one by λ and the other by λ^{-1}. It is convenient to use from the start the negative sign for interaction terms (such as the term $-\frac{g}{6}\phi^3$) so that the minus sign of the counterterms is automatic for the monomials in I and the external structure of the component graphs is the evaluation at zero external momenta divided by the coupling constant.

The main result of [**26**], [**166**], [**305**] is the fact that this procedure takes care of both the problem of organizing subdivergences and of eliminating non-local terms. It is clear by construction that it accounts for subdivergences. Moreover, the following holds.

THEOREM 1.17. (BPHZ) *1) The coefficients of the pole part of the prepared graph $\overline{R}(\Gamma)$ are given by local terms.*
2) The perturbative expansion of the functional integral with Euclidean Lagrangian

$$\mathcal{L}_E - \sum_{\iota(\Gamma^{(0)})\subset I} \frac{C(\Gamma)}{\sigma(\Gamma)} \tag{1.167}$$

gives the renormalized value of the theory.

To be more specific, in the case of the ϕ^3 theory, the above series takes the form

$$\tfrac{1}{2}\left(1-\sum \frac{C(\Gamma_{(1)})}{\sigma(\Gamma)}\right)(\partial\phi)^2 \quad + \quad \tfrac{m^2}{2}\left(1-\sum \frac{C(\Gamma_{(0)})}{\sigma(\Gamma)}\right)\phi^2$$
$$- \quad \tfrac{g}{6}\left(1+\sum \frac{C(\Gamma)}{\sigma(\Gamma)}\right)\phi^3,$$

where in each case the graphs involved are those with $\iota(\Gamma^{(0)}) \subset I$.
Several of the following sections in this chapter are dedicated to explaining the rich mathematical structure that lies hidden behind the BPHZ formulae (1.162), (1.163), (1.165). Our presentation is based on the work of Connes–Kreimer [**82**], [**83**] and of the authors [**87**], [**89**].

6. The Connes–Kreimer theory of perturbative renormalization

The Connes–Kreimer (CK) theory provides a conceptual understanding of the BPHZ procedure in terms of the Birkhoff factorization of loops in a pro-unipotent complex Lie group associated to a commutative Hopf algebra of Feynman graphs. The main points, which we discuss in detail in the rest of this section, are summarized as follows.

- The Hopf algebra $\mathcal{H}$ of Feynman graphs.
- The BPHZ procedure as a Birkhoff factorization in the Lie group $G(\mathbb{C}) = \mathrm{Hom}(\mathcal{H}, \mathbb{C})$.
- The action of $G(\mathbb{C})$ on the coupling constants of the theory, through formal diffeomorphisms.
- The renormalization group as a 1-parameter subgroup of $G(\mathbb{C})$.

We begin by recalling some general facts about commutative Hopf algebras and affine group schemes. We then present the construction of the Hopf algebra $\mathcal{H}(\mathcal{T})$ of Feynman graphs of a renormalizable quantum field theory $\mathcal{T}$.

6.1. Commutative Hopf algebras and affine group schemes.

The theory of affine group schemes is developed in SGA 3 [**115**]. Whereas affine schemes are dual to commutative algebras, affine group schemes are dual to commutative Hopf algebras (for an introductory text see also [**290**]). We recall here some basic facts that we need later.

Definition 1.18. *Let k be a field of characteristic zero. A commutative Hopf algebra $\mathcal{H}$ over k is a commutative algebra with unit over k, endowed with a (not necessarily cocommutative) coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes_k \mathcal{H}$, a counit $\varepsilon : \mathcal{H} \to k$, which are k-algebra morphisms and an antipode $S : \mathcal{H} \to \mathcal{H}$ which is a k-algebra antihomomorphism. These satisfy the "co-rules"*

$$\begin{aligned} &(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta && : \mathcal{H} \to \mathcal{H} \otimes_k \mathcal{H} \otimes_k \mathcal{H}, \\ &(id \otimes \varepsilon)\Delta = id = (\varepsilon \otimes id)\Delta && : \mathcal{H} \to \mathcal{H}, \\ &m(id \otimes S)\Delta = m(S \otimes id)\Delta = 1\,\varepsilon && : \mathcal{H} \to \mathcal{H}, \end{aligned} \tag{1.168}$$

where we use m to denote multiplication in $\mathcal{H}$.

Suppose given a commutative Hopf algebra $\mathcal{H}$ as above. There is an associated covariant functor G from the category $\mathcal{A}_k$ of unital k-algebras to the category $\mathcal{G}$ of groups. It assigns to a unital k-algebra A the group

$$G(A) = \mathrm{Hom}_{\mathcal{A}_k}(\mathcal{H}, A). \tag{1.169}$$

Thus, elements of the group $G(A)$ are k-algebra homomorphisms

$$\phi : \mathcal{H} \to A\,, \quad \phi(x\,y) = \phi(x)\phi(y), \quad \forall x, y \in \mathcal{H}\,, \quad \phi(1) = 1\,.$$

The product in $G(A)$ is dual to the coproduct of $\mathcal{H}$, that is,

$$\phi_1 \, * \, \phi_2 \, (x) = \langle \phi_1 \otimes \phi_2 \,, \; \Delta(x) \rangle\,. \tag{1.170}$$

Similarly, the inverse and the unit of $G(A)$ are determined by the antipode and the co-unit of $\mathcal{H}$. The co-rules imply that these operations define a group structure on $G(A)$. The functor G is an *affine group scheme.*

One can give the following general definition.

Definition 1.19. *An affine group scheme G is a covariant representable functor from the category of commutative algebras over k to the category of groups.*

In fact, the functor G of (1.169) is certainly representable (by $\mathcal{H}$) and, conversely, any covariant representable functor from the category of commutative algebras over k to groups is an affine group scheme G, represented by a commutative Hopf algebra, uniquely determined up to canonical isomorphism (cf. e.g. [**290**]).

Here are some simple examples of affine group schemes.

- The additive group $G = \mathbb{G}_a$ corresponds to the Hopf algebra $\mathcal{H} = k[t]$ with coproduct $\Delta(t) = t \otimes 1 + 1 \otimes t$.
- The multiplicative group $G = \mathbb{G}_m$ is the affine group scheme obtained from the Hopf algebra $\mathcal{H} = k[t, t^{-1}]$ with coproduct $\Delta(t) = t \otimes t$.

- The group of roots of unity μ_n is the kernel of the homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ given by raising to the n-th power. It corresponds to the Hopf algebra $\mathcal{H} = k[t]/(t^n - 1)$.
- The affine group scheme $G = \mathrm{GL}_n$ corresponds to the Hopf algebra

$$\mathcal{H} = k[x_{i,j}, t]_{i,j=1,\dots,n}/(\det(x_{i,j})t - 1),$$

with coproduct $\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j}$.

The latter example is quite general. In fact, if $\mathcal{H}$ is finitely generated as an algebra over k, then the corresponding affine group scheme G is a linear algebraic group over k, and can be embedded as a Zariski closed subset in some GL_n.

More generally, we have the following result (cf. e.g. [**231**] Proposition 4.13 and [**290**]).

LEMMA 1.20. *Let $\mathcal{H}$ be a commutative positively graded connected Hopf algebra. There exists a family $\mathcal{H}_i \subset \mathcal{H}$, $i \in \mathcal{I}$, indexed by a partially ordered set, where the $\mathcal{H}_i$ are finitely generated algebras over k satisfying the following properties:*

(1) *$\Delta(\mathcal{H}_i) \subset \mathcal{H}_i \otimes \mathcal{H}_i$, for all $i \in \mathcal{I}$.*
(2) *$S(\mathcal{H}_i) \subset \mathcal{H}_i$, for all $i \in \mathcal{I}$.*
(3) *For all $i, j \in \mathcal{I}$, there exists a $k \in \mathcal{I}$ such that $\mathcal{H}_i \cup \mathcal{H}_j \subset \mathcal{H}_k$, and $\mathcal{H} = \cup_i \mathcal{H}_i$.*

Then the affine group scheme G of $\mathcal{H}$ is of the form

$$G = \varprojlim_i G_i, \tag{1.171}$$

where the G_i are the linear algebraic groups dual to $\mathcal{H}_i$.

Thus, in general, such an affine group scheme is a projective limit of unipotent linear algebraic groups G_i, we say that G is a pro-unipotent affine group scheme.

Recall that an element X in a Hopf algebra $\mathcal{H}$ is said to be group-like if $\Delta(X) = X \otimes X$ and is said to be primitive if $\Delta(X) = X \otimes 1 + 1 \otimes X$. If G denotes the affine group scheme of a commutative Hopf algebra $\mathcal{H}$, then a group-like element $X \in \mathcal{H}$ determines a homomorphism $G \to \mathbb{G}_m$, which, at the level of Hopf algebras, is given by $\phi : k[t, t^{-1}] \to \mathcal{H}$ with $\phi(t) = X$. Similarly, a primitive element $X \in \mathcal{H}$ corresponds to a homomorphism $G \to \mathbb{G}_a$.

There is a notion of Lie algebra for an affine group scheme. It is also defined as a functor.

DEFINITION 1.21. *The Lie algebra of an affine group scheme G is a covariant functor $\mathfrak{g} = \mathrm{Lie}\, G$ from the category $\mathcal{A}_k$ of commutative k-algebras to the category $\mathcal{L}_k$ of Lie algebras over k,*

$$A \mapsto \mathfrak{g}(A)\,, \tag{1.172}$$

where $\mathfrak{g}(A)$ is the Lie algebra of linear maps $L : \mathcal{H} \to A$ satisfying

$$L(X\,Y) = L(X)\,\varepsilon(Y) + \varepsilon(X)\,L(Y)\,, \quad \forall X\,, Y \in \mathcal{H}\,, \tag{1.173}$$

where ε is the counit of $\mathcal{H}$.

The Lie bracket of two elements in $\mathfrak{g}(A)$ is given by

$$[L_1,\, L_2](X) = \langle L_1 \otimes L_2 - L_2 \otimes L_1, \Delta(X) \rangle\,. \tag{1.174}$$

Normally, the datum $\mathrm{Lie}\, G$ of the Lie algebra is not sufficient to reconstruct the affine group scheme G. One can see this already in the simplest case: the affine group schemes $\mathbb{G}_m$ and $\mathbb{G}_a$ have the same Lie algebra. There is, however, a particular class of cases in which the knowledge of the Lie algebra is sufficient to reconstruct the Hopf algebra. This follows from the Milnor–Moore Theorem [**231**] Theorem 5.18, which asserts that a primitively generated Hopf algebra over a field k of characteristic zero is the enveloping algebra of the Lie algebra of primitive elements.
Recall that, given a graded connected Hopf algebra $\mathcal{H}$, one obtains a dual Hopf algebra $\mathcal{H}^\vee$ by reversing all the arrows. More precisely we assume that the graded pieces $\mathcal{H}_n$ are finite-dimensional k-vector spaces, and we define $\mathcal{H}^\vee$ as the graded dual of $\mathcal{H}$, i.e. an element of $\mathcal{H}^\vee$ is a finite linear combination of homogeneous linear forms on $\mathcal{H}$. It follows from [**231**] Proposition 3.1, that $\mathcal{H}^\vee$ is a graded Hopf algebra, and also that the bidual $(\mathcal{H}^\vee)^\vee$ is canonically isomorphic to $\mathcal{H}$. We can now state the following corollary of the Milnor–Moore Theorem:

THEOREM 1.22. *Let $\mathcal{H}$ be a commutative Hopf algebra over a field k of characteristic zero. Assume that $\mathcal{H}$ is positively graded and connected, $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$, with $\mathcal{H}_0 = k$, and that the graded pieces $\mathcal{H}_n$ are finite-dimensional k-vector spaces. Let $\mathcal{H}^\vee$ be the dual Hopf algebra and let $\mathcal{L}$ denote the Lie algebra of primitive elements in $\mathcal{H}^\vee$. Then there is a canonical isomorphism of Hopf algebras*

$$\mathcal{H} \cong U(\mathcal{L})^\vee, \tag{1.175}$$

where $U(\mathcal{L})$ is the universal enveloping algebra of $\mathcal{L}$. Moreover, $\mathcal{L} = \mathrm{Lie}\ G(k)$ as a graded Lie algebra.

PROOF. Let $\mathcal{H}^\vee$ be the graded dual of $\mathcal{H}$. By [**231**] Corollary 4.18, $\mathcal{H}^\vee$ is primitively generated. Thus the Milnor–Moore Theorem ([**231**] Theorem 5.18) shows that $\mathcal{H}^\vee \cong U(\mathcal{L})$. The result follows from the isomorphism $\mathcal{H} \cong (\mathcal{H}^\vee)^\vee$. □

REMARK 1.23. Note that without the finiteness hypothesis, the dual $\mathcal{H}^\vee$ of a Hopf algebra $\mathcal{H}$ is ill defined since, if one considers arbitrary linear forms L on $\mathcal{H}$, there is no guarantee that the functional $X \otimes Y \mapsto L(XY)$ can be expressed as a finite linear combination of simple tensors $L_1 \otimes L_2$. This difficulty disappears if one restricts to linear forms for which this finiteness holds (at any level), and this is automatic for primitive elements, as they fulfill by hypothesis the equality

$$L(XY) = L(X)\,\varepsilon(Y) + \varepsilon(X)\,L(Y)\,, \quad \forall X, Y \in \mathcal{H}\,.$$

REMARK 1.24. In our set-up we are only considering Hopf algebras which are commutative rather than "graded-commutative" as in [**231**]. This should not create confusion since one can multiply the grading by 2 to apply the results of [**231**]. In fact, in our setting one should really think of the grading as being an even grading: this will become clearer in comparison with the Lie algebras of motivic Galois groups of mixed Tate motives (see Section 8), where the grading corresponds to the weight filtration, which is naturally parameterized by even integers. A reason for regarding the grading by number of internal lines of graphs (cf. Proposition 1.30 below) as an *even* grading comes from the fact that it is customary in physics to think of internal lines of graphs as a *pair* of half-lines, see for instance [**170**], [**19**]. In fact we saw in §3.1 that the graphs are obtained from pairings of half-lines. Thus the grading is actually given by the number of half-lines that contribute to the internal lines of the graph. In the mathematical literature, this formulation of graphs in terms of collections of half-lines was variously used (cf. [**189**], [**142**]).

The result of Theorem 1.22 can also be used to obtain an explicit description of the Lie algebra of an affine group scheme G, using the primitive elements in the dual Hopf algebra. This is the form in which it is used in the Connes-Kreimer theory.

LEMMA 1.25. *Let $\mathcal{H}$ be a commutative graded connected Hopf algebra, $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$. There is an action of the multiplicative group $\mathbb{G}_m$ on $\mathcal{H}$ and an associated semidirect product of affine group schemes $G^* = G \rtimes \mathbb{G}_m$.*

PROOF. Let Y denote the generator of the grading, namely the linear operator on $\mathcal{H}$ that satisfies $Y(X) = nX$ for all $X \in \mathcal{H}_n$. This defines an action of the multiplicative group $\mathbb{G}_m$ on $\mathcal{H}$ by setting, for all $u \in \mathbb{G}_m$,

$$u^Y(X) = u^n X \quad \forall X \in \mathcal{H}_n. \tag{1.176}$$

Thus, we can consider the affine group scheme obtained as the semidirect product

$$G^* = G \rtimes \mathbb{G}_m, \tag{1.177}$$

and we have a corresponding homomorphism $G^* \to \mathbb{G}_m$. The action (1.176) is also defined on the dual Hopf algebra $\mathcal{H}^\vee$. One then obtains an explicit description of the Lie algebra of G^* in terms of the Lie algebra of G. Namely, the Lie algebra of G^* has an additional generator Z_0 such that

$$[Z_0, X] = Y(X) \quad \forall\, X \in \mathrm{Lie}\, G\,. \tag{1.178}$$

□

6.2. The Hopf algebra of Feynman graphs: discrete part.

The first main step of the CK theory is to associate to a given renormalizable quantum field theory $\mathcal{T}$ a Hopf algebra $\mathcal{H}(\mathcal{T})$ over $k = \mathbb{C}$, where the coproduct reflects the combinatorics of the BPHZ preparation formula (1.162). In this section we describe a simplified version of the Hopf algebra of Feynman graphs (the "discrete part") where we only consider graphs responsible for divergences. We discuss in §6.3 the full Hopf algebra of Feynman graphs, where we take into account arbitrary 1PI graphs. We let as above J denote the set of all monomials in the Lagrangian.

DEFINITION 1.26. *For a given renormalizable quantum field theory $\mathcal{T}$, the* discrete Hopf algebra of Feynman graphs $\mathcal{H}(\mathcal{T})$ *is the free commutative algebra over $\mathbb{C}$ generated by pairs (Γ, w) with $\Gamma \in \mathrm{Graph}(\mathcal{T})$ a 1PI graph, and $w \in J$ a monomial with degree equal to the number of external lines of Γ. The coproduct is defined on generators as*

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma, \tag{1.179}$$

where the set $\mathcal{V}(\Gamma)$ is as in Definition 1.14.

Notice that by construction the meaning of γ in (1.179) is the product of the components $\tilde{\gamma}_i$ where each is endowed with the element $\chi(\gamma_i) \in J$. In other words,

$$\gamma = \prod (\tilde{\gamma}_i, \chi(\gamma_i)).$$

The external lines of Γ/γ are the same as for Γ, hence in (1.179) the notation Γ/γ stands for the pair $(\Gamma/\gamma, w)$ with the same w as for Γ.

Notice how $\mathcal{H}(\mathcal{T})$ is strongly dependent on the physical theory $\mathcal{T}$, both in the generators and in the structure of the coproduct, where the class of subgraphs $\mathcal{V}(\Gamma)$ also depends on $\mathcal{T}$. We see in §7.6 below that there is a universal Hopf algebra that relates naturally to all the $\mathcal{H}(\mathcal{T})$ and encodes the renormalization procedure canonically for all the physical theories.

THEOREM 1.27. ([**82**]) *The commutative algebra of Definition* 1.26 *with the coproduct* (1.179) *is a Hopf algebra* $\mathcal{H}(\mathcal{T})$.

PROOF. Let us prove that Δ is coassociative, i.e. that

$$(\Delta \otimes \mathrm{id})\,\Delta = (\mathrm{id} \otimes \Delta)\,\Delta\,. \tag{1.180}$$

Since both sides of (1.180) are algebra homomorphisms from $\mathcal{H}$ to $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$, it will be enough to check that they give the same result on 1PI graphs Γ. It is suitable to enlarge the definition of $\mathcal{V}(\Gamma)$ to $\bar{\mathcal{V}}(\Gamma)$ which includes the cases $\gamma = \emptyset$ and $\gamma = \Gamma^{(1)}_{\mathrm{int}}$, allowing the graphs $\tilde{\gamma}$ or Γ/γ to be empty, in which case we represent them by the symbol $1 \in \mathcal{H}(\mathcal{T})$. We write

$$\gamma \preceq \Gamma \Leftrightarrow \gamma \in \bar{\mathcal{V}}(\Gamma) \tag{1.181}$$

and

$$\gamma \prec \Gamma \Leftrightarrow \gamma \in \bar{\mathcal{V}}(\Gamma)\,, \gamma \neq \Gamma^{(1)}_{\mathrm{int}}. \tag{1.182}$$

With this notation, the coproduct of Γ takes the form

$$\Delta\,\Gamma = \sum_{\gamma \preceq \Gamma} \tilde{\gamma} \otimes \Gamma/\gamma, \tag{1.183}$$

where by definition $\tilde{\gamma}$ is given by the product of its components

$$\tilde{\gamma} = \prod \tilde{\gamma}_j. \tag{1.184}$$

Applying $\Delta \otimes \mathrm{id}$ on both sides of (1.183) we get

$$(\Delta \otimes \mathrm{id})\,\Delta\,\Gamma = \sum_{\gamma \preceq \Gamma} \Delta\,\tilde{\gamma} \otimes \Gamma/\gamma. \tag{1.185}$$

Since Δ is a homomorphism, one has

$$\Delta(\tilde{\gamma}) = \prod \Delta(\tilde{\gamma}_j).$$

Using (1.183), this can be written as

$$\Delta(\tilde{\gamma}) = \sum_{\gamma' \preceq \tilde{\gamma}} \tilde{\gamma}' \otimes \tilde{\gamma}/\gamma', \tag{1.186}$$

where the notation $\preceq$ has been extended to non-connected graphs. What matters here is that a subgraph of $\tilde{\gamma}$ is given by a collection of subgraphs of its components, with the limit cases in $\bar{\mathcal{V}}$ allowed. Since things happen independently in each component, the product of the sums is equal to the sum of the products and one gets (1.186). We can think of γ' as a collection of subsets γ'_j of the sets of internal lines of the components $\tilde{\gamma}_j$. Thus, we can view the union of the γ'_j as a subset γ' of the set of internal lines of Γ. Let us check that, for $\gamma' \subset \gamma$, we have

$$\gamma' \in \bar{\mathcal{V}}(\Gamma) \Leftrightarrow \gamma'_j \in \bar{\mathcal{V}}(\tilde{\gamma}_j)\,, \quad \forall j. \tag{1.187}$$

Here, for each component γ_j of γ, one lets $\gamma'_j = \gamma_j \cap \gamma'$. First, a connected component δ of γ' is contained in one (and only one) of the γ_j. Let us show that the graph $\tilde{\delta}$ is the same relative to $\tilde{\gamma}_j$ or to Γ. Its set of internal lines is δ in both cases. Its set of vertices is the set of vertices of $|\delta|$, which does not change in passing from $\tilde{\gamma}_j$ to Γ. The set of external lines at a vertex v is the disjoint union of the $\partial_i^{-1}(v) \cap \delta^c$. One has to show that this is unchanged in passing from $\tilde{\gamma}_j$ to Γ. Since a line ℓ of Γ such that $\partial_i \, \ell = v$ is a line of $\tilde{\gamma}_j$, one gets the result. It follows that the graph $\tilde{\delta}$ does not depend on whether it is taken relative to $\tilde{\gamma}_j$ or to Γ and we can now write

$$(\Delta \otimes \mathrm{id}) \, \Delta \, \Gamma = \sum_{\gamma' \preceq \gamma \preceq \Gamma} \tilde{\gamma}' \otimes \tilde{\gamma}/\gamma' \otimes \Gamma/\gamma, \tag{1.188}$$

where the parameters γ and γ' vary among the subsets of $\Gamma^{(1)}_{\mathrm{int}}$ that fulfill

$$\gamma \in \bar{\mathcal{V}}(\Gamma) \,, \quad \gamma' \in \bar{\mathcal{V}}(\Gamma) \,, \quad \gamma' \subset \gamma. \tag{1.189}$$

We can write (using (1.183)) the coproduct of Γ as

$$\Delta \, \Gamma = \sum_{\gamma' \preceq \Gamma} \tilde{\gamma}' \otimes \Gamma/\gamma', \tag{1.190}$$

hence

$$(\mathrm{id} \otimes \Delta) \, \Delta \, \Gamma = \sum_{\gamma' \preceq \Gamma} \tilde{\gamma}' \otimes \Delta \, (\Gamma/\gamma'). \tag{1.191}$$

Thus, to prove (1.180) it suffices to show that, for fixed $\gamma' \in \bar{\mathcal{V}}(\Gamma)$, one has

$$\Delta \, (\Gamma/\gamma') = \sum_{\gamma \preceq \Gamma, \, \gamma \supset \gamma'} \tilde{\gamma}/\gamma' \otimes \Gamma/\gamma. \tag{1.192}$$

We let $\Gamma' = \Gamma/\gamma'$ be the contracted graph. The set of internal lines of Γ' is the complement of γ' in $\Gamma^{(1)}_{\mathrm{int}}$. Thus, subsets $\gamma \supset \gamma'$ correspond to subsets of $(\Gamma')^{(1)}_{\mathrm{int}}$ by the map $\gamma \mapsto \rho(\gamma) = \gamma'' = \gamma \backslash \gamma'$. One has (using (1.183)),

$$\Delta \, \Gamma' = \sum_{\gamma'' \preceq \Gamma'} \tilde{\gamma}'' \otimes \Gamma'/\gamma''. \tag{1.193}$$

It remains to show that the map ρ fulfills

$$\gamma \preceq \Gamma \Leftrightarrow \rho(\gamma) \preceq \Gamma' \,, \quad \tilde{\gamma}/\gamma' = \tilde{\rho(\gamma)} \,, \quad \Gamma/\gamma = \Gamma'/\rho(\gamma). \tag{1.194}$$

We take a subset $\gamma \supset \gamma'$ and we let

$$\pi : |\Gamma| \to |\Gamma/\gamma'| \tag{1.195}$$

be the continuous surjection of (1.161). It is not injective but, if $\pi(x) = \pi(y)$, then x and y are in the same connected component of $|\gamma'|$. Since we have $\gamma' \subset \gamma$, each connected component of $|\gamma'|$ is inside a component of $|\gamma|$. Thus, if $\pi(x) = \pi(y)$, then they are in the same connected component of $|\gamma|$. We let γ_i be the components of γ. The argument above shows that the $\pi(|\gamma_i|)$ are disjoint. The lines of $\pi(|\gamma_i|)$ are labelled by the complement $\gamma''_i = \gamma_i \backslash \gamma'$ of $\gamma' \cap \gamma_i$ in γ_i. Thus, γ''_i is a connected component of $|\gamma''|$, where $\rho(\gamma) = \gamma'' = \gamma \backslash \gamma'$.

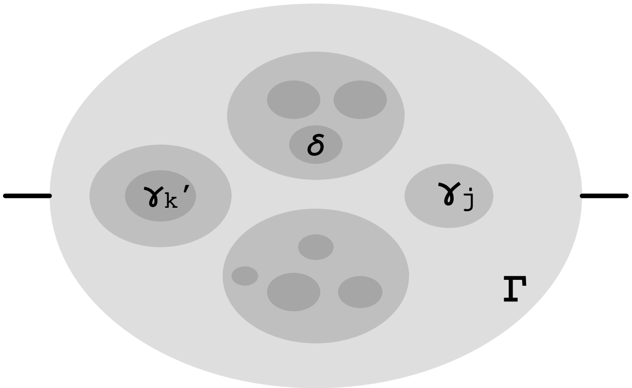


FIGURE 26. The pair $\gamma' \subset \gamma$, components δ of γ' are the darker ones

Conversely, every connected component $|\gamma''_j|$ of $|\gamma''|$ is obtained in this way and its inverse image $\pi^{-1}(|\gamma''_j|)$ is connected and is a component $|\gamma_i|$ of $|\gamma|$. The corresponding graph $\tilde{\gamma}_i$ is obtained by inserting at some vertices of $\tilde{\gamma}''_j$ the corresponding 1PI graphs associated to the components of γ'. This operation preserves the property of being 1PI by Proposition 1.15. Moreover, the collapsed graph of a 1PI graph is still 1PI. This makes it possible to check, using Definition 1.14, that $\gamma \preceq \Gamma \Leftrightarrow \rho(\gamma) \preceq \Gamma'$. Moreover, the connected components of $|\gamma|$ and $|\gamma''|$ correspond bijectively under π. For each component γ_i of γ, one has $\tilde{\gamma}_i/\gamma' = \tilde{\gamma}''_i$ with $\gamma''_i = \gamma_i \backslash \gamma'$. To check this one needs to show that the external lines correspond. This follows from the equality $\gamma = \gamma' \cup \gamma''$.

