Personal Notes for Commutative Algebra by P. Nelson

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Contents

About the Course

The course website is https://metaphor.ethz.ch/x/2017/hs/401-3132-00L/.

The topic includes

- Basics about rings, ideals and modules
- Localization
- Primary decomposition
- Integral dependence and valuations
- Noetherian rings
- Completions
- Basic dimension theory

Prerequisite:

Rings, homomorphism, ideals, quotient rings, zero divisors, prime/maximal ideals, fields.

Convention: Ring, we mean a commutative ring with identity. In particular for a ring homomorphism $f : R \to S$. We have $f(1_R) = 1_S$. Remark: we allow 1=0 but then R=0. Caution, by definition $1 \neq 0$ in a field .

1 Rings, ideals, radicals

1.1 Lecture 1. Motivation and Basics by Paul Steinmann

In differential geometry, we have the theorem of level sets:

Theorem 1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$. If $0 \in \mathbb{R}^n$ is a regular value of f then $f^{-1}(0)$ is a submanifold.*

In algebraic geometry, we look at $f^{-1}(0)$ for polynomial f. More precisely, fix an algebraic-closed field $\mathbb{K}$ and an integer $n > 0$, consider the ring $R := \mathbb{K}[x_1, ..., x_n]$. Def: For a subset $S \subset R$ we define the **affine algebraic variety** by

$$V(S) := \{x \in \mathbb{K}^n | \forall f \in S,\ f(x) = 0 \subset \mathbb{K}^n\} \tag{1}$$

Remark 1.2. *With the affine algebraic varieties defined above, we have:*

- $V(\emptyset) = \mathbb{K}^n$
- $V(\{1\}) = \emptyset$
- *For an non empty collection of subsets* $(S_i)_{i\in I}$ $S_i \subset R$ *we have*

$$\cap_{i\in I} V(S_i) = V(\cup_{i\in I} S_i)$$

- S *and* S' *are subsets in* R

$$V(S) \cup V(S') = V(\{fg|f \in S, g \in S'\})$$

as a consequence, $(V(S))_{S\subset R}$ *form the closed sets of a topology on* $\mathbb{K}^n$ *called* ***Zariski topology***.

Example 1.3. *n=2 ,* $R = \mathbb{K}[X_1, X_2]$
$V(\{X_1\})$ *is the* X_2 *axis in* $\mathbb{K}^2$
$V(\{X_2 - X_1^2\})$ *is the parabola in* $\mathbb{K}^2$

Definition 1.4. *Conversely for all subset* $X \subset \mathbb{K}^n$*, consider*

$$I(X) := \{f \in R|\forall x \in X : f(x) = 0\} \subset R.$$

Remark 1.5. *Fact: For* S *in* R *and* X *subset in* $\mathbb{K}^n$*, we have,*

- $S \subset I(V(S))$
- $X \subset V(I(X))$
- *For* $S \subset S' \subset$ *in* R*, we have* $V(S) \supset V(S')$
- *For* $X \subset X' \subset \mathbb{K}^n$*, we have* $I(X) \supset I(X')$
- $I(X) \subset R$ *is an ideal.*

Definition 1.6. *The* ***radical of an ideal*** $a \subset R$ *is* $rad(\mathfrak{a}) := \{a \in R|\exists n \geq 1$ *s.t.* $a^n \in \mathfrak{a}\} \subset R$ *An ideal* $\mathfrak{a} \subset R$ *with* $rad(\mathfrak{a})$ *is called* ***radical***.

Remark 1.7. *Fact, for every ideal* $\mathfrak{a} \subset R$ *we have* $\mathfrak{a} \subset rad(\mathfrak{a})$.
$rad(\mathfrak{a})$ *is an ideal, proof in exercise.*
For $X \subset \mathbb{K}^n$ *the ideal* $I(X)$ *is radical.*

Theorem 1.8. *(The Hilbert's Nullstelensatz) For any ideal* $\mathfrak{a} \subset R$ *we have*

$$I(V(\mathfrak{a})) = rad(\mathfrak{a}).$$

An important consequence of the theorem:
the maps V and I induce the one to one correspondence between

$$\{\text{radical ideals in the polynomial ring}\} \Longleftrightarrow \{\text{affine algebraic varieties}\}$$

and this correspondence inverse the inclusion.

