$ $\Rightarrow ast = bt \in p$, $st \in S \subseteq A - p$. $\Rightarrow a \in p$
$t \in S$

$(*2)$: $\frac{a}{s}, \frac{b}{t}$ $(a, b \in p, s, t \in S)$

let $\frac{a}{s} \cdot \frac{b}{t} \in \tau_* p$, i.e. $\frac{ab}{st} = \frac{c}{u}$ $(u \in S, c \in p)$,

$\exists v \in S$. $ab\underbrace{uv}_{\in S} = st\underbrace{cv}_{\in p} \in p \Rightarrow ab \in p \Rightarrow a \in p$ or $b \in p$

$\Rightarrow \frac{a}{s}$ or $\frac{b}{t} \in p$.

- Examples

$S = S_f = \{f^n : n \geq 0\}$. $S^{-1}A = A_f = A[\frac{1}{f}]$.

$p \cap S_f \neq \emptyset \Leftrightarrow \exists n. f^n \in p \Leftrightarrow f \in p$

i.e. $\mathrm{spec}(A_f) \cong \{p \in \mathrm{Spec}(A), f \notin p\}$.

EX: $A = \mathbb{Z}$, $f = 2$. $A_f = \mathbb{Z}[\frac{1}{2}]$. $\{$prime ideals in $\mathbb{Z}[\frac{1}{2}]\} \cong \{$prime ideals do not $\ni 2\}$ in $\mathbb{Z}$.

? EX: $A = \mathbb{C}[x,y]$ $\{$maximal ideals in $A\} \longleftrightarrow \mathbb{C}^2$
$\longleftrightarrow (x,y) \longleftrightarrow$ ~~$f(x)$~~ $f \in \mathbb{C}[x,y]$, $f(x,y) = 0$.

PID(prime $\Leftrightarrow$ maximal)

- $f \in \mathbb{C}[x,y]$

maximal ideals in $A_f = \mathbb{C}[x,y,\frac{1}{f}] \longleftrightarrow$

? max'l $m \subseteq \mathbb{C}[x,y]$, $f \notin m \longleftrightarrow \{(x,y) \in \mathbb{C}^2, f(x,y) \neq 0\}$

$\langle x_0 - x, y_0 - y \rangle \longleftarrow x_0, y_0$

Spec A : $\cancel{A_f}$ Spec A_f

- $f, g \in A$. $A \to A_f \to A_{fg}$? commute

EX: A: integral domain $A_f \subseteq \cancel{Af}\ A_{fg}$ $(f, g \neq 0)$

$\mathrm{Frac}(A) = \bigcup_f A_f \ (f \neq 0)$ $\mathrm{Frac}(A) = (A - \{0\})^{-1} A$

$\forall p \in \mathrm{Spec}(A_f) \subseteq \mathrm{Spec}(A)$ $\{f \in A, f \notin p\} = \{f \in A, f(p) \neq 0\}$ when $f(p) \in k(p)$ is the image of f

$A_f \longmapsto k(p)$: $\frac{a}{f^n}$

- Recall. A: local $\Leftrightarrow \exists! m \in \mathrm{Spec}_m(A) \Leftrightarrow \exists$ maximal m, $\forall m \subseteq A^\times$

$\Leftrightarrow A - m \subseteq A^\times$

- $p \subseteq A$, $S_p = A - p$, $S_p^{-1} A = A_p$ (prime)

<u>Prop</u> ① $\mathrm{Spec}(A_p) \cong \{q \in \mathrm{Spec}(A), q \subseteq p\}$

② A_p : local ring, with maximal ideal $P_p = \tau_*(p)$

A_p: localization of A at p.

pf: ① $q \cap S_p = \Phi$, $q \cap (A - p) = \Phi$, $q \subseteq p$.

② τ_* : inclusion preserving (by def)

$\forall p_0$: prime, $\tau_*(p_0) \subseteq P_p$, any ideal is contained in some maximal ideal. $A \neq \{0\}$

Hence P_p : is the unique maximal ideal

<u>EX</u>. $p = \langle p \rangle \subseteq \mathbb{Z} = A$, $A_p = \mathbb{Z}_{(p)}$

local ring, $\mathrm{Spec}(\mathbb{Z}_{(p)}) \cong \{q \in \mathrm{Spec}\,\mathbb{Z} : q \subseteq p\}. = \{(0), (p)\}$

- Residual Field. $\mathbb{Z}_{(p)} / P_p \cong \mathbb{Z}/(p)$

In general, $A_p / P_p = k(p)$

- EX $A = \{f$: holomorphic in $\mathbb{C} \to \mathbb{C}\}$ or $\Omega \in \cancel{\subseteq}\ \mathbb{C}$, open

germs of f at p.

germ at p: equivalence class $[(U, f)]$ of pairs (U, f), U: open nbhd of p in Ω, $f: U \mapsto \mathbb{C}$, holomorphic

$(U_1, f_1) \sim (U_2, f_2)$ IFF $f_1 = f_2$ on some $p \in V \subseteq_{\text{open}} (U_1 \cap U_2)$

Fix $p \in \Omega$. $\mathfrak{p} = \{f \in A.\ f(p) = 0\}$

A is a local ring with maximal ideal $\mathfrak{p}$.

Check $A - \mathfrak{p} \subseteq A^\times$. $f(p) \neq 0$, $\exists$ open neigh of p on which $\frac{1}{f(p)}$ is also holomorphic.

- $A = \mathbb{C}[x,y]$, $\mathfrak{p} = (y)$, $A_\mathfrak{p}$.

  $\mathrm{Spec}(A_\mathfrak{p}) \cong \{ q \in \mathrm{spec}(A),\ q \subseteq (y) \}$.

  $q = (y)$

  $q = (0)$ (zero is the only prime ideal properly contained in y)

  $A_\mathfrak{p}$: local ring with two primes, residual field $\cong \mathbb{C}(x)$

- $A = \mathbb{C}[x,y]$ $\mathfrak{p} = (x,y)$ then $\mathrm{Spec}(A_\mathfrak{p}) \cong \{\} \cup \{(f) : 0 \neq f \in \mathbb{C}[x,y]$, irreducible, $f(0,0) = 0\} \cup \{(0)\}$.

*

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Localization of Modules

$S^{-1}A$ module

$S \subseteq A$, M, $\mapsto S^{-1}M$ $\quad (m_1, s_1) \sim (m_2, s_2)$: $\exists t \in M$, s.t. $tm_1 s_2 = tm_2 s_1$

$\frac{m}{s} = [(m,s)]$

Lem. $S^{-1}A \otimes_A M \xrightarrow{\cong} S^{-1}M$ $\quad (S^{-1}A, M) \xrightarrow{\varphi} S^{-1}M$ bilinear. (Extension of Scalar).

Note: $S^{-1}M$ is $S^{-1}A$-module.

Pf:

$\frac{a}{s} \otimes m \longmapsto \frac{am}{s}$

$\frac{1}{s} \otimes m \longleftarrow \frac{m}{s}$

Inverses, check well-defined.

$f: M \longmapsto N$ (A-linear) $\quad S^{-1}f : S^{-1}M \longmapsto S^{-1}N$, $\frac{m}{s} \longmapsto \frac{f(m)}{s}$

$S^{-1}(f \circ g) = S^{-1}f \circ S^{-1}g$ $\Rightarrow$ well-defined morphism.

$\mathfrak{p} \in \mathrm{Spec}(A) \mapsto A_\mathfrak{p}$, $M_\mathfrak{p} = S_\mathfrak{p}^{-1} M \cong A_\mathfrak{p} \otimes_A M$

Next Class: local properties, # M depend only upon $M_\mathfrak{p}$ $\forall \mathfrak{p} \in \mathrm{Spec}(A)$.

Mod Localization — Lec 5-1

$\forall S^{-1}: (A\text{-mod}) \longmapsto (S^{-1}A\text{-mod})$ exact :

If $M' \xrightarrow{f} M \xrightarrow{g} M''$ exact, then $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ exact.

Pf : $g \circ f = 0 \Rightarrow S^{-1}g \circ S^{-1}f = S^{-1}(g \circ f) = 0$.

$\ker(S^{-1}g) \supset \operatorname{Im}(S^{-1}f)$.

Atiyah — Now check. $\ker(S^{-1}g) \subset \operatorname{Im}(S^{-1}f)$.

$\frac{x}{s} \in \ker(S^{-1}g)$, $\frac{x}{s} \mapsto \frac{g(x)}{s} = \frac{0}{1}$,

$\exists t \in S$, s.t. $tg(x) = 0$. $\Rightarrow g(tx) = 0$, $tx \in \ker(g) = \operatorname{Im}(f)$, $tx = f(x')$

$\exists$ $\frac{f(x')}{s'} = \frac{x}{s}$ i.e. $\exists t' \in S$, s.t. $t's f(x') = t's' x$

Thus $\frac{tx}{ts} = \frac{f(x')}{ts \in S} \in \operatorname{Im}(S^{-1}f)$

$\frac{x}{s} =$

Cor : $S^{-1}A$ is a flat A-mod.

By def. M flat$_{A\text{-mod}}$ iff ~~$\forall N \xrightarrow{inj} M \otimes_A N$~~ $\forall N \hookrightarrow P$, $N \otimes M \hookrightarrow P \otimes M$.

Want. $\forall N \hookrightarrow_A P$, $0 \to S^{-1}A N \otimes_A S^{-1}A \hookrightarrow P \otimes_A S^{-1}A \to 0$

$\qquad\qquad S^{-1}N \longrightarrow S^{-1}P \to 0$.

Cor/Lem : S^{-1} commutes with $\begin{cases} \text{finite sums / intersections} \\ \text{kernel} \quad \ker(S^{-1}M \to S^{-1}N) = S^{-1}\ker(M \to N) \\ \text{quotients} \\ \otimes \quad (S^{-1}M \otimes_{S^{-1}A} S^{-1}N) \cong S^{-1}(M \otimes_A N) \end{cases}$

Zero, local property.

TFAE ① $M = 0$ ② $M_p = 0$, $\forall$ prime ③ $M_m = 0$ $\forall$ maxi

Check $M_m = 0 \Rightarrow M = 0$
($\forall$ m maxi)

Claim 1. $x \in M$, $x \neq 0 \Leftrightarrow \operatorname{Ann}(x) \neq (1)$ $\quad \begin{cases} x \neq 0 \Leftrightarrow 1 \cdot x \neq 0 \Leftrightarrow 1 \notin \operatorname{Ann}(x) \\ \Leftrightarrow (1) \neq \operatorname{Ann}(x) \quad \text{ideal} \end{cases}$

Claim 2 : m : maxi, $x \notin \ker(M \to M_m) \Leftrightarrow \operatorname{Ann}(x) \subseteq m$.

$\Updownarrow$ $\nexists s \in A - m$, $sx = 0$. $\Leftrightarrow$ ~~$\operatorname{Ann}(x) \cap S = \Phi$~~

$\operatorname{Ann}(x) \cap (A - m) = \Phi \Leftrightarrow \operatorname{Ann}(x) \subseteq m$.

Pf: ISTS (it suffices to show)

$M \neq 0 \Rightarrow \exists m$ max, $M_m \neq 0$.

$0 \neq x \in M \Rightarrow \operatorname{Ann}(x) \neq (1) \Rightarrow \exists$ max $m \supseteq \operatorname{Ann}(x)$

$\Rightarrow x \notin \ker(M \to M_m) \Rightarrow M_m \neq 0$.

WMA (we may assume)

Inject / Surj is Local

TFAE ① $M \xrightarrow{\phi} N$ surjective

② $M_p \xrightarrow{\phi_p} N_p$ surjective $\forall p$ prime

③ $M_m \xrightarrow{\phi_m} N_m$ surjective $\forall m$ max

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Pf: $M \to N \to K = M/N \to 0$ exact

$M_p \to N_p \to K_p \to 0$ exact

ϕ surjective $\Leftrightarrow K=0$.

$\Leftrightarrow K_p = 0$.

previous lemma.

Flatness Local. $A \curvearrowright M$; M: A-mod TFAE

(a) $A \curvearrowright M$, flat (b) $A_p \curvearrowright M_p$ $\forall p$ flat (c) $A_m \curvearrowright M_m$, $\forall m$. flat

Pf: (c) ⇒ (a) let $N \hookrightarrow P$ ($N \to P$ injective A-linear map)

want $N \otimes M \hookrightarrow P \otimes M$

$\Leftarrow (N \otimes M)_m \hookrightarrow (P \otimes M)_m$, $\forall m$ max

$\quad\quad \| \quad\quad\quad\quad \|$

$N_m \otimes_{A_m} M_m \hookrightarrow P_m \otimes_{A_m} M_m$, $\forall m$ max.

$\Leftarrow N_m \hookrightarrow P_m \quad \Leftrightarrow N \hookrightarrow P$.

M_m: flat

Def / Lem A (Ring), TFAE.

① A satisfies the ascending chain condition on ideals:

$\forall$ sequence of ideals ~~$a_1 \subsetneq a_2 \subsetneq \cdots \subsetneq a_n \subsetneq \cdots$~~ $a_1 \subseteq a_2 \subseteq \cdots \subseteq a_n \subseteq \cdots$

Atiyah stabilizes ($\exists n_0$, s.t. $\forall n \geq n_0$. $a_n = a_{n_0}$)

e.g. $A = \mathbb{Z}$, a_3 $a_n \supseteq \cdots a_2 \supseteq a_1$

$b_n | \cdots | b_2 | b_1$

② Every ideal of A is fin generated.

③ {ideals in A} satisfies the maxi proper:

Every subset contains a maximal element.

i.e. $\forall S \neq \emptyset$ subsets of ideals, ~~$\exists$ maximal ideal containing all~~

$\exists \mathfrak{a} \in S$, $\forall \mathfrak{b} \in S$, $\mathfrak{b} \not\supsetneq \mathfrak{a}$?

not strict inclusion. $\exists \mathfrak{a} \in S$, $\forall \mathfrak{b} \in S$, $\underline{\mathfrak{b} \not\supsetneq \mathfrak{a}}$

$\not\supsetneq$

$\mathfrak{a}$

Def A: called Noetherian. then

$\forall \mathfrak{a} \in S$, $\exists \mathfrak{b} \in S$, $\underline{\mathfrak{b} \supsetneq \mathfrak{a}}$

(a) $\Rightarrow$ (b) $\mathfrak{a}$, not finite generated.

Inductively construct. $x_1, \cdots x_n, \in \mathfrak{a}$.

$\mathfrak{a}$: ideal, then $(x_1) \neq 0$, $\mathfrak{a} \supsetneq (x_1)$

take $x_2 \in \mathfrak{a} - (x_1)$, ~~[illegible]~~ $\mathfrak{a} \supsetneq (x_1, x_2) \supsetneq (x_1)$.

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... Then we have $(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots \subsetneq (x_1, \cdots, x_n)$

ACC MAX Contradiction

(a) $\Rightarrow$ (c) If MAX fails, then ACC fails

$\mathfrak{a}_1, \mathfrak{a}_2, \cdots \in S$.

$\exists$ sequence $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \mathfrak{a}_3 \cdots$.

MAX ACC

(c) $\Rightarrow$ (a) If ACC fails.

$\exists \mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \cdots$

FG ACC

(b) $\Rightarrow$ (a) Let $a_1 \subseteq a_2 \cdots$, want $\exists n_0, \forall \mathfrak{a}_n = \mathfrak{a}_{n_0}, \forall n \geq n_0$.

Know: every ideal is fin gen.

$\mathfrak{a} = \bigcup_n \mathfrak{a}_n$, is indeed an ideal.

$\mathfrak{a} = \langle x_1, \cdots, x_r \rangle$. $x_i \in \mathfrak{a}_{n_j}$

$n_0 = \max(n_1, \cdots, n_r)$

Then $x_1, \cdots x_r \in \mathfrak{a}_{n_0}$. ... $\square$

Def/Lem M: A-mod. (a) M: ACC on submodules

(b) Every submodule is fin't genera (c) M: MAX on submodules

Def M: Noetherian A-mod.

Generalization of Noetherian Ring.

SES (Short exact sequence).

Lem: $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ SEM. Inf fr kerg

M: Noeth $\Leftrightarrow$ M', M'': Noeth

Bosch

Pf. "$\Leftarrow$" ACC $N_1 \subseteq N_2 \subseteq \cdots$ $\leftarrow$ submodules of M.

Show $\exists n_0$, s.t. $N_n = N_{n_0}$, $\forall n \geq n_0$

$N_j'' = \mathrm{Im}(N_j)$ in M'', then $N_1'' \subseteq N_2'' \subseteq \cdots$

by ACC for M'', then $N_{n_0}'' = N_n''$ $\forall n \geq n_0$

$N_1' \subseteq N_2' \subseteq \cdots$ $N_j' = M' \cap N_j$ (Use $M' \subseteq M$ inject) ($N_n' = N_{n_0}'$)

Need if $N_i \subseteq N_j \subseteq M$, w/ $N_i'' = N_j''$, $N_i' = N_j'$ then $N_i = N_j$

~~Theorem~~: A: Noetherian $\longrightarrow A[x]$: Noetherian.
(Hilbert basis theorem)

Cor. ① A: Noeth $\Rightarrow A[x_1,\dots,x_n]$ Noeth, $A[x_1,\dots,x_n]/\mathfrak{a}$: Noeth.
$\forall \mathfrak{a} \in A[x_1,\dots,x_n]$

Pf: Let $\mathfrak{a} \subseteq A[x]$, want $\mathfrak{a}$: f.g.

Consider the ideal $\mathcal{L}$ Gen by "leady coff of elements of $\mathfrak{a}$".
i.e. $ax^n + \cdots$

$\mathcal{L} \subseteq A$, f.g. $\mathcal{L} = (t_1,\dots,t_r)$ $t_j \in \mathcal{L}$

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$\exists f_1,\dots f_r \in \mathfrak{a}$, $f_j = t_j x^{n_j} + \cdots (\dots)$,

$N = \max(n_1,\dots,n_r)$, $M = \bigoplus_{j=0}^{N} Ax^j \subseteq A[x]$
$\circlearrowleft A$

$A \circlearrowleft M \cap \mathfrak{a}$ f.g. (M: Noeth $\overset{(*)}{\Leftarrow}$ A: Noeth)

(*) $M \cong A^n$, $0 \to A \to A^n \to A^{n-1} \to 0$
Induct on n, use previous lem.

Claim $\mathfrak{a} = (f_1,\dots,f_r) + \underbrace{M \cap \mathfrak{a}}_{\text{f.g.}}$.

Pf: $\subseteq$: ✓
$\supseteq$: $f \in \mathfrak{a}$, consider seperly $cx^s + \cdots$ $(s \geq N, s \leq N)$.

Lec 5-2

Pf: $\forall \mathfrak{a} \subset A[x]$, want $\mathfrak{a}$: f.g.
$\mathfrak{a}' = \{$leading coeff of $f \in \mathfrak{a}\}$, ideal $\subset A$, $\mathfrak{a}'$: f.g.

$\mathfrak{a}' = (a_1,\dots,a_r)$
$\exists f_1 = a_1 x^{n_1} + \cdots$, $f_r = a_r x^{n_r} + \cdots \in \mathfrak{a}$
$n \geq \max\{n_1,\dots,n_r\}$.
let $f = ax^n + \cdots + (\dots) \in \mathfrak{a}$, w/ $n \geq \max(n_1,\dots,n_r)$ $\leftarrow N$

$a = c_1a_1 + \cdots + c_ra_r$.
(*) $f - \underbrace{(c_1 x^{n-n_1} f_1 + \cdots + c_r x^{n-n_r} f_r)}_{\in (f_1,\dots,f_r) \subseteq \mathfrak{a}} = (\underbrace{a - \sum c_j a_j}_{=0}) x^n + \cdots$

$M_n := \bigoplus_{j=0}^{n} Ax^j \cap \mathfrak{a}$ is f.g. gene. by $\langle x^1,\dots,x^n \rangle$
M_N: f.g.

(*) $\Rightarrow$ $\mathfrak{a} \subseteq M_N + (f_1,\dots,f_r) \subseteq \mathfrak{a}$
i.e. $\mathfrak{a} = M_N + (f_1,\dots,f_r)$

App.

• $A[x_1, \ldots, x_n]/\mathfrak{a}$, Noeth $\Leftarrow$ A: Noeth $\quad (\forall \mathfrak{a} \subset A[x_1, \ldots, x_n])$

Recall : $S \subseteq \mathbb{C}[x_1, \ldots, x_d]$, $V(S)$ $\longleftrightarrow$ Noetherian.

$V(S) = V(\langle S \rangle)$

Hilbert basis theorem $\Rightarrow \forall V$ (varieties), $\exists$ finite $S \subseteq \mathbb{C}[x_1, \ldots, x_d]$, s.t. $V = V(S)$

i.e. Any ~~fin~~ system of polynomial equations is the same as some finite system.

Pf: S, $\mathfrak{a} = \langle S \rangle$.

~~$\mathfrak{a}$~~ $\mathbb{C}$: Noeth $\Rightarrow \mathbb{C}[x_1, \ldots, x_n]$: Noeth.

$\mathfrak{a}$: f.g.

$V = V(S) = V(\mathfrak{a}) = V\langle f_1, \ldots, f_r \rangle$

Non-example.

$A = \mathbb{C}[x_1, \ldots, x_n, \ldots]$ not Noeth.

$m = (x_1, \ldots, x_n, \ldots)$ not f.g.

If $S \subseteq m$, finite, let $S = \{s_1, \ldots, s_n\}$, then let $M_i = \max$ ~~$\mathfrak{a}$~~ i: x_i "$\in$" S_i

then let $N = \max_i \{M_i\} + 1$, x_N is not in S.

Note A : integer domain $\Rightarrow K = \mathrm{Frac}(A)$ is a field

i.e. K Noeth but $A \subseteq K$ ~~not~~ may not be.

Lemma. A : Noeth, $A \xrightarrow{h} B$ $\quad \mathrm{Im}(h)$ Noeth. $\Rightarrow$ $\quad \forall \mathfrak{a} \subseteq A$, ideal

$\Leftarrow \quad A \twoheadrightarrow A/\mathfrak{a}$ (surj).

$\mathrm{Im}(h) \cong A/\mathrm{Ker}(h)$.

i.e. $\underbrace{\text{any homorphic image of}}_{A/\mathfrak{a}}$ A is Noeth.

Pf. $\{$ideals in A contains $\mathfrak{a}\} \cong \{$ideals in $A/\mathfrak{a}\}$

use ~~the f.g~~ /MAX condition.

Localization of Noeth Rings.

A: Noeth, $\quad S \subseteq A$, mult closed, $\Rightarrow S^{-1}A$: Noeth

i.e. A_f, $A_{\mathfrak{p}}$ Noeth if A Noeth.

$\{$ideals in $S^{-1}A\} \longrightarrow \{$ideals in $A\}$

$\mathfrak{b} \longmapsto \tau^*(\mathfrak{b})$ (ideal) $\qquad \mathfrak{b} = \tau_*(\tau^*(\mathfrak{b}))$

use the MAX condition / f.g condition.

Lec 6-1. CA.

Recall: F (field) V (v.s over F) $S \subseteq V$ (linear indep): $\forall$ $\{s_1, \dots, s_n\} \subseteq S$ (distinct).

$\forall c_1, \dots, c_n \in F$. $c_1 s_1 + \dots + c_n s_n = 0 \Rightarrow c_1 = \dots = c_n = 0$.

Thm: $S \subseteq V$, (a) S: line indep | S: maximal (i.e. among lin-indep subsets)

$\Leftrightarrow$ span $S = V$.

(b) $\{v_1, \dots, v_n\} \subseteq V$, maximal linear indep, i.e. basis.

Suppose $\{w_1, \dots, w_m\} \subseteq V$, linear indep, then $m \leq n$.

(c) hence dimension of basis in fini spaces, well-defined. (invariant)

finite
- (d) Every vector space (finite) has a basis.
- (e) Every lin-indep subset $S \subseteq V$ extends to a basis.
- (f) Every $S \subseteq V$ w/ span $S = V$, subtract to a basis.

Field Extension. (Wiki: $\forall e \in E$. $\{e\}$ alge inde IFF e: transcedental over F)

E/F: field extension $S \subset E$ alge indep if

$\forall \{s_1, \dots, s_n\} \subseteq S$ (distinct), $\forall p \in F[x_1, \dots, x_n]$. $p(s_1, \dots, s_n) = 0 \Rightarrow p = 0$.

Thm. (a) $S \subseteq E$, alge indep
then S: maximal $\Leftrightarrow$ $E/F(S)$ algebraic field extension

extension on S ← when S infinite, how is this different from S being finite?

($S = \Phi$ if E/F is algebraic, otherwise $|S| \geq 1$) tr.dg$(\Phi) = 0$.

(b) $\{v_1, \dots, v_n\} \subseteq E$: alge indep, maxim

i.e. ~~union of finite field extension~~

write: tr. basis: transendan basis.

If $\{w_1, \dots, w_m\} \subseteq E$: alge indep then $m \leq n$.

(c) Any 2 tr. basis has the same cardinality

write: Transcedem degree E/F, tr.deg(E/F).

(d) Every E/F has a tr. basis (Requires Zorn's Lemma)

(e) Any alge inde $S \subseteq E$ extends - - -

(f) $S \subseteq E$, $E/F(S)$ algebraic, then $\exists$ tr. basis T for E/F, $T \subseteq S$

need more detailed proof

Pf (a). "$\Rightarrow$" Let $\alpha \in E$, want $F(\alpha, S)/F(S)$ finite.

If $\alpha \in S$, then done; Else, $S \cup \{\alpha\}$ is not alge inde, so.

$\exists s_1, \dots, s_n \in S$, $p \in F[x_1, \dots, x_n]$, $p \neq 0$, between $s_1, \dots, s_n, \alpha$.

$\exists m \geq 1$, $p_0, \dots, p_m \in F[X_1, \dots X_n]$ s.t. OR: $p \in F[X_0, \dots, X_n]$, s.t. $p(\alpha, \dots, s_n) = 0$

$\alpha^m p_m(s_1, \dots, s_n) + \dots + \alpha p_1(s_1, \dots, s_n) + p_0(s_1, \dots, s_m) = 0$.
($p_m \neq 0$)

$\Rightarrow p_m(s_1, \dots, s_n) \neq 0$ (by S: alge indep)

$\Rightarrow [F(\alpha, s_1, \dots, s_m) : F(s_1, \dots, s_m)] \leq m$

α is alge over $F(S)$

"$\Leftarrow$" $E/F(S)$ algebraic, want S: max

Assum $\exists \alpha \in E$, $\alpha \notin S$ s.t. $S \cup \{\alpha\}$ is alge indep.
then α is algebraic over $F(S)$ by assumption.

(☆☆) $\alpha^m + \dfrac{p_{m-1}(s_1, \dots, s_n)}{q_{m-1}(s_1, \dots s_n)} \alpha^{m-1} \dots = 0$, some $s_1, \dots, s_n \in S$. $p_i, q_j \in F[X_1, \dots, X_n]$
($m \geq 1$)

Clear the denominators, to get a nontrivail polynomia relation involy $s_1, \dots, s_n, \alpha$, contradn... $\{\alpha\} \cup S$ alge indep.

PExer: Prob (b)—(f) in the special case of finite transcenden degree. $\boxed{\text{Do!!!}}$

Example: $\mathrm{tr.dg}(\bar{\mathbb{Q}}/\mathbb{Q}) = 0$, $\mathrm{tr.dg}(\mathbb{C}/\mathbb{Q}) = \infty$, $\mathrm{tr.dg}(F(t_1, \dots, t_n)/F) = n$
field

若 $E/F(t_1, \dots, t_n)$ algebraic extension, then $\mathrm{tr.dg}(E/F) = n$.

?

closure or not $K[x_1, \dots, x_n]$

Goal. $\bar{L}/K$, field extension. s.t. L: f.g as K-alge. then L ~~fi~~ finite K-algebra.
$\Rightarrow L/K$ finite.

pf: write $L = \langle x_1, \dots, x_n \rangle$ K-alg

$r := \mathrm{tr.dg}(L/K)$, conclusion $\Leftrightarrow r = 0$. ?

$L = \bigcup_{i \in I} L_i/K$

L/K:

Note:
$\mathrm{tr.dg}(E/F) = 0$
$\Leftrightarrow E/F$ algebraic

Suppose not, then $r \geq 1$.
by (f), then $\{x_1, \dots, x_r\}$ is a tr. basis of L/K.
Each $x_{r+1}, \dots, x_n$ is algebraic over $K(x_1, \dots, x_r) = M$
L/M, finite.

By the Lem below,

$$L \text{ — } \underbrace{K(x_1, \dots, x_r) \text{ — } K}$$

↑ finite field exten, finite ~~K-alg~~ K-mod

↑ is not f.g.

~~Galois~~

Lemma. $A \subseteq B \subseteq C$ rings. Then B : f.g. A-Algebra.

A: Noetherian

C: f.g A-algebra. finite B-mod

pf: $C = \langle y_1, \dots, y_m \rangle_{B\text{-mod}}$. $C = \langle x_1, \dots, x_n \rangle_{A\text{-alg}}$

$x_i = \sum_j b_{ij} \cdot y_j$, $b_{ij} \in B$

(☆) $\underbrace{y_i \cdot y_j}_{\text{as elements in } B} = \sum_k b_{ijk} y_k$. (As a element of B as an A-algebra)

$B_0 := A[\{b_{ij}\} \cup \{b_{ijk}\}] \subseteq B$

B_0: f.g A-algebra. $\underset{(HBT)}{\Rightarrow}$ B_0: Noeth.

$C = \{\text{poly in } \{x_j\} \text{ w/ coeff in } A\} \underset{\text{Substitute (☆)}}{=} \{\text{linear combinat of the } y_j \text{ w/ coeff in } B_0\}$.

$\Rightarrow$ C is a finite B_0-mod

$\Rightarrow$ C: Noethen B_0-mod.

$\Rightarrow$ $\forall$ submodule fin gen. Then $B_0 \subseteq B \subseteq C$ is f.g.

B } f.g
$|$
B_0 } f.g
$|$
A
} f.g A-alg

Relation to $r(\mathfrak{a}) = I(V(\mathfrak{a}))$.

$k = \bar{k}$ (alge closure), $\mathfrak{a} = k[t_1, \dots, t_d]$, $V(\mathfrak{a}) := \{x \in k^d, f(x) = 0, \forall f \in \mathfrak{a}\}$

$I(S) = \{f \in A : f(x) = 0, \forall x \in S\}$. $r(\mathfrak{a}) = \{f \in A;\ f^n \in \mathfrak{a} \text{ for some } n\}$.

pf: ① $r(\mathfrak{a}) \subseteq I(V(\mathfrak{a}))$. $f \in r(\mathfrak{a})$, $f^n \in \mathfrak{a}$, $f^n|_{V(\mathfrak{a})} = 0$. $\Rightarrow$ $f|_{V(\mathfrak{a})} = 0$
k: field.

② If $f \notin r(\mathfrak{a})$, want $f \notin I(V(\mathfrak{a}))$

$r(\mathfrak{a}) = \bigcap_{i \in I} p_i \ (p_i \supseteq \mathfrak{a})$

$\exists p_i$ s.t. $p_i \supseteq \mathfrak{a}$, $p_i \not\ni f$, then $\bar{f} \neq 0 \in A/p_i$

$(A/p)_{\bar{f}} = (A/p)[\frac{1}{\bar{f}}] \neq 0$.

Let $m \subseteq (A/p)_{\bar{f}}$, maximal ideal, Let $B = (A/p)_{\bar{f}}$

$L = B/m$, field.

L is fin. gen k-alge.

then L/k fin $\underset{k = \bar{k}}{\Rightarrow}$ $L = k$

Set $X = (x_1, \dots, x_d)$, x_j = image in $L \supseteq k$, of $t_j \in L$.

Check $f(x) \neq 0$, $X \in V(\mathfrak{a})$. End of Proof □

Primary Decomposition. Lec 6-2 (see later) ✓

A: PID, ~~$\mathfrak{a} \subseteq A$~~, $\mathfrak{a} \in A$, uniquely write $\mathfrak{a} = \varepsilon\, p_1^{r_1} \cdots p_k^{r_k}$ (ε: unit; p_i: prime, distinct)

$(\mathfrak{a}) = (p_1^{r_1}) \cap \cdots \cap (p_k^{r_k})$ decomposition of $(\mathfrak{a})$. ($(p_i^{r_i})$: primary)

Def A: ring, $\mathfrak{q} \subseteq A$ ideal, primary, IFF $\forall$ zero divisor in $A/\mathfrak{q}$ is nilpotent.

i.e. $\forall t \in A/\mathfrak{q}$, $ts \in \mathfrak{q}$ $(s \notin \mathfrak{q})$, $\exists n$, s.t. $t^n \in \mathfrak{q}$.

Prime $\Rightarrow$ Primary

Comp $\mathfrak{p}$: prime if $\nexists A/\mathfrak{p}$: zero div $\neq 0$

Def Decomposible. $\mathfrak{a} \subseteq A$.

IFF $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ of primary ideals.

i.e. $\mathfrak{a}$ has a primary decomposition.

Prop: A: Noetherian, $\forall \mathfrak{a} \subseteq A$ decomposible.

Lem. Noetherian Induction. TFAE.

(I) $S \subseteq \{$ideals of $A\}$ has a max

(II) $S \subseteq \{$ideals of $A\}$ s.t. (a)+(b) hold.

(a) $(1) \in S$ (b) $\forall \mathfrak{a}$, if $\mathfrak{b} \supsetneq \mathfrak{a} \Rightarrow \mathfrak{b} \in S$, then $\mathfrak{a} \in S$

Then $S = \{$ideals of $A\}$.

Math Induction $S \subseteq \mathbb{N}$

TFAE (I) S has a min

(II) $(1 \in S,\ n \in S \Rightarrow n+1 \in S)$ $\Rightarrow S = \mathbb{N}$

char$(F) = 0$ $\Rightarrow \not\exists\, 1 \neq 0$ ...

char$(F) = \infty$

$p \supset \boxed{\cap}$

$p \supset \square$

~~$(\bar{0}) = \bar{r}_1 \wedge \bar{r}_2$~~

$(\bar{0}) = \cancel{\bar{r}_1 \wedge \bar{r}_2}$

$b_1 \wedge b_2$

p^{λ_1} p^{λ_2} p^{λ_3}

~~$t y^{n+1} \in \mathfrak{a}$~~

$\forall r$ $t \cdot y^{n+1} r \in \mathfrak{a}$

$(t \cdot y^n r) y$

$A \longrightarrow A/\mathfrak{p}$

Ass$(\mathfrak{a})$

$\mathrm{rad}(\mathfrak{a} : x)$, prime ($x \in A$)

$\mathrm{rad}(\mathfrak{a} : x)$ prime

or

$\exists j.$ $\forall x$ $x \in \mathfrak{q}_j$ ~~$x \notin \mathfrak{q}_i\ (\forall i \neq j)$~~

$x \notin \mathfrak{q}_i$

P4 (Bosch)

Def A a ring. A-Algebra: B (ring) w/ $f: A \to B$ homomorphism

E.X. Any ring B is a $\mathbb{Z}$ Algebra $\mathbb{Z}: z \in \mathbb{Z} \mapsto z \cdot 1_B$.