One then needs to show that $\Gamma'/\gamma'' = \Gamma/\gamma$, that is, that $(\Gamma/\gamma')/\gamma'' = \Gamma/\gamma$. In both cases the set of internal lines is the complement of $\gamma = \gamma' \cup \gamma''$ in $\Gamma^{(1)}_{\rm int}$. Let us look at vertices. The set of vertices of Γ/γ is the disjoint union of $\Gamma^{(0)} \backslash \gamma$ with the set of connected components γ_i of γ. In other words, it is the quotient of $\Gamma^{(0)}$ by the identification of vertices in the same component γ_i of γ. In the same way the set of vertices of $(\Gamma/\gamma')/\gamma''$ is obtained as a quotient of $\Gamma^{(0)}$. One first identifies vertices in the same component of γ' and then of γ''. Under the above correspondence between components γ_i of γ and components of γ'', one gets the desired identification of the sets of vertices. The external legs are unaltered in the collapsing process, hence one gets $\Gamma/\gamma = \Gamma'/\rho(\gamma)$ as required.

We have neglected to take the map χ into account in the above discussion and we explain how to handle it now. Indeed, a subgraph is not just given as a subset $\gamma \subset \Gamma^{(1)}_{\rm int}$ but it also consists of the datum of a map χ from the set of connected components of $|\gamma|$ to J. All the sums considered here above involve summations over compatible choices of χ. Thus, in (1.188) one has to sum over all choices of a χ for γ and a χ'_i for the $\gamma'_i \subset \gamma_i$, where the γ_i are the components of γ. The only requirement is that, if $\gamma'_i = \gamma_i$, then $\chi'_i(\gamma'_i) = \chi(\gamma_i)$. In any case, giving χ and the family χ'_i is the same as giving a χ' for the subset $\gamma' \subset \Gamma^{(1)}_{\rm int}$, and a χ'' for the subgraph γ/γ' of Γ'. Thus, the sums involved in (1.188) and (1.191) give the same result.

This ends the proof of the coassociativity. One still needs to show the existence of the antipode for $\mathcal{H}(\mathcal{T})$. This can be obtained ([**82**]) by an inductive procedure. In fact, this follows from the existence (cf. Proposition 1.30) of gradings of $\mathcal{H}(\mathcal{T})$ given by maps from $\mathrm{Graph}(\mathcal{T})$ to $\mathbb{N}^*$ such that

$$\sum \delta(\gamma_i) + \delta(\Gamma/\gamma) = \delta(\Gamma)\,, \quad \forall \gamma \in \mathcal{V}(\Gamma). \tag{1.196}$$

This implies that, for a monomial $X \in \mathcal{H}(\mathcal{T})$, the coproduct $\Delta(X)$ can be written in the form

$$\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X'', \tag{1.197}$$

where the terms X' and X'' are of strictly lower degree in the chosen grading satisfying (1.196). The equation $m(S \otimes 1)\Delta = \varepsilon 1$ can then be solved inductively for the antipode S using the formula

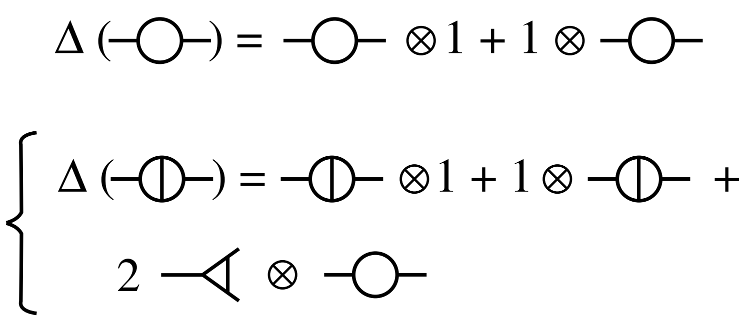
$$S(X) = -X - \sum S(X')X''. \tag{1.198}$$

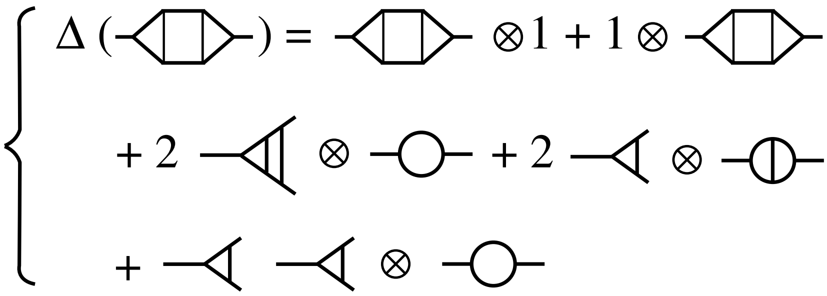
□

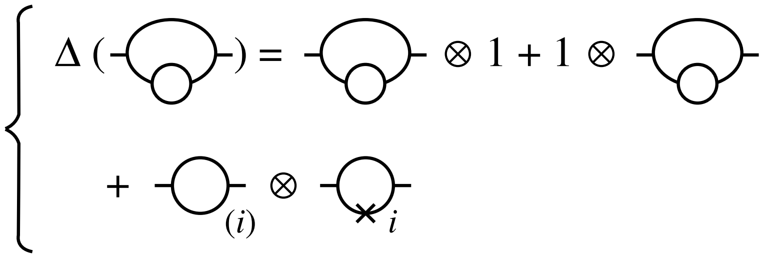
It is useful to see the Hopf algebra of Definition 1.26 in the simplest concrete example. As we did previously, we often use as an example the theory $\mathcal{T} = \phi^3_6$, which has the Lagrangian density (1.137) in dimension $D = 6$. This example is not very physical, both because of the dimension and because the potential does not have a stable equilibrium. However, it is a very convenient example, because it is sufficiently easy (a scalar theory with a very simple potential) and at the same time sufficiently generic from the point of view of renormalization (it is not super-renormalizable).

In this case, the conditions on the class of subgraphs $\mathcal{V}(\Gamma)$ of Definition 1.14 can be rephrased as follows. A subgraph $\gamma \subset \Gamma$ in $\mathcal{V}(\Gamma)$ is a non-trivial (non-empty as well as its complement) subset $\gamma \subset \Gamma^{(1)}_{\mathrm{int}}$ such that the components γ_i of γ are 1PI graphs with two or three external lines, together with a choice of index $\chi(\gamma_i) \in \{0, 1\}$ for each component with two external lines. By Proposition 1.15 the graph Γ/γ is a 1PI graph. The map χ is used to give a label $\in \{0, 1\}$ to the two-point vertices of the contracted graph. Thus the above general discussion agrees with that of [**82**].

The following are examples of an explicit calculation of the coproduct for graphs Γ of the ϕ^3_6 theory, taken from [**82**].







As one can see clearly in the examples above, the coproduct has an interesting property of "linearity on the right", which is expressed more precisely in the following result.

PROPOSITION 1.28. ([**82**]) *Let $\mathcal{H}_1$ be the linear subspace of $\mathcal{H}$ generated by 1 and by the 1PI graphs. Then, for all $\Gamma \in \mathcal{H}_1$, the coproduct satisfies*

$$\Delta(\Gamma) \in \mathcal{H} \otimes \mathcal{H}_1 .$$

We have introduced the Hopf algebra $\mathcal{H}(\mathcal{T})$ in Definition 1.26 as a Hopf algebra over $\mathbb{C}$. However, for possible arithmetic applications, it is useful to know that, in fact, the Hopf algebras of Feynman graphs can be defined over $\mathbb{Q}$.

REMARK 1.29. *The definition of the coproduct and antipode in Definition* 1.26 *continue to make sense if we consider $\mathcal{H}(\mathcal{T})$ be be the free commutative algebra over $\mathbb{Q}$ generated by the 1PI graphs.*

We now discuss several natural gradings of the Hopf algebra $\mathcal{H}(\mathcal{T})$.

PROPOSITION 1.30. *The Hopf algebra $\mathcal{H}(\mathcal{T})$ is graded by the loop number,* $\deg(\Gamma) = b_1(\Gamma)$*, extended by*

$$\deg(\Gamma_1 \cdots \Gamma_r) = \sum_i \deg(\Gamma_i) \quad \textit{and} \quad \deg(1) = 0. \tag{1.199}$$

It also has a grading by the line number

$$\ell(\Gamma) = \#\Gamma^{(1)}_{\rm int} \quad \textit{and} \quad \ell(\prod_j \Gamma_j) = \sum_j \ell(\Gamma_j). \tag{1.200}$$

The grading by line number has the property that the graded components of $\mathcal{H}(\mathcal{T})$ are finite-dimensional.

PROOF. For any subgraph $\gamma \in \mathcal{V}(\Gamma)$, one has

$$\ell(\Gamma) = \sum_i \ell(\gamma_i) + \ell(\Gamma/\gamma), \tag{1.201}$$

since the set of internal lines of Γ/γ is the complement of γ in $\Gamma^{(1)}_{\mathrm{int}}$ while the internal lines of the components are the disjoint subsets γ_i. There are only finitely many graphs with a given number of internal lines. This gives the required finiteness for the dimensions of the graded components. Let us check that the prescription

$$v(\Gamma) = \#\Gamma^{(0)} - 1 \quad \text{and} \quad v(\prod_j \Gamma_j) = \sum_j v(\Gamma_j) \tag{1.202}$$

defines another possible grading on $\mathcal{H}(\mathcal{T})$. We need to show that

$$v(\Gamma) = \sum_i v(\gamma_i) + v(\Gamma/\gamma). \tag{1.203}$$

The graph Γ/γ is obtained by collapsing each of the connected components γ_i to a single vertex, i.e. by replacing the $v(\tilde{\gamma}_i) + 1$ vertices of γ_i with a single vertex. Thus, (1.203) follows. Since both ℓ and v are gradings so is their difference $\ell - v$, which gives the grading by loop number b_1. □

Notice that, with respect to the grading v by number of vertices (1.202), the Hopf algebra $\mathcal{H}(\mathcal{T})$ is in general not connected, i.e. in general the degree-zero component of $\mathcal{H}(\mathcal{T})$ is not just a copy of the field of scalars. One can see this in the example of the ϕ^4 graph of Figure 22.

With the grading given by the loop number, the Hopf algebra $\mathcal{H}(\mathcal{T})$ is graded connected, namely $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$, with $\mathcal{H}_0 = \mathbb{C}$. However, with respect to this choice of grading, the graded components $\mathcal{H}_n$ are in general not finite-dimensional. In fact, one can insert two-point vertices without changing the loop number.

The graded components $\mathcal{H}_n$ are finite-dimensional for the grading ℓ of Proposition 1.30 associated to the counting of the number of internal lines, as shown above. Moreover, for this grading the Hopf algebra is connected, i.e. $\mathcal{H}_0 = \mathbb{C}$.

Thus, as in Lemma 1.20, we have a corresponding affine group scheme that is a projective limit of linear algebraic groups. The affine group scheme is prounipotent, since $\mathcal{H}$ is positively graded connected.

DEFINITION 1.31. *The group of* diffeographisms $\mathrm{Difg}(\mathcal{T})$ *of a renormalizable theory* $\mathcal{T}$ *is the pro-unipotent affine group scheme of the Hopf algebra* $\mathcal{H}(\mathcal{T})$ *of Feynman graphs.*

The terminology “diffeographisms” is motivated by another result of the CK theory, which we discuss in §6.5 below, according to which $\mathrm{Difg}(\mathcal{T})$ acts on the coupling constants of the theory through formal diffeomorphisms tangent to the identity.

The Lie algebra of $\mathrm{Difg}(\mathcal{T})$ is identified in [**82**] using the Milnor–Moore theorem (Theorem 1.22).

THEOREM 1.32. ([**82**]) *The Lie algebra* $\mathrm{Lie}\,\mathrm{Difg}(\mathcal{T})$ *has a canonical linear basis given by pairs* (Γ, w) *consisting of a 1PI graph* Γ *and an element* $w \in J$ *with degree the number of external lines of* Γ. *The Lie bracket is given by*

$$[(\Gamma, w), (\Gamma', w')] = \sum_{v,\, \iota(v)=w'} (\Gamma \circ_v \Gamma', w) - \sum_{v',\, \iota(v')=w} (\Gamma' \circ_{v'} \Gamma, w') \tag{1.204}$$

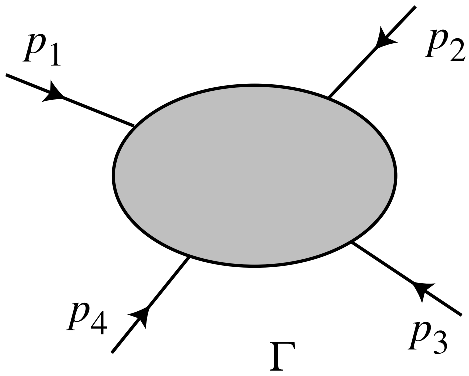


FIGURE 27. The value $U(\Gamma(p_1, \ldots, p_N))$ depends on the *incoming* momenta.

where $\Gamma \circ_v \Gamma'$ *denotes the graph obtained by inserting* Γ' *in* Γ *at the vertex* v *of* Γ.

The Lie bracket is uniquely defined from the Milnor–Moore Theorem. In general the notation $\Gamma \circ_v \Gamma'$ is ambiguous since there may be several ways of inserting Γ' in Γ at the vertex v due to the ordering of the external legs, and one has to take into account the corresponding combinatorial factor. We refer to [**276**] for the needed precise discussion of this point.

REMARK 1.33. The Hopf algebra $\mathcal{H}(\mathcal{T})$ simplifies even further when we take the massless case of the ϕ^3 theory in $D = 6$ dimensions (which is consistent in dimension 6). In that case the monomials in the Lagrangian are distinguished by their degrees and one can drop the w as well as the maps ι and χ of the above discussion. Moreover there is no need to introduce the new vertices $\times_{(i)}$ since the only relevant one, namely $\times_{(1)}$, brings in a k^2 which yields the same result as if one eliminates the new vertex altogether (the resulting loss of a propagator is compensated for exactly by the removal of the k^2 vertex). The grading by loop number is now finite in each degree and the counting ℓ of the number of internal lines is no longer a grading.

6.3. The Hopf algebra of Feynman graphs: full structure.

So far, we have not been taking into account the fact that Feynman graphs have an external structure, $\Gamma(p_1, \ldots, p_N)$ (cf. Figure 27). We now describe the full Hopf algebra of Feynman graphs, following [**82**].

We need to input the external data in the structure of the algebra. For a Feynman graph Γ of the theory with N external legs, we define the set

$$E_\Gamma := \Big\{(p_i)_{i=1,\ldots,N} \;;\; \sum p_i = 0\Big\} \tag{1.205}$$

of possible external momenta subject to the conservation law. Consider then the space of smooth functions $C^\infty(E_\Gamma)$ and let $C_c^{-\infty}(E_\Gamma)$ denote the space of distributions dual to $C^\infty(E_\Gamma)$.

DEFINITION 1.34. *An external structure on a Feynman graph* Γ *with* N *external legs is a distribution* $\sigma \in C_c^{-\infty}(E_\Gamma)$.

It is often sufficient to consider distributions $\sigma : C^\infty(E_\Gamma) \to \mathbb{C}$ that are δ-functions. However, this is in general not enough. In fact, one should keep in mind the non-linear dependence of $U(\Gamma(p_1, \ldots, p_N))$ on the external momenta.

We choose a set of distributions σ_ι, indexed by the elements $\iota \in J$, so that the σ_ι form a dual basis to the basis of monomials in the full Lagrangian, i.e. they fulfill the analog of (1.166).

We then consider the disjoint union $E := \cup_\Gamma E_\Gamma$, over 1PI graphs Γ and the corresponding direct sum of the spaces of distributions $C_c^{-\infty}(E_\Gamma)$, which we denote by

$$C_c^{-\infty}(E) = \oplus_\Gamma C_c^{-\infty}(E_\Gamma).$$

We obtain the following description of the full Hopf algebra of Feynman graphs.

Definition 1.35. *For a given renormalizable quantum field theory $\mathcal{T}$, the* full Hopf algebra of Feynman graphs *$\tilde{\mathcal{H}}(\mathcal{T})$ is the symmetric algebra on the linear space of distributions $C_c^{-\infty}(E)$,*

$$\tilde{\mathcal{H}}(\mathcal{T}) = \mathrm{Sym}(C_c^{-\infty}(E))\,. \tag{1.206}$$

The coproduct on $\tilde{\mathcal{H}}(\mathcal{T})$ is defined on generators (Γ, σ), with Γ a 1PI graph of $\mathcal{T}$ and $\sigma \in C_c^{-\infty}(E_\Gamma)$, by the formula

$$\Delta(\Gamma, \sigma) = (\Gamma, \sigma) \otimes 1 + 1 \otimes (\Gamma, \sigma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} (\gamma, \sigma_{\chi(\gamma)}) \otimes (\Gamma/\gamma, \sigma). \tag{1.207}$$

In the formula (1.207), the subgraphs γ belong to the class $\mathcal{V}(\Gamma)$ of Definition 1.14. The notation $(\gamma, \sigma_{\chi(\gamma)})$ is a shorthand for

$$\prod (\tilde{\gamma}_i, \sigma_{\chi(\gamma_i)})$$

where the product is over the components of γ and χ is the map from the set of components of γ to J of Definition 1.14. The contracted graph Γ/γ has the same external lines as the graph Γ so that one can endow it with the external structure σ.

The proof that Definition 1.35 indeed gives a Hopf algebra, cf. [**82**], is a direct extension of the above proof of Theorem 1.27.

The full Hopf algebra $\tilde{\mathcal{H}}(\mathcal{T})$ also has an associated affine group scheme $\widetilde{\mathrm{Difg}}(\mathcal{T})$. The relation of $\widetilde{\mathrm{Difg}}(\mathcal{T})$ to the group of diffeographisms $\mathrm{Difg}(\mathcal{T})$ of the discrete Hopf algebra $\mathcal{H}(\mathcal{T})$ is through a semidirect product

$$\widetilde{\mathrm{Difg}}(\mathcal{T}) = \mathrm{Difg}_{\mathrm{ab}}(\mathcal{T}) \rtimes \mathrm{Difg}(\mathcal{T}), \tag{1.208}$$

where $\mathrm{Difg}_{\mathrm{ab}}(\mathcal{T})$ is abelian. This can be seen at the level of the Lie algebra.

Consider the linear space

$$\mathcal{L} = C^\infty(E), \tag{1.209}$$

with $E := \cup_\Gamma E_\Gamma$, the disjoint union over 1PI graphs Γ. We denote an element of $\mathcal{L}$ as a family $L = (f^\Gamma)$ with $f^\Gamma \in C^\infty(E_\Gamma)$, indexed by 1PI graphs Γ. An element L in $\mathcal{L}$ determines a linear form Z_L on the Hopf algebra $\tilde{\mathcal{H}}(\mathcal{T})$, which is non-zero only for elements of

$$C_c^{-\infty}(E) \subset \mathrm{Sym}(C_c^{-\infty}(E))$$

and pairs as follows with the generators (Γ, σ),

$$\langle Z_L, (\Gamma, \sigma) \rangle := \sigma(f^\Gamma), \tag{1.210}$$

where f^Γ is the component in $C^\infty(E_\Gamma)$ of L. The form $Z_L : \tilde{\mathcal{H}}(\mathcal{T}) \to \mathbb{C}$ satisfies the condition (1.173)

$$Z_L(XY) = Z_L(X)\varepsilon(Y) + \varepsilon(X) Z_L(Y), \quad \forall X, Y \in \tilde{\mathcal{H}}(\mathcal{T}),$$

hence it defines an element in $\mathrm{Lie}\,\widetilde{\mathrm{Difg}}(\mathcal{T})$. The bracket then corresponds to the commutator $[Z_{L_1}, Z_{L_2}] = Z_{L_1} * Z_{L_2} - Z_{L_2} * Z_{L_1}$, where the product is obtained by transposing the coproduct of $\tilde{\mathcal{H}}(\mathcal{T})$,

$$\langle Z_{L_1} * Z_{L_2}, (\Gamma, \sigma) \rangle = \langle Z_{L_1} \otimes Z_{L_2}, \Delta(\Gamma, \sigma) \rangle.$$

When we evaluate $[Z_{L_1}, Z_{L_2}]$ on (Γ, σ) the only terms of (1.207) which contribute are those coming from subgraphs $\gamma \in \mathcal{V}(\Gamma)$ which have only one component. For such subgraphs $\gamma \in \mathcal{V}_c(\Gamma)$ the map χ is just an element $w \in J$ whose degree is the number of external lines of $\tilde{\gamma}$. One gets

$$\langle [Z_{L_1}, Z_{L_2}], (\Gamma, \sigma) \rangle = \sum_{\gamma \in \mathcal{V}_c(\Gamma),\, w} \sigma_w(f_1^\gamma)\, \sigma(f_2^{\Gamma/\gamma}) - \sigma_w(f_2^\gamma)\, \sigma(f_1^{\Gamma/\gamma})$$

One then has a description of the Lie algebra $\mathrm{Lie}\,\widetilde{\mathrm{Difg}}(\mathcal{T})$, identified with the Lie algebra of primitive elements in the dual Hopf algebra $\tilde{\mathcal{H}}(\mathcal{T})^\vee$.

PROPOSITION 1.36. *The Lie algebra* $\mathrm{Lie}\,\widetilde{\mathrm{Difg}}(\mathcal{T})$ *is the linear space* $\mathcal{L}$ *of* (1.209) *endowed with the Lie bracket*

$$[L_1, L_2]^\Gamma = \sum_{\gamma \in \mathcal{V}_c(\Gamma),\, w} \sigma_w(f_1^\gamma)\, f_2^{\Gamma/\gamma} - \sigma_w(f_2^\gamma)\, f_1^{\Gamma/\gamma} \tag{1.211}$$

where $L_j = (f_j^\Gamma)$.

The Lie algebra $\mathrm{Lie}\,\widetilde{\mathrm{Difg}}(\mathcal{T})$ is the semidirect product of an abelian Lie algebra $\mathcal{L}_{\mathrm{ab}}$ with the Lie algebra $\mathrm{Lie}\,\mathrm{Difg}(\mathcal{T})$. Elements $L = (f^\Gamma)$ of the abelian subalgebra $\mathcal{L}_{\mathrm{ab}}$ are characterized by the condition

$$L = (f^\Gamma) \in \mathcal{L}_{\mathrm{ab}} \Leftrightarrow \sigma_w(f^\gamma) = 0\,, \quad \forall w \in J \tag{1.212}$$

for any graph γ whose number of external lines is the degree of $w \in J$. We refer the reader to [**82**] for more details.

6.4. BPHZ as a Birkhoff factorization.

We now come to the main result of the CK theory of perturbative renormalization. We begin by describing the Birkhoff factorization of loops.

DEFINITION 1.37. *Let* $C = \partial\Delta$ *denote a circle around the point* $z = 0 \in \mathbb{C}$, *with* Δ *a small disk centered at* $z = 0$. *We denote by* $C_\pm$ *the two components of* $\mathbb{P}^1(\mathbb{C}) \setminus C$, *with* $0 \in C_+$ *and* $\infty \in C_-$. *Let* $G(\mathbb{C})$ *be a connected complex Lie group. Given a smooth loop* $\gamma : C \to G(\mathbb{C})$, *we say that* γ *admits a Birkhoff factorization if it can be written as a product*

$$\gamma(z) = \gamma_-(z)^{-1}\, \gamma_+(z), \quad \forall z \in C\,, \tag{1.213}$$

where the $\gamma_\pm$ *are boundary values of holomorphic functions* $\gamma_\pm : C_\pm \to G(\mathbb{C})$, *with* $\gamma_-(\infty) = 1$.

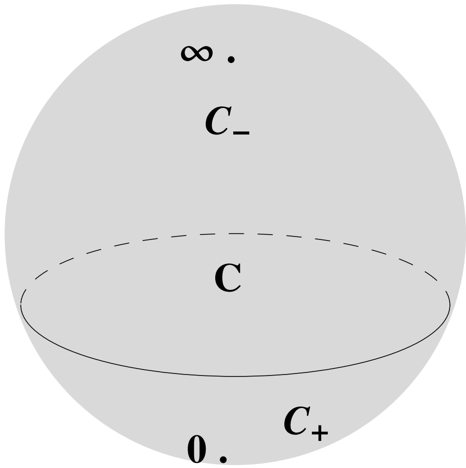


FIGURE 28. Birkhoff decomposition

In general, not all loops admit a Birkhoff factorization. This can be seen in the classical case of loops in $\mathrm{GL}_n(\mathbb{C})$. In this case, the factorization problem is closely related to the classification of holomorphic bundles on $\mathbb{P}^1(\mathbb{C})$. In fact, a loop $\gamma : C \to GL_n(\mathbb{C})$ has in general a factorization of the form

$$\gamma(z) = \gamma_-(z)^{-1}\,\lambda(z)\,\gamma_+(z), \tag{1.214}$$

where $\gamma_\pm$ are boundary values of holomorphic maps $\gamma_\pm : C_\pm \to GL_n(\mathbb{C})$, $\gamma_-(\infty) = 1$, while the middle term λ is a homomorphism of S^1 to the subgroup of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$,

$$\lambda(z) = \begin{pmatrix} z^{k_1} & & & \\ & z^{k_2} & & \\ & & \ddots & \\ & & & z^{k_n} \end{pmatrix}, \tag{1.215}$$

with integers k_i as exponents. The holomorphic maps $\gamma_\pm$ give the local frames for a trivialization of the restrictions to the open sets $C_\pm$ of a holomorphic vector bundle E on $\mathbb{P}^1(\mathbb{C})$. The bundle E is obtained by gluing these trivializations using the transition function $\lambda : C \to \mathrm{GL}_n(\mathbb{C})$. This recovers in terms of factorizations of loops (cf. [**247**]) the Grothendieck decomposition [**153**] of E as a sum of line bundles

$$E = L_1 \oplus \cdots \oplus L_n\,, \tag{1.216}$$

with Chern classes $c_1(L_i) = k_i$. Up to a permutation of the indices this decomposition is unique and the integers k_i give a complete invariant of the holomorphic vector bundle E.

We are interested in the case when Δ is an infinitesimal disk around $z = 0$ and C an infinitesimal loop. In this case, we can translate the formulation of the Birkhoff factorization (1.213) into the language of affine group schemes.

We assume that the Lie group $G(\mathbb{C})$ is the set of complex points of the affine group scheme G of a commutative Hopf algebra $\mathcal{H}$ over $\mathbb{C}$. Namely, we have $G(\mathbb{C}) = \mathrm{Hom}_{\mathcal{A}_{\mathbb{C}}}(\mathcal{H}, \mathbb{C})$. We denote by

$$K = \mathbb{C}(\{z\}) = \mathbb{C}\{z\}[z^{-1}] \tag{1.217}$$

the field of convergent Laurent series, that is, germs of meromorphic functions at the origin. We regard K as a unital commutative $\mathbb{C}$-algebra, and consider the K-points of G,

$$G(K) = \mathrm{Hom}_{\mathcal{A}_{\mathbb{C}}}(\mathcal{H}, K). \tag{1.218}$$

We can think of elements in $G(K)$ as describing loops $\gamma(z)$ on an infinitesimal circle (or an infinitesimal punctured disk Δ^*) around $z = 0$. Similarly, consider

$$\mathcal{O} = \mathbb{C}\{z\}, \tag{1.219}$$

the ring of convergent power series, or germs of holomorphic functions at $z = 0$. Again, we can consider the corresponding group

$$G(\mathcal{O}) = \mathrm{Hom}_{\mathcal{A}_{\mathbb{C}}}(\mathcal{H}, \mathcal{O}), \tag{1.220}$$

which describes loops $\gamma(z)$ that extend holomorphically to $z = 0$. Finally, we consider the ring

$$\mathcal{Q} = z^{-1}\mathbb{C}[z^{-1}], \quad \text{and the unital ring} \quad \tilde{\mathcal{Q}} = \mathbb{C}[z^{-1}], \tag{1.221}$$

and the group

$$G(\tilde{\mathcal{Q}}) = \mathrm{Hom}_{\mathcal{A}_{\mathbb{C}}}(\mathcal{H}, \tilde{\mathcal{Q}}). \tag{1.222}$$

We can impose a normalization condition corresponding to $\gamma_-(\infty) = 1$ for an element $\phi \in G(\tilde{\mathcal{Q}})$, by requiring that $\varepsilon_- \circ \phi = \varepsilon$, where ε_- is the augmentation in the ring $\tilde{\mathcal{Q}}$ and ε is the counit of the Hopf algebra $\mathcal{H}$.