Example 1.9. *For any point $x = (x_1, ..., x_n) \in \mathbb{K}^n$ the ideal*

$$I(x) = \mathfrak{m}_x := (X_1 - x_1, ..., X_n - x_n)$$

is maximal.

Proof. If not, then there exists an ideal $\mathfrak{a} \subset R$ s.t.

$$R \supsetneq \mathfrak{a} \supsetneq \mathfrak{m}_x,$$

but then by the Nullstellensatz,

$$\emptyset \subsetneq V(\mathfrak{a}) \subsetneq V(\mathfrak{m}_x) = \{x\},$$

which makes the contradiction. □

Weak Nullstellensatz the ideals m_x is are precisely the maximal ideals of $\mathbb{K}[x_1, ..., x_n]$, where $\mathbb{K}$ needs to be algebraically closed

Example 1.10. $\mathbb{K} = \mathbb{R}, n = 1$. *$X^2 + 1$ is irreducible in $\mathbb{R}[X]$. And $\mathbb{R}[X]/(X^2 + 1) \cong \mathbb{C}$ is maximal. Consequence, we have a bijection*

$$\{\textit{max ideals of } R \textit{ polynomial ring } \mathbb{K}[X_1, ..., X_n]\} \Longleftrightarrow \{\textit{Points in } \mathbb{K}^n\}$$

Let A be a ring. Remember
An element $a \in A$ is **nilpotent** if there $\exists n > 1 \in \mathbb{Z}$ s.t. $a^n = 0$.
An element $a \in A$ is a **zero divisor** if there is an element $b \in A, b \neq 0$ s.t. $ab = 0$.
Fact: every nilpotent element is a zero divisor but not conversely.

Example 1.11. *take $(0, 1) \in A \times A$ then $(0, 1) \cdot (1, 0) = (0, 0)$*

Definition 1.12. *The ideal $N : rad((0))$ is called the **nil radical** of A.*

Then we have:

1. $\mathcal{N}$ is the set of all nilpotent elements of A

2. $A/\mathcal{N}$ has no nilpotent elements.

Proof. 1. From definitions. 2. Let $x \in A$ s.t. $\bar{x} \in A/\mathcal{N}$ is nilpotent. Let $n > 0$ s.t. $\bar{x}^n = 0$ then $x^n \in \mathcal{N}$ Thus there exists $k > 0$ s.t. $(x^n)^k = 0$ hence $x^{nk} = 0$, $x \in \mathcal{N}$. □

Proposition 1.13. *The nil radical of A is the intersection of all prime ideals of A.*

Proof. Denote by $\mathcal{N}'$ the intersection of all prime ideals of A. For any nilpotent element $f \in A$ with $n > 0$ s.t. $f^n = 0$, We have $f^n \in \mathfrak{p}$ for every prime ideal $\mathfrak{p}$. Hence $f \in \mathfrak{p}$ We conclude $f \in \mathcal{N}'$ Conversely, suppose $f \in A$ is not nilpotent Define $\Sigma := \{\mathfrak{a} \subset A \text{ ideals} | \forall n > 0 : f^n \notin \mathfrak{a}\}$ We will apply Zorn's lemma. We have

1. $(0) \in \Sigma$, so Σ is nonempty,
2. Σ is partially ordered by inclusion.
3. For any chain $(a_i)_{i \in I} \subset \Sigma$, the set $\mathfrak{a} := \cup_{i \in I} a_i$ is an ideal and

for all $n > 0$, we have $f^n \notin \mathfrak{a}$, hence $\mathfrak{a} \in \Sigma$. By Zorn's lemma we conclude that there is a maximal element $\mathfrak{p} \in \Sigma$. We show that $\mathfrak{p}$ is a prime ideal.

For any $x, y \notin \mathfrak{p}$, consider the ideals $\mathfrak{p} + (x), \mathfrak{p} + (y)$. They strictly contain $\mathfrak{p}$ and are thus not in Σ. Let $n, m > 0$ s.t. $f^n \in (x), f^m \in \mathfrak{p} + (y)$. We conclude that $f^{n+m} \in \mathfrak{p} + (xy)$, so $\mathfrak{p} + (xy) \notin \Sigma$. Hence $xy \notin \mathfrak{p}$, which means, $\mathfrak{p}$ is a prime ideal so $f \notin \mathcal{N}'$. □

Remember let $f : A \to B$ be a ring morphism. And $\mathfrak{p} \subset B$ a prime ideal . Then $f^{-1}(\mathfrak{p})$ is a prime ideal of A. Caution: Not true for maximal ideals in general.