Def. A: field F. $\forall f: F \to B$ nonzero, injective $F \hookrightarrow B$. (If $F \xrightarrow{h} B$ is nonzero?) (0, 1) → Ker: ideal.

(*1) i.e. F-algebra B, $\cong$ B ring containing F as a field.

e.g. B: field of characteristic p (0/prime) ~~$A=0$~~ $(1+\cdots+1=0)$. for ring with (p: least). P no nontrivial zero divisors.

(*2) $\mathbb{Q} \to B$ ($z \in \mathbb{Z}$ ✓, $\frac{1}{z}$ ✓)

$p=0$, B: $\mathbb{Q}$-algebra

$p>0$: B is $\mathbb{F}_p$-algebra $=\{0,\dots,p-1\}$

Def. B: f.g. A-Algebra if $\exists x_1,\dots,x_n \in B$, s.t. $B = \langle f(A), x_1, \dots, x_n \rangle$

Theorem A: Noeth, B: f.g. Algebra, then B: Noeth. (*1)

Def. $A \xrightarrow{f} B$, $A \xrightarrow{g} C$, A-Algebras.
morphism of A-Algebras. ~~:=~~ $h: B \to C$. s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$$
commutes

i.e. $h(f(a)b) = g(a)h(b)$ $\underset{b=1}{\Rightarrow}$ $h(f(a)) = g(a)$

$= h(f(a))h(b)$ $\Leftarrow$

$? \langle f(A), x_1, \dots, x_n \rangle$

Def. B: f.g. A-Algebra IFF $\exists n \geq 0$, $\exists h: A[x_1,\dots,x_n] \to B$ surjective. morphism of A-Algebras.

(*3)

Pf (*1) $\Rightarrow$ $A[x_1,\dots,x_n]$ Noeth, h surjective. then B: homomorphic image of Noeth $\Rightarrow$ Noeth

~~$\Leftarrow$ B: A-algebra, B (finite)~~

$A \xrightarrow{f} B$ { $a \in A$, $b \in B$, $ab = f(a)b$; $\exists b_1,\dots,b_n$, s.t. $\forall b \in B$, $b = a_1b_1 + \cdots + a_nb_n = f(a_1)b_1 + \cdots + f(a_n)b_n$ }

Def B: A-algebra B (finite) if f.g. as A-mod

B: finite A-alg $\Leftrightarrow$ $\exists$ surjection of A-mod. $A^n \to B$. $\Leftrightarrow$ $\exists x_1,\dots,x_n \in B$, $B = \{$finite combination of $x_1,\dots,x_n$ in $A\}$

$\Downarrow$

B. f.g. A-alg $\Leftrightarrow$ $\exists$ surj ···· $A[x_1 \cdots x_n] \to B$. $\Leftrightarrow$ $\exists x_1,\dots,x_n$, $B = \{$poly $A[x_1,\dots,x_n]\}$

$A = \mathbb{Z}$

B	finite	f.g
$\mathbb{Z}$	✓	✓
$\frac{1}{2}\mathbb{Z}$	✓	N/A
$\mathbb{Z}[\frac{1}{2}]$	×	✓
$\mathbb{Q}$	×	×

(*1) $R \xrightarrow{h} S$: ring homomorphism. (char(S) | char(R))

(*2) with homomorphism maps $1_a \mapsto 1_B$, $\frac{1}{m_a} = \frac{1}{1_a + \cdots + 1_a} \mapsto \frac{1}{1_B + \cdots + 1_B}$

(*3) $A \neq \langle f(A), x_1, \dots, x_n \rangle$

Theorem. K : field, $K \subseteq L$, L : field.

(L : k-Algebra.)

L : f.g. K-algebra. $K[x_1, \cdots, x_n] \xrightarrow{\text{surject}} L$ $\quad$ L/K : v.s. $\dim < \infty$

Then L : finite K-algebra. i.e. L/K is a finite field extension.

Cor 1. maxi ideals of $A = \mathbb{C}[x_1, \cdots, x_n]$

$= \{ m_X = (x_1 - X_1, \cdots, x_d - X_d), X \in \mathbb{C}^d \}$. $\quad X = (X_1, \cdots, X_d)$

pf (Theor. $\Rightarrow$ Cor): $m \subseteq A$, max

$L = A/m$ field.

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook] \arrow[rr, bend right, "j \text{ (homomphs.)}"'] & \overset{A}{\mathbb{C}[x_1 \cdots x_n]} \arrow[r, "q"] & \overset{A/m}{L} \end{tikzcd}$$

Note: L is f.g. $\mathbb{C}$-alger, generated by $q(x_1) \cdots q(x_d)$.

$\Rightarrow L/j(\mathbb{C})$: finite field extnsn.

$L \cong \mathbb{C}$ ($\mathbb{C}$: alg closed)

$X = (j^{-1}(q(x_1)), \cdots, j^{-1}(q(x_d))) \in \mathbb{C}^d$.

Check $m = m_X$.

Cor 2. $d \geq 1$, $\mathbb{C}(x_1, \cdots, x_d)$ is not f.g $\mathbb{C}$-alge
$= \mathrm{Frac}(\mathbb{C}[x_1, \cdots, x_d])$

pf : $K = \mathbb{C}$, $L = \mathbb{C}(x_1, \cdots, x_d)$

L/k : not finite field extn $\Rightarrow L$: not f.g. K-alg.

Works if $\mathbb{C} = K$ (field)

pf (Theorem) $f_1, \cdots, f_n \in \cancel{\mathbb{C}}K(x_1, \cdots, x_d)$

$$f_1 = \frac{g_1}{h_1}, \quad \cdots, \quad f_n = \frac{g_n}{h_n}$$

$u := 1 + x_1 h_1 \cdots h_n$, $\Rightarrow u$: coprime to h_j ($\forall j$)

(Euclid's proof that $\mathbb{Z}$ has ∞ many primes.

$\Rightarrow \frac{1}{u} \notin K[f_1, \cdots, f_n]$ (u : coprime to. h_j)

Lec 6-2.

- A : Noetherian, $\mathfrak{a} \subseteq A$ (ideal) $\Rightarrow$ $\mathfrak{a}$: decomposible. (*1)
- $q \subseteq A$ ideal, primary **Def**: $\forall \bar{r} \in A/q$ zero divisor. $\Rightarrow$ $\bar{r}$ nilpotent.

  i.e. $\forall r, \exists s$ $(s \notin q)$ st. $rs \in q \Rightarrow \exists n$ st. $r^n \in q$

  * Prime $\Rightarrow$ Primary. **Prime**: $\forall \bar{r} \in A/q$, zero divisor $\Rightarrow \bar{r} = 0$.

  i.e. $\forall r, \exists s$ $(s \notin q)$ st $rs \in q \Rightarrow r \in q$

- Decomposible. $\exists\ \mathfrak{a} = q_1 \cap \cdots \cap q_n$. q_i : primary
- **Lem** Noetherian Induction : TFAE — Axiom, Theorem.

  (I) $S \subseteq \{\text{ideals of } A\}$, S has a maximal element

  (II) $S \subseteq \{\text{ideals of } A\}$, st. (i) $(1) \in S$, (ii) $\forall \mathfrak{a} \subseteq A$ (ideals), if $\mathfrak{b} \supsetneq \mathfrak{a}$, $\mathfrak{b} \in S$, then $\mathfrak{a} \in S$

  then $S = \{\text{ideals of } A\}$.

- (*1) Pf:

  Def 1). $\mathfrak{r} \subseteq A$ (ideal) irreducible **Def**: $\forall \mathfrak{r}', \mathfrak{r}''$ (ideal) s.t. $\mathfrak{r}' \cap \mathfrak{r}'' = \mathfrak{r}$, then $\mathfrak{r} = \mathfrak{r}'$ or $\mathfrak{r} = \mathfrak{r}''$

  Lem 2) A : Noetherian. Then ~~Primary~~ irreducible $\Rightarrow$ primary.

  Lem 3) A : Netherian. Then all ideals in A are finite. intersection of irreducible ideals.

  | e.g. $(2) \cap (3) = (6)$ — not irreducible
  | In $\mathbb{Z}$, $\langle p^\lambda \rangle$, $\lambda \geq 0$, p: prime number is irreducible.

  Pf Lem 2).

  ~~$\mathfrak{a}$: irreducible~~

  ~~Lem 4) $(\mathfrak{a} : y^n) = (\mathfrak{a} : y^{n+1}) \Rightarrow (\mathfrak{a} : y) \cap (y^{n+1}) \subseteq \mathfrak{a}$. $\mathfrak{a}$: ideal, $y \in A$~~

  ~~Pf. $\mathfrak{a}$: irreducible~~

  Lem 4). $(\mathfrak{a} : y^n) = (\mathfrak{a} : y^{n+1}) \Rightarrow (\mathfrak{a} : y) \cap (y^n) \subseteq \mathfrak{a}$. ($\mathfrak{a}$: ideal, $y \in A$)

  Pf: let $z \in (\mathfrak{a} : y) \cap (y^n)$, $z = y^n \cdot r_0$, $\exists r_0 \in A$. Want: $z \in \mathfrak{a}$

  $\forall r \in A$, $z(yr) = y^{n+1} r r_0 \in \mathfrak{a}$. / $z \cdot (y) = r_0 (y^{n+1}) \subseteq \mathfrak{a}$

  Know. $\forall t_1$ s.t. $t_1 (y^n) \subseteq \mathfrak{a} \Rightarrow t_1 (y^{n+1}) \subseteq \mathfrak{a}$

  $\forall t_2$ s.t. $t_2 (y^{n+1}) \subseteq \mathfrak{a} \Rightarrow t_2 (y^n) \subseteq \mathfrak{a}$

  i.e. $r_0 \in (\mathfrak{a} : y^{n+1}) = (\mathfrak{a} : y^n)$ (*2)

  hence $r_0 (y^n) \subseteq \mathfrak{a}$, $z = y^n \cdot r_0 \in \mathfrak{a}$.

  Notice: by (*2) we only need $(\mathfrak{a} : y^{n+1}) \subseteq (\mathfrak{a} : y^n)$

  Thus the Lem 4) can be changed as $(\mathfrak{a} : y^n) \supseteq (\mathfrak{a} : y^{n+1})$, but $(\mathfrak{a} : y^n) \subseteq (\mathfrak{a} : y^{n+1})$ always, thus the condition is not superfluous.

  (CTD). let p : irreducible. now let $s \notin p$. $ys \in p$ Want $\exists n$, $y^n \in p$.

  Now A/p, hence $\bar{s} \neq \bar{0}$, $\bar{y}\bar{s} = \bar{0}$, Want $\exists n$, $\bar{y}^n = \bar{0}$

  $(\mathrm{Ann}(\bar{y}^n))_{n=1}^{\infty}$ is a increasing chain of ideals in A/p, since A : Noetherian, A/p : Noetherian

  $\exists n_0$ st. $\mathrm{Ann}(\bar{y}^{n_0}) = \mathrm{Ann}(\bar{y}^{n_0+1}) \cdots$

  by Lem 4) $\mathrm{Ann}(\bar{y}) \cap (\bar{y}^{n_0}) \subseteq (\bar{0})$, hence $\mathrm{Ann}(\bar{y}) \cap (\bar{y}^{n_0}) = (\bar{0})$

  p : irreducible in A $\Rightarrow$ $(\bar{0})$ irreducible in A/p (check!)

hence either $\mathrm{Ann}(\bar y) = (\bar 0)$ or $(\bar y^{n}) = (\bar 0)$

$\Downarrow$ $\bar s = \bar 0$ (contra to assmp) $\Downarrow$ Want ✓ □

Pf Lem 3). S^c = {ideals in A : not finite intersection of irreducible ideals}

Want $S^c = \Phi$.

If not, $\exists\, \mathfrak{a}$: maximal element in S^c. (M.A.X condition of Noetherian)

$\mathfrak{a} \neq (1)$. $\mathfrak{a}$: not irreducible, then $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$, $\mathfrak{b}, \mathfrak{c}$: ideals, $\mathfrak{b} \supsetneq \mathfrak{a}$, $\mathfrak{c} \supsetneq \mathfrak{a}$

(1) : irreducible.

~~[for example, we can take $\mathfrak{b}$: maximal ideal~~

[Note: any ideal can be write as intersection of two larger ideals, i.e. such "decomposition" is not empty. Since $\mathfrak{a}$ is irreducible, then such $\mathfrak{b}, \mathfrak{c}$ must exist]

Since $\mathfrak{b}, \mathfrak{c} \supsetneq \mathfrak{a}$, $\mathfrak{b}, \mathfrak{c} \in S$, then $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ will be in S. Contra. □

(CTD *1). Directly from Lem 2), 3).

• Lem 1). A (ring). $\mathfrak{q}$: primary. $\mathfrak{p} = \mathrm{rad}(\mathfrak{q})$.

① $\mathfrak{p}$: prime ② least prime ideal containing $\mathfrak{q}$.

Pf: ① let $xy \in \mathfrak{p}$ $(x \notin \mathfrak{p})$. $\exists n$, s.t. $(xy)^n = x^n y^n \in \mathfrak{q}$ $(x^n \notin \mathfrak{q})$. then $\exists m$, s.t. $(y^n)^m \in \mathfrak{q}$, $\Rightarrow y \in \mathfrak{p}$

② Recall. r : ideal, $\mathrm{rad}(r) = \bigcap_{i\in I} P_i$, P_i : prime ideal $\supseteq r$

• Def 2). $\mathfrak{q}$: primary. $\mathfrak{p} = \mathrm{rad}(\mathfrak{q})$. Call $\mathfrak{q}$: $\mathfrak{p}$-primary.

• Lem 3). $\mathfrak{q}_1, \cdots, \mathfrak{q}_n$: $\mathfrak{p}$-primary (i.e. primary with the same radical $\mathfrak{p}$).

then $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$: $\mathfrak{p}$-primary.

Pf: ① primary. let $\mathfrak{q} = \bigcap_{i=1}^{n} \mathfrak{q}_i$, let $xy \in \mathfrak{q}$, want $y \in \mathfrak{q}$. $(x \notin \mathfrak{p}, x^m \notin \mathfrak{q}, \forall m)$

$xy \in \mathfrak{q}_i\ (\forall i)$, $x \notin \mathfrak{p}$, then $\forall i$, $\forall m$, $x^m \notin \mathfrak{q}_i$, thus $y \in \mathfrak{q}_i$, $\forall i$.

$y \in \mathfrak{q}$. ✓.

② $\mathrm{Rad}(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n) = \mathrm{Rad}(\mathfrak{q}_1) \cap \cdots \cap \mathrm{Rad}(\mathfrak{q}_n) = \mathfrak{p}$

Definition of Primary.

$\forall xy \in \mathfrak{q}$, $y \in \mathfrak{q}$, ⋯

$y \notin \mathfrak{q}$, $x^m \in \mathfrak{q}$

$\forall xy \in \mathfrak{q}$ $x^m \in \mathfrak{q}$

$(\forall m, x^m \notin \mathfrak{q}) \Rightarrow y$

• Rmk : $\mathfrak{p}$: prime. ① $\mathfrak{p}$-primary ideals need not be a power of $\mathfrak{p}$.

② A power of $\mathfrak{p}$ need not be primary.

• Lem 4) $\mathfrak{q} \subseteq A$ (ideal), $\mathrm{rad}(\mathfrak{q}) = \mathfrak{m}$, $\mathfrak{m}$: maximal. $\Rightarrow$ $\mathfrak{q}$: primary.

Pf: $\mathfrak{m}/\mathfrak{q}$ be the ideal in $A/\mathfrak{q}$ formed by mapping $\mathfrak{m} \mapsto \mathfrak{m}/\mathfrak{q}$.

then $\mathrm{Nil}(A/\mathfrak{q}) = \mathfrak{m}/\mathfrak{q}$, since $\mathfrak{m}$: maximal, $\mathfrak{m}/\mathfrak{q}$: maximal.

$\mathrm{Nil}(A/\mathfrak{q})$: intersection of all ~~maximal~~ prime ideals in $A/\mathfrak{q}$.

Thus any maximal ideal will contain $\mathrm{Nil}(A/\mathfrak{q}) = \mathfrak{m}/\mathfrak{q}$.

Thus $\mathfrak{m}/\mathfrak{q}$: only maximal ideal in $A/\mathfrak{q}$. i.e. $(A/\mathfrak{q}, \mathfrak{m}/\mathfrak{q})$ local ring.

Thus ~~either~~ let $\bar t \in A/\mathfrak{q}$, either $\bar t \in \mathfrak{m}/\mathfrak{q}$, i.e. $\exists n$ s.t. $\bar t^{n} = 0/\mathfrak{q}$ $(t^n \in \mathfrak{q})$

or $\bar t \notin \mathfrak{m}/\mathfrak{q}$, then $\bar t$: unit (not a zero divisor in this nonzero ring).

Hence. $\mathfrak{q}$: primary.

• Lem 5). by 4). If $\mathfrak{m}$: maximal in A, then $\mathfrak{m}^n$: $\mathfrak{m}$-primary.

Pf: $\mathrm{rad}(\mathfrak{m}^n) = \bigcap_{i=1}^{n} \mathrm{rad}(\mathfrak{m})^n = \mathrm{rad}(\mathfrak{m}) = \mathfrak{m}$

Lec 6-2 ((TD1)

- Examples for Lem 4), 5).

① $k[x,y]$, $m=(x,y)$, maximal, m^n: primary

② $q=(x^2,y) \subseteq k[x,y]$ m-primary. ?

③ $\mathbb{C}[x]$. $\mathfrak{a}=\left(\prod_{j=1}^{J}(x-z_j)^{n_j}\right)$, $(z_1,\dots,z_J)$ distinct in $\mathbb{C}$

$\mathfrak{a}=q_1\cap\cdots\cap q_J$, $q_j=((x-z_j)^{n_j})$, $p_j=\mathrm{rad}(q_j)=(x-z_j)$

④ $q_1=((x,y))^2=(x^2,xy,y^2)\subseteq k[x,y]$. $p_1=\mathrm{rad}(q_1)=(x,y)$

$q_2=(y)$, $p_2=\mathrm{rad}(q_2)=(y)$. $\mathfrak{a}=q_1\cap q_2=(xy,y^2)$

- Minimal Primary Decomposition. $\mathfrak{a}=q_1\cap\cdots\cap q_n$, $p_j=\mathrm{rad}(q_j)$

(a) If $p_i=p_j$, $i\neq j$. then replace q_i with $q_i\cap q_j$ and delete q_j (notice, $q_i\cap q_j$ are also primary)

(b) If $q_j\supseteq\left(\bigcap_{i\neq j}q_i\right)$, then delete q_j

A primary decomposition s.t. $p_i\neq p_j$ and $\forall k$, $q_k\not\supseteq\left(\bigcap_{i\neq k}q_i\right)$ is minimal.

- $\mathrm{Ass}(\mathfrak{a})$, $\mathfrak{a}$: ideal. $\mathrm{Ass}(\mathfrak{a})$ associated ideals of $\mathfrak{a}$ Def $\mathrm{Ass}(\mathfrak{a})=\{$prime ideals of the form $\mathrm{rad}(\mathfrak{a}:x)$, $x\in A\}$.

note: $\forall r\in\mathfrak{a}$ $r\in(\mathfrak{a}:x)\subseteq\mathrm{rad}(\mathfrak{a}:x)$
i.e. $\mathfrak{a}\subseteq\mathrm{rad}(\mathfrak{a}:x)$

- Theorem. 1). $\mathfrak{a}=q_1\cap\cdots\cap q_n$ minimal primary decomposition.

then $\mathrm{Ass}(\mathfrak{a})=\{p_1,\dots,p_n\}$, $p_i=\mathrm{rad}(q_i)$

Thus though minimal primary decomposition may not be unique, the radicals of the "divisors" are unique.

Pf: Lem 2) q: p-primary, $x\in A$. then.

① $x\in q\Rightarrow(q:x)=(1)$ ② $x\notin q\Rightarrow(q:x)$: p-primary

③ $x\notin p\Rightarrow(q:x)=q$

Lem 2) $\Rightarrow$ Theorem 1).

Pf: $\mathrm{rad}(\mathfrak{a}:x)_{x\in A}=\mathrm{rad}\left(\left(\bigcap_{j=1}^{n}q_j\right):x\right)=\bigcap_{j=1}^{n}(\mathrm{rad}(q_j:x))=\bigcap_{x\notin q_j}p_j$

① let $p=\mathrm{rad}(\mathfrak{a}:x)$: prime. (thus $p\neq(1)$) then $\exists j$, s.t. $p\supset p_j$. also clearly $p_j\supset p$.

thus $\exists j$ s.t. $p=p_j$. Hence $\mathrm{Ass}(\mathfrak{a})\subseteq\{p_1,\dots,p_n\}$.
$\in$ ← distinct by minimal decomposition

② Also, by minimal decomposition, $\forall j$, $\exists x\in A$, s.t. $x\notin q_j$, $x\in q_i(i\neq j)$

(Otherwise, $\exists j$, $\forall x$, s.t. $x\in q_j$ or $x\notin q_i$ ($\exists i\neq j$)
then if $x\in\bigcap_{i\neq j}q_i$, $x\in q_j$. contra)

Thus, $\forall j$, $\exists x\in A$, s.t. $\mathrm{rad}(\mathfrak{a}:x)=p_j$.

Thus $\mathrm{Ass}(\mathfrak{a})=\{p_1,\dots,p_n\}$. □

Pf of Lem 2). (See Next Note)

Lec 7-1 ✓

Lem 2). $\mathfrak{q}$: $\mathfrak{p}$-primary. $x \in A$ then.

(i) $x \in \mathfrak{q} \Rightarrow (\mathfrak{q}:x) = (1)$ (ii) $x \notin \mathfrak{q}$, $\Rightarrow (\mathfrak{q}:x)$ is $\mathfrak{p}$-primary ideal, i.e. $\mathrm{rad}(\mathfrak{q}:x) = \mathfrak{p}$

(iii) $x \notin \mathfrak{p} \Rightarrow (\mathfrak{q}:x) = \mathfrak{q}$.

Check: (ii) primary, let $yz \in (\mathfrak{q}:x)$, if $y \notin (\mathfrak{q}:x)$, then (Want) $\exists n$ s.t. $z^n \in (\mathfrak{q}:x)$

$xyz \in \mathfrak{q}$, $xy \notin \mathfrak{q}$, by $\mathfrak{q}$ primary, $\exists n$ s.t. $z^n \in \mathfrak{q}$. thus $(\mathfrak{q}:x)$ primary.

Want $r(\mathfrak{q}:x) = \mathfrak{p}$. let $y^n \in (\mathfrak{q}:x)$ want $y \in \mathfrak{p}$. (i.e. $r(\mathfrak{q}:x) \subseteq \mathfrak{p}$).

$xy^n \in \mathfrak{q}$, $x \notin \mathfrak{q} \Rightarrow y \in r(\mathfrak{q}) = \mathfrak{p}$. $((\mathfrak{q}:x) \supseteq \mathfrak{q}, r(\mathfrak{q}:x) \supseteq \mathfrak{p})$ ($r(\mathfrak{q}:x) \supseteq \mathfrak{p}$ clearly)

(iii). "$\supseteq$" clearly. ~~$((\mathfrak{q}:x) \supseteq \mathfrak{q}, r(\mathfrak{q}:x) \supseteq r(\mathfrak{q}) = \mathfrak{p})$~~

"$\subseteq$" suppose $y \in (\mathfrak{q}:x)$, i.e. $xy \in \mathfrak{q}$, also $x \notin \mathfrak{p}$. thus $\forall n$, $x^n \notin \mathfrak{q}$.

by def of primary of $\mathfrak{q}$, $y \in \mathfrak{q}$. □

Lem 5)

- A: Noether, $\mathfrak{q}$: $\mathfrak{p}$-primary $\Rightarrow \exists x \in A$ s.t. $(\mathfrak{q}:x) = \mathrm{rad}(\mathfrak{q})$. $(x \notin \mathfrak{q})$ ~~$(\mathfrak{q}:x)=(1) \Rightarrow Ax = (x)$~~ □

pf: $\mathfrak{p}$, f.g ideal. $\forall x \in \mathfrak{p}$, thus $\exists n_0$ s.t. $x^{n_0} \in \mathfrak{q}$. thus some $\mathfrak{p}^m \subseteq \mathfrak{q}$ (take the generators of $\mathfrak{p}$) (See *1 foot)

let n be such minimal $\mathfrak{p}^n \subseteq \mathfrak{q}$ $(n \geq 1)$, then $\mathfrak{p}^{n-1} \not\subseteq \mathfrak{q}$, i.e. $\exists x \in \mathfrak{p}^{n-1}$, $x \notin \mathfrak{q}$ $(n=1, \mathfrak{q}=\mathfrak{p}, (\mathfrak{q}:x) = \mathfrak{q} = \mathfrak{p}\ \forall x \notin \mathfrak{p})$

Then $(\mathfrak{q}:x) = \mathfrak{p}$. { "$\subseteq$" by $(\mathfrak{q}:x)$ $\mathfrak{p}$-primary
"$\supseteq$" if $y \in \mathfrak{p}$, then $xy \in \mathfrak{p}^n \subseteq \mathfrak{q} \Rightarrow y \in (\mathfrak{q}:x)$ □ $(x \in \mathfrak{p}^{n-1}, y \in \mathfrak{p})$

- Example. k: field, $A = k[t]$. $\mathfrak{q} = (t^N)$ $(N \geq 1)$. $\mathfrak{p} = (t)$

Take $x \in A$, let $x = ct^n + c't^{n+1} + \cdots$ $(c \neq 0, n \geq 0)$, call $n = \mathrm{ord}_t(x)$ (least order in x)

~~$(\mathfrak{q}:x) = \mathrm{rad}(\mathfrak{q})$~~

$(\mathfrak{q}:x) = (t^m)$ where $m = \max(N-n, 0)$ $\left[\frac{x}{1} \in A_{\mathfrak{p}}, \left(\frac{x}{1}\right) = \left(\frac{t^n}{1}\right) \right.$

Check

$x \in \mathfrak{q} \Leftrightarrow n \geq N \Leftrightarrow m = 0 \Leftrightarrow (\mathfrak{q}:x) = (1)$

$x \notin \mathfrak{q} \Leftrightarrow m \geq 1 \Rightarrow (\mathfrak{q}:x)$ $\mathfrak{p}$-primary. (prime $\Rightarrow$ primary, $\mathrm{rad}(t^m)_{m \geq 1} = (t)$)

i.e. $x \in t^{N-1}$, $x \notin (t^N) \Rightarrow (\mathfrak{q}:x) = \mathfrak{p} = (t)$

$x \notin \mathfrak{p} \Leftrightarrow n=0 \Leftrightarrow m = N \Rightarrow (\mathfrak{q}:x) = \mathfrak{q}$

- Given / M.P.D, $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$. $x \in A$. $(\mathfrak{a}:x) = \bigcap_j (\mathfrak{q}_j : x)$
min prim decomp.
$\mathrm{rad}(\mathfrak{a}:x) = \bigcap_j \mathrm{rad}(\mathfrak{q}_j : x) = \bigcap_{j: \mathfrak{q}_j \not\ni x} \mathrm{rad}(\mathfrak{q}_j : x)$
By M.P.D, we may find. ~~$x \notin \mathfrak{q}_i$~~ $\longrightarrow x \notin \mathfrak{q}_i$, $x \in \bigcap_{i \neq j} \mathfrak{q}_j$, $\forall i$
~~$(\mathfrak{q}_i \not\supseteq \bigcap_{i \neq j} \mathfrak{q}_j)$~~ $\longrightarrow (\mathfrak{q}_i \not\supseteq \bigcap_{i \neq j} \mathfrak{q}_j)$, $\forall i$
$\mathrm{rad}(\mathfrak{a}:x) = \mathfrak{p}_i$ [by Lem 2)] $\Rightarrow \mathfrak{p}_i \in \mathrm{Ass}(\mathfrak{a})$
Conversely, if $\mathfrak{p}$: prime, $\mathfrak{p} = \mathrm{rad}(\mathfrak{a}:x)$, $\exists x$.
then $\mathfrak{p} = \bigcap_{j: \mathfrak{q}_j \not\ni x} \mathfrak{p}_j$, then $\mathrm{Ass}(\mathfrak{a}) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, $\mathfrak{p} = \mathfrak{p}_j$ for some j. □

This is showing Lem 2) $\Rightarrow$ Thoem. See the previous Notes.

- Moreover, if A: Noether, we may find for each i, an element x_i with $(\mathfrak{a}:x_i) = \mathfrak{p}_i$.

by applying the final part of the last lema. □

- Example. $\{z_1, \ldots, z_n\} \subseteq k$, k: field, $\mathfrak{a} = \cap \mathfrak{q}_j$, $\mathfrak{q}_j = \left((t - z_j)^{N_j}\right)$, $A = k[t]$

(*) Use the example to test the Lemmas above.

$r(\mathfrak{a}) = \bigcap_j \mathfrak{p}_j$ (prime radicals) if $\mathfrak{a} = \bigcap_j \mathfrak{q}_j$ with M.P.D.

(*1 foot) $\mathfrak{p}$: gen by $(x_1, \ldots, x_n)$ s.t. $x_i^{k_i} \in \mathfrak{q}$, let $k = \sum_{i=1}^n k_i$, then $\mathfrak{p}^k$ = gen by $\{(x_1^{\lambda_1} \cdots x_n^{\lambda_n}) \mid \sum \lambda_i = k\} = S$

because let $r = \sum_{i = \mathrm{fin}} a^i r_1^i \cdots r_k^i \in \mathfrak{p}$, $(a^i \in A, r_j^i \in \mathfrak{p})$ then $r = \sum_{i = \mathrm{fin}} a^i (b_1 x_1 + \cdots + b_n x_n) \times \cdots \times (d_1 x_1 + \cdots + d_n x_n)$; and $S \subseteq \mathfrak{q}$, thus $\mathfrak{p}^k \subseteq \mathfrak{q}$

Lec 7-1.

- zero-divisor mod $\mathfrak{a}$. $Z(\mathfrak{a}) = \{x \in A,\ \exists y \in A - \mathfrak{a},$ s.t. $xy \in \mathfrak{a}\}$ $(A/\mathfrak{a})$ $= \bigcup_{y \in A-\mathfrak{a}} (\mathfrak{a}:y)$ (?) $\bigcup_{y\in A-\mathfrak{a}} \mathrm{rad}(\mathfrak{a}:y)$

- Prop. $\mathfrak{a} = \bigcap_j \mathfrak{q}_j$, M.P.D. then $Z(\mathfrak{a}) = \bigcup_j \mathfrak{p}_j$

  Pf (?) If $x^n \in (\mathfrak{a}:y)$, $\exists y \in A-\mathfrak{a}$, then $x^n y \in \mathfrak{a}$, then let n be minimal.
  then $x \cdot \underbrace{x^{n-1}y}_{\notin \mathfrak{a}} \in \mathfrak{a}$, $x \in (\mathfrak{a} : \underbrace{x^{n-1}y}_{\notin\mathfrak{a}})$ $\Rightarrow$ (?) i.e. ~~$x \in \mathrm{rad}(\mathfrak{a}:x^{n-1}y)$~~ ~~$x \in \mathrm{rad}(\mathfrak{a}:y)$~~ $x \in (\mathfrak{a}:x^{n-1}y)$

Pf Prop : $\mathrm{rad}(\mathfrak{a}:y) = \bigcap_{j:\ \mathfrak{q}_j \not\ni y} \mathfrak{p}_j$, ① $Z(\mathfrak{a}) \subseteq \bigcup \mathfrak{p}_j$

let $x \in Z(\mathfrak{a})$ want $x \in \mathfrak{p}_j$, $\exists j$.

$\Updownarrow$

$\bigcup_{y \in A-\mathfrak{a}} \mathrm{rad}(\mathfrak{a}:y) = \bigcup \bigcap_{y \notin \mathfrak{q}_j} \mathfrak{p}_j$ $\Leftrightarrow$ $(\mathfrak{a}:x) \neq \mathfrak{a}$. $(\mathfrak{a}:x) = \bigcap_j \underbrace{(\mathfrak{q}_j : x)}_{x \notin \mathfrak{q}_j,\ \mathfrak{p}_j\text{-primary}}$

If $x \notin \mathfrak{p}_j$, $\forall j$.

then $(\mathfrak{q}_j : x) = \mathfrak{q}_j$, $\forall j$.

$(\mathfrak{a}:x) = \bigcap_j \mathfrak{q}_j = \mathfrak{a}$, contra to $x \in Z(\mathfrak{a})$

② If $x \notin Z(\mathfrak{a})$, want $x \notin \mathfrak{p}_j$, $\forall j$.
show $\bigcup \mathfrak{p}_j \subseteq Z(\mathfrak{a})$

$\forall j$, $\exists y_j$, s.t. $\mathrm{rad}(\mathfrak{a}:y_j) = \mathfrak{p}_j$.

Necessarily $y \notin \mathfrak{a}$, so if $x \in \mathfrak{p}_j$, then $x \in \mathrm{rad}(\mathfrak{a}:y) \subseteq Z(\mathfrak{a})$ □

Example (*) $0 \neq x \in A \rightarrow n_j := \mathrm{ord}_{t-z_j}(x) :=$

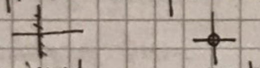
then $x \in Z(\mathfrak{a}) \Leftrightarrow \exists j,\ n_j \geq 1$.

$x \in \mathrm{rad}(\mathfrak{a}) \Leftrightarrow \forall j,\ n_j \geq 1$.

