Localization techniques in quantum field theories

Vasily Pestun[1], Maxim Zabzine[2], Francesco Benini[3,4], Tudor Dimofte[5], Thomas T. Dumitrescu[6], Kazuo Hosomichi[7], Seok Kim[8], Kimyeong Lee[9], Bruno Le Floch[10], Marcos Mariño[11], Joseph A. Minahan[2], David R. Morrison[12], Sara Pasquetti[13], Jian Qiu[2,14,15], Leonardo Rastelli[16], Shlomo S. Razamat[17], Silvu S. Pufu[18], Yuji Tachikawa[19], Brian Willett[20], Konstantin Zarembo[21,2]

[1]*Institut des Hautes Études Scientifique, France*
[2]*Department of Physics and Astronomy, Uppsala University, Sweden*
[3]*International School for Advanced Studies (SISSA), via Bonomea 265, 34136 Trieste, Italy*
[4]*Blackett Laboratory, Imperial College London, London SW7 2AZ, United Kingdom*
[5]*Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, ON N2J 2Y5, Canada*
(on leave) Department of Mathematics, University of California, Davis, CA 95616, USA
[6]*Department of Physics, Harvard University, Cambridge, MA 02138, USA*
[7]*Department of Physics, National Taiwan University, Taipei 10617, Taiwan*
[8]*Department of Physics and Astronomy & Center for Theoretical Physics, Seoul National University, Seoul 151-747, Korea*
[9]*School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea*
[10]*Princeton Center for Theoretical Science, Princeton University, Princeton, NJ 08544, USA*
[11]*Département de Physique Théorique et Section de Mathématiques Université de Genève, Genève, CH-1211 Switzerland*
[12]*Departments of Mathematics and Physics, University of California, Santa Barbara Santa Barbara, CA 93106 USA*
[13]*Dipartimento di Fisica, Università di Milano-Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy*
[14]*Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany*
[15] *Department of Mathematics, Uppsala University, Box 480, SE-75106 Uppsala, Sweden*
[16] *C. N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, NY 11794-3840, USA*
[17] *Department of Physics, Technion, Haifa, 32000, Israel*
[18]*Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA*
[19]*Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwa, Chiba 277-8583, Japan*
[20]*KITP/UC Santa Barbara*
[21]*Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden*

Contents

0.1 Summary . . . 6
0.2 Individual chapters . . . 7
0.3 Volume structure . . . 12
References . . . 12

1 Introduction to localization in quantum field theory (by Vasily Pestun and Maxim Zabzine) 14
1.1 Main idea and history . . . 15
1.2 Localization in diverse dimensions . . . 18
1.3 Applications of the localization technique . . . 29
References . . . 31

2 Review of localization in geometry (by Vasily Pestun) 35
2.1 Equivariant cohomology . . . 36
2.2 Classifying space and characteristic classes . . . 37
2.3 Weil algebra . . . 38
2.4 Weil model and Cartan model of equivariant cohomology . . . 41
2.5 Equivariant characteristic classes in Cartan model . . . 43
2.6 Standard characteristic classes . . . 45
2.7 Index formula . . . 49
2.8 Equivariant integration . . . 50
2.9 Equivariant index formula (Dolbeault and Dirac) . . . 55
2.10 Equivariant index and representation theory . . . 58
2.11 Equivariant index for differential operators . . . 61
2.12 Equivariant cohomological field theories . . . 65
References . . . 67

3 Supersymmetric localization in two dimensions (by Francesco Benini and Bruno Le Floch) 69
3.1 Introduction . . . 70
3.2 $\mathcal{N}$=(2, 2) gauge theories on spheres . . . 73
3.3 Other $\mathcal{N}$=(2, 2) curved-space results . . . 89
3.4 Elliptic genera of $\mathcal{N}$=(2, 2) and $\mathcal{N}$=(0, 2) theories . . . 103
3.5 Dualities . . . 116

3.6 Conclusion . . . 128
References . . . 129

4 Gromov–Witten invariants and localization (by David R. Morrison) 138
4.1 Introduction . . . 139
4.2 Kähler potentials and 2-sphere partition functions . . . 140
4.3 Metrics on conformal manifolds . . . 143
4.4 Quantum corrections to the Kähler potential . . . 151
4.5 The two-sphere partition function and Gromov–Witten invariants . . . 157
4.6 Evaluating the partition function . . . 160
4.7 Appendix. $\mathcal{N} = (2, 2)$ supersymmetry in two dimensions . . . 165
References . . . 166

5 An introduction to supersymmetric field theories in curved space (by Thomas T. Dumitrescu) 173
5.1 Introduction . . . 174
5.2 Four-dimensional $\mathcal{N} = 1$ theories . . . 182
5.3 Three-dimensional $\mathcal{N} = 2$ theories . . . 194
References . . . 197

6 Localization on three-dimensional manifolds (by Brian Willett) 202
6.1 Introduction . . . 203
6.2 Supersymmetry on the 3-sphere . . . 208
6.3 Localization of the partition function on the 3−sphere . . . 219
6.4 Lens spaces . . . 234
6.5 $S^2 \times S^1$ partition function . . . 238
6.6 Applications . . . 246

7 Localization at large N in Chern–Simons–matter theories (by Marcos Mariño) 269
7.1 Introduction . . . 270
7.2 The ABJM matrix model . . . 271
7.3 Generalizations . . . 316
7.4 Conclusions and outlook . . . 323
References . . . 325

8 The F-Theorem and F-Maximization (by Silviu S. Pufu) 332
8.1 Introduction . . . 333
8.2 The F-coefficients of various CFTs . . . 335
8.3 $\mathcal{N} = 2$ SCFTs and F-maximization . . . 345
8.4 Conclusion . . . 358
References . . . 359

9 Perturbative and nonperturbative aspects of complex Chern-Simons Theory (by Tudor Dimofte) 365
9.1 Introduction 366
9.2 Warmup: quantization of $\mathcal{H}(T^2)$ 368
9.3 Complex Chern-Simons theory as a TQFT? 374
9.4 Three connections to three-manifold topology 377
9.5 The Quantum Modularity Conjecture 386
References 389

10 $\mathcal{N} = 2$ SUSY gauge theories on S^4 (by Kazuo Hosomichi) 395
10.1 Introduction 396
10.2 Construction of theories 399
10.3 Partition function 407
10.4 Supersymmetric observables 415
References 424

11 Localization and AdS/CFT Correspondence (by Konstantin Zarembo) 429
11.1 Introduction 429
11.2 $\mathcal{N} = 4$ Super-Yang-Mills theory 430
11.3 Circular Wilson loop 431
11.4 Higher representations and D-branes 438
11.5 Wavy lines, cusp and latitude 447
11.6 Operator Product Expansion 455
11.7 Massive theory 462
11.8 Conclusions 466
References 467

12 A brief review of the 2d/4d correspondences (by Yuji Tachikawa) 473
12.1 Introduction 474
12.2 Two two-dimensional theories 475
12.3 A class of four-dimensional theories 487
12.4 Two supersymmetric backgrounds 492
12.5 Two correspondences and an interpretation 500
12.6 Future directions 507
References 512

13 The supersymmetric index in four dimensions (by Leonardo Rastelli and Shlomo S. Razamat) 517
13.1 Introduction 518
13.2 Definition of the index 519
13.3 Index of sigma models 525
13.4 Index of gauge theories 527
13.5 Index spectroscopy 532

13.6 Dualities and Identities 536
13.7 Limits 540
13.8 Other topics and open problems 551
References 552

14 Review of localization for 5d supersymmetric gauge theories (by Jian Qiu and Maxim Zabzine) 557
14.1 Introduction 558
14.2 The basic setup 560
14.3 The cohomological complex 564
14.4 Analysis of the fixed points 571
14.5 Gauge fixing and the determinant 576
14.6 Asymptotics and comparison with flat space 582
14.7 Appendix. Geometrical setting 587
References 595

15 Matrix models for 5d super Yang-Mills (by Joseph A. Minahan) 598
15.1 Introduction 599
15.2 Matrix model for $\mathcal{N}=1$ $5d$ Yang-Mills with matter 600
15.3 $\mathcal{N}=1^*$ $5d$ super Yang-Mills 604
15.4 Supergravity comparisons 607
15.5 Discussion 609
References 610

16 Holomorphic blocks and the 5d AGT correspondence (by Sara Pasquetti) 613
16.1 Introduction 614
16.2 Compact manifolds and Holomorphic Block factorisation 615
16.3 Chiral 5d AGT 631
16.4 3d & 5d partition functions as q-deformed correlators 642
16.5 Appendix 647
References 650

17 Indices for 6 dimensional superconformal field theories (by Seok Kim and Kimyeong Lee) 657
17.1 Introduction 658
17.2 Coulomb branch indices for the self-dual strings 661
17.3 Superconformal indices of the 6d $(2,0)$ theories 668
17.4 Discussions 689
17.5 Appendix. Off-shell supergravity analysis on S^5 691
References 694

0.1 Summary

This is the summary of the special volume “Localization techniques in quantum field theories” which contains 17 individual chapters.[1] The focus of the volume is on the localization technique and its applications in supersymmetric gauge theories. Although the ideas of equivariant localization in quantum field theory go back 30 years, here we concentrate on the recent surge in the subject during the last ten years. This subject develops rapidly and thus it is impossible to have a fully satisfactory overview of the field. This volume took about two and a half years in making, and during this period some important new results have been obtained, and it was hard to incorporate all of them. However we think that it is important to provide an overview and an introduction to this quickly developing subject. This is important both for the young researchers, who just enter the field and to established scientists as well. We have tried to do our best to review the main results during the last ten years.

The volume has two types of chapters, some chapters concentrate on the localization calculation in different dimensions by itself, and other chapters concentrate on the major applications of the localization result. Obviously, such separation is sometimes artificial. The chapters are ordered roughly according to the dimensions of the corresponding supersymmetric theories. First, we try to review the localization calculation in given dimension, and then we move to the discussion of the major applications.

The volume covers the localization calculations for the supersymmetric theories in dimensions 2,3,4 and 5. The volume discusses the applications of these calculations for theories living up to dimension 6, and for string/M theories. We have to apologize in advance for omitting from the review the new and important calculations which have appeared during last couple of years.

This volume is intended to be a single volume where the different chapters cover the different but related topics within a certain focused scope. Some chapters depend on results presented in a different chapter, but the dependency is not a simple linear order.

The whole volume, when published, could be cited as

> V. Pestun and M. Zabzine, eds.,
> “Localization techniques in quantum field theories”,
> *Journal of Physics A* (2016)

The arXiv preprint version can be accessed from arXiv summary entry which lists all authors and links to all 17 individual contributions, the corresponding citation would be

> arXiv:1608.02952

An individual contribution can be cited by its chapter number, for exampe

> S. Pufu, “The F-Theorem and F-Maximization,”
> Chapter 8 in V. Pestun and M. Zabzine, eds.,

[1] The preprint version is available at http://pestun.ihes.fr/pages/LocalizationReview/LocQFT.pdf

"Localization techniques in quantum field theories",
Journal of Physics A (2016)

and accessed on arXiv and cited by the arXiv number

arXiv:1608.02960.

0.2 Individual chapters

Below we summarize the content of each individual contribution/chapter:

Chapter 1: "Introduction to localization in quantum field theory" (Vasily Pestun and Maxim Zabzine)

This is the introductory chapter to the whole volume, outlining its scope and reviewing the field as a whole. The chapter discusses shortly the history of equivariant localization both in finite and infinite dimensional setups. The derivation of the finite dimensional Berline-Vergne-Atiyah-Bott formula is given in terms of supergeometry. This derivation is formally generalized to the infinite dimensional setup in the context of supersymmetric gauge theories. The result for supersymmetric theories on spheres is presented in a uniform fashion over different dimensions, and the related index theorem calculations are reviewed. The applications of localization techniques are listed and briefly discussed.

Chapter 2: "Review of localization in geometry" (Vasily Pestun)

This chapter is a short summary of the mathematical aspects of the Berline-Vergne-Atiyah-Bott formula and Atiyah-Singer index theory. These tools are routinely used throughout the volume. The chapter reviews the definition of equivariant cohomology, and its Weyl and Cartan models. The standard characteristic classes and their equivariant versions are reviewed. The equivariant integration is discussed and the mathematical derivation of the Berline-Vergne-Atiyah-Bott formula is explained. The Atiyah-Singer index theorems and their equivariant versions are briefly reviewed.

Chapter 3: "Supersymmetric localization in two dimensions" (Francesco Benini and Bruno Le Floch)

This chapter concentrates on the localization techniques for 2d supersymmetric gauge theories and on the major applications of 2d localization results. The main example is the calculation of the partition function for $\mathcal{N} = (2, 2)$ gauge theory on S^2. Two different approaches are presented, the Coulomb branch localization (when the result is written as an integral over the Cartan subalgebra) and the Higgs branch localization (when the answer is written as a sum). Briefly $\mathcal{N} = (2, 2)$ gauge theories on other curved backgrounds are discussed, and the calculation for the hemisphere is presented. The important calculation of the partition function for $\mathcal{N} = (2,2)$ and $\mathcal{N} = (2,0)$ theories on the torus is presented, this quantity is known as the elliptic genus. The result is written in terms of the Jeffrey-Kirwan residue, which is a higher dimensional analog of the residue operation. The mathematical aspects of the Jeffrey-Kirwan residue operation are briefly explained. As the main application of the localization calculation in 2d, some dualities are discussed; in particular mirror symmetry and Seiberg duality.

Chapter 4: "Gromov-Witten invariants and localization" (David Morrison)

This chapter concentrates on an important application of 2d localization calculation, see Chapter 3. The chapter provides a pedagogical introduction to the relation between the genus 0 Gromov-Witten invariants (counting of holomorphic maps) and the localization of 2d gauged linear sigma models. The relation is based on the conjecture which connects the partition function of $\mathcal{N} = (2,2)$ gauge theories on S^2 with the Zamolodchikov metric on the conformal manifold of the theory. This relation allows to deduce the Gromov-Witten invariants on the Calabi-Yau manifold from the partition function on S^2 of the corresponding linear sigma model. This chapter explains this conjecture and reviews the main step of the calculation.

Chapter 5: "An Introduction to supersymmetric field theories in curved space" (Thomas Dumitrescu)

This chapter addresses the problem of defining rigid supersymmetric theories on curved backgrounds. The systematic approach to this problem is based on the Festuccia-Seiberg work on organizing the background fields into off-shell supergravity multiplets. The chapter concentrates in details on two major examples, $\mathcal{N} = 1$ supersymmetric theories in 4d and $\mathcal{N} = 2$ supersymmetric theories in 3d. The full classification of supersymmetric theories on curved backgrounds can be given for the theories with four or fewer supersymmetry in four or fewer dimensions.

Chapter 6: "Localization on three-dimensional manifolds" (Brian Willett)

This chapter provides an introduction to the localization technique for 3d supersymmetric gauge theories. The 3d $\mathcal{N} = 2$ supersymmetric theories are introduced and their formulation on curved space is briefly discussed, this is closely related to Chapter 5. The calculation of the partition function on S^3 is presented in great details with the final result presented as an integral over the Cartan sublagebra of the Lie algebra of the gauge group. The calculation on the lens spaces, on $S^2 \times S^1$ and different applications of these calculations are also discussed. The dualities between different gauge theories are briefly discussed. The factorization of the result into holomorphic blocks is also considered, and in this context the Higgs branch localization is discussed.

Chapter 7: "Localization at large N in Chern-Simons-matter theories" (Marcos Mariño)

The result of the localization calculation in 3d is given in terms of matrix integrals, see Chapter 6. These matrix integrals are complicated and it is not easy to extract information from this answer. This chapter is devoted to the study of 3d matrix models and extracting physical information from them. The chapter concentrates on the famous ABJM model which plays a crucial role in the AdS/CFT correspondence. The M-theory expansion for the ABJM model is discussed in details and the relation to topological strings is presented.

Chapter 8: "The F-Theorem and F-Maximization" (Silviu Pufu)

The partition function on S^3 for $\mathcal{N} = 2$ supersymmetric gauge theories is written as matrix integrals which depend on the different parameters of the theory, see Chapter 6. This chapter studies the properties of the free energy (minus the logarithm of the sphere partition function), which is regarded as the measure of the degrees of freedom in the theory. In particular the chapter states and explains the F-theorem and F-maximization principles for 3d theories. The F-theorem is a 3d analogue of the Zamolodchikov's c-theorem in 2d and the a-theorem in 4d. For 3d theories the F-theorem makes a precise statement about the idea that the number of degrees of freedom decreases along the RG flow.

Chapter 9: "Perturbative and nonperturbative aspects of complex Chern-Simons Theory" (Tudor Dimofte)

This chapter discusses another important application for the localization calculation in 3d. The chapter starts by briefly reviewing some basic facts about the complex Chern-Simons theory, the main interest is the Chern-Simons theory for $SL(N, \mathbb{C})$. There is a short discussion of the 3d/3d correspondence, which states that the partition function of the complex Chern-Simons theory on M is the same as the partition function of a specific supersymmetric gauge theory (whose field content depends on M) on the lens space. The chapters finishes with a discussion of the quantum modularity conjecture.

Chapter 10: "$\mathcal{N} = 2$ SUSY gauge theories on $\mathbf{S}^4$" (Kazuo Hosomichi)

This chapter gives a detailed exposition of the calculation of the partition function and other supersymmetric observables for $\mathcal{N} = 2$ supersymmetric gauge theories on S^4, both round and squashed. Using off-shell supergravity, the construction of $\mathcal{N} = 2$ supersymmetric theories on squashed S^4 is presented. The localization calculation is performed and the determinants are explicitly evaluated using index theorems (review in Chapter 2). The inclusion of supersymmetric observables (Wilson loops, 't Hooft operators and surface operators) into the localization calculation on S^4 is discussed.

Chapter 11: "Localization and AdS/CFT Correspondence" (Konstantin Zarembo)

One of the major application of the localization calculation on S^4 (see Chapter 10) is the application to AdS/CFT. This chapter is devoted to the study of the matrix models which appear in the calculation on S^4 and its application to the AdS/CFT correspondence. Localization offers a unique laboratory for the AdS/CFT correspondence, since we are able to explore the supersymmetric gauge theory in non-perturbative domain. Using holography the localization computation can be compared to string theory and supergravity calculations.

Chapter 12: "A brief review of the 2d/4d correspondences" (Yuji Tachikawa)

From the perspective of the $\mathcal{N} = (0, 2)$ self-dual 6d theory, this chapter explains the 2d/4d correspondence (AGT), considering the 6d theory on a product of 2d and 4d manifold. This correspondence relates the 4d computations for supersymmetric gauge theories of class $\mathcal{S}$, obtained by compactification of the 6d theory on the 2d manifold, to 2d computations in 2d theory obtained by compactification of 6d theory on the 4d manifold. The chapter starts by reviewing basic facts about 2d q-deformed Yang-Mills theory and the Liouville theory. The main building block of rank 1 theories considered in the chapter is the trifundamental multiplet coupled with $SU(2)$ gauge fields. The partition function on $S^1 \times S^3$ for such a 4d theory is computed in 2d by q-deformed Yang-Mills and the partition function on S^4 is computed in 2d by the Liouville theory.

Chapter 13: "The supersymmetric index in four dimensions" (Leonardo Rastelli and Shlomo Razamat)

This chapter studies the partition function on $S^3 \times S^1$ for $\mathcal{N} = 1$ superymmetric theories in 4d, also known as the 4d supersymmetric index. The chapter starts by defining the supersymmetric index and reviewing combinatorial tools to compute it in theories with a Lagrangian description. After illustrating some basic properties of the index in the simple setting of supersymmetric sigma models, the chapter turns to the discussion of the index of supersymmetric gauge theories, emphasizing physical applications. The index contains useful information about the spectrum of shortened multiplets, and how to extract this information is discussed in some detail. The most important application of the index, as a powerful tool for checking non-perturbative dualities between supersymmetric gauge theories, is illustrated in several examples. The last part of the chapter considers several interesting limits of the supersymmetric index.

Chapter 14: "Review of localization for 5d supersymmetric gauge theories" (Jian Qiu and Maxim Zabzine)

The chapter provides the introduction to localization calculation for $\mathcal{N} = 1$ supersymmetric gauge theories on toric Sasaki-Einstein manifolds, for example on a five-sphere S^5. The chapter starts by recalling basic facts about supersymmetry and supersymmetric gauge theories in flat 5d space. Then the construction of the supersymmetric gauge theory on the Sasaki-Einstein manifolds is explicitly given. Using the field redefinition, the supersymmetry transformations are rewritten in terms of differential forms, thus making geometrical aspects of the localization more transparent. For toric Sasaki-Einstein manifolds the localization calculation can be carried out completely, the calculation of determinants is given and the full partition function is conjectured. The chapter ends with comments about deducing the flat space results from the curved result.

Chapter 15: "Matrix models for 5d super Yang-Mills" (Joseph Minahan)

The result of 5d localization calculation is given in terms of complicated matrix models, see Chapter 14. This chapter studies the resulting matrix models. The basic properties of the matrix models are described and the 't Hooft limit is analyzed for $\mathcal{N} = 1^*$ theory (a vector multiplet plus a hypermultiplet in the adjoint representation). For large 't Hooft coupling the free energy behaves as N^3 for $U(N)$ gauge theory and the corresponding supergravity analysis is performed. This analysis support the idea that the non-perturbative completion of 5d theory is the 6d $\mathcal{N} = (2,0)$ superconformal field theory.

Chapter 16: "Holomorphic blocks and the 5d AGT correspondence" (Sara Pasquetti)

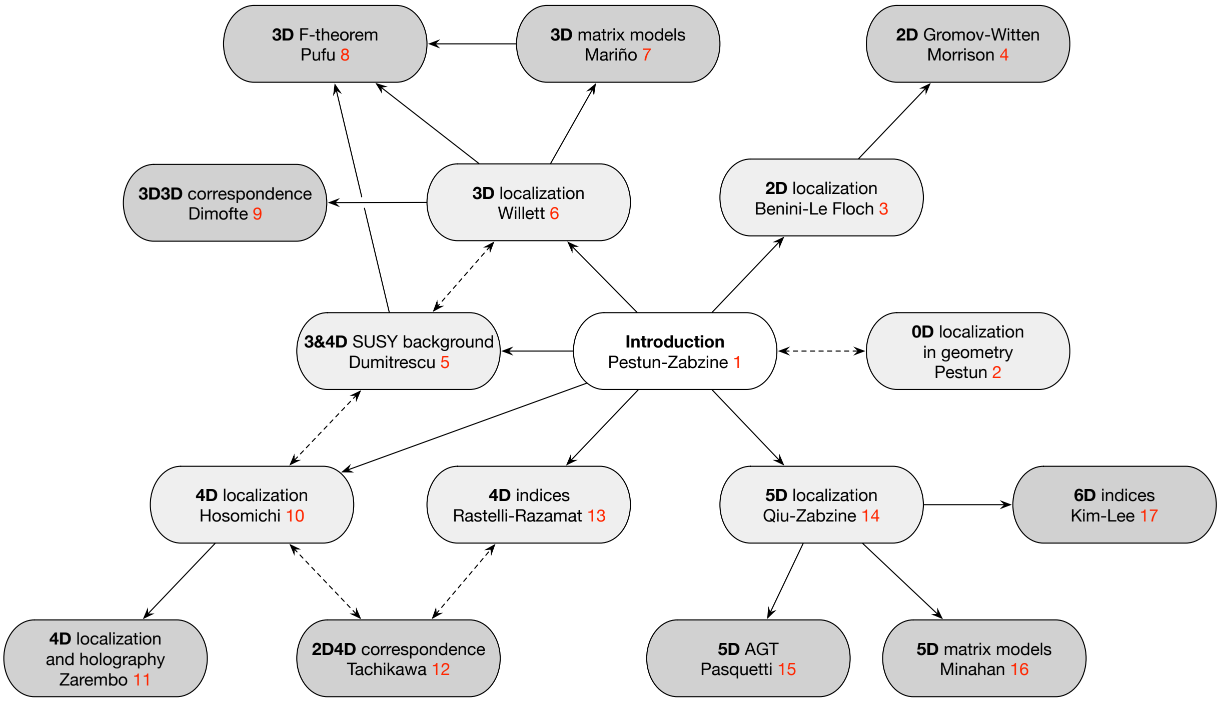
This chapter further studies partition functions from 2d to 5d. In particular it concentrates on the idea that the partition function on a compact manifold can be built up from basic blocks, so-called holomorphic blocks. The main point is that the geometric decomposition of the compact manifold should have its counterpart in the appropriate decomposition of the partition function. These factorization properties are reviewed in different dimensions. The rest of the chapter concentrates on a 5d version of the AGT correspondence.

Chapter 17: "Indices for 6 dimensional superconformal field theories" (Seok Kim and Kimyeong Lee)

This chapter deals with the 6d $(2,0)$ superconformal field theory. This theory cannot be accessed directly, but it is related to many other supersymmetric gauge theories, e.g. it is believed to be the UV-completion of maximally supersymmetric 5d gauge theory. The relation between 5d partition function and 6d supersymmetric index is discussed in details in this chapter.