Proposition 1.14. *Let $\mathfrak{a} \subset A$ be an ideal, $\pi : A \to A/\mathfrak{a}$ There is a one to one correspondence between ideals of $A/\mathfrak{a}$ and ideals in A which contain $\mathfrak{a}$ via $\mathfrak{c} = \pi^{-1}(\mathfrak{b})$*

Corollary 1.15. *Let $\mathfrak{a} \subset A$ be an ideal, then $rad(\mathfrak{a})$ is the intersection of all prime ideals which contain $\mathfrak{a}$.*

Proof. consider the homomorphism $\pi : A \to A/\mathfrak{a}$ Then $rad(\mathfrak{a}) = \pi^{-1}(\mathcal{N}_{A/\mathfrak{a}})$. By the above proposition $\mathcal{N}_{A/\mathfrak{a}}$ is the intersection of all prime ideals of $A/\mathfrak{a}$. By the correspondence we conclude the statement. □

Definition 1.16. *The **Jacobson Radical** $\mathcal{R}$ of A is the intersection of all maximal ideals in A.*

Proposition 1.17. *We have $x \in \mathcal{R} \iff \forall y \in A : 1 - xy$ is a unit.*

Proof. "$\Longrightarrow$" let $x \in \mathcal{R}$ and $y \in A$ s.t. $1-xy$ is not a unit. Then $1-xy \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset A$. But $x \in \mathcal{R} \subset \mathfrak{m}$, hence $1 \in \mathfrak{m}$ contradiction.

"$\Longleftarrow$" let $x \notin \mathcal{R}$ then $x \notin \mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset A$. Since $\mathfrak{m}$ is maximal we conclude that $(x) + \mathfrak{m} = A$. Hence there exists $y \in A,\ u \in \mathfrak{m}$ s.t. $xy + u = 1$. We conclude that $1 - xy \in \mathfrak{m}$, so in particular , $1 - xy$ is not a unit. $\square$

1.2 Lecture 2. local rings, coprime ideals, ideal quotients by Paul Steinmann

Definition 1.18. *A ring A is called a* ***local ring*** *if A admits precisely one maximal ideal;*

Example 1.19.

- *Every field is a local ring with maximal ideal $\mathfrak{m} = 0$, because every nonzero element is a unit.*
- *$\mathbb{K}[[X]]$ is the ring of formal power series over a field $\mathbb{K}$, it has a unique maximal ideal (X). One can check that every element with nonzero constant term is invertible. i.e. $(a_0(1-g))^{-1} = a_0^{-1}(1 + g + g^2 + ...)$*

Proposition 1.20.

- *Let A be a ring and $\mathfrak{m} \neq (1)$ is an ideal of A s.t. every $x \in A - \mathfrak{m}$ is a unit of A, then A is a local ring with maximal ideal $\mathfrak{m}$.*
- *Let A be ring and $\mathfrak{m} \subset A$ is a maximal ideal s.t. any element of $1 + \mathfrak{m} = \{1 + a | a \in \mathfrak{m}\}$ is a unit in A. Then A is a local ring.*

Proof. For first part, every proper ideal consists of non-units, hence is contained in $\mathfrak{m}$. In other words, an element is a unit iff it is not contained in any maximal ideal. For the second part, let $x \in A - \mathfrak{m}$. Since $\mathfrak{m}$ is maximal , we have $(x) + \mathfrak{m} = (1)$, hence , $\exists y \in A, t \in \mathfrak{m}$, s.t. $xy + t = 1$, which implies $xy = 1 - t \in 1 + \mathfrak{m}$. Thus ,$xy$ is a unit which implies that x is a unit, Now use the first part. $\square$

Definition 1.21. *A ring A is called* ***semilocal*** *if A admits finitely many maximal ideals.*

Example 1.22.