NB. $n_j :=$ order of vanishing of x at z_j, $n_j \geq c \Leftrightarrow$ first c taylor coeff of x at z_j all vanish

- Def $\mathrm{Ass}(\mathfrak{a}) \ni \mathfrak{p}$, is either minimal/isolated OR embeded.
  ① $\mathfrak{p}$: minimal element of $\mathrm{Ass}(\mathfrak{a})$ ② else. (then $V(\mathfrak{p})$ will be embeded in some $V(\mathfrak{p}')$) ?
  strict?
  yes, if not strict. $\mathfrak{p}_1 \supseteq \mathfrak{p}_2$, $I(V(\mathfrak{p}_1)) = \mathfrak{p}_1 = I(V(\mathfrak{p}_2)) = \mathfrak{p}_2$ contradiction.

ex $\mathfrak{p}_1 = (x)$, $\mathfrak{p}_2 = (x,y)$. $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2^2 = (xy, x^2)$



minimal embeded.

- Theorem. In any MPD, $\mathfrak{a} = \bigcap \mathfrak{q}_j$, $\{\mathfrak{q}_j : \mathfrak{p}_j \text{ minimal}\} = S$. S depends only on $\mathfrak{a}$.
  more precisely, if $\mathfrak{p}_j$ minimal, then $\mathfrak{q}_j = \tau^* \tau_*(\mathfrak{a})$ with $\tau: A \to A_{\mathfrak{p}_j}$

- $S \subseteq A$, multiplic closed. recall: $\mathfrak{p}$ = prime, $\mathfrak{p} \cap S \neq \emptyset$, then $\tau_*(\mathfrak{p}) = (1)$
  $\mathfrak{p} \cap S = \emptyset$, then $\tau^* \tau_*(\mathfrak{p}) = \mathfrak{p}$
  $\tau_*(\mathfrak{p})$: prime.

  - $\tau^*\tau_*(\mathfrak{a}) = \bigcup_{s \in S} (\mathfrak{a}:s)$

  $x \in \tau^*\tau_*(\mathfrak{a}) \Leftrightarrow \frac{x}{1} \in \tau_*\mathfrak{a} = \{\frac{y}{s} : y \in \mathfrak{a}, s \in S\}$, ① suppose $\frac{x}{1} = \frac{y}{s}$, $\exists y \in \mathfrak{a}, s \in S$.
  then $t(xs - y) = 0$, $\exists t \in S$, $\Rightarrow stx = yt \in \mathfrak{a}$, $\Rightarrow x \in (\mathfrak{a} : \underbrace{st}_{\in S})$ □
  ② $x \in (\mathfrak{a}:s)$ $\exists s \in S$, then $xs =: y \in \mathfrak{a}$. then $\frac{x}{1} = \frac{y}{s} \in \tau^*\tau_*\mathfrak{a}$.

Lem $\mathfrak{q} \subseteq A$: primary, $S \subseteq A$ multiplic closed. $\mathfrak{p} = \mathrm{rad}(\mathfrak{q})$.
$\mathfrak{p} \cap S \neq \emptyset \Rightarrow \tau_* \mathfrak{q} = (1)$ | $\mathfrak{p} \cap S = \emptyset$, then $\tau_*\mathfrak{q}$: $\tau_*\mathfrak{p}$-primary.
$\tau^*\tau_*\mathfrak{q} = \mathfrak{q}$.

Note: $S \cap \mathfrak{p} = \emptyset \Leftrightarrow S \cap \mathfrak{q} = \emptyset$ $\{$ $\Rightarrow$ trivial
$\Leftarrow$ if $S \cap \mathfrak{p} \neq \emptyset$, $t \in \mathfrak{p} \cap S$. $t^m \in \mathfrak{q} \cap S$. thus $\mathfrak{q} \cap S \neq \emptyset$.

Lec 7-2 (PLD)

Pf: ① let $\mathfrak{p}\cap S \neq \Phi$, $s \in \mathfrak{p}\cap S$, $\tau_*(q) = \{\frac{x}{s} : x\in q, s\in S\}$.
Want $1 \in \tau_*(q)$. i.e. $\exists s'$ st. $s's = xs$. So: does the job. let $x = s_0^n$ (s.th $s_0^n \in q$).

② Suppose $\mathfrak{p}\cap S = \Phi$. the $\mathrm{rad}(\tau_* q) = \tau^*(\mathrm{rad} q) = \tau_* \mathfrak{p}$
let $\frac{x}{s}, \frac{y}{t} \in S^{-1}A$, suppose $(\frac{x}{s})\frac{y}{t} \in \tau_* q$, $\frac{y}{t} \notin \tau_* q$.
Want some $(\frac{x}{s})^n \in \tau_* q$.

~~③ ...~~ Note, we may assume $q = (0)$, since localization is exact and commute with taking quotients, i.e. $S^{-1}(A/q) = S^{-1}A/\tau_*(q)$
Want (i) τ: injective (ii) $\tau_*(0) = (0)_{S^{-1}A}$ is primary.
now let $S\cap q = \Phi$, $q = (0)$, i.e. $0 \notin S$.
(i) τ: injective, thus $\tau^*(0) = \underbrace{\tau^* \tau_*(0) = 0}_{\text{injective}}$. thus $\tau^* \tau_*(q) = q$
(ii)

Pf (i): S contains no (nonzero) zero divisors. $\Leftrightarrow$ τ injective
Note: $0 \subseteq A$ primary $\underset{\text{def}}{\Leftrightarrow}$ zero-divisors $\subseteq$ nilpotents in A.
thus $\underset{(0)\text{ is primary (here } (0)=q)}{\Leftrightarrow}$ S contains no nilpotents. $\Leftrightarrow$ $\underset{(S\cap \mathrm{rad}(0)=\phi)}{S\cap\mathfrak{p} = \phi}$ $\Leftrightarrow$ $\underset{(S\cap(0)=\phi)}{S\cap q = \Phi}$ $\Leftrightarrow$ $0\notin S$

Pf (ii). Check $\{\text{zero divisors in } S^{-1}A\} = \{\frac{x}{s} : x\in A, s\in S, \text{ st. } \exists\, y/t \in S^{-1}A, \text{ nonzero, st. } \frac{x}{s}\frac{y}{t} = \frac{0}{1}\}$
$= \{\frac{x}{s} : x\in A, s\in S, \text{ st } \exists y \in A, yt\neq 0, \forall t\in S, \exists u\in S \text{ with } xyu = 0\}$.
$\subseteq \{\frac{x}{s} : x\in A, \text{zero divisor}, s\in S\} \underset{\text{primary}}{=} \{\frac{x}{s} : s\in S, x: \text{nilpotent in } A\}$.
$= \mathrm{Nil}(S^{-1}A)$ (because localization commutes with radicals.
Thus $\{\text{zero divisors in } S^{-1}A\} \subseteq \mathrm{Nil}(S^{-1}A)$, $(0)_{S^{-1}A}$: primary. □ Lem

- Theorem $A \supseteq \mathfrak{a} = \cap q_j$, MPD. if $\mathfrak{p}_j \in \mathrm{Ass}'(\mathfrak{a}) = \{\text{minimal elements of Ass}(\mathfrak{a})\}$
if $\mathfrak{p}_j \in \mathrm{Ass}'(\mathfrak{a})$, then $q_j = \tau^* \tau_*(\mathfrak{a})$ $= \{\text{isolated elements of Ass}(\mathfrak{a})\}$
where $\tau : A \to A_{\mathfrak{p}_j}$.

Thus Def. call q_j the $\mathfrak{p}_j$-primary component of $\mathfrak{a}$.

- Lem: $\mathfrak{a}$, decomposible, then $\mathrm{Ass}'(\mathfrak{a}) = \{\text{minimal primes containing } \mathfrak{a}\} = \{\mathfrak{p} : \text{prime}, \mathfrak{p} \supseteq \mathfrak{a}, \text{ if } \mathfrak{p} \supseteq \underset{(\text{prime})}{\mathfrak{p}'} \supseteq \mathfrak{a} \text{ then } \mathfrak{p}' = \mathfrak{p}\}$.

Pf Lem: If $\mathfrak{p}$ (prime) $\supseteq \mathfrak{a}$, then $\exists j$, st. $\mathfrak{p} \supseteq \mathfrak{p}_j$ because
note $\underset{\mathfrak{p}}{r(\mathfrak{p})} \supseteq r(\mathfrak{a}) = \bigcap_{j=1}^{n} \mathfrak{p}_j$, by a theorem, $\mathfrak{p} \supseteq \mathfrak{p}_j$.
Hence if $\mathfrak{p} \in \mathrm{Ass}'(\mathfrak{a})$, and if $\mathfrak{p}' \subseteq \mathfrak{p}$, $\mathfrak{p}' \supseteq \mathfrak{a}$,

CA Lec 7-2 (CTD)

<u>Pf Theorem</u>: $\tau_j^* \tau_{j*}(\mathfrak{a}) = \underbrace{\tau_j^* \tau_{j*}\left(\bigcap_{i=1}^{n} q_i\right)}_{(*1)} = \bigcap_{i=1}^{n} \tau_j^* \tau_{j*}(q_i)$ (*1) finite intersections commute with τ_*, τ^*.

$S = A - p_j$, $S \cap q_j = \Phi$. $\tau_j : A \to A_{p_j}$

$i=j$. $\begin{cases} q_j, & i=j, \text{(general fact)} \\ (1), & i \neq j, \text{(this requires the minimal property, i.e. } S \cap q_i \neq \Phi \end{cases}$

$\Updownarrow$

If $x \in p_i$, $\notin p_j$, then $x^n \notin q_j$, $x^n \notin p_j$. $\Rightarrow$ $p_i \nsubseteq p_j$ $\Rightarrow$ $q_i \nsubseteq p_j$

$\Uparrow$ by minimal

<u>Def</u>. A-mod M <u>Artin / Artinian</u> IFF either of the follwg true.

(i). DCC, $M \geq M_1 \geq M_2 \geq \cdots \geq M_n \geq \cdots$ then $\exists n_0$ s.t. $M_n = M_{n_0}$, $\forall n \geq n_0$

(ii) MIN. every nonempty collection of submodules has a minimal element.

pf: (i) $\Leftrightarrow$ (ii) Same as Noetherian.

<u>Def</u>. A: Artin as a ring IFF

A: Artin as an A-mod.

Also A: Artin IFF

A: ~~DD~~ DCC on ideals

OR MIN on ideals.

<u>Lem</u> $0 \to M' \to M \to M'' \to 0$ SES

M: Artin IFF M', M'': Artin.

<u>Cor</u> Any finitely generated module over an Artin ring is Artin.

~~Cor~~

Cor: A: Artin $\Rightarrow$ $A/\mathfrak{a}$: Artin, $\forall \mathfrak{a}$ ideal

EX: • $\mathbb{Z}$: not Artin. $(2^2) \supsetneq (2^3) \cdots$

• Any finite ring : Artin + Noetherian e.g. $\mathbb{Z}/n\mathbb{Z}$.

• Any finite product of Artin rings is Artin.

k: ~~finite~~ field. $m = (x_1, \cdots, x_n) \leq k[x_1, \cdots x_n] = A$

then $\underline{A/m^l}$: Artin, $\forall l \geq 0$

fin dim Vector Space.

• $k[x]/[x^l]$ is Artin, $\forall l \geq 0$

• $k[x^2, x^3]/[x^6]$

• $k[x]$ not Artin $(x) \supsetneq (x^2) \supsetneq \cdots$

Lem: A: Artin, then every prime is maximal

also, A has only finitely many prime ideals

then $Jac(A) = Nil(A)$

Pf: p: prime, $B = A/p$, Artin.

① B: integral domain, ~~Want~~

Want: B field.

Let $0 \neq x \in B$, Want: $x \in B^\times$

let $(x) \geq (x^2) \geq (x^3) \geq \cdots$

then $\exists n$, s.t. $(x^n) = (x^{n+1})$.

i.e. $x^n = ux^{n+1}$, $u \in B$,

then $(ux-1)x^n = 0$, thus $ux = 1$. $(x \neq 0)$.

thus $x \in B^\times$.

③ let $m_1, m_2, \cdots$ (distinct) maximal ideals

$m_1 \geq (m_1 \cap m_2) \geq \cdots$.

$\exists n_0$, s.t. $\bigcap_{i=1}^{n_0} m_i = \bigcap_{i=1}^{n} m_i$, $\forall n \geq n_0$

thus $m_n \geq m_1 \cap \cdots \cap m_{n_0}$, $\forall n \geq n_0$.

by "prime", then $m_n \geq m_j$, $\exists j \leq n_0$

by "maximal", then $m_n = m_j$. $\exists j \leq n_0$.

Thus there is only finitely many distinct

(If not, use the ~~following~~ above trick and get a contradiction)

□

Prop: A : Artin, $\Rightarrow$

$\mathcal{N} = \mathrm{Nil}(A)$ is nilpotent.

i.e. $\exists n \geq 0$, $\mathcal{N}^n = (0)$.

Note, let $A = \bigoplus_{i=1}^{n} k[x_i]/(x_i^i)$

$N = \bigoplus_{i=1}^{n} (\overline{x_i})$
$\hookrightarrow \subseteq k[x_i]/(x_i^i)$

then $N^n = (0)$.

$\mathcal{N}^{n-1} \neq (0)$.

If $n < \infty$, then A, Artin

then A : not Artin, and $\mathcal{N}$ not Nilpotent

Proof: $J = \mathrm{Nil}(A) = \mathrm{Jac}(A)$.

$J \supseteq J^2 \supseteq J^3 \cdots$

$\exists n$ st $J^n = J^{n+1}$ Want $J^n = (0)$

let $I = J^n$

We have $JI = I$, $J^n I = I$, if I : finitely generated.

then by Nakayama, $I = (0)$.

Suppose $I \neq (0)$.

b : minimal element of $\{$ ideals $b \subseteq I$, $J^n b \neq (0) \}$

then $\exists\ 0 \neq x \in b$ with $J^n x \neq (0)$

then $(x) \subseteq b \subseteq I$, $J(x) \neq (0)$, by minimality, $b = (x)$.

then $J^n(x) = J^{n+1}(x) = J \cdot J^n(x)$

Want: $J(x) = (x)$.

If not, then $J(x)$: nonzero submodule of x.

with $\underset{= J^n(x)}{J^n J(x)} \neq 0$ by minimality of (x).

then $J(x) = (x)$. (x) is finitely gen

then by NAK,

1^{st} Uniqueness Theorem of Primary Decomposition

Theorem 1) (Lec 6-2).

$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ M.P.D, then $Ass(\mathfrak{a}) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\} = \mathfrak{p}_i = rad(\mathfrak{q}_i)$

Recall: $Ass(\mathfrak{a}) = \{ \mathfrak{p} = r(\mathfrak{a}:x), x \in A \mid \mathfrak{p}: \text{prime} \}$.

Note: $\mathfrak{p}_i$: distinct by minimal primary decomposition (MPD). $\quad\quad$ $\emptyset$ (Pf omitted (later)

Lem 2). $\mathfrak{q}: \mathfrak{p}$-primary, $x \in A$. then

① $x \in \mathfrak{q}$, $(\mathfrak{q}:x) = (1)$

② $x \notin \mathfrak{q}$, $(\mathfrak{q}:x) = \mathfrak{p}$-primary

③ $x \notin \mathfrak{p}$, $(\mathfrak{q}:x) = \mathfrak{q}$.

Pf: (i) ✓

(ii) <primary> let $yz \in (\mathfrak{q}:x)$. $y \notin (\mathfrak{q}:x)$

then $(xy)z \in \mathfrak{q}$, with $xy \notin \mathfrak{q}$.

then $z^n \in \mathfrak{q}$, $\exists n$. ✓.

<rad $= \mathfrak{p}$>.

$(\mathfrak{q}:x) \supseteq \mathfrak{q}$, $r(\mathfrak{q}:x) \supseteq r(\mathfrak{q}) = \mathfrak{p}$.

let $y \in r(\mathfrak{q}:x)$, i.e. $y^n \in (\mathfrak{q}:x)$. $\exists n$.

i.e. $xy^n \in \mathfrak{q}$, $(x \notin \mathfrak{q})$, $\exists m$, $(y^n)^m \in \mathfrak{q}$.

thus $y \in r(\mathfrak{q}) = \mathfrak{p}$ hence $r(\mathfrak{q}:x) \subseteq \mathfrak{p}$.

(iii) $(\mathfrak{q}:x) \supseteq \mathfrak{q}$ ✓

let $y \in (\mathfrak{q}:x)$, $xy \in \mathfrak{q}$, $(x \notin \mathfrak{p})$,

~~thus $\exists n$, $y^n \notin \mathfrak{q}$~~ $\forall n$, $x^n \notin \mathfrak{q}$.

then $y \in \mathfrak{q}$. $(\mathfrak{q}:x) \subseteq \mathfrak{q}$ ✓ $\quad\square$

Lem 5).

A: Noetherian, $\mathfrak{q}: \mathfrak{p}$-primary.

$\Rightarrow \exists x \in A$, s.t. $(\mathfrak{q}:x) = rad(\mathfrak{q})$.

Pf: Since $(\mathfrak{q}:x) = rad(\mathfrak{q}) \neq (1)$ (prime)

$x \notin \mathfrak{q}$.

(i) $(\mathfrak{q}:x) \subseteq \mathfrak{p}$. by Lem 2).

$rad(\mathfrak{q}:x) = \mathfrak{p}$. ✓

(ii) $(\mathfrak{q}:x) \supseteq \mathfrak{p}$.

Claim ① $\exists m$, s.t. $\mathfrak{p}^m \subseteq \mathfrak{q}$.

Bec: let $\mathfrak{p} = \langle x_1, \cdots, x_n \rangle$ generators.

let $k_1, \cdots, k_n$, s.t. $x_i^{k_i} \in \mathfrak{q}$.

$k = \sum k_i$. $\mathfrak{p}^k$ generated by T_1

$T_1 = \{ (x_1^{\partial_1} \cdots x_n^{\partial_n}) \mid \sum \partial_i = k \}$

~~Bec~~

Bec: let $r \in \mathfrak{p}^k$,

$r = \sum_{\text{finite}} a^j r_1^j \cdots r_k^j \quad (r_1^j, \cdots r_k^j \in \mathfrak{p})(a^j \in A)$

$r = \sum_{\text{finite}} a^j (t_{1,1}^j x_1 + \cdots + t_{1,n}^j x_n) \cdots (t_{k,1}^j x_1 + \cdots + t_{k,n}^j x_n)$

$= \sum_{\text{finite}} \boxtimes\, x_1^{\partial_1} \cdots x_n^{\partial_n}$ where $\sum \partial_i = k$

$\forall x_1^{\partial_1} \cdots x_n^{\partial_n} \in \mathfrak{q}$,

thus $\mathfrak{p}^k \subseteq \mathfrak{q}$

Now, let $\mathfrak{p}^{n_0}$ has the least n_0, s.t. $\mathfrak{p}^{n_0} \subseteq \mathfrak{q}$.

i.e. $\mathfrak{p}^{n_0} \subseteq \mathfrak{q}$, $\mathfrak{p}^{n_0 - 1} \not\subseteq \mathfrak{q}$ $\quad (n_0 \geq 1)$

Note: $\mathfrak{a}^0$ (zero power of an ideal) $= (1)$.

$\therefore \exists x \in \mathfrak{p}^{n_0-1}$ s.t. $x \notin \mathfrak{q}$.

If $y \in \mathfrak{p}$, then $xy \in \mathfrak{p}^{n_0-1} \cdot \mathfrak{p} = \mathfrak{p}^{n_0} \subseteq \mathfrak{q}$.

~~$x \notin \mathfrak{q}$, thus $y \in$~~ $y \in (\mathfrak{q}:x)$

hence $\mathfrak{p} \subseteq (\mathfrak{q}:x)$ $\quad\square$

Pf Theorem 1).

(i) show $\forall i$, $\mathfrak{p}_i \in Ass(\mathfrak{a})$

i.e. $\exists x_i$, s.t. ~~$(\mathfrak{q}_i:x)$~~ $r(\mathfrak{a}:x_i) = \mathfrak{p}_i$. $\forall i$

$r(\mathfrak{a}:x) = r((\bigcap \mathfrak{q}_i) : x))$

$= r(\bigcap(\mathfrak{q}_i:x)) = \bigcap_{i=1}^n r(\mathfrak{q}_i:x))$

If $x \in \mathfrak{q}_i$, $r(\mathfrak{q}_i:x) = r(1) = (1)$

$r(\mathfrak{a}:x)) = \bigcap_{x \notin \mathfrak{q}_i} r(\mathfrak{q}_i:x))$. if such x exists.

by M.P.D, $\forall i$, $\mathfrak{q}_i \not\supseteq (\bigcap_{i \neq j} \mathfrak{q}_j)$

i.e. $\forall i$, $\exists x_i$, s.t. $x \in \bigcap_{i \neq j} \mathfrak{q}_j$, $x_i \notin \mathfrak{q}_i$.

Hence $\forall i$, $\exists x_i$, s.t. $r(\mathfrak{a}:x_i) = r(\mathfrak{q}_i:x_i)) = \mathfrak{p}_i$

(ii) Show if $\mathfrak{p} = r(\mathfrak{a}:x))$, is prime, then

$\exists i$, s.t. $\mathfrak{p} = \mathfrak{p}_i$.

~~Then~~ $\mathfrak{p} = \bigcap_{i=1}^n r((\mathfrak{q}_i:x))$ [by Lem 2)] $= \bigcap_{x \notin \mathfrak{p}_i} \mathfrak{p}_i$

by a theorem, $\mathfrak{p} \supseteq \mathfrak{p}_i$, $\exists i$, $x \notin \mathfrak{p}_i$

also $\forall i$, $\mathfrak{p}_i \supseteq \mathfrak{p}$ $(\forall i, x \notin \mathfrak{p}_i)$

thus $\exists i$, $\mathfrak{p} = \mathfrak{p}_i$. $\quad\square$

Corollary:

A: Noetherian $\Rightarrow \forall i, \exists x_i$, s.t. $(\mathfrak{a} : x_i) = \mathfrak{p}_i$.

Pf: by Lem 5), $\forall i, \exists x_i$, s.t. $(\mathfrak{q}_i : x_i) = \mathfrak{p}_i$. (*1)

$(\mathfrak{a} : x) = \bigcap_{i=1}^{n} (\mathfrak{q}_i : x)$

$= \bigcap_{x \notin \mathfrak{q}_i} (\mathfrak{q}_i : x)$ if ~~such~~ x exist.

Want, such x_i in (*1), $x_i \notin \mathfrak{q}_i$, $x_i \in \left(\bigcap_{j \neq i} \mathfrak{q}_j\right)$

Example

Def Zero-divisor mod $\mathfrak{a} \subseteq A$

$Z(\mathfrak{a}) = \{ x \in A \mid [x] \text{ zero divisor in } A/\mathfrak{a} \}$.

$= \{ x \in A \mid \exists y \notin \mathfrak{a}, \text{ s.t. } xy \in \mathfrak{a} \}$

$= \bigcup_{y \in A - \mathfrak{a}} (\mathfrak{a} : y)$

(*1) $= \bigcup_{y \in A - \mathfrak{a}} \operatorname{rad}(\mathfrak{a} : y)$.

Pf (*1): show $\forall y \in (A - \mathfrak{a})$, $(\mathfrak{a} : y) = \operatorname{rad}(\mathfrak{a} : y)$.

~~let~~ $\cancel{x^n \in (\mathfrak{a} : y)}$, ~~then~~ $\cancel{x^n y \in \mathfrak{a}}$. $\cancel{(n \geq 1)}$

~~let~~ $\cancel{x^{n_0}}$ ~~be of the least no s.t.~~ $\cancel{x^{n_0} y \in \mathfrak{a}}$

~~then~~ $\cancel{x \cdot \underbrace{x^{n_0 - 1} y}_{\notin \mathfrak{a}} \in \mathfrak{a}}$.

① $\forall (\mathfrak{a} : y)$, $(\mathfrak{a} : y) \subseteq \operatorname{rad}(\mathfrak{a} : y)$. ✓

② let $x \in \bigcup_{y \in A - \mathfrak{a}} \operatorname{rad}(\mathfrak{a} : y)$,

i.e. $\exists y \in A - \mathfrak{a}$, s.t. $x \in \operatorname{rad}(\mathfrak{a} : y)$

i.e. $\exists y \ldots, \exists n$ s.t. $x^n y \in \mathfrak{a}$.

let x^{n_0} be of least no s.t. $x^{n_0} y \in \mathfrak{a}$. $n_0 \geq 1$.

then $x \cdot \underbrace{x^{n_0 - 1} y}_{\notin \mathfrak{a}} \in \mathfrak{a}$,

hence $\exists y' = x^{n_0 - 1} y \notin \mathfrak{a}$, s.t.

$x \in (\mathfrak{a} : y') \subseteq \bigcup_{y \in A - \mathfrak{a}} (\mathfrak{a} : y)$.

hence $\bigcup_{y \in A - \mathfrak{a}} \operatorname{rad}(\mathfrak{a} : y) \subseteq \bigcup_{y \in A - \mathfrak{a}} (\mathfrak{a} : y)$ ✓ □

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Lec 7-1 Revisited.
& Lec 7-2

· Proposition.

$\mathfrak{a}=\bigcap_{j=1}^{n} q_j$. M.P.D $\Rightarrow$, then $Z(\mathfrak{a})=\bigcup_{j=1}^{n} P_j$

pf: ① $\bigcup_{j=1}^{n} P_j \subseteq Z(\mathfrak{a})$

Want, ~~If $x \notin Z(\mathfrak{a})$, then $x \notin P_j, \forall j$~~

If $\exists j$, s.t. $x \in P_j$, then $x \in Z(\mathfrak{a})$

Let $x \in P_j$, $\exists y_j$, s.t. $\mathrm{rad}(\mathfrak{a} : y_j) = p_j$.

(by the Theorem that, $\mathrm{Ass}(\mathfrak{a}) = \{p_1, \dots, p_n\}$,

~~let~~ thus $y_j \notin p_j$, $y_j \notin \mathfrak{a} = \bigcap_{j=1}^{n} q_j$

hence $x \in p_j = \mathrm{rad}(\mathfrak{a} : y_j)$ $(y_j \notin \mathfrak{a})$

thus $x \in Z(\mathfrak{a})$ ✓

② $Z(\mathfrak{a}) \subseteq \bigcup_{j=1}^{n} P_j$

Want $x \in Z(\mathfrak{a}) = \bigcup_{y \notin \mathfrak{a}} \mathrm{rad}(\mathfrak{a} : y)$, then

$\exists j$ st. $x \in P_j$.

then $x \in \mathrm{rad}(\mathfrak{a} : y)$ $\exists y \notin \mathfrak{a}$.

~~where $\mathrm{rad}(\mathfrak{a} : y)$ is prime because,~~

~~$(\mathfrak{a} : y)$ is primar~~

~~Want $\mathrm{rad}(\mathfrak{a} : y)$ = prime.~~

Note $x \in Z(\mathfrak{a})$, then $(\mathfrak{a} : x) \supsetneq \mathfrak{a}$ (*1)

If $x \notin P_j, \forall j$, then $(\mathfrak{a} : x) = \mathfrak{a}$ (*2)

with (*1)(*2) we can proof by contradiction.

(*1). $x \in Z(\mathfrak{a}) \Rightarrow$

$\Leftrightarrow$ $\exists y \notin \mathfrak{a}$, s.t. $x \in \mathrm{rad}(\mathfrak{a} : y)$.

$\Leftrightarrow$ i.e. $\exists y \notin \mathfrak{a}$, $\exists n$, s.t. $x^n y \in \mathfrak{a}$.

let n_0 be the least... s.t. $x^{n_0} y \in \mathfrak{a}$ $(n_0 \geq 1)$

then $x^{n_0 - 1} y \notin \mathfrak{a}$,

$\Rightarrow$ thus $\exists t = x^{n_0-1} y$, s.t.

$t \notin \mathfrak{a}$, $tx \in \mathfrak{a}$, i.e. $t \in (\mathfrak{a} : x)$

$(\Leftarrow)$ hence $(\mathfrak{a} : x) \supsetneq \mathfrak{a}$

(*2) $(\mathfrak{a} : x) = \mathfrak{a}$

$\forall j$, $x \notin p_j$, $\Rightarrow$ then $(q_j : x) = q_j$, $\forall j$

$\Rightarrow$ $(\mathfrak{a} : x) = \bigcap_{j=1}^{n} (q_j : x) = \bigcap_{j=1}^{n} q_j = \mathfrak{a}$

□

· Example.

· Def isolated / minimal, embeded.

$\forall p \in \mathrm{Ass}(\mathfrak{a})$,

if p minimal element of $\mathrm{Ass}(\mathfrak{a})$, i.e.

$\forall p' \in \mathrm{Ass}(\mathfrak{a})$, $p \not\supsetneq p'$ (p is not strictly less →)

then p : isolated / minimal

else, p is embeded.

· Rmk. If p embeded, i.e. $\exists p' \in \mathrm{Ass}(\mathfrak{a})$,

s.t. $p' \supsetneq p$, then $V(p') \subsetneq V(p)$.

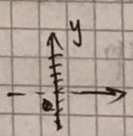
Bec: if $V(p') = V(p)$, then

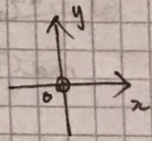
$I(V(p')) = I(V(p))$, i.e. $p = p'$.

EX: $A = k[x, y]$, $p_1 = (x)$, $p_2 = (x, y)$

~~let $\mathfrak{a} = p_1 \cap p_2$~~ $p_2 \supsetneq p_1$

isolated embeded


$\Leftarrow V(p_1)$

$V(p_2) \Rightarrow$


· Theorem (2^{nd} Uniqueness)

$\mathfrak{a} = \bigcap_{i=1}^{n} q_i$, M.P.D.

If p_i = isolated, then

$q_i = \tau_i^{*}\tau_{i*}(\mathfrak{a})$.

where $\tau_i : A \longmapsto A_{p_i}$

· Lem: q: p-primary

$S \subseteq A$ multiplicatively closed.

$\tau : A \to S^{-1}A$.

then

① $S \cap p \neq \emptyset$, $\Rightarrow$ then $\tau_*(q) = (1)$

② $S \cap p = \emptyset$ $\Rightarrow$ then $\tau_*(q)$: $\tau_*(p)$-primary

also $\tau^*\tau_*(q) = q$

Pf: First Note:

$S \cap p = \emptyset$ IFF $S \cap q = \emptyset$ (*1)

(*1): $\Rightarrow$ ✓

$\Leftarrow$ if $\exists s \in S \cap p$, then $\exists n$ s.t. $s^n \in q$.

$s^n \in S$ by multiplic closed.

then $S \cap q \neq \emptyset$ ✓

① If $s \in S \cap q$, then $s \mapsto \frac{s}{1} \in \tau_*(q)$

since $\tau_*(q)$ ideal in $S^{-1}A$,

$\underset{\in\tau_*(q)}{\frac{s}{1}} \cdot \underset{\in S^{-1}A}{\frac{1}{s}} = \frac{1}{1} \in \tau_*(q)$.

thus $\tau_*(q) = (1)$.

② ~~$\tau^*\tau_*(q) = q$ is true for general ideals in A.~~

$\mathrm{rad}(\tau_*(q)) = \tau_*(\mathrm{rad}(q)) = \tau_*(p)$

in general rules.

Want $\tau_*(q)$: primary.

Note: $\tau_*(\tau^*(\mathfrak{b})) = \mathfrak{b}$ for general ideals.

but not the other way around.

(i) Want $\tau^*(\tau_*(q)) = q$.

Enough: $\tau^*(\tau_*(q)) \subseteq q$ (*2)

Bec $\tau^*(\tau_*(q)) \supseteq q$ in general

$t \in \tau^*(\tau_*(q)) \Leftrightarrow \frac{t}{1} \in \tau_*(q) = \left\{ \frac{x}{s} \,\middle|\, x \in q,\ s \in S \right\}$

$\Leftrightarrow \exists s' \in S$, s.t. $t\underline{ss'} = \underbrace{xs'}_{\in q}$

$ss' \in S$, thus $ss' \notin p$, $(\forall m, (ss')^m \notin q)$

thus $t \in q$. (by the primarity of q)

(*2) ✓

(ii) Want $\tau_*(q)$ = primary.

Use the "zero notation".

Note:

A (ring), $\mathfrak{b} \subseteq A$ (ideal), $S \subseteq A$, multiplic closed.

Then $S^{-1}(A/\mathfrak{b}) \cong (S^{-1}A)/\tau_*(\mathfrak{b})$

as ring homorphisms.

by identifying $\frac{[x]}{[s]} \xmapsto{\text{as}} \frac{x}{s} + \tau_*(\mathfrak{b})$

Check:

① ~~well-def linearity~~ well-defined.

If $x' - x = b \in \mathfrak{b}$, then

$\frac{x'}{s} - \frac{x}{s} = \frac{b}{s} \in \tau_*(\mathfrak{b})$

If $s' - s = b \in \mathfrak{b}$, then

$\frac{x}{s'} - \frac{x}{s} = \frac{-xb}{s(s+b)} = \frac{-x}{s''} \cdot \frac{b}{1} \in \tau_*(\mathfrak{b})$

② $\frac{[1]}{[1]} \longrightarrow \frac{1}{1} + \tau_*(\mathfrak{b})$

③ $\frac{[x_1]}{[s_1]} + \frac{[x_2]}{[s_2]} = \frac{[x_1 s_2 + x_2 s_1]}{[s_1 s_2]} \mapsto \frac{x_1}{s_1} + \frac{x_2}{s_2} + \tau_*(\mathfrak{b})$

④ $\frac{[x_1]}{[s_1]} \cdot \frac{[x_2]}{[s_2]} \mapsto \frac{x_1}{s_1} \cdot \frac{x_2}{s_2} + \tau_*(\mathfrak{b})$

⑤ surjectivity ✓

injectivity, if $\frac{x_1}{s_1} \in \tau_*(\mathfrak{b})$, then

$\exists b \in \mathfrak{b}, s \in S$, s.t. $\frac{x_1}{s_1} = \frac{b}{s}$.

thus $\frac{[x_1]}{[s_1]} = \frac{[b]}{[s]} = \frac{[0]}{[1]}$ ✓

Hence:

$S^{-1}(A/q) \cong (S^{-1}A)/\tau_*(q)$.

Note, $A \supseteq \mathfrak{a}$, $\mathfrak{a}$ primary IFF $\langle 0 \rangle$ primary in $A/\mathfrak{a}$.

OR IFF zero divisors in $A/\mathfrak{a}$ are nilpotents.

Thus:

$\tau_*(q)$ primary IFF zero divisors in $(S^{-1}A)/\tau_*(q)$ are nilpotents.

IFF zero divisors in $S^{-1}(A/q)$ are nilpotents.

Since q : primary, ~~A/q~~ $\langle 0 \rangle$ primary in A/q.