0.3 Volume structure

The different chapters are related to each other and the relation is not a simple linear relation, which can be shown by their ordering in the volume. Below we provide the graphical relation between different chapters. This diagram[2] gives the general idea.



References

[1] V. Pestun and M. Zabzine, "Introduction to localization in quantum field theory," *Journal of Physics A* **xx** (2016) 000, 1608.02953.

[2] V. Pestun, "Review of localization in geometry," *Journal of Physics A* **xx** (2016) 000, 1608.02954.

[3] F. Benini and B. Le Floch, "Supersymmetric localization in two dimensions," *Journal of Physics A* **xx** (2016) 000, 1608.02955.

[4] D. Morrison, "Gromov-Witten invariants and localization," *Journal of Physics A* (2016) , 1608.02956.

[5] T. Dumitrescu, "An Introduction to Supersymmetric Field Theories in Curved Space," *Journal of Physics A* **xx** (2016) 000, 1608.02957.

[6] B. Willett, "Localization on three-dimensional manifolds," *Journal of Physics A* **xx** (2016) 000, 1608.02958.

[7] M. Mariño, "Localization at large N in Chern-Simons-matter theories," *Journal of Physics A* **xx** (2016) 000, 1608.02959.

[8] S. Pufu, "The F-Theorem and F-Maximization," *Journal of Physics A* **xx** (2016) 000, 1608.02960.

[2]Special thanks to Yuji Tachikawa for the final design of the diagram

[9] T. Dimofte, “Perturbative and nonperturbative aspects of complex Chern-Simons Theory,” *Journal of Physics A* **xx** (2016) 000, 1608.02961.

[10] K. Hosomichi, “$\mathcal{N} = 2$ SUSY gauge theories on S^4,” *Journal of Physics A* **xx** (2016) 000, 1608.02962.

[11] K. Zarembo, “Localization and AdS/CFT Correspondence,” *Journal of Physics A* **xx** (2016) 000, 1608.02963.

[12] Y. Tachikawa, “A brief review of the 2d/4d correspondences,” *Journal of Physics A* **xx** (2016) 000, 1608.02964.

[13] L. Rastelli and S. Razamat, “The supersymmetric index in four dimensions,” *Journal of Physics A* **xx** (2016) 000, 1608.02965.

[14] J. Qiu and M. Zabzine, “Review of localization for 5D supersymmetric gauge theories,” *Journal of Physics A* **xx** (2016) 000, 1608.02966.

[15] J. Minahan, “Matrix models for 5D super Yang-Mills,” *Journal of Physics A* **xx** (2016) 000, 1608.02967.

[16] S. Pasquetti, “Holomorphic blocks and the 5d AGT correspondence,” *Journal of Physics A* **xx** (2016) 000, 1608.02968.

[17] S. Kim and K. Lee, “Indices for 6 dimensional superconformal field theories,” *Journal of Physics A* **xx** (2016) 000, 1608.02969.

Chapter 1

Introduction to localization in quantum field theory

Vasily Pestun[1] and Maxim Zabzine[2]

[1]*Institut des Hautes Études Scientifique, France*
[2]*Department of Physics and Astronomy, Uppsala University, Sweden*
pestun@ihes.fr, Maxim.Zabzine@physics.uu.se

Abstract

This is the introductory chapter to the volume. We review the main idea of the localization technique and its brief history both in geometry and in QFT. We discuss localization in diverse dimensions and give an overview of the major applications of the localization calculations for supersymmetric theories. We explain the focus of the present volume.

Contents

1.1 Main idea and history **15**
1.2 Localization in diverse dimensions **18**
1.2.1 Topological Yang-Mills 21
1.2.2 Even dimensions 23
1.2.3 Odd dimensions 27
1.3 Applications of the localization technique **29**
References **31**

1.1 Main idea and history

According to the English dictionary[1] the word *localize* means to make local, fix in or assign or restrict to a particular place, locality. Both in mathematics and physics the word "localize" has multiple meanings and typically physicists with different backgrounds mean different things by localization. This volume is devoted to the extension of the Atiyah-Bott localization formula (and related statements, e.g. the Duistermaat-Heckman formula and different versions of the fixed-point theorem) in differential geometry to an infinite dimensional situation of path integral, and in particular in the context of supersymmetric quantum field theory. In quantum field theory one says "supersymmetric localization" to denote such computations. In this volume we concentrate on the development of the supersymmetric localization technique during the last ten years, 2007-2016.

In differential geometry the idea of localization can be traced back to 1926 [2], when Lefschetz proved the fixed-point formula which counts fixed points of a continuous map of a topological space to itself by using the graded trace of the induced map on the homology groups of this space. In the 1950's, the Grothendieck-Hirzebruch-Riemann-Roch theorem expressed in the most general form the index of a holomorphic vector bundle (supertrace over graded cohomology space) in terms of certain characteristic classes. In the 1960's, the Atiyah-Singer index theorem solved the same problem for an arbitrary elliptic complex.

In 1982 Duistermaat and Heckman [3] proved the following formula

$$\int_M \frac{\omega^n}{n!} e^{-\mu} = \sum_i \frac{e^{-\mu(x_i)}}{e(x_i)} \,, \tag{1.1.1}$$

where M is a symplectic compact manifold of dimension $2n$ with symplectic form ω and with a Hamiltonian $U(1)$ action whose moment map is μ. Here x_i are the fixed points of the $U(1)$ action and they are assumed to be isolated, and $e(x_i)$ is the product of the weights of the $U(1)$ action on the tangent space at x_i. Later independently in 1982 Berline and Vergne [4] and in 1984 Atiyah and Bott [5] generalized the Duistermaat-Heckman formula to the case of a general compact manifold M with a $U(1)$ action and an integral $\int \alpha$ of an equivariantly-closed form α, that is $(d + \iota_V)\alpha = 0$, where $V(x)$ is the vector field corresponding to the $U(1)$ action. The Berline-Vergne-Atiyah-Bott formula reads as

$$\int_M \alpha = \sum_i \frac{\pi^n \alpha_0(x_i)}{\sqrt{\det(\partial_\mu V^\nu(x_i))}} \,, \tag{1.1.2}$$

where it is assumed that x_i are isolated fixed points of the $U(1)$ action, and α_0 is the zero-form component of α. The Berline-Vergne-Atiyah-Bott formula has multiple generalizations, to the case of non-isolated fixed locus, to supermanifolds, to the holomorphic case, etc. The more detailed overview of this formula and its relation to equivariant cohomology is given in Chapter 2. Here we will concentrate on conceptual issues and our discussion is rather schematic.

[1] E.g. http://www.dictionary.com, based on the Random House Dictionary, ©Random House, Inc. 2016

Let us review the proof of the Berline-Vergne-Atiyah-Bott formula (1.1.2). We will use the language of supergeometry, since it is easier to generalize to the infinite dimensional setup. Consider the odd tangent bundle ΠTM where x^μ are coordinates on M and ψ^μ are odd coordinates on the fiber (i.e., they transform as dx^μ). Functions $f(x,\psi)$ correspond to differential forms and the integration measure $d^n x\ d^n\psi$ on ΠTM is canonically defined. Assume that there is a $U(1)$ action on compact M with the corresponding vector field $V^\mu(x)\partial_\mu$. Define the following "supersymmetry transformations"

$$\begin{aligned} \delta x^\mu &= \psi^\mu \\ \delta \psi^\mu &= V^\mu(x) \end{aligned} \tag{1.1.3}$$

which correspond to the equivariant differential $d + \iota_V$. We are interested in computation of the integral

$$Z(0) = \int\limits_{\Pi TM} \alpha(x,\psi)\ d^n x\ d^n\psi \tag{1.1.4}$$

for $\alpha(x,\psi)$ a "supersymmetric observable", i.e. an equivariantly closed form $\delta\alpha(x,\psi) = 0$. We can deform the integral in the following way

$$Z(t) = \int\limits_{\Pi TM} \alpha(x,\psi)\ e^{-t\delta W(x,\psi)}\ d^n x\ d^n\psi\ , \tag{1.1.5}$$

where $W(x,\psi)$ is some function. Using the Stokes theorem, one can show that the integral $Z(t)$ is independent of t, provided that $\delta^2 W = 0$. For example, we can choose $W = V^\mu g_{\mu\nu}\psi^\nu$ with $g_{\mu\nu}$ being a $U(1)$-invariant metric. If $Z(t)$ is independent of t, then we can calculate the original integral at $t = 0$ at another value of t, in particular we can send t to infinity

$$Z(0) = \lim_{t\to\infty} Z(t) = \lim_{t\to\infty} \int\limits_{\Pi TM} \alpha(x,\psi)\ e^{-t\delta W(x,\psi)}\ d^n x\ d^n\psi\ . \tag{1.1.6}$$

Thus using the saddle point approximation for $Z(t)$ we can calculate the exact value of $Z(0)$. If we choose $W = V^\mu g_{\mu\nu}\psi^\nu$ with the invariant metric and perform the calculation we arrive at the formula (1.1.2). Let us outline the main steps of the derivation. In the integral (1.1.6)

$$\delta W = V^\mu g_{\mu\nu} V^\nu + \partial_\rho(V^\mu g_{\mu\nu})\psi^\rho\psi^\nu \tag{1.1.7}$$

and thus in the limit $t \to \infty$ the critical points x_i of the $U(1)$ action dominate, $V(x_i) = 0$. Let us consider the contribution of one isolated point x_i, and for the sake of clarity let's assume that $x_i = 0$. In the neighbourhood of this critical point 0, we can rescale coordinates as follows

$$\sqrt{t}x = \tilde{x}\ ,\ \ \sqrt{t}\psi = \tilde{\psi}\ , \tag{1.1.8}$$

so that the integral expression (1.1.6) becomes

$$Z(0) = \lim_{t\to\infty} \int\limits_{\Pi TM} \alpha\left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}}\right)\ e^{-t\delta W\left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}}\right)}\ d^n\tilde{x}\ d^n\tilde{\psi}\ , \tag{1.1.9}$$

where we have used the property of the measure on ΠTM, $d^n x\ d^n \psi = d^n \tilde{x}\ d^n \tilde{\psi}$. Now in (1.1.9) we may keep track of only leading terms which are independent of t. In the exponent with δW only the quadratic terms are relevant

$$\delta W = H_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu + S_{\mu\nu} \tilde{\psi}^\mu \tilde{\psi}^\nu \ , \tag{1.1.10}$$

where the concrete form of the matrices H and S is irrelevant. In the limit $t \to \infty$ the "supersymmetry transformations" (1.1.3) are naturally linearized

$$\begin{aligned} \delta \tilde{x}^\mu &= \tilde{\psi}^\mu \\ \delta \tilde{\psi}^\mu &= \partial_\nu V^\mu(0) \tilde{x}^\nu \ , \end{aligned} \tag{1.1.11}$$

and the condition $\delta^2 W = 0$ now implies

$$H_{\mu\nu} = S_{\mu\sigma} \partial_\nu V^\sigma(0) \ . \tag{1.1.12}$$

Now in the integral (1.1.9) we have to take the limit $t \to \infty$ and perform the gaussian integral in even and odd coordinates

$$Z(0) = \alpha(0,0) \frac{\pi^{\dim M/2} \mathrm{Pf}(S)}{\sqrt{\det H}} \tag{1.1.13}$$

and using (1.1.12) we arrive at

$$Z(0) = \alpha(0,0) \frac{\pi^{\dim M/2}}{\sqrt{\det \partial_\mu V^\nu(0)}} \ . \tag{1.1.14}$$

If we repeat this calculation for every fixed point, we arrive at the Berline-Vergne-Atiyah-Bott formula (1.1.2). This is the actual proof for a $U(1)$ action on a compact M. In principle the requirement of a $U(1)$ action can be relaxed to V being Killing vector on a compact M, since in the derivation we only use the invariance of the metric to construct the appropriate W. For non-compact spaces, one can use the Berline-Vergne-Atiyah-Bott formula as a suitable definition of the integral, for example to introduce the notion of equivariant volume etc. There are many generalizations of the above logic, for example one can construct the holomorphic version of the equivariant differential with the property $\delta^2 = 0$ etc.

This setup can be formally generalized to the case where M is an infinite dimensional manifold.

Indeed, we can regard this as the definition of the infinite dimensional integral, provided that the formal properties are preserved. However, in the infinite dimensional case, the main challenge is to make sure that all steps of the formal proof can be suitably defined, for example the choice of a suitable W may become a non-trivial problem. In the infinite dimensional situation the matrix $\partial_\nu V^\mu(0)$ in (1.1.11) turns into a differential operator and the (super)-determinant of this differential operator should be defined carefully.

The most interesting applications of these ideas come from supersymmetric gauge theories. In this case, one tries to recognise the supersymmetry transformations together with the

BRST-symmetry coming from the gauge fixing as some type of equivaraint differential (1.1.3) acting on the space of fields (an infinite dimensional supermanifold).

In the context of the infinite-dimensional path integral, the localization construction was first proposed by Witten in his work on supersymmetric quantum mechanics [7]. In that case the infinite dimensional manifold M is the loop space LX of an ordinary smooth manifold X. In the simplest case, the $U(1)$ action on LX comes from the rotation of the loop. Similar ideas were later applied to two-dimensional topological sigma model [8] and four dimensional topological gauge theory [9]. In the 1990's the ideas of localization were widely used in the setup of cohomological topological field theories, e.g. see [10] for nice applications of these ideas to two-dimensional Yang-Mills theory. Further development on supersymmetric localization is related to the calculation of Nekrasov's partition function, or equivariant Donaldson-Witten theory [11], based on earlier works [12–15].

The focus of this volume is on the developments starting from the work [16], where the exact partition function and the expectation values of Wilson loops for $\mathcal{N}=2$ supersymmetric gauge theories on S^4 were calculated. In [16] the 4d $\mathcal{N}=2$ theory was placed on S^4, preserving 8 supercharges, and the supersymmetry transformations together with BRST-transformations were recognized as the equivariant differential on the space of fields. The zero modes were carefully treated by Atiyah-Singer index theorem, and the final result was written as a finite-dimensional integral over the Cartan algebra of the Lie algebra of the gauge group. Later this calculation was generalized and extended to other types of supersymmetric theories, other dimensions and geometries. These exact results provide a unique laboratory for the study of non-perturbative properties of gauge theories. Some contributions to this volume provide an overview of the actual localization calculation in concrete dimensions, for concrete class of theories, while other contributions look at the applications of the results and discuss their physical and mathematical significance.

1.2 Localization in diverse dimensions

In order to apply the localization technique to supersymmetric theories one needs to resolve a number of technical and conceptual problems. First of all, one needs to define a rigid supersymmetric theory on curved manifolds and understand what geometrical data goes into the construction. The old idea was that rigid supersymmetry on curved manifolds requires an existence of covariantly constant spinors which would correspond to the parameters in the supersymmetry transformations. The next natural generalization would be if the supersymmetry parameters satisfy the Killing spinor equations [17]. For example, all spheres admit Killing spinors and thus supersymmetric gauge theories can be constructed on spheres. However, a more systematic view on supersymmetric rigid theories on curved manifolds has been suggested in [18] giving background values to auxiliary fields in the supergravity. (More recently an approach of topological gravity was explored in [19, 20].) This approach allows in principle to analyze rigid supersymmetric theories on curved manifolds, although the analysis appears to be increasingly complicated as we deal with higher dimensions and more supersymmetry. At the moment we know how to place on a curved manifold the supersymmetric theories, which in flat space have four or fewer supercharges, in dimension 2,3 and 4 for both Euclidean and Lorentzian signatures [21–24]. For other cases only partial results are available. For example, in four dimensions the situation for theories with eight supercharges remains open, see e.g. [25–27]. Situation is similar in five dimensions, see e.g. [28–30] and in six dimensions [31]; see also [32, 33] in the context of superspace treatment of rigid supergravity. Thus despite the surge in the activity the full classification of supersymmetric theories on curved manifolds remains an open problem. Rigid supersymmetric theories on curved manifolds are discussed in Chapter 5.

Moreover, in order to be able to carry the localization calculation explicitly and write the result in closed form, we need manifolds with enough symmetries, for example with a rich toric action. Again we do not know the full classification of curved manifolds that allow both a toric action and a rigid supersymmetric gauge theory. In 3d we know how to localize the theories with 4 supercharges on S^3, on lens spaces L_p and on $S^2 \times S^1$. In 4d the situation becomes more complicated, we know how to localize the theories with 8 supercharges on S^4 and with 4 supercharges on $S^3 \times S^1$, but the general situation in 4d remains to be understood. In 5d there exists an infinite family of toric Sasaki-Einstein manifold (S^5 is one of them) for which the result up to non-perturbative contributions can be written explicitly for the theories with 8 supercharges. Notice, however, that this is not the most general 5d manifolds which admit the rigid supersymmetry, e.g. a bit separated example is $S^4 \times S^1$. In 6d the nearly Kähler manifolds (e.g., S^6) will allow the theories with 16 supercharges and in 7d the toric Sasaki-Einstein manifolds (e.g., S^7) will allow the theories with 16 supercharges.

The best studied examples are the supersymmetric gauge theories on spheres S^d, which we are going to review briefly since they provide the nice illustration for the general results. The first results were obtained for S^4 in [16], for S^3 in [35], for S^2 in [36,37], for S^5 in [38–40] and finally for S^6 and S^7 were addressed in [41]. These calculations were generalized and extended to the squashed S^3 [42,43], to the squashed S^4 [27,44], the squashed S^5 [45–47] and the result for the squashed S^6 and S^7 was already suggested in [41]. There is also an attempt in [48] to analytically continue the partition function on S^d to generic complex values of d.

Let us describe the result for different spheres in a uniform fashion. We consider the general case of squashed spheres.

The odd and even dimensional spheres S^{2r-1} and S^{2r} lead to two types of special functions called S_r and Υ_r that are used to present the result.

The main building block of these functions is the multiple inverse Gamma function $\gamma_r(x|\epsilon_1, \ldots, \epsilon_r)$, which is a function of a variable x on the complex plane $\mathbb{C}$ and r complex parameters $\epsilon_1, \ldots, \epsilon_r$. This function is defined as a ζ-regularized product

$$\gamma_r(x|\epsilon_1, ..., \epsilon_r) = \prod_{n_1,...,n_r=0}^{\infty} (x + n_1\epsilon_1 + \cdots + n_r\epsilon_r), \tag{1.2.1}$$

The parameters ϵ_i should belong to an open half-plane of $\mathbb{C}$ bounded by a real line passing trough the origin. The unrefined version of γ_r is defined as

$$\gamma_r(x) = \gamma_r(x|1, ..., 1) = \prod_{k=0}^{\infty} (x + k)^{\frac{(k+1)(k+2)\ldots(k+r-1)}{(r-1)!}} \ . \tag{1.2.2}$$

The Υ_r-function, obtained from the localization on S^{2r}, is defined as

$$\Upsilon_r(x|\epsilon_1,...,\epsilon_r) = \gamma_r\left(x|\epsilon_1,...,\epsilon_r\right)\gamma_r\left(\sum_{i=1}^{r}\epsilon_i - x|\epsilon_1,...,\epsilon_r\right)^{(-1)^r} . \tag{1.2.3}$$

These functions form a hierarchy with respect to a shift of x by one of ϵ-parameters

$$\Upsilon_r(x+\epsilon_i|\epsilon_1,...,\epsilon_i,...,\epsilon_r) = \Upsilon_{r-1}^{-1}(x|\epsilon_1,...,\epsilon_{i-1},\epsilon_{i+1},...,\epsilon_r)\Upsilon_r(x|\epsilon_1,...,\epsilon_i,...,\epsilon_r) \tag{1.2.4}$$

The unrefined version of Υ_r is defined as follows

$$\Upsilon_r(x) = \Upsilon_r(x|1,...,1) = \prod_{k\in\mathbb{Z}}(k+x)^{\mathrm{sgn}(k+1)\frac{(k+1)(k+2)...(k+r-1)}{(r-1)!}} . \tag{1.2.5}$$

The S_r-function, called multiple sine, obtained from localization on S^{2r-1}, is defined as

$$S_r(x|\epsilon_1,...,\epsilon_r) = \gamma_r(x|\epsilon_1,...,\epsilon_r)\gamma_r\left(\sum_{i=1}^{r}\epsilon_i - x|\epsilon_1,...,\epsilon_r\right)^{(-1)^{r-1}} . \tag{1.2.6}$$

See [49] for exposition and further references. These functions also form a hierarchy with respect to a shift of x by one o thef ϵ-parameters

$$S_r(x+\epsilon_i|\epsilon_1,...,\epsilon_i,...,\epsilon_r) = S_{r-1}^{-1}(x|\epsilon_1,...,\epsilon_{i-1},\epsilon_{i+1},...,\epsilon_r)S_r(x|\epsilon_1,...,\epsilon_i,...,\epsilon_r). \tag{1.2.7}$$

Notice that $S_1(x|\epsilon) = 2\sin(\frac{\pi x}{\epsilon})$ is a periodic function. Thus S_1 is periodic by itself, S_2 is periodic up to S_1^{-1}, S_3 is periodic up to S_2^{-1} etc. The unrefined version of multiple sine is defined as

$$S_r(x) = S_r(x|1,...,1) = \prod_{k\in\mathbb{Z}}(k+x)^{\frac{(k+1)(k+2)...(k+r-1)}{(r-1)!}} . \tag{1.2.8}$$

The result for a vector multiplet with $4, 8$ and 16 supercharges placed on a sphere S^2, S^4 and S^6 respectively is given in terms of Υ_r functions as follows

$$Z_{S^{2r}} = \int_{\mathfrak{t}} da \prod_{w\in R_{\mathrm{ad}\,\mathfrak{g}}} \Upsilon_r(iw\cdot a|\epsilon)\; e^{P_r(a)} + \cdots\ , \tag{1.2.9}$$

where the integral is taken over the Cartan subalgebra of the gauge Lie algebra $\mathfrak{g}$, the w are weights of the adjoint representation of $\mathfrak{g}$ and $P_r(a)$ is the polynomial in a of degree r,

$$P_r(a) = \alpha_r \operatorname{Tr}(a^r) + \cdots + \alpha_2 \operatorname{Tr}(a^2) + \alpha_1 \operatorname{Tr}(a). \tag{1.2.10}$$

The polynomial $P_r(a)$ is coming from the classical action of the theory. The parameters α_i are related to the Yang-Mills coupling, the Chern-Simons couplings and the FI couplings.

The sphere S^{2r} admits T^r action with two fixed points, and the parameters $\epsilon_1, \ldots, \epsilon_r$ are the squashing parameters for S^{2r} (at the same time $\epsilon_1, \ldots, \epsilon_r$ are equivariant parameters for the T^r action).

For S^2, the dots are non-perturbative contributions coming from other localization loci with non-trivial magnetic fluxes (review in Chapter 3). For S^4, the dots correspond to the contributions of point-like instantons over the north and south poles computed by the Nekrasov instanton partition function (review in Chapter 10). For the case of S^6 the expression corresponds to maximally supersymmetric theory on S^6, and the nature of the dots remains to be understood.

The partition function of the vector multiplet with 4, 8, or 16 supercharges on the odd-dimensional spheres S^3, S^5 and S^7, or S^{2r-1} with $r = 2, 3, 4$, is given by

$$Z_{S^{2r-1}} = \int_{\mathfrak{t}} da \prod_{w \in R_{\mathrm{ad}\,\mathfrak{g}}} S_r(iw \cdot a|\epsilon)\; e^{P_r(a)} + \cdots \,, \tag{1.2.11}$$

where now ϵ-parameters are equivariant parameters of the $T^r \subset SO(2r)$ toric action on S^{2r-1}.

For S^3 the dots are absent and the expression (1.2.11) provides the full results for $\mathcal{N} = 2$ vector multiplet on S^3 (review in Chapter 6). For S^5 the formula (1.2.11) provides the result for $\mathcal{N} = 1$ vector multiplet (review in Chapter 14). The theory on S^7 is unique and it corresponds to the maximally supersymmetric Yang-Millls in 7d with 16 supercharges.

For the case of S^5 and S^7 the dots are there and they correspond to the contributions around non-trivial connection satisfying certain non-linear PDEs. There are some natural guesses about these corrections, but there are no systematic derivation and no understanding of them, especially for the case of S^7.

Our present discussion can be summarized in the following table:

dim	multiplet	#super	special function	references	derivation
S^2	$\mathcal{N} = 2$ vector	4	$\Upsilon_1(x\|\epsilon)$	[36, 37]	Chapter 3
S^3	$\mathcal{N} = 2$ vector	4	$S_2(x\|\epsilon)$	[35]	Chapter 6
S^4	$\mathcal{N} = 2$ vector	8	$\Upsilon_2(x\|\epsilon)$	[16]	Chapter 10
S^5	$\mathcal{N} = 1$ vector	8	$S_3(x\|\epsilon)$	[38–40]	Chapter 14
S^6	$\mathcal{N} = 2$ vector	16	$\Upsilon_3(x\|\epsilon)$	[41]	
S^7	$\mathcal{N} = 1$ vector	16	$S_4(x\|\epsilon)$	[41]	

The contribution of matter multiplet (chiral multiplet for theories with 4 supercharges and hypermultiplets for theories with 8 supercharges) can be expressed in terms of the same special functions, see next section.