In terms of these data, we restate the Birkhoff factorization problem of Definition 1.37 in the following form.

Definition 1.38. *An element $\phi \in G(K)$ admits a Birkhoff factorization if it can be written as a product*

$$\phi = (\phi_- \circ S) * \phi_+ \tag{1.223}$$

with $\phi_+ \in G(\mathcal{O})$ and $\phi_- \in G(\tilde{\mathcal{Q}})$ satisfying $\varepsilon_- \circ \phi_- = \varepsilon$, and with S the antipode of $\mathcal{H}$.

The following result of [**82**] shows that, in the case of interest for renormalization, i.e. when G is the pro-unipotent affine group scheme of a graded connected commutative Hopf algebra $\mathcal{H}$, all loops admit a Birkhoff factorization. In fact, this is not only an existence result, but it provides an explicit recursive formula for the factorization.

Theorem 1.39. ([**82**]) *Let $\mathcal{H}$ be a positively graded connected commutative Hopf algebra, $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$ with $\mathcal{H}_0 = \mathbb{C}$. Then every element $\phi \in G(K) = \mathrm{Hom}_{\mathcal{A}_{\mathbb{C}}}(\mathcal{H}, K)$ admits a unique Birkhoff factorization, as in* (1.223)*. The factorization is given recursively by the identities*

$$\phi_-(X) = -T\left(\phi(X) + \sum \phi_-(X')\phi(X'')\right), \tag{1.224}$$

where T is the projection on the pole part of the Laurent series, and

$$\phi_+(X) = \phi(X) + \phi_-(X) + \sum \phi_-(X')\phi(X''), \tag{1.225}$$

where, for $X \in \mathcal{H}_n$, the X' and X'' are the lower degree terms that appear in the coproduct

$$\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''. \tag{1.226}$$

Proof. We reproduce here the proof from [**82**]. The augmentation ε vanishes on $\mathcal{H}_n$ for $n > 0$, so that the augmentation ideal of $\mathcal{H}$, namely $\mathcal{H}^{>} = \operatorname{Ker}\varepsilon$, is the direct sum of the $\mathcal{H}_n$ for $n > 0$. For $X \in \mathcal{H}_n$ the coproduct $\Delta(X)$ can be written as a linear combination of tensor products $X' \otimes X''$ of homogeneous elements. The equalities

$$(\mathrm{Id} \otimes \varepsilon) \circ \Delta = \mathrm{Id}\,, \quad (\varepsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id}$$

show that $\Delta(X)$ is of the form (1.226).
One needs to show that the map $\phi_- : \mathcal{H} \to K$ defined by (1.224) is indeed a homomorphism. The main point is to show, by induction, that it is multiplicative. For all X, Y of positive degrees $\deg X > 0$, $\deg Y > 0$ one has

$$\begin{aligned} \Delta(XY) = XY \otimes 1 + 1 \otimes XY + X \otimes Y + Y \otimes X + XY' \otimes Y'' + \\ Y' \otimes XY'' + X'Y \otimes X'' + X' \otimes X''Y + X'Y' \otimes X''Y''\,. \end{aligned} \tag{1.227}$$

Thus, we obtain the following expression for $\phi_-(XY)$:

$$\begin{aligned} \phi_-(XY) = -T(\phi(XY)) - T(\phi_-(X)\,\phi(Y) + \phi_-(Y)\,\phi(X) + \\ \phi_-(XY')\,\phi(Y'') + \phi_-(Y')\,\phi(XY'') + \phi_-(X'Y)\,\phi(X'') \\ + \phi_-(X')\,\phi(X''Y) + \phi_-(X'Y')\,\phi(X''Y''))\,. \end{aligned} \tag{1.228}$$

Now we use the fact that ϕ is a homomorphism and the induction hypothesis that ϕ_- is multipicative, $\phi_-(AB) = \phi_-(A)\,\phi_-(B)$, for $\deg A + \deg B < \deg X + \deg Y$. (Notice that this is automatic if $\deg A = 0$ or $\deg B = 0$ since the algebra is connected and $\phi_-(1) = 1$ so that there is no problem in starting the induction.) We then rewrite (1.228) as

$$\begin{aligned} \phi_-(XY) = \quad & -T(\phi(X)\,\phi(Y) + \phi_-(X)\,\phi(Y) \\ & +\phi_-(Y)\,\phi(X) + \phi_-(X)\,\phi_-(Y')\,\phi(Y'') \\ & +\phi_-(Y')\,\phi(X)\,\phi(Y'') + \phi_-(X')\,\phi_-(Y)\,\phi(X'') \\ & +\phi_-(X')\,\phi(X'')\,\phi(Y) \\ & +\phi_-(X')\,\phi_-(Y')\,\phi(X'')\,\phi(Y''))\,. \end{aligned} \tag{1.229}$$

We need to compare this with $\phi_-(X)\phi_-(Y)$. We can compute the latter using the fact that the operator of projection onto the polar part satisfies the relation

$$T(f)\,T(h) = -T(fh) + T(T(f)\,h) + T(f\,T(h))\,. \tag{1.230}$$

By applying (1.230) to

$$f = \phi(X) + \phi_-(X')\,\phi(X'') \quad \text{ and } \quad h = \phi(Y) + \phi_-(Y')\,\phi(Y''),$$

we obtain

$$
\begin{aligned}
&\phi_-(X)\,\phi_-(Y) = \\
&-T\Big((\phi(X)+\phi_-(X')\,\phi(X''))\,(\phi(Y)\,+\phi_-(Y')\,\phi(Y'')) \Big) \\
&+T\Big(T\big(\phi(X)+\phi_-(X')\,\phi(X'')\big)\,(\phi(Y)\,+\phi_-(Y')\,\phi(Y'') \Big) \\
&+T\Big((\phi(X)+\phi_-(X')\,\phi(X''))\,T\big(\phi(Y)+\phi_-(Y')\,\phi(Y'')\big) \Big).
\end{aligned}
\tag{1.231}
$$

Using the fact that $T(f) = -\phi_-(X)$, $T(h) = -\phi_-(Y)$, we can then rewrite (1.231) in the form

$$
\begin{aligned}
&\phi_-(X)\,\phi_-(Y) = \\
&-T\Big(\phi(X)\,\phi(Y) + \phi_-(X')\,\phi(X'')\,\phi(Y) \\
&+\phi(X)\,\phi_-(Y')\,\phi(Y'')\ +\phi_-(X')\,\phi(X'')\,\phi_-(Y')\,\phi(Y'') \Big) \\
&-T\Big(\phi_-(X)(\phi(Y)+\phi_-(Y')\,\phi(Y'') \Big) \\
&-T\Big((\phi(X)+\phi_-(X')\,\phi(X''))\,\phi_-(Y) \Big)\,.
\end{aligned}
\tag{1.232}
$$

Finally, we compare (1.229) with (1.232). Both expressions contain eight terms of the form $-T(a)$ and one can check that they match in pairs. This proves that ϕ_- defined by the recursive formula (1.224) is indeed an algebra homomorphism. It is then clear that the ϕ_+ defined by (1.225) is also an algebra homomorphism, since both ϕ and ϕ_- are. It is clear by construction that $\phi_- \in G(\tilde{\mathcal{Q}})$ since it is a polar part, and that $\phi_+ \in G(\mathcal{O})$. It is also easy to check that one has $\phi_+ = \phi_- * \phi$, since one can see the right-hand side of (1.225) as the pairing $\langle \phi_- \otimes \phi, \Delta(X)\rangle = \phi_- * \phi\,(X)$. ∎

One can then apply the result of Theorem 1.39 to the specific case of the Hopf algebra of Feynman graphs. One finds that the formulae (1.224) and (1.225) for the recursive Birkhoff factorization of loops in the group of diffeographisms are exactly the recursive formulae of the BPHZ renormalization procedure.

THEOREM 1.40. ([**82**]) *Let $\mathcal{T}$ be a renormalizable theory and $\tilde{\mathcal{H}}(\mathcal{T})$ the Hopf algebra of Feynman graphs, with the grading by loop number. Then the formulae* (1.224) *and* (1.225) *for the Birkhoff factorization of loops $\phi \in \widetilde{\mathrm{Difg}}(\mathcal{T})(K) = \mathrm{Hom}_{\mathcal{A}_{\mathbb{C}}}(\tilde{\mathcal{H}}(\mathcal{T}), K)$ are given, respectively, by the formulae* (1.163) *and* (1.165) *of the BPHZ perturbative renormalization.*

PROOF. Consider the data $U^z(\Gamma(p_1,\ldots,p_N))$ of the unrenormalized values of Feynman graphs, regularized by applying DimReg. We can view U as a homomorphism $U : \tilde{\mathcal{H}}(\mathcal{T}) \to K$, which assigns to a generator (Γ,σ) of $\tilde{\mathcal{H}}(\mathcal{T})$ the Laurent series

$$
h(z) = \langle \sigma, U^z(\Gamma(p_1,\ldots,p_N))\rangle. \tag{1.233}
$$

Here we view $U^z(\Gamma(p_1,\dots,p_N)) = f_z(p_1,\dots,p_N)$ as a family of smooth functions $f_z \in C^\infty(E_\Gamma)$, with $z \in \Delta^*$ varying in an infinitesimal punctured disk around $z=0$. Thus, we can pair the distribution σ with f_z and obtain $h(z) = \langle \sigma, f_z\rangle \in K = \mathbb{C}(\{z\})$. It is then clear that, upon setting $\phi_- = C$ and $\phi_+ = R$ one can identify the formulae (1.163) and (1.165) with (1.224) and (1.225). □

Notice that it is fine here to consider the grading by loop number because for the inductive procedure connectedness of the Hopf algebra suffices and one does not need it to be finite-dimensional in each degree.

We can express the result of Theorem 1.40 equivalently in terms of loops γ. We say that the data $U^z(\Gamma(p_1,\dots,p_N))$ define a loop $\gamma(z)$ in the pro-unipotent Lie group of complex points of $\widetilde{\mathrm{Difg}}(\mathcal{T})$. This loop has a Birkhoff factorization

$$\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z),$$

where the "negative piece" $\gamma_-(z)$ gives the counterterms and the "positive part" $\gamma_+(z)$ gives the renormalized value as the evaluation $\gamma_+(0)$.

We now restrict attention to the discrete Hopf algebra $\mathcal{H}(\mathcal{T})$ for the simple reason that the non-trivial counterterms in the full Hopf algebra already come from the Birkhoff factorization of the restriction to $\mathcal{H}(\mathcal{T})$. In fact, notice that terms of the form

$$(\gamma, \sigma_{\chi(\gamma)}) \otimes (\Gamma/\gamma, \sigma)$$

in the coproduct formula of the full Hopf algebra $\widetilde{\mathcal{H}}(\mathcal{T})$ are identified with elements in $\mathcal{H}(\mathcal{T})$ through the choice of the dual basis σ_ι. Thus, if one knows the value of the restriction of $\gamma_-(z)$ to $\mathcal{H}(\mathcal{T})$, the renormalized value of an arbitrary graph does not require the computation of any new counterterm. It is obtained, after preparation by the known counterterms (from $\mathcal{H}(\mathcal{T})$), by a simple subtraction of the pole part (which is non-zero only if the degree of divergence is positive).

THEOREM 1.41. *Let $\mathcal{T}$ be a renormalizable theory and $\mathcal{H}(\mathcal{T})$ the discrete Hopf algebra of Feynman graphs, with its affine group scheme* $\mathrm{Difg}(\mathcal{T})$. *Then the formulae* (1.224) *and* (1.225) *for the Birkhoff factorization of loops applied to the homomorphism U give respectively the counterterms and the renormalized value for the theory $\mathcal{T}'$ with new interaction vertices associated to monomials of the non-interacting part of the Lagrangian of $\mathcal{T}$.*

PROOF. We let as above J be the set of all monomials in the Lagrangian of $\mathcal{T}$ and I the set of interaction monomials. For the graphs $\Gamma \in \mathrm{Graph}\,\mathcal{T}$ such that $\iota(\Gamma) \subset I$ the statement follows from Proposition 1.40.

The formulae (1.224) and (1.225) depend upon the definition of subgraphs of the theory, through the fact that they involve the coproduct in the Hopf algebra. We need to show that our definition of subgraphs has the right properties so that (1.224) and (1.225) give the renormalization of the extended Feynman graphs of the theory (as in Definition 1.11) with $\iota(\Gamma^{(0)}) \subset J$, viewing these graphs as Feynman graphs of the new theory $\mathcal{T}'$.

The main point is that the new theory is still renormalizable and does not require any new type of counterterms in order to be renormalized. This is proved using dimensional analysis. When all the coupling constants are positive one gets a finite number of possible counterterms (cf. [**62**] §3.3.3).

For instance, in the theory ϕ^3 in dimension D the field ϕ has physical dimension $D/2-1$ and the coupling constant g has physical dimension $3-D/2$ (cf. §2.2 above). As long as $D \leq 6$, the theory is renormalizable and it remains such with the new vertices $\overset{0}{-\!\!\times\!\!-}$ and $\overset{1}{-\!\!\times\!\!-}$. Moreover, when we renormalize a graph of that new theory, we do not introduce new counterterms. The three types of counterterms needed are already present from $\mathcal{T}$.

Now start from $\Gamma \in \mathrm{Graph}(\mathcal{T})$ and view it as a graph of the new theory $\mathcal{T}'$. Then, in order to renormalize it, we first prepare it by adding the counterterms corresponding to the subdivergences. We know that the only subgraphs responsible for subdivergences of Γ have at most three external legs (in general they belong to the set J), hence they correspond to subgraphs in the sense of Definition 1.14 for the theory $\mathcal{T}$. According to this definition, we use already all the elements of J as labels for the connected components of the subgraphs. This means that, when we collect the terms, we get the prepared graph as if we were applying BPHZ for the new theory $\mathcal{T}'$. One then proceeds by induction to show that in (1.224) we get the counterterm for Γ as if we were applying BPHZ for the new theory $\mathcal{T}'$. The same holds for the renormalized value. It is crucial, in doing this, that we did not need to add new vertices since the propagator of the new theory remains the same. $\square$

6.5. Diffeographisms and diffeomorphisms.

Another important result of the CK theory, which we discuss in this section, shows that the group of diffeographisms of a renormalizable theory $\mathcal{T}$ maps to the group of formal diffeomorphisms tangent to the identity of the space of coupling constants of the theory.

More precisely, for a renormalizable theory $\mathcal{T}$, consider the complex vector space V with a basis labeled by the coupling constants. We let $\mathrm{Diff}(V)$ be the group of formal diffeomorphisms of V tangent to the identity at $0 \in V$ and $\mathcal{H}_{\mathrm{diff}}(V)$ be its Hopf algebra. We work over the ground field $\mathbb{C}$. There is a Hopf algebra homomorphism $\Phi : \mathcal{H}_{\mathrm{diff}}(V) \to \mathcal{H}(\mathcal{T})$ and a dual group homomorphism $\mathrm{Difg}(\mathcal{T}) \to \mathrm{Diff}(V)$. This map to formal diffeomorphisms explains the terminology "diffeographisms" for the group scheme $\mathrm{Difg}(\mathcal{T})$.

The map Φ is constructed by assigning to the coefficients of the expansion of formal diffeomorphisms the coefficients in $\mathcal{H}(\mathcal{T})$ of the expansion of the effective coupling constants of the theory as formal power series in the bare coupling constants. It is not immediate to show that Φ is indeed a Hopf algebra homomorphism ([**83**]).

We state this result more precisely in the simplest case where $\mathcal{T}$ is the massless ϕ^3_6, where there is only one coupling constant, and one can write a completely explicit formula for the map Φ.

Proposition 1.42. ([**83**]) *Let $\mathcal{T}$ be the massless ϕ^3_6 theory and let $\mathcal{H}_{\mathrm{diff}}(\mathbb{C})$ be the Hopf algebra of formal diffeomorphisms of $\mathbb{C}$ tangent to the identity. Then the following holds.*

(1) *The effective coupling constant has a power series expansion of the form*

$$g_{\text{eff}} = \frac{\left(g + \displaystyle\sum \; g^{2\ell+1} \frac{\Gamma}{\sigma(\Gamma)} \right)}{\left(1 - \displaystyle\sum \; g^{2\ell} \frac{\Gamma}{\sigma(\Gamma)} \right)^{3/2}}, \tag{1.234}$$

where the coefficients are elements in the Hopf algebra $\mathcal{H}(\mathcal{T})$, with $\sigma(\Gamma)$ the symmetry factor of the graph as in (1.98).

(2) *The expansion* (1.234) *induces a Hopf algebra homomorphism $\Phi : \mathcal{H}_{\text{diff}}(\mathbb{C}) \to \mathcal{H}(\phi_6^3)$ with $\Phi(a_n) = \alpha_n$.*

The expansion (1.234) is obtained by considering the effect of adjusting the coupling constants in the Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{g}{6}\phi^3$$

of the massless ϕ_6^3 theory by

$$\frac{1}{2}(\partial\phi)^2(1 - \delta Z) - \frac{g + \delta g}{6}\phi^3.$$

The correction $\frac{1}{2}(\partial\phi)^2(1 - \delta Z)$ can be absorbed in a rescaling $\phi \mapsto \phi(1 - \delta Z)^{1/2}$ so that one obtains

$$\frac{1}{2}(\partial\phi)^2 - \frac{1}{6}(g + \delta g)(1 - \delta Z)^{-3/2}\,\phi^3.$$

The counterterms responsible for the term δZ come from all the graphs with two external legs, while the counterterms involved in the term δg come from graphs with three external legs (cf. the BPHZ formulae (1.162), (1.163), (1.165)), hence the explicit form of the expansion (1.234).

The Hopf algebra $\mathcal{H}_{\text{diff}}(\mathbb{C})$ of formal diffeomorphisms of $\mathbb{C}$ tangent to the identity has generators the coordinates a_n of

$$\varphi(x) = x + \sum_{n \geq 2} a_n(\varphi) x^n,$$

for φ a formal diffeomorphism satisfying $\varphi(0) = 0$ and $\varphi'(0) = \text{id}$. The coproduct $\Delta(a_n)$ is given by

$$\langle \Delta(a_n), \varphi_1 \otimes \varphi_2 \rangle = a_n(\varphi_2 \circ \varphi_1).$$

One obtains an algebra homomorphism $\Phi : \mathcal{H}_{\text{diff}}(\mathbb{C}) \to \mathcal{H}(\phi_6^3)$ by setting $\Phi(a_n) = \alpha_n$, where the $\alpha_n \in \mathcal{H}(\phi_6^3)$ are the coefficients of the series $g_{\text{eff}} = g + \sum_{n \geq 2} \alpha_n, g^n$, which satisfy $\alpha_{2n} = 0$. We refer the reader to [**83**] §4 for the proof that the algebra homomorphism Φ is also comultiplicative.

The fact that there is a group representation $\text{Difg}(\mathcal{T}) \to \text{Diff}(V)$ means that it is possible to lift at the level of diffeographisms various aspects of the theory of renormalization. We will see in §6.6 below that the beta function and the action of the renormalization group can be formulated naturally at the level of the group scheme of diffeographisms.

This result was generalized from the case of massless ϕ^3_6 theory to other theories by Cartier and by Krajewski.

6.6. The renormalization group.

Recall how we remarked in §4 that, in fact, the regularized unrenormalized values

$$U^z(\Gamma(p_1,\ldots,p_N))$$

depend on an additional mass parameter μ, as in (1.135). This induces a corresponding dependence on the parameter μ of the loop $\gamma_\mu(z)$ that encodes the unrenormalized values as in Proposition 1.40. For a graph Γ with loop number L this dependence is simply given by a multiplicative factor of μ^{zL} when working in dimension $D-z$. In the example of the theory $\mathcal{T}=\phi^3_6$, this dependence on μ is described explicitly in terms of the Feynman rules in [**87**] §2.7. Our convention here agrees with that of [**87**] on the relevant Hopf algebra of graphs with two or three external legs, which are the only ones with non-zero counterterms. We discuss here two general properties of $\gamma_\mu(z)$; see Propositions 1.43 and 1.44 below.

Consider the 1-parameter group of automorphisms

$$\theta_t \in \mathrm{Aut}(\mathrm{Difg}(\mathcal{T})), \quad \forall t\in\mathbb{C} \tag{1.235}$$

implementing the grading by loop number, namely, with infinitesimal generator

$$\frac{d}{dt}\,\theta_t\,|_{t=0} = Y \tag{1.236}$$

given by the grading operator

$$Y(X) = n\,X, \quad \forall X\in\mathcal{H}^\vee_n(\mathcal{T}), \tag{1.237}$$

so that

$$\theta_t(X) = e^{nt}\,X, \quad \forall X\in\mathcal{H}^\vee_n(\mathcal{T})\,,\ t\in\mathbb{C}. \tag{1.238}$$

We let the grading θ_t act by automorphisms of both $\mathcal{H}=\mathcal{H}(\mathcal{T})$ and the dual algebra $\mathcal{H}^\vee$ so that

$$\langle\theta_t(u),x\rangle = \langle u,\theta_t(x)\rangle\,, \quad \forall x\in\mathcal{H}\,,\ \forall u\in\mathcal{H}^\vee\,. \tag{1.239}$$

We obtain the following result about the dependence of $\gamma_\mu(z)$ on the parameter μ.

PROPOSITION 1.43. ([**83**]) *Let $\gamma_\mu(z)$ be the loop in the pro-unipotent Lie group of diffeographisms that encodes the regularized unrenormalized values $U^z_\mu(\Gamma)$. Let θ_t be the 1-parameter family of automorphisms* (1.235)*. Then, for all $t\in\mathbb{R}$ and for all $z\in\Delta^*$, the loop $\gamma_\mu(z)$ satisfies the scaling property*

$$\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)). \tag{1.240}$$

PROOF. Since both sides of (1.240) are homomorphisms from $\mathcal{H}$ to K it is enough to check the equality on the generators Γ and one can use (1.239). It corresponds to the fact that the dependence on μ in the unrenormalized value of the graph Γ (1.135) is through a power μ^{zL}, with L the loop number $L=b_1(\Gamma)$, which comes from formally replacing the integration variables d^Dk by $\mu^z d^{D-z}k$ in the "integration in dimension $D-z$" of (1.132). ☐

The following result ([**62**], §5.8 and §7.1) will play a very important role in §7 below.

PROPOSITION 1.44. ([**83**]) *Let $\gamma_\mu(z)$ be the loop in the pro-unipotent Lie group of diffeographisms that encodes the regularized unrenormalized values $U_\mu^z(\Gamma)$. Let*

$$\gamma_\mu(z) = \gamma_{\mu^-}(z)^{-1}\gamma_{\mu^+}(z)$$

be the Birkhoff factorization of Theorem 1.39. Then the negative part of the Birkhoff factorization is independent of the mass parameter μ,

$$\frac{\partial}{\partial\mu}\gamma_{\mu^-}(z) = 0\,. \tag{1.241}$$

PROOF. Let us give the proof in some detail. First notice that we are dealing with general graphs with vertices in J, rather than the restricted graphs with vertices in I. By Theorem 1.41, when evaluated on a graph $\Gamma \in \mathrm{Graph}\,\mathcal{T}$, the negative part $\gamma_{\mu^-}(z)$ of the Birkhoff factorization is the same as the counterterms of that graph in the BPHZ procedure of the new theory $\mathcal{T}'$ with the new vertices labeled by monomials in $J \setminus I$. Notice that these new vertices have coupling constants of positive physical dimension (in the sense of §2.2 above), which are independent of μ. Thus, we are reduced to the analysis of the μ-dependence for counterterms in a renormalizable theory (with couplings of physical dimensions ≥ 0). The proof that the counterterms are independent of μ is done in three steps, as follows.

- The counterterms depend in a polynomial manner on the mass parameters of the theory (μ not included).
- Only powers of $\log\mu$ can appear in the counterterms.
- By dimensional analysis the counterterms do not depend on μ.

We refer to [**62**] §5.8.1 for the proof of the first statement. It is based on repeated differentiation with respect to these parameters and an inductive argument assuming that the counterterms for the subdivergences are already polynomial. This behavior is specific to DimReg and MS and would not apply, for instance, in the mass-shell renormalization scheme.

That only powers of $\log\mu$ can appear in the counterterms follows again by an inductive argument since the counterterms are obtained by extracting the pole part of an expression with new dependence in μ of the form μ^{zL} which one expands in powers of z and $\log\mu$.

The argument of a log is necessarily dimensionless since when we apply the minimal subtraction we are subtracting quantities which have the same dimension. Thus the massive parameter μ can only appear in the form of products of $\log(p^2/\mu^2)$ or of $\log(M^2/\mu^2)$, where M is a mass parameter of the theory. However, we know from the discussion above that momenta p or such mass parameters can only appear as polynomials, hence such expressions of logarithmic form are excluded and μ cannot appear in the counterterms. $\square$

DEFINITION 1.45. *In the following, let $G(\mathbb{C})$ be the pro-unipotent affine group scheme associated to a positively graded connected commutative Hopf algebra $\mathcal{H}$. Let $L(G(\mathbb{C}), \mu)$ denote the space of $G(\mathbb{C})$-valued loops $\gamma_\mu(z)$, defined on an infinitesimal punctured disk $z \in \Delta^*$ around the origin, and satisfying the scaling condition* (1.240) *and the property* (1.241) *that the negative part of the unique Birkhoff factorization of $\gamma_\mu(z)$ is independent of μ.*

The results discussed in the rest of this section apply to any $\gamma_\mu \in L(G(\mathbb{C}), \mu)$, hence in particular to the case when $G(\mathbb{C})$ is the group of complex points of $\mathrm{Difg}(\mathcal{T})$ and $\gamma_\mu(z)$ encodes the regularized unrenormalized values $U^z_\mu(\Gamma)$ of the theory.

DEFINITION 1.46. *Suppose given $\gamma_\mu(z) \in L(G(\mathbb{C}), \mu)$. The residue is defined as*

$$\mathrm{Res}_{z=0}\,\gamma := -\left(\frac{\partial}{\partial u}\,\gamma_-\left(\frac{1}{u}\right)\right)_{u=0}. \tag{1.242}$$

The corresponding beta function is then given by

$$\beta := Y\,\mathrm{Res}\,\gamma, \tag{1.243}$$

where Y is the grading operator (1.237).

Since (1.242) depends only on γ_-, the residue is independent of μ by Proposition 1.44. The beta function β of (1.243) is an element in the Lie algebra $\mathrm{Lie}\,G$.