OR i.e. zero divisors in A/q are nilpotents

Now let $\frac{[x_1]}{[s_1]} \cdot \frac{[x_2]}{[s_2]} = \frac{[0]}{[1]}$

with $\frac{[x_2]}{[s_2]} \neq \frac{[0]}{[1]}$, i.e. $\forall [s_3]$, $[x_2 s_3] \neq [0]$

then $\exists [s_4]$, s.t. $[x_1]\underbrace{[x_2 s_4]}_{\neq [0]} = [0]$

thus $[x_1]$ zero divisor, thus $[x_1]$ nilpotent.

then $\frac{[x_1]}{[s_1]}$ also nilpotent.

P91(CTD) Lec7-1 & Lec7-2 Revisited pf Theorem : 2^{nd} Uniqueness.

Hence $S^{-1}(A/\mathfrak{q})$ ~~pr~~: zero divisors ⇒ nilpotents,

thus $\tau_*(\mathfrak{q})$ primary.

□

• Lem.

$\mathfrak{a}$ = decomposible.

then $\mathrm{Ass}'(\mathfrak{a}) = \underbrace{\{\text{minimal primes contang } \mathfrak{a}\}}_{}$ $= T$

$\overset{\text{i.e.}}{=} \{\mathfrak{p} \geq \mathfrak{a} \text{ prime} \mid \text{If } \mathfrak{p} \geq \underset{\text{prime}}{\mathfrak{p}'} \geq \mathfrak{a}, \text{ then } \mathfrak{p} = \mathfrak{p}'\}$

Recall:

$\mathrm{Ass}'(\mathfrak{a}) = \{\text{minimal elements of } \mathfrak{a}\ \mathrm{Ass}(\mathfrak{a})\}$

Pf:

Let $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$, $r(\mathfrak{a}) = \bigcap_{i=1}^{n} \mathfrak{p}_i$

① $\mathrm{Ass}'(\mathfrak{a}) \subseteq T$

If $\mathfrak{p}_j \in \mathrm{Ass}'(\mathfrak{a})$, then $\mathfrak{p}_j$: prime, $\mathfrak{p}_j \geq \mathfrak{a}$.

Want $\mathfrak{p}_j \geq \mathfrak{p}' \geq \mathfrak{a}$, then $\mathfrak{p}_j = \mathfrak{p}'$.

$\mathfrak{p}' = r(\mathfrak{p}') \geq r(\mathfrak{a}) = \bigcap_{i=1}^{n} \mathfrak{p}_i$

thus $\exists i$, s.t. $\mathfrak{p}' \geq \mathfrak{p}_i$.

~~also, $\forall i$, $\mathfrak{p}_i$, $\mathfrak{p}_i \geq \mathfrak{p}'$~~ Then $\mathfrak{p}_j \geq \mathfrak{p}' \geq \mathfrak{p}_i$

~~thus $\mathfrak{p}' = \mathfrak{p}_i$, $\exists i$.~~ by minimality of $\mathfrak{p}_j$,

~~thus $\mathfrak{p} \geq \mathfrak{p}' = \mathfrak{p}_i \geq \mathfrak{a}$.~~ $\mathfrak{p}_j = \mathfrak{p}' = \mathfrak{p}_i$

~~by minimality in $\mathrm{Ass}'(\mathfrak{a})$~~ thus $\mathfrak{p}_j \in T$.

~~$\mathfrak{p} = \mathfrak{p}_i = \mathfrak{p}'$~~

② $T \subseteq \mathrm{Ass}'(\mathfrak{a})$

let $\mathfrak{p} \in T$,

then $\mathfrak{p} = r(\mathfrak{p}) \geq r(\mathfrak{a}) = \bigcap_{j=1}^{n} \mathfrak{p}_j$

thus $\exists i$, $\mathfrak{p} \geq \mathfrak{p}_i \geq \mathfrak{a}$

by minimality in T, $\mathfrak{p} = \mathfrak{p}_i$

thus we want to show if $\mathfrak{p}_j \in \mathrm{Ass}(\mathfrak{a})$,

$\mathfrak{p}_j \subseteq \mathfrak{p}_i$, then $\mathfrak{p}_j = \mathfrak{p}_i$.

this also follows from minimality in T

thus $\mathfrak{p} = \mathfrak{p}_i \in \mathrm{Ass}'(\mathfrak{a})$ □

· $\underset{\text{ideal}}{\mathfrak{a} \leq A}$, $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ M.P.D.

~~If $\mathfrak{q}_j \in \mathrm{Ass}'(\mathfrak{a})$, then~~

If $\mathfrak{p}_j \in \mathrm{Ass}'(\mathfrak{a})$, then

$\mathfrak{q}_j = \tau_j^* \tau_{j*}(\mathfrak{a})$, where $\tau_j : A \to A_{\mathfrak{p}_j}$

such $\mathfrak{q}_j$: the $\mathfrak{p}_j$-primary component of $\mathfrak{a}$.

~~pf~~:
pf:

Claim: $\mathfrak{p}_j$: minimal / isolated, ~~then~~

$S_j = A - \mathfrak{p}_j$, then

$S_j \cap \mathfrak{q}_i = \emptyset \iff \mathfrak{q}_i = \mathfrak{q}_j$ $(i=j)$ (*1)

(i) ⇐ ✓

(ii) ⇒ $S_j \cap \mathfrak{q}_i = \emptyset$, then $\mathfrak{q}_i \subseteq \mathfrak{p}_j$

then $\mathfrak{p}_i = r(\mathfrak{q}_i) \subseteq r(\mathfrak{p}_j) = \mathfrak{p}_j$

by minimality of $\mathfrak{p}_j$, $\mathfrak{p}_i = \mathfrak{p}_j$

by distinctivity of M.P.D, $i=j$.

Now. $\tau_j^* \tau_{j*}(\mathfrak{a}) = \bigcap_{i=1}^{n} \tau_j^* \tau_{j*}(\mathfrak{q}_i)$ general rules

by (*1). $= \mathfrak{q}_j$

□

§ Artin Rings.

Lec 7-1 & 7-2 Revisited

- **Def** Artin / Artinian Modules

IFF the following equivalent conditions true:

(i) "DCC". $\forall$ chain $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$

$\exists n_0$ s.t. $\forall n \geq n_0$ $M_n = M_{n_0}$

(ii) "MIN" $\forall$ nonempty ~~subset~~ set of submodules, has a minimal element.

Pf: Same as Noetherian Conditions.

- **Artin Rings**

IFF Artinian as a ~~submodule~~ A-mod.

(A: the ring itself)

IFF "DCC", $\forall$ chain $I_1 \supseteq I_2 \supseteq \cdots$

$\exists n_0$, s.t. $\forall n \geq n_0$ $I_n = I_{n_0}$

IFF "MIN" $\forall$ nonempty sub-set of ideals, has a minimal element.

- **Lem** Artinian & SES

Given $0 \to M_1 \to M_2 \to M_3 \to 0$ SES on Modules

Then M_2: Artin IFF M_1, M_3: Artin

Pf: Omitted by Lecturer.

- **Lem** Direct Sum of Artin Rings

Direct Sum of Artin Rings is Artin.

- **Lem** Finitely generated R-mods with R Artin

M: R-mod, R: Artin Then M: Artin

Pf: by the previous 2 Lemmas.

Get a SES

$$0 \to \ker\phi \xrightarrow{\hookrightarrow} R^n \xrightarrow[\text{surjective}]{\phi} M \to 0$$

R^n: direct sum of R's, Artin

Get: M, $\ker\phi$ both Artin.

- **Lem** $A/\mathfrak{a}$ Artin

A: Artin, $\mathfrak{a}$: $\forall$ ideal $\subseteq A$.

Then $A/\mathfrak{a}$: Artin

Pf: Get SES

$$0 \to \mathfrak{a} = \ker(\pi) \xrightarrow{\hookrightarrow} A \xrightarrow[\text{surjective}]{\pi} A/\mathfrak{a} \to 0$$

Thus $\mathfrak{a}$ & $A/\mathfrak{a}$ both Artin.

Example Non Artin Rings

- $\mathbb{Z}$, $\exists (2) \supsetneq (2^2) \supsetneq (2^3) \cdots$

strictly decreasing, infinite length.

Example Finite Rings

$\forall$ finite ring = both Artin + Noetherian.

Recall: a finitely generated module over a field is simply a vector space.

~~Example~~

Lem Finite Dim V.S.: Artin

A finitely generated k mod with k: field, is Artin because a strictly decreasing chain of submodules are subspaces with strictly decreasing dim.

Example Finite Dim V.S.: Artin

k: field, $A = k[x_1, \cdots, x_n]$, k-mod.

$m = (x_1, \cdots, x_n)$ ~~are~~ is composed of the zero polynomial and polynomials with at least degree 1.

$\forall \ell \geq 0$, m^ℓ is composed of zero and polynomials of degree at least ℓ ($m^0 = A = (1)$)

$k[x_1, \cdots, x_n]/m^\ell$ is generated by these representatives: polynomials with degree less than ℓ, which can be generated with $(0, x_1, x_2, \cdots, x_n, x_1x_2, \cdots, x_{n-1}x_n, x_n^2, \cdots$, ~~$x_1$~~ $\cdots)$, which is finite.

Thus A/m^ℓ: finitely generated k-mod

Thus Artin.

Examples

$k[x^2, x^3] = (x^2, x^3) \subseteq k[x]$.

$k[x^2, x^3]/((x^6))$ Artin.

$k[x]$ not Artin. $(x) \supsetneq (x^2) \supsetneq (x^3) \cdots$

Lem Artin Ring, Primes VS Maximals

Given A: Artin, then prime ideals are maximal. $Jac(A) = Nil(A)$.

Also, A has finitely many distinct prime/maximal ideals. i.e. A: semi-local

Pf:

① $\mathfrak{p}$: prime, $A/\mathfrak{p}$: integral domain

Want $A/\mathfrak{p}$: ~~maximal~~ field

Take $x \in A$, $\langle x \rangle$ in $A/\mathfrak{p}$, $x \notin \mathfrak{p}$

$(\langle x \rangle) \supseteq (\langle x^2 \rangle) \supseteq (\langle x^3 \rangle) \supseteq \cdots$

by $A/\mathfrak{p}$: Artin.

$\exists n$ s.t. $(\langle x^n \rangle) = (\langle x^{n+1} \rangle)$

i.e. $\exists \langle u \rangle$ s.t.

$\langle u \rangle \langle x^{n+1} \rangle = \langle x^n \rangle = \langle u \rangle \cdot \langle x^n \rangle \langle x \rangle$

Since $A/\mathfrak{p}$: integral domain.

Get $\langle u \rangle \langle x \rangle = 1$

Thus $\forall \langle x \rangle \neq \langle 0 \rangle$, $\langle x \rangle \in (A/\mathfrak{p})^\times$ (units)

Then $A/\mathfrak{p}$: field.

② Assume A has infinitely many prime/max (distinct): $m_1, m_2, \cdots$

Then $m_1 \supseteq (m_1 \cap m_2) \supseteq (m_1 \cap m_2 \cap m_3) \cdots$

by Artinian,

$\exists n$ s.t. $\bigcap_{i=1}^{n+1} m_i = \bigcap_{i=1}^{n} m_i$

Then $m_{n+1} \overset{\supseteq}{\not\supseteq} (\bigcap_{i=1}^{n} m_i)$

by "prime", $m_{n+1} \supseteq m_j$, $\exists j \leq n$

by "maximal", $m_{n+1} = m_j$, $\exists j \leq n$

Contradiction to the assumption. □

Recall NKY (Nakayama)

M: fin gen R-mod. ~~If~~ $\mathfrak{a} \subset Jac(R)$ ideal

$\mathfrak{a}M = M$ then $M = 0$

~~Green~~

(*1) Lem Commutative Artinian Rings are Noetherian

A: Artin Ring. Then A: Noetherian

Pf: I: ideal Want I: finitely generated.

If I not fin gen, let S_0 be the ~~set of~~ generators, of the set of generators ? How to make sure on the minimal property

then get $\{x_1, x_2, \cdots, x_n, \cdots\}$ a infinite list of distinct elements in S_0, let $S_0' =$ ~~$\{x_1, \cdots, x_n, \cdots\} - S$~~ $S/\{x_1, \cdots x_n \cdots\}$

Then let $S_n\ (n \geq 1) = \{x_n, x_{n+1}, \cdots\}$

Then $I = \langle S_0' \cup S_1 \rangle \supsetneq \langle S_0' \cup S_2 \rangle \cdots \supsetneq \langle S_0' \cup S_n \rangle \cdots$

is a strictly decreasing chain of ideals in A.

Contradiction. □

(*1) Lem A: Artin then $Nil(A)$ "nilpotent"

A: Artin, then $\exists n$, s.t. $Nil(A)^n = (0)$. (some $n \geq 0$)

Pf: ~~Get~~ Let $J = Nil(A) = Jac(A)$

Get $J \supseteq J^2 \supseteq \cdots \supseteq J^n \cdots$

by "Artin", $\exists n$, s.t. $J^n = J^{n+1}$

Let $I = J^n$. then $JI = J$, I: fin gen by the above Lemma

~~If I~~ by NKY. $I = 0$ □

(*1) Note: Artin $\Rightarrow$ Noetherian is prove in P97, where it uses Artin $\Rightarrow Nil(A)$ "nilpotent", thus here in the lemma should use something else.

CA Exercise Class W8

EX.1. $A = k[x,y,z]$.

$\mathfrak{a}_1 = (x, yz)$ not primary

$\mathfrak{a} = (x, y^3 z)$ not primary $= (x, y^3) \cap \underline{(x,z)}$ prime thus primary.

$k[x,y,z]/(x,y^3) \cong k[y,z]/(y^3)$

The ~~$f(y,z) \in y^3 \cdot g(y,z)$~~ map $x \mapsto 0$ is an isomorphism.

(※) $\exists g(y,z) \notin (y^3)$, then $f(y,z)\cdot g(y,z) \in (y^3)$

$\Downarrow$

Want $f^n(y,z) \in (y^3)$.

$f = y \cdot h(y,z)$

thus $f^3(y,z) \in (y)^3$ ✓

Here (※) shows that (x,y^3) primary.

EX 2. $A = k[x,y,z]$. $\mathfrak{a} = (xy, xz) = (x) \cap (y,z)$.

$x=0$ / $y=z=0$.

EX 3. $A = k[x,y,z]$, $\mathfrak{a} = (x(x-y), xz) = (x) \cap (x-y, z)$

$x=0$ / $x=y, z=0$ let $u = (x-y)$.

~~EX 4~~

then $A/\mathfrak{a} = k[x,u,z]/(u,z)$ is prime.

$k[x,y,z]/(x-y,z) \underset{z \mapsto 0}{\cong} k[x,y]/(x-y) \cong k[x]$.

~~$(x-y) \mapsto 0$~~.

sent $x \mapsto x$, $y \mapsto x$, then $(x-y) \mapsto 0$.

$\ker(\varphi) = \{p \in k[x,y] \mid$ ~~$(x-y, 1)$~~ $((x-y)) \ni p\}$

$(1, x, y) \mapsto k_1 + k_2 x + k_3 y = k_1 + (k_2+k_3)x = 0$

$= ((x-y))$

$k_1 = 0$, $k_2 + k_3 = 0$

EX 4 $A = k[x,y,z,w]$. $\mathfrak{a} = (x) \cap (y) \cap (z-w, z^2)$

$x=0$ / $y=0$ / $z=w$, $z=0$. $\to$ primary.

~~$z^2 \ldots z6$~~

EX5. $A = k[x,y,z]$. $\mathfrak{a} =$ ~~$(x) \cap (xy) \cap$~~ $(x^3, xy) \cap (x^2, z)$ ✓ primary

$x=0$ / $x=0, y=0$

$=$ ~~$(x) \cap (xy) \cap (x^2,y)$~~ $(x) \cap (x^3, y) \cap (x^2, z)$ ✓ primary

EX6. $\mathfrak{a} = (x^2-y, 2x^2+xz-z^2)$

$x^2 = y$, $2x^2 \to (x+1)(2x-z)$.

$= (u \cdot v w)$

$u = x^2-y$, $(x+z)(x-z)$

$= (u,v) \cap (u,w) = \underline{(x^2-y, x+z) \cap (x^2-y, 2x-z)}$ primary

Artinian Rings.

$A = \mathbb{Z}$, p: prime number, $M = \mathbb{Z}(\frac{1}{p})/\mathbb{Z}$ is a $\mathbb{Z}$-mod.

Claim: Artinian by not Noetherian.

pf: Submodules of M: generated by $\frac{1}{p^n} + \mathbb{Z}$

let $(a, p) = 1$, then a submodule containing $\frac{a}{p^n} + \mathbb{Z}$, ∅

coprime

by $\exists \bar{a} \in \mathbb{Z}$, s.t. $a\bar{a} \equiv 1 \mod p^n$.

$\bar{a}(\frac{a}{p^n} + \mathbb{Z}) = \frac{\bar{a}a}{p^n} + \mathbb{Z} = \frac{1}{p^n} + \mathbb{Z}$.

then $(\frac{1}{p} + \mathbb{Z}) \subsetneq (\frac{1}{p^2} + \mathbb{Z}) \subsetneq (\frac{1}{p^3} + \mathbb{Z}) \cdots$

thus not Noetherian.

Every descending chain looks like.

$(\frac{1}{n_1} + \mathbb{Z}) \supseteq (\frac{1}{n_2} + \mathbb{Z}) \supseteq (\frac{1}{n_3} + \mathbb{Z}) \cdots$

$\forall i$, $(\frac{1}{n_{i+1}} + \mathbb{Z}) \subseteq (\frac{1}{n_i} + \mathbb{Z})$

i.e. $\exists a_i$ s.t. $\frac{a_i}{n_i} + \mathbb{Z} = \frac{1}{n_{i+1}} + \mathbb{Z}$.

$a_i + \mathbb{Z} = \frac{n_i}{n_{i+1}} + \mathbb{Z} = \mathbb{Z}$

thus $n_{i+1} \mid n_i$.

Lec 8-1

Krull dim of a nonzero ring A

$\dim(A) = \sup\{r \geq 0 \mid \exists$ chain of primes in A of length r: $p_0 \subsetneq p_1 \subsetneq p_2 \subsetneq \cdots\}$

Examples: k: field. (0) only prime.

$\dim(k) = 0$

$\mathbb{Z}$, $\dim(\mathbb{Z}) = 1$

$k[x,y,z]$, k: field, $\dim(k[x,y,z]) = 3$

use lem: $\dim R[x_1, \ldots, x_n] = \dim R + n$

- $\dim(A) = 0 \iff$ primes are maximal

Theorem Artin vs Noetherian

A: Artin $\iff$ A: Noetherian $+ \dim(A) = 0$

pf: ~~...~~

~~A: Noetherian, $\mathfrak{a}$: ideal.~~

Lem[1] A: Noetherian, $\mathfrak{a}$: ideal.

$\exists n \geq 0$, s.t. $r(\mathfrak{a})^n \subseteq \mathfrak{a}$

pf: $r(\mathfrak{a})$ fin gen $r(\mathfrak{a}) = (x_1, \ldots, x_r)$

Let N large, s.t. $x_j^N \in \mathfrak{a}$, $\forall j = 1, \ldots, r$.

$\forall x \in r(\mathfrak{a})$, $x = \sum_{j=1}^{r} a_j x_j$

$x^n = (\sum a_j x_j)^n$ = A-linear combination of $x_1^{n_1} \cdots x_r^{n_r}$ s.t. $\sum_{j=1}^{n} n_j = n$

Let n large enough i.e. $n > r \cdot N$.

then $\exists j$ s.t. $n_j \geq N$.

then $x^n \in \mathfrak{a}$.

This should be ~~$\prod x_j$~~ $\prod_{k=1}^{n} x^k$ with $x^k \in r(\mathfrak{a})$
(See P99 for more)

Supp A1

Corollary: A: Noetherian, q: primary

$p = r(q)$, $\exists n \geq 0$, s.t. $q \supseteq p^n$

$\square$

Lem[2] $(0) \subseteq A$, (0) is a finite product of maximal ideals then A: Artinian $\iff$ A: Noeth

pf: $(0) = m_1 \cdots m_r$. $k_j = A/m_j$ field.

$M_0 = A$, $M_1 = m_1$, $M_2 = m_1 m_2$, $\cdots$ $M_r = m_1 \cdots m_r = (0)$

M_j / M_{j+1} $(j = 0, \ldots, r-1)$ is a k_{j+1}-vector space.

$\{A$-submodules of $M_j/M_{j+1}\}$ $\leftrightarrow$ $\{k_{j+1}$ vector subspaces of $M_j/M_{j+1}\}$

In general, if V: vector space over k.

then V: Artin $\iff \dim_k(V) < \infty \iff V$: Noetherian?

M_j/M_{j+1}: Artinian $\iff$ M_j/M_{j+1} Noetherian.

Lem[3] $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r = \{0\}$.

Chain of modules over a ring A. then

M: Noetherian / Artin $\iff \forall j = 1, \ldots, r-1$.

M_j/M_{j+1} Noether/Artin

pf: Induction on r.

$r = 0$, ✓

$r \geq 1$.

$0 \to M_1 \to \underset{M_0}{M} \to M_0/M_1 \to 0$.

M_1: Noether / Artin $\iff$

M_j/M_{j+1}, $j \geq 1$: Noetherian / Artin

M: Noether / Artin $\iff$

~~M_j~~ & M_1 & M_0/M_1 Noether / Artin.

$\iff M_0/M_1$ Noether / Artin.

pf Theorem "$\Leftarrow$"

(i) Noether + dim = 0 $\Rightarrow$ (0) is a fin prod...

then by lem 2, A: Artin.

(ii) Artin $\Rightarrow$ $(0) \cdots$ fin prod.

then by lem 2, A: Noeth (+ dim = 0)

(i) $\Rightarrow$

A: Artin, let $\{m_1, \dots, m_r\}$ set of prime/maximals

then $(\underbrace{m_1 \cap \cdots \cap m_r}_{Jac(A)})^N \supseteq (m_1 \cdots m_r)^N$

$= (0)$

(ii) $\Rightarrow$ A: Noeth, ~~A~~ $\dim(A) = 0$

$\Downarrow$ each p_j: maximal.

Every p_j: isolated/minimal

let $(0) = \bigcap q_j$ (M.P.D)

~~{primes in A}~~ $= \{p_1, \dots, p_r\}$ all maximal ?

$(p_1 \cdots p_r)^N \subseteq (\underbrace{p_1 \cap \cdots \cap p_r}_{Nil(A) = r(0)})^N \subseteq (0)$

$\uparrow$ by lemma 1.

Artin local rings.

A: Artin Then A: local $\Leftrightarrow$ primary.

pf. $\Rightarrow$ (A, m) local.

$m = Jac(A) = Nil(A)$

$\exists n$, s.t. $m^n = (0)$.

Def: A (ring) primary.

iff (0) primary.

Def: A (ring) prime

iff (0) prime iff A: integral doma

$A - m = A^\times$

$m = \{\text{zero divisors}\} = \{\text{nilpotents}\} = \{\text{non-units}\}$

$\Leftarrow$. (0) primary. $p = r(0)$: prime, thus maximal

the smallest prime is p, thus unique prime

p: unique maximal p in A.

then A: local.

~~A/(0)~~ : ~~Artin~~ V_p

Q: Artin integral domains?

(0) prime $\Rightarrow$ (0) maximal $\Rightarrow$ A: field

Then A: fields.

Prop: (A, m) Noetherian local ring

then. (i) $m^n \neq m^{n+1}$ $\forall n$.

(ii) $\exists n_0$ st. $m^{n_0} = 0$, ~~A~~ (A: Artin)

pf: $\neg$(i), then $m^n = m^{n+1}$, $\exists n$.

$m\mathfrak{a} = \mathfrak{a}$, $\mathfrak{a} = m^n$.

NAK ($\mathfrak{a}$: fin gen, $m = Jac(A)$)

the $\mathfrak{a} = (0)$.

~~(ii)~~

$p \subseteq A$, prime.

then $m^n \subseteq p \subseteq m$

$r(m^n) \subseteq r(p) \subseteq m$

$\quad \| \qquad = p$

$\cap r(m) = m$

$m = p$.

then m: unique prime., $\dim(A) = 0$.

then A: Artin

Ex

- $\mathbb{Z}/(p^n)$: Artin local.
- $k[[x]]$: Noeth local, $m = (x)$
  not Artin local.
- $k[[x]]/(x^n)$ Artin local

m: principal

- $k[x^2, x^3]/(x^{10})$: Artin local. $m = (x^2, x^3)$ not principal
- $k[x] = \{\sum_{i=0}^\infty a_i x^i, a_i \in k, a_i = 0 \ \forall i = \text{long enough}\}$.

$k[[x]] = \{\sum_{i=0}$
formal power series

local, $m = (x)$, $f \in k[[x]]$,

$f = a_0 + a_1 x + \cdots$, $a_0 \neq 0$, then f^{-1} exist in $k[[x]]$

- Theorem:

Every Artin ring is a finite direct product of local Artin rings, unique up to isomorphisms / reordering.

pf: A: Artin $\Rightarrow$ A: Noeth, $\dim = 0$.

$\Rightarrow \exists\ (0) = \bigcap q_j$ MPD.

$p_j = \mathrm{rad}(q_j)$ maximal

$\exists n,\ q_j \supseteq p_j^n\ \forall j$.

p_j: maximal $\Rightarrow$ p_j pairwise coprime.

$\overset{(☆)}{\Rightarrow}$ q_j pairwise coprime.

$\Leftrightarrow\ A/\bigcap_j q_j \to \prod_j A/q_j$ isomorphism.

(Chinese Remainder Theorem)

then $A = A/\bigcap_j q_j$.

A/q_j : primary, Artin then local Artin

Uniqueness>.

$\phi: A \xrightarrow{\cong} \prod_i A_i$ (Artin local) (finite)

$$\begin{tikzcd} \phi_i : A \arrow[r, "\phi_i"] \arrow[dr, "\phi"'] & A_i \\ & \prod A_i \arrow[u, "\text{projection}"'] \end{tikzcd}$$

$q_i' := \ker(\phi_i)$

then $A/q_i' \cong A_i$ (Artin, local $\Rightarrow$ primary)

thus q_i': primary ?

ϕ: isom $\Rightarrow$ $(0) = \bigcap_i q_i'$

Claim. This is a MPD of (0).

this follows from the surjectivity of ϕ

But A: Artin, so every associated prime of (0) is minimal/isolated.

So every primary ~~compound~~ component is unique.

thus $\{q_j\} = \{q_j'\}$ □

(supp*1).

$r(\mathfrak{a}) = (x_1, \dots, x_r)$.

let $k_1, \dots, k_r$ s.t. $x_i^{k_i} \in \mathfrak{a}$.

let $k = \Sigma k_i$, then $x_i^k \in \mathfrak{a}\ \forall i$

$\forall$ --- Check the same proof on P84

see (*1) foot note there. It's a general argument.

CA Ex W9.

. A: Artinian $\Leftrightarrow$ A: D.C.C.

Use (i) A: Artinian $\Rightarrow$ Spec A, finite set

(ii) K: field, A: F.G. K-Algebra

TFAE ① A: Artinian

② A: Finite K-Algebra.

(A: Finite K-Vector Space)

The two are on Exercise Sheet.

$f: A \to B$ ring homomorphism

take B as an A module.

(i.e. $\forall a \in A$, $b \in B$, then $a \cdot b = f(a) b$

f: finite if B is a finite A-Algebra.

$f^*: \mathrm{Spec}\, B \to \mathrm{Spec}\, A$

$q \mapsto f^{-1}(q)$.

Prop: If f finite, then $\forall p \in \mathrm{Spec} A$.

$(f^*)^{-1}(p) = \{ q \in \mathrm{spec} B,\ f^*(q) = p\}$ is a finite set

Pf:

Goal: ~~find~~ find an Artinian ring C.

s.t. $(f^*)^{-1}(p) = \mathrm{spec}\ C$.

Start: $p \in \mathrm{Spec} A$. $S = A - p$

$f_p: A_p \to B_p = f(S)^{-1} B$

$\mathrm{Spec}\, A_p = \{ p' \in \mathrm{spec}\, A,\ p' \cap S = \phi \}$

$\mathrm{Spec}\, B_p = \{ q \in \mathrm{Spec}\, B,\ q \cap f(S) = \phi \}$

$\Updownarrow$

$f^{-1}(q) \cap S = \phi$

$\Updownarrow$

$f^{-1}(q) \subseteq p$.

$(f_p^*)^{-1}(p) = (f^*)^{-1}(p)$

$$= k_p \text{ (residue field)}. \qquad C$$

$$k_p = A_p / pA_p \longrightarrow (B_p) / f(pA_p) \cdot B_p$$

$\uparrow$

Noetherian

$\mathrm{Spec}(A_p / pA_p) = \{p\}$.

$\mathrm{Spec}\ C = \{ q \in \mathrm{Spec}\, B_p,\ q \supseteq f(pA_p) \}$

$= \{ q \in \mathrm{Spec}\, B_p : f_p^*(q) = pA_p \}$

$\cong \{\cdots (f^*)^{-1}(p)$

C: finite k_p- Algebra.

B: finite is a finite A-Algebra

$\Downarrow$

$C = B \otimes_A k_p$ is a finite k_p-Algebra

$\Downarrow$

C: Artinian

$\Downarrow$

Spec C is a finite set

(A, m) Noetherian local ring.

M: F.G. A-mod.

TFAE. ① M: free ② M: flat

③ $m \otimes M \hookrightarrow mM$ injective

$a \otimes x \mapsto ax$

pf: (i) $\Rightarrow$ (ii). ✓

(ii) $\Rightarrow$ (iii)

$m \hookrightarrow A$ injection

$m \otimes M \hookrightarrow A \otimes M$ injection

$a \otimes x \mapsto a \otimes x$.

$A \otimes M \cong M$

$a \otimes x \mapsto ax$

$g: m \otimes M \hookrightarrow A \otimes M \to M$

thus g: injective inclusion.

(iii) $\Rightarrow$ (i)

$K = A/m$, $\forall N$, A-module

$N_K = N/mN = N \otimes_A K$,

M: F.G. let M_K is a finite K-V.S.

then $\{x_1, \dots, x_n\} \subset M$, st.

$\bar{x}_1, \dots, \bar{x}_n$ is a basis of M_K

Nakayama $\Rightarrow$ $M = \langle x_1, \dots, x_n \rangle$

~~$M = \langle x_1, \dots, x_n \rangle$~~

$$\begin{array}{ccccccccc} & & 0 & & & & 0 & & \\ & & \downarrow & & & & \downarrow & & \\ 0 \to & m\ker(\varphi) & \to & mA^m & \to & mM & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \ker(\varphi) & \hookrightarrow & A^m & \xrightarrow{\varphi} & M & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ (*) \quad 0 \to & \ker_K(\varphi) & \to & A^m_K & \to & M_K & \to 0 \\ & \downarrow & & \downarrow & \cong & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

Claim (*) exact.

$\Rightarrow \ker_K(\varphi) = 0 = \ker(\varphi) / m\ker(\varphi)$.

thus $m\ker(\varphi) = \ker(\varphi)$, Want $\ker(\varphi)$ F.G

(Bec A: Noeth $\Rightarrow$ A^m: Noeth)

$0 \to mE \xrightarrow{\phi} mA^n \xrightarrow{\varphi} mM \to 0$

know $\mathrm{Im}(\phi) \subseteq \ker(\varphi)$

Want $\mathrm{Im}(\phi) \supseteq \ker(\varphi)$.

$$\begin{array}{ccccccc} 0 \to & m\otimes E & \xrightarrow{m\otimes i} & m\otimes A^m & \xrightarrow{m\otimes\varphi} & m\otimes M & \to 0 \\ & \downarrow & \circlearrowleft & \downarrow g & \circlearrowleft & \swarrow & \\ & mE & \to & mA^m & \to & mM & \to 0 \end{array}$$

g: isomorphism.

$m\otimes A^n \cong (m\otimes A)^n \cong m^m = mA^n$

(See Lec 8-2 Later)

Lec 9-1 CA.

Theorem 1. $\bigcap_{i\geq 0} \mathfrak{a}^i M = 0 \Leftarrow$ A: Noeth, $\mathfrak{a} \subset \mathrm{Jac}(A)$, M: F.G. A-mod

Artin Rees:

$\mathfrak{a}^i M \cap M' = \mathfrak{a}^{i-k}(\mathfrak{a}^k M \cap M')$, $\exists k, \forall i \geq k$

$\Leftarrow$ A: Noeth, $\mathfrak{a}$: ideal, M: F.G

$M' \subset M$: submod.

TFAE.

(i) (M_i) stable

(ii) $\widetilde{M}$: F.G $\widetilde{A}$-mod

$\Leftarrow$ A: Noeth, $\mathfrak{a}$: ideal, $(M_i)_{i\geq 0}$: $\mathfrak{a}$-filtration

where $\widetilde{A} = \bigoplus_{i\geq 0} \mathfrak{a}^i$, $\widetilde{M} = \bigoplus_{i\geq 0} M_i$

Lem: $\widetilde{A}$ = Noeth $\Leftarrow$ A: Noeth

pf: $\mathfrak{a}: (x_1, \dots, x_r)$ f.g A-mod

$\widetilde{A}$: f.g as A-algebra by $x_1, \dots, x_r \in \mathfrak{a}^1$.

Pf Artin Rees:

$M_i = \mathfrak{a}^i M$, then $(M_i)_i$ stable $\mathfrak{a}$-filtration

$M_i' = \mathfrak{a}^i M \cap M'$, $\mathfrak{a}$-filtration

Then showing that (M_i') stable will imply Artin Rees.

then ~~we~~ (M_i') stable iff

$\widetilde{M}' = \bigoplus_{i\geq 0} M_i'$ is F.G. $\widetilde{A}$-mod.

$\widetilde{M} = \bigoplus_{i\geq 0} M_i$.

$\widetilde{M}' \subseteq \widetilde{M}$ submod as $\widetilde{A}$

(M_i): stable $\Leftrightarrow$ $\widetilde{M}$: f.g $\widetilde{A}$-mod

$\Downarrow$ $\widetilde{A}$: Noeth

$\widetilde{M}$: Noethe

$\Downarrow$

$\widetilde{M}'$: F.G. $\widetilde{A}$-mod.

Corollary 2 $\bigcap_{i\geq 0} \mathfrak{a}^i = 0$. $\begin{pmatrix} A: \text{Noeth} \\ \mathfrak{a} \in \mathrm{Jac}(A) \end{pmatrix}$

Corollary 3 A: Noeth, local with $\mathfrak{m}$

$\bigcap_{i\geq 0} \mathfrak{m}^i = 0$

Exercise: Derive Theorem 1 from Corollary 3.