The detailed discussion of the localization calculation on the spheres and other manifolds can be found in different contributions in this volume, 2d is discussed in Chapter 3, 3d in Chapter 6, 4d in Chapter 10, 5d in Chapter 14.

Next we can schematically explain the above result.

1.2.1 Topological Yang-Mills

We recall that $\mathcal{N} = 1$ super Yang-Mills theory is defined in dimension $d = 3, 4, 6, 10$ and that the algebraic structure of supersymmetry transformations is related to an isomorphism that one can establish between $\mathbb{R}^{d-2}$ and the famous four division algebras:

SYM	algebra	S	$\dim \mathsf{S}$	top SYM	equations	#equations	quotient
3d	$\mathbb{R}$	S	2	1d	—	0	—
4d	$\mathbb{C}$	S	4	2d	$F = 0$	1	Kahler
6d	$\mathbb{H}$	$S^+ \otimes \mathbb{C}^2$	8	4d	$F = - \star F$	3	hyperKahler
10d	$\mathbb{O}$	S^+	16	8d	$F = - \star (F \wedge \Omega)$	7	octonionic

In this table S denotes the $2^{\lfloor d/2 \rfloor}$-dimensional Dirac spinor representation of $Spin(d)$ group. The S^+ denotes the chiral (Weyl) spinor representation of $Spin(d)$. In all cases, one uses Majorana spinors in Lorenzian signature, or *holomorphic Dirac*[2] spinors in Euclidean signature. Notice the peculiarity of the 6d case where one uses chiral $Sp(1)$-doublet spinors with $\mathbb{C}^2$ being the fundamental representation of the $Sp(1) \simeq SU(2)$ R-symmetry, and that in the 10d case one uses a single copy of the chiral spinor representation. The number $\dim \mathsf{S}$ is often referred as the *the number of the supercharges in the theory.*

Also, it is well known that the $\mathcal{N} = 1$ under the dimensional reduction to the dimension $d - 2$ produces the 'topological' SYM which localizes to the solutions of certain first order (BPS type) elliptic equations on the gauge field strength of the curvature listed in the table. The 1d topYM is, of course, the trivial theory, with empty equations, since there is no room for the curvature 2-form in a 1-dimensional theory. The equation for 2d topYM is the equation of zero curvature, for 4d topYM it is the instanton equation of self-dual curvature (defined by the conformal structure on the 4d manifold), and for 8d topYM it is the equation of the octonionic instanton (defined by the Hodge star $\star$ operator and the Cayley 4-form Ω on a $Spin(7)$-holonomy 8d manifold).

The corresponding linearized complexes are

topYM	linearized complex	fiber dimensions
$\mathbb{R}$: 1d	$\Omega^0 \to \Omega^1$	$\mathbf{1} \to \mathbf{1}$
$\mathbb{C}$: 2d	$\Omega^0 \to \Omega^1 \to \Omega^2$	$\mathbf{1} \to \mathbf{2} \to \mathbf{1}$
$\mathbb{H}$: 4d	$\Omega^0 \to \Omega^1 \to \Omega^2_+$	$\mathbf{1} \to \mathbf{4} \to \mathbf{3}$
$\mathbb{O}$: 8d	$\Omega^0 \to \Omega^1 \to \Omega^2_{\mathrm{oct}}$	$\mathbf{1} \to \mathbf{8} \to \mathbf{7}$

(1.2.12)

Here Ω^p is a shorthand for $\Omega^p(X) \otimes \mathrm{ad}\, \mathfrak{g}$, that is the space of $\mathfrak{g}$-valued differential p-forms on X, where $\mathfrak{g}$ is the Lie algebra of the gauge group and X is the space-time manifold. In the 4d theory the space Ω^2_+ denotes the space of self-dual 2-forms that satisfy the instanton equation $F = - \star F$, and in the 8d theory the space Ω^2_{oct} is the space of 2-forms that satisfy the octonionic instanton equation $F = - \star (F \wedge \Omega)$.

In these complexes, the first term Ω^0 describes the tangent space to the infinite-dimensional group of gauge transformations on X, the second term Ω^1 describes the tangent space to the affine space of gauge connections on X, and the last term (Ω^2 for 2d, Ω^2_+ for 4d, Ω^2_{oct} for 8d) describes that space where the equations are valued.

If space-time X is invariant under an isometry group T, the topological YM can be treated equivariantly with respect to the T-action. The prototypical case is the equivariant

[2]This means that the spinors ψ in Euclidean signature are taken to be complex, but algebraically speaking, only ψ appears in the theory but not its complex conjugate $\bar{\psi}$

Donaldson-Witten theory, or 4d topYM in Ω-background defined on $\mathbb{R}^4$ equivariantly with respect to $T = SO(4)$, generating the Nekrasov partition function [11]. Special functions, like the Υ-function defined by infinite products like (1.2.3) are infinite-dimensional versions of the equivariant Euler class of the tangent bundle to the space of all fields appearing after localization of the path integral by Atiyah-Bott fixed point formula (see Chapter 2 section 8.1 for more details). The equivariant Euler class can be determined by computing first the equivariant Chern class (index) of the linearized complex describing the tangent space of the topological YM theory. The T-equivariant Chern class (index) for the equation elliptic complex

$$D : \cdots \to \Gamma(E_k, X) \to \Gamma(E_{k+1}, X) \to \dots \tag{1.2.13}$$

on space X made from sections of vector bundles $E_\bullet$, can be conveniently computed by the Atiyah-Singer index theorem

$$\operatorname{ind}_T(D) = \sum_{x\in X^T} \frac{\sum_k (-1)^k \operatorname{ch}_T(E_k)|_x}{\det_{T_x X}(1-t^{-1})} \tag{1.2.14}$$

where X^T is the fixed point set of T on X (see Chapter 2 section 11.1 for details).

1.2.2 Even dimensions

First we will apply the Atiyah-Singer index theorem (review in Chapter 2 section 11.1) for the *complexified* complexes (1.2.12) on $X = \mathbb{R}^d$ for $d = 2, 4, 8$ topological YM with respect to the natural $SO(d)$ equivariant action on $\mathbb{R}^d$ with fixed point $x = 0$.

For $d = 2r$ and $r = 1, 2, 4$ we pick the Cartan torus $T^r = U(1)^r$ in the $SO(2r)$ with parameters $(t_1, \dots, t_r) \in U(1)^r$. The denominator in the Atiyah-Singer index theorem is

$$\det(1-t^{-1})|_{\mathbb{R}^{2r}} = \prod_{s=1}^{n} (1-t_s)(1-t_s^{-1}) \tag{1.2.15}$$

The numerator is obtained by computing the graded trace over the fiber of the equation complex at the fixed point $x = 0$.

For equivariant 2d topYM on $\mathbb{R}^2$ (coming from SYM with 4 supercharges):

$$\operatorname{ind}_T(D, \mathbb{R}^2, \Omega^0_{\mathbb{C}} \to \Omega^1_{\mathbb{C}} \to \Omega^2_{\mathbb{C}})_{2\mathrm{d}} = \frac{1-(t_1+t_1^{-1})+1}{(1-t_1)(1-t_1^{-1})} = \frac{1}{1-t_1} + \frac{1}{1-t_1^{-1}} \tag{1.2.16}$$

For equivariant 4d topYM on $\mathbb{R}^4$ (coming from SYM with 8 supercharges):

$$\begin{aligned}\operatorname{ind}_T(D, \mathbb{R}^4, \Omega^0_{\mathbb{C}} \to \Omega^1_{\mathbb{C}} \to \Omega^{2+}_{\mathbb{C}})_{4\mathrm{d}} &= \frac{1-(t_1+t_1^{-1}+t_2+t_2^{-1}) + (1+t_1t_2+t_1^{-1}t_2^{-1})}{(1-t_1)(1-t_1^{-1})(1-t_2)(1-t_2^{-1})} \\ &= \frac{1}{(1-t_1)(1-t_2)} + \frac{1}{(1-t_1^{-1})(1-t_2^{-1})}\end{aligned} \tag{1.2.17}$$

For equivariant 8d topYM on $\mathbb{R}^8$ (coming from SYM with 16 supercharges), to preserve the Cayley form and the octonionic equations coming from the $Spin(7)$ structure, the 4 parameters (t_1, t_2, t_3, t_4) should satisfy the constraint $t_1t_2t_3t_4 = 1$. The weights on 7-dimensional bundle, whose sections are $\Omega^2_{\text{oct},\mathbb{C}}$, can be computed from the weights of the chiral spinor bundle S^+ modulo the trivial bundle. The chiral spinor bundle S^+ can be identified (after a choice of complex structure on X) as $S^+ \simeq (\oplus_{p=0}^{2}\Lambda^{2p}T^{0,1}_X) \otimes K^{\frac{1}{2}}$ where K is the canonical bundle on $X = \mathbb{R}^8 \simeq \mathbb{C}^4$ equivariantly trivial with respect to the T^3 action parametrized by (t_1, t_2, t_3, t_4) with $t_1t_2t_3t_4 = 1$. Then

$$\begin{aligned}\text{ind}_T(D, \mathbb{R}^8, \Omega^0_{\mathbb{C}} \to \Omega^1_{\mathbb{C}} \to \Omega^2_{\text{oct},\mathbb{C}})_{8\text{d}} &= \frac{(1 - (\sum_{s=1}^{4}(t_s + t_s^{-1})) + (1 + \sum_{1\le r<s\le 4} t_rt_s)}{\prod_{s=1}^{4}(1 - t_s)(1 - t_s^{-1})} = \\ &= \frac{1}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_4)}, \qquad t_1t_2t_3t_4 = 1 \end{aligned} \tag{1.2.18}$$

It is also interesting to consider the dimensional reduction of the 8d topYM (coming from the SYM with 16 supercharges) to the 6d theory. The numerator in the index is computed in the same way as (1.2.18), but the denominator is changed to the 6d determinant, hence we find

$$\begin{aligned}\text{ind}_T(D, \mathbb{R}^6, \Omega^0_{\mathbb{C}} \to \Omega^1_{\mathbb{C}} \to \Omega^{2,\text{oct}}_{\mathbb{C}})_{6\text{d reduction}} &= \frac{(1 - (\sum_{s=1}^{4}(t_s + t_s^{-1})) + (1 + \sum_{1\le r<s\le 4} t_rt_s)}{\prod_{s=1}^{3}(1 - t_s)(1 - t_s^{-1})} = \\ &= \frac{1 - t_4^{-1}}{(1 - t_1)(1 - t_2)(1 - t_3)} \overset{t_1t_2t_3t_4=1}{=} \\ &= \frac{1}{(1 - t_1)(1 - t_2)(1 - t_3)} + \frac{1}{(1 - t_1^{-1})(1 - t_2^{-1})(1 - t_3^{-1})} \end{aligned} \tag{1.2.19}$$

From equations (1.2.16)(1.2.17)(1.2.19) we see that the index for the complexified vector multiplet of the 2d theory (4 supercharges), 4d theory (8 supercharges) and 6d theory (16 supercharges) on $\mathbb{R}^{2r}$ can be uniformly written in the form

$$\text{ind}_T(D, \mathbb{R}^{2r}, \text{vector}_{\mathbb{C}}) = \frac{1 + (-1)^r \prod_{s=1}^{r} t_s}{\prod_{s=1}^{r}(1 - t_s)} = \frac{1}{\prod_{s=1}^{r}(1 - t_s)} + \frac{1}{\prod_{s=1}^{r}(1 - t_s^{-1})} \qquad r = 1, 2, 3 \tag{1.2.20}$$

Hence, the equivariant index of the complexified vector multiplet in 2,4 and 6 dimensions on flat space is equivalent to the index of the Dolbeault complex plus its dual, because (see review in Chapter 2 section 9)

$$\text{ind}_T(\bar{\partial}, \mathbb{C}^{2r}, \Omega^{0,\bullet}) = \frac{1}{\prod_{s=1}^{r}(1 - t_s^{-1})} \tag{1.2.21}$$

The vector multiplet is in a real representation of the equivariant group: each non-zero weight eigenspace appears together with its dual. Generally, the index of a real representation has the form

$$f(t_1, \ldots, t_r) + f(t_1^{-1}, \ldots, t_r^{-1}) \tag{1.2.22}$$

The equivariant Euler class in the denominator of the Atiyah-Bott localization formula (Chapter 2 section 8.1 and section 12) is defined as the Pfaffian rather then the determinant, hence each pair of terms in the equivariant index, describing a weight space and its dual, corresponds to a single weight factor in the equivariant Euler class. The choice between two opposite weights leads to a sign issue, which depends on the choice of the orientation on the infinite-dimensional space of all field modes. A careful treatment leads to interesting sign factors discussed in details for example in Chapter 3.

A natural choice of orientation leads to the holomorphic projection of the vector multiplet index (1.2.20) in 2, 4 and 6 dimensions by picking only the first term in (1.2.20) so that

$$\mathrm{ind}_T(D, \mathbb{R}^{2r}, \mathrm{vector}_{\mathbb{C}})_{\mathrm{hol}} = \frac{1}{\prod_{s=1}^r (1-t_s)} \qquad r = 1,2,3 \tag{1.2.23}$$

The supersymmetric Yang-Mills with 4, 8 and 16 supercharges can be put on the spheres S^2, S^4 and S^6 as was done in [16], [36], [37], [41] and reviewed in Chapter 3 and Chapter 10.

A certain generator Q_ϵ of the global superconformal group can be used for the localization computation. This generator Q_ϵ is represented by a conformal Killing spinor ϵ on a sphere S^{2r}, and satisfies $Q_\epsilon^2 = R$ where R is a rotation isometry. There are two fixed points of R on an even-dimensional sphere, usually called the north and the south poles. It turns out that the equivariant elliptic complex of equations, describing the equations of the topological YM, is replaced by a certain equivariant transversally elliptic complex of equations. Near the north pole this complex is approximated by the equivariant topological YM theory (theory in Ω-background), and near the south pole by its conjugate.

The index of the transversally elliptic operator can be computed by the Atiyah-Singer theorem, see for the complete treatement [54], application [16], Chapter 2 or Chapter 10. The result is that the index is contributed by the two fixed point on the sphere S^{2r}, with a particular choice of the distribution associated to the rational function, in other words with a particular choice of expansion in positive or negative powers of t_s, denoted by $[]_+$ or $[]_-$ respectively (see Chapter 2 section 11.1):

$$\mathrm{ind}_T(D, S^{2r}, \mathrm{vector}_{\mathbb{C}})_{\mathrm{hol}} = \left[\frac{1}{\prod_{s=1}^r (1-t_s)}\right]_+ + \left[\frac{1}{\prod_{s=1}^r (1-t_s)}\right]_- \qquad r = 1,2,3 \tag{1.2.24}$$

So far we have computed only the space-time geometrical part of the index. Now, suppose that the multiplet is tensored with a representation of a group G (like the gauge symmetry, R-symmetry or flavour symmetry), and let $L_\xi \simeq \mathbb{C}$ be a complex eigenspace in representation of G with eigenweight $\xi = e^{ix}$. Then

$$\mathrm{ind}_{T\times G}(D, S^{2r}, \mathrm{vector}_{\mathbb{C}} \otimes L_\xi)_{\mathrm{hol}} = \left[\frac{\xi}{\prod_{s=1}^r (1-t_s)}\right]_+ + \left[\frac{\xi}{\prod_{s=1}^r (1-t_s)}\right]_- \tag{1.2.25}$$

Now let ϵ_s and x be the Lie algebra parameters associated with the group parameters t_s and ξ as

$$t_s = \exp(i\epsilon_s), \qquad \xi = \exp(ix) \tag{1.2.26}$$

By definition, let $\Upsilon_r(x|\epsilon)$ be the equivariant Euler class (Pfaffian) of the graded vector space of fields of a vector multiplet on S^{2r} with the character (index) defined by (1.2.25)

$$\Upsilon_r(x|\epsilon) = \mathrm{eu}_{T\times G}(D, S^{2r}, \mathrm{vector}_{\mathbb{C}} \otimes L_\xi)_{\mathrm{hol}}|_{t_s=e^{i\epsilon_s},\xi=e^{ix}} \tag{1.2.27}$$

Explicitly, converting the infinite Taylor sum series of (1.2.25)

$$\left[\frac{\xi}{\prod_{s=1}^r(1-t_s)}\right]_+ + \left[\frac{\xi}{\prod_{s=1}^r(1-t_s)}\right]_- = \sum_{n_1=0,\cdots,n_r=0}^{\infty} \xi(t_1^{n_1}\dots t_r^{n_r} + (-1)^r t_1^{-1-n_1}\cdots t_r^{-1-n_r}) \tag{1.2.28}$$

into the product of weights we find the infinite-product definition of the $\Upsilon_r(x|\epsilon)$ function

$$\Upsilon_r(x|s) \stackrel{reg}{=} \prod_{n_1=0,\dots,n_r=0}^{\infty} \left(x + \sum_{s=1}^r n_s\epsilon_s\right)\left(\epsilon - x + \sum_{s=1}^r n_s\epsilon_s\right)^{(-1)^r} \tag{1.2.29}$$

where $\stackrel{reg}{=}$ denotes Weierstrass or ζ-function regularization and

$$\epsilon = \epsilon_1 + \cdots + \epsilon_r \tag{1.2.30}$$

The analysis for the scalar multiplet (the chiral multiplet in 2d for the theory with 4 supercharges or the hypermultiplet in 4d for the theory with 8 supercharges) is similar. On equivariant $\mathbb{R}^{2r}$ the corresponding complex for the scalar multiplet is the Dirac operator $S^+ \to S^-$, which differs from the Dolbeault complex by the twist by the square root of the canonical bundle, hence

$$\mathrm{ind}_T(D, \mathbb{R}^{2r}, \mathrm{scalar})_{\mathrm{hol}} = -\frac{\prod_{s=1}^r t_s^{\frac{1}{2}}}{\prod_{s=1}^r(1-t_s)} \qquad r = 1, 2 \tag{1.2.31}$$

On the sphere S^{2r}, again, one takes the contribution from the north and the south pole approximated locally by $\mathbb{R}^{2r}$ with opposite orientations, and gets

$$\mathrm{ind}_T(D, S^{2r}, \mathrm{scalar})_{\mathrm{hol}} = -\left[\frac{\prod_{s=1}^r t_s^{\frac{1}{2}}}{\prod_{s=1}^r(1-t_s)}\right]_+ - \left[\frac{\prod_{s=1}^r t_s^{\frac{1}{2}}}{\prod_{s=1}^r(1-t_s)}\right]_- \qquad r = 1, 2 \tag{1.2.32}$$

Hence, the equivariant Euler class of the graded space of sections of the scalar multiplet is obtained simply by a shift of the argument of the Υ-function and inversion

$$\mathrm{eu}_{T\times G}(D, S^{2r}, \mathrm{scalar} \otimes L_\xi)_{\mathrm{hol}}|_{t_s=e^{i\epsilon_s},w=e^{ix}} = \Upsilon_r\left(x + \frac{\epsilon}{2}\right)^{-1} \tag{1.2.33}$$

As computed in [16], [36], [37], [41] and reviewed in Chapter 3 and Chapter 10, the localization by the Atiyah-Bott formula brings the partition function of supersymmetric Yang-Mills with 4, 8 and 16 supercharges on the spheres S^2, S^4 and S^6 to the form of an integral over the imaginary line contour in the complexified Lie algebra of the Cartan torus of the gauge group (the zero mode of one of the scalar fields in the vector multiplet). The integrand is a product of the classical factor induced from the classical action and the determinant factor (the inverse of the equivariant Euler class of the tangent space to the space of fields) which has been computed above in terms of the Υ_r-function. Hence, for $r = 1, 2, 3$ we get perturbatively exact result of the partition function in the form of a finite-dimensional integral over the Cartan subalgebra of the Lie algebra of the gauge group (generalized matrix model)

$$Z_{S^{2r},\text{pert}} = \int_{\mathfrak{t}_G} da \frac{\prod_{w \in R_{\text{ad}\,\mathfrak{g}}} \Upsilon_r(iw \cdot a|\epsilon)}{\prod_{w \in R_{G\times F}} \Upsilon_r(iw \cdot (a, m) + \frac{\epsilon}{2}|\epsilon)} e^{P(a)} \tag{1.2.34}$$

Hence $Z_{S^{2r},\text{pert}}$ is the contribution to the partition function of the trivial localization locus (all fields vanish except the zero mode a of one of the scalars of the vector multiplet and some auxliary fields). The $Z_{S^{2r},\text{pert}}$ does not include the non-perturbative contributions. The factor $e^{P(a)}$ is induced by the classical action evaluated at the localization locus. The product of Υ_r-functions in the numerator comes from the vector multiplet and it runs over the weights of the adjoint representation. The product of Υ_r-functions in the denominator comes from the scalar multiplet (chiral or hyper), and it runs over the weights of a complex representation R_G of the gauge group G in which the scalar multiplet transforms. In addition, by taking the matter fields multiplets to be in a representation of a flavor symmetry F, the mass parameters $m \in \mathfrak{t}_F$ can be introduced naturally. For $r = 3$ the denominator is empty, because the 6d gauge theory with 16 supercharges is formed only from the gauge vector multiplet.

The non-perturbative contributions come from other localization loci, such as magnetic fluxes on S^2, or instantons on S^4, and their effect modifies the equivariant Euler classes presented as Υ_r-factors in (1.2.34) by certain rational factors. The 4d non-perturbative contributions are captured by fusion of Nekrasov instanton partition function with its conjugate [11, 16]. See 2d details in Chapter 3 and 4d details in Chapter 10.

Much before localization results on gauge theory on S^4 were obtained, the Υ_2 function prominently appeared in Zamolodchikov-Zamolodchikov paper [55] on structure functions of 2d Liouville CFT. The coincidence was one of the key observations by Alday-Gaiotto-Tachikawa [56] that led to a remarkable 2d/4d correspondence (AGT) between correlators in Liouville (Toda) theory and gauge theory partition functions on S^4, see review in Chapter 12.

1.2.3 Odd dimensions

Next we discuss the odd dimensional spheres (in principle, this discussion is applicable for any simply connected Sasaki-Einstein manifold, i.e. the manifold X admits at least two Killing spinors). After field redefinitions, which involve the Killing spinors, the integration space for odd dimensional supersymmetric gauge theories with the gauged fixing fields can be represented as the following spaces

$$\begin{aligned}
&\text{3d}: && \mathcal{A}(X, \mathfrak{g}) \times \Pi\Omega^0(X, \mathfrak{g}) \times \Pi\Omega^0(X, \mathfrak{g}) \times \Pi\Omega^0(X, \mathfrak{g}) \\
&\text{5d}: && \mathcal{A}(X, \mathfrak{g}) \times \Pi\Omega^{2,+}_H(X, \mathfrak{g}) \times \Pi\Omega^0(X, \mathfrak{g}) \times \Pi\Omega^0(X, \mathfrak{g}) \\
&\text{7d}: && \mathcal{A}(X, \mathfrak{g}) \times \Omega^{3,0}_H(X, \mathfrak{g}) \times \Pi\Omega^{2,+}_H(X, \mathfrak{g}) \times \Pi\Omega^0(X, \mathfrak{g}) \times \Pi\Omega^0(X, \mathfrak{g})
\end{aligned} \tag{1.2.35}$$

where in all cases there are common last two factors $\Pi\Omega^0(X,\mathfrak{g}) \times \Pi\Omega^0(X,\mathfrak{g})$ coming from the gauge fixing. The space $\mathcal{A}(X,\mathfrak{g})$ is the space of connections on X with the Lie algebra $\mathfrak{g}$. The Sasaki-Einstein manifold is a contact manifold and the differential forms can be naturally decomposed into vertical and horizontal forms using the Reeb vector field R and the contact form κ. The horizontal plane admits a complex structure and thus the horizontal forms can be decomposed further into (p,q)-forms. For two forms we define the space $\Omega_H^{2,+}$ as $(2,0)$-forms plus $(0,2)$-forms plus forms proportional to $d\kappa$. Thus for 5d $\Omega_H^{2,+}$ is the space of standard self-dual forms in four dimensions (rank 3 bundle), and for 7d forms in $\Omega_H^{2,+}$ obey the hermitian Yang-Mills conditions in six dimensions (rank 7 bundle: 3 complex components and 1 real). By just counting degrees of freedom one can check that the 3d case corresponds to an $\mathcal{N}=2$ vector multiplet (4 supercharges), the 5d case to an $\mathcal{N}=1$ vector multiplet (8 supercharges) and 7d to $\mathcal{N}=1$ maximally supersymmetric theory (16 supercharges). The supersymmetry square Q_ϵ^2, which acts on this space, is given by the sum of Lie derivative along the Reeb vector field R and constant gauge transformations: $Q_\epsilon^2 = \mathcal{L}_R + ad_a$. Around the trivial connection, after some cancelations, the problem boils down to the calculation of the following superdeterminant

$$Z_{S^{2r-1}} = \int_{\mathfrak{t}_G} da \ \mathrm{sdet}_{\Omega_H^{(\bullet,0)}(X,\mathfrak{g})}(\mathcal{L}_R + ad_a)\ e^{P_r(a)} + \cdots \ , \tag{1.2.36}$$

and this is a uniform description for Sasaki-Einstein manifolds in 3d, 5d and 7d. In 3d the only simply connected Sasaki-Einstein manifold is S^3, while in 5d and 7d there are many examples of simply connected Sasaki-Einstein manifolds (there is a rich class of the toric Sasaki-Einstein manifolds). The determinant can be calculated in many alternative ways, and the result depends on X.