- *$\mathbb{Z}$ is not semilocal.*
- *Let $m \in \mathbb{Z}$. Then $\mathbb{Z}/(m\mathbb{Z})$ is a semilocal ring with maximal ideals $d\mathbb{Z}/m\mathbb{Z}$ for prime number $d|m$.*
- *In particular, for $p \in \mathbb{Z}$ prime , $\mathbb{Z}/p\mathbb{Z}$ is local ring.*

Reminder: Let $\mathfrak{a}, \mathfrak{b} \subset A$ be ideals their sum is

$$\mathfrak{a} + \mathfrak{b} := \{a + b | a \in \mathfrak{a}, b \in \mathfrak{b}\},$$

Which is the smallest ideal containing $\mathfrak{a} \cup \mathfrak{b}$. Also infinite sums $(\mathfrak{a}_i)_{i \in I} \subset A$ ideals,

$$\sum_{i \in I} \mathfrak{a}_i := \left\{ \sum_{i \in I} x_i | x_i \in \mathfrak{a}_i x_i = 0 \text{for almost all i} \right\}$$

And we also have

$$\mathfrak{a} \cdot \mathfrak{b} \textit{ or } \mathfrak{a}\mathfrak{b} = \left\{ \sum_{i \in I} x_i y_i | x_i \in \mathfrak{a}, y_i \in \mathfrak{b}, \text{all but finitely many terms are } 0 \right\}.$$

Definition 1.23. *Two ideals $\mathfrak{a}, \mathfrak{b} \subset A$ are called **coprime**[1] if $\mathfrak{a} + \mathfrak{b} = (1)$*

Remark 1.24. *If $\mathfrak{a}, \mathfrak{b} \subset A$ are coprime ideals then $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b}$.*
For general ideals $\mathfrak{a}, \mathfrak{b} \subset A$:

$$(\mathfrak{a} + \mathfrak{b}) \cdot (\mathfrak{a} \cap \mathfrak{b}) \subset \mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}.$$

However, for coprime ideals, we also have $\mathfrak{a}\mathfrak{b} \supset \mathfrak{a} \cap \mathfrak{b}$, because $1 = a + b$ for $a \in \mathfrak{a}, b \in \mathfrak{b}$, then $\forall x \in \mathfrak{a} \cap \mathfrak{b}$ we have $x = x \cdot 1 = x(a + b) = xa + xb \in \mathfrak{a} \cdot \mathfrak{b}$.

Proposition 1.25. *Let $\mathfrak{a}_1, ..., \mathfrak{a}_n \subset A$ be ideals, denote $\varphi : A \to \prod_{i \in I}^n (A/\mathfrak{a}_i)$ for the canonical homomorphism.*
(i) if $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime for $i \neq j$, then $\prod_{i=1}^n \mathfrak{a}_i = \cap_{i=1}^n \mathfrak{a}_i$.
(ii)φ is surjective iff $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime for $i \neq j$.
(iii) φ is injective iff $\cap_{i=1}^n \mathfrak{a}_i = (0)$.

[1] In some literature, it is called **comaximal**

Proof. (iii) Note that $ker\varphi = \cap_{i=1}^n \mathfrak{a}_i$.
(i) by induction on n. For $n = 2$ it is checked above. Suppose $n > 2$ let $\mathfrak{b} := \prod_{i=1}^{n-1} \mathfrak{a}_i = \cap_{i=1}^{n-1} \mathfrak{a}_i$ Since $\mathfrak{a}_i + \mathfrak{a}_n = (1)$ for $1 \leq i \leq n-1$. We have $x_i + y_i = 1$ for some $x_i \in \mathfrak{a}_i, y_i \in \mathfrak{a}_n$ Thus $\prod_{i=1}^{n-1} x_i = \prod_{i=1}^{n-1}(1 - y_i) \equiv 1 \mod \mathfrak{a}_n$ We conclude that $\mathfrak{a}_n + \mathfrak{b} = (1)$, s.t.

$$\prod_{i=1}^{n} \mathfrak{a}_i = \mathfrak{b}\mathfrak{a}_n = \mathfrak{a} \cap \mathfrak{a}_n = \cap_{i=1}^n \mathfrak{a}_i$$