Q: $\mathfrak{p} \in \mathrm{Spec}(A)$? $\mathrm{Ker}(A \to A_\mathfrak{p})$

Recall: n^{th} symbolic power $\mathfrak{p}^{(n)}$ $\underline{\text{def}}$

$\mathfrak{p}^{(n)} = \tau^*((\tau_*\mathfrak{p})^n)$. for $\tau: A \to A_\mathfrak{p}$.

$= \tau^*(\tau_*(\mathfrak{p}^n))$

(bc $\forall a, b$, $\tau_*(ab) = \tau_*(a)\tau_*(b)$

$\forall \tau: A \to S^{-1}A$.

$\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal

Corollary 4.

A: Noeth, $\mathfrak{p}$: prime.

$\Rightarrow \mathrm{ker}(A \to A_\mathfrak{p}) = \bigcap_{i\geq 0} \mathfrak{p}^{(i)}$

Pf: $\mathrm{ker}(A \to A_\mathfrak{p}) = \tau^*((0))$

$=$ by Cor 2 $= \tau^*(\bigcap_{i\geq 0} \cancel{\mathfrak{p}^i}) \bigcap_{i\geq 0} \tau_*(\mathfrak{p}^i)$

$= \bigcap_{i\geq 0} \tau^*(\cancel{\mathfrak{p}^i})(\tau_*(\mathfrak{p}^i))$

Note $(A_\mathfrak{p}, \tau_*(\mathfrak{p}))$ local, Noeth.

$= \bigcap_{i\geq 0} \mathfrak{p}^{(i)}$

Lem 5.

A: Noeth, local integral domain

(Special case of Krull Dimension Theorem)

$\underline{\text{NB}}$: $\underset{\text{prime}}{(0)} \subseteq \underset{\text{maximal}}{\mathfrak{m}} \subseteq A$

TFAE:

(i) $\exists \mathfrak{p} \subseteq A$ prime, $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$

(ii) $\forall f \in \mathfrak{m}$, $\exists \mathfrak{p}$: prime st. $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$ and $f \in \mathfrak{p}$.

Example: $A = k[[x, y]]$ ~~$= E$~~ with $\mathfrak{m} = (x, y)$

$(0) \subseteq \mathfrak{m}$. then $\exists \mathfrak{p} = (x)$ or (y)

$(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$

also then by $\underline{\text{Lem 5}}$, $\forall f \in \mathfrak{m}$, $\exists \mathfrak{p} \ni f$

st. $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$

Pf: (ii) $\Rightarrow$ (i) $0 \in \mathfrak{m}$, hence clear

~~(i) $\Rightarrow$ (ii)~~

$\underline{\text{Def}}$ $\dim(A) = \sup\{t \geq 0.\ \exists$ chain of primes $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_t \subseteq A\}$.

$\underline{\text{Def}}$. For $\mathfrak{p} \subsetneq A$ (prime)

$\underset{\text{height}}{\mathrm{ht}(\mathfrak{p})} = \sup\{t \geq 0 : \exists$ chain of primes $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_t = \mathfrak{p} \subseteq A\}$.

$\underline{\text{Def}}$ coheight.

$\mathrm{coht}(\mathfrak{p}) = \sup\{t \geq 0.\ \exists$ chain of primes $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_t \subseteq A\}$.

Lem 5' A: Noeth integral domain

$f \neq 0, \in A$, then

Then any $\underline{\text{minimal prime } \mathfrak{p}_0}$ of (f)

(least prime containing $\mathfrak{p}_0$,

$\mathfrak{p}_0 \in \mathrm{Ass}'((f))$)

satisfies $\mathrm{ht}(\mathfrak{p}_0) \leq 1$. / f non unit, $\mathrm{ht}(\mathfrak{p}) = 1$.

$\underline{\text{Lem}}$ $\mathrm{ht}(\mathfrak{p}) = \dim(A_\mathfrak{p})$

$\mathrm{coht}(\mathfrak{p}) = \dim(A/\mathfrak{p})$.

Lem 5 $\to$ Lem 5'.

assume A local with $\mathfrak{p}_0$ maximal

$A_{\mathfrak{p}_0}$: local, $\tau_*(\mathfrak{p}_0)$ maximal

replace f by $f/1 = \tau(f)$

Then $\{\text{prime } \mathfrak{p} \subseteq \mathfrak{p}_0\} \Longleftrightarrow \{\text{prime of } A_{\mathfrak{p}_0}\}$.

$\{\text{prime } \mathfrak{p} \ni f\} \Longleftrightarrow \{\text{prime}: \mathfrak{p} \ni f/1\}$

Let $(A, \mathfrak{p}_0)$: Noeth local domain.

$\mathfrak{m} = \mathfrak{p}_0$. Then $\mathfrak{m} \in \mathrm{Ass}'((f))$.

i.e. $\nexists$ prime $\mathfrak{p} \in f$ st. $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$

$\mathfrak{p} \ni f \neq 0$.

then $f \in \mathfrak{p} \subsetneq \mathfrak{m}$, then $\mathfrak{m}$ would not be a minimal prime of (f).

Want $\mathrm{ht}(\mathfrak{m}) = 1$.

$\Leftrightarrow (0) \subsetneq \mathfrak{m}$ ($\exists f \neq 0$, non unit)

$\nexists$ prime $\mathfrak{p}: (0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$. $\square$

(Lem 5 $\to$ Lem 5')

(i.e. Lem 5: Contrapositive of Lem 5')

Pf Lem(5) (i) $\Rightarrow$ (ii)

(ii) $\Leftrightarrow$ $\forall f$: $\dim(A/(f)) = 1$

$\begin{cases} m: \text{ maximal in } A/(f) \\ p \subsetneq m: \text{ prime ideal in } A/(f). \end{cases}$

$\neg$(ii) $\Leftrightarrow$ $A/(f)$: Artin (Noetherian A)

$\Rightarrow \exists k, \forall i \geq k, \ p^{(k)} + (f) = p^{(i)} + (f)$

$(p^{(k)} + (f))/(f)$ is indeed a descending chain in $A/(f)$.

$\neg$(ii) $\Rightarrow$ $\underset{\exists f \cdot p}{\underline{p: \text{prime}, f}}$ s.t. ~~$p \neq f$~~ $p \subseteq m$,

$\neg$(ii) $\Rightarrow$ $\exists f$, $\forall p$, prime if $((0) \subsetneq (p) \subsetneq m)$ fails then $p \not\ni f$.

(either $p \not\ni f$ or not $(0) \subsetneq (p) \subsetneq m$)

from (i), $(0) \subsetneq p \subsetneq m$, thus

want $f \notin p$ ~~...~~.

Know $f \notin p \subseteq m$ Want $p = (0)$

As above, $\exists k$, $\forall i \geq k$, $p^{(k)} \subseteq p^{(i)} + f$ (★)

Claim $p^{(k)} = p^{(i)} + f \cdot p^{(k)}$

Pf: "$\supseteq$" ✓

"$\subseteq$" Let $x \in p^{(k)}$, by (★)

$\exists y \in p^{(i)}, z \in A$, $x = y + fz$

$\Rightarrow z \in (p^{(k)} : f) = p^{(k)}$

bec $p^{(k)}$ is p-primary and $p \not\ni f$

Claim $\Rightarrow$ $\underset{\in m}{f}\, p^{(k)}/p^{(i)} = p^{(k)}/p^{(i)}$

by NAK, $p^{(k)}/p^{(i)} = 0$.

$\Rightarrow \forall i \geq k$, $p^{(i)} = p^{(k)}$.

$p^{(k)} = \bigcap_{i \geq 0} p^{(i)} = \ker(A \to A_p)$

$\qquad\qquad\qquad\qquad \| \quad A$: domain

$\qquad\qquad\qquad\quad \{0\}$

$p^k = \{0\} \underset{A:\text{ domain}}{\Longrightarrow} p = (0)$ as desired.

$\square$

Recall from Linear Algebra:

- V: V.S. on field of $\dim < \infty$

l : nonzero linear functional. $l: V \to k$.

then $\dim(\ker(l)) = \dim(V) - 1$

Krull's Dim Theorem is a variant for polynomials.

~~If V ∈~~

Corollary: $0 \neq (f) \in \mathbb{C}[x_1, \ldots, x_N]$

$Z(f)$: zero set of (f)

irreducible component of $Z(f)$ has dimension $N-1$.

$Z(f) \leftrightarrow$ primes $p \supset (f)$

$\cup$ irredu components $\leftrightarrow$ minimal primes $p \supseteq (f)$

"$\mathrm{Codim}_{\mathbb{C}^N}(X) = 1$" $\Leftrightarrow$ $ht(p) = 1$.

Lec 8-2 CA

- Krull Intersection Theorem

~~G~~Given: A: Noetherian, $\mathfrak{a} \subset \mathrm{Jac}(A)$, M: F.G A-mod

$\Rightarrow \bigcap_{i\geq 0} \mathfrak{a}^i M = \{0\}$.

- Artin Rees Lemma.

Cond: $\exists k, \forall i \geq k$, $\mathfrak{a}^i M \cap M' = \mathfrak{a}^{i-k}(\mathfrak{a}^k M \cap M')$

Given: A (Noeth) $\mathfrak{a} \subseteq A$ (ideal)

M (F.G A-mod), $M' \subseteq M$ (submod)

- Pf ~~Artin~~ Krull Intersection Theorem:

$M' = \bigcap_{i\geq 0} \mathfrak{a}^i M$

M: Noeth (A: Noeth + M: F.G. A-mod)

$M' \subseteq M$ submod, F.G. A-mod.

Want $\mathfrak{a}M' = M'$

then apply NAK $M' = \{0\}$.

It Remains to show $\mathfrak{a}M' \supseteq M'$.

Actually Artin Rees Shows both.

By Artin Rees,

$\exists k$ s.t. $\forall i \geq k$.

$\mathfrak{a}^i M \cap M' = \mathfrak{a}^{i-k}(\mathfrak{a}^k M \cap M')$

Then $\mathfrak{a}^{k+1} M \cap M' = \mathfrak{a}(\mathfrak{a}^k M \cap M')$

$M' = \mathfrak{a}M'$, as desired.

- Rmk: Pf of Artin Rees Lemma requires more definitions and results.

- Formal Power series. (Wikipedia)

A F.P.S is ~~an~~ a sequence over F (field) or over R (ring)

Define: (+) addition

$(a_0, a_1, \cdots) + (b_0, b_1, \cdots) = (a_0+b_0, a_1+b_1, \cdots)$

($\cdot$) multiplication

$(a_0, a_1, \cdots) \cdot (b_0, b_1, \cdots) = (a_0 b_0, a_0 b_1 + a_1 b_0, \cdots)$

thus we form a ring of F.P.S over a commutative ring R.

Write as $R[[x]]$.

Example:

$\mathbb{Z}[[x]]$, $(1+x) = (1, 1, 0, \cdots) = a$

$(1 - x + x^2 - x^3 \cdots) = (1, -1, 1, -1, \cdots) = b$

$a \cdot b = (1, 0, 0, 0, \cdots) = \mathbb{1}$

hence a, b are inverses of each other.

Rmk:

the multiplication is called Cauchy Product.

- An Example: Krull's Intersection Theorem Won't Work Generally.

K: field, $A = \bigcup_{n\geq 1} K[[x^{\frac{1}{n}}]]$?

This is called:

F.P.S with ~~rational~~ $\mathbb{Q}_{\geq 0}$ exponents.

Note, n=1, $K[[x]] \subseteq A$

A: local with $\mathfrak{m} := \{a \mid a = \sum_{i\in\mathbb{Q}_{\geq 0}} c_i x^i, c_0 \geq 0\}$

$K \cong A/\mathfrak{m}$

Now: $\mathfrak{m} = \mathrm{Jac}(A)$.

by a Corollary of Krull's Intersection Theorem

$\bigcap_{i\geq 0} \mathfrak{m}^i = \mathfrak{m}$ (*1).

show (*1).

$\mathfrak{m} = \{(x^a)_{a\in\mathbb{Q}>0}\}_K$

($\mathfrak{m}$: generated by the set over K)

$\forall i \geq 1$, $x^a = (x^{\frac{a}{i}})^i \in \mathfrak{m}^i$

thus $\bigcap_{i\geq 0} \mathfrak{m}^i = \mathfrak{m}$.

this gives a counterexample.

Def. I: monoid := set with binary operation
- Closed
- Associativity
- Identity

e.g. $(\mathbb{Z}_{\geq 0}, +)$

Def Graded Ring.

Notice: a ring can be graded by any monoid, but most frequently graded by $(\mathbb{Z}_{\geq 0}, +)$ so here use the def restricted to $(\mathbb{Z}_{\geq 0}, +)$.

A (ring),

$A = \bigoplus_{i\geq 0} A_i$ as Abelian Group

$A_i \cdot A_j \subseteq A_{i+j}$ with this multiplicative property.

Lem. $A_0 \subseteq A$ subring.

i.e. $1 \in A_0$.

let $1 \in A_i$.

$A_i \cdot A_j \subseteq A_{i+j}$, $\forall j$

thus $1 \cdot A_j \subseteq (A_j \cap A_{i+j})$, $\forall j$

either $A_j = \{0\}$, $\forall j$, then the ring A is trivial

OR. $A_j = A_{i+j}$, $\forall j$, and $i=0$

bec $A = \bigoplus_{i\geq 0} A_i$ direct sum

Ex $A = k[x_1, \dots, x_n]$

$A_i = \{$homogenous polynomials of degree $i\}$ (in the general sense)

$= \{\sum \boxtimes \cdot x_1^{a_1} \cdots x_n^{a_n}$ s.t. $\sum a_j = i\}$.

then $A = \bigoplus_{i\geq 0} A_i$ is graded

bec $A_i \cdot A_j \subseteq A_{i+j}$, $\forall i, j$.

Ex $A = k[x_1, \dots, x_n]$, $I = \bigoplus_{i=1}^{n} \mathbb{Z}$ (monoid)

$A_{(i_1, \dots, i_n)} = \{\boxtimes \cdot x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}\}$

$A = \bigoplus_{(i_1,\dots,i_n)\in I} A_{(i_1, \dots, i_n)}$ is graded.

$A_{(i_1,\dots,i_n)} \cdot A_{(j_1,\dots,j_n)} \subseteq A_{(i_1+j_1, \dots, i_n+j_n)}$

Def: $r_i \in A_i$, homogenous elements of degree i

Note: 0 is of degree n, $\forall n \geq 0$.

Def Graded Module M.

$A = \bigoplus_{i\geq 0} A_i$ graded, M: A-mod.

Graded IFF

$M = \bigoplus_{i\geq 0} M_i$ as Abelian Groups

$A_i M_j \subseteq M_{i+j}$ with this multiplicative property. (M_j: A_0-mod)

Rmk:

A(ring) as an A-mod is graded IFF

A is graded as a ring.

Def Graded Submod $M' \subseteq M$

$M' \subseteq M = \bigoplus_{i\geq 0} M_i$ (graded)

M' graded IFF

$M' = \bigoplus_{i\geq 0} M' \cap M_i$ as Abelian Groups.

Def Graded ideal.

A(ring), $\mathfrak{a}$ (ideal) Graded IFF

$\mathfrak{a}$ is a graded submod of the A-mod: A

i.e. $A = \bigoplus_{i\geq 0} A_i$ graded

$\mathfrak{a} = \bigoplus_{i\geq 0} \mathfrak{a} \cap A_i$.

Non Ex

$A = k[x, y]$, graded by $(\mathbb{Z}_{\geq 0}, +)$

$\mathfrak{a} = ((x^2 + y))$ is not graded.

bec $\mathfrak{a} \cap A_i = \{0\}$, $\forall i$.

Lem M: graded over A

$M = \bigoplus_{i\geq 0} M_i$, $A = \bigoplus_{i \geq 0} A_i$.

$M' \subseteq M$ submod, Graded IFF

M' generated by homogenous elements.

Further:

If M': F.G, M': graded, then

M' generated by finitely many

homogenous elements.

Pf: ① $M' \subseteq M$ graded.

$M' = \bigoplus_{i \geq 0} M' \cap M_i$

then M' generated by homogenous elements.

② Let $M' = \{(x_\lambda)_{\lambda \in \Lambda}\}_A$

let $x_\lambda = x_{i_\lambda}$ s.t. $x_{i_\lambda} \in M_i$.

Let $T = \bigoplus_{i \geq 0} M' \cap M_i$ as Abelian Groups

the sum is direct bec $(M_i)_i$ "independent".

$T \subseteq M'$ clearly.

$M' \subseteq T$ bec, $\forall m' \in M'$.

$m' = \sum a_\lambda \cdot x_\lambda$, $a_\lambda \in A$.

$a_\lambda = \sum a_{\lambda, j_\lambda}$ with $a_{\lambda, j_\lambda} \in A_j$.

$m' = \sum_{\lambda, j_\lambda} a_{\lambda, j_\lambda} x_\lambda$

$a_{\lambda, j_\lambda} \cdot x_\lambda = a_{\lambda, j_\lambda} \cdot x_{i_\lambda} \in M_{i+j} \cap M'$

thus $m' \in T$

③ M': F.G, graded.

$M' = (\{x_1, \dots, x_n\})_A$

$M' = \bigoplus_{i \geq 0} M' \cap M_i$

$x_i = \sum y_{i, k(i)}$, $y_{i,k(i)} \in M_k \cap M'$

$M' = (\{ (y_{i,k(i)})_{i,k} \})_A$

thus M': generated by finitely many

homogenous elements.

Def. $M' \subseteq M$, M', M: graded

$M = \bigoplus_{i \geq 0} M_i$, $A = \bigoplus_{i \geq 0} A_i$.

Naturally graded M/M':

$M/M' = \bigoplus_{i \geq 0} (M_i / (M_i \cap M'))$.

i.e. $\dfrac{\bigoplus_{i \geq 0} M_i}{\bigoplus_{i \geq 0} M_i \cap M'} \cong \bigoplus_{i \geq 0} \dfrac{M_i}{(M_i \cap M')}$

~~**Def** $\hat{A} = \bigoplus_{i \geq 0} \mathfrak{a}^i$, $\mathfrak{a} \leq A$ (ideal)~~

Cambridge – CA – Lec 10 – Filtration,

Graded Rings & Completions.

Def. $\mathbb{Z}$-filtration on A (ring):

$(A_i)_{i \in \mathbb{Z}}$, subgroups of A (Abelian)

$A_i \subset A_{i+1}$, $\forall i \in \mathbb{Z}$

$A_i A_j \subset A_{i+j}$, $\forall i, j \in \mathbb{Z}$

$1 \in A_0$

$A = \bigcup_{i \in \mathbb{Z}} A_i$

Call A: filtered ring.

$\mathbb{Z}$-filtration is

separated IFF $\bigcap_{i \in \mathbb{Z}} A_i = 0$

positive IFF $A_i = 0$, $\forall i < 0$

negative IFF $A_i = A$, $\forall i \geq 0$.

Rmk $A_0 \subseteq A$ subring.

$A_i \leq A_0$ ideal $\forall i < 0$

Examples.

(1) $\underset{\text{ideal}}{I} \subsetneq A$, $A_i = I^{-i}$ if $i < 0$, $A_i = A$ if $i \geq 0$.

This is a filtration, called:

I-adic filtration.

(2) $A = \mathbb{C}[x]$, $\lambda \in \mathbb{C}$, $A_i = (x - \lambda)^{-i} \mathbb{C}[x]$, $i < 0$

$A_i = \mathbb{C}[x]$ $i \geq 0$.

$A_i = \{$functions with a zero of order $\geq -i\}$

(3) $A = \mathbb{C}[X]$, $\lambda \in \mathbb{C}$.

? $A_i \sim$ {functions with a pole of order at most i at λ}.

(4) $A = \mathbb{Z}$, p: prime. $A = (p^{-i})$, $i<0$. $A_i = \mathbb{Z}$, $i \geq 0$

(5) $A = \mathbb{C}[x_1, \dots, x_n]$, $A_i = 0$, $i<0$.

A_i = span of all monomials of total degree at most i for $i \geq 0$.

Def. A: filtered, M: A-mod.

M-filtered IFF. $(M_i)_i$ Abelian Subgroup $\subseteq M$

s.t. $M_i \subseteq M_{i+1}$, $\forall i \in \mathbb{Z}$

$A_i M_j \subseteq M_{i+j}$, $\forall i,j \in \mathbb{Z}$

$\bigcup_{i \in \mathbb{Z}} M_i = M$

The filtration is

Separated IFF $\bigcap_{i \in \mathbb{Z}} M_i = \{0\}$.

Ex $M = (X)_A$ (generated by the set X)

$M_i = A_i X$, $A = \bigoplus_{i \in \mathbb{Z}} A_i$

$\bigoplus_{i \in \mathbb{Z}} (M_i)$ is a filtration of M.

In Particular, $A = (1)_A$ is a filtered A-mod.

Def. A filtration is good if $\exists m_1, \dots, m_r \in M$.

$k_1, \dots, k_r \in \mathbb{Z}$ s.t. $M_j = \sum_{i=1}^{n} A_{j-k_i} m_i$, $\forall j \in \mathbb{Z}$

Lem. $(M_n), (M'_n)$ good filtrations of M.

$\Rightarrow \exists n_0 \in \mathbb{N}$ s.t.

$M_{n-n_0} \subset M'_n \subset M_{n+n_0}$, $\forall n \in \mathbb{Z}$.

Pf: let $m_1, \dots, m_r \in M$, $k_1, \dots, k_r \in \mathbb{Z}$.

s.t. $\forall j \in \mathbb{Z}$, $M_j = \sum_{i=1}^{r} A_{j-k_i} m_i$

$\exists t_1, \dots, t_r$ s.t. $m_i \in M'_{k_i + t_i}$

(let $m_i \in M$, $m_i \in M'_j$, $\exists j$. take t_i large enough so that $M'_{k+t_i} \supseteq M'_j$)

let $t = \max(t_1, \dots, t_r)$

$m_i \in M'_{k_i+t}$, $\forall i$,

$M_n \subset M'_{n+t}$, $\forall n \in \mathbb{Z}$

~~$(M_n = \sum_{i=1}^{r} A_{n-k_i} m_i \subseteq \sum A_{n-k_i} M_{k_i+t}$~~

by ~~M_n~~ symmetry, $\exists t'$ s.t.

$M'_n \subset M_{n+t'}$, $\forall n \in \mathbb{Z}$.

let $n_0 = \max\{t, t'\}$. then

$M_{n-n_0} \subset M'_n \subset M_{n+n_0}$, $\forall n \in \mathbb{Z}$. ∎

Rmk. Analogy.

Any two norms defined on finite dim vector spaces are equivalent.

Prop. A: graded ring, $A_n = 0$, $\forall n > 0$.

then TFAE

(i) A: Noeth

(i') A_0: Noeth, A: F.G A_0-algebra.

Def. A: filtered. Rees ring of A

$\tilde{A} = \bigoplus_{i\in\mathbb{Z}} A_i \cdot t^i \subset A[t,t^{-1}]$

$\tilde{A}$ is graded.

If A is filtered I-adically.

~~i.e. $A = \bigoplus_{i\in\mathbb{Z}} \bigoplus_{i\le 0}$~~

i.e. $A_i = I^{-i}$ $(i\le 0)$ $A_i = A$ $(i\ge 0)$

$A = \bigcup_{i\in\mathbb{Z}} A_i$. $A_i \subseteq A_{i+1}$

$A_iA_j \subseteq A_{i+j}$. $1\in A_0$.

Then $\tilde{A} = B[t]$. $B = \bigoplus_{i\le 0} I^{-i}t^i$

Lem A: filtered I-adically.

A: Noeth. $\{x_1,\cdots,x_n\}$ generates I.

then B: Noeth, bec

$B_0 = A$, B is generated by $\{x_i t\}$ as an A-algebra.

$\tilde{A} = B[t]$ is Noetherian.

Def. A: filtered, M: filtered A-mod.

Rees Mod of M:

$\tilde{M} := \bigoplus_{i\in\mathbb{Z}} M_i t^i \subseteq M[t,t^{-1}]$

is graded, as an $\tilde{A}$-mod.

Rmk $a\in\tilde{A}$, $a = \sum_{i\in\mathbb{Z}} a_i t^i$ $a_i\in A_i$

$m\in\tilde{M}$, $m = \sum_{i\in\mathbb{Z}} m_i \cdot t^i$, $m_i\in M_i$

$a.m = \sum_{ij} a_i m_j t^{i+j} = \sum_{i,j} M'_{i+j} t^{i+j}$

Lem A: filtered, M: filtered A-mod

Then the filtration (of M) is good IFF

$\tilde{M}$: F.G. as an $\tilde{A}$-mod.

Lec 9-2

$\mathrm{Coht}(\mathfrak{a}) = \sup\{\mathrm{coht}(\mathfrak{p}),\ \mathfrak{p}\supseteq\mathfrak{a}\}$

Theorem $\Leftrightarrow$ A: Noeth, $a\in A$ (non zerodivisor/unit)

$ht((a)) = 1$

Pf Lem: A: Noeth, $(0) =$ decomposible $\bigcap \mathfrak{q}_i$ M.P.D.

Recall $\{$zero divisors in $A\} = \bigcup_i \mathfrak{p}_i$

Comp ($\bigcap_i \mathfrak{p}_i = \mathrm{Nil}(A)$)

$a\notin\mathfrak{p}_i$, $\forall i$.

Let i s.t. $\mathfrak{p}\supseteq\mathfrak{p}_i$

($\exists$ bec $\{\mathfrak{p}_i\}\supseteq\{$minimal primes of $A\}$)

$\left.\begin{array}{l} a\in\mathfrak{p} \\ a\notin\mathfrak{p}_i\end{array}\right\} \Rightarrow \mathfrak{p}_i\subsetneq\mathfrak{p} \Rightarrow ht(\mathfrak{p})\ge 1.$

Now want $ht(\mathfrak{p})\le 1$.

If $\exists$ longer chain s.t. $\mathfrak{p}''\subsetneq\mathfrak{p}'\subsetneq\mathfrak{p}$.

We may assume $\mathfrak{p}''$ minimal, and changing the index if necessary, that $\mathfrak{p}'' = \mathfrak{p}_i$

Now replace A by $A/\mathfrak{p}_i$,

$\mathfrak{p}'$ by $\mathfrak{p}'/\mathfrak{p}_i$,

$\mathfrak{p}$ by $\mathfrak{p}/\mathfrak{p}_i$,

a by $a+\mathfrak{p}_i$

~~Then $A/\mathfrak{p}_i$: Noeth Integral Domain.~~

Then A: Noeth, Integral Domain

$a\ne 0$, $a\in\mathfrak{p}$, $\mathfrak{p}\supseteq(a)$ minimal

$A/\mathfrak{p}_i$: Noeth, Integral Domain

~~a: $a\ne 0$~~, $a\in\mathfrak{p}$, $a\notin\mathfrak{p}_i$, $\mathfrak{p}/\mathfrak{p}_i\supseteq a+\mathfrak{p}_i$ minimal.

thus $ht(\mathfrak{p}/\mathfrak{p}_i)$ in $A/\mathfrak{p}_i = 1$.

Contradiction to $\mathfrak{p}''\subsetneq\mathfrak{p}'\subsetneq\mathfrak{p}$. $\square$

Geometric Interpretation.

$k = \bar{k}$, $A = k[x_1,\cdots,x_n]/\mathfrak{q}$, $\mathfrak{q}$: prime.

X: Irreducible variety in k^n $:=$

$V(\mathfrak{q}) = \{x : f(x) = 0\ \forall f\in\mathfrak{q}\}$

$\dim(A) \Leftrightarrow \dim(X) := \sup\{t\ge 0,\ \exists$ chain of irreducible subvarieties

$X = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_t\}$

prime in $A \longleftrightarrow$ (prime $\mathfrak{p}\supseteq\mathfrak{q}$) $\longleftrightarrow$

$Y\subseteq X$: irreducible Subvariety

The above correspondence: 1-1, inclusion reserving

~~preserving~~

$ht(\mathfrak{p}) \leftrightarrow \mathrm{codim}_X(Y) = \sup\{t\ge 0,\ \exists$ chain... $X = X_0\supsetneq\cdots\supsetneq X_t = Y\}$

---

e.g. $A = \bigoplus A_i$ (finite) $\pi_i: A\to A_i$

$\mathfrak{p}_i = \mathfrak{q}_i = \ker(\pi_i) = (x,x,\cdots,0,x,\cdots)$ i th.

$(0) = \bigcap_{i=1}^{n}\mathfrak{q}_i$

Lec 9-2

Theorem 1. (Krull P.I Theorem)

R: Noeth, $a \in R$ (not unit / zero divisor)

~~Thm~~. $\mathfrak{p} \in \mathrm{Ass}^1((a))$ i.e. ($\mathfrak{p} \supseteq (a)$ minimal)

then $ht(\mathfrak{p}) = 1$

$\mathfrak{a} \subseteq A$, any ideal. $\Longleftrightarrow$ $Z \subseteq X$, closed subvariety.
$r(\mathfrak{a}) = \mathfrak{a}$.

($\Rightarrow$ $\mathfrak{q}_i = \mathfrak{p}_i$ in MPD)

$\mathfrak{p}_i \in \mathrm{Ass}^1(\mathfrak{a})$ $\Longleftrightarrow$ irreducible components $V(\mathfrak{p}_i) = V(\mathfrak{q}_i)$
($\mathfrak{p}_i \supseteq \mathfrak{a}$ minimal) $\quad Y_i \subseteq Z \quad Y_i =$

$\mathrm{Codim}_X(Z) = \inf_{Y_i:\text{ irreducible components}} \mathrm{codim}_X(Y_i)$

$\mathrm{Coht}(\mathfrak{p}) = \dim(A/\mathfrak{p}) \Longleftrightarrow \dim(Y)$

$\mathrm{coht}(\mathfrak{a}) = \dim(A/\mathfrak{a}) = \sup_i \mathrm{coht}(\mathfrak{p}_i)$

$\Longleftrightarrow \dim Z = \sup_i \dim(Y_i)$

Krull ~~Dim~~ P.I. Theorem

$\Rightarrow$ Irreducible Component of a hypersurface in X has codim 1.

$\downarrow$

$a \in k[x_1, \dots, x_n]$, ~~$V((a)) = I$~~ $V((a))$

$ht(\mathfrak{a}) = ht((a)) = \mathrm{codim}_X(Z) = 1$

$\underset{\text{minimal}}{\mathfrak{p}_i \supseteq \mathfrak{a}}$, $ht(\mathfrak{p}) = 1 \Longleftrightarrow \mathrm{codim}_X(Y_i) = 1$

- Krull Dimension Theorem

R: Noeth. $\underset{\text{(ideal)}}{\mathfrak{a} \subseteq R}$ generated by $(a_1, \dots, a_r)$ $(r \geq 1)$

then $ht(\mathfrak{p}) \leq r$ if $\mathfrak{p}$: minimal prime divisor of ~~(a)~~ $\mathfrak{a}$

pf: Induction on $r \geq 1$.

$\exists\, \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_t = \mathfrak{p}$.

Want $t \leq r$.

Replace A by $A_{\mathfrak{p}}$.

Reduced to the case $(A, \mathfrak{p})$ Noeth local ~~with~~

$\Rightarrow$ $\left.\begin{array}{l}\mathfrak{p}\text{ - maximal in } A \\ \text{minimal prime} \supseteq \mathfrak{a}\end{array}\right\} \Rightarrow$ $\mathfrak{p}$ is the only prime $\supseteq \mathfrak{a}$.

$\mathfrak{p}_t = \mathfrak{p} \underset{\text{minimal}}{\supseteq} \mathfrak{a} \Rightarrow \mathfrak{p}_{t-1} \not\supseteq \mathfrak{a}$.

$\Rightarrow$ $\exists$ generator of $\mathfrak{a}$ not in $\mathfrak{p}_{t-1}$

suppose $a_r \notin \mathfrak{p}_{t-1}$

~~Thus $(a_r) + \mathfrak{p}_{t-1} \subsetneq \mathfrak{p}_t = \mathfrak{p}$~~

We may assume by enlarging the chain if necessary, that there are no prime between $\mathfrak{p}_{t-1}$, $\mathfrak{p}_t$

(A: Noeth $\Rightarrow$ $\exists$ maximal prime $\mathfrak{q}$.
s.t. $\mathfrak{p}_{t-1} \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_t$, then add $\mathfrak{q}$ to our chain)

then $\underset{\mathfrak{p}_{t-1} \subsetneq}{\mathfrak{p}_{t-1} + (a_r)} \subseteq \mathfrak{p}_t$

Thus $\mathfrak{p} = \mathfrak{p}_t$ the only prime containing $\mathfrak{p}_{t-1} + (a_r)$

$\mathrm{rad}(\mathfrak{p}_{t-1} + (a_r)) = \mathfrak{p} \supseteq \mathfrak{a} \ni a_i$, $\forall i = 1, \dots, r-1$

$\exists N \geq 1$, s.t. $a_i^N \underset{(*2)}{=} a_i' + a_r y_i \in \mathfrak{p}_{t-1} + (a_r)$

Def $\mathfrak{a}' = (a_1', \dots, a_{r-1}') \subseteq \mathfrak{p}_{t-1}$.

Want $\mathfrak{p}_{t-1} \in \mathrm{Ass}^1(\mathfrak{a}')$

If this true, then by induction.

$t-1 \leq r-1$

hence $t \leq r$.

Let $\mathfrak{p}' \in \mathrm{Ass}^1(\mathfrak{a}')$ s.t. $\mathfrak{p}' \subseteq \underbrace{\mathfrak{p}_{t-1} \subsetneq \mathfrak{p}_t}_{\text{no primes between}} = \mathfrak{p}$
(such $\mathfrak{p}'$ exists)

To show that $\mathfrak{p}' = \mathfrak{p}_{t-1}$.

TSTS $ht(\underset{\subseteq A/\mathfrak{p}'}{\mathfrak{p}/\mathfrak{p}'}) \leq 1$

$\overline{a_r} :=$ Image of $a_r \in A/p'$

by Krull P.I. ISTS $p/p' \in \mathrm{Ass}'(\overline{a_r})$

$\Rightarrow$ $\Updownarrow$

$p = \mathrm{rad}(p' + (a_r))$ (*1)

show (*1). $p = \mathrm{rad}(a) = \mathrm{rad}(a' + (a_r))$ by (*2)

$\subseteq \mathrm{rad}(p' + (a_r)) \subseteq p$.