If X is a sphere S^{2r-1}, the determinant in (1.2.36), equivalently, the inverse equivariant Euler class of the normal bundle to the localization locus in the space of all fields, can be computed from the equivariant Chern character, or the index, of a certain transversally elliptic operator $D = \pi^*\bar\partial$ induced from the Dobeault operator $\bar\partial$ by the Hopf fibration projection $\pi : S^{2r-1} \to \mathbb{CP}^{r-1}$.

The index, or equivariant Chern character, is easy to compute by the Aityah-Singer fixed point theorem (see the details in Chapter 2 section 11.2). The result is

$$\mathrm{ind}_T(D, S^{2r-1}) = \sum_{n=-\infty}^{\infty} \mathrm{ind}_T(\bar\partial, \mathbb{CP}^{r-1}, \mathcal{O}(n)) = \left[\frac{1}{\prod_{k=1}^r (1-t_k)}\right]_+ + \left[\frac{(-1)^{r-1} t_1^{-1}\dots t_r^{-1}}{\prod_{k=1}^r (1-t_k^{-1})}\right]_- \tag{1.2.37}$$

Converting the additive equivariant Chern character to the multiplicative equivariant Euler character, we find the definition of the multiple sine function

$$S_r(x|\epsilon) = \mathrm{eu}_{T\times G}(S^{2r-1}, D\otimes L_\xi)_{\mathrm{hol}}|_{t_s = e^{i\epsilon_s}, \xi = e^{ix}} \tag{1.2.38}$$

where L_ξ is a 1-dimensional complex eigenspace with character ξ. Explicitly

$$S_r(x|\epsilon) \overset{reg}{=} \prod_{n_1=0,\dots,n_r=0}^{\infty} \left(x + \sum_{s=1}^r n_s\epsilon_s\right)\left(\epsilon - x + \sum_{s=1}^r n_s\epsilon_s\right)^{(-1)^{r-1}} \tag{1.2.39}$$

and this leads to the formula (1.2.11) for the perturbative part of the partition function of a vector multiplet on S^{2r-1}.

For $r = 2, 3$ we can also treat a scalar supermultiplet (a chiral multiplet for the theory with 4 supercharges or a hypermultiplet for the theory with 8 supercharges). The corresponding complex is described by an elliptic operator $\pi^* \not{D}$ for $\pi : S^{2r-1} \to \mathbb{CP}^{r-1}$, where $\not{D}$ is the Dirac operator $S^+ \to S^-$ on $\mathbb{CP}^{r-1}$. The Dirac complex is isomorphic to the Dolbeault complex by a twist by a square root of the canonical bundle. Because of the opposite statistics, there is also an overall sign factor like in (1.2.32).

Finally, the contribution of both vector multiplet in representation $R_{\mathrm{ad}\,\mathfrak{g}}$ and scalar multiplet in representation $R_{G\times F}$ to the perturbative part of the partition function is computed by the finite-dimensional integral over the localizationl locus $\mathfrak{t}_G$ with the following integrand made of S_r functions

$$Z_{S^{2r-1},\mathrm{pert}} = \int_{\mathfrak{t}_G} da \frac{\prod_{w\in R_{\mathrm{ad}\,\mathfrak{g}}} S_r(iw\cdot a|\epsilon)}{\prod_{w\in R_{G\times F}} S_r(iw\cdot(a,m)+\frac{\epsilon}{2}|\epsilon)}\, e^{P(a)} \tag{1.2.40}$$

Here F is a possible flavor group of symmetry, and $m \in \mathfrak{t}_F$ is a mass parameter.

For reviews of 3d localization see Chapter 6, Chapter 7, Chapter 8, Chapter 9 and for reviews of 5d localization see Chapter 14, Chapter 15, Chapter 16.

The case of $S^n \times S^1$ is built from the trigonometric version of S^n-result.

The trigonometric version of the Υ_r-function (1.2.29) is given by

$$H_r(x|\epsilon_1,\ldots,\epsilon_r) = \prod_{n_1,\ldots,n_r=0}^{\infty} \left(1-e^{2\pi i x}e^{2\pi i \vec{n}\vec{\epsilon}}\right)\left(1-e^{2\pi i(\sum_{i=1}^{r}\epsilon_i - x)}e^{2\pi i\vec{n}\vec{\epsilon}}\right)^{(-1)^r} . \tag{1.2.41}$$

The trigonometric version of the multiple sine function S_r (1.2.39) is given by the multiple elliptic gamma function

$$G_r(x|\epsilon_1,\ldots,\epsilon_r) = \prod_{n_1,\ldots,n_r=0}^{\infty} \left(1-e^{2\pi i x}e^{2\pi i \vec{n}\vec{\epsilon}}\right)\left(1-e^{2\pi i(\sum_{i=1}^{r}\epsilon_i - x)}e^{2\pi i\vec{n}\vec{\epsilon}}\right)^{(-1)^{r-1}} . \tag{1.2.42}$$

where G_1 corresponds to the θ-function, G_2 corresponds to the elliptic gamma function.

The partition function on $S^r \times S^1$ has an interpretation as a supersymmetric index, namely a graded trace over the Hilbert space. The review of supersymmetric index in 2d is in Chapter 3, in 4d is in Chapter 13 and in 6d is in Chapter 17.

1.3 Applications of the localization technique

The localization technique can be applied only to a very restricted set of supersymmetric observables, e.g. partition functions, supersymmetric Wilson loops etc. Unfortunately, the localization technique does not allow us to calculate correlators of generic local operators. However, the supersymmetric localization offers a unique opportunity to study the full non-perturbative answer for these restricted class of observables and this is a powerful tool to inspect interacting quantum field theory. As one can see from the previous section, the localization results are given in terms of complicated finite dimensional integrals. Thus one has to develop techniques to study these integrals and learn how to deduce the relevant physical and mathematical information. Some of the reviews in this volume are dedicated to the study of the localization results (sometimes in various limits) and to the applications of these results in physics and mathematics.

The original motivation of [16] was to prove the Erickson-Semenoff-Zarembo and Drukker-Gross conjecture, which expresses the expectation value of supersymmetric circular Wilson loop operators in $\mathcal{N}=4$ supersymmetric Yang-Mills theory in terms of a Gaussian matrix model, see review in Chapter 11. This conjecture was actively used for checks of AdS/CFT correspondence. After more general localization results became available, they were also used for stronger tests of AdS/CFT.

On the AdS side, it is relatively easy to perform the calculation, since it requires only classical supergravity. However, on the gauge theory side, we need the full non-perturbative result in order to be able to compare it with the supergravity calculation. The localization technique offers us a unique opportunity for non-perturbative checks of AdS/CFT correspondence. A number of reviews are devoted to the use of localization for AdS/CFT correspondence: for $\text{AdS}_4/\text{CFT}_3$ see review in Chapter 7 and Chapter 8, for $\text{AdS}_5/\text{CFT}_4$ see review in Chapter 11, for $\text{AdS}_7/\text{CFT}_6$ see review in Chapter 15 and Chapter 17. The localization results for spheres (1.2.34) and (1.2.40) gave rise to new matrix models which had not been investigated before. One of the main problems is to find out how the free energy (the logarithm of the partition function) scales in the large N-limit. In 3d there is an interesting scaling $N^{3/2}$, and the analysis of the partition function on S^3 for the ABJM model is related to different subjects such as topological string, see review in Chapter 7. On the other hand, the 5d theory establishes a rather exotic scaling N^3 for the gauge theory, and it supports the relation of the 5d theory to 6d $(2,0)$ superconformal field theory, see review in Chapter 17.

Once we start to calculate the partition functions on different manifolds (e.g., S^r and $S^{r-1}\times S^1$), we start to realize the composite structure of the answer. Namely the answer can be built from basic objects called holomorphic blocks, this is discussed in details for 2d, 3d, 4d and 5d theories in Chapter 6 and Chapter 16. Besides, it seems that in odd dimensions the partition function may serve as a good measure for the number of degrees of freedom. This can be made more precise for the partition function on S^3 which measures the number of degrees of freedom of the supersymmetric theory. Thus one can study how it behaves along the RG flow, see Chapter 8.

Another interesting application of localization appears in the context of the BPS/CFT-correspondence [66], in which BPS phenomena of 4d gauge theories are related to 2d conformal field theory or its massive, lattice, or integrable deformation. A beautiful and precise realization of this idea is the Alday-Gaiotto-Tachikawa (AGT) correspondence which relates 4d $\mathcal{N}=2$ gauge theory of class $\mathcal{S}$ to Liouville (Toda) CFT on some Riemann surface C. A 4d $\mathcal{N}=2$ gauge theory of class $\mathcal{S}$ is obtained by compactification of 6d $(2,0)$ tensor self-dual theory on C. For a review of this topic see Chapter 12.

The 3d/3d version of this correspondence is reviewed in Chapter 9 and 5d version is reviewed in Chapter 16.

The 2d supersymmetric non-linear sigma models play a prominent role in string theory and mathematical physics, but it is hard to perform direct calculations for non-linear sigma model. However some gauged linear sigma models (2d supersymmetric gauge theories) flow to non-linear sigma model. This flow allows to compute some quantities of non-linear sigma models, such as genus 0 Gromov-Witten invariants (counting of holomorphic maps from $S^2 \simeq \mathbb{CP}^1$ to a Calabi-Yau target) by localization in 2d gauge theories on S^2. See review in Chapter 4 and Chapter 3.

Other important applications of localization calculations are explicit checks of QFT dualities. Sometimes QFT theories with different Lagrangians describe the same physical system and have the same physical dynamics, a famous example is Seiberg duality [68]. The dual theories may look very different in the description by gauge group and matter content, but have the same partition functions, provided approriate identification of the parameters. Various checks of the duality using the localization results are reviewed in Chapter 3, Chapter 6, Chapter 8 and Chapter 13.

Acknowledgement: We thank all authors who contributed to this volume: Francesco Benini, Tudor Dimofte, Thomas T. Dumitrescu, Kazuo Hosomichi, Seok Kim, Kimyeong Lee, Bruno Le Floch, Marcos Mariño, Joseph A. Minahan, David Morrison, Sara Pasquetti, Jian Qiu, Leonardo Rastelli, Shlomo S. Razamat, Silvu S. Pufu, Yuji Tachikawa and Brian Willett. We are grateful for their cooperation and for their patience with long completion of this project, for many suggestions and helpful ideas. Special thanks to Joseph A. Minahan and Bruno Le Floch for a meticulous proofreading of this introduction, and to Guido Festuccia for comments.

The research of V.P. is supported by ERC grant QUASIFT. The research of M.Z. is supported by Vetenskapsrådet under grant #2014- 5517, by the STINT grant and by the grant "Geometry and Physics" from the Knut and Alice Wallenberg foundation.

References

[1] V. Pestun and M. Zabzine, eds., *Localization techniques in quantum field theory*, vol. xx. Journal of Physics A, 2016. 1608.02952. http://pestun.ihes.fr/pages/LocalizationReview/LocQFT.pdf.

[2] S. Lefschetz, "Intersections and transformations of complexes and manifolds," *Trans. Amer. Math. Soc.* **28** (1926) no. 1, 1–49. http://dx.doi.org/10.2307/1989171.

[3] J. J. Duistermaat and G. J. Heckman, "On the variation in the cohomology of the symplectic form of the reduced phase space," *Invent. Math.* **69** (1982) no. 2, 259–268. http://dx.doi.org/10.1007/BF01399506.

[4] N. Berline and M. Vergne, "Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante," *C. R. Acad. Sci. Paris Sér. I Math.* **295** (1982) no. 9, 539–541.

[5] M. F. Atiyah and R. Bott, "The moment map and equivariant cohomology," *Topology* **23** (1984) no. 1, 1–28. http://dx.doi.org/10.1016/0040-9383(84)90021-1.

[6] V. Pestun, "Review of localization in geometry," *Journal of Physics A* **xx** (2016) 000, 1608.02954.

[7] E. Witten, "Supersymmetry and Morse theory," *J. Differential Geom.* **17** (1982) no. 4, 661–692 (1983). http://projecteuclid.org/euclid.jdg/1214437492.

[8] E. Witten, "Topological sigma models," *Comm. Math. Phys.* **118** (1988) no. 3, 411–449. http://projecteuclid.org/euclid.cmp/1104162092.

[9] E. Witten, "Topological quantum field theory," *Comm. Math. Phys.* **117** (1988) no. 3, 353–386. http://projecteuclid.org/euclid.cmp/1104161738.

[10] E. Witten, "Two-dimensional gauge theories revisited," *J. Geom. Phys.* **9** (1992) 303–368, arXiv:hep-th/9204083 [hep-th].

[11] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," *Adv. Theor. Math. Phys.* **7** (2003) no. 5, 831–864. http://projecteuclid.org/euclid.atmp/1111510432.

[12] A. Losev, N. Nekrasov, and S. L. Shatashvili, "Issues in topological gauge theory," *Nucl. Phys.* **B534** (1998) 549–611, arXiv:hep-th/9711108 [hep-th].

[13] G. W. Moore, N. Nekrasov, and S. Shatashvili, "Integrating over Higgs branches," *Commun. Math. Phys.* **209** (2000) 97–121, arXiv:hep-th/9712241 [hep-th].

[14] A. Lossev, N. Nekrasov, and S. L. Shatashvili, "Testing Seiberg-Witten solution," arXiv:hep-th/9801061 [hep-th]. http://alice.cern.ch/format/showfull?sysnb=0266564.

[15] A. Losev, G. W. Moore, N. Nekrasov, and S. Shatashvili, "Four-dimensional avatars of two-dimensional RCFT," *Nucl. Phys. Proc. Suppl.* **46** (1996) 130–145, arXiv:hep-th/9509151 [hep-th].

[16] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," *Commun.Math.Phys.* **313** (2012) 71–129, arXiv:0712.2824 [hep-th].

[17] M. Blau, "Killing spinors and SYM on curved spaces," *JHEP* **11** (2000) 023, arXiv:hep-th/0005098 [hep-th].

[18] G. Festuccia and N. Seiberg, "Rigid Supersymmetric Theories in Curved Superspace," *JHEP* **06** (2011) 114, arXiv:1105.0689 [hep-th].

[19] C. Imbimbo and D. Rosa, "Topological anomalies for Seifert 3-manifolds," *JHEP* **07** (2015) 068, arXiv:1411.6635 [hep-th].

[20] J. Bae, C. Imbimbo, S.-J. Rey, and D. Rosa, "New Supersymmetric Localizations from Topological Gravity," *JHEP* **03** (2016) 169, arXiv:1510.00006 [hep-th].

[21] C. Klare, A. Tomasiello, and A. Zaffaroni, "Supersymmetry on Curved Spaces and Holography," *JHEP* **08** (2012) 061, arXiv:1205.1062 [hep-th].

[22] T. T. Dumitrescu, G. Festuccia, and N. Seiberg, "Exploring Curved Superspace," *JHEP* **08** (2012) 141, arXiv:1205.1115 [hep-th].

[23] D. Cassani, C. Klare, D. Martelli, A. Tomasiello, and A. Zaffaroni, "Supersymmetry in Lorentzian Curved Spaces and Holography," *Commun. Math. Phys.* **327** (2014) 577–602, arXiv:1207.2181 [hep-th].

[24] C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, "Supersymmetric Field Theories on Three-Manifolds," *JHEP* **05** (2013) 017, arXiv:1212.3388 [hep-th].

[25] C. Klare and A. Zaffaroni, "Extended Supersymmetry on Curved Spaces," *JHEP* **10** (2013) 218, arXiv:1308.1102 [hep-th].

[26] D. Butter, G. Inverso, and I. Lodato, "Rigid 4D $\mathcal{N} = 2$ supersymmetric backgrounds and actions," *JHEP* **09** (2015) 088, arXiv:1505.03500 [hep-th].

[27] V. Pestun, "Localization for $\mathcal{N} = 2$ Supersymmetric Gauge Theories in Four Dimensions," in *New Dualities of Supersymmetric Gauge Theories*, J. Teschner, ed., pp. 159–194. 2016. arXiv:1412.7134 [hep-th]. http://inspirehep.net/record/1335346/files/arXiv:1412.7134.pdf.

[28] Y. Pan, "Rigid Supersymmetry on 5-dimensional Riemannian Manifolds and Contact Geometry," *JHEP* **05** (2014) 041, arXiv:1308.1567 [hep-th].

[29] Y. Imamura and H. Matsuno, "Supersymmetric backgrounds from 5d N=1 supergravity," *JHEP* **07** (2014) 055, arXiv:1404.0210 [hep-th].

[30] Y. Pan and J. Schmude, "On rigid supersymmetry and notions of holomorphy in five dimensions," *JHEP* **11** (2015) 041, arXiv:1504.00321 [hep-th].

[31] H. Samtleben, E. Sezgin, and D. Tsimpis, "Rigid 6D supersymmetry and localization," *JHEP* **03** (2013) 137, arXiv:1212.4706 [hep-th].

[32] S. M. Kuzenko, "Supersymmetric Spacetimes from Curved Superspace," *PoS* **CORFU2014** (2015) 140, arXiv:1504.08114 [hep-th].

[33] S. M. Kuzenko, J. Novak, and G. Tartaglino-Mazzucchelli, "Symmetries of curved superspace in five dimensions," *JHEP* **10** (2014) 175, arXiv:1406.0727 [hep-th].

[34] T. Dumitrescu, "An Introduction to Supersymmetric Field Theories in Curved Space," *Journal of Physics A* **xx** (2016) 000, 1608.02957.

[35] A. Kapustin, B. Willett, and I. Yaakov, "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter," *JHEP* **03** (2010) 089, arXiv:0909.4559 [hep-th].

[36] F. Benini and S. Cremonesi, "Partition Functions of $\mathcal{N} = (2, 2)$ Gauge Theories on S^2 and Vortices," *Commun. Math. Phys.* **334** (2015) no. 3, 1483–1527, arXiv:1206.2356 [hep-th].

[37] N. Doroud, J. Gomis, B. Le Floch, and S. Lee, "Exact Results in D=2 Supersymmetric Gauge Theories," *JHEP* **05** (2013) 093, arXiv:1206.2606 [hep-th].

[38] J. Kallen and M. Zabzine, "Twisted supersymmetric 5D Yang-Mills theory and contact geometry," *JHEP* **05** (2012) 125, arXiv:1202.1956 [hep-th].

[39] J. Kallen, J. Qiu, and M. Zabzine, "The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere," *JHEP* **08** (2012) 157, arXiv:1206.6008 [hep-th].

[40] H.-C. Kim and S. Kim, "M5-branes from gauge theories on the 5-sphere," *JHEP* **05** (2013) 144, arXiv:1206.6339 [hep-th].

[41] J. A. Minahan and M. Zabzine, "Gauge theories with 16 supersymmetries on spheres," *JHEP* **03** (2015) 155, arXiv:1502.07154 [hep-th].

[42] N. Hama, K. Hosomichi, and S. Lee, "SUSY Gauge Theories on Squashed Three-Spheres," *JHEP* **05** (2011) 014, arXiv:1102.4716 [hep-th].

[43] Y. Imamura and D. Yokoyama, "N=2 supersymmetric theories on squashed three-sphere," *Phys. Rev.* **D85** (2012) 025015, arXiv:1109.4734 [hep-th].

[44] N. Hama and K. Hosomichi, "Seiberg-Witten Theories on Ellipsoids," *JHEP* **09** (2012) 033, arXiv:1206.6359 [hep-th]. [Addendum: JHEP10,051(2012)].

[45] G. Lockhart and C. Vafa, "Superconformal Partition Functions and Non-perturbative Topological Strings," arXiv:1210.5909 [hep-th].

[46] Y. Imamura, "Perturbative partition function for squashed S^5," arXiv:1210.6308 [hep-th].

[47] H.-C. Kim, J. Kim, and S. Kim, "Instantons on the 5-sphere and M5-branes," arXiv:1211.0144 [hep-th].

[48] J. A. Minahan, "Localizing gauge theories on S^d," *JHEP* **04** (2016) 152, arXiv:1512.06924 [hep-th].

[49] A. Narukawa, "The modular properties and the integral representations of the multiple elliptic gamma functions," *Advances in Mathematics* **189** (2004) no. 2, 247 – 267. http://www.sciencedirect.com/science/article/pii/S0001870803003670.

[50] F. Benini and B. Le Floch, "Supersymmetric localization in two dimensions," *Journal of Physics A* **xx** (2016) 000, 1608.02955.

[51] K. Hosomichi, "$\mathcal{N} = 2$ SUSY gauge theories on S^4," *Journal of Physics A* **xx** (2016) 000, 1608.02962.

[52] B. Willett, "Localization on three-dimensional manifolds," *Journal of Physics A* **xx** (2016) 000, 1608.02958.

[53] J. Qiu and M. Zabzine, "Review of localization for 5D supersymmetric gauge theories," *Journal of Physics A* **xx** (2016) 000, 1608.02966.

[54] M. F. Atiyah, *Elliptic operators and compact groups.* Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 401.

[55] A. B. Zamolodchikov and A. B. Zamolodchikov, "Structure constants and conformal bootstrap in Liouville field theory," *Nucl. Phys.* **B477** (1996) 577–605, arXiv:hep-th/9506136 [hep-th].

[56] L. F. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories," *Lett.Math.Phys.* **91** (2010) 167–197, arXiv:0906.3219 [hep-th].

[57] Y. Tachikawa, "A brief review of the 2d/4d correspondences," *Journal of Physics A* **xx** (2016) 000, 1608.02964.

[58] M. Mariño, "Localization at large N in Chern-Simons-matter theories," *Journal of Physics A* **xx** (2016) 000, 1608.02959.

[59] S. Pufu, "The F-Theorem and F-Maximization," *Journal of Physics A* **xx** (2016) 000, 1608.02960.

[60] T. Dimofte, "Perturbative and nonperturbative aspects of complex Chern-Simons Theory," *Journal of Physics A* **xx** (2016) 000, 1608.02961.

[61] J. Minahan, "Matrix models for 5D super Yang-Mills," *Journal of Physics A* **xx** (2016) 000, 1608.02967.

[62] S. Pasquetti, "Holomorphic blocks and the 5d AGT correspondence," *Journal of Physics A* **xx** (2016) 000, 1608.02968.

[63] L. Rastelli and S. Razamat, "The supersymmetric index in four dimensions," *Journal of Physics A* **xx** (2016) 000, 1608.02965.

[64] S. Kim and K. Lee, "Indices for 6 dimensional superconformal field theories," *Journal of Physics A* **xx** (2016) 000, 1608.02969.

[65] K. Zarembo, "Localization and AdS/CFT Correspondence," *Journal of Physics A* **xx** (2016) 000, 1608.02963.

[66] N. Nekrasov, "On the BPS/CFT correspondence," *Lecture at the University of Amsterdam string theory group seminar* (2004) .

[67] D. Morrison, "Gromov-Witten invariants and localization," *Journal of Physics A* (2016) , 1608.02956.

[68] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," *Nucl. Phys.* **B435** (1995) 129–146, arXiv:hep-th/9411149 [hep-th].

Chapter 2

Review of localization in geometry

Vasily Pestun

Institut des Hautes Études Scientifique, France
pestun@ihes.fr

Abstract

Review of localization in geometry: equivariant cohomology, characteristic classes, Atiyah-Bott formula, Atiyah-Singer equivariant index formula, Mathai-Quillen formalism

Contents

2.1 Equivariant cohomology **36**
2.2 Classifying space and characteristic classes **37**
2.3 Weil algebra **38**
2.4 Weil model and Cartan model of equivariant cohomology . . . **41**
2.4.1 Cartan model 41
2.4.2 Weil model 43
2.5 Equivariant characteristic classes in Cartan model **43**
2.6 Standard characteristic classes **45**
2.6.1 Euler class 45
2.6.2 Euler class of vector bundle and Mathai-Quillen form 46
2.6.3 Chern character 48
2.6.4 Chern class 48
2.6.5 Todd class 48
2.6.6 The $\hat{A}$ class 49
2.7 Index formula **49**
2.8 Equivariant integration **50**
2.8.1 Thom isomorphism and Atiyah-Bott localization 51
2.8.2 Duistermaat-Heckman localization 52

2.8.3 Gaussian integral example 53
2.8.4 Example of a two-sphere 54
2.9 Equivariant index formula (Dolbeault and Dirac) 55
2.10 Equivariant index and representation theory 58
2.11 Equivariant index for differential operators 61
2.11.1 Atiyah-Singer equivariant index formula for elliptic complexes . . . 61
2.11.2 Atiyah-Singer index formula for a free action G-manifold 62
2.11.3 General Atiyah-Singer index formula 63
2.12 Equivariant cohomological field theories 65
References 67

Foundations of equivariant de Rham theory have been laid in two papers by Henri Cartan [2] [3]. The book by Guillemin and Sternberg [4] covers Cartan's papers and treats equivariant de Rham theory from the perspective of supersymmetry. See also the book by Berline-Vergne [5], the lectures by Szabo [6] and by Cordes-Moore-Ramgoolam [7], and Vergne's review [8].