The renormalization group is obtained as a 1-parameter subgroup F_t of $G(\mathbb{C})$ as follows.

PROPOSITION 1.47. ([**83**]) *Suppose given $\gamma_\mu(z) \in L(G(\mathbb{C}), \mu)$. Then the following properties hold.*

(1) *The loop $\gamma_-(z)\,\theta_{tz}(\gamma_-(z)^{-1})$ is regular at $z = 0$.*
(2) *The limit*

$$F_t = \lim_{z\to 0} \gamma_-(z)\,\theta_{tz}(\gamma_-(z)^{-1}) \tag{1.244}$$

defines a 1-parameter subgroup of $G(\mathbb{C})$. Viewed as a homomorphism

$$F_t \in \mathrm{Hom}_{\mathcal{A}_\mathbb{C}}(\mathcal{H}, \mathbb{C}),$$

it has the property that $F_t(X)$ is polynomial in t, for all $X \in \mathcal{H}$.

(3) *The infinitesimal generator of F_t is the beta function* (1.243).
(4) *Let $\gamma_{\mu^+}(z)$ be the positive part of the Birkhoff factorization of $\gamma_\mu(z)$. It satisfies*

$$\gamma_{e^t\mu^+}(0) = F_t\,\gamma_{\mu^+}(0)\,, \quad \forall t \in \mathbb{R}. \tag{1.245}$$

PROOF. (1) Since $\gamma_-(z)$ is the negative part in the Birkhoff factorization of $\gamma_\mu(z)$ and of $\theta_{-tz}(\gamma_\mu(z)) = \gamma_{e^{-t}\mu}(z)$, the elements $\gamma_-(z)\,\gamma_\mu(z)$ and $y(z) := \gamma_-(z)\,\theta_{-tz}(\gamma_\mu(z))$ are regular at $z = 0$. It follows that $\theta_{tz}(y(z)) = \theta_{tz}(\gamma_-(z))\,\gamma_\mu(z)$ is regular at $z = 0$, hence so is the ratio

$$\gamma_-(z)\,\theta_{tz}(\gamma_-(z)^{-1}) = (\gamma_-(z)\,\gamma_\mu(z))(\theta_{tz}(\gamma_-(z))\,\gamma_\mu(z))^{-1}\,.$$

(2) Thus, for any $t \in \mathbb{R}$, the limit

$$\lim_{z\to 0} \langle \gamma_-(z)\,\theta_{tz}(\gamma_-(z)^{-1}), X\rangle \tag{1.246}$$

exists, for any $X \in \mathcal{H}$. Here we view the element $\gamma_-(z)\,\theta_{tz}(\gamma_-(z)^{-1})$ of $G(\mathbb{C})$ as a homomorphism $\mathcal{H} \to \mathbb{C}$.

The 1-parameter family θ_t of automorphisms of G (cf. (1.235)) also acts as automorphisms of the Hopf algebra $\mathcal{H}$, satisfying by definition

$$\langle \theta_t(\gamma), X\rangle = \langle \gamma, \theta_t(X)\rangle\,, \quad \forall X \in \mathcal{H}\,, \quad \forall \gamma \in G(\mathbb{C})\,. \tag{1.247}$$

Thus, for $X \in \mathcal{H}$, we obtain

$$\langle \gamma_-(z)\,\theta_{tz}(\gamma_-(z)^{-1}), X\rangle = \langle \gamma_-(z)^{-1} \otimes \gamma_-(z)^{-1}, (S \otimes \theta_{tz})\,\Delta(X)\rangle\,. \tag{1.248}$$

Upon writing the coproduct as a sum of homogeneous elements

$$\Delta(X) = \sum X_{(1)} \otimes X_{(2)},$$

we obtain (1.248) as a sum of terms of the form

$$\langle \gamma_-(z)^{-1}, S\,X_{(1)} \rangle \, \langle \gamma_-(z)^{-1}, \theta_{tz}\, X_{(2)} \rangle = P_1\left(\frac{1}{z}\right) e^{ktz}\, P_2\left(\frac{1}{z}\right), \tag{1.249}$$

for polynomials P_1, P_2. We know, by the existence of the limit (1.246), that the sum of these terms is holomorphic at $z = 0$. Thus, by expanding e^{ktz} in a Taylor series, we obtain that

$$\langle F_t, X \rangle = \lim_{z\to 0} \langle \gamma_-(z)\, \theta_{tz}(\gamma_-(z)^{-1}), X \rangle$$

is a polynomial in t.

To see that F_t is a one-parameter subgroup of $G(\mathbb{C})$, we endow $G(\mathbb{C})$ with the topology of pointwise convergence

$$\gamma_n \to \gamma \quad \text{iff} \quad \langle \gamma_n, X \rangle \to \langle \gamma, X \rangle, \quad \forall X \in \mathcal{H}\,. \tag{1.250}$$

The fact that $G(\mathbb{C})$ is a topological group follows from the equality

$$\langle \gamma\, \gamma', X \rangle = \sum \langle \gamma, X_{(1)} \rangle \, \langle \gamma', X_{(2)} \rangle$$

and similarly for the inverse. Using (1.247) and the definition of F_t we have

$$\lim_{z\to 0} \theta_{sz}(\gamma_-(z)\, \theta_{tz}(\gamma_-(z)^{-1})) = F_t, \tag{1.251}$$

hence we obtain

$$\begin{aligned} F_{s+t} &= \lim_{z\to 0} \; \gamma_-(z)\, \theta_{(s+t)z}\,(\gamma_-(z)^{-1}) \\ &= \lim_{z\to 0} \; \gamma_-(z)\, \theta_{sz}(\gamma_-(z)^{-1})\, \theta_{sz}(\gamma_-(z)\, \theta_{tz}(\gamma_-(z)^{-1})) = F_s\, F_t\,, \end{aligned}$$

so that $F_{s+t} = F_s F_t$, for all $s, t \in \mathbb{R}$.

(3) This will follow from Lemma 1.48, cf. Corollary 1.49 below.

(4) The value $\gamma_\mu^+(0)$ is the regular value of $\gamma_-(z)\, \gamma_\mu(z)$ at $z = 0$. Similarly, $\gamma_{e^t\mu}^+(0)$ is the regular value of $\gamma_-(z)\, \theta_{tz}(\gamma_\mu(z))$, or equivalently of $\theta_{-tz}(\gamma_-(z))\, \gamma_\mu(z)$, at $z = 0$. We know that the ratio satisfies

$$\theta_{-tz}(\gamma_-(z))\, \gamma_-(z)^{-1} \to F_t$$

when $z \to 0$, hence we obtain the result. □

We conclude this section by the following result of [**83**], which will be the starting point for the main topic of the next Section, the Riemann–Hilbert correspondence underlying perturbative renormalization.

Lemma 1.48. *Suppose given $\gamma_\mu(z) \in L(G(\mathbb{C}), \mu)$ and let $\gamma_-(z)$ be the negative piece of the Birkhoff factorization, which can be written as*

$$\gamma_-(z)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{d_n}{z^n}, \tag{1.252}$$

with coefficients $d_n \in \mathcal{H}^\vee$. Then the coefficients d_n satisfy the relations

$$Y\,(d_{n+1}) = d_n\, \frac{d}{dt} F_t|_{t=0} \quad \forall n \geq 1, \quad \textit{and} \quad Y(d_1) = \frac{d}{dt} F_t|_{t=0}\,. \tag{1.253}$$

PROOF. We first show that, for all $X \in \mathcal{H}$, we have

$$\langle \frac{d}{dt} F_t|_{t=0}, X\rangle = \lim_{z\to 0} z\langle \gamma_-(z)^{-1} \otimes \gamma_-(z)^{-1}, (S \otimes Y)\,\Delta(X)\rangle. \tag{1.254}$$

We know from (1.248) and (1.244) that, for $z \to 0$, we have

$$\langle \gamma_-(z)^{-1} \otimes \gamma_-(z)^{-1}, (S \otimes \theta_{tz})\,\Delta(X)\rangle \to \langle F_t, X\rangle. \tag{1.255}$$

We denote by Ξ the expression on the left-hand side of (1.255). We know by (1.249) that it is a finite sum $\Xi = \sum P_k(z^{-1})\, e^{ktz}$, where the P_k are polynomials. As before, we can replace the exponentials e^{ktz} by their Taylor expansions up to some order N in z and obtain polynomials in t with coefficients that are Laurent series in z. Since we know the expression is regular at $z = 0$ for any value of t, these polynomials have, in fact, coefficients that are regular at $z = 0$, i.e. that are polynomials in z. Thus, the left-hand side of (1.255) is a uniform family of holomorphic functions of t, for t varying in a disk around $t = 0$. The derivative $\partial_t \Xi$ at $t = 0$ converges to $\partial_t \langle F_t, X\rangle|_{t=0}$ when $z \to 0$, so that we have

$$z\,\langle \gamma_-(z)^{-1} \otimes \gamma_-(z)^{-1}, (S \otimes Y)\,\Delta(X)\rangle \to \langle \frac{d}{dt} F_t|_{t=0}, X\rangle. \tag{1.256}$$

Notice then that the function

$$z \mapsto z\,\langle \gamma_-(z)^{-1} \otimes \gamma_-(z)^{-1}, (S \otimes Y)\,\Delta(X)\rangle \tag{1.257}$$

is holomorphic for $z \in \mathbb{C}\backslash\{0\}$ and extends holomorphically to $z = \infty \in \mathbb{P}_1(\mathbb{C})$, since $\gamma_-(\infty) = 1$, so that $Y(\gamma_-(\infty)) = 0$ (see (1.259)). Moreover, by the previous argument, it is also holomorphic at $z = 0$, which implies that it is a constant. This gives, for all $X \in \mathcal{H}$,

$$\langle \gamma_-(z)^{-1} \otimes \gamma_-(z)^{-1}, (S \otimes Y)\,\Delta(X)\rangle = \frac{1}{z}\,\langle \frac{d}{dt} F_t|_{t=0}, X\rangle\,,$$

which is equivalent to

$$\langle \gamma_-(z)\, Y(\gamma_-(z)^{-1}), X\rangle = \frac{1}{z}\,\langle \frac{d}{dt} F_t|_{t=0}, X\rangle.$$

Thus, one obtains the identity

$$Y(\gamma_-(z)^{-1}) = \frac{1}{z}\;\gamma_-(z)^{-1}\,\frac{d}{dt} F_t|_{t=0}\,. \tag{1.258}$$

Using (1.252), one obtains

$$Y(\gamma_-(z)^{-1}) = \sum_{n=1}^{\infty} \frac{Y(d_n)}{z^n} \tag{1.259}$$

and

$$\frac{1}{z}\,\gamma_-(z)^{-1}\,\frac{d}{dt} F_t|_{t=0} = \frac{1}{z}\,\frac{d}{dt} F_t|_{t=0} + \sum_{n=1}^{\infty} \frac{d_n}{z^{n+1}}\,\frac{d}{dt} F_t|_{t=0}.$$

These identities applied to the two sides of (1.258) yield the desired relations (1.253). $\square$

COROLLARY 1.49. *The beta function defined in* (1.243) *of Definition* 1.46 *is the infinitesimal generator of the renormalization group F_t of Proposition* 1.47*, namely*

$$\frac{d}{dt} F_t|_{t=0} = \beta. \tag{1.260}$$

PROOF. It is sufficient to see that in the equation (1.253) we have $Y(d_1) = \frac{d}{dt}F_t|_{t=0}$ and that d_1 is by construction the residue $\mathrm{Res}_{z=0}\gamma$. The minus sign in (1.242) accounts for the use of $\gamma_-(z)$ instead of its inverse $\gamma_-(z)^{-1}$. Thus, by (1.243) we have $Y(d_1) = \beta$, which gives (1.260). □

7. Renormalization and the Riemann–Hilbert correspondence

> *"La parenté de plus en plus manifeste entre le groupe de Grothendieck–Teichmüller d'une part, et le groupe de renormalisation de la Théorie Quantique des Champs n'est sans doute que la première manifestation d'un groupe de symétrie des constantes fondamentales de la physique, une espèce de groupe de Galois cosmique!"*
>
> Pierre Cartier, [**42**]

In this passage, written in 2000, Cartier conjectured the existence of a hidden group of symmetries, closely related in structure to certain arithmetic Galois groups, and acting on the constants of physical theories in a way related to the action of the renormalization group.

In this section, which is based on the result of our work [**87**], [**89**], we verify this conjecture, by identifying Cartier's "cosmic Galois group" as a universal group of symmetries that organizes the structure of the divergences in perturbative quantum field theory. We show that the group can also be realized as a motivic Galois group, and has therefore precisely the type of arithmetic nature expected by Cartier.

This group acts on the constants of physical theories. In fact, it maps to the group of diffeographisms of any given renormalizable theory by a representation determined by the beta function of the theory. Therefore, it also maps to the group of formal diffeomorphisms of the coupling constants by the results of §6.5 above.

We will also see that the renormalization group sits naturally as a 1-parameter subgroup of this universal group and this provides an interpretation of the renormalization group as Galois symmetries.

The main steps, which we are going to discuss in detail in the rest of this section, are summarized as follows.

- The data of counterterms as iterated integrals (Gross–'t Hooft relations).
- A geometric formulation: flat equisingular connections.
- The local Riemann–Hilbert correspondence: differential Galois group through the Tannakian formalism.
- The universal singular frame and universal symmetries.

7.1. Counterterms and time-ordered exponentials.

The first result we describe here (Theorem 1.58 below) is a refinement of Theorem 2 of [**83**], which proved the analog in the context of the CK theory of a well known result of D. Gross ([**151**] §4.5) and 't Hooft (unpublished). The latter states that the counterterms in perturbative renormalization only depend on the beta function of the theory (Gross–'t Hooft relations).

We first need to discuss the mathematical formulation of the *time-ordered exponential.*

We assume that $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$ is a positively graded connected commutative Hopf algebra over $\mathbb{C}$, with G its affine group scheme. We have $\mathfrak{g} = \operatorname{Lie} G$ as in Definition 1.21. When the $\mathcal{H}_n$ are finite-dimensional vector spaces, the Lie algebra $\mathfrak{g}$ is related to the dual Hopf algebra $\mathcal{H}^\vee$ as in Theorem 1.22.

Let H be a Lie group and $t \mapsto \alpha(t)$ a smooth map from the interval $[a,b]$ to the Lie algebra of H.

Dyson's time-ordered exponential $\mathrm{T}e^{\int_a^b \alpha(t)\,dt}$ is a notation for the parallel transport in the trivial H-principal bundle $[a,b]\times H$ with the left action of H, endowed with the connection associated to the 1-form $\alpha(t)\,dt$ with values in the Lie algebra of H. In other words $h(u) = \mathrm{T}e^{\int_a^u \alpha(t)\,dt} \in H$ is the solution of the differential equation

$$dh(u) = h(u)\,\alpha(u)du\,, \quad h(a) = 1\,. \tag{1.261}$$

A mathematical definition can be given as an iterated integral. This type of formalism was developed in the topological context in [**55**], [**56**] (Chen's iterated integral) and in the operator algebra context in [**3**] (Araki's expansional). We give here a formulation adapted to the context of affine group schemes.

Definition 1.50. *Given a $\mathfrak{g}(\mathbb{C})$-valued smooth function $\alpha(t)$, with $t \in [a,b] \subset \mathbb{R}$, the time-ordered exponential (also called the expansional) is defined as*

$$\mathrm{T}e^{\int_a^b \alpha(t)\,dt} := 1 + \sum_1^\infty \int_{a\leq s_1\leq\cdots\leq s_n\leq b} \alpha(s_1)\cdots\,\alpha(s_n)\; ds_1\cdots ds_n\,, \tag{1.262}$$

with the product taken in $\mathcal{H}^\vee$, and with $1 \in \mathcal{H}^\vee$ the unit corresponding to the counit ε of $\mathcal{H}$.

One has the following result, which in particular shows that the expansional only depends on the 1-form $\alpha(t)dt$.

Proposition 1.51. *The time-ordered exponential* (1.262) *satisfies the following properties:*

(1) *When paired with any $X \in \mathcal{H}$ the sum* (1.262) *is finite.*
(2) *The iterated integral* (1.262) *defines an element of $G(\mathbb{C})$.*
(3) *The expansional* (1.262) *is the value $g(b)$ of the unique solution $g(t) \in G(\mathbb{C})$ with initial condition $g(a) = 1$ of the differential equation*

$$dg(t) = g(t)\,\alpha(t)\,dt\,. \tag{1.263}$$

(4) *The iterated integral is multiplicative over the sum of paths:*

$$\mathrm{T}e^{\int_a^c \alpha(t)\,dt} = \mathrm{T}e^{\int_a^b \alpha(t)\,dt}\,\mathrm{T}e^{\int_b^c \alpha(t)\,dt}. \tag{1.264}$$

(5) *Let $\rho : G(\mathbb{C}) \to H$ be a homomorphism to a Lie group H; then $\rho(\mathrm{T}e^{\int_a^b \alpha(t)\,dt})$ is the parallel transport in the principal bundle $[a,b]\times H$ endowed with the connection associated to the 1-form $\rho(\alpha(t))\,dt$.*
(6) *The inverse in $G(\mathbb{C})$ of the element* (1.262) *is of the form*

$$(\mathrm{T}e^{\int_a^b \alpha(t)\,dt})^{-1} = \mathrm{T}'e^{-\int_a^b \alpha(t)\,dt}, \tag{1.265}$$

with

$$\mathrm{T}'e^{\int_a^b \alpha(t)\,dt} := 1 + \sum_1^\infty \int_{a\leq s_1\leq\cdots\leq s_n\leq b} \alpha(s_n)\cdots\,\alpha(s_1)\; ds_1\cdots ds_n\,. \tag{1.266}$$

PROOF. (1) The elements $\alpha(t) \in \mathfrak{g}(\mathbb{C})$, viewed as linear forms on $\mathcal{H}$, vanish on any element of degree 0. Thus, for $X \in \mathcal{H}$ of degree n, one has

$$\langle \alpha(s_1) \cdots \alpha(s_m),\, X \rangle \;=\; 0 \quad \forall m > n\,,$$

so that

$$\langle \mathrm{T}e^{\int_a^b \alpha(t)\, dt}, X \rangle \tag{1.267}$$

is a finite sum.

(2) First notice that (1.267) satisfies

$$\partial_t \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X \rangle = \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds} \alpha(t), X \rangle. \tag{1.268}$$

In fact, differentiating each term in the finite sum (1.267) in the variable t amounts to fixing the last integration variable $s_n = t$ in

$$\int_{a \le s_1 \le \cdots \le s_n \le t} \langle \alpha(s_1) \cdots \alpha(s_n),\, X \rangle \; ds_1 \cdots ds_n.$$

In order to show that (1.262) defines an element in $G(\mathbb{C})$, we need to check that for all $X, Y \in \mathcal{H}$,

$$\langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, XY \rangle = \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X \rangle \; \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, Y \rangle. \tag{1.269}$$

We show this for homogeneous X and Y, by induction on the sum of their degrees. Using (1.268), we write

$$\begin{aligned} \partial_t \,\langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X\, Y \rangle = &\quad \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds} \alpha(t), X\, Y \rangle \\ = &\quad \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds} \otimes \alpha(t), \Delta(X)\, \Delta(Y) \rangle\,. \end{aligned}$$

We then write the coproduct in the form

$$\Delta(X) = X_{(1)} \otimes X_{(2)} = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$$

where only terms of lower degree appear in the last sum. Using the derivation property (1.173) of $\alpha(t) \in \mathfrak{g}(\mathbb{C})$, one gets that it pairs trivially with all the products $X_{(2)}\, Y_{(2)}$ except when $Y_{(2)} = 1$ or $X_{(2)} = 1$. This gives

$$\begin{aligned} \partial_t \,\langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X\, Y \rangle = &\quad \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X_{(1)}\, Y \rangle \; \langle \alpha(t), X_{(2)} \rangle \\ + &\quad \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X\, Y_{(1)} \rangle \; \langle \alpha(t), Y_{(2)} \rangle\,. \end{aligned}$$

We can then apply the induction hypothesis and obtain

$$\begin{aligned} \partial_t \,\langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X\, Y \rangle \; &= \partial_t \left(\langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X \rangle \right) \; \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, Y \rangle \\ &+ \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X \rangle \; \partial_t \left(\langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, Y \rangle \right). \end{aligned}$$

If we denote

$$F(t) := \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X\, Y \rangle - \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, X \rangle \; \langle \mathrm{T}e^{\int_a^t \alpha(s)\, ds}, Y \rangle$$

we obtain $\partial_t F(t) = 0$, for all t. Thus, since for $t = a$ we clearly have $F(a) = 0$, we obtain (1.269).

(3) This then follows immediately from (1.268).

(4) The property (1.264) follows from (3), since both sides satisfy equation (1.263) and agree for $c = b$, hence they are equal.

(5) This follows from (3) and (1.261).
(6) Notice that one can apply the notion of expansional to the opposite algebra. This amounts to reversing the order of the terms. Thus, (1.262) gets replaced by (1.266). As in (3) above, one gets that, for $t > 0$, the expression $g(t) = \mathrm{T}' e^{\int_0^t \alpha(u)\, du}$ is the unique solution to the ODE

$$dg(t) = \alpha(t)\, g(t)\, dt\,. \tag{1.270}$$

This shows that the inverse of a time-ordered exponential with T is given by the time-ordered exponential with T', which gives (1.265). □

Consider now an open domain $\Omega \subset \mathbb{R}^2$ and, for $(s,t) \in \Omega$, let

$$\varpi = \alpha(s,t) ds + \eta(s,t) dt \tag{1.271}$$

be a *flat* $\mathfrak{g}(\mathbb{C})$-valued connection. The flatness condition means that we have

$$\partial_s\, \eta - \partial_t\, \alpha + [\alpha, \eta] = 0. \tag{1.272}$$

The time-ordered exponential satisfies the following property

Proposition 1.52. *Let ϖ be a flat $\mathfrak{g}(\mathbb{C})$-valued connection on $\Omega \subset \mathbb{R}^2$ as above, and let $\gamma : [0,1] \to \Omega$ be a path in Ω. Then the time-ordered exponential $\mathrm{T} e^{\int_0^1 \gamma^* \varpi}$ only depends on the homotopy class $[\gamma]$ of paths with endpoints $a = \gamma(0)$ and $b = \gamma(1)$.*

This follows from Proposition 1.51, using the fact that $G(\mathbb{C})$ is a projective limit of Lie groups.

Recall that a differential field (K, δ) is a field K endowed with a derivation satisfying $\delta(f + h) = \delta(f) + \delta(h)$ and $\delta(fh) = f\delta(h) + \delta(f)h$, for all $f, h \in K$. The field of constants is the subfield $\{f \in K | \delta(f) = 0\}$.

Definition 1.53. *Let (K, δ) be a differential field with differentiation $f \mapsto f' := \delta(f)$ and with field of constants $\mathbb{C}$. The logarithmic derivative on the group $G(K) = \mathrm{Hom}_{\mathcal{A}_{\mathbb{C}}}(\mathcal{H}, K)$ is given by*

$$D(g) := g^{-1}\, g' \in \mathfrak{g}(K), \quad \forall g \in G(K), \tag{1.273}$$

where $g' = \delta(g)$ is the linear map $g' : \mathcal{H} \to K$,

$$g'(X) = \delta(g(X)), \quad \forall X \in \mathcal{H}.$$

It is not hard to check that (1.273) indeed defines an element in $\mathfrak{g}(K)$. In fact, $D(g) : \mathcal{H} \to K$ is the linear map

$$\langle D(g), X \rangle = g^{-1} \star g'\,(X) = \langle g^{-1} \otimes g', \Delta X \rangle$$

and this satisfies

$$\langle D(g), X\, Y \rangle = \langle D(g), X \rangle\, \varepsilon(Y) + \varepsilon(X)\, \langle D(g), Y \rangle, \quad \forall X, Y \in \mathcal{H}.$$

In the following, we assume that $K = \mathbb{C}(\{z\})$, the differential field of convergent Laurent series, as in (1.217), with $\delta(f) = \frac{d}{dz} f$. Later, we will also consider the case where $K = \mathbb{C}((z)) = \mathbb{C}[[z]][z^{-1}]$ is the field of formal Laurent series.

LEMMA 1.54. *Let $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$ be a positively graded connected commutative Hopf algebra over $\mathbb{C}$, with G its affine group scheme. Let $\mathcal{H}(i) \subset \mathcal{H}$ be an increasing sequence of finitely generated commutative Hopf subalgebras, with $\mathcal{H} = \cup\mathcal{H}(i)$. The affine group scheme G is the projective limit (1.171) of the linear algebraic groups G_i dual to the finitely generated commutative Hopf subalgebras $\mathcal{H}(i) \subset \mathcal{H}$. Suppose given an element $\varpi \in \mathfrak{g}(K)$, for $K = \mathbb{C}(\{z\})$.*

(1) *There is a well defined monodromy representation*

$$M_i(\varpi) : \mathbb{Z} \to G_i(\mathbb{C}).$$

(2) *The condition of trivial monodromy $M(\varpi) = 1$ is well defined in $G(\mathbb{C})$.*

PROOF. (1) For $\varpi \in \mathfrak{g}(K)$, and a fixed G_i, consider the corresponding element $\varpi_i \in \mathfrak{g}_i(K)$ obtained by restriction to $\mathcal{H}(i) \subset \mathcal{H}$. Since $\mathcal{H}(i)$ is finitely generated, ϖ_i depends on only finitely many elements in K, hence there is a radius $\rho_i > 0$ such that all these finitely many Laurent series elements converge in a punctured disk Δ_i^* of this radius around $z = 0$. Thus, we can choose a base point $z_i \in \Delta_i^*$. Composing with the evaluation $f \to f(z)$ for $z \in \Delta_i^*$ we get a flat $\mathfrak{g}_i(\mathbb{C})$-valued connection $\omega_i = \varpi_i(z)dz$ on Δ_i^*. For any smooth path $\gamma : [0,1] \to \Delta_i^*$ the expression

$$M_i(\varpi)(\gamma) := \mathrm{T}e^{\int_0^1 \gamma^*\omega_i} \in G_i(\mathbb{C}) \tag{1.274}$$

is well defined. By Proposition 1.52 and the flatness of the connection ω_i, viewed as a connection in two real variables, (1.274) depends only on the homotopy class of the loop γ in Δ_i^* with $\gamma(0) = \gamma(1) = z_i$. Thus, it defines a monodromy representation

$$M_i(\varpi) : \pi_1(\Delta_i^*, z_i) \to G_i(\mathbb{C}). \tag{1.275}$$

(2) By construction, the conjugacy class of M_i does not depend on the choice of the base point. When passing to the projective limit, one has to take care of the change of base points, but the condition of *trivial monodromy*

$$M(\varpi) = 1 \Leftrightarrow M_i(\varpi) = 1\,, \quad \forall i \tag{1.276}$$

is well defined at the level of the projective limit G of the groups G_i.

□

The monodromy encodes the obstruction to the existence of solutions to the differential equation $D(g) = \varpi$, for $g \in \mathfrak{g}(K)$.

EXAMPLE 1.55. *Let $\mathbb{G}_a$ be the additive group. Let as above $K = \mathbb{C}(\{z\})$ with $\delta(f) = \frac{d}{dz}f$. One has $\mathbb{G}_a(K) = K$, $D(f) = \delta(f) = f'$, and the residue of $\varpi \in K$ is a non-trivial obstruction to the existence of solutions of the equation $D(f) = \varpi$.*

The following result shows that the monodromy is in fact the only obstruction.