$\Rightarrow p = \mathrm{rad}(p' + (a_r))$ hence (*1)

Corollary 3. A: Noeth. $a \subseteq A$ ideal.

$ht(a) < \infty$.

$ht(A) < \infty$

Corollary 4 (A, m): Noeth, local.

$k = A/m$: field.

Note: $\forall$ A-mod: M, M/mA is a k-v.s.

then $\dim(A) \le \dim_k(m/m^2)$

pf: let $a_1, \cdots a_r$ is a basis of m/m^2 ($r = \dim_k(m/m^2)$)

Let $\widehat{a_1}, \cdots, \widehat{a_r} \in m$ be lifts of a_i.

NAK $\Longrightarrow m = (\widehat{a_1}, \cdots, \widehat{a_r})$

pf: $M = m$, $N = M/(\widehat{a_1}, \cdots, \widehat{a_r})$

if $mN = N$. then $N = 0$

$M = (\widehat{a_1}, \cdots, \widehat{a_r})$

$\Rightarrow ht(m) \le r$.

$\dim(A) = ht(m) \le r$. (by A: local)

· Corollary 5.

A: Noeth, $a \subseteq A$ ideal $r = ht(a)$

then $\exists a_1, \cdots, a_r \in A$ s.t. ~~$a = \ldots$~~

$ht(a) = ht(a_1, \ldots, a_r) = r$

pf: ISTS by induction

For $s \le r$, if we can find $a_1, \cdots, a_{s-1} \in a$ ~~$a_1, \ldots, a_{s-1}$~~

with $ht((a_1, \cdots, a_{s-1})) = s-1$.

then $\exists a_s \in a$ s.t. $ht((a_1, \cdots, a_s)) = s$

Let $\underbrace{(a_1, \cdots, a_{s-1})}_{b}$ with MPD $= \bigcap_{i=1}^{n} q_i$

$ht(b) = s-1$, $p_i = \mathrm{rad}(q_i)$

$ht(b) = s-1$

ISTS is $a \not\subseteq \bigcup p_i$

(b.c. $\forall a_s \in \bigcup p_i - a$ will give.

$ht((a_1, \cdots, a_s)) \le s$ by Krull

$\ge s$ by M.P.D.

⟨Lec 10-1⟩

· Theorem.

Converse To Krull's Dimension Theorem

Lemma 10 (Bosch).

(i) A: Noeth, $a \subseteq A$ ideal, $ht(a) = r$.

~~$\exists a_1, \cdots, a_{s-1}$ s.t. $ht(a)$~~

$\exists a_1, \cdots, a_{s-1}$ s.t. $ht(a_1, \cdots, a_{s-1}) = s-1$, $s \le r$.

then $\exists a_s \in A$, s.t. $ht(a_1, \cdots, a_s) = s$.

OR Lec: $\forall s \in 1, \cdots, r$, $\exists x_1, \cdots, x_s \in a$ s.t.

$ht(x_1, \ldots, x_s) = s$.

(ii) $p_0 \subsetneq \cdots \subsetneq p_r$, $\exists x_1, \cdots, x_r \in p_r$.

s.t. $p_i \supseteq (x_1, \cdots, x_i)$ minimal.

(iii) $\forall p$: prime, ~~&~~ $ht(p) = r$ is a minimal prime of some ideal $(a_1, \cdots, a_r)$.

(i) $\Rightarrow$ (ii)

pf: (i)

$ht(a) = r$, $\Rightarrow \exists$ chain $p_0 \subsetneq \cdots \subsetneq p_s$,

$p_i \supseteq a$.

Induct on s, $x_1, \cdots, x_s$ s.t.

$p_i \supseteq (x_1, \cdots, x_s)$ minimal $\forall i \le s$.

then $ht(p_i) \le i$ (by Theorem 0), $ht(p_i) \ge i$ (by the existing ch[ain])

Initiation, $s = 1$.

Consider the minimal primes $\{q_j\} = \mathrm{Ass}'(x_1, \cdots x_i)$

Claim $p_{i+1} \not\subseteq \bigcup_{j=1} q_j$

If not, q_j: prime, $p_{i+1} \subseteq q_j$, $\exists j$.

then $ht(q_j) \le i$ by Krull, $ht(p_{i+1}) \ge i+1$.

Contradiction.

Let $x_{i+1} \in \mathfrak{p}_{i+1} - \bigcup_{j=1}^{i} q_j$

Then $\mathfrak{p}_{i+1} \supseteq (x_1, \dots, x_{i+1})$

Want $\mathfrak{p}_{i+1} \in \mathrm{Ass}((x_1, \dots, x_{i+1}))$

Corollary:

(A, m) Noeth, local, $\dim(A) = ht(m) < \infty$

$\exists\, x_1, \dots, x_n \in m$, s.t. m is minimal over $(x_1, \dots, x_n)$.

$m = rad(x_1, \dots x_n)$ and $(x_1, \dots, x_n)$ is m-primary.

Recall, any ideal whose radical is maximal is necessarily a primary ideal.

Def. $d = \dim(R)$, (R, m) Noeth local.

$(x_1, \dots, x_d)$: m-primary.

then $(x_1, \dots, x_n)$: system of parameters.

IFF 1/3 (TFAE) holds:

(i) $m \supseteq (x_1, \dots, x_n)$ minimal

(ii) $m = rad((x_1, \dots, x_n))$

(iii) $(x_1, \dots, x_n)$: m-primary.

Theorem. (A, m) Noeth, local.

Let $x_1, \dots x_r \in m$, TFAE

(i) $x_1, \dots, x_r$ can be extended to the system of parameters of m.

(ii) $\dim(A/(x_1, \dots, x_r)) = \dim(A) - r$.

(iii) $ht((x_1, \dots, x_r)) = r$

Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

(i) $\Leftrightarrow$ (ii).

$y_1, \dots, y_s \in m$. ~~$\bar{A} = A/(y_1, \dots)$~~

$\bar{A} = A/(x_1, \dots, x_r)$. $\bar{m} \subseteq \bar{A}$: image of m.

$(\bar{A}, \bar{m})$: Noeth, local.

$\bar{y}_1, \dots, \bar{y}_s \in \bar{A}$.

How can $\{x_1, \dots, x_r, y_1, \dots, y_s\}$ form a system of parameters.

$\Updownarrow$

$r+s = \dim(A)$, $(x_1, \dots, x_r, y_1, \dots, y_s)$: m-prim

$\Updownarrow$

$r+s = \dim(A)$, $r(x_1, \dots, x_r, y_1, \dots, y_s) = m$

$\Updownarrow$ OR m: only prime containg $(x_1, \dots, y_s)$

$r+s = \dim(A)$, $\bar{m}$: only prime cntng $(\bar{y}_1, \dots, \bar{y}_s)$

$\Updownarrow$

$r+s = \dim(A)$, ~~$r(\bar{y}_1, \dots, \bar{y}_s)$~~ $(\bar{y}_1, \dots, \bar{y}_s)$ m-primary.

- $\{\bar{y}_1, \dots, \bar{y}_s\}$ system of generators of $(\bar{A}, \bar{m})$.

$\Updownarrow$

$s = \dim(\bar{A})$, $(\bar{y}_1, \dots, \bar{y}_s)$: $\bar{m}$-primary.

- $\exists\, y_1, \dots, y_s$ s.t. $(x_1, \dots y_s)$ m-primary

$\Downarrow$

$\dim(A) \le r+s$

- $\exists\, \bar{y}_1, \dots \bar{y}_s$ s.t. $(\bar{y}_1, \dots, \bar{y}_s)$: $\bar{m}$-primary,

$\Rightarrow \dim(\bar{A}) \le s$.

(i) $\Rightarrow$. $\exists\, y_1, \dots, y_s$ s.t. $(x_1, \dots, y_s)$ system of gen

$r+s = \dim(A)$ ~~$\Rightarrow d$~~

Suppose $\exists\, t < s$, $z_1, \dots, z_t \in A$

$(\bar{z}_1, \dots, \bar{z}_t)$ m-primary. (this happens if $\dim(\bar{A}) < s$)

Then $(x_1, \dots, x_r, z_1, \dots z_t)$ = m-primary.

$\Rightarrow \dim(A) \le r+t < r+s$. contradi.

$\square$

(ii) $\Rightarrow$. $s := \dim(A) - r \overset{(ii)}{=} \dim(\bar{A})$

$\exists\, y_1, \dots y_s \in A$ s.t. $\{\bar{y}_1, \dots \bar{y}_s\}$ is a system of parameters. $\Rightarrow \{\bar{y}_1, \dots, \bar{y}_s\}$ $\bar{m}$-primary.

$\Rightarrow (x_1, \dots y_s) = m$-primary

and $s + r = \dim(A)$.

thus $(x_1, \dots, x_s)$ is extendable.

CTD Lec 10-1.

Col: (A,m) Noeth, local.

$a\in A$: nonzero divisor.

then $\dim(A/(a)) = \dim(A) - 1$.

$ht((a)) = 1$, P.I. theorem. Krull.

by the theorem above:

$\dim(A/(a)) = \dim(A) - 1$

(bec any (a) is contained in m) ∎

- A: Noeth $\Rightarrow \dim A[x_1,\dots,x_n] = \dim(A) + n$

Pf: we may assume $n=1$ and iterate with A replaced by $A[x_1,\dots,x_{n-1}]$ to show for $A[x_1,\dots,x_n]$.

① $\dim A[x] \geq \dim A + 1$.

$p_0 \subsetneq \dots \subsetneq p_n$ in A $n = \dim(A)$

Consider. $p_0 A[x] \subsetneq \dots \subsetneq p_n A[x] \subsetneq T$

$p_i A[x]$ prime bec $A[x]/p_i A[x] \cong [A/p_i][x]$ integral doma.

$T = \underbrace{p_n A[x] + x\cdot A[x]}$.

prime bec $A[x]/T \cong A/p_n$

② $\dim A[x] \leq \dim A + 1$ ⟨Lec 10-2⟩

$\dim A[x] = \sup_{m:\text{maximal}} ht(m)$.

ISTS: $\forall m$: maximal, $ht(m) \leq r+1$

where $r = \dim A$.

We may assume $r < \infty$.

Consider $\cancel{m \cap A}$ $\subseteq A[x]$, prime in A $p = m\cap A$

Localize at p: $S = S_p = A - p$

$S^{-1}A = A_p$: local with maximal ideal pA_p

$S^{-1}(A[x]) \cong (S^{-1}A)[x] = A_p[x]$.

Then $S^{-1}m \subseteq S^{-1}A[x]$ remains a maximal ideal and $ht(S^{-1}m) = ht(m)$

bec localization preserves primes & inclusions.

We may assume (A,p): Noetherian local ring, $m \subseteq A[x]$: maximal, $m\cap A = p$

$r = \dim(A) < \infty$

Want $ht(m) \leq r+1$.

ISTS: construct $\mathfrak{a} = (a_1,\dots,a_{r+1})_{A[x]}$.

$r(\mathfrak{a}) = m$.

Know $r = \dim(A) = ht(p)$

then $\exists x_1,\dots,x_r \in A$ st. p is the only prime of A containing $(x_1,\dots,x_r)$.

i.e. $\mathrm{rad}((x_1,\dots,x_r)) = p$.

Consider: $A[x] \longrightarrow A[x]/pA[x] \cong \underbrace{(A/p)[x]}_{\text{P.I.D.}}$

Note: $m \supseteq pA[x]$

$m \longmapsto \overline{m}$: maximal.

then $\overline{m} = (\overline{f})$, for some $\overline{f} \in (A/p)[x]$

then let $f\in m$, $f \mapsto \overline{f}$ by quoting $pA[x]$.

<u>Claim:</u> m the only prime q that contains $x_1,\dots,x_r, f$

Indeed, $q\cap A$ is a prime containing $x_1,\dots,x_r$

$q\cap A = p$.

$q \supseteq pA[x]$, then $\overline{q} \subseteq A[x]/pA[x]$ contains $\overline{f}$.

hence $\overline{q} = \overline{m}$ hence $q = m$. ∎

- One other example of dimension computation

$k[[x,y]] := \{$formal power series in x,y over $k\}$

$= \{\sum_{i,j\geq 0} a_{ij} x^i y^j\}$

$k[[x,y]]$ Noeth local ring with (x,y) maximal

let $A = k[[x,y]]/(x^2, xy)$. $x=0$

$\dim A = 1$

$A/(x) \cong k[[y]]$.

$A/(x,y) \cong k$

both k, $k[[y]]$ integral domains.

$(x) \subsetneq (x,y)$ chain of primes.

$\dim(A) \geq 1$

$A \supseteq m = (x,y)$, i.e. m is the $x=\overline{x}, y=\overline{y}$

$m = (\overline{x},\overline{y})_A = (x,y)_A$

Claim $\mathrm{rad}((y)) = m$.

Pf: $x^2 = 0 \in (y)$. $y^2 \in (y)$

$\Rightarrow m \subseteq \mathrm{rad}((y)) \Rightarrow m = \mathrm{rad}((y))$

hence $\dim A \leq 1$.

- Lem. $k = \bar{k}$. (say $k = \mathbb{C}$)

$A = k[x_1, \dots, x_n]$

Let $m \subseteq A$ be a maximal ideal.

then $m = (x_1 - x_1, \dots, x_n - x_n)$

for $(x_1, \dots, x_n) \in k^n$.

then $ht(m) = n$.

and A_m is a local ring of dim n with mA_m has ~~exactly~~ n generators.

Pf: mA_m has n generators.

$\dim(A_m) = ht(mA_m) = ht(m) \le \dim(A) = n$

$ht(m) \ge n$ bec

$m = p_n \supsetneq \cdots \supsetneq p_0$, $p_i = (x_1 - x_1, \dots, x_i - x_i)$

- (A, m): Noeth local. $d = \dim(A) = ht(m)$

Lem: ① $k = A/m$, $d \le \dim_{A/m}(m/m^2)$

② TFAE.

(i) m: $m = (x_1, \dots, x_d)$

(ii) $d = \dim_{A/m}(m/m^2)$

- Def: If (i)/(ii) hold, then (A) regular.

EX: $k[x_1, \dots, x_n]$ is regular.

Pf:

(i) $\Rightarrow$ (ii).

$m = (x_1, \dots, x_d)$, then $m/m^2 = (\bar{x}_1, \dots, \bar{x}_d)$

so $\dim_{A/m}(m/m^2) \le d$.

(ii) $\Rightarrow$ (i).

same proof to ①.

- Motivation.

Show: $\forall A$: integral domain A: F.G as algebra over $k \subseteq A$, k: field.

then $\dim(A) = tr.deg_k(\mathrm{Frac}(A))$

EX: $A = k[x_1, \dots, x_n]$

this already true.

Want: Connect to $k[x_1, \dots, x_n]$, this requires more intro.

- Integral Extension

$x^n + a_1 x^{n-1} + \cdots + a_n = 0$. $(*)$

algebraic if $(*)$ satisfied. (Generalize To Rings)

Def: $A \subseteq B$, rings, $x \in B$ integral over A.

if $\exists n \ge 1$, $a_1, \dots, a_n \in A$. s.t. $(*)$ holds

- B: integral over A if $\forall x \in B$ is integral over A.

Note: x, y: integral over A then $x \pm y$, xy integral over A.

- Key Lem $A \subseteq B$ rings TFAE. $\forall x \in B$

① x: integral over A

② $A[x]$ is a F.G. A-mod.
i.e. $\exists e_1, \dots, e_n \in A[x]$. st. $A[x] = \sum A e_i$

③ $\exists$ Subring. C st. $A[x] \subseteq C \subseteq B$.
C: F.G A-mod

④ $\exists$ faithful $A[x]$-mod M which is F.G. as an A-mod.

Faithful: if $y \in A[x]$, s.t. $ym = 0$ $\forall m \in M$ then $y = 0$.

EX: $\frac{1}{2} \in \mathbb{Q}$ is not integral over $\mathbb{Z}$.

bec $\mathbb{Z}(\frac{1}{2})$ is not F.G. $\mathbb{Z}$-mod.

Scanned by CamScanner

Pf: (i)⇒(ii') if x has (*).

$A[x] = \sum_{i=1}^{n-1} \cancel{Ax} \, Ax^i$

(ii')⇒(iii) $C = A[x]$.

(iii)⇒(iv) : $M := C$

(iv)⇒(i) : $M = \sum_{i=1}^{n} Ae_i$, $e_i \in M$

M : A-mod, F.G.

~~?~~ M: faithful.

$y \in A[x]$. let $M = \langle e_1, \dots, e_n \rangle$.

~~Want if~~

Want : x: integral over A.

$\exists a_{ij}$, $x e_1 = a_{11} e_1 \dots$ $\quad x e_j \in M$.

$\vdots$

$x e_n = a_{n1} e_1 \dots$

$\sum_j (\delta_{ij} x - a_{ij}) e_j = 0$, $\forall i$.

$\Delta_{ij} = \delta_{ij} x - a_{ij}$.

$e = (e_1, \dots, e_n)^T \in M^n$

$\Delta \cdot e = (0, \dots, 0)^T \in M^n$.

$\Delta^{ad} \cdot \Delta \cdot e = (0, \dots, 0)^T \in M^n$

$\Delta^{ad} \cdot \Delta = \begin{pmatrix} \det(\Delta) & & \\ & \ddots & \\ & & \det(\Delta) \end{pmatrix}$

$\Rightarrow (\det(\Delta) e_1, \dots, \det(\Delta) e_n) = (0, \dots, 0)$

$\det(\Delta) m = 0$, $\forall m \in M$

thus $\det(\Delta) = 0 \in A[x]$.

but $\det(\Delta) = x^n + \dots$

thus x: integral over A.

□

Exercise Class Week 11.

• $R = \mathbb{C}[x, y, z]/(xy, xz)$

$m_1 = (x-1, y, z)$, $m_2 (x, y-1, z)$

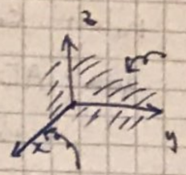
$ht(m_1) = ?$, $ht(m_2) =$

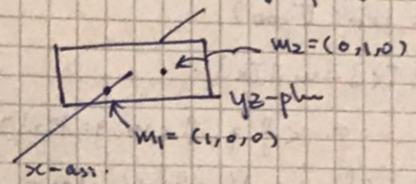
R (xy, xz)

$x = 0$ / $y = z = 0$.

$\Downarrow$

x-axis & yz-plane.





the picture suggests $ht(m_1) = 1$, $ht(m_2) = 2$.

Pf: ① $ht(m_1) = \dim R_{m_1}$.

In R_{m_1}, ~~$\bar{y} = \bar{z} = \bar{0}$~~ $\bar{y} = \bar{z} = 0$.

$\overline{x-1} \in m_1$, $x \notin m_1$

$\Rightarrow \bar{x} \in R_{m_1}^{\times}$, $\overline{xy} = 0$, $\overline{xz} = 0$ $\Rightarrow$ $\bar{y} = 0$, $\bar{z} = 0$. in R_{m_1}

thus $R_{m_1} \cong \mathbb{C}[T]_{(T-1)}$

$ht(m_1) = \dim R_{m_1} = \dim \mathbb{C}[T]_{(T-1)}$

$= ht(T-1)$ in $\mathbb{C}[T] = 1$

② Omitted.

• A: general ring, $\dim A + 1 \le \dim A[x] \le 2 \dim A + 1$

Pf: $\exists A$: $\dim A = 1$, $\dim A[x] = 3$.

• K: field, π: transcendental over K.

$A = \{ f \in K(\pi)[[y]] : f(0) \in K \}$.

i.e. $A = \{ a_0 + \sum_{i \ge 1} a_i y^i \mid a_i \in K(\pi)_{i \ge 1}, a_0 \in K \}$.

A : not Noeth.

$\operatorname{Spec} A = \{ (0), (\{ \sum_{i \ge 1} a_i y^i, a_i \in K(\pi) \}) \}$.

$= \{(0)\} \cup (\{ f \in K(\pi)[[y]], f(0) = 0 \})$

$\| $

m (maximal in A).

$A[x]$, (0) is prime.

take m, $m[x]$, $A[x]/m[x] \cong (A/m)[x]$ (not field)

thus $m[x]$ is not maximal in $A[x]$,

$(0) \subsetneq \dots \subsetneq \underset{\text{(prime)}}{m[x]} \subsetneq m_{A[x]}$

Consider $A[x] \xrightarrow{V_\pi} A$, by evaluation $x \mapsto \pi$

$k_0 = \ker(V_\pi)$, $k_0 \in \operatorname{spec} A[x]$.

Because $A[x]/k_0 \cong A$. (integral domain)

Want $k_0 \subseteq m[x]$

$V_{y=0}: A[x] \longmapsto \{g \in k(\pi)[x] : g(0) \in K\}$

that send $y \mapsto 0$

$m[x] = \ker(\pi_{y=0})$

$m[x] = mA[x]$.

$= \langle f \cdot g .\ f \in m, g \in A[x] \rangle$

$V_{y=0}(f \cdot g) = \underbrace{V_{y=0}(f)}_{0} \cdot V_{y=0}(g) = 0$.

thus $m[x] \subseteq \ker(V_{y=0})$

Now. $g \in \ker(\pi_{y=0})$

$g = \sum_{i=1}^{k} f_i(y) x^i$

$f_i(y) = \underset{\underset{K}{\uparrow}}{a_{0,i}} + \sum_{j=1}^{\infty} \underset{\underset{k(\pi)}{\uparrow}}{a_{j,i}\, y^j}$

Thus $0 = \sum_{i=0}^{k} a_{0,i}\, x^i$.

thus $a_{0,i} = 0$, $\forall i$.

thus $g = \sum_{i=1}^{k} f_i(y) x^i$

$f_i(y) = \sum_{j=1}^{\infty} a_{j,i}\, y^j$

Hence. $g \in m[x]$

Let $g \in \ker(~~\pi~~\ V_\pi) = k_0$.

$V_{y=0}(V_\pi(g)) = 0$ ~~$= V_\pi(V_{y=0}(g))$~~

$V_\pi(V_{y=0}(g)) = V_\pi(\sum_{i=1}^{k} a_{i,0} x^i) = \sum_{i=1}^{k} a_{i,0} \pi^i$ $(*1)$

$g = \sum_{i=1}^{k} f_i(y) x^i$, $f_i(y) = \cdots$

Thus $V_\pi(V_{y=0}(g)) = V_{y=0}(V_\pi(g)) = 0$

hence $a_{i,0} = 0$, $\forall i$

thus $V_{y=0}(g) = 0$ (see $*1$)

$g \in m[x]$. hence $k_0 \subseteq m[x]$.

$0 \subseteq k_0 \subseteq m[x] \subsetneq m$.

Prove strict inclusion.

① $V_\pi(y\pi - yx) = y\pi - \pi y = 0$.

② $y \in m[x]$, $y \notin k_0$

Check $f \in A^* \Leftrightarrow f(0) \neq 0$.

$m = \langle \pi^k y ; k \in \mathbb{Z} \rangle$.

Lec 10-1 ~~Contents.~~

- ~~"Converse" to Krull Dimension Theorem.~~ ~~(Bosch P77~~

Lec 9-2. ~~Contents.~~

- ~~R: Noeth, $\mathfrak{a} \subseteq R$ ideal,~~ $\mathfrak{a} = (a_1, \cdots, a_r)$,
~~$\forall \mathfrak{p} \in \mathrm{Ass}'(\mathfrak{a})$,~~ $ht(\mathfrak{p}) \leq r$. ~~(Atiyah, Bosch)~~

Lec 11-1.

~~$A \subseteq B \subseteq C$~~

Lem[2]. $A \subseteq B \subseteq C$, x: integral over A
$\Rightarrow$ x: integral over B. $\square$

Lem[3]. $A \subseteq B \subseteq C$. C: finite B-mod.
B: finite A-mod

$\Rightarrow$ C: finite A-mod.

pf: $B[X_1, \cdots, X_n] \twoheadrightarrow C$ surj
$A[y_1, \cdots, y_m] \to B$. surj
$(A[y_1, \cdots, y_m])[x_1, \cdots, x_n]) = A[z_1, \cdots, z_{m+n}) \to C$ surj.

$\square$

Lem[4]. $A \subseteq B$: rings. $x_1, \cdots, x_n \in B$, integral over A.
then ① $A[x_1, \cdots, x_n]$: finite over A. (F-G. A-mod)
~~$\forall x_1, \cdots, x_n$, x_i: integral over A.~~
② $A[x_1, \cdots, x_n]$ is ~~fini~~ integral over A.

Lem^5. $A \subseteq B$, TFAE.

(i) B: integral over A, F.G as A-alge

(ii) B: finite as A-mod, (F.G.)

Pf: (i) $\Rightarrow$ $B = A[x_1, \cdots, x_n]$ for some $x_j \in B$.

$\Rightarrow$ by Lem^4, B: F.G. A-mod.

(ii) $\Rightarrow$ (i) by Lem^1, (iii) $\Rightarrow$ (i)

Lem^6. $A \subseteq B$ rings, $\bar{A} = \{x \in B : x \text{ integral over } A\}$.

$\bar{A}$: A-subalgebra of B.

Pf: If $x, y \in B$ are integral over A, then

Lem^4 $\Rightarrow$ $A[x,y]$: ~~f.~~ F.G A-mod, integral over A.

So xy, $a_1x + a_2y$ $\in A[x,y]$ are integral over A.

$\forall a_1, a_2 \in A$

Thus $\bar{A}$ = A-subalgebra.

Lem^7. $A \subseteq B \subseteq C$, B: integral over A,

C: integral over B $\Rightarrow$ C: integral over A

Pf: $x \in C$, let $x^n + b_1 x^{n-1} + \cdots + b_n = 0$ for $b_1, \cdots, b_n \in B$

$B_0 = A[b_1, \cdots b_n]$ is F.G. A-mod.

by Lem^5, B_0: integral over A, F.G. A-mod.

x: integral over B_0. thus $B_0[x]$: F.G B_0-mod.

Then $B_0[x] \supseteq B_0 \supseteq A$

$\Rightarrow$ $B_0[x]$ F.G. A-mod.
finite fout

by Lem^1, $\underline{B_0[x]}$. $\underline{A[x] \subseteq B_0[x]} \subseteq C$

(bec $A \subseteq B_0$.)

thus x integral over A.

Cor 8: ~~$A \subseteq B$, rings.~~

Def: $A \subseteq B$, $\bar{A}$ integral closure of A in B.

A: integral closed in B if $\bar{A} = A$.

Cor 8: $\bar{A}$ is integral closed in B, $\bar{\bar{A}} = \bar{A}$.

Pf: Restatement of Cor 8.

$\left\{ \begin{array}{l} A \subseteq B \text{ rings}, \quad x \in B \text{ integral over } \bar{A}. \\ \text{Then } x \in \bar{A}. \end{array} \right.$

① $\bar{A} \subseteq \bar{\bar{A}}$
② $\bar{\bar{A}} \subseteq \bar{A}$ bec
$\bar{\bar{A}} \supseteq \bar{A} \supseteq A$
integral, integral

i.e. if x integral over $\bar{A}$, then x integral over A.
$(x^n + \bar{a}_1 x^{n-1} + \dots + \bar{a}_n = 0)$ $(\bar{a}_i^{n_i} + \dots = 0)$

Lem 9. $A \subseteq B$, $\mathfrak{b} \subseteq B$: ideal, $\mathfrak{a} = A \cap \mathfrak{b}$.

If: B integral over A, then $B/\mathfrak{b}$ integral over $A/\mathfrak{a}$

Pf: $x + \mathfrak{b} \in B/\mathfrak{b}$, want.

$x^n + \dots = 0$ (x integral over A)
(coefficients in A)

$\bar{x}^n + \dots = 0$ (take quotient of x)
(coefficients in A)

$\bar{x}^n + \dots = 0$
(take quotient of coefficients). $\square$

Lem 10. $S \subseteq A \subseteq B$, then B integral over A
multiplicative subring

$\Rightarrow S^{-1}B$ integral over $S^{-1}A$

Pf: $\frac{x}{s} \in S^{-1}B$, $x \in B$. $s \in S$.

$x^n + a_1 x^{n-1} + \dots + a_n = 0$ $(a_j \in A)$

$\left(\frac{x}{s}\right)^n + \frac{a_1}{s}\left(\frac{x}{s}\right)^{n-1} + \dots + \frac{a_n}{s^n} = 0.$

$\frac{x}{s}$: integral over $S^{-1}A$ $\square$

Def A: domain, (ID). $K = \mathrm{Frac}(A)$

A: normal IFF A: integrally closed in K.

(normal can be ~~used~~ called as " integrally closed" in txt).

A: normal IFF $x \in \overline{A} \subseteq K \Rightarrow x \in A$.

A: normal IFF $x \in \overline{A} \subseteq K \Rightarrow x \in A$.

Lem''. $\mathbb{Z}$ is normal. ($\Rightarrow \frac{1}{2}$ not integral over $\mathbb{Z}$)

pf: Let $x \in \mathbb{Q}^\times$, $x = r/s$, $\gcd(r,s)=1$, $r,s \in \mathbb{Z}$, $r \neq 0$, $s \neq 0$

suppose $a_1, \dots, a_n \in \mathbb{Z}$, $x^n + a_1 x^{n-1} + \dots + a_n = 0$

multiply by s^n, $r^n = -(a_1 r^{n-1} s + \dots + a_n s^n)$

thus $r^n \equiv 0 \pmod{s}$, thus $s \mid r^n$,

since $\gcd(r,s)=1$, thus $s \in \mathbb{Z}^\times$

$\Rightarrow x \in \mathbb{Z}$.

Lem'' Any UFD is normal. (Same proof)

e.g. $\mathbb{Z}$, $k[x_1, \dots, x_n]$.

EX. $A = k[x^2, x^3] \subseteq K = \mathrm{Frac}(A) = k(x)$ $\underline{k[x]}$

bec $(x^3/x^2) = (x)$.

A: not normal.

bec $x \in K$, $x^2 - (x^2) = 0$, but $x \notin A$.

as in A

EX. $k[x(x-1), x^2(x-1)]$ is not normal

~~EX~~ Prop: A: ~~(normal)~~ domain, $K = \mathrm{Frac}(A)$.

L/K: algebraic field extension, ~~then B: normal~~. check

$B = \overline{A} \subseteq L$, then B: normal $\mathrm{Frac}(B) = L$

□

e.g. $K = \mathbb{Q}$, L/K: finite extension. $A = \mathbb{Z}$

L: number field. $B = \mathcal{O}_L$: ring of integers in L.

e.g. $L = \mathbb{Q}(\sqrt{-3})$, $\mathcal{O}_L = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}] \supsetneq \mathbb{Z}[\sqrt{-3}]$

thus $\mathbb{Z}[\sqrt{-3}]$ is not normal.

· Def. A domain, $A^{norm} :=$ integral closure of A in $\mathrm{Frac}(A)$

A^{norm} is the normalization of A.

A^{norm} is ~~normaly~~ normal

Ex. $k[x^2, x^3]^{norm} = k[x]$.

$\mathbb{Z}[\sqrt{-3}]^{norm} = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$,

· Lem[12]. $A \subseteq B$, B integral over A.

(i) (A: field $\Leftrightarrow$ B: field.) Given A, B both domains

(ii) $\mathfrak{q} \subseteq B$ prime, thus $\mathfrak{p} = \mathfrak{q} \cap A$: prime (in A)

Then $\mathfrak{q}$: maximal $\Leftrightarrow$ $\mathfrak{p}$: maximal

pf: (i) "$\Rightarrow$" let $x \neq 0$ $x \in B$.

let $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ with $a_1, \ldots, a_n$ in A.

let n be minimal such equation exists.

$a_n \neq 0$, otherwise x can be cancelled since B: domain

then $x^{n-1} + a_1 x^{n-2} + \cdots + \frac{a_n}{x} = 0$. (*1)

thus if A: field, $a_n \in A^\times$, $\frac{1}{x} = -(\frac{x^{n-1}}{a_n} + \frac{a_1}{a_n} x^{n-2} + \cdots)$

(*1) as hold in $\mathrm{Frac}(B)$.

then $\frac{1}{x} = -(\frac{x^{n-1}}{a_n} + \frac{a_1}{a_n} x^{n-2} + \cdots)$ is actually in B.

(i) "$\Leftarrow$". B: field, $x \in A - \{0\} \subseteq B - \{0\}$.

$\frac{1}{x} \in B$ (B: field), integral over A so.

$(\frac{1}{x})^n + a_1 (\frac{1}{x})^{n-1} + \cdots + a_n = 0$.

for some $a_j \in A$.

multiply by x^{n-1},

$$\frac{1}{x} + \underbrace{a_1 + a_2 x + \cdots + a_n x^{n-1}}_{= a \in A} = 0.$$

(ii) Apply $A/\mathfrak{p} \subseteq B/\mathfrak{q}$ with (i).

Lec 11-2.

- **Cor** (A, m) local, $A \subseteq B$ integral extension.

 Then $\{m' \in B : m' \text{ maximal}\} = \{m' \in B : m' \cap A = m\}$

pf: follows directly from Lem 12 (ii) of Lec 11-1. $\square$

- Theorem 1. $A \subseteq B$ integral extension.

(i) $\forall p \in \text{spec}(A)$, $\exists q \in \text{spec}(B)$.

 s.t. $p = q \cap A$.

 i.e. $\text{spec}(B) \to \text{spec}(A)$ by

 $q \mapsto q \cap A$ is surjective.

Def: $A \subseteq B$, integral extensions, $q \in \text{Spec}(B)$ "lies over" $p \in \text{spec}(A)$ iff $q \cap A = p$.

(ii) Incompatibility. Primes in B lying over a fixed prime in A is incompatible.

 i.e. if $q, q' \in \text{spec}(B)$ s.t. $q \cap A = q' \cap A$, then

 $q \subseteq q' \Rightarrow q = q'$.

 i.e. if $q, q' \in \text{spec}(B)$, $q \subsetneq q'$, then.

 $q \cap A \subsetneq q' \cap A$.

(iii) ~~$\forall q \subseteq q$~~ $\forall p \subseteq p' \in \text{spec}(A)$. $\exists q \subseteq q' \in \text{spec}(B)$

 s.t. $p = q \cap A$, $p' = q' \cap A$.

 Equivalently, ~~$A \subseteq B$ int~~

 $p_1 \subseteq \cdots \subseteq p_n \in \text{spec}(A)$

 $\exists q_1 \subseteq \cdots \subseteq q_n \in \text{spec}(B)$, s.t. $q_i \cap A = p_i$, $\forall i$.