2.1 Equivariant cohomology

Let G be a compact connected Lie group. Let X be a G-manifold, which means that there is a defined action $G \times X \to X$ of the group G on the manifold X.

If G acts freely on X (all stabilizers are trivial) then the space X/G is an ordinary manifold on which the usual cohomology theory $H^\bullet(X/G)$ is defined. If the G action on X is free, the G-equivariant cohomology groups $H^\bullet_G(X)$ are defined to be the ordinary cohomology $H^\bullet(X/G)$.

If the G action on X is not free, the naive definition of the equivariant cohomology $H^\bullet_G(X)$ fails because X/G is not an ordinary manifold. If non-trivial stabilizers exist, the corresponding points on X/G are not ordinary points but fractional or stacky points.

A proper topological definion of the G-*equivariant cohomology* $H_G(X)$ sets

$$H^\bullet_G(X) = H^\bullet(X \times_G EG) = H^\bullet((X \times EG)/G)) \tag{2.1.1}$$

where the space EG, called *universal bundle* [9, 10] is a topological space associated to G with the following properties

1. The space EG is contractible
2. The group G acts freely on EG

Because of the property (1) the cohomology theory of X is isomorphic to the cohomology theory of $X \times EG$, and because of the property (2) the group G acts freely on $X \times EG$ and hence the quotient space $(X \times_G EG)$ has a well-defined ordinary cohomology theory.

2.2 Classifying space and characteristic classes

If X is a point pt, the ordinary cohomology theory $H^\bullet(pt)$ is elementary

$$H^n(pt,\mathbb{R}) = \begin{cases} \mathbb{R}, & n=0 \\ 0, & n>0 \end{cases} \tag{2.2.1}$$

but the equivariant cohomology $H^\bullet_G(pt)$ is less trivial. Indeed,

$$H^\bullet_G(pt) = H^\bullet(EG/G) = H^\bullet(BG) \tag{2.2.2}$$

where the quotient space $BG = EG/G$ is called *classifying space.*

The terminology *universal bundle* EG and *classifying space* BG comes from the fact that any smooth principal G-bundle on a manifold X can be induced by a pullback f^* of the universal principal G-bundle $EG \to BG$ using a suitable smooth map $f: X \to BG$.

The cohomology groups of BG are used to construct *characteristic classes* of principal G-bundles.

Let $\mathfrak{g} = Lie(G)$ be the real Lie algebra of a compact connected Lie group G. Let $\mathbb{R}[\mathfrak{g}]$ be the space of real valued polynomial functions on $\mathfrak{g}$, and let $\mathbb{R}[\mathfrak{g}]^G$ be the subspace of Ad_G invariant polynomials on $\mathfrak{g}$.

For a principal G-bundle over a base manifold X the Chern-Weil morphism

$$\begin{aligned} \mathbb{R}[\mathfrak{g}]^G &\to H^\bullet(X,\mathbb{R}) \\ p &\mapsto p(F_A) \end{aligned} \tag{2.2.3}$$

sends an adjoint invariant polynomial p on the Lie algebra $\mathfrak{g}$ to a cohomology class $[p(F_A)]$ in $H^\bullet(X)$ where $F_A = \nabla_A^2$ is the curvature 2-form of any connection ∇_A on the G-bundle. The cohomology class $[p(F_A)]$ does not depend on the choice of the connection A and is called the *characteristic class* of the G-bundle associated to the polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$.

The main theorem of Chern-Weil theory is that the ring of characteristic classes $\mathbb{R}[\mathfrak{g}]^G$ is isomorphic to the cohomology ring $H^\bullet(BG)$ of the classifying space BG: the Chern-Weil morphism (2.2.3) is an isomorphism

$$\mathbb{R}[\mathfrak{g}]^G \xrightarrow{\sim} H^\bullet(BG,\mathbb{R}) \tag{2.2.4}$$

For the circle group $G = S^1 \simeq U(1)$ the universal bundle ES^1 and classifying space BS^1 can be modelled as

$$ES^1 \simeq S^{2n+1}, \qquad BS^1 \simeq \mathbb{CP}^n \qquad \text{at} \qquad n \to \infty \tag{2.2.5}$$

Then the Chern-Weil isomorphism is explicitly

$$\mathbb{C}[\mathfrak{g}]^G \simeq H^\bullet(\mathbb{CP}^\infty,\mathbb{C}) \simeq \mathbb{C}[\epsilon] \tag{2.2.6}$$

where $\epsilon \in \mathfrak{g}^\vee$ is a linear function on $\mathfrak{g} = Lie(S^1)$ and $\mathbb{C}[\epsilon]$ denotes the free polynomial ring on one generator ϵ. The $\epsilon \in H^2(\mathbb{CP}^\infty, \mathbb{C})$ is negative of the first Chern class c_1 of the universal bundle

$$-c_1(\gamma) = \epsilon = \frac{1}{2\pi\sqrt{-1}} \operatorname{tr}_{\mathbf{1}} F_A(\gamma) \tag{2.2.7}$$

where $\operatorname{tr}_{\mathbf{1}}$ denotes trace of the curvature two-form $F_A = dA + A \wedge A$ in the fundamental complex 1-dimensional representation in which the Lie algebra of $\mathfrak{g} = Lie(S^1)$ is represented by $\imath\mathbb{R}$. The cohomological degree of ϵ is

$$\deg \epsilon = \deg F_A(\gamma) = 2 \tag{2.2.8}$$

Generally, for a compact connected Lie group G we reduce the Chern-Weil theory to the maximal torus $T \subset G$ and identify

$$\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{t}]^{W_G} \tag{2.2.9}$$

where $\mathfrak{t}$ is the Cartan Lie algebra $\mathfrak{t} = Lie(T)$ and W_G is the Weyl group of G.

For example, if $G = U(n)$ the Weyl group $W_{U(n)}$ is the permutation group of n eigenvalues $\epsilon_1, \dots \epsilon_n$. Therefore

$$H^\bullet(BU(n), \mathbb{C}) = \mathbb{C}[\mathfrak{g}]^{U(n)} \simeq \mathbb{C}[\epsilon_1, \dots, \epsilon_n]^{W_{U(n)}} \simeq \mathbb{C}[c_1, \dots, c_n] \tag{2.2.10}$$

where $(c_1, \dots, c_n)$ are elementary symmetrical monomials called Chern classes

$$c_k = (-1)^k \sum_{i_1 \leq \dots \leq i_k} \epsilon_{i_1} \dots \epsilon_{i_k} \tag{2.2.11}$$

The classifying space for $G = U(n)$ is

$$BU(n) = \lim_{k \to \infty} \operatorname{Gr}_n(\mathbb{C}^{k+n}) \tag{2.2.12}$$

where $\operatorname{Gr}_n(V)$ denotes the space of n-planes in the vector space V.

To summarize, if G is a connected compact Lie group with Lie algebra $\mathfrak{g} = Lie(G)$, maximal torus T and its Lie algebra $\mathfrak{t} = Lie(T)$, and Weyl group W_G, then it holds

$$\boxed{H^\bullet_G(pt, \mathbb{R}) \simeq H^\bullet(BG, \mathbb{R}) \simeq \mathbb{R}[\mathfrak{g}]^G \simeq \mathbb{R}[\mathfrak{t}]^{W_G}} \tag{2.2.13}$$

2.3 Weil algebra

The cohomology $H^\bullet(BG, \mathbb{R})$ of the classifying space BG can also be realized in the Weil algebra

$$\mathcal{W}_\mathfrak{g} := \mathbb{R}[\mathfrak{g}[1] \oplus \mathfrak{g}[2]] = \Lambda\mathfrak{g}^\vee \otimes S\mathfrak{g}^\vee \tag{2.3.1}$$

Here $\mathfrak{g}[1]$ denotes shift of degree so that elements of $\mathfrak{g}[1]$ are Grassmann. The space of polynomial functions $\mathbb{R}[\mathfrak{g}[1]]$ on $\mathfrak{g}[1]$ is the anti-symmetric algebra $\Lambda\mathfrak{g}^\vee$ of $\mathfrak{g}^\vee$, and the space of polynomial functions $\mathbb{R}[\mathfrak{g}[2]]$ on $\mathfrak{g}[2]$ is the symmetric algebra $S\mathfrak{g}^\vee$ of $\mathfrak{g}^\vee$.

The elements $c \in \mathfrak{g}[1]$ have degree 1 and represent the connection 1-form on the universal bundle. The elements $\phi \in \mathfrak{g}[2]$ have degree 2 and represent the curvature 2-form on the universal bundle. An odd differential on functions on $\mathfrak{g}[1] \oplus \mathfrak{g}[2]$ can be described as an odd vector field δ such that $\delta^2 = 0$. The odd vector field δ of degree 1 represents de Rham differential on the universal bundle

$$\begin{aligned} \delta c &= \phi - \frac{1}{2}[c, c] \\ \delta \phi &= -[c, \phi] \end{aligned} \tag{2.3.2}$$

which follows from the standard relations between the connection A and the curvature F_A

$$\begin{aligned} dA &= F_A - \frac{1}{2}[A, A] \\ dF_A &= -[A, F_A] \end{aligned} \tag{2.3.3}$$

This definition implies $\delta^2 = 0$. Indeed,

$$\begin{aligned} \delta^2 c &= \delta\phi - [\delta c, c] = -[c, \phi] - [\phi - \frac{1}{2}[c, c], c] = 0 \\ \delta^2 \phi &= -[\delta c, \phi] + [c, \delta\phi] = -[\phi - \frac{1}{2}[c, c], \phi] - [c, [c, \phi]] = 0 \end{aligned} \tag{2.3.4}$$

Given a basis T_α on the Lie algebra $\mathfrak{g}$ with structure constants $[T_\beta, T_\gamma] = f^\alpha_{\beta\gamma} T_\alpha$ the differential δ has the form

$$\begin{aligned} \delta c^\alpha &= \phi^\alpha - \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \\ \delta \phi^\alpha &= -f^\alpha_{\beta\gamma} c^\beta \phi^\gamma \end{aligned} \tag{2.3.5}$$

The differential δ can be decomposed into the sum of two differentials

$$\delta = \delta_{\mathrm{K}} + \delta_{\mathrm{BRST}} \tag{2.3.6}$$

with

$$\begin{aligned} \delta_{\mathrm{K}} \phi &= 0, & \delta_{\mathrm{BRST}} \phi &= -[c, \phi] \\ \delta_{\mathrm{K}} c &= \phi, & \delta_{\mathrm{BRST}} c &= -\frac{1}{2}[c, c] \end{aligned} \tag{2.3.7}$$

The differential δ_{BRST} is the BRST differential (Chevalley-Eilenberg differential for Lie algebra cohomology with coefficients in the Lie algebra module $S\mathfrak{g}^\vee$). The differential δ_{K} is the Koszul differential (de Rham differential on $\Omega^\bullet(\Pi\mathfrak{g})$).

The field theory interpretation of the Weil algebra and the differential (2.3.6) was given in [11] and [12].

The Weil algebra $\mathcal{W}_\mathfrak{g} = \mathbb{R}[\mathfrak{g}[1] \oplus \mathfrak{g}[2]]$ is an extension of the Chevalley-Eilenberg algebra $CE_\mathfrak{g} = \mathbb{R}[\mathfrak{g}[1]] = \Lambda\mathfrak{g}^\vee$ by the algebra $\mathbb{R}[\mathfrak{g}[2]]^G = S\mathfrak{g}^\vee$ of symmetric polynomials on $\mathfrak{g}$

$$CE_\mathfrak{g} \leftarrow \mathcal{W}_\mathfrak{g} \leftarrow S\mathfrak{g}^\vee \tag{2.3.8}$$

which is quasi-isomorphic to the algebra of differential forms on the universal bundle

$$G \to EG \to BG \tag{2.3.9}$$

The duality between the Weil algebra $\mathcal{W}_{\mathfrak{g}}$ and the de Rham algebra $\Omega^\bullet(EG)$ of differential forms on EG is provided by the Weil homomorphism

$$\mathcal{W}_{\mathfrak{g}} \to \Omega^\bullet(EG) \tag{2.3.10}$$

after a choice of a connection 1-form $A \in \Omega^1(EG) \otimes \mathfrak{g}$ and its field strength $F_A \in \Omega^2(EG) \otimes \mathfrak{g}$ on the universal bundle $EG \to BG$.

Indeed, the connection 1-form $A \in \Omega^1(EG) \otimes \mathfrak{g}$ and field strength $F \in \Omega^2(EG) \otimes \mathfrak{g}$ define maps $\mathfrak{g}^\vee \to \Omega^1(EG)$ and $\mathfrak{g}^\vee \to \Omega^2(EG)$

$$\begin{aligned} c^\alpha &\mapsto A^\alpha \\ \phi^\alpha &\mapsto F^\alpha \end{aligned} \tag{2.3.11}$$

The cohomology of the Weil algebra is trivial

$$H^n(\mathcal{W}_{\mathfrak{g}}, \delta, \mathbb{R}) = \begin{cases} \mathbb{R}, & n = 0 \\ 0, & n > 0 \end{cases} \tag{2.3.12}$$

corresponding to the trivial cohomology of $\Omega^\bullet(EG)$.

To define G-equivariant cohomology we need to consider G action on EG. To compute $H^\bullet_G(pt) = H^\bullet(BG)$, consider $\Omega^\bullet(BG) = \Omega^\bullet(EG/G)$.

For any principal G-bundle $\pi : P \to P/G$ the differential forms on P in the image of the pullback π^* of the space of differential forms on P/G are called *basic*

$$\Omega^\bullet(P)_{\text{basic}} = \pi^*\Omega^\bullet(P/G) \tag{2.3.13}$$

Let L_α be the Lie derivative in the direction of a vector field α generated by a basis element $T_\alpha \in \mathfrak{g}$, and i_α be the contraction with the vector field generated by T_α.

An element $\omega \in \Omega^\bullet(P)_{\text{basic}}$ can be characterized by two conditions

1. ω is invariant on P with respect to the G-action: $L_\alpha \omega = 0$
2. ω is horizontal on P with respect to the G-action: $i_\alpha \omega = 0$

In the Weil model the contraction operation i_α is realized as

$$\begin{aligned} i_\alpha c^\beta &= \delta_\alpha^\beta \\ i_\alpha \phi^\beta &= 0 \end{aligned} \tag{2.3.14}$$

and the Lie derivative L_α is defined by the usual relation

$$L_\alpha = \delta i_\alpha + i_\alpha \delta \tag{2.3.15}$$

From the definition of $\Omega^\bullet(P)_{\text{basic}}$ for the case of $P = EG$ we obtain

$$H^\bullet_G(pt) = H^\bullet(BG, \mathbb{R}) = H^\bullet(\Omega^\bullet(EG)_{\text{basic}}, \mathbb{R}) = H^\bullet(\mathcal{W}_{\mathfrak{g}}, \delta, \mathbb{R})_{\text{basic}} = (S\mathfrak{g}^\vee)^G \tag{2.3.16}$$

2.4 Weil model and Cartan model of equivariant cohomology

The isomorphism

$$H(BG, \mathbb{R}) = H(EG, \mathbb{R})_{\text{basic}} = H(\mathcal{W}_{\mathfrak{g}}, \delta, \mathbb{R})_{\text{basic}} \tag{2.4.1}$$

suggests to replace the topological model for G-equivariant cohomologies of real manifold X

$$H_G(X, \mathbb{R}) = H((X \times EG)/G, \mathbb{R}) \tag{2.4.2}$$

by the *Cartan model*

$$H_G(X, \mathbb{R}) = H((\Omega^\bullet(X) \otimes S\mathfrak{g}^\vee)^G, \mathbb{R}) \tag{2.4.3}$$

or by the equivalent algebraic *Weil model*

$$H_G(X, \mathbb{R}) = H((\Omega^\bullet(X) \otimes \mathcal{W}_{\mathfrak{g}})_{\text{basic}}, \mathbb{R}) \tag{2.4.4}$$

2.4.1 Cartan model

Here $(\Omega^\bullet(X) \otimes S\mathfrak{g}^\vee)^G$ denotes the G-invariant subspace in $(\Omega^\bullet(X) \otimes S\mathfrak{g}^\vee)$ under the G-action induced from G-action on X and adjoint G-action on $\mathfrak{g}$.

It is convenient to think about $(\Omega^\bullet(X) \otimes S\mathfrak{g}^\vee)$ as the space

$$\Omega^{\bullet,0}_{C^\infty,\text{poly}}(X \times \mathfrak{g}) \tag{2.4.5}$$

of smooth differential forms on $X \times \mathfrak{g}$ of degree 0 along $\mathfrak{g}$ and polynomial along $\mathfrak{g}$.

In (T_a) basis on $\mathfrak{g}$, an element $\phi \in \mathfrak{g}$ is represented as $\phi = \phi^\alpha T_\alpha$. Then (ϕ^α) is the dual basis of $\mathfrak{g}^\vee$. Equivalently ϕ^α is a linear coordinate on $\mathfrak{g}$.

The commutative ring $\mathbb{R}[\mathfrak{g}]$ of polynomial functions on the vector space underlying $\mathfrak{g}$ is naturally represented in the coordinates as the ring of polynomials in generators $\{\phi^\alpha\}$

$$\mathbb{R}[\mathfrak{g}] = \mathbb{R}[\phi^1, \dots, \phi^{\operatorname{rk}\mathfrak{g}}] \tag{2.4.6}$$

Hence, the space (2.4.5) can be equivalently presented as

$$\Omega^{\bullet,0}_{C^\infty,\text{poly}}(X \times \mathfrak{g}) = \Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}] \tag{2.4.7}$$

Given an action of the group G on any manifold M

$$\rho_g : m \mapsto g \cdot m \tag{2.4.8}$$

the induced action on the space of differential forms $\Omega^\bullet(M)$ comes from the pullback by the map $\rho_{g^{-1}}$

$$\rho_g : \omega \mapsto \rho^*_{g^{-1}}\omega, \qquad \omega \in \Omega^\bullet(M) \tag{2.4.9}$$

In particular, if $M = \mathfrak{g}$ and $\omega \in \mathfrak{g}^\vee$ is a linear function on $\mathfrak{g}$, then (2.4.9) is the *co-adjoint action* on $\mathfrak{g}^\vee$.

The invariant subspace $(\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}])^G$ forms a complex with respect to the *Cartan differential*

$$d_G = d \otimes 1 + i_\alpha \otimes \phi^\alpha \tag{2.4.10}$$

where $d : \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ is the de Rham differential, and $i_\alpha : \Omega^\bullet(X) \to \Omega^{\bullet-1}(X)$ is the operation of contraction of the vector field on X generated by $T_\alpha \in \mathfrak{g}$ with differential forms in $\Omega^\bullet(X)$.

The Cartan model of the G-equivariant cohomology $H_G(X)$ is

$$H_G(X) = H\left((\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}])^G, d_G\right) \tag{2.4.11}$$

To check that $d_G^2 = 0$ on $(\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}])^G$ we compute d_G^2 on $\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}]$ and find

$$d_G^2 = L_\alpha \otimes \phi^\alpha \tag{2.4.12}$$

where $L_\alpha : \Omega^\bullet(X) \to \Omega^\bullet(X)$ is the Lie derivative on X

$$L_\alpha = d i_\alpha + i_\alpha d \tag{2.4.13}$$

along vector field generated by T_α.

The infinitesimal action by a Lie algebra generator T_a on an element $\omega \in \Omega^\bullet(X) \otimes R[\mathfrak{g}]$ is

$$T_\alpha \cdot \omega = (L_\alpha \otimes 1 + 1 \otimes L_\alpha) \cdot \omega \tag{2.4.14}$$

where $L_\alpha \otimes 1$ is the geometrical Lie derivative by the vector field generated by T_α on $\Omega^\bullet(X)$ and $1 \otimes L_a$ is the coadjoint action on $\mathbb{R}[\mathfrak{g}]$

$$L_\alpha = f_{\alpha\beta}^\gamma \phi^\beta \frac{\partial}{\partial \phi^\gamma} \tag{2.4.15}$$

If ω is a G-invariant element, $\omega \in (\Omega^\bullet(X) \otimes R[\mathfrak{g}])^G$, then

$$(L_\alpha \otimes 1 + 1 \otimes L_\alpha)\omega = 0 \tag{2.4.16}$$

Therefore, if $\omega \in (\Omega^\bullet(X) \otimes R[\mathfrak{g}])^G$ it holds that

$$d_G^2 \omega = (1 \otimes \phi^\alpha L_\alpha)\omega = \phi^\alpha f_{\alpha\beta}^\gamma \phi^\beta \frac{\partial \alpha}{\partial \phi^c} = 0 \tag{2.4.17}$$

by the antisymmetry of the structure constants $f_{\alpha\beta}^\gamma = -f_{\beta\alpha}^\gamma$. Therefore $d_G^2 = 0$ on $(\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}])^G$.

The grading on $\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}]$ is defined by the assignment

$$\deg d = 1 \quad \deg i_{v_\alpha} = -1 \quad \deg \phi^\alpha = 2 \tag{2.4.18}$$

which implies

$$\deg d_G = 1 \tag{2.4.19}$$

Let

$$\Omega^n_G(X) = \oplus_k (\Omega^{n-2k} \otimes \mathbb{R}[\mathfrak{g}]^k)^G \tag{2.4.20}$$

be the subspace in $(\Omega(X) \otimes \mathbb{R}[\mathfrak{g}])^G$ of degree n according to the grading (2.4.18).

Then

$$\dots \stackrel{d_G}{\to} \Omega^n_G(X) \stackrel{d_G}{\to} \Omega^{n+1}_G(X) \stackrel{d_G}{\to} \dots \tag{2.4.21}$$

is a differential complex. The equivariant cohomology groups $H^\bullet_G(X)$ in the Cartan model are defined as the cohomology of the complex (2.4.21)

$$H^\bullet_G(X) \equiv \operatorname{Ker} d_G / \operatorname{Im} d_G \tag{2.4.22}$$

In particular, if $X = pt$ is a point then

$$H^\bullet_G(pt) = \mathbb{R}[\mathfrak{g}]^G \tag{2.4.23}$$

in agreement with (2.3.16).

If x^μ are coordinates on X, and $\psi^\mu = dx^\mu$ are Grassman coordinates on the fibers of ΠTX, we can represent the Cartan differential (2.4.10) in the notations more common in quantum field theory traditions

$$\begin{aligned} \delta x^\mu &= \psi^\mu \\ \delta \psi^\mu &= \phi^\alpha v^\mu_\alpha \end{aligned} \qquad \delta\phi = 0 \tag{2.4.24}$$

where v^μ are components of the vector field on X generated by a basis element T_α for the G-action on X. In quantum field theory, the coordinates x^μ are typically coordinates on the infinite-dimensional space of bosonic fields, and ψ^μ are typically coordinates on the infinite-dimensional space of fermionic fields.

2.4.2 Weil model

The differential in Weil model can be presented in coordinate notations similar to (2.4.24) as follows

$$\begin{aligned} \delta x^\mu &= \psi^\mu + c^\alpha v^\mu_\alpha & \qquad \delta c^\alpha &= \phi^\alpha - \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \\ \delta\psi^\mu &= \phi^\alpha v^\mu_\alpha + \partial_\nu v^\mu_\alpha c^\alpha \psi^\nu & \qquad \delta\phi^\alpha &= -f^\alpha_{\beta\gamma} c^\beta \phi^\gamma \end{aligned} \tag{2.4.25}$$

In physical applications, typicallly c is the BRST ghost field for gauge symmetry, and Weil differential is the sum of a supersymmetry transformation and BRST transformation, for example see [13].

2.5 Equivariant characteristic classes in Cartan model

For a reference see [14] and [15].

Let G and T be compact connected Lie groups.

We consider a T-equivariant G-principal bundle $\pi : P \to X$. This means that an equivariant T-action is defined on P compatible with the G-bundle structure of $\pi : P \to X$. One can take that G acts from the right and T acts from the left.

The compatibility means that T-action on the total space of P

- commutes with the projection map $\pi : P \to X$
- commutes with the G action on the fibers of $\pi : P \to X$

Let $D_A = d + A$ be a T-invariant connection on a T-equivariant G-bundle P. Here the connection A is a $\mathfrak{g}$-valued 1-form on the total space of P (such a connection always exists by the averaging procedure for compact T).

