Oral Exam for Commutative Algebra, Fall 2017

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1. For elements $f_1, \dots, f_n$ of a ring A, show that the following are equivalent:
   (a) $(f_1, \dots, f_n) = A$.
   (b) For each nonzero module M over A, at least one of the localized modules $M[1/f_i]$ is nonzero.

2. Let M, N be nonzero finitely-generated modules over a local ring $(A, \mathfrak{m})$. Show that $M \otimes_A N$ is nonzero.

3. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $k := A/\mathfrak{m}$ denote its residue field. Let $f_1, \dots, f_r \in \mathfrak{m}$. Set $\overline{A} := A/(f_1, \dots, f_r)$. Let $\overline{\mathfrak{m}} \subset \overline{A}$ denote the image of $\mathfrak{m}$.
   (a) Show that $\dim_k(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) \geq \dim(\overline{A}) \geq d - r$.
   (b) Assume that A is regular. Let $\overline{f_1}, \dots, \overline{f_r} \in \mathfrak{m}/\mathfrak{m}^2$ denote the images of $f_1, \dots, f_r$. Show that the following are equivalent:
      i. $\overline{A}$ is regular of dimension $d - r$.
      ii. $\overline{f_1}, \dots, \overline{f_r}$ are linearly independent over k.

4. Let $(A, \mathfrak{m})$ be a Noetherian local domain of dimension one. Let $x, y \in A$ with $x \neq 0$ and $y \in \mathfrak{m}$. Show that $ax = y^n$ for some $a \in A$ and $n \in \mathbb{Z}_{\geq 1}$.

5. State precisely the following major results from the course: existence and uniqueness of primary decomposition, lying over, going up/down, Krull dimension theorems, characterization of dimension via systems of parameters, Noether normalization, characterization of DVR's among one-dimensional local Noetherian domains. State the following theorems precisely, and explain how their proofs reduce to the major results mentioned previously:
   (a) $\dim(k[x_1, \dots, x_n]) = n$ for a field k
   (b) "dimension equals transcendence degree"
   (c) ideals in a Dedekind domain factor uniquely into primes