PROPOSITION 1.56. *Let $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$ be a positively graded connected commutative Hopf algebra over $\mathbb{C}$, with G its affine group scheme. Suppose given $\varpi \in \mathfrak{g}(K)$ with trivial monodromy, $M(\varpi) = 1$. Then there exists a solution $g \in G(K)$ of the equation*

$$D(g) = \varpi\,. \tag{1.277}$$

PROOF. We view, as above, G as the projective limit of the G_i and use the same notations as above. We treat the case of a fixed G_i first. We let

$$g_i(z) = \mathrm{T}e^{\int_{z_i}^{z} \omega_i}\,, \tag{1.278}$$

where by the right-hand side we mean the time-ordered exponential $\mathrm{T}e^{\int_0^1 \gamma^* \omega_i}$ for any smooth path in Δ_i^* from the base point z_i to z. The condition of trivial monodromy ensures that the result does not depend on the choice of the path γ. Let $\mathcal{H}(i)$ be as in Lemma 1.54. One needs to show that, for any $X \in \mathcal{H}(i) \subset \mathcal{H}$, the evaluation

$$h_i(z) = \langle g_i(z), X \rangle$$

is a convergent Laurent series in Δ_i^*. By construction the same property holds for $\varpi_i(z)$. Moreover, by Proposition 1.51 (1) we know that, when pairing with $X \in \mathcal{H}(i)$ only finitely many terms of the infinite sum (1.262) defining $g_i(z)$ contribute. Thus, it follows that $z^{N_i} h_i(z)$ is bounded in Δ_i^*, for sufficiently large N_i. Moreover, by (1.263) of Proposition 1.51, with $2\overline{\partial} = \partial_t + i\partial_s$ and $z = t + is$, we obtain $\overline{\partial} h_i = 0$, which gives $h_i \in K$. Thus $g_i \in G_i(K)$.

Finally (1.263) of Proposition 1.51 shows that $g_i \in G_i(K)$ is a solution of

$$D(g_i) = \varpi_i \,. \tag{1.279}$$

Given two solutions g and h of (1.279), one gets that $g\,h^{-1}$ satisfies $D(gh^{-1}) = 0$. Thus since the field of constants of K is $\mathbb{C}$, it follows that there exists an element $a \in G_i(\mathbb{C})$ such that $g = ah$. Now since the $\mathcal{H}(i) \subset \mathcal{H}$ are Hopf subalgebras of $\mathcal{H}$, the canonical projection

$$p_i : G_{i+1}(\mathbb{C}) \to G_i(\mathbb{C})$$

is surjective. Thus, in order to pass to the projective limit, one constructs by induction a projective system of solutions $g_i \in G_i(K)$, where one modifies the solution in $G_{i+1}(K)$ by left multiplication by an element of $G_{i+1}(\mathbb{C})$ so that it projects onto g_i. $\square$

EXAMPLE 1.57. *It is crucial in Proposition* 1.56 *to assume that $\mathcal{H}$ is positively graded connected. Let $\mathbb{G}_m$ be the multiplicative group. Let as above $K = \mathbb{C}(\{z\})$ with $\delta(f) = \frac{d}{dz} f$. One has $\mathbb{G}_m(K) = K^*$, $D(f) = f^{-1}\,\delta(f)$, and with $\varpi = \frac{1}{z^2} \in K$ the equation $D(f) = \varpi$ has trivial monodromy with formal solution given by*

$$f(z) = e^{-1/z}$$

but no solution in $\mathbb{G}_m(K) = K^*$.

With these general results about time-ordered exponentials in place, we come to the Gross–'t Hooft relations. In our context, the fact that the counterterms in perturbative renormalization depend only on the beta function is a consequence of the following explicit form of the negative piece of the Birkhoff factorization as a time-ordered exponential.

THEOREM 1.58. *Suppose given $\gamma_\mu(z) \in L(G(\mathbb{C}), \mu)$. Then the negative piece $\gamma_-(z)$ of the Birkhoff factorization has the explicit form*

$$\gamma_-(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)\, dt}. \tag{1.280}$$

PROOF. We write $\gamma_-(z)^{-1} = 1 + \sum_{n=1}^\infty \frac{d_n}{z^n}$, as in (1.252) of Lemma 1.48. To obtain the result we first show that the coefficients d_n are given by the formula (cf. [**83**])

$$d_n = \int_{s_1 \geq s_2 \geq \cdots \geq s_n \geq 0} \theta_{-s_1}(\beta)\, \theta_{-s_2}(\beta) \cdots \theta_{-s_n}(\beta)\; ds_1 \cdots ds_n \,. \tag{1.281}$$

To check this, notice first that, for $n = 1$, the expression (1.281) reduces to

$$d_1 = \int_0^\infty \theta_{-s}(\beta)\, ds\,. \tag{1.282}$$

This follows from the equality

$$Y^{-1}(X) = \int_0^\infty \theta_{-s}(X)\, ds, \tag{1.283}$$

for all $X \in \mathcal{H}$ with $\varepsilon(X) = 0$, where ε is the counit of $\mathcal{H}$. In fact, given α and α' in $\mathcal{H}^\vee$ satisfying $\alpha' = Y(\alpha)$ and $\langle \alpha, 1\rangle = \langle \alpha', 1\rangle = 0$, the identity (1.283) implies

$$\alpha = \int_0^\infty \theta_{-s}(\alpha')\, ds\,.$$

This gives (1.282) when applied to $\beta = Y(d_1)$ of Corollary 1.49. The general formula (1.281) then follows by induction, applying the relations between the coefficients d_n proved in Lemma 1.48 and the fact that the 1-parameter family θ_t generated by the grading of $\mathcal{H}$ also acts as automorphisms of $\mathcal{H}^\vee$.

To get (1.280) from (1.281), we use (6) of Proposition 1.51. Using (1.281) we get

$$\gamma_-(z)^{-1} = \mathrm{T}' e^{\frac{1}{z}\int_0^\infty \theta_{-s}(\beta)\, ds}$$

and the required equality (1.280) follows from (1.265). □

Remark 1.59. *One does not have to worry about the convergence issue in dealing with the iterated integrals* (1.281) *in* (1.280).

In fact, when evaluating (1.281) on elements $X \in \mathcal{H}$, one obtains for $\langle d_n, X\rangle$ the expression

$$\int_{s_1 \geq \cdots \geq s_n \geq 0} \langle \beta \otimes \cdots \otimes \beta, \theta_{-s_1}(X_{(1)}) \otimes \theta_{-s_2}(X_{(2)}) \otimes \cdots \otimes \theta_{-s_n}(X_{(n)})\rangle ds_1 \cdots ds_n,$$

where we used the notation

$$\Delta^{(n-1)}(X) = \sum X_{(1)} \otimes X_{(2)} \otimes \cdots \otimes X_{(n)}\,.$$

The convergence of the iterated integral is exponential. In fact, we have the estimate $\langle \beta, \theta_{-s}(X_{(i)})\rangle = O\left(e^{-s}\right)$ for $s \to +\infty$. If X is homogeneous of degree $\deg(x)$ and if $n > \deg(x)$, then at least one of the $X_{(i)}$ has degree 0, and $\langle \beta, \theta_{-s}(X_{(i)})\rangle = 0$ so that $\langle d_n, X\rangle$ vanishes. This implies that, when pairing $\gamma_-(z)^{-1}$ with $X \in \mathcal{H}$, only finitely many terms contribute to

$$\langle \gamma_-(z)^{-1}, X\rangle = \varepsilon(X) + \sum_{n=1}^\infty \frac{1}{z^n}\, \langle d_n, X\rangle\,.$$

This remark will be useful when we compare the non-formal case, where $K = \mathbb{C}(\{z\})$, with the formal case, where $K = \mathbb{C}((z))$ (cf. §7.6 below).

7.2. Flat equisingular connections.

The main advantage of introducing the formalism of time-ordered exponentials as iterated integrals, as in Definition 1.50, is their property of being characterized as solutions of a differential equation (1.263), as in Proposition 1.51. In fact, this makes it possible to translate data described by time-ordered exponentials into a class of differential equations.

Following this idea, we obtain in this section a geometric characterization of the divergences in renormalizable quantum field theory, in terms of a suitable class of differential systems.

As a first step, we give a more precise characterization of the class $L(G(\mathbb{C}),\mu)$ of loops $\gamma_\mu(z)$ satisfying the two conditions (1.240) and (1.241), as in Definition 1.45.

For instance, given any loop $\gamma_{\rm reg}(z)$ which is regular at $z=0$, one can easily obtain an element $\gamma_\mu \in L(G(\mathbb{C}),\mu)$ by setting $\gamma_\mu(z)=\theta_{z\log\mu}(\gamma_{\rm reg}(z))$. The following result shows that the only remaining piece of data that is necessary, in order to specify completely the loops that belong to the class $L(G(\mathbb{C}),\mu)$, is the choice of an element β in the Lie algebra of the affine group scheme G.

THEOREM 1.60. *Suppose given a loop $\gamma_\mu(z)$ in the class $L(G(\mathbb{C}),\mu)$. Then the following properties hold.*

(1) *The loop $\gamma_\mu(z)$ has the form*

$$\gamma_\mu(z) = \mathrm{T}e^{-\frac{1}{z}\int_\infty^{-z\log\mu}\theta_{-t}(\beta)\,dt}\;\theta_{z\log\mu}(\gamma_{\rm reg}(z)), \tag{1.284}$$

for a unique $\beta\in\mathfrak{g}(\mathbb{C})$, with $\gamma_{\rm reg}(z)$ a loop regular at $z=0$.

(2) *The Birkhoff factorization of $\gamma_\mu(z)$ has the form*

$$\begin{aligned} \gamma_{\mu^+}(z) &= \mathrm{T}e^{-\frac{1}{z}\int_0^{-z\log\mu}\theta_{-t}(\beta)\,dt}\,\theta_{z\log\mu}(\gamma_{\rm reg}(z))\,,\\ \gamma_-(z) &= \mathrm{T}e^{-\frac{1}{z}\int_0^{\infty}\theta_{-t}(\beta)\,dt}\,. \end{aligned} \tag{1.285}$$

(3) *Conversely, given an element $\beta\in\mathfrak{g}(\mathbb{C})$ and a regular loop $\gamma_{\rm reg}(z)$, the expression* (1.284) *gives an element $\gamma_\mu\in L(G(\mathbb{C}),\mu)$.*

PROOF. (1) Let $\gamma_\mu(z)$ be a loop in $L(G(\mathbb{C}),\mu)$, with Birkhoff factorization $\gamma_\mu(z) = \gamma_-(z)^{-1}\gamma_{\mu^+}(z)$. Consider the loop

$$\alpha_\mu(z) := \theta_{z\log\mu}(\gamma_-(z)^{-1}). \tag{1.286}$$

This satisfies the scaling property (1.240) by construction,

$$\alpha_{e^s\mu}(z) = \theta_{sz}(\alpha_\mu(z)). \tag{1.287}$$

Consider the ratio $\alpha_\mu(z)^{-1}\gamma_\mu(z)$. This still satisfies the scaling property (1.240). Moreover, the loop $\alpha_\mu(z)^{-1}\gamma_\mu(z)$ is regular at $z=0$. In fact, this holds for $\mu=1$ and hence all values of μ by the scaling property (1.240). Thus, it can be written in the form

$$\alpha_\mu(z)^{-1}\gamma_\mu(z) = \theta_{z\log\mu}(\gamma_{\rm reg}(z)), \tag{1.288}$$

with $\gamma_{\rm reg}(z)$ a regular loop.

Notice then that, using (1.286) and (1.280) of Theorem 1.58, we obtain

$$\alpha_\mu(z)^{-1} = \mathrm{T}e^{-\frac{1}{z}\int_{-z\log\mu}^{\infty}\theta_{-t}(\beta)\,dt}. \tag{1.289}$$

Thus, using (1.264), we obtain (1.284).

(2) Given $\gamma_\mu \in L(G(\mathbb{C}),\mu)$, we write it in the form (1.284) as above. Thus, we have

$$\gamma_\mu(z)^{-1} = \theta_{z\log\mu}(\gamma_{\rm reg}(z))^{-1}\mathrm{T}e^{-\frac{1}{z}\int_{-z\log\mu}^{\infty}\theta_{-t}(\beta)\,dt}.$$

We write the second term, using (1.264), as the product

$$\mathrm{T}e^{-\frac{1}{z}\int_{-z\log\mu}^{0}\theta_{-t}(\beta)\,dt}\,\mathrm{T}e^{-\frac{1}{z}\int_{0}^{\infty}\theta_{-t}(\beta)\,dt}.$$

Thus, we have obtained the expression

$$\gamma_\mu(z)^{-1} = \theta_{z\log\mu}(\gamma_{\rm reg}(z))^{-1}\ \mathrm{T}e^{-\frac{1}{z}\int_{-z\log\mu}^{0}\theta_{-t}(\beta)\,dt}\gamma_-(z),$$

where $\gamma_-(z)$ is a regular function of $1/z$ with $\gamma_-(\infty)=1$. We then need to check the regularity at $z=0$ of the remaining part

$$\mathrm{T}e^{-\frac{1}{z}\int_{0}^{-z\log\mu}\theta_{-t}(\beta)\,dt}\theta_{z\log\mu}(\gamma_{\rm reg}(z)).$$

It is enough to show that we have

$$\lim_{z\to 0}\mathrm{T}e^{-\frac{1}{z}\int_{0}^{-sz}\theta_{-t}(\beta)\,dt} = e^{s\beta}. \tag{1.290}$$

To see this we take the straight path from 0 to $-sz$ in the form $\pi(u) = -szu$ for $u\in[0,1]$ and write

$$\mathrm{T}e^{-\frac{1}{z}\int_{0}^{-sz}\theta_{-t}(\beta)\,dt} = \mathrm{T}e^{\int_0^1\alpha(u)\,du}$$

where the form $\alpha(u)\,du$ is the pullback by π of the form $-\frac{1}{z}\,\theta_{-t}(\beta)\,dt$. So far both s and z are fixed. One has

$$\alpha(u)\,du = -\frac{1}{z}\,\theta_{szu}(\beta)\,(-sz)\,du = s\,\theta_{szu}(\beta)\,du$$

This gives

$$\mathrm{T}e^{-\frac{1}{z}\int_{0}^{-sz}\theta_{-t}(\beta)\,dt} = \mathrm{T}e^{s\int_0^1\theta_{szu}(\beta)\,du}$$

which converges when $z\to 0$ to $e^{s\beta}$. We then obtain the Birkhoff factorization in the form (1.285).

(3) Given a choice of an element $\beta\in\mathfrak{g}(\mathbb{C})$ and a regular loop $\gamma_{\rm reg}(z)$, the expression (1.284) satisfies the scaling property (1.240) by construction. The formula (1.285) for the Birkhoff factorization also shows clearly that the negative piece is independent of μ so that (1.241) is also satisfied, hence $\gamma_\mu\in L(G(\mathbb{C}),\mu)$. $\square$

In the rest of this section, we use the classification result of Theorem 1.60, together with the properties of the time-ordered exponential, to give a geometric reformulation of the class $L(G(\mathbb{C}),\mu)$ in terms of a class of differential systems associated to a family of singular flat connections.

We introduce some notation. Let (K,δ) be a differential field with field of constants $\mathbb{C}$. We will assume $K=\mathbb{C}(\{z\})$ with $\delta(f)=f'$. We have $\mathcal{O}=\mathbb{C}\{z\}$, the subring of convergent power series as in (1.219).

We denote by $\Omega^1(\mathfrak{g})$ the space of 1-forms on K with values in $\mathfrak{g}$. These are all of the form $A(z)\,dz$ for some $A\in\mathfrak{g}(K)$. The differential

$$d:K\to\Omega^1$$

is given by $df=\delta(f)\,dz=\frac{df}{dz}\,dz$.

The logarithmic derivative D on $G(K)$, as in (1.273) of Definition 1.53, induces an operator (which, with a slight abuse of notation, we still call D)

$$D : G(K) \to \Omega^1(\mathfrak{g}), \qquad Df = f^{-1}\, df. \tag{1.291}$$

This satisfies the property

$$D(fh) = Dh + h^{-1}\, Df\, h. \tag{1.292}$$

One can then consider differential equations of the form

$$Df = \varpi \tag{1.293}$$

for $\varpi \in \Omega^1(\mathfrak{g})$.

We can think of the element $\varpi \in \Omega^1(\mathfrak{g})$ as a connection on the trivial principal G-bundle over an infinitesimal punctured disk Δ^* around $z = 0$. In fact, a connection on the trivial principal G-bundle $\Delta^* \times G$ is specified by the restriction of the connection form to $\Delta^* \times 1$, i.e. by a $\mathfrak{g}$-valued 1-form ϖ on Δ^*, which is the same as an element in $\Omega^1(\mathfrak{g})$. We use the principal bundle point of view in §7.3 below. For the moment, we simply define a connection as a $\mathfrak{g}$-valued 1-form ϖ.

We then have the following natural equivalence relation for connections $\varpi \in \Omega^1(\mathfrak{g})$.

DEFINITION 1.61. *Two connections ϖ and ϖ' are equivalent iff they are gauge conjugate by an element regular at $z = 0$, i.e.*

$$\varpi' = Dh + h^{-1}\varpi\, h, \qquad \textit{with} \;\; h \in G(\mathcal{O}). \tag{1.294}$$

As in Lemma 1.54, the condition of trivial monodromy $M(\varpi) = 1$ is well defined and it ensures the existence of solutions to (1.293). Suppose given a solution $f \in G(K)$ to (1.293). We know, by the same argument as used in Theorem 1.39, that loops $f \in G(K)$ have a unique Birkhoff factorization

$$f = f_-^{-1} f_+, \tag{1.295}$$

with $f_+ \in G(\mathcal{O})$ and $f_- \in G(\tilde{\mathcal{Q}})$ satisfying $\varepsilon_- \circ f_- = \varepsilon$ as in Definition 1.38.

The following result then explains in more detail the meaning of the equivalence relation of Definition 1.61.

PROPOSITION 1.62. *Suppose given two connections ϖ and ϖ' in $\Omega^1(\mathfrak{g})$ with trivial monodromy. Then ϖ and ϖ' are equivalent, in the sense of* (1.294)*, iff there exist solutions to the corresponding equations* (1.293) *that have the same negative pieces of the Birkhoff factorization,*

$$\varpi \sim \varpi' \Longleftrightarrow f_-^{\varpi} = f_-^{\varpi'}, \tag{1.296}$$

for some f^{ϖ} and $f^{\varpi'}$ in $G(K)$ satisfying $D(f^{\varpi}) = \varpi$ and $D(f^{\varpi'}) = \varpi'$.

PROOF. By Lemma 1.54, we know that there exist solutions, since the monodromy is trivial. We first show that ϖ is equivalent to the 1-form $D((f_-^{\varpi})^{-1})$. We have $f^{\varpi} = (f_-^{\varpi})^{-1} f_+^{\varpi}$, hence the product rule (1.292) gives the required equivalence, since $f_+^{\varpi} \in G(\mathcal{O})$. This shows that, if $f_-^{\varpi} = f_-^{\varpi'}$ for some choices of solutions, then ϖ and ϖ' are equivalent.

Conversely, assume that ϖ and ϖ' are equivalent, i.e. $\varpi' = Dh + h^{-1}\varpi\, h$. We take a solution f^{ϖ} and let $f^{\varpi'}$ be given by

$$f^{\varpi'} = f^{\varpi}\, h\,,$$

for the above $h \in G(\mathcal{O})$. It follows from (1.292) that $D(f^{\varpi'}) = \varpi'$. The uniqueness of the Birkhoff factorization then gives $f^{\varpi}_{-} = f^{\varpi'}_{-}$. □

Definition 1.63. *Two elements f_j of $G(K)$ have the same singularity iff*

$$f_1^{-1}\, f_2 \in G(\mathcal{O}) \tag{1.297}$$

Equivalently this means that the negative parts of their Birkhoff decompositions are the same.

From the point of view of renormalization, introducing the equivalence relation of Definitions 1.61 and 1.63 means that one is interested in retaining only the information on the behavior of the divergences, which is encoded in the counterterms given by the negative piece $\gamma_-(z)$ of the Birkhoff factorization of Proposition 1.40.

In order to make this geometric setting suitable to treat the data of perturbative renormalization, i.e. the class of loops $L(G(\mathbb{C}), \mu)$, we need to account for the mass parameter μ in the geometry.

This can be done by considering a principal $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$-bundle B,

$$\mathbb{G}_m \to B \xrightarrow{\pi} \Delta\,, \tag{1.298}$$

over the infinitesimal disk Δ. We let $P = B \times G$ be the trivial principal G-bundle over the base space B.

We denote by V the fiber over $z = 0$,

$$V = \pi^{-1}(\{0\}) \subset B,$$

and by B^0 its complement

$$B^0 = B \setminus V \subset B.$$

We also fix a base point $y_0 \in V$. We let $P^0 = B^0 \times G$ denote the restriction to B^0 of the bundle P.

Remark 1.64. The physical meaning of these data is the following. In the case of a renormalizable theory $\mathcal{T}$, the group scheme is $G = \mathrm{Difg}(\mathcal{T})$. The disk Δ corresponds to the complexified dimension $z \in \Delta$ of DimReg. The principal bundle B over Δ encodes all the possible choices of normalization for the "integral in dimension $D-z$" of (1.135), hence it accounts for the presence of the mass parameter μ. The choice of a base point $y_0 \in V$ corresponds to fixing the value of the Planck constant $\hbar$. One needs to be careful and refrain from considering the fiber $\pi^{-1}(\{z\}) \subset B$ as the set of possible values of μ. It is rather the set of possible values of $\mu^z\, \hbar$. In fact in all the above discussion we restricted to $\mu \in \mathbb{R}_+^*$ and the fact that $\log \mu$ makes sense did play a role. The role of the choice of a unit of mass μ is the same as the choice of a section $\sigma : \Delta \to B$ (up to order one).

We have an action of $\mathbb{G}_m$ on B, which we write as

$$b \mapsto u(b), \quad \forall u \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*\,.$$

Since the affine group scheme G is dual to a graded Hopf algebra $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$, we also have an action of $\mathbb{G}_m$ on G, as in Lemma 1.25, induced by the action u^Y of

(1.176) on the Hopf algebra, where Y is the grading operator on $\mathcal{H}$. We denote the action of $\mathbb{G}_m$ on G also by u^Y, as the meaning is clear. Thus, we have a $\mathbb{G}_m$-action on $P = B \times G$ given by

$$u(b,g) = (u(b), u^Y(g)), \quad \forall u \in \mathbb{G}_m\,. \tag{1.299}$$

We explain in §7.3 below how this action, which as such is not given by automorphisms of a G-bundle, in fact comes from a $\mathbb{G}_m$-equivariant principal bundle over B.

We now introduce the class of connections that will be our main object of interest, as they encode geometrically the properties of the class of loops $L(G(\mathbb{C}), \mu)$.

Definition 1.65. *We say that a flat connection ϖ on P^0 is* equisingular *iff ϖ is $\mathbb{G}_m$-invariant and for any solution γ, $D\gamma = \varpi$, the restrictions of γ to sections $\sigma : \Delta \to B$ with $\sigma(0) = y_0$ have the same singularity.*

We can fix a choice of a (non-canonical) regular section $\sigma : \Delta \to B$ of the fibration (1.298), with $\sigma(0) = y_0$. Then the first condition of Definition 1.65 can be written in the form

$$\varpi(z, u(v)) = u^Y(\varpi(z,v)), \quad \forall u \in \mathbb{G}_m, \tag{1.300}$$

with $v = (\sigma(z), g)$, for $z \in \Delta$ and $g \in G$.

The second condition states that, if σ_1 and σ_2 are two sections of B as above, with $\sigma_1(0) = y_0 = \sigma_2(0)$, then

$$\sigma_1^*(\gamma) \sim \sigma_2^*(\gamma), \tag{1.301}$$

that is, the pullbacks of γ by these sections are equivalent on $\Delta^* \times G$, through the equivalence (1.297) of Definition 1.63. This means that $\sigma_1^*(\gamma)$ and $\sigma_2^*(\gamma)$ have the same singularity at the origin $z = 0$, hence the use of the term “equisingular” for ϖ. (cf. Figure 29.)

The meaning of this condition is that the pullbacks of a solution have the same negative pieces of the Birkhoff factorization, independent of the choice of the section and hence of the parameter μ, in our physical interpretation of the geometric data. In the proof of Theorem 1.67 below, we will see much more precisely this relation between equisingular connections and loops in $L(G(\mathbb{C}), \mu)$.

We introduce the following notion of equivalence for connections on P^0.

Definition 1.66. *Two connections ϖ and ϖ' on P^0 are equivalent iff*

$$\varpi' = Dh + h^{-1}\varpi h, \tag{1.302}$$

with h a G-valued $\mathbb{G}_m$-invariant map regular in B.

We again fix a choice of a (non-canonical) regular section $\sigma : \Delta \to B$, with $\sigma(0) = y_0$. For simplicity of notation, we write $\Delta \times \mathbb{G}_m$ for the base space, identified with B through the trivialization given by the section σ.

The main result of this section is the following classification of flat equisingular connections.

Theorem 1.67. *Consider the space $P = B \times G$ as above. There is a bijective correspondence between equivalence classes of flat equisingular G-connections ϖ on P^0 and elements $\beta \in \mathfrak{g}(\mathbb{C})$, with the following properties:*

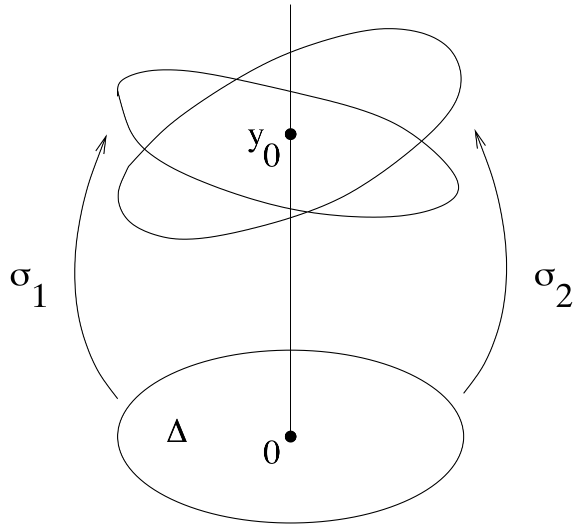


FIGURE 29. Equisingular connections have the property that, when approaching the singular fiber, the type of singularity of $\sigma^*\gamma$, for $D\gamma = \varpi$, does not depend on the section σ along which one restricts γ, but only on the value of the section at $z = 0$.

(1) *Given a flat equisingular connection ϖ, there is a unique $\beta \in \mathfrak{g}(\mathbb{C})$ such that $\varpi \sim D\gamma$ for*

$$\gamma(z,v) = \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta)\,\frac{du}{u}}, \tag{1.303}$$

with the integral performed on the straight path $u = tv$, $t \in [0,1]$.

(2) *This correspondence is independent of the choice of a local regular section $\sigma : \Delta \to B$ with $\sigma(0) = y_0$.*

PROOF. (1) We show that, for a given flat equisingular G-connection ϖ, there exists a unique element $\beta \in \mathfrak{g}(\mathbb{C})$ such that ϖ is equivalent, in the sense of Definition 1.66, to the flat equisingular connection $D\gamma$, with $\gamma(z,v)$ as in (1.303).