Then we define the T-equivariant connection

$$D_{A,T} = D_A + \epsilon^a i_{v_a} \tag{2.5.1}$$

and the T-equivariant curvature

$$F_{A,T} = (D_{A,T})^2 - \epsilon^a \otimes \mathcal{L}_{v_a} \tag{2.5.2}$$

where ϵ^a are coordinates on the Lie algebra $\mathfrak{t}$ (like the coordinates ϕ^a on the Lie algebra $\mathfrak{g}$ in the previous section defining Cartan model of G-equivariant cohomology), which is in fact is an element of $\Omega^2_T(X) \otimes \mathfrak{g}$

$$F_{A,T} = F_A - \epsilon^a \otimes \mathcal{L}_{v_a} + [\epsilon^a \otimes i_{v_a}, 1 \otimes D_A] = F_A + \epsilon^a i_{v_a} A \tag{2.5.3}$$

Let X^T be the T-fixed point set in X. If the equivariant curvature $F_{A,T}$ is evaluated on X^T, only the vertical component of i_{v_a} contributes to the formula (2.5.3) and v_a pairs with the vertical component of the connection A on the T-fiber of P given by $g^{-1}dg$. The T-action on G-fibers induces the homomorphism

$$\rho : \mathfrak{t} \to \mathfrak{g} \tag{2.5.4}$$

and let $\rho(T_a)$ be the images of T_a basis elements of $\mathfrak{t}$.

An ordinary characteristic class for a principal G-bundle on X is $[p(F_A)] \in H^{2d}(X)$ for a G-invariant degree d polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$. Here F_A is the curvature of any connection A on the G-bundle.

In the same way, a T-equivariant characteristic class for a principal G-bundle associated to a G-invariant degree d polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$ is $[p(F_{A,T})] \in H^{2d}_T(X)$. Here $F_{A,T}$ is the T-equivariant curvature of any T-equivariant connection A on the G-bundle.

Restricted to T-fixed points X^T the T-equivariant characteristic class associated to polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$ is

$$p(F_A + \epsilon^a \rho(T_a)) \tag{2.5.5}$$

In particular, if V is a representation of G and p is the Chern character of the vector bundle V, then if X is a point, the equivariant Chern characters is an ordinary character of the space V as a G-module.

2.6 Standard characteristic classes

For a reference see the book by Bott and Tu [16].

2.6.1 Euler class

Let $G = SO(2n)$ be the special orthogonal group which preserves a Riemannian metric $g \in S^2V^\vee$ on an oriented real vector space V of $\dim_{\mathbb{R}} V = 2n$.

The Euler characterstic class is defined by the adjoint invariant polynomial

$$\mathrm{Pf} : \mathfrak{so}(2n, \mathbb{R}) \to \mathbb{R} \tag{2.6.1}$$

of degree n on the Lie algebra $\mathfrak{so}(2n)$ called *Pfaffian* and defined as follows. For an element $x \in \mathfrak{so}(2n)$ let $x' \in V^\vee \otimes V$ denote representation of x on V (fundamental representation), so that x' is an antisymmetric $(2n) \times (2n)$ matrix in some orthonormal basis of V. Let $g \cdot x' \in \Lambda^2 V^\vee$ be the two-form associated by g to x' , and let $v_g \in \Lambda^{2n} V^\vee$ be the standard volume form on V associated to the metric g, and $v_g^* \in \Lambda^{2n} V$ be the dual of v_g. By definition

$$\mathrm{Pf}(x) = \frac{1}{n!} \langle v_g^*, (g \cdot x')^{\wedge n} \rangle \tag{2.6.2}$$

For example, for the 2×2-blocks diagonal matrix

$$\mathrm{Pf} \begin{pmatrix} 0 & \epsilon_1 & \dots & \dots & 0 & 0 \\ -\epsilon_1 & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & \epsilon_n \\ 0 & 0 & \dots & \dots & -\epsilon_n & 0 \end{pmatrix} = \epsilon_1 \dots \epsilon_n \tag{2.6.3}$$

For an antisymmetric $(2n) \times (2n)$ matrix x', the definition implies that $\mathrm{Pf}(x)$ is a degree n polynomial of matrix elements of x which satisfies

$$\mathrm{Pf}(x)^2 = \det x \tag{2.6.4}$$

Let P be an $SO(2n)$ principal bundle $P \to X$.

In the standard normalization the Euler class $\mathrm{e}(P)$ is defined in such a way that it takes values in $H^{2n}(X, \mathbb{Z})$ and is given by

$$\mathrm{e}(P) = \frac{1}{(2\pi)^n} [\mathrm{Pf}(F)] \tag{2.6.5}$$

For example, the Euler characteristic of an oriented real manifold X of real dimension $2n$ is an integer number given by

$$\mathrm{e}(X) = \int_X \mathrm{e}(T_X) = \frac{1}{(2\pi)^n} \int_X \mathrm{Pf}(R) \tag{2.6.6}$$

where R denotes the curvature form of the tangent bundle T_X.

In quantum field theories the definition (2.6.2) of the Pfaffian is usually realized in terms of a Gaussian integral over the Grassmann (anticommuting) variables θ which satisfy $\theta_i\theta_j = -\theta_j\theta_i$. The definition (2.6.2) is presented as

$$\mathrm{Pf}(x) = \int d\theta_{2n} \dots d\theta_1 \exp(-\frac{1}{2}\theta_i x_{ij}\theta_j) \tag{2.6.7}$$

By definition, the integral $[d\theta_{2n} \dots d\theta_1]$ picks the coefficient of the monomial $\theta_1 \dots \theta_{2n}$ of an element of the the Grassman algebra generated by θ.

2.6.2 Euler class of vector bundle and Mathai-Quillen form

See Mathai-Quillen [17] and Aityah-Jeffrey [18].

The Euler class of a vector bundle can be presented in a QFT formalism. Let E be an oriented real vector bundle E of rank $2n$ over a manifold X.

Let x^μ be local coordinates on the base X, and let their differentials be denoted $\psi^\mu = dx^\mu$.

Let h^i be local coordinates on the fibers of E. Let ΠE denote the superspace obtained from the total space of the bundle E by inverting the parity of the fibers, so that the coordinates in the fibers of ΠE are odd variables χ^i. Let g_{ij} be the matrix of a Riemannian metric on the bundle E. Let A^i_μ be the matrix valued 1-form on X representing a connection on the bundle E.

Using the connection A we can define an odd vector field δ on the superspace $\Pi T(\Pi E)$, or, equivalently, a de Rham differential on the space of differential forms $\Omega^\bullet(\Pi E)$. In local coordinates (x^μ, ψ^μ) and (χ^i, h^i) the definition of δ is

$$\begin{aligned} \delta x^\mu &= \psi^\mu & \delta\chi^i &= h^i - A^i_{j\mu}\psi^\mu\chi^j \\ \delta\psi^\mu &= 0 & \delta h^i &= \delta(A^i_{j\mu}\psi^\mu\chi^j) \end{aligned} \tag{2.6.8}$$

Here $h^i = D\chi^i$ is the *covariant* de Rham differential of χ^i, so that under the change of framing on E given by $\chi^i = s^i_j\tilde{\chi}^j$ the h^i transforms in the same way, that is $h^i = s^i_j\tilde{h}^j$.

The odd vector field δ is nilpotent

$$\delta^2 = 0 \tag{2.6.9}$$

and is called de Rham vector field on $\Pi T(\Pi E)$.

Consider an element α of $\Omega^\bullet(\Pi E)$ defined by the equation

$$\alpha = \frac{1}{(2\pi)^{2n}} \exp(-t\delta V) \tag{2.6.10}$$

where $t \in \mathbb{R}_{>0}$ and

$$V = \frac{1}{2}(g_{ij}\chi^i h^j) \tag{2.6.11}$$

Notice that since h^i has been defined as $D\chi^i$ the definition (2.6.10) is coordinate independent.

To expand the definition of α (2.6.10) we compute

$$\delta(\chi, h) = (h - A\chi, h) - (\chi, dA\chi - A(h - A\chi)) = (h, h) - (\chi, F_A\chi) \tag{2.6.12}$$

where we suppresed the indices i, j, the d denotes the de Rham differential on X and F_A the curvature 2-form on the connection A

$$F_A = dA + A \wedge A \tag{2.6.13}$$

The Gaussian integration of the form α along the vertical fibers of ΠE gives

$$\frac{1}{(2\pi)^{2n}} \int [dh][d\chi] \exp(-\frac{1}{2}\delta(\chi, h)) = \frac{1}{(2\pi)^n}\mathrm{Pf}(F_A) \tag{2.6.14}$$

which agrees with definition of the integer valued Euler class (2.6.5). The representation of the Euler class in the form (2.6.10) is called the Gaussian *Mathai-Quillen representation* of the Thom class.

The Euler class of the vector bundle E is an element of $H^{2n}(X, \mathbb{Z})$. If $\dim X = 2n$, the number obtained after integration of the fundamental cycle on X

$$e(E) = \int_{\Pi T(\Pi E)} \alpha \tag{2.6.15}$$

is an integer Euler characterstic of the vector bundle E.

If $E = TX$ the equation (2.6.15) provides the Euler characteristic of the manifold X in the form

$$e(X) = \frac{1}{(2\pi)^{\dim X}} \int_{\Pi T(\Pi TX)} \exp(-t\delta V) \overset{t \to 0}{=} \frac{1}{(2\pi)^{\dim X}} \int_{\Pi T(\Pi TX)} 1 \tag{2.6.16}$$

Given a section s of the vector bundle E, we can deform the form α in the same δ-cohomology class by taking

$$V_s = \frac{1}{2}(\chi, h + \sqrt{-1}s) \tag{2.6.17}$$

After integrating over (h, χ) the the resulting differential form on X has factor

$$\exp(-\frac{1}{2t}s^2) \tag{2.6.18}$$

so it is concentraited in a neigborhood of the locus $s^{-1}(0) \subset X$ of zeroes of the section s.

In this way the Poincare-Hopf theorem is proven: given an oriented vector bundle E on an oriented manifold X, with $\operatorname{rank} E = \dim X$, the Euler characteristic of E is equal to the number of zeroes of a generic section s of E counted with orientation

$$e(E) = \sum_{x \in s^{-1}(0) \subset X} \operatorname{sign} \det ds|_x \tag{2.6.19}$$

where $ds|_x : T_x \to E_x$ is the differential of the section s at a zero $x \in s^{-1}(0)$. The assumption that s is a generic section implies that $\det ds|_x$ is non-zero.

For a short reference on the Mathai-Quillen formalism see [19].

2.6.3 Chern character

Let P be a principal $GL(n,\mathbb{C})$ bundle over a manifold X. The Chern character is an adjoint invariant function

$$\mathrm{ch} : \mathfrak{gl}(n,\mathbb{C}) \to \mathbb{C} \tag{2.6.20}$$

defined as the trace in the fundamental representation of the exponential map

$$\mathrm{ch} : x \mapsto \mathrm{tr}\, e^x \tag{2.6.21}$$

The exponential map is defined by formal series

$$\mathrm{tr}\, e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{tr}\, x^n \tag{2.6.22}$$

The eigenvalues of the $gl(n,\mathbb{C})$ matrix x are called *Chern roots*. In terms of the Chern roots the Chern character is

$$\mathrm{ch}(x) = \sum_{i=1}^{n} e^{x_i} \tag{2.6.23}$$

2.6.4 Chern class

Let P be a principal $GL(n,\mathbb{C})$ bundle over a manifold X. The Chern class c_k for $k \in \mathbb{Z}_{>0}$ of $x \in \mathfrak{gl}(n,\mathbb{C})$ is defined by expansion of the determinant

$$\det(1+tx) = \sum_{k=0}^{n} t^n c_n \tag{2.6.24}$$

In particular

$$c_1(x) = \mathrm{tr}\, x, \qquad c_n(x) = \det x \tag{2.6.25}$$

In terms of Chern roots c_k is defined by elementary symmetric monomials

$$c_k = \sum_{1 \le i_1 < i_2 \dots < i_k \le n} x_{i_1} \dots x_{i_n} \tag{2.6.26}$$

Remark on integrality. Our conventions for characteristic classes of $GL(n,\mathbb{C})$ bundles differ from the frequently used conventions in which Chern classes c_k take value in $H^{2k}(X,\mathbb{Z})$ by a factor of $(-2\pi\sqrt{-1})^k$. In our conventions the characteristic class of degree $2k$ needs to be multiplied by $\frac{1}{(-2\pi\sqrt{-1})^k}$ to be integral.

2.6.5 Todd class

Let P be a principal $GL(n,\mathbb{C})$ bundle over a manifold X. The Todd class of $x \in \mathfrak{gl}(n,\mathbb{C})$ is defined to be

$$\mathrm{td}(x) = \det \frac{x}{1-e^{-x}} = \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} \tag{2.6.27}$$

where det is evaluated in the fundamental representation. The ratio evaluates to a series expansion involving Bernoulli numbers B_k

$$\frac{x}{1-e^{-x}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} B_k x^k = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \dots \tag{2.6.28}$$

2.6.6 The $\hat{A}$ class

Let P be a principal $GL(n,\mathbb{C})$ bundle over a manifold X. The $\hat{A}$ class of $x \in GL(n,\mathbb{C})$ is defined as

$$\hat{A} = \det \frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} = \prod_{i=1}^{n} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \tag{2.6.29}$$

The $\hat{A}$ class is related to the Todd class by

$$\hat{A}(x) = \det e^{-\frac{x}{2}} \operatorname{td} x \tag{2.6.30}$$

2.7 Index formula

For a holomorphic vector bundle E over a complex variety X of $\dim_{\mathbb{C}} X = n$ the index $\operatorname{ind}(\bar{\partial}, E)$ is defined as

$$\operatorname{ind}(\bar{\partial}, E) = \sum_{k=0}^{n} (-1)^k \dim H^k(X, E) \tag{2.7.1}$$

The localization theorem in K-theory gives the index formula of Grothendieck-Riemann-Roch-Hirzebruch-Atiyah-Singer relating the index to the Todd class

$$\boxed{\operatorname{ind}(\bar{\partial}, E) = \frac{1}{(-2\pi\sqrt{-1})^n} \int_X \operatorname{td}(T_X^{1,0}) \operatorname{ch}(E)} \tag{2.7.2}$$

Similarly, the index of Dirac operator $\not{D} : S^+ \otimes E \to S^- \otimes E$ from the positive chiral spinors S^+ to the negative chiral spinors S^-, twisted by a vector bundle E, is defined as

$$\operatorname{ind}(\not{D}, E) = \dim\ker \not{D} - \dim\operatorname{coker} \not{D} \tag{2.7.3}$$

and is given by the Atiyah-Singer index formula

$$\boxed{\operatorname{ind}(\not{D}, E) = \frac{1}{(-2\pi\sqrt{-1})^n} \int_X \hat{A}(T_X^{1,0}) \operatorname{ch}(E)} \tag{2.7.4}$$

Notice that on a Kahler manifold the Dirac complex

$$\not{D} : S^+ \to S^-$$

is isomorphic to the Dolbeault complex

$$\dots \to \Omega^{0,p}(X) \xrightarrow{\bar\partial} \Omega^{0,p+1}(X) \to \dots$$

twisted by the square root of the canonical bundle $K = \Lambda^n (T^{1,0}_X)^\vee$

$$\not{D} = \bar\partial \otimes K^{\frac{1}{2}} \tag{2.7.5}$$

consistently with the relation (2.6.30) and the Riemann-Roch-Hirzebruch-Atiyah-Singer index formula

Remark on 2π and $\sqrt{-1}$ factors. The vector bundle E in the index formula (2.7.2) can be promoted to a complex

$$\to E^\bullet \to E^{\bullet+1} \to \tag{2.7.6}$$

In particular, the $\bar\partial$ index of the complex $E^\bullet = \Lambda^\bullet (T^{1,0})^\vee$ of $(\bullet, 0)$-forms on a Kahler variety X equals the Euler characteristic of X

$$e(X) = \operatorname{ind}(\bar\partial, \Lambda^\bullet (T^{1,0})^\vee) = \sum_{q=0}^{n} \sum_{p=0}^{n} (-1)^{p+q} \dim H^{p,q}(X) \tag{2.7.7}$$

We find

$$\operatorname{ch} \Lambda^\bullet (T^{1,0})^\vee = \prod_{i=1}^{n} (1 - e^{-x_i}) \tag{2.7.8}$$

where x_i are Chern roots of the curvature of the n-dimensional complex bundle $T^{1,0}_X$. Hence, the Todd index formula (2.7.2) gives

$$\mathrm{e}(X) = \frac{1}{(2\pi\sqrt{-1})^n} \int c_n(T^{1,0}_X) \tag{2.7.9}$$

The above agrees with the Euler characteristic (2.6.6) provided it holds that

$$\det(\sqrt{-1} x_{\mathfrak{u}(n)}) = \operatorname{Pf}(x_{\mathfrak{so}(2n)}) \tag{2.7.10}$$

where $x_{\mathfrak{so}(2n)}$ represents the curvature of the $2n$-dimensional real tangent bundle T_X as $2n \times 2n$ antisymmetric matrices, and $x_{\mathfrak{u}(n)}$ represents the curvature of the complex holomorphic n-dimensional tangent bundle $T^{(1,0)}_X$ as $n \times n$ anti-hermitian matrices. That (2.7.10) holds is clear from the (2 × 2 representation of $\sqrt{-1}$

$$\sqrt{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.7.11}$$

2.8 Equivariant integration

See the paper by Atiyah and Bott [20].

2.8.1 Thom isomorphism and Atiyah-Bott localization

A map

$$f : F \to X$$

of manifolds induces a natural pushfoward map on the homology

$$f_* : H_\bullet(F) \to H_\bullet(X)$$

and pullback on the cohomology

$$f^* : H^\bullet(X) \to H^\bullet(F)$$

In the situation when there is Poincaré duality between homology and cohomology we can construct pushforward operation on the cohomology

$$f_* : H^\bullet(F) \to H^\bullet(X) \tag{2.8.1}$$

We can display the pullback and pushforward maps on the diagram

$$H^\bullet(F) \underset{f^*}{\overset{f_*}{\rightleftarrows}} H^\bullet(X) \tag{2.8.2}$$

For example, if F and X are compact manifolds and $f : F \hookrightarrow X$ is the inclusion, then for the pushforward map $f_* : H^\bullet(F) \to H^\bullet(X)$ we find

$$f_* 1 = \Phi_F \tag{2.8.3}$$

where Φ_F is the cohomology class in $H^\bullet(X)$ which is Poincaré dual to the manifold $F \subset X$: for a form α on X we have

$$\int_F f^*\alpha = \int_X \Phi_F \wedge \alpha \tag{2.8.4}$$

If X is the total space of the orthogonal vector bundle $\pi : X \to F$ over the oriented manifold F then $\Phi_F(X)$ is called the Thom class of the vector bundle X and $f_* : H^\bullet(F) \to H^\bullet(X)$ is the Thom isomorphism: to a form α on F we associate a form $\Phi \wedge \pi^*\alpha$ on X. The important property of the Thom class Φ_F for a submanifold $F \hookrightarrow X$ is

$$f^*\Phi_F = \mathrm{e}(\nu_F) \tag{2.8.5}$$

where $\mathrm{e}(\nu_F)$ is the Euler class of the normal bundle to F in X. Combined with (2.8.3) the last equation gives

$$f^* f_* 1 = \mathrm{e}(\nu_F) \tag{2.8.6}$$

as a map $H^\bullet(F) \to H^\bullet(F)$.

More generally, if $f : F \hookrightarrow X$ is an inclusion of a manifold F into a manifold X the Poincaré dual class Φ_F is isomorphic to the Thom class of the normal bundle of F in X.

Now we consider T-equivariant cohomologies for a compact abelian Lie group T acting on X. Let $F = X^T$ be the set of T fixed points in X. Then the equivariant Euler class $\mathrm{e}_T(\nu_F)$ is invertible, therefore the identity map on $H_T^\bullet(X)$ can be presented as

$$1 = f_* \frac{1}{\mathrm{e}_T(\nu_F)} f^* \tag{2.8.7}$$

Let $\pi^X : X \to pt$ be the map from a manifold X to a point pt. The pushforward operator $\pi^X_* : H_T^\bullet(X) \to H_T^\bullet(pt)$ corresponds to the integration of the cohomology class over X. The pushforward is functorial. For maps $F \xrightarrow{f} X \xrightarrow{\pi^X} pt$ we have the composition $\pi^X_* f_* = \pi^F_*$ for $F \xrightarrow{\pi^F} pt$. So we arrive to the Atiyah-Bott integration formula

$$\pi^X_* = \pi^F_* \frac{f^*}{\mathrm{e}_T(\nu_F)} \tag{2.8.8}$$

or more explicitly

$$\boxed{\int_X \alpha = \int_F \frac{f^* \alpha}{\mathrm{e}_T(\nu_F)}} \tag{2.8.9}$$

2.8.2 Duistermaat-Heckman localization

A particular example where the Atiyah-Bott localization formula can be applied is a symplectic space on which a Lie group T acts in a Hamiltonian way. Namely, let (X, ω) be a real symplectic manifold of $\dim_{\mathbb{R}} X = 2n$ with symplectic form ω and let compact connected Lie group T act on X in Hamiltonian way, which means that there exists a function, called *moment map* or Hamiltonian

$$\mu : X \to \mathfrak{t}^\vee \tag{2.8.10}$$

such that

$$d\mu_a = -i_a \omega \tag{2.8.11}$$

in some basis (T_a) of $\mathfrak{t}$ where i_a is the contraction operation with the vector field generated by the T_a action on X.

The degree 2 element $\omega_T \in \Omega^\bullet(X) \otimes S\mathfrak{t}^*$ defined by the equation

$$\omega_T = \omega + \epsilon^a \mu_a \tag{2.8.12}$$

is a d_T-closed equivariant differential form:

$$d_T \omega_T = (d + \epsilon^a i_a)(\omega + \epsilon^b \mu_b) = \epsilon^a d\mu_a + \epsilon^a i_a \omega = 0 \tag{2.8.13}$$

This implies that the mixed-degree equivariant differential form

$$\alpha = e^{\omega_T} \tag{2.8.14}$$

is also d_T-closed, and we can apply the Atiyah-Bott localization formula to the integral

$$\int_X \exp(\omega_T) = \frac{1}{n!}\int_X \omega^n \exp(\epsilon^a \mu_a) \tag{2.8.15}$$

For $T = SO(2)$ so that $\mathrm{Lie}(SO(2)) \simeq \mathbb{R}$ the integral (2.8.15) is the typical partition function of a classical Hamiltonian mechanical system in statistical physics with Hamiltonian function $\mu : X \to \mathbb{R}$ and inverse temperature parameter $-\epsilon$.

Suppose that $T = SO(2)$ and that the set of fixed points X^T is discrete. Then the Atiyah-Bott localization formula (2.8.9) implies

$$\boxed{\frac{1}{n!}\int_X \omega^n \exp(\epsilon^a \mu_a) = \sum_{x \in X^T} \frac{\exp(\epsilon^a \mu_a)}{\mathrm{e}_T(\nu_x)}} \tag{2.8.16}$$

where ν_x is the normal bundle to a fixed point $x \in X^T$ in X and $\mathrm{e}_T(\nu_x)$ is the T-equivariant Euler class of the bundle ν_x.

The rank of the normal bundle ν_x is $2n$ and the structure group is $SO(2n)$. In notations of section 2.5 we evaluate the T-equivariant characteristic Euler class of the principal G-bundle for $T = SO(2)$ and $G = SO(2n)$ by equation (2.5.5) for the invariant polynomial on $\mathfrak{g} = \mathfrak{so}(2n)$ given by $p = \frac{1}{(2\pi)^n}\mathrm{Pf}$ according to definition (2.6.5).

2.8.3 Gaussian integral example

To illustrate the localization formula (2.8.16) suppose that $X = \mathbb{R}^{2n}$ with symplectic form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy_i \tag{2.8.17}$$

and $SO(2)$ action

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \mapsto \begin{pmatrix} \cos w_i\theta & -\sin w_i\theta \\ \sin w_i\theta & \cos w_i\theta \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \tag{2.8.18}$$

where $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ parametrizes $SO(2)$ and $(w_1, \dots, w_n) \in \mathbb{Z}^n$.

The point $0 \in X$ is the *fixed point* so that $X^T = \{0\}$, and the normal bundle $\nu_x = T_0 X$ is an $SO(2)$-module of real dimension $2n$ and complex dimension n that splits into a direct sum of n irreducible $SO(2)$ modules with weights $(w_1, \dots, w_n)$.