 Moreover, by (ii). $\forall p \subsetneq p' \in \text{spec}(A)$, $\exists q \subsetneq q' \in \text{Spec}(B) \cdots$

 Equivalently, $p_1 \subsetneq \cdots \subsetneq p_n \in \text{Spec}(A)$

 $\exists q_1 \subsetneq \cdots \subsetneq q_n \in \text{Spec}(B)$, s.t. $q_i \cap A = p_i$, $\forall i$.

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Cor $A \subseteq B$ integral extension.

(i) $\dim(A) = \dim(B)$

(ii) $\dim(A/\mathfrak{a}) = \dim(B/\mathfrak{b})$ given $\underset{\text{ideal}}{\mathfrak{b} \subseteq B}$, $\mathfrak{a} = \mathfrak{b} \cap A$.

(iii) $ht(\mathfrak{b}) \le ht(\mathfrak{a})$ given $\underset{\text{ideal}}{\mathfrak{b} \subseteq B}$, $\mathfrak{a} = \mathfrak{b} \cap A$.

Pf: (i) Clearly from Thm 1

(ii) Notice $A/\mathfrak{a} \subseteq B/\mathfrak{b}$ is again integral extension.

(iii) Let $\mathfrak{p}$ be the minimal prime divisor of $\mathfrak{a}$, then let $\mathfrak{q}$ lying over $\mathfrak{p}$, $ht(\mathfrak{q}) \le ht(\mathfrak{p})$ by "taking intersections", (actually, $ht\,\mathfrak{p} = ht\,\mathfrak{q}$), thus $ht(\mathfrak{b}) \le ht(\mathfrak{q}) \le \underbrace{ht(\mathfrak{p})}_{\forall \text{ minimal divisor}}$, thus $ht(\mathfrak{b}) \le ht(\mathfrak{a})$.

Pf Theorem 1:

(i) $\mathfrak{p} \in \operatorname{Spec}(A)$, $\underset{\circ}{\underset{\#}{A_\mathfrak{p}}} \subseteq \underset{\circ}{\underset{\#}{B_\mathfrak{p}}}$: integral.

$(A_\mathfrak{p}, \mathfrak{p}A_\mathfrak{p})$ local.

maximal ideals of $B_\mathfrak{p}$ $\leftrightarrow$ primes of $B_\mathfrak{p}$ lying over $\mathfrak{m}$. (maximals)

$$\begin{array}{ccccc} ③\ \mathfrak{q} = \tau^{-1}(\mathfrak{n}) \in B & \xrightarrow{\tau} & B_\mathfrak{p} & \supseteq & \mathfrak{n} : \text{maximal } (\exists)\ ② \\ | & & \downarrow & & \\ \mathfrak{p} \in A & \xrightarrow[①]{} & A_\mathfrak{p} & \ni & \mathfrak{m} = \mathfrak{p}A_\mathfrak{p}. \end{array}$$

$\updownarrow$

primes of B over $\mathfrak{p}$.

$\updownarrow$

maximals of $B_\mathfrak{p}$.

Thus (ii) is proved as well.

(iii) let $\mathfrak{p} \subseteq \mathfrak{p}' \subseteq A$, consider, let ~~$\mathfrak{q}' \cap A = \mathfrak{p}'$~~, $\mathfrak{q} \cap A = \mathfrak{p}$.

$\mathfrak{p}'/\mathfrak{p} \subseteq A/\mathfrak{p} \subseteq B/\mathfrak{q}$

By lying over, $\exists\, Q \in B/\mathfrak{q}$ s.t. $Q \cap (A/\mathfrak{p}) = \mathfrak{p}'/\mathfrak{p}$.

then $Q = \mathfrak{q}'/\mathfrak{q}$ for some $\mathfrak{q}' \in B$.

$\square$

Galois Transitivity.

Def: normal extension L/K of fields, L/K: algebraic.
is an extension st. each f: irreducible $\in K[x]$.
that has ≥ 1 root in L, splits completely in L.

Def: L/K Galois: normal & seperable.

Def: L/K seperable if $\forall \alpha \in L$ is seperable over K

Def: $\alpha \in L$ seperable over K if.
$$\#\mathrm{Hom}_K(K(\alpha), \bar{K}) = \dim_K K(\alpha).$$
(note: "$\leq$" holds in general)

OR $\alpha \in L$ seperable over K if
$f \in K[x]$, minimal polynomial for α.
(i.e. $f(\alpha)=0$, $\deg(f) = \min$)
has no repeated roots.

NB: $\mathrm{char}(K)=0 \Rightarrow$ extension is seperable.

EX: $K := \mathbb{F}_p(t)$, $L := K(t^{1/p^n})$ is not seperable.
$\#\mathrm{Hom}_K(K(\alpha), \bar{K}) = 1$, $\forall \alpha \in L$. purely inseperable

Fact: L/K: normal extension, $G := \mathrm{Aut}(L/K)$
$L^G := \{\alpha \in L, g(\alpha) = \alpha, \forall g \in G\}$.
$$L \supseteq L^G \supseteq K$$
Galois purely inseperable.

"Lang Algebra"

Theorem 2. A: normal domain, $K = \mathrm{Frac}(A)$

L/K: normal extension of fields.

$B := $ Integral closure of A in L.

Then $G := \mathrm{Aut}(L/K)$ acts transitively on the primes of B lying above a given prime of A.

(i) $g \in G$, the restriction of g to B induces an automorphisms $g: B \to B$

(ii) $\forall g \in G$, ~~$\mathfrak{q} \cap A$~~ $\mathfrak{q} \in \mathrm{spec}(B)$, $g(\mathfrak{q}) \in \mathrm{spec}(B)$ and $\mathfrak{q} \cap A = g(\mathfrak{q}) \cap A$.

(iii) $\forall \mathfrak{q}, \mathfrak{q}' \in \mathrm{spec}(B)$ with $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$

$\exists g \in G$ s.t. $g(\mathfrak{q}) = \mathfrak{q}'$

Pf: (i) $g \in G$, $x \in B$, want: $g(x) \in B$.

let $x^n + a_1 x^{n-1} + \dots + a_n = 0$, $a_j \in A$.

$0 = g(x^n + a_1 x^{n-1} + \dots + a_n) = g(x)^n + \dots + g(a_n)$

thus $g(x) \in B$. (Do this also for g^{-1}, get $g \circ g^{-1} = g^{-1} \circ g = id$)

(ii) $g \in G$, $\mathfrak{q} \in \mathrm{spec}(B)$

$g(\mathfrak{q} \cap A) = g(\mathfrak{q}) \cap g(A)$ since $g \in \mathrm{Aut}(L/K)$, $g(A) = A$.

$= g(\mathfrak{q}) \cap g A$.

(iii). First let L/K: finite, then $\# G < \infty$

let $\mathfrak{q}, \mathfrak{q}' \in \mathrm{spec}(B)$ s.t. $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$.

claim $\mathfrak{q}' \subseteq \bigcup_{g \in G} g(\mathfrak{q})$ (This is sufficient) $\hookrightarrow \exists g \in G$, $\mathfrak{q}' \subseteq g(\mathfrak{q})$ $\Rightarrow \mathfrak{q}' = g(\mathfrak{q})$

Pf Claim:

$x \in \mathfrak{q}'$, set $y := \prod_{g \in G} g(x) \in L^G$

By "Galois theory", $\exists n \geq 1$, s.t. $y^n \in K$.

Moreover, since $g(x) \in B$, and B integral over A.

We see that y^n integral over A, and belong to K.

Since A is normal, get that $y^n \in A \cap \mathfrak{q} \subseteq \mathfrak{q}$

Since $\mathfrak{q}$: prime. $y^n = \prod_{g \in G} g(x)^n \subseteq \bigcap_{g \in G} g(x)^n$ (#G finite)

Then $g(x) \in \mathfrak{q}$, $\exists g \in G$, hence

$x \in g^{-1}(\mathfrak{q})$. then $\mathfrak{q}' \subseteq \bigcup_{g \in G} g^{-1}(\mathfrak{q}) = \bigcup_{g \in G} g(\mathfrak{q})$

Lec 12-1.

- Zorn's Lem, $(\mathcal{A}, \leq)$ partially ordered, nonempty.

  s.t. $\forall$ chain (totally ordered subset) $C \subseteq \mathcal{A}$

  has an upper bound (in $\mathcal{A}$).

  Then there is a maximal element.

Pf ctd. Theorem 2 (iii)

use this to deduce the infinite case from the finite case. Consider a ~~subdivision~~ subextension

$L/E/K$. with E/K normal.

Then $R_E := B \cap E$ is the integral closure of A in E.

$\mathfrak{q}_1 \cap R_E = \mathfrak{q}_1 \cap E$, $\mathfrak{q}_2 \cap R_E = \mathfrak{q}_2 \cap E$

primes of R_E that lies over $\mathfrak{p}$.

Let $\mathcal{A} := \{ (E, g) \mid E$ as above, $g \in Aut(E/K)$

s.t. $g(\mathfrak{q}_1 \cap E) = \mathfrak{q}_2 \cap E$

- Want: $\exists\, g \in \mathrm{Aut}(L/K)$ s.t. $(L, g) \in \mathcal{A}$.

Note: $\mathcal{A} \neq \emptyset$ bec $(K, \mathrm{id}) \in \mathcal{A}$.

Def $\leq$ to be: $(E, g) \leq (E', g')$ IFF $E \subseteq E'$, $g'|_E = g$

Let $\mathcal{C} = \{(E_i, g_i)\}_{i \in I} \subseteq \mathcal{A}$ be a chain.

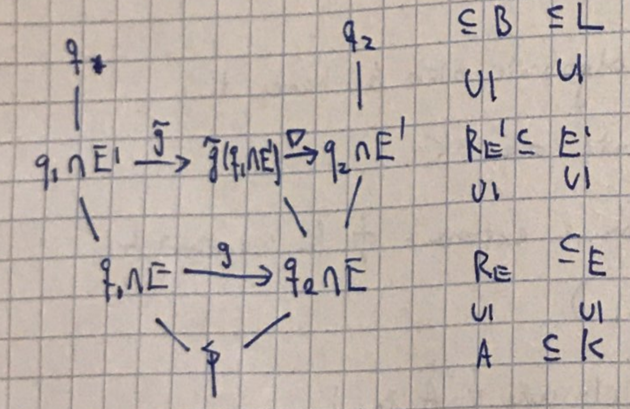
Then: $\mathcal{C}$ has an upper bound (E, g) s.t.

$E = \bigcup E_i$, then $\exists!\, g \in \mathrm{Aut}(E/K)$ s.t. $g|_{E_j} = g_j$

Check that $(E, g) \in \mathcal{A}$ ✓.

By Zorn, $\exists$ maximal $(E, g) \in \mathcal{A}$

Argue by contradiction, if not theorem, $\exists$ finite normal extension E'/E, s.t. $E' \subseteq L$ and $E' \supsetneq E$ (Take any $a \in L$, $a \notin E$. Let $F \in E[X]$ be the minimal polynomial of a over E. Take $E' :=$ field obtained by adjoining to E all roots of F in L.)



Let $\tilde{g} \in \mathrm{Aut}(E'/K)$ be any extension of g (such $\tilde{g}$ exists)

Then $\tilde{g}(q_1 \cap E')$ lies over $q_2 \cap E$

Since the theorem hold for E'/E (finite extension)

$\exists\, \nabla \in \mathrm{Aut}(E'/E)$ s.t. $\nabla(\tilde{g}(q_1 \cap E')) = q_2 \cap E'$

Set $g' = \nabla \circ \tilde{g} \in \mathrm{Aut}(E'/K)$

- Clearly $g'(q_1 \cap E') = q_2 \cap E'$ so $(E', g') \in \mathcal{A}$, $E \subsetneq E'$, $g'|_E = g$

  This contradicts the fact (E, g) maximal.

- Going Down Property.

  $A \subseteq B$ ring extension has G.D.P. IFF

  $\forall$ primes, $p' \subseteq p \subseteq A$, $q \subseteq B$, s.t. $q \cap A = p$.

  $\exists$ prime $q' \subseteq q$ with $q' \cap A = p'$.

  It's also true that, $\forall p' \subsetneq p$ with $\begin{matrix} q \\ | \\ p \end{matrix}$, $\exists q' \subsetneq q$.

- See P239 of Eisenbud OR P32-33. Matsumura for non example of G.D. in ring extensions.

- Theorem. $A \subseteq B$ domains. with A: normal.

  B: integral over A. Then $A \subseteq B$ has G.D.

pf: $K = \mathrm{Frac}(A)$. $L_1 = \mathrm{Frac}(B)$

L_1/K algebraic field extension

Let $\frac{x}{y} \in L_1$, $x \in B$, $y \neq 0 \in B$.

x, y both algebraic over A, hence K.

Thus x/y algebraic over K.

L any normal algebraic extension of K containing L_1.

(e.g. $L = \bar{K} \supseteq L_1$)

Let $C :=$ integral closure of A in L.

$=$ integral closure of B in L.

$$\begin{array}{ccccccc}
\exists Q' \subsetneq \exists Q_1 & & C & \subseteq & L \\
 & & \cup & & \cup \\
 & q & B & \subseteq & L_1 \\
 & & \cup & & \cup \\
p' \subsetneq & p & \subseteq A & \subseteq & K
\end{array}$$

$G = \mathrm{Aut}(L/K)$ acts transitively on fibres of $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$

$\exists Q'$ by lying over.

$\exists Q_1$ by going up.

$\exists Q \subseteq C$ lying over p.

Then $\exists g \in \mathrm{Aut}(L/K)$, $g(Q_1) = Q$

Let $q' := g(Q') \cap B$

Since $g(Q_1) = Q$, then $g(Q') \cap B = q' \subseteq q$.

Then q' lies over p' bec g fixes A.

$$q' \cap A = g(Q') \cap A = g(Q' \cap A) = g(p') = p'. \quad \square$$

· Recall, $\forall A \subseteq B$ integral, $\forall b \in B$, $a := b \cap A$.

then (i) $\dim(A) = \dim(B)$

(ii) $\dim(A/a) = \dim(B/b)$. i.e. $\mathrm{coht}(a) = \mathrm{coht}(b)$

(iii) $ht(b) \leq ht(a)$

· Cor. $A \subseteq B$ domains, A: normal, B: Integral over A

$b \subseteq B$, $a = A \cap b$.

Then $ht(a) = ht(b)$

pf: $ht(b) \geq ht(a)$

$ht(c) := \inf\{ht(p) \mid p \supseteq c\}$

ISTS: $\forall q$ minimal divisor of b.

$\exists p \supseteq a$, that $ht(q) \geq ht(p)$.

Take $p = q \cap A$.

Let $p_0 \subsetneq \cdots \subsetneq p_r = p$ be a chain of primes

Given q lies over p, $\exists q_{r-1}$ is by G.D.

Apply induction, then $ht(q) \geq ht(p)$

· Def. "k-domain" A : k field, A k-alge, A domain

"F.G k-domain" A: if F.G as k-alge.

Then $k[x_1, \ldots, x_n] \longrightarrow A$

$A \cong k[x_1, \ldots, x_n]/a$ iff A "F.G. k-domain".

· Theorem 3. Noetherian Normalization.

A : F.G. k-domain, Set $d = \dim A) \in \mathbb{Z}_{\geq 0}$.

Then $\exists$ injective morphism $k[X_1, \dots, X_d] \xrightarrow{\mathcal{R}} A$.

Denote $k[X] = k[X_1, \dots, X_d]$.

$\mathcal{R}$ is integral IFF A integral over $k[X_1, \dots, X_d]$.

with $\mathcal{R}$: morphism of k-algebras.

Notes: $\mathcal{R}$ is determined by $x_1 = \mathcal{R}(X_1), \dots, x_n = \mathcal{R}(X_d)$

$\mathcal{R}$: injective $\Leftrightarrow$ $x_1, \dots, x_d \in A$ algebraicly independen /k

· Theorem 3. Noetherian Normalization

~~If~~ A : F.G. ~~k-alge~~ k-domain. domain Not Needed.

Then $\exists$ $k[X_1, \dots, X_d] \xrightarrow{\mathcal{K}} A$ injective morphisms of k-alge

$\mathcal{K}$: integral.

Then $\exists$ alge indep $x_1, \dots, x_d \in A$. s.t.

A : Integral over the k-subalgebra.

$A_0 = k[x_1, \dots, x_d] \subseteq A$.

· Cor 4. Let A : F.G k-domain $K = \mathrm{Frac}(A)$

Then $\dim(A) = \mathrm{trangrd}_k(K)$

Pf: Let $d := \dim(A)$. $A_0 = k[x_1, \dots, x_d]$ as above.

$K_0 = \mathrm{Frac}(A_0)$, by $A_0 \cong k[X_1, \dots, X_d)$

$K_0 = \mathrm{Frac}(A_0) = k(X_1, \dots, X_d)$

A : integral over A_0, by proof of Theorem 2.

K : algebraic over K_0.

thus ~~tra~~ $\mathrm{trdeg}_k(K) = \mathrm{trdeg}_k(K_0) = d$

Lec12-2

· If $K: k[X_1, \cdots, X_n] \hookrightarrow A$. injective & integral.
then automatically $n = \dim(A)$.
bec integral extensions preserve dimension.

· NN, Key Lem.

A = k-alge, $x_1, \ldots, x_n \in A$, $0 \neq f \in k[X_1, \cdots, X_n]$.
(free polynomial ring)

$w := f(x_1, \cdots, x_n) \in k[x_1, \cdots, x_n] \subseteq A$
(k-subalgebra)

Then $\exists\, z_1, \cdots, z_{n-1} \in k[x_1, \cdots, x_n]$.

s.t. $k[x_1, \ldots, x_n]$ integral over $k[z_1, \cdots, z_{n-1}, w]$.

NN $\Leftarrow$ Key Lem.

$\mathfrak{a} = \ker(k[X_1, \cdots, X_n] \to k[x_1, \cdots, x_n])$
substitution

$k[x_1, \ldots, x_n] \cong k[X_1, \ldots, X_n]/\mathfrak{a}$

$x_1, \ldots, x_n$: generators.

If alge independent, then done

Else let $f \neq 0$, $f(x_1, \ldots, x_n) = 0$

use key Lem get $k[x_1, \ldots, x_n]$ integral over $k[z_1 \ldots z_{n-1}]$.

Iterate and get $\exists\, d \leq n-1$, $k[x_1, \ldots, x_n]$ integral over $k[u_1, \ldots, u_d)$.
with $u_1, \ldots u_d$ alge indep

(notice if down to 1 element nonzero, then trivially independent)

Key Lem Pf.

$n=2, w=0$: $A = k[x,y]/(f)$, $f \neq 0 \in k[X, Y]$

Want $\exists\, z \in A$ s.t. A integral over $k[z]$.

- Example. $f = xy - 1$.

  A : not integral over $k[x]$

  $k[x] \subsetneq A = \dfrac{k[x,y]}{(xy-1)} \cong k[x, \frac{1}{x}] \subseteq k(x) \cong k(X)$

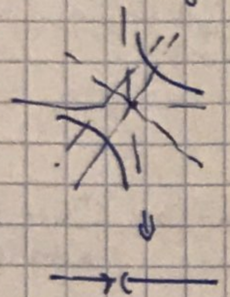
  $\cong$

  $k[X]$ normal (integrally closed in $k(x)$)

- Take $z = x + y$ / $z = y - ax$. (a : unit).

  then A integral over $k[z]$. (for $f = xy - 1$).

- Spec A $\{(x,y), xy = 1\}$. Spec $k[y]$



- $f = f_n + \cdots + f_0$ f_i : homog. of i

  if $y - ax \nmid f_n$, then A integral over $k[y-ax]$

  $x' = x$, $y' = y - ax$.

  reduce to $a = 0$.

- Cor if k : infinite, then Baby case holds.
- How about k : finite. ?

  EX. $k = \mathbb{F}_2$, $f = y(y-x) - 1$

  $z = y - x^s$, $s \geq 2$.

  $f = f_n + \cdots + f_0$ (x,y) $y = z + x^s$ $\to (x,z)$

- Pf (Nagata)

$$f = \sum_{\alpha \in I} C_\alpha X^\alpha \qquad C_\alpha \in k^\times$$

$X^\alpha \hookrightarrow x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$I = \mathbb{Z}^n_{\geq 0}$

$z_j = x_j - x_n^{s_j} \quad j = 1, \cdots, n-1.$ $\quad s_j \geq 0$ to be choosen later

$$f(x_1, \cdots, x_n) = \sum C_\alpha (x_n^{s_1} + z_1)^{\alpha_1} \cdots (x_n^{s_{n-1}} + z_{n-1})^{\alpha_{n-1}} x_n^{s_n}$$

$$= x^{l(\alpha)} + \cdots, \quad l(\alpha)$$

Let $S_j = s^j \quad s = \max_{j=1,\cdots,n} |\alpha_j| + 1$

Then if $\alpha \neq \beta$, then $l(\alpha) \neq l(\beta)$.

Let $l(\alpha)$: maximal in $\alpha \in I$.

$w = C_\alpha x_n^{l(\alpha)} + \cdots, (\ )$

Thus x_n: integral over $k[z_1, \cdots, z_{n-1}, w]$.

each $x_j = x_n^{s_j} + z_j$, integral over $k[z_1, \cdots z_{n-1}, w]$

thus $\cdots$ □

- Theorem A: F.G k-domain.

$\mathfrak{p} \in \operatorname{Spec}(A) \qquad \operatorname{ht}(\mathfrak{p}) + \operatorname{coht}(\mathfrak{p}) = \dim(A)$

§1.1 Hartshone. Algebraic Geometry.

Pf: $n = \dim(A) \Rightarrow$ (NN) $\exists\, x_1 \cdots x_n \in A$.

alge indep s.t. A: integral over $A_0 = k[x_1, \cdots, x_n] \cong k[X_1, \cdots X_n]$

$A_0 \subseteq A$ integral.

Let $\mathfrak{p}_0 = A_0 \cap \mathfrak{p} \in \operatorname{Spec} A_0$.

$\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}_0)$, $\operatorname{coht}(\mathfrak{p}) = \operatorname{coht}(\mathfrak{p}_0)$

~~Replace A by A_0,~~

let $A = K[X_1, \dots X_n]$

If $\mathfrak{p} = (0)$, ✓

$\mathfrak{p} \neq (0)$, suppose $\mathfrak{p} \ni x_n$ (case simplified to this later).

$\overline{A} = A/(x_n) \cong K[X_1, \dots, X_{n-1}]$

$\overline{\mathfrak{p}} = \mathfrak{p}/(x_n)$.

$0 \subsetneq (x_n) \subseteq \mathfrak{p}$ so $ht(\mathfrak{p}) \geq ht(\overline{\mathfrak{p}}) + 1$

$\left(\begin{array}{l} 0 = \overline{\mathfrak{q}}_0 \subsetneq \cdots \overline{\mathfrak{q}}_r = \overline{\mathfrak{p}}, \\ 0 \subsetneq (x_n) = \mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r = \mathfrak{p}. \end{array} \right)$

Also $coht(\overline{\mathfrak{p}}) = \dim(\overline{A}/\overline{\mathfrak{p}}) = coht(\mathfrak{p})$

$\overline{A}/\overline{\mathfrak{p}} \cong A/\mathfrak{p}$

$\dim(\overline{A}) = n-1 = \dim(A) - 1$.

$\Leftarrow ht(\mathfrak{p}) + coht(\mathfrak{p}) \geq \dim(A)$

$ht(\mathfrak{p}) \geq ht(\overline{\mathfrak{p}}) + 1$, $coht(\mathfrak{p}) = coht(\overline{\mathfrak{p}})$, $\dim(A) = \dim(\overline{A}) + 1$

Argue by induction on n with in $x_n \in \mathfrak{p}$.

. $0 \neq f \in \mathfrak{p}$. (Key Lemma NN)

$\exists\, z_1, \dots, z_{n-1}$ s.t. $A = K[x_1, \dots x_n]$ ~~is~~ integral over

$A_0 = [z_1, \dots z_{n-1}, f]$.

Then $trdg(A_0) = trdg(A) = n$.

thus A_0: polynomial ring, normal

Now, $\mathfrak{p}_0 := \mathfrak{p} \cap A_0$,

$\dim A_0 = \dim A$. $ht(\mathfrak{p}_0) = ht(\mathfrak{p})$, $coht(\mathfrak{p}_0) = coht(\mathfrak{p})$

and get back to the previous case.

Lem. A . F.G K-domain
$\mathfrak{p} \subsetneq \mathfrak{p}'$ adjacent
$ht(\mathfrak{p}') = ht(\mathfrak{p}) + 1$.

Pf. Thm 4 to $(A, \mathfrak{p})$, $(A, \mathfrak{p}')$, $(\bar{A}, \bar{\mathfrak{p}}')$ $\quad \bar{A} = A/\mathfrak{p}$.

$ht(\overline{\mathfrak{p}'}) = 1$.

$$ht(\mathfrak{p}) + coht(\mathfrak{p}) = \dim(A)$$
$$coht(\mathfrak{p}) = \dim(A/\mathfrak{p}) = \bar{A}.$$

$$ht(\mathfrak{p}') + coht(\mathfrak{p}') = \dim(A)$$
$$coht(\mathfrak{p}') = coht(\overline{\mathfrak{p}'})$$

$$ht(\overline{\mathfrak{p}'}) + coht(\overline{\mathfrak{p}'}) = \dim(\bar{A})$$

Week 13 CA Ex Class.

Sheet 11. Ex3.

$A \subseteq B$ integral, $n \subseteq B$ maximal, $m = n \cap A$. maximal

Qu: B_n integral over A_m?

Counterexample For the Going Down property.

- $A \subseteq B$ domains, A: normal. ($\bar{A} \subseteq Frac(A)$, $\bar{A} = A$) B integral over A, then $A \subseteq B$ satisfies the going down property.

$\forall$ prime ideals $\mathfrak{p}' \subseteq \mathfrak{p} \subseteq A$ and $\mathfrak{q} \subseteq B$, s.t. $\mathfrak{q} \cap A = \mathfrak{p}$.
$\mathfrak{q}$ prime.

$\exists\, \mathfrak{q}' \subseteq \mathfrak{q}$ prime s.t. $\mathfrak{q}' \cap A = \mathfrak{p}'$

Pic:
$$\begin{array}{ccccc} (\exists)\,\mathfrak{q}' & \subseteq & \mathfrak{q} & \subseteq & B \\ | & & | & & \\ \mathfrak{p}' & \subseteq & \mathfrak{p} & \subseteq & A \end{array}$$

EX (GD fails)

$A = \mathbb{Z}$, $B = \mathbb{Z}[X]/[X^2-X, 2X] \cong \mathbb{Z}[x]$ where x is the image of X in B.

$q = (2, x-1) \subseteq B$ prime

$B/q = \cancel{\mathbb{Z}[x]}\ \mathbb{Z}[x]/[2, x-1] \cong \mathbb{F}_2[x]/(x-1)$

integral domain, bec $(x-1)$ prime in $\mathbb{F}_2[x]/\cancel{(\ \ )}$.

$p = q \cap \mathbb{Z} = (2)$

$p' = (0) \subseteq \mathbb{Z}$ prime.

Claim, $\nexists$ prime ideal $q' \subseteq B$ s.t. $q' \cap \mathbb{Z} = p' = (0)$, $q' \subseteq q$

Suppose q' exists,

$2 \notin (0) = q' \cap \mathbb{Z}$, thus $2 \notin q'$.

$2x = 0 \in q'$, q' prime $\Rightarrow x \in q' \subseteq q = (2, x-1)$

contradiction ■

Q ① B: not a domain. ✓

② B: Integral over $\mathbb{Z}$.

bec x: integral over $\mathbb{Z}$ ✓

$x^2 - x = 0$

~~③ B: normal.~~ ✗

EX $\frac{1+\sqrt{5}}{2}$

$\left(\frac{1+\sqrt{5}}{2}\right)^2 = 3 + \frac{5}{2} + \sqrt{5}$

$2x = 1+\sqrt{5}$

ETH CA Lec13-1

- Corollary (From Last Time)

Every maximal chain of primes in A has length dim(A).

Where: A : F.G k-domain

Rmk: false for general rings, see Exercise for example.

Pf: $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ max chain

$\Rightarrow \mathfrak{p}_0 = (0)$, $\mathfrak{p}_r$: maximal

$\Rightarrow ht(\mathfrak{p}_0) = 0$, $ht(\mathfrak{p}_r) = \dim(A)$.

also $ht(\mathfrak{p}_i / \mathfrak{p}_{i-1}) = 1$

$\Rightarrow ht(\mathfrak{p}_i) = ht(\mathfrak{p}_{i-1}) + 1$

by induction, $ht(\mathfrak{p}_r) = r$, thus $r = \dim(A)$ □

- Recall Last Time Theorem.

k : field A : F.G. k-domain.

(i) $\dim(A) = tr.deg_k(Frac(A))$

(ii) $ht(\mathfrak{p}) + coht(\mathfrak{p}) = \dim(A)$, $\forall \mathfrak{p} \in Spec(A)$

$\Rightarrow ht(m) = \dim(A)$, $\forall m \leq A$ maximal.

(iii) If $\mathfrak{p} \subsetneq \mathfrak{p}'$ adjacent, i.e. $ht(\mathfrak{p}'/\mathfrak{p}) = 1$, then $ht(\mathfrak{p}') = ht(\mathfrak{p}) + 1$.

- Corollary (Another Proof Nullstellensatz, version new)

K/k : field extension, K : FG k-alge.

Then K/k : finite.

Pf: ISTS K/k : algebraic (integral).

(From: integral + F.G. $\Leftrightarrow$ finite)

Want! $tr.deg_k(K) = 0$

but $tr.deg_k(K) = \dim(K) = 0$. □

Scanned by CamScanner

New Section:

Valuation Rings + Normality.

- Recall

$A = k[X,Y]/(XY-1)$ x, y denote $\overline{X}, \overline{Y}$

A: not ~~normal~~ integral over $k[X]$. (A is normal)

with Pf: $A \hookrightarrow \cancel{k[}k(X)$

Check Lec 12-2

$X, Y \mapsto X, \frac{1}{X}$

$\cancel{k(X) \text{ is normal, hence } A \cong k[X, \tfrac{1}{X}]}$ not

Another Pf this Lecture.

- Def. $\forall\, 0 \neq f \in A$ written as

$a_{-n}X^{-n} + \cdots + a_m X^m$ with $a_j \in k$, $a_{-n} \neq 0$.

set $v(f) = -n \in \mathbb{Z}$.

Notice $v(f_1 f_2) = v(f_1) + v(f_2)$

$v(f_1 + f_2) \geq \min(v(f_1), v(f_2))$

If $v(f_1) \neq v(f_2)$, then $v(f_1+f_2) = \min(v(f_1), v(f_2))$

- Now the other Pf:

suppose $f \in k[X, \frac{1}{X}]$, $f \notin k[X]$ is integral over $k[X]$

then $v(f) \leq -1$.

$f^n + a_1 f^{n-1} + \cdots + a_0 = 0$ with $a_j \in k[X]$

Note $v(a_j) \geq 0$.

$v(a_j f^{n-j}) \geq v(f^{n-j}) = (n-j)v(f)$

thus $v(a_1 f^{n-1} + \cdots + a_n) \geq (n-1)v(f)$

but then $v(f^n) = n v(f) < v(-(a_1 f^{n-1} + \cdots + a_n))$

Contradicting. $f^n + a_1 f^{n-1} + \cdots + a_n = 0$. $\square$

· Def. K : field. $(G, 0, +, \leq)$ totally ordered Ab Group (Abelian)! with (*1) (e.g. $\mathbb{Z}$)

A map $v: K^\times \to G$ is a valuation if

(i) $v(f_1 f_2) = v(f_1) + v(f_2)$

(ii) $v(f_1 + f_2) \geq$ ~~min v(f1), v(f2)~~ $\min(v(f_1), v(f_2))$.

if $v(f_1) \neq v(f_2)$, $v(f_1 + f_2) = \min(v(f_1) + v(f_2))$

· Note, $A = \{x \in K^\times : v(x) \geq 0\} \cup \{0\}$.

A is a subring of K (*2)

this is the Valuation Ring of v, K.

· (*1) : $x \leq y \Rightarrow x + z \leq y + z, \forall z \in G$. see (*2)

· $m := \{x \in K^\times :$ ~~v(x)~~ $v(x) > 0\} \cup \{0\}$ is maximal ideal.

(A, m) local.

$A^\times = \{x \in K^\times : v(x) = 0\}$ (*3)

Pf: Check (*3) if $x \in A$, $v(x) = 0$, than $\exists x^{-1}$ s.t. $x \cdot x^{-1} = 1$ in $K^\times$

$v(x) = 0 \Rightarrow v(\frac{1}{x}) + v(x) = v(1) = 0$.

$\frac{1}{x} \in A$, so $x \in A^\times$ ~~thus~~, also check the other direction.

(A, m) : local, $A - m \subseteq A^\times$. (One only needs to check m is an ideal)

· Def. K : field, $A \subseteq K$, valuation ring.

If $\exists v : K^\times \to G$ is a valuation that induces A.

and G : value group of v.

· (*2) by Def(i), $v(1) = v(-1) = 0$, use (*1) twice one get $v(-1) = 0$.

Check $\forall x, y \in A$, $x + y \in A$, $(-1) \cdot x \in A$, $xy \in A$.

Notice, $V(x) =$ ~~v(x^-1)~~ $V(-x)$

• Lem Valuation Rings are normal.

Pf: let $f \in K - A$ integral over A.

$f^n + a_1 f^{n-1} + \dots + a_n = 0$.

Each $v(a_j) \geq 0$ ($a_j \in A$).

but $v(f) < 0$ (bec $f \notin A$)

Check Atiyah P65 for Pf

• Custom, One extends $v : K^\times \to G$ to $v : K \to G \cup \{\infty\}$.

by $v(0) = \infty$. ($\forall g \in G$, $v(g) < \infty$.)

Then $v \mapsto A := \{x \in K ; v(x) \geq 0\}$.

$m := \{x \in K : v(x) > 0\}$.

• Example. $K = k(X) = \mathrm{Frac}(k[X])$

$f \in K^\times$, $f = X^n \frac{u}{v}$ with $u, v \in k[X]$. $X \nmid uv$ in $k[X]$

then $v(f) = n$

Check EX $v(X^2 + X) = v(X(1+X)) = 1$.

$\min(v(X), v(X^2)) = 1$.

• NonExample. $k[X^2, X^3] \subseteq k(X)$ is not a valuation ring.

bec $k[X^2, X^3]$ is not normal.

So does $k[X(X-1), X^2(X-1)]$ not valuation ring, not normal.