This will be achieved in three main steps. We first need to prove the vanishing of the monodromies $M_{\Delta^*}(\varpi)$ and $M_{\mathbb{C}^*}(\varpi)$ associated to the two generators of the fundamental group $\pi_1(B)$. This is necessary to ensure the existence of a solution to the equation $D\gamma = \varpi$. The next step is to get around the problem of the choice of the base point. This will be done by showing that an invariant connection automatically extends to the product $\Delta^* \times \mathbb{C}$ while the restriction to $\Delta^* \times \{0\}$ vanishes. Thus one can take arbitrarily a base point in $\Delta^* \times \{0\}$ and get a canonical formula for solutions corresponding to such a specific choice of base point. We then use the equisingularity condition and apply the result of Theorem 1.60 to the restriction of γ to a section of B over Δ.

Using the trivialization given by $\sigma : \Delta \to B$, we can write the connection ϖ on P^0 in terms of $\mathfrak{g}(\mathbb{C})$-valued 1-forms on B^0 as

$$\varpi = A(z,v)\,dz + B(z,v)\,\frac{dv}{v},$$

where $A(z,v)$ and $B(z,v)$ are $\mathfrak{g}$-valued functions and $\frac{dv}{v}$ is the fundamental 1-form of the principal $\mathbb{C}^*$-bundle B.

By the first condition (1.300) of equisingularity, we have

$$\varpi(z,uv) = u^Y(\varpi(z,v)),$$

which shows that the coefficients A and B of ϖ are determined by their restriction to $v = 1$. Thus, we can write ϖ as

$$\varpi(z,u) = u^Y(a(z))\,dz + u^Y(b(z))\,\frac{du}{u}, \tag{1.304}$$

for some $a, b \in \mathfrak{g}(K)$.

The flatness condition means that these coefficients satisfy

$$\frac{db}{dz} - Y(a) + [a,b] = 0. \tag{1.305}$$

The positivity of the integral grading Y on $\mathcal{H}$ shows that the connection ϖ extends to a flat connection on the product $\Delta^* \times \mathbb{C}$. Indeed for any element $a \in \mathfrak{g}(K)$ and $X \in \mathcal{H}$ the function

$$u \mapsto \langle u^Y(a), X\rangle$$

is a polynomial $P(u)$ with $P(0) = 0$. This shows that the term $u^Y(b)\,\frac{du}{u}$ in (1.304) is regular at $u = 0$. Moreover the restriction of the connection ϖ to $\Delta^* \times \{0\}$ is equal to zero, since $u^Y(a) = 0$ for $u = 0$.

Consider then the monodromy $M_{\{z_0\}\times\mathbb{C}^*}(\varpi)$ for a fixed value $z_0 \in \Delta^*$. This is trivial, because the connection ϖ extends to the simply connected domain $\{z_0\} \times \mathbb{C}$.

Notice that here we are only working with an infinitesimal disk Δ, while the argument given above works for a disk of finite radius, but in fact we can proceed as in Lemma 1.54, writing the group scheme G as an inverse limit of G_i with finite-dimensional $\mathcal{H}(i)$. We can compute the monodromy $M_{\{z_0\}\times\mathbb{C}^*}(\varpi)$ at this finite level G_i, with $z_0 \in \Delta_i^*$, using the argument above. We can then use the fact that the vanishing of the monodromy is a well defined condition in the projective limit.

Consider the monodromy $M_{\Delta^*\times\{u\}}(\varpi)$. Notice that we can choose to compute this monodromy for the value $u = 0$, but the restriction of the connection to $\Delta^* \times \{0\}$ is identically equal to 0, so this monodromy also vanishes.

We have shown that the connection ϖ has trivial monodromy along the two generators of $\pi_1(B^0) = \mathbb{Z}^2$. Thus, we know that the equation $D\gamma = \varpi$ admits solutions, and in fact we can write explicitly a solution as in Proposition 1.56, taking as the base point $(z_0, 0) \in \Delta^* \times \{0\}$. Namely, we consider a path in $\Delta^* \times \{0\}$ from $(z_0, 0)$ to $(z, 0)$ and then the straight path (z, tv), $t \in [0,1]$. By Proposition 1.52, this gives a solution of the form

$$\gamma(z,v) = \mathrm{T}e^{\int_0^v u^Y(b(z))\,\frac{du}{u}}\ , \tag{1.306}$$

with $b(z)$ as in (1.304) and with the integral performed on the straight path $u = tv$, for $t \in [0,1]$.

Notice that we have constructed an *invariant* solution of the equation $D\gamma = \varpi$, i.e. one has

$$\gamma(z,wu) = w^Y\gamma(z,u). \tag{1.307}$$

It is not true in general that an invariant connection on an equivariant bundle admits invariant solutions, precisely because of the problem of the choice of an invariant base point. This problem is dealt with above by the choice of a base point in $\Delta^* \times \{0\}$.

Let $\gamma(z) = \gamma(z,v)|_{v=1}$. By construction $\gamma(z)$ satisfies

$$\gamma(z,u) = u^Y \gamma(z). \tag{1.308}$$

Since $\gamma(z,v)$ is a solution of $D\gamma = \varpi$, the loop $\gamma(z)$ also satisfies

$$\gamma(z)^{-1} d\gamma(z) = a(z)\, dz \quad \text{and} \quad \gamma(z)^{-1} Y \gamma(z) = b(z)\,. \tag{1.309}$$

Let us denote by σ_s the section $\sigma_s(z) = (z, e^{sz})$, for $z \in \Delta$. The restriction of (1.308) to the section σ_s can be written in the form

$$\gamma_s(z) = \theta_{sz}\gamma(z). \tag{1.310}$$

The second condition (1.301) of equisingularity for ϖ implies that the negative parts of the Birkhoff factorization $\gamma_s(z) = \gamma_{s^-}(z)^{-1}\gamma_{s^+}(z)$ satisfy

$$\frac{\partial}{\partial s}\gamma_{s^-}(z) = 0. \tag{1.311}$$

It is immediate to compare (1.310) and (1.311) with the conditions (1.240) and (1.241) of Definition 1.45. Thus, in particular, we have obtained that a flat equisingular connection ϖ on P^0 determines a loop $\gamma_s(z)$ in the class $L(G(\mathbb{C}),\mu)$, for $\mu = e^s$.

We can then apply the classification result for loops in $L(G(\mathbb{C}),\mu)$, proved in Theorem 1.60. We obtain that there exists a unique element $\beta \in \mathfrak{g}(\mathbb{C})$ and a regular loop $\gamma_{\rm reg}(z)$ such that

$$\gamma(z,1) = \mathrm{T}e^{-\frac{1}{z}\int_\infty^0 \theta_{-t}(\beta)\, dt}\, \gamma_{\rm reg}(z)\,. \tag{1.312}$$

We obtain from (1.312) the expression

$$\gamma(z,v) = v^Y(\mathrm{T}e^{-\frac{1}{z}\int_\infty^0 \theta_{-t}(\beta)\, dt})\, v^Y(\gamma_{\rm reg}(z)). \tag{1.313}$$

Since v^Y is an automorphism, we also have

$$v^Y(\mathrm{T}e^{-\frac{1}{z}\int_\infty^0 \theta_{-t}(\beta)\, dt}) = \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta)\,\frac{du}{u}}, \tag{1.314}$$

with the second integral taken on the straight path $u = tv$, for $t \in [0,1]$. Thus, we can write (1.313) as

$$\gamma(z,v) = \left(\mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta)\,\frac{du}{u}}\right) v^Y(\gamma_{\rm reg}(z)). \tag{1.315}$$

Thus, using $h = v^Y(\gamma_{\rm reg}(z))$ as the regular loop realizing the equivalence (1.302), we obtain, as stated, the equivalence of flat equisingular connections

$$\varpi \sim D(\mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta)\,\frac{du}{u}}).$$

We now need to understand how the class of the solution (1.313) depends upon $\beta \in \mathfrak{g}$. Consider an equivalence $\varpi' = Dh + h^{-1}\varpi h$, with h a G-valued $\mathbb{G}_m$-invariant map regular in B, as in Definition 1.66. Since h is $\mathbb{G}_m$-invariant one has $h(z,u) = u^Y(h(z,1)$ and since we are dealing with group elements it extends regularly to $h(z,0) = 1$. Thus both the solutions γ_j and the map h are normalized to be 1 on

$\Delta^* \times \{0\}$ and the equivalence generates a relation between invariant solutions of the form

$$\gamma_2(z,u) = \gamma_1(z,u)\, h(z,u), \tag{1.316}$$

with h regular. Thus, for $j = 1,2$, the loops

$$\gamma_j(z,1) = \mathrm{T}e^{-\frac{1}{z}\int_\infty^0 \theta_{-t}(\beta_j)\,dt}$$

will have the same negative pieces of the Birkhoff factorization, $\gamma_{1^-} = \gamma_{2^-}$. This implies $\beta_1 = \beta_2$, by the equality of the residues at $z = 0$.

Finally, suppose given an element $\beta \in \mathfrak{g}(\mathbb{C})$ and consider the loop

$$\gamma(z,v) = \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta)\,\frac{du}{u}}. \tag{1.317}$$

We need to show that $\varpi = D\gamma$ is equisingular. The equivariance condition (1.300) is satisfied by construction since γ is invariant. To see that the second condition (1.301) is also satisfied, we first note that it is enough to check it for one solution γ of the equation $D\gamma = \varpi$. Indeed, any other solution is of the form $\gamma' = g\,\gamma$ with $g \in G(\mathbb{C})$. The relation between the negative parts of the Birkhoff decompositions is the conjugacy by g and this preserves the equisingularity condition. Thus we just need to check the latter on the solution γ of the form (1.317). We let $v(z) \in \mathbb{C}^*$ be a regular function of $z \in \Delta$, with $v(0) = 1$ and consider the section $v(z)\sigma(z)$ instead of the chosen trivialization $\sigma(z)$. The restriction of the solution γ to this section is given by the $G(\mathbb{C})$-valued loop

$$\gamma_v(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^{v(z)} u^Y(\beta)\,\frac{du}{u}}. \tag{1.318}$$

Then the Birkhoff factorization $\gamma_v(z) = \gamma_{v^-}(z)^{-1}\gamma_{v^+}(z)$ is given by

$$\gamma_{v^-}(z)^{-1} = \mathrm{T}e^{-\frac{1}{z}\int_0^1 u^Y(\beta)\,\frac{du}{u}}, \quad \text{and} \quad \gamma_{v^+}(z) = \mathrm{T}e^{-\frac{1}{z}\int_1^{v(z)} u^Y(\beta)\,\frac{du}{u}}. \tag{1.319}$$

In fact, we have

$$\gamma_v(z) = \left(\mathrm{T}e^{-\frac{1}{z}\int_0^1 u^Y(\beta)\,\frac{du}{u}}\right)\left(\mathrm{T}e^{-\frac{1}{z}\int_1^{v(z)} u^Y(\beta)\,\frac{du}{u}}\right), \tag{1.320}$$

where the first term in the product is a regular function of z^{-1} and gives a polynomial in z^{-1} when paired with any element of $\mathcal{H}$, while the second term is a regular function of z, as one can see using the Taylor expansion of $v(z)$ at $z = 0$, with $v(0) = 1$.

(2) This last argument, which we used to prove the equisingularity of $D\gamma$, describes explicitly the effect of changing the choice of the trivialization $\sigma : \Delta \to B$ by a factor given by a regular function $v(z) \in \mathbb{C}^*$. Thus, the same argument also gives us as a consequence the independence of the the choice of the trivialization of B.

□

We have the following result, which follows from the previous argument.

COROLLARY 1.68. *Let ϖ be a flat equisingular connection. Suppose given two such choices of trivialization $\sigma_j : \Delta \to B$, with $\sigma_2 = \alpha\,\sigma_1$, for some $\alpha(z) \in \mathbb{C}^*$ regular at $z = 0$ with $\alpha(0) = 1$. Then the regular values $\gamma_{\mathrm{reg}}(y_0)_j$ of solutions of the differential systems associated to the connections $\sigma_i^*(\varpi)$ are related by*

$$\gamma_{\mathrm{reg}}(y_0)_2 = e^{-s\beta}\,\gamma_{\mathrm{reg}}(y_0)_1, \tag{1.321}$$

with

$$s = \frac{d\alpha(z)}{dz}|_{z=0} \, .$$

REMARK 1.69. In the physics context, the relation (1.321) expresses the ambiguity inherent in the renormalization group action as coming from the fact that there is no preferred choice of a local regular section σ of B.

We have obtained, by Theorem 1.67 and Theorem 1.60, an explicit correspondence between the data $\gamma_-(z)$ for loops $\gamma_\mu(z)$ in $L(G(\mathbb{C}), \mu)$ and equivalence classes of flat equisingular connections. In particular, in the physically relevant case where $G = \mathrm{Difg}(\mathcal{T})$, this makes it possible to encode geometrically the data of the counterterms (the divergences) of the renormalizable theory $\mathcal{T}$ as a class of equisingular flat connections over the space P described above.

7.3. Equivariant principal bundles and the group $G^* = G \rtimes \mathbb{G}_m$.

The formalism of the above discussion was based on $\mathfrak{g}(\mathbb{C})$-valued differential forms on the base space B. We now relate more precisely this discussion with connections on $\mathbb{G}_m$-equivariant principal bundles over B. This will involve the affine group scheme $G^* = G \rtimes \mathbb{G}_m$ of Lemma 1.25.

To fix the notation, we recall that, given a principal bundle

$$\pi : P \to B \, , \quad \text{with} \quad P \times H \to P \, , \quad (\xi, h) \mapsto R_h(\xi) = \xi \, h, \tag{1.322}$$

with structure group H acting on the right, a connection is specified by a 1-form α on P with values in the Lie algebra $\mathfrak{h}$ of H and such that

$$\alpha|_{\pi^{-1}(b)} = h^{-1} dh \, , \quad \text{with} \quad R_a^*(\alpha) = \, (\mathrm{Ad}\, a^{-1})(\alpha), \tag{1.323}$$

where the Maurer-Cartan form $h^{-1}dh$ makes sense in the fibers $\pi^{-1}(b) \sim H$ independently of the choice of a base point. Given a section $\xi : B \to P$ and a connection on P with connection form α, the pullback $\varpi = \nabla(\xi) = \xi^*(\alpha)$ is a 1-form on B with values in the Lie algebra $\mathfrak{h}$. For any smooth map k from B to H, one has

$$\nabla(\xi \, k) = \, k^{-1} \, dk + \, k^{-1} \, \nabla(\xi) \, k. \tag{1.324}$$

In particular, the knowledge of the 1-form $\varpi = \nabla(\xi) = \xi^*(\alpha)$ uniquely determines the connection ∇. Moreover, if we fix the section ξ and the connection ∇, then looking for a flat section $\eta = \xi \, k$, with $\nabla \eta = 0$, is the same thing as trying to find a smooth map k from B to H such that

$$k^{-1} \, dk + \, k^{-1} \, \nabla(\xi) \, k = \, 0. \tag{1.325}$$

Equivalently, in terms of $h = k^{-1}$ and $\varpi = \nabla(\xi)$, it means solving the equation

$$h^{-1} \, dh = \, \varpi. \tag{1.326}$$

Let now $H_1 \subset H$ be a closed subgroup that acts on the base B. We consider the trivial H-principal bundle

$$P = B \times H \, , \quad \text{with} \;\; R_a(b, h) = \, (b, ha) \, , \quad \forall b \in B \, , \;\; h \in H \, , \;\; a \in H. \tag{1.327}$$

PROPOSITION 1.70. *1) The action*

$$h_1\,(b,h) = \,(h_1\,b, h_1\,h) \tag{1.328}$$

of H_1 on P turns this H-principal bundle into an H_1-equivariant H-principal bundle.
2) Let ∇ be a connection on P, and $\varpi = \nabla(\xi)$, where ξ is the section $\xi(b) = (b, 1)$. Then, for any $h_1 \in H_1$, the pull-back ∇' of the connection ∇ by the action of h_1 on P is the connection given by

$$\varpi' = \nabla'(\xi) = h_1^{-1}\,h_1^*(\varpi)\,h_1\,. \tag{1.329}$$

PROOF. 1) One needs to check that the projection π is equivariant, which is clear, and that the left action of h_1 commutes with the right action of the group H, which is also clear.
2) Let α be the connection 1-form of the connection ∇. Let $h_1 \in H_1$. By construction the connection 1-form α' of the pullback ∇' is the pullback of α by the left multiplication L of (1.328). Thus, we obtain

$$\nabla'(\xi) = \xi^*(\alpha') = \,\xi^*(L^*(\alpha)) = \,(L \circ \xi)^*(\alpha).$$

One has

$$L \circ \xi(b) = (h_1\,b, h_1) = \,R_{h_1}\,\xi(h_1\,b)\,, \quad \forall b \in B.$$

By (1.323) $R_{h_1}^*(\alpha) = \,(\mathrm{Ad}\,h_1^{-1})(\alpha)$, which gives the required equality (1.329). □

PROPOSITION 1.71. *1) A connection ∇ on P is H_1-invariant iff $\varpi = \nabla\xi$ satisfies the condition*

$$h_1^*(\varpi) = \,h_1\,\varpi\,h_1^{-1}\,, \quad \forall h_1 \in H_1\,. \tag{1.330}$$

2) Let γ be a smooth map from B to H such that

$$\gamma(h_1\,b) = \,h_1\,\gamma(b)\,h_1^{-1}\,, \quad \forall h_1 \in H_1\,. \tag{1.331}$$

then the gauge equivalence

$$\varpi \mapsto \varpi' = \gamma^{-1}d\gamma + \gamma^{-1}\,\varpi\,\gamma \tag{1.332}$$

preserves the H_1-invariance of connections.

PROOF. 1) This follows from (1.329).
2) Let us assume that ϖ fulfills (1.330). We show that ϖ' also does. One has

$$h_1^*(\gamma^{-1}\,\varpi\,\gamma) = \,h_1^*(\gamma)^{-1}\,h_1^*(\varpi)\,h_1^*(\gamma) = \,h_1\,\gamma^{-1}\,\varpi\,\gamma\,h_1^{-1}$$

and since $dh_1 = 0$ one has

$$h_1^*(\gamma^{-1}\,d\gamma) = \,h_1\,\gamma^{-1}\,d\gamma\,h_1^{-1},$$

so that the required invariance follows. □

We now specialize Propositions 1.70 and 1.71 to the subgroup

$$H_1 = \mathbb{G}_m \subset H = G^* = G \rtimes \mathbb{G}_m$$

and to connections with values in the Lie subalgebra

$$\mathfrak{g} = \,\mathrm{Lie}(G) \subset \mathfrak{g}^* = \,\mathrm{Lie}(G^*) \tag{1.333}$$

The adjoint action of the subgroup $H_1 = \mathbb{G}_m$ gives the grading u^Y on $\mathfrak{g} = \,\mathrm{Lie}(G)$.
A comment about notation: since we are interested in the relative situation of the inclusion $B^0 \subset B$, we adopt the notation X^0 to denote the restriction to $B^0 =$

$B \setminus V \subset B$ of an object X (bundle, connection, section, etc) defined on B. In particular we let $\tilde{P} = B \times G^*$ be the trivial principal G^*-bundle over the base space B, and $\tilde{P}^0$ its restriction to $B^0 = B \setminus V \subset B$,

$$\tilde{P}^0 = B^0 \times G^* \tag{1.334}$$

We have the following result.

PROPOSITION 1.72. *Let $\mathbb{G}_m \to B \xrightarrow{\pi} \Delta$ denote the $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$-bundle over the infinitesimal disk Δ of* (1.298). *We let $\tilde{P} = B \times G^*$ be the trivial principal G^*-bundle over the base space B.*
1) The action

$$u\,(b,k) = \,(ub, uk) \tag{1.335}$$

of $\mathbb{G}_m$ on $\tilde{P}$ turns this G^*-principal bundle into a $\mathbb{G}_m$-equivariant G^*-principal bundle.
2) Let $\xi(b) = (b,1)$. Then the map

$$\nabla \mapsto \nabla(\xi) \tag{1.336}$$

is an isomorphism between $\mathfrak{g}$-valued $\mathbb{G}_m$-invariant connections on $\tilde{P}^0$ and $\mathfrak{g}$-valued connections on B^0 fulfilling (1.300).
3) Let γ be a regular map from B to G such that

$$\gamma(ub) = \,u\,\gamma(b)\,u^{-1}\,, \quad \forall u \in \mathbb{G}_m\,. \tag{1.337}$$

Then the transformation L_γ, given by $L_\gamma(b,\,k) = (b, \gamma(b)k)$, is an automorphism of the $\mathbb{G}_m$-equivariant G^*-principal bundle $\tilde{P}$.
4) The gauge equivalence of connections under the automorphisms L_γ corresponds under the isomorphism (1.336) *to the equivalence of Definition* 1.66.

PROOF. 1) The construction of the $\mathbb{G}_m$-equivariant G^*-principal bundle follows from Proposition 1.70.
2) The invariance condition on connections given by (1.300) is identical to (1.330).
3) By construction L_γ is an automorphism of G^*-principal bundle. Condition (1.337) means that it commutes with the action of $\mathbb{G}_m$.
4) Let α be the connection form of the first connection. The connection form of the transformed connection is $L_\gamma^*(\alpha)$, so that the actions on sections are related by $\nabla' = \nabla \circ L_\gamma$. Then $\varpi' = \nabla'(\xi) = \nabla(\tilde{\gamma})$ where $\tilde{\gamma}(b) = (b,\,\gamma(b))$. Thus, we write

$$\varpi' = \,\gamma^{-1}\,d\gamma + \,\gamma^{-1}\,\varpi\,\gamma.$$

The invariance condition on gauge equivalences given by (1.337) is identical to the one used for the equivalence of equisingular connections in Definition 1.66. $\square$

It is important to understand geometrically the meaning of the restriction to $\mathfrak{g}$-valued connections in part 2) of Proposition 1.71 and to maps to G (instead of G^*) in part 3) of the same proposition. In both cases one needs to express the triviality of the image under the canonical morphism

$$\epsilon : G^* = G \rtimes \mathbb{G}_m \to \mathbb{G}_m. \tag{1.338}$$

We let $\epsilon(\tilde{P})$ be the $\mathbb{G}_m$-equivariant $\mathbb{G}_m$-principal bundle obtained from $\tilde{P}$ and the homomorphism ϵ. We denote by $\tilde{\epsilon}$ the bundle map $\tilde{\epsilon} : \tilde{P} \to \epsilon(\tilde{P})$.

LEMMA 1.73. *1) A connection ∇ on $\tilde{P}^0$ is $\mathfrak{g}$-valued iff it is compatible through $\tilde{\epsilon}$ with the trivial connection on $\epsilon(\tilde{P})$, i.e. iff $\epsilon(\nabla) = d$.*
2) Suppose given an invariant flat connection ∇ on $\tilde{P}^0$ with $\epsilon(\nabla) = d$ as above. For any section η with $\nabla\,\eta = 0$, consider the unique bundle isomorphism between the restrictions of $\tilde{P}^0$ to sections $\sigma : \Delta \to B$ with $\sigma(0) = y_0$, determined by the trivialisation η. For such an invariant flat connection ∇, consider the corresponding $\nabla(\xi)$, defined as in (1.336)*. This $\nabla(\xi)$ is equisingular iff the bundle isomorphism defined by η is regular on Δ.*
3) An invariant bundle automorphism of the $\mathbb{G}_m$-equivariant G^*-principal bundle $\tilde{P}$ comes from a map γ as in (1.337) *iff it induces the identity automorphism on $\epsilon(\tilde{P})$.*

PROOF. 1) The compatibility with the trivial connection means that the image by $\tilde{\epsilon}$ of horizontal vectors is horizontal. Let $\xi(b) = (b, 1)$ and $\varpi = \nabla(\xi)$. The compatibility holds iff $\epsilon(\varpi(X)) = 0$ for every tangent vector, i.e. iff ϖ, which is a priori $\mathfrak{g}^*$-valued, is in fact $\mathfrak{g}$-valued.
2) Let $\varpi = \nabla(\xi)$. A solution $\gamma(z, v) \in G$ of the equation $D\gamma = \varpi$ gives a flat section $\eta(b) = (b, \gamma^{-1}(b))$ of $\tilde{P}^0$. Let $\sigma_j : \Delta \to B$ with $\sigma_j(0) = y_0$. The bundle isomorphism between the $\sigma_j^* \tilde{P}^0$, associated to the trivialisation η, is given by left multiplication by a map $z \mapsto k(z) \in G^*$, such that

$$k(z)\,\gamma^{-1}(\sigma_1(z)) = \gamma^{-1}(\sigma_2(z)), \quad \forall z \in \Delta^*.$$

Thus, one has $k = \sigma_2^*(\gamma)^{-1}\,\sigma_1^*(\gamma)$ and k is regular iff the $\sigma_j^*(\gamma)$ have the same singularity.
This shows that, if a flat section η for ∇ fulfills the condition of statement 2) of the lemma, then $\varpi = \nabla(\xi)$ is equisingular. Conversely, if $\varpi = \nabla(\xi)$ is equisingular, then we get a flat section η of $\tilde{P}^0$ which fulfills the condition of 2) of the lemma.
In fact, any other flat section is of the form $\eta' = \eta g$ for some $g \in G^*(\mathbb{C})$, and this does not alter the regularity of the bundle isomorphism associated to the trivialization.
3) A bundle automorphism is given by the left multiplication by a map γ from B to G^*. It is invariant iff (1.337) holds, and it induces the identity automorphism iff $\epsilon \circ \gamma = 1$. This means that γ takes values in $G = \operatorname{Ker} \epsilon$. ☐

DEFINITION 1.74. *We say that ∇ is equisingular when it fulfills the hypothesis of 2) of Lemma* 1.73.

Thus, we have translated into this more geometric language the notion of equisingular connection of Definition 1.65.

We remarked in the proof of Theorem 1.67 that the flat invariant connections extend automatically to the "compactification" of the base B obtained by passing from $\mathbb{C}^*$ to $\mathbb{C}$. We now state this result in more geometric terms. We define the "compactification" of the base B as

$$\underline{B} = B \times_{\mathbb{G}_m} \mathbb{G}_a. \tag{1.339}$$

This is still a $\mathbb{G}_m$-space, since one has a natural action of $\mathbb{G}_m$ on the right, but it is no longer a principal $\mathbb{G}_m$-bundle, since this action admits a whole set of fixed points given by

$$B_{\mathrm{fix}} = B \times_{\mathbb{G}_m} \{0\} \subset \underline{B} = B \times_{\mathbb{G}_m} \mathbb{G}_a. \tag{1.340}$$

As above, we use the notation X^0 to indicate the restriction of a bundle X to B^0.

LEMMA 1.75. *1) The bundle $\tilde{P}$ extends canonically to a $\mathbb{G}_m$-equivariant G^*-principal bundle $\underline{\tilde{P}}$ on $\underline{B}$.*
2) Any invariant connection ∇ on $\tilde{P}^0$ with $\epsilon(\nabla) = d$ extends canonically to an invariant connection $\underline{\nabla}$ on $\underline{\tilde{P}}^0$. The restriction of $\underline{\nabla}$ to $B_{\rm fix}$ is the trivial connection d.
3) An invariant bundle automorphism γ of the $\mathbb{G}_m$-equivariant G^*-principal bundle $\tilde{P}$ with $\epsilon(\gamma) = 1$ extends to an invariant bundle automorphism of $\underline{\tilde{P}}$ whose restriction to $B_{\rm fix}$ is the identity.