(ii) "$\Longrightarrow$", Suppose φ is surjective. Let $i \neq j$, There exists an element $x \in A$ s.t. $\varphi(x) = (0, ..., 0, 1, 0, ..., 0)$, nonzero only at the i-th entry. Thus $x \equiv 1 \mod \mathfrak{a}_i$ and $x \equiv 0 \mod \mathfrak{a}_j$. So $1 = (1 - x) + x \in \mathfrak{a}_i + \mathfrak{a}_j$.
"$\Longleftarrow$" We show that for all $k \in \{1, ..., n\}$ there exists an element $x \in A$ s.t. $\varphi(x) = (0, ..0, 1, 0..0)$, nonzero at the k-th entry. Let $k \in \{1, ..., n\}$. For every $j \in \{1, ..., n\} \backslash \{k\}$. We have $\mathfrak{a}_k + \mathfrak{a}_j = (1)$, and thus there are elements $u_j \in \mathfrak{a}_k, v_j \in \mathfrak{a}_j$ s.t. $u_j + v_j = 1$. Define $x := \prod_{i \neq k} v_i$. Then $x \equiv 0 \mod \mathfrak{a}_j$, $\forall j \neq k$ and $x = \prod_{i \neq k}(1 - u_i) \equiv 1 \mod \mathfrak{a}_k$. Hence, $\varphi(x) = (0, .., 0, 1, 0, ..., 0)$ nonzero in the k-th entry.
As a result, if each pair $\mathfrak{a}_i, \mathfrak{a}_j$ is coprime, we have

$$A/\left(\prod_{i=1}^{n} \mathfrak{a}_i\right) \cong \prod_{i=1}^{n} (A/\mathfrak{a}_i).$$

□

Proposition 1.26. *Le t $\mathfrak{a}, \mathfrak{b} \subset A$ be ideals s.t. $rad(\mathfrak{a}), rad(\mathfrak{b})$ are coprime. Then $\mathfrak{a}, \mathfrak{b}$ are coprime.*

Proof. In fact, we have

$$rad(\mathfrak{a} + \mathfrak{b}) = rad(rad(\mathfrak{a}) + rad(\mathfrak{b})) = rad((1)) = (1)$$

Details in the exercise sheet. □

Proposition 1.27.
(i) Let $\mathfrak{p}_1, ..., \mathfrak{p}_n \subset A$ prime ideals and let $\mathfrak{a} \subset A$ be an ideal which is contained in $\cup_{i=1}^n \mathfrak{p}_i$ then $\mathfrak{a} \subset \mathfrak{p}_j$ for some j.
(ii)Let $\mathfrak{a}_1, ..., \mathfrak{a}_n \subset A$ be ideals and $\mathfrak{p} \subset A$ a prime ideal s.t. $\mathfrak{p} \supset \cap_{i=1}^n \mathfrak{a}_i$. Then $\mathfrak{p} \supset \mathfrak{a}_i$ for some i. If $\mathfrak{p} = \cap_{i=1}^n \mathfrak{a}_i$, then $\mathfrak{p} = \mathfrak{a}_i$ for some i .

Proof. Induction on n. For $n = 1$, easily checked. For $n > 1$. Assume that $\mathfrak{a} \not\subset \mathfrak{p}_i$ for all $1 \leq i \leq n$. We show $\mathfrak{a} \not\subset \cup_{i=1}^n \mathfrak{p}_i$. By induction hypothesis we know that $\forall k, \mathfrak{a} \not\subset \cup_{i \neq k}^n \mathfrak{p}_i$, so there exists $x_k \in \mathfrak{a}$ s.t. $x_k \notin \mathfrak{p}_i$, $\forall i \neq k$.We choose an x_k for each $\mathfrak{p}_k$ in the above manner. If $x_k \notin \mathfrak{p}_k$ for some k , then we are done. If not, then $x_k \in \mathfrak{p}_k$ for all k. Consider $y := \sum_{k=1}^n \prod_{j \neq k} x_j$. We have $y \in \mathfrak{a}$ and $y \equiv \prod_{j \neq k} x_j \mod \mathfrak{p}_k, \forall k$. Since $x_j \notin \mathfrak{p}_k$ for $j \neq k$ and $\mathfrak{p}_k$ is a prime ideal, we conclude that $y \notin \mathfrak{p}_k$ for all k hence $\mathfrak{a} \not\subset \cup_{i=1}^n \mathfrak{p}_i$.
(ii) Suppose for all $i \in \{1, ..., n\}$ we have $\mathfrak{p} \not\supset \mathfrak{a}_i$. Then there are $x_i \in \mathfrak{a}_i$ with $x_i \notin \mathfrak{p}$ for all i. And thus $\prod_{i=1}^n x_i \in \prod_{i=1}^n \mathfrak{a}_i \subset \cap_{i=1}^n \mathfrak{a}_i$. Since $\mathfrak{p}$ is a prime ideal $\prod_{i=1}^n x_i \notin \mathfrak{p}$, hence $\mathfrak{p} \not\supset \cap_{i=1}^n \mathfrak{a}_i$. If $\mathfrak{p} = \cap_{i=1}^n \mathfrak{a}_i \subset \mathfrak{a}_k$ for all k, which produce the last part. □