ETH CA Lec13-1

· Lem. Let $A \subseteq K$ subring. TFAE

(i) A is a Valuation Ring

(ii) $x \in K - A \Rightarrow \frac{1}{x} \in A$. $(x \neq 0)$

(i.e. $\forall x \in K$, either $x \in A$ OR $\frac{1}{x} \in A$ OR both)

(both units of A)

Pf. (i) $\Rightarrow$ (ii) $v(x) + v(\frac{1}{x}) = 0$.

(ii) $\Rightarrow$ (i)

$G = K^\times / A^\times$ (multiplicative group quotient)

$\sigma : K^\times \to G$

Def order $xA^\times \leq yA^\times \Leftrightarrow y \in xA$. ($\cancel{xy}$ $yx^{-1} \in A$ equivalently)

then $\leq$ is a total order. bec:

Let $xA^\times, yA^\times \in G$, let $x, y \in K^\times$ as representatives

If $x/y \in A$. then $x \in yA$.

thus $yA^\times \leq xA^\times$.

Else $\frac{x}{y} \in K - A$ so $\frac{y}{x} \in A$ by (ii)

thus $xA^\times \subseteq yA^\times$.

Equality $\Leftrightarrow xA = yA \Leftrightarrow xA^\times = yA^\times$. (*1)

Check σ satisfies the valuation criteria, (*2)

and A is induces by σ.

$A' = \{x \in K^\times : \sigma(x) \geq "0"\} \cup \{"0"\}$.

with $"0" = 1 \cdot A^\times$ (multiplicative identity), if $\sigma(x) \geq \sigma(1)$
$= \sigma(1)$

thus $x \in 1 \cdot A = A$. | also if $x \in A \cdots$

thus $A' \subseteq A$. $\square$

(*1) also need to check $xA^\times \leq yA^\times \Rightarrow xzA^\times \leq yzA^\times$ $(xz z^{-1} y^{-1} \in A)$ ✓

(*2) $\sigma(x+y)$, assume $\sigma(x) \leq \sigma(y)$, $yx^{-1} \in A$, then $(x+y)x^{-1} = 1 + yx^{-1} \in A$, $\sigma(x+y) \geq \sigma(x)$.

• Theorem.

$A \subseteq K$ (field). $\overline{A}$ (integral closure).

$\overline{A} = \bigcap_{B:\boxtimes} B$, $B:\boxtimes$, B is a valuation ring in K containing A.

pf: " $\subseteq$ " $x \in \overline{A}$, then $\forall B$, $x \in \overline{B} = B$ (by B: normal)

" $\supseteq$ ". If $x \notin \overline{A}$, then $\exists B \supseteq A$, $\in \boxtimes$

s.t. $x \notin B$.

thus $f \in A[\frac{1}{f}]$, else $f = a_0 + \frac{a_1}{f} + \cdots + \frac{a_n}{f^n}$.

then multiply by f^n get f integral over A

thus $\frac{1}{f} \notin A[\frac{1}{f}]^{\times}$

$\exists \mathfrak{p} \subseteq A[\frac{1}{f}]$ prime s.t. $\frac{1}{f} \in \mathfrak{p}$.

Now quote Lem.

• Lem $A' \subseteq K$ (field) $\mathfrak{p} \in \mathrm{Spec}(A')$

then $\exists$ VR: B, $A'_{\mathfrak{p}} \subseteq B \subseteq K$.

s.t. $\mathfrak{m}$ (maximal) $\subseteq B$ lies over $\mathfrak{p}$.

$\mathfrak{m} \cap A' = \mathfrak{p}$. ~~then~~.

• Lem $\Rightarrow$ Theorem. " $\supseteq$ "

(Let $\mathfrak{p}$ be the unique prime ideal)$\times$

$\exists$ B : VR with $(B, \mathfrak{m})$ s.t. $B \supseteq A_{\frac{1}{f}}$, $\cancel{\mathfrak{m} \cap \mathfrak{p} =}$ $\mathfrak{p} = \mathfrak{m} \cap A$.

then $\frac{1}{f} \in \mathfrak{m}$, $v(\frac{1}{f}) > 0$, $v(f) < 0$, thus $f \notin B$.

pf Lem:

$\mathcal{A} := \{$rings B with $A_p \subseteq B \subseteq K$, $pB \neq B\}$.

Satisfies hypothesis of Zorn's Lem.

Take B: maximal element.

Claim B: VR with $m \cap A = p$.

CA ETH Lec13-2.

· Example of Valuation Rings.

$K = k((x))$, $A = k[[x]]$, $v(\sum a_i x^i) = \min\{i, a_i \neq 0\}$.

· Theorem. $\bar{A} = \bigcap_{B: VR, A \subseteq B \subseteq K} B$.

· Lem. $A \subseteq K$ subring, $p \in \mathrm{Spec}(A)$.

~~$\mathcal{A} := \{B \mid B: VR, A_p \subseteq B \subseteq K\}$~~ X

$\mathcal{A} := \{B \mid A_p \subseteq B \subseteq K, pB \neq B\}$. Let B: max in $\mathcal{A}$

then (i) B: VR with (B, m) local.

(ii) $m \cap A = p$.

pf: (i) $\Rightarrow$ (ii).

$p \subseteq (pB) \cap A \subseteq m \cap A \subseteq m \cap A_p \cap A \subseteq pA_p \cap A = p$.

(i) (pf) $x \in K - B$, want $\frac{1}{x} \in B$.

$B[x] \supsetneq B$ by maximality of B.

~~$pB \neq B$~~. $pB[x] = B[x]$.

Let $1 \in B[x]$,

$1 = \sum_{i=0\to n} b_i x^i$, $b_i \in pB$.

Then we know B is local:

let m: maximal in B s.t. $pB \subseteq m \subseteq B$

(exist bec $pB \neq B$).

then $pB_m \subseteq mB_m \neq B_m$ bec $B_m \neq \{0\}$

by maximality of B, $B_m = B$

thus B local with maximal ideal m.

(Also $\mathrm{rad}(pB) = m$).

$(1-b_0)x^{-n} = \sum_{i=1,\dots,n} b_i x^{i-n}$

$1-b_0 \equiv 1\ (pB)$, $\equiv 1\ (m)$, thus $1-b_0 \in B^\times$.

$\frac{1}{x}$: integral over B.

$B[\frac{1}{x}]$: integral over B.

by lying over.

$\exists\ n \in \mathrm{Spec}(B[\frac{1}{x}])$.

$n \cap B = m$.

$pB[\frac{1}{x}] \subseteq mB[\frac{1}{x}] \subseteq nB[\frac{1}{x}] \neq B[\frac{1}{x}]$.

by maximality of B, $B[\frac{1}{x}] = B$.

$\blacksquare$

Up CA ETH Lec 13-2 CTD

· Def. Discrete valuation, $G \cong \mathbb{Z}$, $v \neq 0$.

Normalized v: surjection, $v(K^\times) = \mathbb{Z}$.

(In general, $v(K^\times) = n\mathbb{Z}$, $\exists n$,

thus $\frac{1}{n}v$ is normalized).

· DVR (discrete valuation ring).

VR attached to a (normalized) discrete valuation.

· EX. $A = k[x]$, $v(x^n \cdot \frac{u}{v}) = n$, $x \nmid uv$, $u, v \in k[x]$.

$\subseteq k(x)$

· EX. $A = k[[x]]$. v as before.

· EX. $\mathbb{Z}_{(p)}$: DVR, $:= \{x = p^n \cdot \frac{a}{b} \in \mathbb{Q}$, $a, b \in \mathbb{Z}$, $b \neq 0$,

$p \nmid ab$, $n \in \mathbb{Z}_{\geq 0}\}$.

$v(x) = n$.

· Non Example. $A = \bigcup_{n \geq 1} k[[x^{\frac{1}{n}}]] \ni f = \sum_{i \in \mathbb{Q}} a_i x^i$.

$v(f) := \min\{i : a_i \neq 0\}$

$v: K^\times \to \mathbb{Q} \not\cong \mathbb{Z}$.

· $A \subseteq K$, DVR, normalized.

$v: K \to \mathbb{Z} \cup \{\infty\}$.

$\mathfrak{p} = \{x \in K : v(x) \geq 1\}$.

A uniformizer is an element ~~ϖ~~ ϖ with $v(\varpi) = 1$.

· Lem Each element $x \in K^\times$ is uniquely of the form $x = u \cdot \varpi^n$, $u \in A^\times$, $n \in \mathbb{Z}$.

(then $v(x) = n$, $n \geq 0 \Leftrightarrow x \in A$, $n \geq 1 \Leftrightarrow x \in \mathfrak{p}$)

pf: let $n = v(x)$, $u = x \cdot \varpi^{-n}$

$v(u) = v(x) + v(\varpi^{-n}) = 0$.

thus $u \in A^\times$.

· Lem. Ideals in A are (0), and (ϖ^n)

Moreover $(\varpi^n) = \mathfrak{p}^n$, $\forall n \in \mathbb{Z}_{\geq 0}$.

Note: any uniformizers differ by a unit.

$= \mathfrak{p} - \mathfrak{p}^2$

pf: ① Check (ϖ^n) ideal, $\forall n$.

② If $\mathfrak{a} \subseteq A$: arbitrary ideal.

$n := \min\{v(x) : 0 \neq x \in \mathfrak{a}\}$.

$\mathfrak{a} \subseteq (\varpi^n)$, $\forall y \in \mathfrak{a}$, $v(y) \geq n =$

.

$\mathfrak{a} \supseteq (\varpi^n)$. $\exists x \in \mathfrak{a}$ with $v(x) = n$.

.

.

③ $\mathfrak{p} = (\varpi)$, and $\mathfrak{p}^n = (\varpi^n)$.

let ϖ', $\mathfrak{p} = (\varpi') = (\varpi)$, thus differ by units.

④ Rmk $\forall 0 \neq x \in A$,

$v(x) = \min\{n \geq 0 : x \in \mathfrak{p}^n\}$.

Corollary. A : DVR. A : Noetherian.

bec. A : P.I.D.

$\dim(A) = 1$.

$\mathrm{Spec}(A) = \{(0), \mathfrak{p}\}$.

Corollary A : regular, (i.e. $\dim_k(\mathfrak{p}/\mathfrak{p}^2) = \dim(A)$)

by NAK, $k = A/\mathfrak{p}$, $\dim_k(\mathfrak{p}/\mathfrak{p}^2) = \#$ (minimal) generators of

Corollary. Nonzero A-submodule of K

forms a group $\cong \mathbb{Z}$.

they are of the form $A\varpi^n$, $n \in \mathbb{Z}$.

$A\varpi^m \cdot A\varpi^n = A\varpi^{m+n}$.

- A : Noetherian local domain, $\dim(A) = 1$.

When is A : DVR.

- Non Example. $A = k[[x^2, x^3]] := \{f : \sum_{i=0}^{\infty} a_i x^i, a_i \in k, a_1 = 0\}$.

$m = (x^2, x^3)$

Not regular, not normal.

- Theorem.

A : Noetherian, local, domain $\dim(A) = 1$, $(A, \mathfrak{p})$

TFAE. ① A : DVR

② A : normal

③ A : P.I.D.

④ A : regular

⑤ $\mathfrak{p}$: principal, $\exists \varpi \in A$ s.t. $\mathfrak{p} = (\varpi)$ (P.I.D.$^-$)

⑥ $\exists \varpi \in A$, every nonzero ideal in A is of the form (ϖ^n) $\exists n \geq 0$. (P.I.D.$^+$)

Pf: normal $\overset{\checkmark}{\Leftarrow}\!\!\Rightarrow$ DVR $\Rightarrow$ PID$^+$ $\overset{\checkmark}{\Rightarrow}$ PID $\overset{\checkmark}{\Rightarrow}$ PID$^-$ $\overset{\checkmark}{\Longleftrightarrow}$ regular (Def)

Note: general properties.

- $\mathfrak{p}^n \neq \mathfrak{p}^{n+1}$ $\forall n$ (otherwise A: Artin, with $\dim A = 1$)
- $\bigcap_{n\geq 0} \mathfrak{p}^n = (0)$ by Krull Intersection.
- $\forall x \in K := \mathrm{Frac}(A)$, $\exists n \geq 0$ s.t. $\mathfrak{p}^n x \subseteq A$.

Pf: $x = y/z$, $0 \neq y, z \in A$.

ISTS, $\mathfrak{p}^n \subseteq Az$.

$z \neq 0 \Rightarrow A/(z)$ Artin.

(Indeed, $\mathfrak{p}$ the only prime of A containing (z))
(bec $\dim(A) = 1$.)

$A/(z)$ local Artin, with maximal ideal $\mathfrak{p}/(z)$

$\Rightarrow$ nilpotent, i.e. $\exists n \geq 0$, $(\mathfrak{p}/(z))^n = (0)$.

i.e. $\mathfrak{p}^n \subseteq (z)$.

Pf CTD: PID$^+$ $\Rightarrow$ DVR.

A: VR, Let $x \in K - A$, WTS $x^{-1} \in A$.

$x = y/z$. $0 \neq y, z \in A$.

by PID$^+$, $(y) = (\varpi^m)$, $(z) = (\varpi^n)$.

so $x = u\varpi^{m-n}$, $\exists u \in A^\times$, $x \notin A \Rightarrow m - n < 0$.

then $x^{-1} = u^{-1}\varpi^{n-m}$, $n - m \geq 1 \geq 0$.

$\Rightarrow x^{-1} \in A$.

Sublemma: In a VR, every f.g. ideal is principal.
Also, a VR with principal maximal ideal is a DVR.
In a particular, a Noetherian VR is a DVR.

CA Ex Week 14.

Up

- Example 12.3. (PS)

Noether's Normalization.

A : F.G. k-alge, then $\exists x_1, \cdots, x_n \in A$ s.t.

① $x_1, \ldots, x_n$: algebraically independent

② A : integral over $k[x_1, \cdots, x_n]$.

① $\Leftrightarrow$ $k[X_1, \cdots, X_n]$ by substituting $X_i \to x_i$ is injective.

12.3(a) $A = k[X, Y]/(X \cdot Y) = k[x, y]$, $x = \bar{X}$, $y = \bar{Y}$

A is integral over $k[x+y]$.

e.g. $\cancel{x^2 + x +}$ $x^2 - (x+y)x = 0$,

$y^2 - (x+y)y = 0$.

Thus. A integral over $k[x+y]$

and $x+y$ algebraically independent trivially.

12.3(c). $A = k[X, Y, Z]/((XY-1) \cap (Y, Z))$

Similarly, $A = k[x, y, z]$

Key Lem. A: k-alge, $x_1, \cdots, x_n \in A$.

$0 \neq f \in k[X_1, \cdots, X_n]$. Set $w = f(x_1, \cdots, x_n) \in A$

then $\exists z_1, \cdots, z_{n-1} \in k[x_1, \cdots, x_n]$ st.

$k[x_1, \cdots, x_n]$ integral over $k[z_1, \cdots, z_{n-1}, w]$.

Pf (Core) $z_j = \cancel{x_j}\ x_j - x_n^{s_j}$

$s_j = s^j$ with $s = \max\limits_{\alpha \in I,\ j = 1, \cdots, n} |\alpha_j| + 1$

with I s.t. $f = \sum\limits_{\alpha \in I} c_\alpha X^\alpha$

Back to 12.3(c).

$f(X, Y, Z) = (XY-1) \cdot Z$ (or $\cancel{XY}$ $(XY-1)Y$ will work)

$\in k(X, Y, Z)$

$w = f(x,y,z) = 0$. , $f = \Sigma\, XYZ - Z$

with $s = \max_{\substack{j \in 1,\dots,n \\ a_j \in I}} |a_j| + 1$, $s = 2$.

$z_1 = X - Z^2$, $z_2 = Y - Z^4$.

Key Lem $\Rightarrow$ A integral over $k[z_1, z_2]$.

Are z_1, z_2 algebraically independent.

Want $\dim k[z_1, z_2] = \dim A = 2$. (? why sufficient, why z_1, z_2 then alge indep)

$A = k[X,Y,Z] / \underbrace{(XY-1) \cap (Y,Z)}_{\mathfrak{a}}$.

$ht(\mathfrak{a}) = \inf_{\mathfrak{p} \supseteq \mathfrak{a}} ht(\mathfrak{p})$, let $\mathfrak{p} = (XY-1) \supseteq \mathfrak{a}$.

$\mathfrak{p}$ is prime bec $XY-1$ is irreducible.

$ht(\mathfrak{p}) = \sup\{\dots\} = 1$. , thus $ht(\mathfrak{a}) \le 1$.

$ht(\mathfrak{a}) \ge 1$ bec ($\mathfrak{a} \ne 0$).

$coht(\mathfrak{a}) = \sup_{\mathfrak{p} \supseteq \mathfrak{a}} coht(\mathfrak{p}) = \sup\{n \ge 0 \mid \exists\, \mathfrak{a} \subseteq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n\}$.

Again, take $\mathfrak{p}_0 = (XY-1)$.

$\mathfrak{a} \subseteq \mathfrak{p}_0 \subsetneq (XY-1, X-1) = (X-1, Y-1) \subsetneq \underset{\substack{\| \\ (X-1,\, Y-1,\, Z)}}{(XY-1, X-1, Z)}$

$coht(\mathfrak{a}) \ge 2$.

by $ht(\mathfrak{a}) + coht(\mathfrak{a}) \le \dim(k[X,Y,Z])$.

thus $coht(\mathfrak{a}) = 2$.

$\dim A = coht(\mathfrak{a}) = 2$.

12.5. R (ring). A R-alge (i.e. $R \to A$ morphism, A as A-mod.)

$\mathfrak{p} \subseteq R$ prime ideal $A_\mathfrak{p} = \{\frac{a}{b} \mid a \in A, b \in R \setminus \mathfrak{p}\}$.

Note: $\mathfrak{p}^e \subseteq A$ might not be prime, $A \setminus \mathfrak{p}$ not multipl closed

A

Valuation EX.

Every $0 \neq f \in k[x, \frac{1}{x}]$ can be written as

$f = a_n x^n + \cdots + a_m x^m \quad a_j \in \mathbb{Z}, \ m, n \in \mathbb{Z}. \ n \leq m.$

$v(f) := n \in \mathbb{Z}$

Google: "eth math gruppe 1" doodle.

CA ETH Lec 14-1.

. CTD Proof of Theorem Charatarizing DVR with others.

Recall Sublemma.

① $(A, \mathfrak{p})$: Noeth local, domain, $\dim A = 1$.

$K = \mathrm{Frac}(A)$. $\Rightarrow$ $\forall x \in K$, $\exists n \geq 0$ s.t. $x\mathfrak{p}^n \subseteq A$

② $(A, \mathfrak{p})$: Noeth local domain, $K = \mathrm{Frac}(A)$.

Let $y \in K$ st. $y\mathfrak{p} \subseteq A$, then either

(i) $y^{-1} \in \mathfrak{p}$ OR (ii) y: integral over A

Pf: ① If $y\mathfrak{p} = A$, then $yt = 1$ $\exists t \in \mathfrak{p}$.

then $y^{-1} = t \in \mathfrak{p}$.

② If $y\mathfrak{p} \subsetneq A$, thus ~~$y\mathfrak{p} \subsetneq \mathfrak{p}$~~. $y\mathfrak{p} \subseteq \mathfrak{p}$.

$\mathfrak{p}$: ~~A~~ F.G. thus y: integral over A

bec it admits a faithful f.g. module, i.e. $\mathfrak{p}$.

Theorem normal $\Rightarrow$ DVR.

Sublemma:

③ A: VR, Noeth $\Leftrightarrow$ A: field OR A: DVR.

$\Leftarrow$ ✓ Clear (DVR satisfies ACC)

$\Rightarrow$. Suppose $(A, \mathfrak{p})$ VR. Noeth, with $\mathfrak{p} = (\varpi_1, \cdots, \varpi_n)$

with $v: K \to G \cup \{\infty\}$ valuation, $K = \mathrm{Frac}(A)$

$A = K$ then A: field

Else $v(K^\times) \neq \{0\}$.

thus A contains nonzero non units.

$\mathfrak{p} \neq (0)$, $v(\varpi_j) > 0$.

Let $g := \min(v(\varpi_1), \dots, v(\varpi_n))$

Say $g = v(\varpi_1)$. $\varpi_1 \neq 0$. Then $v(\frac{\varpi_j}{\varpi_1}) \geq 0$.

$\varpi_j \in A\varpi_1$, thus $\mathfrak{p} = (\varpi_1)$

Now given $x \in A - \{0\}$, by Krull's Theor.

$\cap \mathfrak{p}^n = (0)$ choose $n \geq 0$ s.t. $x \in \mathfrak{p}^n$, $x \notin \mathfrak{p}^{n+1}$.

$\mathfrak{p}^n = (\varpi_1^n)$, $\mathfrak{p}^{n+1} = (\varpi_1^{n+1})$

then $\frac{x}{\varpi^n} := u \in A - \mathfrak{p} = A^\times$

so $(x) = \mathfrak{p}^n$, $v(x) = ng$.

thus $v: K^\times \to \mathbb{Z}g \cong \mathbb{Z}$.

thus A: DVR.

(This pf also shows that $\text{PID}^- \Rightarrow \text{PID}^+ \Rightarrow \text{DVR}$).

Pf of theorem.

as we saw last time, $\exists$ VR: B s.t. $A \subseteq B \subseteq K$.

s.t. $(B, \mathfrak{m})$, $\mathfrak{m} \cap A = \mathfrak{p}$

Let $v: K \to G \cup \{\infty\}$ induces B.

thus $v(\mathfrak{p}) > 0$. Let $x \in B$, $x \in K$.

s.t. $\exists n \geq 0$ s.t. $x\mathfrak{p}^n \subseteq A$, let n be such minimal.

If $n = 0$, then $x \in A$, done.

Else $n \geq 1$, $x\mathfrak{p}^{n-1} \not\subseteq A$, $\exists y \in x\mathfrak{p}^{n-1}$, $y \notin A$.

then $y\mathfrak{p} \subseteq x\mathfrak{p}^n \subseteq A$

then either $y^{-1} \in \mathfrak{p}$ or y: integral over A.

Then $v(y^{-1}) > 0$, $y \in \mathfrak{p}^{n-1}x$, $v(x) \geq 0$, $v(\mathfrak{p}) > 0$, $\Rightarrow v(y) \geq 0$

contradiction.

If (iii) then since A: normal, $y \in A$, contradiction

Thus $n=0$. $x \in A$, thus $B = A$, A: DVR.

· Def: Dedekind Domain. IFF Noetherian Domain, Normal, $\dim = 1$

IFF A: domain, Noetherian.

A: integrally closed in $K = \mathrm{Frac}(A)$

Every nonzero prime is maximal. $A \neq K$.

· Theorem. A local Dedekind domain is a DVR.

· Lem: Any Localization s.t. $S^{-1}A \neq K$ of a Dedekind domain is a Dedekind domain.

I.P. $(A, \mathfrak{p})$ Dedekind with $\mathfrak{p}$ any prime.

$A_\mathfrak{p}$: Dedekind domain.

Pf: Just need to check for any domain,

A: normal $\Leftrightarrow$ $A_\mathfrak{p}$: normal, $\forall$ prime $\Leftrightarrow$ A_m: normal, $\forall$ max

If A: normal, let $x \in K$ integral over $A_\mathfrak{p}$.

$x^n + a_1 x^{n-1} + \cdots + a_n = 0$, $a_j = \frac{b_j}{s_j}$, $b_j \in A$, $s_j \in S - \mathfrak{p}$.

let $s = s_1 \cdots s_n \in A - \mathfrak{p}$. $y = sx$, $x = y/s$.

thus $s a_j \in A b_j \subseteq A$.

$y^n + \cancel{sa_1} \, s a_1 y^{n-1} + \cdots + s^n a_n = 0$.

since A: normal, $y \in A$. thus $x = y/s \in A_\mathfrak{p}$.

If $\forall m$, A_m normal.

let $x \in K$, integral over A.

thus $\forall m$, $x^{N_m} + a_{m,1} x^{N_m - 1} + \cdots + a_{m, N_m} = 0$

thus $x \in A_m$, $\forall m$.

then $\forall m$, $\exists s_m \in A - m$ ~~s.t. $s_m x = y_m$ integral over A~~

s.t. $s_m x \in A$.

Thus $\sum_{m:\text{maximal}} A s_m \underset{\nsubseteq m}{\sim} \nsubseteq m$, $\forall m$.

thus $\sum_{m:\text{maximal}} A s_m = A \ni 1$.

$\exists m_1, \dots, m_n$, $\lambda_1, \dots, \lambda_n \in A$.

s.t. $1 = \sum \lambda_j s_{m_j}$.

then $x = \sum \lambda_j \underbrace{s_{m_j} x}_{\in A}$, thus $x \in A$.

The Above only shows for A_p (A_m), same argument works for general localization.

Example A: PID, not field, then A: Dedekind domain.

Check: PID $\Rightarrow$ UFD normal.

$\Rightarrow$ Noetherian,

$\forall m \in \operatorname{spec}(A)$ maximal, A_m: Noeth local with principal maximal ideal.

$\Rightarrow \dim A_m \leq 1$, thus $\dim A \leq 1$.

$A \neq$ field, $\dim(A) = 1$.

Lem: A: Dedekind $\Leftrightarrow$ A_m: Dedekind, $\forall m$.

Prop. A: Dedekind, $K := \operatorname{Frac}(A)$.

L/K: finite, seperable. extension of fields.

B: integral closure of A in L.

$\Rightarrow B$: Dedekind.

· Corollary. Ring of integer = integral closure of $\mathbb{Z}$ in L

Number Field = Finite field extension $L/\mathbb{Q}$

Ring of integer : Dedekind.

pf: $A = \mathbb{Z}$, $K = \mathbb{Q}$

pf Prop: $\exists$ K-basis, $v_1, \dots, v_n \in B$ for L.

$L = \bigoplus_{j=1}^{n} K v_j$

Let: $T : L \longrightarrow K$, $x \mapsto \mathrm{trace}(M_x)$.

M_x: K-linear map : $L \to L$ s.t. $y \overset{T}{\mapsto} xy$.

Fact: L/K : seperable $\Leftrightarrow$ K-bilinear Form

$L \times L \ni (x,y) \longmapsto T(x,y)$

is non-degenerate:

it induces an isomorphism

$L \to L^* = \mathrm{Hom}_K(L, K)$

$x \mapsto [y \mapsto T(xy)]$

Let $u_1, \dots, u_n \in L$ dual basis wrt T.

$u_1, \dots, u_n$: K basis of L.

$T(u_i v_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$

Claim $B \subseteq \bigoplus_j A u_j$ ($\Rightarrow$ B: Noetherian)

· (By going up/down, $\dim(B) = \dim(A) = 1$, $\because$ transitivity $\Rightarrow$ B : normal)

pf:

Up CA ETH Lec14-2.

- Reduced the proof to checking that B: Noeth.

For this, ISTS B = F.G. A-mod. (bec A: Noeth)

Note: $\exists u_1, \dots, u_n \in B$: K: basis for L,

$\Rightarrow v_1, \dots, v_n \in L$: K-basis for L, as dual basis bec L/K Seperable.

s.t. $T: L \to K$, $z \mapsto \mathrm{trace}_{L/K}(M_z)$

we have $T(u_i v_j) = \delta_{ij}$

We have used without pf that.

L/K: seperable $\Rightarrow$ bilinear form $L \times L \to K$,

$(x, y) \mapsto T(xy) \in K$. is non-degenerate

(non-degenerate means:

$\nexists\, x \neq 0 \in L$, s.t. $T(xy) = 0$. $\forall y \in L$.

$\Leftrightarrow L \to L^* = \mathrm{Hom}_K(L, K)$ is an isomorphism)

Since this system is linear in x, we can extend

scalars. ISTS: $\nexists\, 0 \neq x \in L \otimes_K K^s$, K^s: seperable closure of K

s.t. $T(x, y) = 0\ \forall y \in L \otimes_K K^s$.

But $L \otimes_K K^s = (K^s)^{[L:K]}$ ($\Leftarrow L/K$ separable)

(Here T: K^s-linear extension of T

$T: L \otimes_K K^s \to K^s$, $t \otimes z \mapsto T(t) z$)

$\rightsquigarrow T: (K^s)^{[L:K]} \to K^s$, $(z_i) \mapsto \sum z_i$.

s.t. with $T: L \to K$, $z \mapsto \mathrm{trace}_{L/K}(M_z)$.

Claim $B \subseteq \bigoplus_i A v_i$ $\quad \sum K v_i = L$

(Then A: Noeth $\Rightarrow$ B: Noeth).

Indeed, let $x \in B$, write $x = \sum x_i v_i$. with $x_i \in K$

Want $x_i \in A$

Indeed, $T(x \cdot u_i) = x_i$.

Want: $Tr(B) \subseteq A$

Case L/K: Galois.

$T(x) = \sum_{\sigma \in Gal(L/K)\ (\sigma B)} x^\sigma$, $x^\sigma = \sigma(x)$ apply the galois extension

$x \in B \Rightarrow$ each $x^\sigma \in B$, by A: normal.

$T(x) \in B \cap K = A$.

For the other cases, we need to pass to the Galois case:

$$\left.\begin{array}{ccc} B_1 & \subseteq & L_1 \\ \cup\!\!| & & \cup\!\!| \\ B & \subseteq & L \\ \cup\!\!| & & \cup\!\!| \\ A & \subseteq & K \end{array}\right\} \text{Galois.}$$

L_1: Galois Closure of L on K.

Our proof shows B_1: F.G A-mod. since A: Noeth,

we concluded that B: Noeth. $\square$

__Lem__ A: Noeth, domain, $\dim = 1$.

Then ~~or~~ every nonzero ideal $\mathfrak{a} \subseteq A$

factors uniquely as a product of prime with.

distinct radicals.

$$\mathfrak{a} = \prod_{i=1}^{n} \mathfrak{q}_i \quad , \mathfrak{p}_i \neq \mathfrak{p}_j, \forall i \neq j \qquad \mathfrak{p}_i = rad(\mathfrak{q}_i)$$

Note: domain with $\dim = 1$ then, every prime nonzero is maximal.

Pf. Existence. let $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ be a M.P.D

since $\mathfrak{a} \neq 0$, then $\mathfrak{p}_i \neq 0$, thus $\mathfrak{p}_i$ all maximal

thus $\mathfrak{p}_i$ pairwisely coprime. $(\mathfrak{p}_i + \mathfrak{p}_j = (1), i \neq j)$

thus $\mathfrak{q}_i$ pairwisely coprime.

then $\cap \mathfrak{q}_i = \prod \mathfrak{q}_i$ (Chinese Remainder Theorem)

thus $\mathfrak{a} = \prod \mathfrak{q}_i$

Uniqueness:

$\mathfrak{a} = \prod \mathfrak{q}_i$, $\mathfrak{p}_i \neq \mathfrak{p}_j$, $\forall i \neq j$.

then for the same reason $\mathfrak{a} = \prod \mathfrak{q}_i = \cap \mathfrak{q}_i$

thus, $\mathfrak{a}$ is a MPD. with each component isolated,

and uniqueness follows from the second uniqueness theorem.

$\square$

Theorem. In a Dedekind domain.

(i) Any power of a nonzero prime is primary

(ii) Any nonzero primary is a power of its radical

(iii) Any nonzero ideal is uniquely a product of prime ideals.

pf: (ii) true if A = local. bec then A = DVR.

In general, $\mathfrak{p}$-primary ideals in A are in a bijection with.

$\mathfrak{p}A_\mathfrak{p}$ – primary ideals in $A_\mathfrak{p}$.

The latter = {nonzero ideals in $A_\mathfrak{p}$} = {~~powers~~ primary of $\mathfrak{p}A_\mathfrak{p}$}.

(iii) is from (i) + (ii).

Def. Let A: domain, $K = \mathrm{Frac}(A)$.

Fraction Ideal. I : submod

s.t. $xI \subseteq A$, $\exists x \in A$, $x \neq 0$.

Example: $I = Ay$, $\exists y \in K$, I = fraction ideal.

In General, any F.G. submodule $I \subseteq K$ is a fractional ideal

$\sum Ay_i = I$, $y_i = \frac{u_i}{v_i}$, then $v_1 \cdots v_n I \subseteq A$.

- If A: Noeth. and I: fractional ideal then I: F.G.

- Def. An invertible ideal is a submodule $I \subseteq K$.
  s.t. $\exists$ submodule $J \subseteq K$ s.t. $IJ = A$.
  where $I \cdot J = \{\sum a_i b_i : a_i \in I, b_i \in J\}$.
  I: invertible, $IJ = A \Rightarrow J = (A : I) = \{x \in K : xI \subseteq A\}$.
  bec $J \subseteq (A:I) = (A:I) I \cdot J \subseteq A \cdot J = J$

Invertible Ideals Form A Group.

Theorem. A: Dedekind Domain, then every nonzero fractional ideals is invertible
The group of nonzero fractional ideals is free on the nonzero primes.

$$\bigoplus_{0 \neq p \subseteq A} \mathbb{Z} \to \{\text{non-zero fractional ideals}\}.$$

$$(n_p)_p \mapsto \prod_p p^{n_p}$$

Pf: Let I: nonzero fractional ideal.
then $xI \subseteq A$, $\exists x \in A$.
If A: local, then $x = \varpi^n u$, $n \geq 0$.
$xI = (\varpi^m)$, $m \geq 0$. $\Rightarrow I = A\varpi^{m-n}$
thus I: invertible
thus $\mathbb{Z} \to \{\text{invertible ideals}\} = \{\text{nonzero fraction ideals}\}$
$n \mapsto p^n = (\varpi^n)$

CA Lec14-2.

Up

- In general, know that I_p: invertible $\forall p \neq 0$.

then $I(A:I) \overset{?}{=} A \Leftarrow (I \cdot (A:I))_p \overset{?}{=} A_p$

$\Leftarrow (I \cdot (A:I))_p = I_p \cdot (A:I)_p \underset{\substack{\Uparrow \\ I : f.g.}}{=} I_p \cdot (A_p : I_p) \underset{\substack{\uparrow \\ I_p : \text{invertible.}}}{=} A_p$

- Def. A: Dedekind.

$\mathcal{Q}(A) = \{$nonzero fractional ideals$\}$.

O_K: integer closure of $\mathbb{Z}$ in K.

Theorem $K/\mathbb{Q}$: finite $\Rightarrow \# \mathcal{Q}(O_K) < \infty$

- $k[[x^2, x^3]]$ not ~~DVR~~. (x^3) not a power of primes

Eisenbug Ch11. (Not Exam)

Localization: Important!