PROOF. 1) Let $\underline{\tilde{P}}$ be the trivial G^*-principal bundle $\underline{\tilde{P}} = \underline{B} \times G^*$. Since the action of $\mathbb{G}_m$ on B extends canonically to an action on $\underline{B}$ we can apply Proposition 1.70 1) and get the required $\mathbb{G}_m$-equivariant G^*-principal bundle.
2) Let us show that the differential form $\varpi = \nabla\,\xi$ on B^0, where ξ is the section $\xi(b) = (b, 1)$, extends to $\underline{B}^0$ by continuity. This extension then uniquely determines the extension of ∇ as a connection on $\underline{\tilde{P}}^0$. We use a (non-canonical) trivialization of B as a $\mathbb{G}_m$-principal bundle, and write $B = \Delta \times \mathbb{G}_m$. The condition $\epsilon(\nabla) = d$ shows that the form ϖ takes values in the Lie subalgebra $\mathfrak{g} \subset \mathfrak{g}*$, where $\mathfrak{g}* = {\rm Lie}(G^*)$. Thus, it is of the form

$$\varpi = \, a(z, v)\, dz \, + \, b(z, v)\, \frac{dv}{v},$$

where $a(z, v)$ and $b(z, v)$ are $\mathfrak{g}$-valued functions and $\frac{dv}{v}$ is the fundamental 1-form of the principal $\mathbb{C}^*$-bundle B. The invariance condition means that

$$a(z, uv) = \, u^Y\, a(z, v)\,, \quad b(z, uv) = \, u^Y\, b(z, v)\,, \quad \forall u \in \mathbb{G}_m(\mathbb{C}).$$

Thus, the strict positivity of the grading Y on $\mathfrak{g}$ shows that $a(z, v)\, dz$ extends by 0 for $v = 0$ and that $u^Y\, b(z, 1)\frac{1}{u}$ extends by continuity to $\underline{B}^0$ (with a not necessarily zero value at $u = 0$) so that $b(z, v)\,\frac{dv}{v}$ extends to $\underline{B}^0$.
3) The invariant bundle automorphism is given by left multiplication L_γ by $\gamma(z, v) \in G$ fulfilling

$$\gamma(z, uv) = \, u^Y \gamma(z, v)\,, \quad \forall u \in \mathbb{G}_m(\mathbb{C}).$$

Since the grading is strictly positive on the kernel of the augmentation ideal of the Hopf algebra of the affine group scheme G, one gets that $u^Y\, \gamma(z, 1)$ extends by continuity to $\gamma(z, 0) = 1$. □

REMARK 1.76. In the physics interpretation of Remark 1.64, the fiber

$$\pi^{-1}(\{z\}) \subset B$$

over z is the set of possible values of $\mu^z\,\hbar$. Thus the compactification $\underline{B}$ corresponds then to adding the classical limit $\hbar = 0$.

7.4. Tannakian categories and affine group schemes.

In the previous sections we have translated the data of perturbative renormalization from loops to equisingular flat connections. This makes it possible to use the Riemann–Hilbert correspondence and classify the equisingular flat connections in terms of representation theoretic data.

We begin by recalling some general facts about the Tannakian formalism. This gives a very general setting in which a given set of data with suitable properties (a neutral Tannakian category) can be shown to be equivalent to the category finite-dimensional linear representations of an affine group scheme.

An abelian category is a category to which the usual tools of homological algebra apply. This is made precise by the following definition (cf. e.g. [**139**], §2.2).

Definition 1.77. *A category $\mathcal{C}$ is an abelian category if the following axioms are satisfied.*

- *For any $X, Y \in \mathrm{Obj}(\mathcal{C})$, the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an abelian group, with respect to which the composition of morphisms is bi-additive.*
- *There is an object $0 \in \mathrm{Obj}(\mathcal{C})$ such that $\mathrm{Hom}_{\mathcal{C}}(0,0)$ is the trivial group.*
- *There are finite products and coproducts, namely, for each $X, X' \in \mathrm{Obj}(\mathcal{C})$ there exists a $Y \in \mathrm{Obj}(\mathcal{C})$ and morphisms*

$$X \xrightarrow{f_1} Y \xleftarrow{f_2} X' \quad \text{and} \quad X \xleftarrow{h_1} Y \xrightarrow{h_2} X',$$

  with $h_1 f_1 = 1_X$, $h_2 f_2 = 1_{X'}$, $h_2 f_1 = 0 = h_1 f_2$, $f_1 h_2 + f_2 h_1 = 1_Y$.
- *For any $X, Y \in \mathrm{Obj}(\mathcal{C})$, every morphism $f : X \to Y$ has a canonical decomposition*

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K',$$

  where $j \circ i = f$, with $K = \mathrm{Ker}(f)$, $K' = \mathrm{Coker}(f)$, and $I = \mathrm{Ker}(k) = \mathrm{Coker}(c)$.

We need to consider more structure, in order to arrive at something that has the right properties to compare with the category of representations of an affine group scheme.

Definition 1.78. *A category $\mathcal{C}$ is k-linear for a field k if, for any $X, Y \in \mathrm{Obj}(\mathcal{C})$, the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a k-vector space. A tensor category over k is a k-linear category $\mathcal{C}$, endowed with a bi-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a distinguished object $1 \in \mathrm{Obj}(\mathcal{C})$ satisfying the following property:*

- *There are functorial isomorphisms*

$$a_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

$$l_X : X \otimes 1 \to X \quad \text{and} \quad r_X : 1 \otimes X \to X.$$

  One assumes in the commutativity property that $c_{Y,X} = c_{X,Y}^{-1}$.

A rigid tensor category is a tensor category $\mathcal{C}$ over k with a duality $\vee : \mathcal{C} \to \mathcal{C}^{op}$, satisfying the following properties:

- *For any $X \in \mathrm{Obj}(\mathcal{C})$ the functor $- \otimes X^\vee$ is left adjoint to $- \otimes X$ and the functor $X^\vee \otimes -$ is right adjoint to $X \otimes -$.*
- *There is an evaluation morphism $\epsilon : X \otimes X^\vee \to 1$ and a unit morphism $\delta : 1 \to X^\vee \otimes X$ satisfying $(\epsilon \otimes 1) \circ (1 \otimes \delta) = 1_X$ and $(1 \otimes \epsilon) \circ (\delta \otimes 1) = 1_{X^\vee}$.*

One assumes that $\mathrm{End}(1) \cong k$.

Remark 1.79. *There is a well-defined notion of dimension for objects in a rigid tensor category, with values in $k \cong \mathrm{End}(1)$, given by $\dim(X) := \mathrm{Tr}(1_X) = \epsilon \circ c_{X^\vee,X} \circ \delta$.*

We follow here the terminology of [**106**] p.165. Elsewhere in the literature the same quantity $\mathrm{Tr}(1_X)$ is referred to as the Euler characteristic. The reason for considering it a "dimension" lies in the result recalled in Remark 1.82 below. For the relation in the motivic context of this notion to the usual Euler characteristic and its effect on the Tannakian property of the category of pure motives see [**2**].

Recall also the following properties of functors.

DEFINITION 1.80. *A functor $\omega : \mathcal{C} \to \mathcal{C}'$ is faithful if, for all $X, Y \in \mathrm{Obj}(\mathcal{C})$, the mapping*

$$\omega : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(\omega(X), \omega(Y)) \tag{1.341}$$

is injective. If $\mathcal{C}$ and $\mathcal{C}'$ are k-linear categories, a functor ω is additive if (1.341) *is a k-linear map. An additive functor ω is exact if, for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$, the corresponding sequence $0 \to \omega(X) \to \omega(Y) \to \omega(Z) \to 0$ in $\mathcal{C}'$ is also exact. A functor $\omega : \mathcal{C} \to \mathcal{C}'$ between k-linear tensor categories is a tensor functor if there are functorial isomorphisms $\tau_1 : \omega(1) \to 1$ and*

$$\tau_{X,Y} : \omega(X \otimes Y) \to \omega(X) \otimes \omega(Y).$$

We further enrich the structure of a rigid tensor category by mapping it to a category of vector spaces through a fiber functor. This is the extra piece of structure that will make it possible to recover the affine group scheme from the category.

DEFINITION 1.81. *Let $\mathcal{C}$ be a k-linear rigid abelian tensor category. Let Vect_K denote the category of finite-dimensional vector spaces over a field K. A fiber functor is an exact faithful tensor functor $\omega : \mathcal{C} \to \mathrm{Vect}_K$. A k-linear rigid abelian tensor category $\mathcal{C}$ is a Tannakian category if it admits a fiber functor $\omega : \mathcal{C} \to \mathrm{Vect}_K$, with K an extension of the field k. The category $\mathcal{C}$ is a neutral Tannakian category if $K = k$.*

REMARK 1.82. *Deligne showed in* [**106**] *that a rigid abelian tensor category $\mathcal{C}$ over a field k of characteristic zero is a Tannakian category if and only if the dimensions* $\dim(X) \in \mathrm{End}(1) = k$ *are non-negative integers for all objects $X \in \mathrm{Obj}(\mathcal{C})$ (cf. Remark* 1.79*).*

The main example of a neutral Tannakian category is the category Rep_G of finite-dimensional linear representations of an affine group scheme G over k, with fiber functor the forgetful functor to vector spaces. The point of the Tannakian formalism is that all neutral Tannakian categories are of the form Rep_G.

Suppose given a neutral Tannakian category $\mathcal{C}$, with fiber functor $\omega : \mathcal{C} \to \mathrm{Vect}_k$. To see that there is an associated affine group scheme, consider first the group $G(k)$ of automorphisms of ω, i.e. invertible natural transformations that are compatible with all the structure on the category $\mathcal{C}$. If K is an extension of k, then there is an induced functor $\omega_K : \mathcal{C} \to \mathrm{Vect}_K$ by $\omega_K(X) = \omega(X) \otimes_k K$ and one can associate to it the group $G(K)$. Similarly, one can make sense of $G(A)$ for A a unital commutative k-algebra and check that one obtains in this way a covariant representable functor to groups $G : \mathcal{A}_k \to \mathcal{G}$, i.e. an affine group scheme.

We have the following result ([**259**], [**114**]).

Theorem 1.83. *Let $\mathcal{C}$ be a neutral Tannakian category, with fiber functor $\omega : \mathcal{C} \to \mathrm{Vect}_k$. Let $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ be the affine group scheme of automorphisms of the fiber functor. Then ω induces an equivalence of rigid tensor categories*

$$\omega : \mathcal{C} \to \mathrm{Rep}_G.$$

The affine group scheme G is determined by the pair $(\mathcal{C}, \omega)$, but when no confusion can arise as to the fiber functor ω, we simply refer to G as the Galois group of $\mathcal{C}$.

Remark 1.84. *If $\mathcal{C}' \subset \mathcal{C}$ is a subcategory of a neutral Tannakian category $\mathcal{C}$, which inherits from $\mathcal{C}$ the structure of neutral Tannakian category, then there is a corresponding homomorphism of group schemes $G \to G'$, where $\mathcal{C} \cong \mathrm{Rep}_G$ and $\mathcal{C}' = \mathrm{Rep}_{G'}$.*

The properties of the affine group scheme G reflect the properties of the neutral Tannakian category $\mathcal{C}$. We mention one property that will play a role in §8 below. A k-linear abelian category $\mathcal{C}$ is semi-simple if there exists $A \subset \mathrm{Obj}(\mathcal{C})$ such that all objects X in A are simple (namely $\mathrm{Hom}(X,X) \simeq k$), with $\mathrm{Hom}(X,Y) = 0$ for $X \neq Y$ in A, and such that every object of $\mathcal{C}$ is isomorphic to a direct sum of objects in A. A linear algebraic group over a field of characteristic zero is reductive if every finite-dimensional linear representation is a direct sum of irreducibles. An affine group scheme is pro-reductive if it is a projective limit of reductive linear algebraic groups. These properties are related as follows.

Remark 1.85. *Let G be the affine group scheme $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ of a neutral Tannakian category $\mathcal{C}$ with fiber functor ω. Then G is pro-reductive if and only if the category is semi-simple.*

We look at a concrete case, to illustrate the result of Theorem 1.83. Consider the category Rep_H of finite dimensional complex linear representations of a group H. This is a neutral Tannakian category with fiber functor the forgetful functor to $\mathrm{Vect}_{\mathbb{C}}$. Theorem 1.83 then shows that there exists an affine group scheme G, dual to a commutative Hopf algebra $\mathcal{H}$ over $\mathbb{C}$, such that the fiber functor gives an equivalence of categories $\mathrm{Rep}_H \cong \mathrm{Rep}_G$. This G is called the *algebraic hull* of H and can be quite non-trivial even in very simple cases. For instance, consider the following example (cf. [**283**]).

Example 1.86. *Consider the group $H = \mathbb{Z}$. In this case $\mathrm{Rep}_H \cong \mathrm{Rep}_G$, where G is the affine group scheme dual to the Hopf algebra $\mathcal{H} = \mathbb{C}[e(q), t]$, for $q \in \mathbb{C}/\mathbb{Z}$, with the relations $e(q_1 + q_2) = e(q_1)e(q_2)$ and the coproduct $\Delta(e(q)) = e(q) \otimes e(q)$ and $\Delta(t) = t \otimes 1 + 1 \otimes t$.*

The Tannakian formalism was initially proposed by Grothendieck as a "linear version" of the theory of fundamental groups. The main idea is that the group of symmetries (fundamental group in the case of covering spaces, Galois group in the case of algebraic equations) always arises as the group of automorphisms of a fiber functor. In the theory of (profinite) fundamental groups, one considers a functor ω from a certain category of finite étale covers to the category of finite sets. The profinite group $G = \mathrm{Aut}(\omega)$ of automorphisms of this functor determines an equivalence between the category of covers and that of finite G-sets. (The reader interested in seeing more about the case of fundamental groups should look at SGA1 [**154**].)

The Tannakian formalism was then developed by Saavedra [**259**] and Deligne–Milne [**114**] (cf. also the more recent [**106**]). The fiber functor ω with values in vector spaces can be thought of as a "linear version" of the case with values in finite sets, and the affine group scheme $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ plays the role of the Galois group (or the fundamental group) in this setting.

In a very different context, a philosophy very similar to the Tannakian formalism can be found in results by Doplicher and Roberts [**124**], obtained with operator algebra techniques in the context of algebraic quantum field theory. There one recovers a global compact gauge group G from its category of finite-dimensional continuous unitary representations. This category is characterized as a monoidal C^*-category whose objects are endomorphisms of certain unital C^*-algebras and whose morphisms are intertwining operators between these endomorphisms. A characterization in terms of integer dimensions similar to the one found by Deligne for Tannakian categories also holds in this context (cf. Remark 1.82).

7.5. Differential Galois theory and the local Riemann–Hilbert correspondence.

For our purposes, we are interested in the Tannakian formalism applied to categories of differential systems, where it has a fundamental role in the context of differential Galois theory and the (local) Riemann–Hilbert correspondence (cf. [**222**], [**284**]).

We recall briefly, in the rest of this section, some general facts about differential Galois theory and the Riemann–Hilbert correspondence, before analyzing the case of equisingular connections in the next section.

Differential Galois theory was discovered by Picard and Vessiot at the beginning of the twentieth century, but it remained for a long time a quite intractable problem to compute the differential Galois group of a given differential equation. All results were confined to the regular singular case, in which it is the Zariski closure of the representation of monodromy, by an old result of Schlesinger. This situation has changed drastically since the work of Deligne, Malgrange, Ramis and Ecalle. In short, Martinet and Ramis discovered a natural generalization of the notion of monodromy that plays in the local analysis of irregular singularities the same role as monodromy does in the regular case.
The fundamental result is that an irregular singular differential equation generates a representation of the wild fundamental group,

$$\pi_1^{\mathrm{wild}} \to \mathrm{GL}_n(\mathbb{C})$$

and this representation classifies the equation up to gauge equivalence. Moreover the differential Galois group of the differential equation is the Zariski closure of the image of π_1^{wild} in $\mathrm{GL}_n(\mathbb{C})$. This holds both in the local and the global case and extends to the irregular case the classical (i.e. regular-singular) Riemann-Hilbert correspondence.

Regular linear differential equation D modulo gauge equivalence	Representation of π_1 in $\mathrm{GL}_n(\mathbb{C})$
$\mathrm{Gal}_K(D)$	Zariski closure of $\mathrm{Im}\,\pi_1$
Linear differential equation D modulo gauge equivalence	Representation of π_1^{wild} in $\mathrm{GL}_n(\mathbb{C})$
$\mathrm{Gal}_K(D)$	Zariski closure of $\mathrm{Im}\,\pi_1^{\mathrm{wild}}$

In the usual setting of differential Galois theory (see e.g. [**284**]), one considers a differential field (K, δ) with field of constants $\mathbb{C}$. The two main cases for the local theory are the formal theory where $K = \mathbb{C}((z))$ and the non-formal theory where $K = \mathbb{C}(\{z\})$.

A system of ordinary differential equations of the form

$$\delta(u) = Au, \tag{1.342}$$

with $A = (a_{ij})$ an $n \times n$ matrix, determines a differential module with $V = K^n$ and the connection ∇ given by $\nabla(e_i) = -\sum a_{ji} e_j$, with the e_i a basis of V over K. The equation (1.342) is then equivalent to the condition $\nabla(\sum_i u_i e_i) = 0$. A linear differential equation

$$Df = 0\,, \quad \text{with} \quad D = \sum_{j=0}^{n} a_j\, \delta^j \tag{1.343}$$

of order n determines a system, using the companion matrix of the polynomial expression of D.

In general, an equation of the form (1.342) will have at most n solutions in K that are linearly independent over the field of constants, but in general it might not have a full set of solutions in K. In that case, one can consider the Picard–Vessiot extension of K determined by (1.342). This is the smallest differential field extension E of K (the derivation of E restricts to that of K), with the same field of constants, such that (1.342) has a set of n independent solutions in E. The fact that the field of constants is algebraically closed ensures the existence of Picard–Vessiot extensions. The differential Galois group of the equation (1.342) is the group $G(E/K)$ of differential automorphisms of the Picard–Vessiot extension E of (1.342) that fix K. Its action on a system of n independent solutions in E determines a natural faithful representation of $G(E/K)$ in $\mathrm{GL}_n(\mathbb{C})$ and its image is an algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$.

To understand the meaning of the differential Galois group let us mention two very simple examples and a general result on the *solvability* of equations.

EXAMPLE 1.87. *Let $a \in K$ be an element which does not belong to the image of δ. Then adjoining the primitive of a, that is, a solution of the inhomogeneous equation $\delta(f) = a$, can be described through the system given by the matrix*

$$A = \begin{bmatrix} 0 & -a \\ 0 & 0 \end{bmatrix}.$$

The Picard-Vessiot extension E is given as the differential field $E = K\langle f \rangle$ obtained by adjoining a formal primitive f of a to K. Since $a \notin \operatorname{Im} \delta$ one checks that f is transcendental over K. The differential Galois group is the group $G(E/K) = \mathbb{G}_a(\mathbb{C})$ acting by $f \mapsto f + c$.

In general, it is quite useful, in order to determine the Picard–Vessiot extension E, to know that it is obtained as the field of quotients of the Picard–Vessiot ring R of the differential system (1.342). This is characterized as follows.

- R is a differential algebra;
- R has no non-trivial differential ideal;
- There exists $F \in \mathrm{GL}_n(R)$, such that $F' + A\,F = 0$;
- R is generated by the coefficients of F.

EXAMPLE 1.88. *For given $a \in K$, consider the equation*

$$\delta(u) = au.$$

Let $R = K[u, u^{-1}]$ be obtained by adjoining to K a formal solution and its inverse. If a is such that R has no non-trivial differential ideal, then R is the Picard–Vessiot ring, and the Picard–Vessiot extension E is the field of quotients. In that case the differential Galois group is the group $G(E/K) = \mathbb{G}_m(\mathbb{C})$ acting by $u \mapsto \lambda u$. In general, the differential Galois group is a subgroup of $\mathbb{G}_m(\mathbb{C})$.

The significance of the differential Galois group is that it makes it possible to recognize whether a given equation is solvable by the elementary steps described in Examples 1.87 and 1.88. One has the following general result:

THEOREM 1.89. *The following conditions are equivalent for a linear differential equation D.*
(i) The equation is solvable by repeated applications of the following steps:

- *Adjunction of primitives (example* 1.87*);*
- *Adjunction of exponentials of primitives (example* 1.88*);*
- *Finite algebraic extensions.*

(ii) The connected component (for the Zariski topology) of the identity in the differential Galois group is solvable.

It is quite difficult to compute a differential Galois group in general, but since it is the Zariski closure of the image of the wild fundamental group, the key tool is the Riemann–Hilbert correspondence.

In order to understand the Riemann–Hilbert correspondence in the general local case, we now come to the Tannakian formalism. We first introduce the category of differential modules.

DEFINITION 1.90. *Let (K, δ) be a differential field with $\mathbb{C}$ as field of constants. The category $\mathcal{D}_K$ of differential modules over K has as objects pairs (V, ∇) of a vector space $V \in \mathrm{Obj}(\mathcal{V}_K)$ and a connection, namely a $\mathbb{C}$-linear map $\nabla : V \to V$ satisfying $\nabla(fv) = \delta(f)v + f\nabla(v)$, for all $f \in K$ and all $v \in V$. A morphism between objects (V_1, ∇_1) and (V_2, ∇_2) is a K-linear map $T : V_1 \to V_2$ satisfying $\nabla_2 \circ T = T \circ \nabla_1$.*

The category $\mathcal{D}_K$ of differential modules over K of Definition 1.90 is a $\mathbb{C}$-linear rigid tensor category, with the tensor product

$$(V_1, \nabla_1) \otimes (V_2, \nabla_2) = (V_1 \otimes V_2, \nabla_1 \otimes 1 + 1 \otimes \nabla_2)$$

and with $(V, \nabla)^\vee$ the K-linear dual of V with the induced connection. One can check that the functor to vector spaces over $\mathbb{C}$ that assigns to a module (V, ∇) its solution space $\mathrm{Ker}\nabla$ is a fiber functor. Thus, Theorem 1.83 shows that there is an affine group scheme G and an equivalence of categories $\mathcal{D}_K \cong \mathrm{Rep}_G$.

In the formal case $K = \mathbb{C}((z))$, the affine group scheme G is a semidirect product $G = \mathcal{T} \rtimes \bar{\mathbb{Z}}$ of the *Ramis exponential torus* $\mathcal{T}$ by the algebraic hull of $\mathbb{Z}$. The Ramis exponential torus corresponds to the symmetries which multiply by a non-zero scalar

$$e^{P(z^{-1/\nu})} \mapsto \lambda_P \, e^{P(z^{-1/\nu})}, \qquad \text{with} \quad \lambda_{P_1+P_2} = \lambda_{P_1} \lambda_{P_2}$$

the formal exponentials of a polynomial in $z^{-1/\nu}$, where ν is a ramification index. This torus is then of the form $\mathcal{T} = \mathrm{Hom}(\mathcal{B}, \mathbb{C}^*)$, where $\mathcal{B} = \cup_{\nu \in \mathbb{N}} \mathcal{B}_\nu$, for $\mathcal{B}_\nu = z^{-1/\nu}\mathbb{C}[z^{-1/\nu}]$. In the non-formal case $K = \mathbb{C}(\{z\})$, the Galois group is given by the *Ramis wild fundamental group*. This has additional generators, which depend upon resummation of divergent series and are related to the Stokes phenomenon. We will not discuss this further here. We refer the interested reader to [**222**] and [**284**] for a more detailed treatment of both the formal and the non-formal case.

It is useful to remark, however, that there are suitable classes of differential systems for which the Stokes phenomena are not present, hence the differential Galois group is the same in the formal and in the non-formal setting (cf. e.g. Proposition 3.40 of [**284**]).

One recovers the differential Galois group of a single equation by considering the affine group scheme associated to the Tannakian subcategory of $\mathcal{D}_K$ generated by the differential module associated to the equation.

Similarly, one can restrict to particular Tannakian subcategories of $\mathcal{D}_K$ and identify them with categories of representations of affine group schemes. All these results can be seen as a particular case of the Riemann–Hilbert correspondence, which is broadly meant as an equivalence of categories between the analytic datum of certain equivalence classes of differential systems and a representation theoretic datum.

An important case is related to Hilbert's 21st problem, or the Riemann–Hilbert problem for regular–singular equations. For simplicity, we discuss here only the local case of equations over an infinitesimal disk $\Delta \subset \mathbb{C}$ around the origin. We refer the reader to [**105**] for a more general treatment.

EXAMPLE 1.91. *Let (K, δ) be the differential field $K = \mathbb{C}((z))$ with $\delta(f) = f'$. A differential equation* (1.342) *is regular-singular if there exists an invertible matrix T with coefficients in K such that $T^{-1}AT - T^{-1}\delta(T) = B/z$, where B is a matrix with coefficients in $\mathbb{C}[[z]]$. The Tannakian subcategory $\mathcal{D}_K^{\mathrm{rs}}$ of $\mathcal{D}_K$ generated by the regular-singular equations is equivalent to $\mathrm{Rep}_{\bar{\mathbb{Z}}}$, where $\bar{\mathbb{Z}}$ is the algebraic hull of $\mathbb{Z}$*

as in Example 1.86*. This corresponds to the fact that a regular singular equation is determined by its monodromy representation, where* $\mathbb{Z} = \pi_1(\Delta^*)$*.*

Deligne proved in [**105**] a much stronger form of the regular-singular Riemann–Hilbert correspondence, where instead of the simple local case of the infinitesimal disk one considers the general global case of a complement of divisors in an algebraic variety. In this case it is still true that the category of regular-singular differential systems is equivalent to the category of finite-dimensional linear representations of the fundamental group. This type of global regular-singular Riemann–Hilbert correspondence admits further generalizations, in the form of an equivalence of derived categories between regular holonomic $\mathcal{D}$-modules and perverse sheaves. We refer the reader to a short survey of this more general viewpoint given in §8 of [**139**].

Here we are interested specifically in identifying the Riemann–Hilbert correspondence for the equivalence classes of flat equisingular connections of §7.2. This case differs from the usual setting recalled here in the following ways.

- Our base space is not just the disk Δ but the $\mathbb{C}^*$-fibration B over Δ, so we leave the category $\mathcal{D}_K$ of differential systems of Definition 1.90.
- The equivalence relation on connections that we consider is through gauge transformations regular at $z = 0$.
- The equisingular connections are not regular-singular, hence we need to work in the setting of the "irregular" Riemann–Hilbert correspondence, where one allows for an arbitrary degree of irregularity, as in the case of $\mathcal{D}_K$ above.
- The Galois group will be the same in both the formal and the non-formal setting.

Recall that we already know from the proof of Theorem 1.67 that flat equisingular connections have trivial monodromy, so that the representation theoretic datum that classifies them is certainly not a monodromy representation. We see that the Galois group has a structure similar to the Ramis exponential torus.

We discuss all this in detail in the next section.

7.6. Universal Hopf algebra and the Riemann–Hilbert correspondence.

The main result of this section is the Riemann–Hilbert correspondence underlying the theory of perturbative renormalization. This will identify a universal affine group scheme $\mathbb{U}^*$ that governs the structure of the divergences of all renormalizable theories and which has the type of properties that Cartier envisioned for his "cosmic Galois group".