We identify *Lie*$(SO(2))$ with $\mathbb{R}$ with basis element $\{1\}$ and coordinate function $\epsilon \in$ *Lie*$(SO(2))^*$. The $SO(2)$ action (2.8.18) is Hamiltonian with respect to the moment map

$$\mu = \mu_0 + \frac{1}{2}\sum_{i=1}^{n} w_i(x_i^2 + y_i^2) \tag{2.8.19}$$

Assuming that $\epsilon < 0$ and all $w_i > 0$ we find by direct Gaussian integration

$$\frac{1}{n!}\int_X \omega^n \exp(\epsilon\mu) = \frac{(2\pi)^n}{(-\epsilon)^n \prod_{i=1}^n w_i} \exp(\epsilon\mu_0) \tag{2.8.20}$$

and the same result by the localization formula (2.8.16) because

$$e_T(\nu_x) = \frac{1}{(2\pi)^n}\mathrm{Pf}(\epsilon\rho(1)) \tag{2.8.21}$$

according to the definition of the T-equivariant class (2.5.5) and the Euler characteristic class (2.6.5), and where $\rho : Lie(SO(2)) \to Lie(SO(2n))$ is the homomorphism in (2.5.4) with

$$\rho(1) = \begin{pmatrix} 0 & -w_1 & \dots & \dots & 0 & 0 \\ w_1 & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & -w_n \\ 0 & 0 & \dots & \dots & w_n & 0 \end{pmatrix} \tag{2.8.22}$$

according to (2.8.18).

2.8.4 Example of a two-sphere

Let (X, ω) be the two-sphere S^2 with coordinates (θ, α) and symplectic structure

$$\omega = \sin\theta d\theta \wedge d\alpha \tag{2.8.23}$$

Let the Hamiltonian function be

$$H = -\cos\theta \tag{2.8.24}$$

so that

$$\omega = dH \wedge d\alpha \tag{2.8.25}$$

and the Hamiltonian vector field be $v_H = \partial_\alpha$. The differential form

$$\omega_T = \omega + \epsilon H = \sin\theta d\theta \wedge d\alpha - \epsilon\cos\theta$$

is d_T-closed for

$$d_T = d + \epsilon i_\alpha \tag{2.8.26}$$

Let

$$\alpha = e^{t\omega_T} \tag{2.8.27}$$

Locally there is a degree 1 form V such that $\omega_T = d_T V$, for example

$$V = -(\cos\theta)d\alpha \tag{2.8.28}$$

but globally V does not exist. The d_T-cohomology class $[\alpha]$ of the form α is non-zero.

The localization formula (2.8.15) gives

$$\int_X \exp(\omega_T) = \frac{2\pi}{-\epsilon}\exp(-\epsilon) + \frac{2\pi}{\epsilon}\exp(\epsilon) \tag{2.8.29}$$

where the first term is the contribution of the T-fixed point $\theta = 0$ and the second term is the contribution of the T-fixed point $\theta = \pi$.

2.9 Equivariant index formula (Dolbeault and Dirac)

Let G be a compact connected Lie group.

Suppose that X is a complex variety and E is a holomorphic G-equivariant vector bundle over X. Then the cohomology groups $H^\bullet(X, E)$ form representation of G. In this case the index of E (2.7.1) can be refined to *an equivariant index* or *character*

$$\operatorname{ind}_G(\bar\partial, E) = \sum_{k=0}^{n} (-1)^k \operatorname{ch}_G H^k(X, E) \tag{2.9.1}$$

where $\operatorname{ch}_G H^i(X, E)$ is the character of a representation of G in the vector space $H^i(X, E)$. More concretely, the equivariant index can be thought of as a gadget that attaches to G-equivariant holomorphic bundle E a complex valued adjoint invariant function on the group G

$$\operatorname{ind}_G(\bar\partial, E)(g) = \sum_{k=0}^{n} (-1)^k \operatorname{tr}_{H^k(X,E)} g \tag{2.9.2}$$

on elements $g \in G$. The sign alternating sum (2.9.2) is also known as *the supertrace*

$$\operatorname{ind}_G(\bar\partial, E)(g) = \operatorname{str}_{H^\bullet(X,E)} g \tag{2.9.3}$$

The index formula (2.7.2) is replaced by the equivariant index formula in which characteristic classes are promoted to G-equivariant characteristic classes in the Cartan model of G-equivariant cohomology with differential $d_G = d + \phi^a i_a$ as in (2.4.10)

$$\operatorname{ind}(\bar\partial, E)(e^{\phi^a T_a}) = \frac{1}{(-2\pi\sqrt{-1})^n} \int_X \operatorname{td}_G(T_X) \operatorname{ch}_G(E) = \int_X e_G(T_X) \frac{\operatorname{ch}_G E}{\operatorname{ch}_G \Lambda^\bullet T_X^\vee} \tag{2.9.4}$$

Here $\phi^a T_a$ is an element of Lie algebra of G and $e^{\phi^a T_a}$ is an element of G, and T_X denotes the holomorphic tangent bundle of the complex manifold X.

If the set X^G of G-fixed points is discrete, then applying the localization formula (2.8.9) to the equivariant index (2.9.4) we find the equivariant Lefshetz formula

$$\boxed{\operatorname{ind}(\bar\partial, E)(g) = \sum_{x \in X^G} \frac{\operatorname{tr}_{E_x}(g)}{\det_{T_x^{1,0}X}(1 - g^{-1})}} \tag{2.9.5}$$

The Euler character is cancelled against the numerator of the Todd character.

Example of $\mathbb{CP}^1$

Let X be $\mathbb{CP}^1$ and let $E = \mathcal{O}(n)$ be a complex line bundle of degree n over $\mathbb{CP}^1$, and let $G = U(1)$ equivariantly act on E as follows. Let z be a local coordinate on $\mathbb{CP}^1$, and let an element $t \in U(1) \subset \mathbb{C}^\times$ send the point with coordinate z to the point with coordinate tz so that

$$\operatorname{ch} T_0^{1,0} X = t \qquad \operatorname{ch} T_\infty^{1,0} X = t^{-1} \tag{2.9.6}$$

where $T_0^{1,0}X$ denotes the fiber of the holomorphic tangent bundle at $z=0$ and similarly $T_\infty^{1,0}X$ the fiber at $z=\infty$. Let the action of $U(1)$ on the fiber of E at $z=0$ be trivial. Then the action of $U(1)$ on the fiber of E at $z=\infty$ is found from the gluing relation

$$s_\infty = z^{-n}s_0 \tag{2.9.7}$$

to be of weight $-n$, so that

$$\operatorname{ch} E|_{z=0} = 1, \qquad \operatorname{ch} E|_{z=\infty} = t^{-n} \tag{2.9.8}$$

Then

$$\operatorname{ind}(\bar\partial, \mathcal{O}(n), \mathbb{CP}^1)(t) = \frac{1}{1-t^{-1}} + \frac{t^{-n}}{1-t} = \frac{1-t^{-n-1}}{1-t^{-1}} = \begin{cases} \sum_{k=0}^n t^{-k}, & n \ge 0 \\ 0, & n=-1, \\ -t\sum_{k=0}^{-n-2} t^k, & n<-1 \end{cases} \tag{2.9.9}$$

We can check against the direct computation. Assume $n \ge 0$. The kernel of $\bar\partial$ is spanned by $n+1$ holomorphic sections of $\mathcal{O}(n)$ of the form z^k for $k=0,\dots,n$, the cokernel is empty by Riemann-Roch. The section z^k is acted upon by $t \in T$ with weight t^{-k}. Therefore

$$\operatorname{ind}_T(\bar\partial, \mathcal{O}(n), \mathbb{CP}^1) = \sum_{k=0}^n t^{-k} \tag{2.9.10}$$

Even more explicitly, for illustration, choose a connection 1-form A with constant curvature $F_A = -\frac{1}{2}in\omega$, denoted in the patch around $\theta=0$ (or $z=0$) by $A^{(0)}$ and in the patch around $\theta=\pi$ (or $z=\infty$) by $A^{(\pi)}$

$$A^{(0)} = -\frac{1}{2}in(1-\cos\theta)d\alpha \qquad A^{(\pi)} = -\frac{1}{2}in(-1-\cos\theta)d\alpha \tag{2.9.11}$$

The gauge transformation between the two patches

$$A^{(0)} = A^{(\pi)} - in\,d\alpha \tag{2.9.12}$$

is consistent with the defining E bundle transformation rule for the sections $s^{(0)}, s^{(\pi)}$ in the patches around $\theta=0$ and $\theta=\pi$

$$s^{(0)} = z^n s^{(\pi)} \qquad A^{(0)} = A^{(\pi)} + z^n dz^{-n}. \tag{2.9.13}$$

The equivariant curvature F_T of the connection A in the bundle E is given by

$$F_T = -\frac{1}{2}in(\omega + \epsilon(1-\cos\theta)) \tag{2.9.14}$$

as can be verified against the definition (2.5.3) $F_T = F + \epsilon i_v A$. Notice that to verify the expression for the equivariant curvature (2.9.14) in the patch near $\theta=\pi$ one needs to take into account contributions from the vertical component $g^{-1}dg$ of the connection A on the total space of the principal $U(1)$ bundle and from the T-action on the fiber at $\theta = \pi$ with weight $-n$.

Then

$$\begin{aligned} \mathrm{ch}(E)|_{\theta=0} &= \exp(F_T)|_{\theta=0} = 1 \\ \mathrm{ch}(E)|_{\theta=\pi} &= \exp(F_T)|_{\theta=\pi} = \exp(-in\epsilon) = t^{-n} \end{aligned} \tag{2.9.15}$$

for $t = \exp(i\epsilon)$ in agreement with (2.9.9).

A similar exercise gives the index for the Dirac operator on S^2 twisted by a magnetic field of flux n

$$\mathrm{ind}(\not{D}, \mathcal{O}(n), S^2) = \frac{t^{n/2} - t^{-n/2}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \tag{2.9.16}$$

where now we have chosen the lift of the T-action symmetrically to be of weight $n/2$ at $\theta = 0$ and of weight $-n/2$ at $\theta = \pi$. Also notice that up to overall multiplication by a power of t related to the choice of lift of the T-action to the fibers of the bundle E, the relation (2.7.5) holds

$$\mathrm{ind}(\not{D}, \mathcal{O}(n), S^2) = \mathrm{ind}(\bar{\partial}, \mathcal{O}(n-1), \mathbb{CP}^1) \tag{2.9.17}$$

because on $\mathbb{CP}^1$ the canonical bundle is $K = \mathcal{O}(-2)$.

Example of $\mathbb{CP}^m$

. Let $X = \mathbb{CP}^m$ be defined by the projective coordinates $(x_0 : x_1 : \cdots : x_m)$ and L_n be the line bundle $L_n = \mathcal{O}(n)$. Let $T = U(1)^{(m+1)}$ act on X by

$$(x_0 : x_1 : \ldots x_m) \mapsto (t_0^{-1} x_0 : t_1^{-1} x_1 : \cdots : t_m^{-1} x_m) \tag{2.9.18}$$

and by t_k^n on the fiber of the bundle L_n in the patch around the k-th fixed point $x_k = 1, x_{i \neq k} = 0$. We find the index as a sum of contributions from $m+1$ fixed points

$$\mathrm{ind}_T(D) = \sum_{k=0}^{m} \frac{t_k^n}{\prod_{j \neq k}(1 - (t_j/t_k))} \tag{2.9.19}$$

For $n \geq 0$ the index is a homogeneous polynomial in $\mathbb{C}[t_0, \ldots, t_m]$ of degree n representing the character on the space of holomorphic sections of the $\mathcal{O}(n)$ bundle over $\mathbb{CP}^m$.

$$\mathrm{ind}_T(D) = \begin{cases} s_n(t_0, \ldots, t_m), \quad n \geq 0 \\ 0, \quad -m \leq n < 0 \\ (-1)^m t_0^{-1} t_1^{-1} \ldots t_m^{-1} s_{-n-m-1}(t_0^{-1}, \ldots, t_m^{-1}), \quad n \leq -m-1 \end{cases} \tag{2.9.20}$$

where $s_n(t_0, \ldots, t_m)$ are complete homogeneous symmetric polynomials. This result can be quickly obtained from the contour integral representation of the sum (2.9.19)

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dz}{z} \frac{z^n}{\prod_{j=0}^{m}(1 - t_j/z)} = \sum_{k=0}^{m} \frac{t_k^n}{\prod_{j \neq k}(1 - (t_j/t_k))}, \tag{2.9.21}$$

If $n \geq -m$ we pick the contour of integration $\mathcal{C}$ to enclose all residues $z = t_j$. The residue at $z = 0$ is zero and the sum of residues is (2.9.19). On the other hand, the same contour integral is evaluated by the residue at $z = \infty$ which is computed by expanding all fractions in inverse powers of z, and is given by the complete homogeneous polynomial in t_i of degree n.

If $n < -m$ we assume that the contour of integration is a small circle around the $z = 0$ and does not include any of the residues $z = t_j$. Summing the residues outside of the contour, and taking that $z = \infty$ does not contribute, we get (2.9.19) with the $(-)$ sign . The residue at $z = 0$ contributes by (2.9.20).

Also notice that the last line of (2.9.20) relates[1] to the first line by the reflection $t_i \to t_i^{-1}$

$$\frac{t_k^n}{\prod_{j\neq k}(1 - t_j/t_k)} = \frac{(-1)^m (t_k^{-1})^{-n-m-1} (\prod_j t_j^{-1})}{\prod_{j\neq k}(1 - t_j^{-1}/t_k^{-1})} \tag{2.9.22}$$

which is the consequence of the Serre duality on $\mathbb{CP}^m$.

2.10 Equivariant index and representation theory

The $\mathbb{CP}^1$ in example (2.9.16) can be thought of as a flag manifold $SU(2)/U(1)$, and (2.9.9) (2.9.16) as characters of $SU(2)$-modules. For index theory on general flag manifolds $G_{\mathbb{C}}/B_{\mathbb{C}}$, that is Borel-Weyl-Bott theorem[2], the shift of the form (2.9.17) is a shift by the Weyl vector $\rho = \sum_{\alpha>0} \alpha$ where α are positive roots of $\mathfrak{g}$.

The index formula with localization to the fixed points on a flag manifold is equivalent to the Weyl character formula.

The generalization of formula (2.9.16) for generic flag manifold appearing from a co-adjoint orbit in $\mathfrak{g}^*$ is called *Kirillov character formula* [21], [22], [23].

Let G be a compact simple Lie group. The Kirillov character formula equates the T-equivariant index of the Dirac operator $\mathrm{ind}_T(D)$ on the G-coadjoint orbit of the element $\lambda + \rho \in \mathfrak{g}^*$ with the character χ_λ of the G irreducible representation with highest weight λ.

The character χ_λ is a function $\mathfrak{g} \to \mathbb{C}$ determined by the representation of the Lie group G with highest weight λ as

$$\chi_\lambda : X \mapsto \mathrm{tr}_\lambda\, e^X, \qquad X \in \mathfrak{g} \tag{2.10.1}$$

Let X_λ be an orbit of the co-adjoint action by G on $\mathfrak{g}^*$. Such orbit is specified by an element $\lambda \in \mathfrak{t}^*/W$ where $\mathfrak{t}$ is the Lie algebra of the maximal torus $T \subset G$ and W is the Weyl group. The co-adjoint orbit X_λ is a homogeneous symplectic G-manifold with the canonical symplectic structure ω defined at point $x \in X \subset \mathfrak{g}^*$ on tangent vectors in $\mathfrak{g}$ by the formula

$$\omega_x(\bullet_1, \bullet_2) = \langle x, [\bullet_1, \bullet_2] \rangle \qquad \bullet_1, \bullet_2 \in \mathfrak{g} \tag{2.10.2}$$

The converse is also true: any homogeneous symplectic G-manifold is locally isomorphic to a coadjoint orbit of G or central extension of it.

[1] Thanks to Bruno Le Floch for the comment

[2] For a short presentation see exposition by J. Lurie at http://www.math.harvard.edu/~lurie/papers/bwb.pdf

The minimal possible stabilizer of λ is the maximal abelian subgroup $T \subset G$, and the maximal co-adjoint orbit is G/T. Such orbit is called *a full flag manifold*. The real dimension of the full flag manifold is $2n = \dim G - \operatorname{rk} G$, and is equal to the number of roots of $\mathfrak{g}$. If the stabilizer of λ is a larger group H, such that $T \subset H \subset G$, the orbit X_λ is called *a partial flag manifold* G/H. A degenerate flag manifold is a projection from the full flag manifold with fibers isomorphic to H/T.

Flag manifolds are equipped with natural complex and Kahler structure. There is an explicitly holomorphic realization of the flag manifolds as a complex quotient $G_{\mathbb{C}}/P_{\mathbb{C}}$ where $G_{\mathbb{C}}$ is the complexification of the compact group G and $P_{\mathbb{C}} \subset G_{\mathbb{C}}$ is a parabolic subgroup. Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ be the standard decomposition of $\mathfrak{g}$ into the Cartan $\mathfrak{h}$ algebra and the upper triangular $\mathfrak{g}_+$ and lower triangular $\mathfrak{g}_-$ subspaces.

The minimal parabolic subgroup is known as Borel subgroup $B_{\mathbb{C}}$, its Lie algebra is conjugate to $\mathfrak{h} \oplus \mathfrak{g}_+$. The Lie algebra of generic parabolic subgroup $P_{\mathbb{C}} \supset B_{\mathbb{C}}$ is conjugate to the direct sum of $\mathfrak{h} \oplus \mathfrak{g}_+$ and a proper subspace of $\mathfrak{g}_-$.

Full flag manifolds with integral symplectic structure are in bijection with irreducible G-representations π_λ of highest weight λ

$$X_{\lambda+\rho} \leftrightarrow \pi_\lambda \tag{2.10.3}$$

This is known as the Kirillov correspondence in geometric representation theory.

Namely, if $\lambda \in \mathfrak{g}^*$ is a weight, the symplectic structure ω is integral and there exists a line bundle $L \to X_\lambda$ with a unitary connection of curvature ω. The line bundle $L \to X_\lambda$ is acted upon by the maximal torus $T \subset G$ and we can study the T-equivariant geometric objects. The Kirillov-Berline-Getzler-Vergne character formula equates the equivariant index of the Dirac operator $\not{D}$ twisted by the line bundle $L \to X_{\lambda+\rho}$ on the co-adjoint orbit $X_{\lambda+\rho}$ with the character χ_λ of the irreducible representation of G with highest weight λ

$$\operatorname{ind}_T(\not{D})(X_{\lambda+\rho}) = \chi_\lambda \tag{2.10.4}$$

This formula can be easily proven using the Atiyah-Singer equivariant index formula

$$\operatorname{ind}_T(\not{D})(X_{\lambda+\rho}) = \frac{1}{(-2\pi i)^n} \int_{X_{\lambda+\rho}} \operatorname{ch}_T(L) \hat{A}_T(T_X) \tag{2.10.5}$$

and the Atiyah-Bott formula to localize the integral over $X_{\lambda+\rho}$ to the set of fixed points $X_{\lambda+\rho}^T$.

The localization to $X_{\lambda+\rho}^T$ yields the Weyl formula for the character. Indeed, the stabilizer of $\lambda+\rho$, where λ is a dominant weight, is the Cartan torus $T \subset G$. The co-adjoint orbit $X_{\lambda+\rho}$ is the full flag manifold. The T-fixed points are in the intersection $X_{\lambda+\rho} \cap \mathfrak{t}$, and hence, the set of the T-fixed points is the Weyl orbit of $\lambda+\rho$

$$X_{\lambda+\rho}^T = \operatorname{Weyl}(\lambda+\rho) \tag{2.10.6}$$

At each fixed point $p \in X_{\lambda+\rho}^T$ the tangent space $T_{X_{\lambda+\rho}}|_p$ is generated by the root system of $\mathfrak{g}$. The tangent space is a complex T-module $\oplus_{\alpha>0}\mathbb{C}_\alpha$ with weights α given by the positive roots of $\mathfrak{g}$. Consequently, the denominator of $\hat{A}_T$ gives the Weyl denominator, the numerator of $\hat{A}_T$ cancels with the Euler class $e_T(T_X)$ in the localization formula, and the restriction of $\mathrm{ch}_T(L) = e^{\omega}$ is $e^{w(\lambda+\rho)}$

$$\frac{1}{(-2\pi i)^n}\int_{X_{\lambda+\rho}} \mathrm{ch}_T(L)\hat{A}(T_X) = \sum_{w\in W} \frac{e^{iw(\lambda+\rho)\epsilon}}{\prod_{\alpha>0}(e^{\frac{1}{2}i\alpha\epsilon} - e^{-\frac{1}{2}i\alpha\epsilon})} \tag{2.10.7}$$

We conclude that the localization of the equivariant index of the Dirac operator on $X_{\lambda+\rho}$ twisted by the line bundle L to the set of fixed points $X^T_{\lambda+\rho}$ is precisely the Weyl formula for the character.

The Kirillov correspondence between the index of the Dirac operator of $L \to X_{\lambda+\rho}$ and the character is closedly related to the Borel-Weyl-Bott theorem.

Let $B_{\mathbb{C}}$ be a Borel subgroup of $G_{\mathbb{C}}$, $T_{\mathbb{C}}$ be the maximal torus, λ an integral weight of $T_{\mathbb{C}}$. A weight λ defines a one-dimensional representation of $B_{\mathbb{C}}$ by pulling back the representation on $T_{\mathbb{C}} = B_{\mathbb{C}}/U_{\mathbb{C}}$ where $U_{\mathbb{C}}$ is the unipotent radical of $B_{\mathbb{C}}$ (the unipotent radical $U_{\mathbb{C}}$ is generated by $\mathfrak{g}_+$). Let $L_\lambda \to G_{\mathbb{C}}/B_{\mathbb{C}}$ be the associated line bundle, and $\mathcal{O}(L_\lambda)$ be the sheaf of regular local sections of L_λ. For $w \in \mathrm{Weyl}_G$ define the action of w on a weight λ by $w * \lambda := w(\lambda+\rho) - \rho$.

The *Borel-Weyl-Bott* theorem is that for any weight λ one has

$$H^{l(w)}(G_{\mathbb{C}}/B_{\mathbb{C}}, \mathcal{O}(L_\lambda)) = \begin{cases} R_\lambda, & w*\lambda \quad \text{is dominant} \\ 0, & w*\lambda \quad \text{is not dominant} \end{cases} \tag{2.10.8}$$

where R_λ is the irreducible G-module with highest weight λ, the w is an element of Weyl group such that $w*\lambda$ is dominant weight, and $l(w)$ is the length of w. We remark that if there exists $w \in \mathrm{Weyl}_G$ such that $w*\lambda$ is dominant weight, then w is unique. There is no $w \in \mathrm{Weyl}_G$ such that $w*\lambda$ is dominant if in the basis of the fundamental weights Λ_i some of the coordinates of $\lambda+\rho$ vanish.

Example

For $G = SU(2)$ the $G_{\mathbb{C}}/B_{\mathbb{C}} = \mathbb{CP}^1$, an integral weight of $T_{\mathbb{C}}$ is an integer $n \in \mathbb{Z}$, and the line bundle L_n is the $\mathcal{O}(n)$ bundle over $\mathbb{CP}^1$. The Weyl weight is $\rho = 1$.

The weight $n \geq 0$ is dominant and the $H^0(\mathbb{CP}^1, \mathcal{O}(n))$ is the $SL(2,\mathbb{C})$ module of highest weight n (in the basis of fundamental weights of $SL(2)$).

For weight $n = -1$ the $H^i(\mathbb{CP}^1, \mathcal{O}(-1))$ is empty for all i as there is no Weyl transformation w such that $w*n$ is dominant (equivalently, because $\rho + n = 0$).

For weight $n \leq -2$ the w is the $\mathbb{Z}_2$ reflection and $w*n = -(n+1) - 1 = -n-2$ is dominant and $H^1(\mathbb{CP}^1, \mathcal{O}(n))$ is an irreducible $SL(2,\mathbb{C})$ module of highest weight $-n-2$.

The relation between Borel-Weil-Bott theorem for $G_{\mathbb{C}}/B_{\mathbb{C}}$ and the Dirac complex on $G_{\mathbb{C}}/B_{\mathbb{C}}$ is that Dirac operator is precisely the Dolbeault operator shifted by the square root of the canonical bundle

$$S^+(X) \ominus S^-(X) = K^{\frac{1}{2}} \sum (-1)^p \Omega^{0,p}(X) \tag{2.10.9}$$

and consequently

$$\mathrm{ind}(X_{\lambda+\rho}, \not{D} \otimes L_{\lambda+\rho}) = \mathrm{ind}(G_{\mathbb{C}}/B_{\mathbb{C}}, \bar{\partial} \otimes L_\lambda) \tag{2.10.10}$$

The Borel-Bott-Weyl theorem has a generalization for partial flag manifolds. Let $P_{\mathbb{C}}$ be a parabolic subgroup of $G_{\mathbb{C}}$ with $B_{\mathbb{C}} \subset P_{\mathbb{C}}$ and let $\pi : G_{\mathbb{C}}/B_{\mathbb{C}} \to G_{\mathbb{C}}/P_{\mathbb{C}}$ denote the canonical projection. Let $E \to G_{\mathbb{C}}/P_{\mathbb{C}}$ be a vector bundle associated to an irreducible finite dimensional $P_{\mathbb{C}}$ module, and let $\mathcal{O}(E)$ the the sheaf of local regular sections of E. Then $\mathcal{O}(E)$ is isomorphic to the direct image sheaf $\pi_*\mathcal{O}(L)$ for a one-dimensional $B_{\mathbb{C}}$-module L and

$$H^k(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathcal{O}(E)) = H^k(G_{\mathbb{C}}/B_{\mathbb{C}}, \mathcal{O}(L))$$

For application of Kirillov theory to Kac-Moody and Virasoro algebra see [24].