Definition 1.28. *Let $\mathfrak{a}, \mathfrak{b} \subset A$ be two ideals. Their* ***ideal quotient*** *is*

$$(\mathfrak{a} : \mathfrak{b}) := \{x \in A | x\mathfrak{b} \subset \mathfrak{a}\}.$$

The ***annihilator*** *of an ideal $\mathfrak{a} \subset A$ is*

$$Ann(\mathfrak{a}) := \{(0) : \mathfrak{a}\}.$$

Notation: For $x \in A$ we write $(a : x) := (a : (x))$.
Fact: (i)The ideal quotient of two ideals is again an ideal.
(ii)The set of zero divisors of A is

$$D = \cup_{x \neq 0} Ann(x) = \cup_{x \neq 0} (Ann(x))$$

Proof. (i) (ii) The first equality is just by definition. The the second equality.

$$D = rad(D) = rad(\cup_{x \neq 0} Ann(x)) = \cup_{x \neq 0} rad(Ann(x)),$$

where we extend rad to arbitrary subsets. □

Properties: Let $\mathfrak{a}, \mathfrak{b} \subset A$ be ideals
(i)$\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$
(ii) $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subset \mathfrak{a}$
(iii)$((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b} \cdot \mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$
(iv) for ideals $(\mathfrak{a}_i)_{i \in I} \subset A$, $(\cap_{i \in I} \mathfrak{a}_i : \mathfrak{b}) = \cap_{i \in I} (\mathfrak{a}_i : \mathfrak{b})$
(v) for ideals $(\mathfrak{b}_i)_{i \in I} \subset A$, $(\mathfrak{a} : \sum_{i \in I} \mathfrak{b}_i) = \cap_{i \in I} (\mathfrak{a} : \mathfrak{b}_i)$.

Definition 1.29. *Let $\mathfrak{a} \subset A$ be an ideal $f : A \to B$ a ring homomorphism. We define the **extension** of $\mathfrak{a}$ by f to be the ideal*

$$\mathfrak{a}^e := f_*(\mathfrak{a}) := Bf(\mathfrak{a})$$

, Which is just the ideal in B generated by $f(a)$
*For an dieal $\mathfrak{b} \subset B$. We define the **contraction** of $\mathfrak{b}$ via f to be the ideal*

$$\mathfrak{b}^c := f^*(\mathfrak{b}) := f^{-1}(\mathfrak{b})$$

Properties: Let $f : A \to B$ be a ring homomorphism , $\mathfrak{a} \subset A$ $\mathfrak{b} \subset B$ ideals. Then :
(i) $\mathfrak{a} \subset f^* f_*(\mathfrak{a}) = \mathfrak{a}^{ec}$,$\mathfrak{b} \supset f_* f^*(\mathfrak{b}) = \mathfrak{b}^{ce}$.
(ii) $f^*(\mathfrak{b}) = f^* f_* f^*(\mathfrak{b}), f_*(\mathfrak{a}) = f_* f^* f_*(\mathfrak{a})$.
(iii) Denote by C the set of contracted ideals in A and by E the set of extended ideals in B, then

$$C = \{\mathfrak{a} \subset A | f^* f_*(\mathfrak{a}) = \mathfrak{a}\},$$

$$E = \{\mathfrak{b} \subset B | f_* f^*(\mathfrak{b}) = \mathfrak{b}\}.$$

And $f_* : C \to E$ is a bijection with inverse f^*.

Proof. For (i), $\mathfrak{a} \subset f^{-1}f(\mathfrak{a}) \subset f^{-1}f_*(\mathfrak{a}) = f^* f_*(\mathfrak{a})$. For (ii) $\mathfrak{b} \supset f(f^{-1}(\mathfrak{b}))$ and $\mathfrak{b}$ is an ideal so $\mathfrak{b} \supset f_* f^*(\mathfrak{b})$. Part (iii) is left as an exercise. □

2 Modules

2.1 Lecture 3. Modules, Exact sequences by Professor Kowalski

Outline of this chapter

- Definition examples and Nakayama's Lemma
- exact sequences , snake lemma
- tensor products
- Algebra over a ring

Roughly speaking, module is "vector spaces for rings". It is closely related to fibre bundles in geometry. For the convention, we still fix commutative ring $\mathcal{A}$ with unit.