We begin by constructing a category of *flat equisingular vector bundles*. We will then see how it relates to the flat equisingular connections for a particular choice of the pro-unipotent affine group scheme G.

We first introduce the notion of a W-connection on a filtered vector bundle. As above we let $B^0 \subset B$ be the complement of the fiber over $z = 0 \in \Delta$ for the space B of (1.298).

DEFINITION 1.92. *Let (E, W) be a filtered vector bundle over B, with an increasing filtration $W^{-n-1}(E) \subset W^{-n}(E)$ and a given trivialization of the associated graded*

$$\mathrm{Gr}_n^W(E) = W^{-n}(E)/W^{-n-1}(E).$$

A W-connection on E is a connection ∇ on the vector bundle $E^0 = E|_{B^0}$, with the following properties :

1. *The connection ∇ is compatible with the filtration, i.e. it restricts to all the $W^{-n}(E^0)$.*
2. *The connection ∇ induces the trivial connection on the associated graded $\mathrm{Gr}^W(E)$.*

Two W-connections ∇_i on E^0 are W-equivalent iff there exists an automorphism T of E, preserving the filtration, that induces the identity on $\mathrm{Gr}^W(E)$ and conjugates the connections, $T \circ \nabla_1 = \nabla_2 \circ T$.

We reformulate conditions (1.300) and (1.301) of equisingularity in this context. Let V be a finite-dimensional $\mathbb{Z}$-graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$, and consider the trivial vector bundle $E = B \times V$. We view it as a $\mathbb{G}_m$-equivariant filtered vector bundle (E, W), with W the weight filtration

$$W^{-n}(V) = \oplus_{m\geq n} V_m, \tag{1.344}$$

and the $\mathbb{G}_m$-action induced by the grading of V.

DEFINITION 1.93. *A flat W-connection ∇ on E is equisingular if it is $\mathbb{G}_m$-invariant and for any fundamental system of solutions of $\nabla\eta = 0$ the associated isomorphism between restrictions of E to sections $\sigma : \Delta \to B$ with $\sigma(0) = y_0$ is regular.*

We construct a category $\mathcal{E}$ of flat equisingular vector bundles as follows.

DEFINITION 1.94. *The objects* $\mathrm{Obj}(\mathcal{E})$ *are given by data $\Theta = [V, \nabla]$. Here V is a finite-dimensional $\mathbb{Z}$-graded vector space and ∇ is an equisingular W-connection on the filtered bundle $E^0 = B^0 \times V$. The brackets $[V, \nabla]$ mean taking the W-equivalence class of the connection ∇. The data (E, ∇) are a flat equisingular vector bundle.*

The morphisms $\mathrm{Hom}_{\mathcal{E}}(\Theta, \Theta')$ *are linear maps $T : V \to V'$ that are compatible with the grading and satisfy the following condition. Consider on the bundle $(E' \oplus E)^*$ the W-connection*

$$\nabla_1 := \begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix}$$

and the W-connection ∇_2, which is the conjugate of ∇_1 by the unipotent matrix

$$\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}. \tag{1.345}$$

Then the requirement for a morphism $T \in \mathrm{Hom}_{\mathcal{E}}(\Theta, \Theta')$ is that these two W-connections are W-equivalent on B,

$$\nabla_2 = \begin{pmatrix} \nabla' & T\nabla - \nabla' T \\ 0 & \nabla \end{pmatrix} \sim \nabla_1 = \begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix}. \tag{1.346}$$

It is worth explaining why it is necessary to use the direct sum $E' \oplus E$ and the matrices in condition (1.346) for the definition of morphisms. This is related to the fact that we are dealing with filtered spaces. The problem is that the category of filtered vector spaces, with morphisms that are linear maps respecting the filtration, is not an abelian category. In fact, suppose given a linear map $f : V \to V'$ of filtered vector spaces, such that $f(W^{-n}(V)) \subset W^{-n}(V')$. In general, there is no relation between the restriction to the image of f of the filtration of V' and the filtration on the quotient of V by $\mathrm{Ker}(f)$, while in an abelian category one would expect an isomorphism between $V/\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$. Thus, we have to refine our class of morphisms to one for which this property is satisfied.

We first explain the relation of Definition 1.94 to the equivalence classes of flat equisingular G-connections considered in Sections 7.2 and 7.3.

LEMMA 1.95. *Let G be the affine group scheme dual to a graded connected Hopf algebra $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$, with finite-dimensional $\mathcal{H}_n$, as in §7.2. Let ϖ be a flat equisingular connection as in Definition 1.65. Let $\xi : G \to \mathrm{GL}(V)$ be a finite-dimensional linear graded representation of G. Then the data (ϖ, ξ) determine an element $\Theta \in \mathrm{Obj}(\mathcal{E})$. Equivalent ϖ determine the same Θ.*

PROOF. Notice that a graded representation of the pro-unipotent G in V can be described equivalently as a graded representation of the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ in V. Since the Lie algebra $\mathfrak{g}$ is positively graded, both representations are compatible with the weight filtration $W^{-n}(V) = \oplus_{m\geq n} V_m$. The induced representation on the associated graded $\mathrm{Gr}^W_n(V) = W^{-n}(V)/W^{-n-1}(V)$ is the identity. We let $\xi^* : G^* \to \mathrm{GL}(V)$ be the natural extension of ξ to $G^* = G \rtimes \mathbb{G}_m$ which when restricted to $\mathbb{G}_m$ is the grading of V. Consider the vector bundle E on B associated to the trivial principal G^*-bundle $\tilde{P}$ by the representation ξ^*. The connection ϖ is a flat equisingular G-connection in the sense of Definition 1.65 and by Lemma 1.73 it gives a flat equisingular connection on the G^*-bundle $\tilde{P}^0$ of (1.334). It induces a connection ∇ on $E^0 = E|_{B^0}$, which using Lemma 1.73 is a flat equisingular W-connection in the sense of Definition 1.93. If ϖ and ϖ' are flat equisingular G-connection on P^0 that are equivalent as in Definition 1.66, then the corresponding W-connections ∇ and ∇' are W-equivalent. ☐

Thus, the category $\mathcal{E}$ of flat equisingular vector bundles of Definition 1.94 provides a universal setting where equivalence classes of flat equisingular G-connections for varying G can be analyzed simultaneously. We will see later how we recover the classes of flat equisingular connections for a particular G from the general case.

We first analyze the main properties of the category $\mathcal{E}$. We begin by introducing the affine group scheme $\mathbb{U}^*$ and the universal singular frame $\gamma_{\mathbb{U}}$.

DEFINITION 1.96. *Let $\mathcal{L}_{\mathbb{U}} = \mathcal{F}(1, 2, 3, \cdots)_\bullet$ denote the free graded Lie algebra generated by elements e_{-n} of degree n, for each $n > 0$. Consider the Hopf algebra*

$$\mathcal{H}_{\mathbb{U}} := U(\mathcal{F}(1, 2, 3, \cdots)_\bullet)^\vee, \tag{1.347}$$

which is the graded dual of the universal enveloping algebra of $\mathcal{L}_{\mathbb{U}}$. We denote by $\mathbb{U}$ the affine group scheme associated to the commutative Hopf algebra $\mathcal{H}_{\mathbb{U}}$, and by $\mathbb{U}^*$ the semidirect product $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$, with the action given by the grading.

As an algebra $\mathcal{H}_{\mathbb{U}}$ is isomorphic to the linear space of noncommutative polynomials in variables f_n, $n \in \mathbb{N}$, with the shuffle product.

The sum

$$e = \sum_{1}^{\infty} e_{-n}\,, \tag{1.348}$$

defines an element of the Lie algebra $\mathcal{L}_{\mathbb{U}}$ of $\mathbb{U}$. Since $\mathbb{U}$ is by construction a pro-unipotent affine group scheme we can lift e to a morphism

$$\mathbf{rg}\; : \mathbb{G}_a \to \mathbb{U}\,, \tag{1.349}$$

of affine group schemes from the additive group $\mathbb{G}_a$ to $\mathbb{U}$. This morphism will play an important role from the point of view of renormalization. In fact, we will see in Theorem 1.106 that it gives a universal and canonical lift of the renormalization group.

Definition 1.97. *The universal singular frame is the loop in the pro-unipotent Lie group $\mathbb{U}(\mathbb{C})$ given by the formula*

$$\gamma_{\mathbb{U}}(z,v) = \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(e)\,\frac{du}{u}}\,. \tag{1.350}$$

We can compute explicitly the coefficients of the universal singular frame as follows.

Proposition 1.98. *The universal singular frame is given by*

$$\gamma_{\mathbb{U}}(-z,v) = \sum_{n\geq 0, k_j>0} \frac{e_{-k_1}e_{-k_2}\cdots e_{-k_n}}{k_1\,(k_1+k_2)\cdots(k_1+k_2+\cdots+k_n)}\, v^{\sum k_j}\, z^{-n} \tag{1.351}$$

Proof. Using (1.348) and (1.262), we obtain that the coefficient of $e_{-k_1}e_{-k_2}\cdots e_{-k_n}$ is given by the expression

$$v^{\sum k_j}\, z^{-n} \int_{0\leq s_1\leq\cdots\leq s_n\leq 1} s_1^{k_1-1}\cdots\, s_n^{k_n-1}\, ds_1\cdots ds_n\,.$$

This yields the desired result. ☐

Remark 1.99. *Notice that the coefficients of $\gamma_{\mathbb{U}}(z,v)$ are all rational numbers. Moreover, the coefficients of* (1.351) *are the same coefficients that appear in the local index formula of* [**92**].

The main result of this section, which gives the Riemann–Hilbert correspondence for the category of flat equisingular bundles, is the following theorem.

Theorem 1.100. *Let $\mathcal{E}$ be the category of equisingular flat bundles of Definition 1.94. Let $\omega : \mathcal{E} \to \mathrm{Vect}_{\mathbb{C}}$ be the functor defined by $\omega(\Theta) = V$, for $\Theta = [V,\nabla]$. Then $\mathcal{E}$ is a neutral Tannakian category with fiber functor ω and is equivalent to the category of representations $\mathrm{Rep}_{\mathbb{U}^*}$ of the affine group scheme $\mathbb{U}^*$ of Definition* 1.96.

Proof. We prove this result in several steps. Let V be a finite-dimensional graded vector space over a field k. (Here we work with $k = \mathbb{C}$, but see Corollary 1.105 below.) Let $G_V(k)$ denote the group of linear transformations $S \subset \mathrm{End}(V)$ compatible with the weight filtration,

$$S\,W_{-n}(V) \subset W_{-n}(V), \tag{1.352}$$

and inducing the identity on the associated graded

$$S|_{\mathrm{Gr}_n^W} = 1. \tag{1.353}$$

The unipotent algebraic group G_V is (non-canonically) isomorphic to the unipotent group of upper triangular matrices. Its Lie algebra is then identified with strictly upper triangular matrices. We then have a direct translation between W-connections and G-valued connections.

PROPOSITION 1.101. *Let (E, ∇) be a flat equisingular vector bundle, with $E = B \times V$. Then the following holds.*

1. *The connection ∇ defines a flat equisingular G_V-valued connection ϖ, for G_V as above.*
2. *All flat equisingular G_V-valued connections are obtained in this way.*
3. *This bijection preserves equivalence, namely equivalent flat equisingular G_V-valued connections correspond to the same object $\Theta = [V, \nabla]$ of $\mathcal{E}$.*

PROOF. Since W-connections are compatible with the filtration and trivial on the associated graded, they are obtained by adding a $\mathrm{Lie}\, G_V$-valued 1-form ϖ to the trivial connection. Similarly, W-equivalence is given by the equivalence of Definition 1.66. □

LEMMA 1.102.

1. *Let $\Theta = [V, \nabla]$ be an object of $\mathcal{E}$. There exists a unique representation $\rho = \rho_\Theta$ of $\mathbb{U}^*$ in V such that the restriction to $\mathbb{G}_m \subset \mathbb{U}^*$ is the grading and*
$$D\rho(\gamma_{\mathbb{U}}) \simeq \nabla, \tag{1.354}$$
where $\gamma_{\mathbb{U}}$ is the universal singular frame.
2. *Given a representation $\rho : \mathbb{U}^* \to \mathrm{GL}(V)$, there exists a connection ∇ on $E^0 = B^0 \times V$, unique up to equivalence, such that $[V, \nabla]$ is an object in $\mathcal{E}$ and ∇ satisfies* (1.354).

PROOF. Let G_V be defined as above. By Proposition 1.101, we know that ∇ defines a flat equisingular G_V-valued connection ϖ. Then, using (1.303) of Theorem 1.67, we know that we have an equivalence
$$\varpi \sim D\left(\mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta)\,\frac{du}{u}}\right),$$
for a unique element $\beta \in \mathrm{Lie}\, G_V$. We can decompose the element β into homogeneous components for the action of the grading, $\beta = \sum_n \beta_n$ with $Y(\beta_n) = n\,\beta_n$. Thus, the element β (and the grading) uniquely determine a representation ρ of $\mathbb{U}^*$ in G_V, where $\rho(e_{-n}) = \beta_n$. This representation satisfies (1.354) by construction. Conversely, given a representation ρ of $\mathbb{U}^*$, we consider the grading associated to the restriction to $\mathbb{G}_m$ and let
$$\gamma(z, v) = \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\rho(e))\,\frac{du}{u}}.$$
The flat equisingular connection $D\gamma$ determines a W-connection ∇ on the vector bundle E^0 with the desired properties. □

We will see in Theorem 1.106 below that the construction of representations of $\mathbb{U}$ in G_V used in the proof of Lemma 1.102 above holds in general and provides the way to recover classes of flat equisingular connections for a given pro-unipotent affine group scheme G from the data of the category $\mathcal{E}$. We now continue with the proof of Theorem 1.100.

LEMMA 1.103. *Let $\Theta = [V, \nabla]$ be an object in $\mathcal{E}$. Then the following holds.*

(1) *For any $S \in \mathrm{Aut}(V)$ compatible with the grading, $S \nabla S^{-1}$ is an equisingular connection.*
(2) *The representation ρ_Θ of Lemma 1.102 satisfies*
$$\rho_{[v, S \nabla S^{-1}]} = S\, \rho_{[V,\nabla]}\, S^{-1}.$$
(3) *The equisingular connections ∇ and $S \nabla S^{-1}$ are equivalent if and only if $[\rho_{(E,\nabla)}, S] = 0$.*

PROOF. (1) The equisingularity condition is satisfied. In fact, the $\mathbb{G}_m$-invariance follows from the compatibility with the grading and the restriction to a section $\sigma : \Delta \to B$ satisfies
$$\sigma^*(S \nabla S^{-1}) = S\, \sigma^*(\nabla)\, S^{-1},$$
so that the second condition for equisingularity is also satisfied.
(2) The second statement follows from Lemma 1.102 and the compatibility of S with the grading. In fact, we have, for an element $\beta \in \mathrm{Lie}\, G_V$,
$$S\, \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta)\frac{du}{u}}\, S^{-1} = \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(S\beta S^{-1})\frac{du}{u}}.$$
(3) The third statement follows immediately from the second, since equivalence corresponds to having the same $\beta \in \mathrm{Lie}\, G_V$, by Theorem 1.67. □

PROPOSITION 1.104. *Let $\Theta = [V, \nabla]$ and $\Theta' = [V', \nabla']$ be objects of $\mathcal{E}$. Let $T : V \to V'$ be a linear map compatible with the grading. Then the following two conditions are equivalent:*

(1) *$T \in \mathrm{Hom}_{\mathcal{E}}(\Theta, \Theta')$;*
(2) *$T \rho_\Theta = \rho_{\Theta'} T$.*

PROOF. Let $S \in \mathrm{Aut}(E' \oplus E)$ be the unipotent automorphism of (1.345). By construction, S is an automorphism of $E' \oplus E$, compatible with the grading. By (3) of the previous Lemma, we have
$$S \begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix} S^{-1} \sim \begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix}$$
if and only if
$$\begin{pmatrix} \beta' & 0 \\ 0 & \beta \end{pmatrix} S = S \begin{pmatrix} \beta' & 0 \\ 0 & \beta \end{pmatrix}.$$
This holds if and only if $\beta' T = T\beta$. □

To complete the proof of Theorem 1.100, we check that the functor $\mathrm{Rep}_{\mathbb{U}^*} \to \mathcal{E}$ given by $\rho \mapsto D\rho(\gamma_U)$ is compatible with the tensor structure. In $\mathcal{E}$ this is given by
$$(V, \nabla) \otimes (V', \nabla') = (V \otimes V', \nabla \otimes 1 + 1 \otimes \nabla').$$
This is compatible with the W-equivalence and with the condition of equisingularity of the connections. The compatibility of $\rho \mapsto D\rho(\gamma_U)$ with tensor products then follows from the formula
$$\mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta\otimes 1 + 1\otimes\beta')\frac{du}{u}} = \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta)\frac{du}{u}} \otimes \mathrm{T}e^{-\frac{1}{z}\int_0^v u^Y(\beta')\frac{du}{u}}.$$
On morphisms, it is sufficient to check the compatibility on $1 \otimes T$ and $T \otimes 1$.
We have shown that the tensor category $\mathcal{E}$ is equivalent to the category $\mathrm{Rep}_{\mathbb{U}^*}$ of finite-dimensional representations of $\mathbb{U}^*$. The fact that $\mathbb{U}^* = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ then follows,

since the fiber functor ω becomes through this equivalence the forgetful functor that assigns to a representation the underlying vector space.

$\square$

Corollary 1.105. *The category $\mathcal{E}$ can also be defined over the field $k = \mathbb{Q}$ and the equivalence of rigid k-linear tensor categories proved in Theorem* 1.100 *still holds for $k = \mathbb{Q}$.*

Proof. In the case where $k = \mathbb{Q}$, we work in the formal setting with the differential field $K = \mathbb{Q}((z))$. We still consider the same geometric setting as before, where the infinitesimal disk $\Delta = \mathrm{Spec}(K)$. All the arguments still go through. In fact, notice that (1.273) gives a rational expression for the operator D. This, together with the fact that the coefficients of the universal singular frame in (1.351) are rational, implies that we can work with a rational ∇. $\square$

In particular, Corollary (1.105) also shows that for the category of flat equisingular bundles the formal and the nonformal theory give the same Galois group $\mathbb{U}^*$. This reflects the fact that, due to the pro-unipotent nature of the affine group scheme, our arguments usually depend upon only finitely many terms in an infinite sum, cf. Remark 1.59.

In the rational case, we can define, for each $n \in \mathbb{Z}$, an object $\mathbb{Q}(n)$ of the category $\mathcal{E}$ of equisingular flat vector bundles where V is given by a one-dimensional $\mathbb{Q}$-vector space placed in degree n, and ∇ is the trivial connection on the associated vector bundle E over B. Then the fiber functor takes the form $\omega = \oplus\, \omega_n$, with

$$\omega_n(\Theta) = \mathrm{Hom}(\mathbb{Q}(n), \mathrm{Gr}^W_{-n}(\Theta))\,. \tag{1.355}$$

We now return to discuss the physical significance of the result of Theorem 1.100 and the role of the affine group scheme $\mathbb{U}^*$ in perturbative renormalization. The following result shows how $\mathbb{U}^*$ is related to the pro-unipotent affine group scheme G. In particular, this holds in the case where $G = \mathrm{Difg}(\mathcal{T})$ is the group of diffeographisms of a renormalizable theory.

Theorem 1.106. *Let G be a pro-unipotent affine group dual to a graded connected commutative Hopf algebra $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$, with finite-dimensional $\mathcal{H}_n$. Then the following properties hold.*

(1) *There exists a canonical bijection between equivalence classes of flat equisingular connections on $\tilde{P}^0$ and graded representations $\mathbb{U} \to G$, or equivalently representations*

$$\rho : \mathbb{U}^* \to G^* = G \rtimes \mathbb{G}_m, \tag{1.356}$$

which are the identity on $\mathbb{G}_m$.

(2) *The universal singular frame $\gamma_{\mathbb{U}}$ provides universal counterterms. Namely, given a loop $\gamma_\mu \in L(G(\mathbb{C}), \mu)$, the universal singular frame maps to $\gamma_-(z)$ under the representation ρ of* (1.356)*.*

(3) *The renormalization group F_t in $G(\mathbb{C})$ described in Proposition 1.47 is obtained as the composite $\rho \circ \mathbf{rg}$, with ρ as in* (1.356) *and $\mathbf{rg} : \mathbb{G}_a \to \mathbb{U}$ as in* (1.349)*.*

PROOF. (1) Theorem 1.67 shows that the equivalence classes of flat equisingular connections are parameterized by the elements $\beta \in \mathrm{Lie}\, G$. We then proceed as we did in the proof of Lemma 1.102. The positivity and integrality of the grading make it possible to write β as an infinite formal sum

$$\beta = \sum_{1}^{\infty} \beta_n \,, \tag{1.357}$$

where, for each n, β_n is homogeneous of degree n for the grading, i.e. $Y(\beta_n) = n\beta_n$. Assigning β and the action of the grading on it is the same as giving a collection of homogeneous elements β_n that fulfill no restriction besides $Y(\beta_n) = n\beta_n$. In particular, there is no condition on their Lie brackets. Thus, assigning such data is the same as giving a graded homomorphism from the affine group scheme $\mathbb{U}$ to G, i.e. a ρ as in (1.356).

(2) This follows from Theorem 1.58, since we have

$$\rho(\gamma_{\mathbb{U}})(z,v) = \mathrm{T}e^{\frac{-1}{z}\int_0^v u^Y(\beta)\frac{du}{u}} \,.$$

(3) By Corollary 1.49 we know that β is the infinitesimal generator of F_t $\square$

COROLLARY 1.107. *The affine group scheme* $\mathbb{U}$ *acts on the coupling constants of physical theories, through the representation* (1.356) *to the group of diffeographisms* $\mathrm{Difg}(\mathcal{T})$ *and then applying Proposition* 1.42 *to obtain*

$$\mathbb{U} \to \mathrm{Difg}(\mathcal{T}) \to \mathrm{Diff}. \tag{1.358}$$

Thus, the affine group scheme $\mathbb{U}$ has the right properties that Cartier expected for his "cosmic Galois group". In §8 we will also discuss the arithmetic nature of $\mathbb{U}$ and its relation to motivic Galois groups. Note that, while the morphism $\mathrm{Difg}(\mathcal{T}) \to \mathrm{Diff}$ is defined over $\mathbb{Q}$ the morphism $\mathbb{U} \to \mathrm{Difg}(\mathcal{T})$ in (1.358) is only defined over $\mathbb{C}$ since it is associated to an equisingular connection with complex coefficients. In the dual map of Hopf algebras, the image of the generators Γ will have coefficients with respect to the generators of $\mathcal{H}_{\mathbb{U}}$ that are transcendental numbers obtained from the residues of the graphs.

The map (1.356) can be seen as a map of Galois groups, from the Galois group of the Tannakian category $\mathcal{E}$ to that of the Tannakian subcategory generated by the flat equisingular vector bundles that come from flat equisingular G-connections on $\tilde{P}^0$. In general, for a given quantum field theory $\mathcal{T}$ the subcategory $\mathcal{E}_{\mathcal{T}}$ of $\mathcal{E}$ given by flat equisingular vector bundles coming from $\mathcal{T}$ will differ from $\mathcal{E}$. The corresponding affine group scheme plays the same role as the Galois group of a given differential equation does with respect to the universal group. Thus it is natural to define the Galois group of $\mathcal{T}$ as follows, with

$$\mathrm{Gal}(\mathcal{T})^* = \mathrm{Gal}(\mathcal{T}) \rtimes \mathbb{G}_m. \tag{1.359}$$

DEFINITION 1.108. *Let* $\mathcal{T}$ *be a given renormalizable quantum field theory. We define the Galois group* $\mathrm{Gal}\,(\mathcal{T})^*$ *as the affine group scheme associated to the subcategory* $\mathcal{E}_{\mathcal{T}}$ *of flat equisingular vector bundles coming from* $\mathcal{T}$*. We let* $\mathrm{Gal}\,(\mathcal{T})$ *be the graded affine group scheme kernel of the canonical morphism* $\mathrm{Gal}\,(\mathcal{T})^* \to \mathbb{G}_m$.

Since we are dealing with a subcategory of $\mathcal{E}$, one has a natural surjection

$$\mathbb{U} \to \mathrm{Gal}\,(\mathcal{T}) \tag{1.360}$$

and the group homomorphism $\rho : \mathbb{U} \to G$ of (1.356) with $G = \mathrm{Difg}(\mathcal{T})$ factors through $\mathrm{Gal}\,(\mathcal{T})$ by (1.360). When there are no divergences in the theory one has $\mathrm{Gal}\,(\mathcal{T}) = \{1\}$ and this can happen even though $\mathrm{Difg}(\mathcal{T})$ is quite large. It is of course desirable to find in our framework an analogue of Theorem 1.89 on the resolution of equations.

Proposition 1.109. *Let $\mathcal{T}$ be a given renormalizable quantum field theory.*
1) The theory $\mathcal{T}$ is finite iff its Galois group is trivial, that is $\mathrm{Gal}\,(\mathcal{T}) = \{1\}$.
2) If the theory $\mathcal{T}$ is super-renormalizable, its Galois group $\mathrm{Gal}\,(\mathcal{T})$ *is finite-dimensional.*

Proof. 1) If the theory is finite then there are no divergences and $\beta = 0$ so that the image of $\mathbb{U}$ in $G = \mathrm{Difg}(\mathcal{T})$ is trivial. The converse also holds, since the divergences are determined by the residues and hence by β.
2) By definition (cf. [**62**] §5.7.3) a theory is super-renormalizable iff only a finite number of graphs need overall counterterms. The prototypical example is $\mathcal{T} = \phi^3_4$. It then follows that only a finite number of homogeneous components β_k are non-zero so that the Lie algebra of $\mathrm{Gal}\,(\mathcal{T})$ is finite-dimensional. ☐

In particular, $\mathrm{Lie}\,\mathrm{Gal}(\mathcal{T})$ is the Lie subalgebra of $\mathrm{Lie}\,\mathrm{Difg}(\mathcal{T})$ generated by the β_n.

8. Motives in a nutshell

In this section we give a brief overview of Grothendieck's theory of motives. The theory is vast and one can easily get into very hard technical aspects, but we will only give an impressionistic sketch, and focus on those aspects that are of direct relevance to the interaction with noncommutative geometry and quantum physics. For the reader who might be interested in a more detailed treatment of the subject, we recommend the recent book by Yves André [**2**], and the two AMS volumes of the 1991 Seattle conference (cf. [**173**]).

8.1. Algebraic varieties and motives.

We describe in this section those aspects of the theory of algebraic varieties that contributed to the formulation of Grothendieck's original idea of a theory of motives.

8.1.1. *Cohomology theories.* With the development of étale cohomology of algebraic varieties, one finds that there are several different viable cohomology theories one can associate to an algebraic variety. Moreover, these theories are related by specific comparison isomorphisms.
If X is a smooth projective algebraic variety over a field k, then for a given separable closure k^{sep} and for $\bar{X}$ the variety over k^{sep} obtained by extension of scalars, one can define the ℓ-adic étale cohomology

$$H^i_{\mathrm{et}}(\bar{X}, \mathbb{Q}_\ell) \tag{1.361}$$

whenever ℓ is prime to the characteristic of k. These are finite-dimensional $\mathbb{Q}_\ell$-vector spaces, which satisfy all the desirable properties of a cohomology theory: Poincaré