2.11 Equivariant index for differential operators

See the book by Atiyah [25].

Let E_k be vector bundles over a manifold X. Let G be a compact Lie group acting on X and the bundles E_k. The action of G on a bundle E induces canonically a linear action on the space of sections $\Gamma(E)$. For $g \in G$ and a section $\phi \in \Gamma(E)$ the action is

$$(g\phi)(x) = g\phi(g^{-1}x), \quad x \in X \tag{2.11.1}$$

Let D_k be linear differential operators compatible with the G action, and let $\mathcal{E}$ be the complex (that is $D_{k+1} \circ D_k = 0$)

$$\mathcal{E} : \Gamma(E_0) \xrightarrow{D_0} \Gamma(E_1) \xrightarrow{D_1} \Gamma(E_2) \to \dots \tag{2.11.2}$$

Since D_k are G-equivariant operators, the G-action on $\Gamma(E_k)$ induces the G-action on the cohomology $H^k(\mathcal{E})$. The equivariant index of the complex $\mathcal{E}$ is the virtual character

$$\mathrm{ind}_G(D) : \mathfrak{g} \to \mathbb{C} \tag{2.11.3}$$

defined by

$$\mathrm{ind}_G(D)(g) = \sum_k (-1)^k \operatorname{tr}_{H^k(\mathcal{E})} g \tag{2.11.4}$$

2.11.1 Atiyah-Singer equivariant index formula for elliptic complexes

If the set X^G of G-fixed points is discrete, the Atiyah-Singer equivariant index formula is

$$\boxed{\mathrm{ind}_G(D) = \sum_{x \in X^G} \frac{\sum_k (-1)^k \operatorname{ch}_G(E_k)|_x}{\det_{T_x X}(1 - g^{-1})}} \tag{2.11.5}$$

For the Dolbeault complex $E_k = \Omega^{0,k}$ and $D_k = \bar{\partial} : \Omega^{0,k} \to \Omega^{0,k+1}$

$$\to \Omega^{0,\bullet} \xrightarrow{\bar{\partial}} \Omega^{0,\bullet+1} \to \tag{2.11.6}$$

the index (2.11.5) agrees with (2.9.5) because the numerator in (2.11.5) decomposes as $\operatorname{ch}_G E \operatorname{ch}_G \Lambda^\bullet T^*_{0,1}$ and the denominator as $\operatorname{ch}_G \Lambda^\bullet T^*_{0,1} \operatorname{ch}_G \Lambda^\bullet T^*_{1,0}$ and the factor $\operatorname{ch}_G \Lambda^\bullet T^*_{0,1}$ cancels out.

For example, the equivariant index of $\bar{\partial} : \Omega^{0,0}(X) \to \Omega^{0,1}(X)$ on $X = \mathbb{C}_{\langle x \rangle}$ under the $T = U(1)$ action $x \mapsto t^{-1}x$ where $t \in T$ is the fundamental character is contributed by the fixed point $x = 0$ as

$$\operatorname{ind}_T(\mathbb{C}, \bar{\partial}) = \frac{1 - \bar{t}}{(1-t)(1-\bar{t})} = \frac{1}{1-t} = \sum_{k=0}^{\infty} t^k \tag{2.11.7}$$

where the denominator is the determinant of the operator $1 - t$ over the two-dimensional normal bundle to $0 \in \mathbb{C}$ spanned by the vectors ∂_x and $\partial_{\bar{x}}$ with eigenvalues t and $\bar{t}$. In the numerator, 1 comes from the equivariant Chern character on the fiber of the trivial line bundle at $x = 0$ and $-\bar{t}$ comes from the equivariant Chern character on the fiber of the bundle of $(0,1)$ forms $d\bar{x}$.

We can compare the expansion in power series in t^k of the index with the direct computation. The terms t^k for $k \in \mathbb{Z}_{\geq 0}$ come from the local T-equivariant holomorphic functions x^k which span the kernel of $\bar{\partial}$ on $\mathbb{C}_{\langle x \rangle}$. The cokernel is empty by the Poincaré lemma. Compare with (2.9.10).

Similarly, for the $\bar{\partial}$ complex on $\mathbb{C}^r$ we obtain

$$\operatorname{ind}_T(\mathbb{C}^r, \bar{\partial}) = \left[\prod_{k=1}^{r} \frac{1}{(1-t_k)} \right]_+ \tag{2.11.8}$$

where $[]_+$ means expansion in positive powers of t_k.

For application to the localization computation on spheres of even dimension S^{2r} we can compute the index of a certain transversally elliptic operator D which naturally interpolates between the $\bar{\partial}$-complex in the neighborhood of one fixed point (north pole) of the r-torus T^r action on S^{2r} and the $\bar{\partial}$-complex in the neighborhood of another fixed point (south pole). The index is a sum of two fixed point contributions

$$\begin{aligned} \operatorname{ind}_T(S^{2r}, D) &= \left[\prod_{k=1}^{r} \frac{1}{(1-t_k)} \right]_+ + \left[\prod_{k=1}^{r} \frac{1}{(1-t_k)} \right]_- \\ &= \left[\prod_{k=1}^{r} \frac{1}{(1-t_k)} \right]_+ + \left[\prod_{k=1}^{r} \frac{(-1)^r t_1^{-1} \dots t_r^{-1}}{(1-t_k^{-1})} \right]_- \end{aligned} \tag{2.11.9}$$

where $[]_+$ and $[]_-$ denotes the expansions in positive and negative powers of t_k.

2.11.2 Atiyah-Singer index formula for a free action G-manifold

Suppose that a compact Lie group G acts freely on a manifold X and let $Y = X/G$ be the quotient, and let

$$\pi : X \to Y \tag{2.11.10}$$

be the associated G-principal bundle.

Suppose that D is a $G \times T$ equivariant operator (differential) for a complex $(\mathcal{E}, D)$ of vector bundles E_k over X as in (2.11.2). The $G \times T$-equivariance means that the complex $\mathcal{E}$ and the operator D are pullbacks by π^* of a T-equivariant complex $\tilde{\mathcal{E}}$ and operator $\tilde{D}$ on the base Y

$$\mathcal{E} = \pi^* \tilde{\mathcal{E}}, \quad D = \pi^* \tilde{D} \tag{2.11.11}$$

We want to compute the $G \times T$-equivariant index $\mathrm{ind}_{G\times T}(D; X)$ for the complex $(\mathcal{E}, D)$ on the total space X for a $G \times T$ transversally elliptic operator D using T-equivariant index theory on the base Y. We can do that using Fourier theory on G (counting KK modes in G-fibers).

Let R_G be the set of all irreducible representations of G. For each irreducible representation $\alpha \in R_G$ we denote by χ_α the character of this representation, and by W_α the vector bundle over Y associated to the principal G-bundle (2.11.10). Then, for each irrep $\alpha \in R_G$ we consider a complex $\tilde{\mathcal{E}} \otimes W_\alpha$ on Y obtained by tensoring $\tilde{\mathcal{E}}$ with the vector bundle W_α over Y. The Atiyah-Singer formula is

$$\mathrm{ind}_{G\times T}(D; X) = \sum_{\alpha \in R_G} \mathrm{ind}_T(\tilde{D} \otimes W_\alpha; Y)\chi_\alpha. \tag{2.11.12}$$

Example of S^{2r-1}

We consider an example immediately relevant for localization on odd-dimensional spheres S^{2r-1} which are subject to the equivariant action of the maximal torus T^r of the isometry group $SO(2r)$. The sphere $\pi : S^{2r-1} \to \mathbb{CP}^{r-1}$ is the total space of the S^1 Hopf fibration over the complex projective space $\mathbb{CP}^{r-1}$.

We will apply the equation (2.11.12) for a transversally elliptic operator D induced from the Dolbeault operator $\tilde{D} = \bar{\partial}$ on $\mathbb{CP}^{r-1}$ by the pullback π^*.

To compute the index of operator $D = \pi^* \bar{\partial}$ on $\pi : S^{2r-1} \to \mathbb{CP}^{r-1}$ we apply (2.11.12) and use (2.9.20) and obtain

$$\mathrm{ind}(D, S^{2r-1}) = \sum_{n=-\infty}^{\infty} \mathrm{ind}_T(\bar{\partial}, \mathbb{CP}^{r-1}, \mathcal{O}(n)) = \left[\frac{1}{\prod_{k=1}^r (1-t_k)}\right]_+ + \left[\frac{(-1)^{r-1} t_1^{-1} \dots t_r^{-1}}{\prod_{k=1}^r (1-t_k^{-1})}\right]_- \tag{2.11.13}$$

where $[]_+$ and $[]_-$ denotes the expansion in positive and negative powers of t_k. See further review in Chapter 1.

2.11.3 General Atiyah-Singer index formula

The Atiyah-Singer index formula for the Dolbeault and Dirac complexes and the equivariant index formula (2.11.5) can be generalized to a generic situation of an equivariant index of transversally elliptic complex (2.11.2).

Let X be a real manifold. Let $\pi : T^*X \to X$ be the cotangent bundle. Let $\{E^\bullet\}$ be an indexed set of vector bundles on X and $\pi^* E^\bullet$ be the vector bundles over T^*X defined by the pullback.

The symbol $\sigma(D)$ of a differential operator $D : \Gamma(E) \to \Gamma(F)$ (2.11.2) is a linear operator $\sigma(D) : \pi^* E \to \pi^* F$ which is defined by taking the highest degree part of the differential operator and replacing all derivatives $\frac{\partial}{\partial x^\mu}$ by the conjugate coordinates p^μ in the fibers of T^*X.

For example, for the Laplacian $\Delta : \Omega^0(X, \mathbb{R}) \to \Omega^0(X, \mathbb{R})$ with highest degree part in some coordinate system $\{x^\mu\}$ given by $\Delta = g^{\mu\nu}\partial_\mu\partial_\nu$ where $g^{\mu\nu}$ is the inverse Riemannian metric, the symbol of Δ is a $\mathrm{Hom}(\mathbb{R}, \mathbb{R})$-valued (i.e. number valued) function on T^*X given by

$$\sigma(\Delta) = g^{\mu\nu} p_\mu p_\nu \tag{2.11.14}$$

where p_μ are conjugate coordinates (momenta) on the fibers of T^*X.

A differential operator $D : \Gamma(E) \to \Gamma(F)$ is elliptic if its symbol $\sigma(D) : \pi^*E \to \pi^*F$ is an isomorphism of vector bundles π^*E and π^*F on T^*X outside of the zero section $X \subset T^*X$.

The index of a differential operator D depends only on the topological class of its symbol in the topological K-theory of vector bundles on T^*X. The Atiyah-Singer formula for the index of the complex (2.11.2) is

$$\boxed{\mathrm{ind}_G(D, X) = \frac{1}{(2\pi)^{\dim_{\mathbb{R}} X}} \int_{T^*X} \hat{A}_G(\pi^* T_X)\, \mathrm{ch}_G(\pi^* E^\bullet)} \tag{2.11.15}$$

Here T^*X denotes the total space of the cotangent bundle of X with canonical orientation such that $dx^1 \wedge dp_1 \wedge dx^2 \wedge dp_2 \dots$ is a positive element of $\Lambda^{\mathrm{top}}(T^*X)$.

Let $n = \dim_{\mathbb{R}} X$. Let π^*T_X denote the vector bundle of dimension n over the total T^*X obtained as pullback of $T_X \to X$ to T^*X. The $\hat{A}_G$-character of π^*T_X is

$$\hat{A}_G(\pi^* T_X) = \det{}_{\pi^* T_X} \left(\frac{R_G}{e^{R_G/2} - e^{-R_G/2}} \right) \tag{2.11.16}$$

where R_G denotes the G-equivariant curvature of the bundle π^*T_X. Notice that the argument of $\hat{A}$ is $n \times n$ matrix where $n = \dim_{\mathbb{R}} T_X$ (real dimension of X) while if general index formula is specialized to Dirac operator on Kahler manifold X as in (2.7.4) the argument of the $\hat{A}$-character is an $n \times n$ matrix where $n = \dim_{\mathbb{C}} T^{1,0}_X$ (complex dimension of X).

Even though the integration domain T^*X is non-compact the integral (2.11.16) is well-defined because of the (G-transversal) ellipticity of the complex π^*E.

For illustration take the complex to be $E_0 \xrightarrow{D} E_1$. Since $\sigma(D) : \pi^*E_0 \to \pi^*E_1$ is an isomorphism outside of the zero section we can pick a smooth connection on π^*E_0 and π^*E_1 such that its curvature on E_0 is equal to the curvature on E_1 away from a compact tubular neighborhood $U_\epsilon X$ of $X \subset T^*X$. Then $\mathrm{ch}_G(\pi^*E^\bullet)$ is explicitly vanishing away from $U_\epsilon X$ and the integration over T^*X reduces to integration over the compact domain $U_\epsilon X$.

It is clear that under localization to the fixed points of the G-action on X the general formula (2.11.16) reduces to the fixed point formula (2.11.5). This is due to the fact that the numerator in the $\hat{A}$-character $\det_{\pi^*T_X} R_G = \mathrm{Pf}_{T_{T^*_X}}(R_G)$ is the Euler class of the tangent bundle $T_{T^*_X}$ to T^*X which cancels with the denominator in (2.8.9), while the restriction of the denominator of (2.11.16) to fixed points is equal to (2.11.16) or (2.11.5), because $\det e^{R_G} = 1$, since R_G is a curvature of the tangent bundle T_X with orthogonal structure group.

2.12 Equivariant cohomological field theories

Certain field theories first have been interpreted as cohomological and topological field theories by Witten, see [27], [28].

Often the path integral for supersymmetric field theories can be represented in the form

$$Z = \int_X \alpha \tag{2.12.1}$$

where X is the superspace (usually of infinite dimension) of all fields of the theory. Moreover, the integrand measure α is closed with respect to an odd operator δ which is typically constructed as a sum of a supersymmetry algebra generator and a BRST charge

$$\delta\alpha = 0 \tag{2.12.2}$$

The integrand is typically a product of an exponentiated action functional S, perhaps with insertion of a non-exponentiated observable $\mathcal{O}$

$$\alpha = e^{-S}\mathcal{O} \tag{2.12.3}$$

so that both S and $\mathcal{O}$ are δ-closed

$$\delta S = 0, \qquad \delta\mathcal{O} = 0. \tag{2.12.4}$$

If X is a supermanifold, such as a total space ΠE of a vector bundle E (over a base Y) with parity inversed fibers, the equivariant Euler characteristic class (Pfaffian) in the Atiyah-Bott formula (2.8.9) is replaced by the graded (super) version of the Pfaffian. The weights associated to fermionic components contribute inversely compared to the weights associated to bosonic components.

Typically, in quantum field theories the base Y of the bundle $E \to Y$ is the space of fields. Certain differential equations (like BPS equations) are represented by a section $s : Y \to E$. The zero set of the section $s^{-1}(0) \subset Y$ are the field configurations which solve the equations. For example, in topological self-dual Yang-Mills theory (Donaldson-Witten theory) the space Y is the infinite-dimensional affine space of all connections on a principal G-bundle on a smooth four-manifold M_4. In a given framing, connections are represented by adjoint-valued 1-forms on M_4, so $Y \simeq \Omega^1(M_4) \otimes \operatorname{ad}\mathfrak{g}$. A fiber of the vector bundle E at a given connection A on the G-bundle on M_4 is the space of adjoint-valued two-forms $\Omega^{2+}(M_4) \otimes \operatorname{ad}\mathfrak{g}$. The section $s : \Omega^1(M_4) \otimes \operatorname{ad}\mathfrak{g} \to \Omega_2$ is represented by the self-dual part of the curvature form

$$A \mapsto F_A^+ \tag{2.12.5}$$

The zeroes of the section $s = 0$ are connections A that are solutions of the equation $F_A^+ = 0$. The integrand α is the Mathai-Quillen representative of the Thom class for the bundle $E \to Y$ like in (2.6.10) and (2.6.17). The integral over the space of all fields $X = \Pi E$ localizes to the integral over the zeroes $s^{-1}(0)$ of the section , which in the Donaldson-Witten example is the moduli space of self-dual connections, called *instanton moduli space*.

The functional integral version of the localization formula of Atiyah-Bott has the same formal form

$$\boxed{\int_X \alpha = \int_F \frac{f^*\alpha}{\mathrm{e}(\nu_F)}} \tag{2.12.6}$$

except that in the quantum field theory version the space X is an infinite-dimensional superspace of fields. The F denotes the localization locus in the space of fields. Let $\Phi_F \subset H^\bullet(X)$ be the Poincaré dual class to F, or Thom class of the inclusion $f : F \hookrightarrow X$ which provides the isomorphism

$$f_* : H^\bullet(F) \to H^\bullet(X) \tag{2.12.7}$$

$$f_* : 1 \mapsto \Phi_F \tag{2.12.8}$$

Let ν_F be the normal bundle to F in X. In quantum feld theory language the space F is called the moduli space or localization locus, and ν_F is the space of linearized fluctuations of fields transversal to the localization locus. The cohomology class of $f^*\Phi_F$ in $H^\bullet(F)$ is equal to the Euler class of the normal bundle ν_F

$$[f^*\Phi_F] = e(\nu_F) \tag{2.12.9}$$

The localization (2.8.9) from X to F exists whenever the locus F is such that there exists an inverse to the Euler class $e(\nu_F)$ of its normal bundle in X. Two examples of such F have been considered above:

(i) if $X = \Pi E$ is the total space of a vector bundle $E \to Y$ with parity inversed fibers, then $F \subset Y \subset X$ can be taken to be the set of zeroes $F = s^{-}(0)$ of a generic section $s : Y \to E$

(ii) If X is a G-manifold for a compact group G, then F can be taken to be $F = X^G$, the set of G-fixed points on X

The formula (2.12.6) is more general than these examples. In practice, in quantum field theory problems, the localization locus F is found by deforming the form α to

$$\alpha_t = \alpha \exp(-t\delta V) \tag{2.12.10}$$

Here $t \in \mathbb{R}$ is a deformation parameter, and V is a fermionic functional on the space of fields, such that δV has a trivial cohomology class (the cohomology class δV is automatically trivial on effectively compact spaces, but on a non-compact space of fields, which usually appears in quantum field theory path integrals, one has to take extra care of the contributions from the boundary at infinity to ensure that δV has trivial cohomology class).

If the even part of the functional δV is positive definite, then by sending the paratemeter $t \to \infty$ we can see that the integral

$$\int_X \alpha \exp(-t\delta V) \tag{2.12.11}$$

localizes to the locus $F \subset X$ where δV vanishes. Such locus F has an invertible Euler class of its normal bundle in X and the localization formula (2.12.6) holds.

In some quantum field theory problems, a compact Lie group G acts on X and δ is isomorphic to an equivariant de Rham differential in the Cartan model of G-equivariant cohomology of X, so that an element $\mathbf{a}$ of the Lie algeba of G appears as a parameter of the partition function Z.

Then the partition function $Z(\mathbf{a})$ can be interpreted as an element of $H^{\bullet}_{\mathbf{G}}(pt)$, and the Atiyah-Bott localization formula can be applied to compute $Z(\mathbf{a})$.

There are are two types of equivariant partition functions.

In the partition functions of the first type $Z(\mathbf{a})$, the variable $\mathbf{a}$ is a *parameter* of the quantum field theory such as a coupling constant, a background field, a choice of vacuum, an asymptotics of fields or a boundary condition. Such a partition function is typical for a quantum field theory on a non-compact space, such as the Nekrasov partition function of equivariant gauge theory on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ [29].

In the partition function of the second type, the variable $\mathbf{a}$ is actually a *dynamical field* of the quantum field theory, so that the complete partition function is defined by integration of the partial partition function $\tilde{Z}(\mathbf{a}) \in H^{\bullet}_G(pt)$

$$Z = \int_{\mathbf{a}\in\mathbf{g}} \mu(\mathbf{a})\tilde{Z}(\mathbf{a}) \tag{2.12.12}$$

where $\mu(a)$ is a certain adjoint invariant volume form on the Lie algebra $\mathfrak{g}$. The partition function Z of second type is typical for quantum field theories on compact space-times reviewed in this volume, such as the partition function of a supersymmetric gauge theory on S^4 [13] reviewed in Chapter 10, or on spheres of other dimensions, see summary of results in Chapter 1.

Acknowledgements

The author is grateful to Bruno Le Floch for a careful proofreading and comments on this chapter. The author is supported by ERC QUASIFT grant.

References

[1] V. Pestun and M. Zabzine, eds., *Localization techniques in quantum field theory*, vol. xx. Journal of Physics A, 2016. 1608.02952. http://pestun.ihes.fr/pages/LocalizationReview/LocQFT.pdf.

[2] H. Cartan, "Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie," *Colloque de topologie (espaces fibrés), Bruxelles, 1950* (1951) 15–27.

[3] H. Cartan, "La transgression dans un groupe de Lie et dans un espace fibré principal," *Colloque de topologie (espaces fibrés), Bruxelles, 1950* (1951) 57–71.

[4] V. W. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*. Mathematics Past and Present. Springer-Verlag, Berlin, 1999. http://dx.doi.org/10.1007/978-3-662-03992-2. With an appendix containing two reprints by Henri Cartan [MR0042426 (13,107e); MR0042427 (13,107f)].

[5] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.

[6] R. J. Szabo, "Equivariant localization of path integrals," arXiv:hep-th/9608068 [hep-th].

[7] S. Cordes, G. W. Moore, and S. Ramgoolam, "Large N 2-D Yang-Mills theory and topological string theory," *Commun. Math. Phys.* **185** (1997) 543–619, hep-th/9402107.

[8] M. Vergne, "Applications of Equivariant Cohomology," math/0607389.

[9] J. Milnor, "Construction of universal bundles. I," *Ann. of Math. (2)* **63** (1956) 272–284.

[10] J. Milnor, "Construction of universal bundles. II," *Ann. of Math. (2)* **63** (1956) 430–436.

[11] H. Kanno, "Weyl Algebra Structure and Geometrical Meaning of BRST Transformation in Topological Quantum Field Theory," *Z. Phys.* **C43** (1989) 477.

[12] J. M. F. Labastida and M. Pernici, "A Gauge Invariant Action in Topological Quantum Field Theory," *Phys. Lett.* **B212** (1988) 56.

[13] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," *Commun.Math.Phys.* **313** (2012) 71–129, arXiv:0712.2824 [hep-th].

[14] R. Bott and L. W. Tu, *Equivariant characteristic classes in the Cartan model*. World Sci. Publ., River Edge, NJ, 2001.

[15] N. Berline and M. Vergne, "The equivariant Chern character and index of G-invariant operators. Lectures at CIME, Venise 1992,".

[16] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, vol. 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[17] V. Mathai and D. Quillen, "Superconnections, Thom classes, and equivariant differential forms," *Topology* **25** (1986) no. 1, 85–110. http://dx.doi.org/10.1016/0040-9383(86)90007-8.

[18] M. F. Atiyah and L. Jeffrey, "Topological Lagrangians and cohomology," *J. Geom. Phys.* **7** (1990) no. 1, 119–136.

[19] S. Wu, "Mathai-Quillen formalism," arXiv:hep-th/0505003 [hep-th].

[20] M. F. Atiyah and R. Bott, "The moment map and equivariant cohomology," *Topology* **23** (1984) no. 1, 1–28.

[21] A. A. Kirillov, *Lectures on the orbit method*, vol. 64 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2004. http://dx.doi.org/10.1090/gsm/064.

[22] B. Kostant, "Quantization and unitary representations. I. Prequantization,".

[23] A. A. Kirillov, "Merits and demerits of the orbit method," *Bull. Amer. Math. Soc. (N.S.)* **36** (1999) no. 4, 433–488. http://dx.doi.org/10.1090/S0273-0979-99-00849-6.

[24] A. Alekseev and S. L. Shatashvili, "Path Integral Quantization of the Coadjoint Orbits of the Virasoro Group and 2D Gravity," *Nucl. Phys.* **B323** (1989) 719.

[25] M. F. Atiyah, *Elliptic operators and compact groups*. Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 401.

[26] V. Pestun and M. Zabzine, "Introduction to localization in quantum field theory," *Journal of Physics A* **xx** (2016) 000, 1608.02953.

[27] E. Witten, "Introduction to cohomological field theories," *Int.J.Mod.Phys.* **A6** (1991) 2775–2792.

[28] E. Witten, "Topological Quantum Field Theory," *Commun.Math.Phys.* **117** (1988) 353.

[29] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," *Adv.Theor.Math.Phys.* **7** (2004) 831–864, arXiv:hep-th/0206161 [hep-th]. To Arkady Vainshtein on his 60th anniversary.

[30] K. Hosomichi, "$\mathcal{N} = 2$ SUSY gauge theories on S^4," *Journal of Physics A* **xx** (2016) 000, 1608.02962.