Definition 2.1. *A **module** M over $\mathcal{A}$ is an Abelian group with a linear action of $\mathcal{A}$ on M, i.e.*

$$\begin{aligned}\mathcal{A} \times M &\to M\\ (a, x) &\mapsto ax\end{aligned}$$

so that

$$\begin{aligned}&a(x + y) = ax + ay\\ &(a + b)x = ax + bx\\ &a(bx) = abx\\ &1x = x\end{aligned}$$

Example 2.2. *1. $\{0\}$ is an $\mathcal{A}$-module*

2. if $\mathcal{A}$ is a field $\mathcal{A}$-module is just $\mathcal{A}$-vector space.

3. $I \subset \mathcal{A}$ ideal; then $I$is an $\mathcal{A}$-module (a submodule of $\mathcal{A}$)

4. $\mathcal{A} = \mathbb{Z}$, an $\mathcal{A}$-module is an abelian group.

Definition 2.3. *M and N are $\mathcal{A}$-modules $f : M \to N$ is **$\mathcal{A}$-linear** if $f(ax + by) = af(x) + bf(y)$. The set of such $\rho : M \to N$ is denoted $Hom_{\mathcal{A}}(M, N)$. It is an $\mathcal{A}$-module with*

$$(f + g)(x) = f(x) + g(x),$$

$$(af)(x) = af(x).$$

If $Q \xrightarrow{h} M \xrightarrow{f} N \xrightarrow{g} P$, then $g \circ f \in Hom_{\mathcal{A}}(M, P)$ and $g \circ (f \circ h) = (g \circ f) \circ h$. Also, $id_M \in Hom_{\mathcal{A}}(M, M)$. In other word, $\mathcal{A}$-module is a category.

Definition 2.4. *$f : M \to N$ is an **isomorphism** iff $\exists g : N \longrightarrow M$ s.t. $g \circ f = id_N$ and $f \circ g = id_M$.*

Remark 2.5. *$Q \to (h)M \to (f)N \to (g)P$, then for any P, we get*

$$f^* : Hom_{\mathcal{A}}(M, P) \to Hom_{\mathcal{A}}(M, P)$$

$$g \mapsto g \circ f$$

and

$$f_* : Hom_{\mathcal{A}}(Q, M) \to Hom_{\mathcal{A}}(Q, N)$$
$$h \mapsto f \circ h$$

They are $\mathcal{A}$-linear, because for example

$$\begin{aligned}(f^*(ah+bg))(x) &= ((ah+bg)\circ f)(x)\\ &= (ah+bg)(f(x))\\ &= ah(f(x))+bg(f(x))\\ &= (af^*(h)+bf^*(g))(x).\end{aligned}$$

Remark 2.6. *Suppose M is an $\mathcal{A}$-module and $N \subset M$ as submodule, then M/N has the structure of $\mathcal{A}$-module such that the canonical projection $\pi : M \to M/N$ is $\mathcal{A}$-linear. $a(x+N) = ax+N$ is well defined because $aN \subset N$.*

Definition 2.7. *$f : M \longrightarrow N$ is a morphism of $\mathcal{A}$-modules.*

- *$Ker(f) = f^{-1}(\{0\}) \subset M$ is a submodule of M.*
- *$Im(f) = f(M) \subset N$ is a submodule of N.*
- *Coker(f)=N/Im(f) is an $\mathcal{A}$-module.*

Remark 2.8. 1. *$ker(f) = 0 \Longleftrightarrow f$ is injective.*

2. *$coker(f) = 0 \Longleftrightarrow f$ is surjective.*

3. *if $f : M \to N$ and $M' \subset ker(f)$, then we get an induced linear map $\bar{f}$, s.t the diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"] & N \\ M/M' \arrow[ur, "\bar{f}"] & \end{tikzcd}$$

commutes. It properly defined by $\bar{f}(x+M') = f(x)$ since $f(M') = \{0\}$ Then we have

$$Im(\bar{f}) = Im(f),$$

and

$$Ker(\bar{f}) = Ker(f)/M'.$$

In particular, if $M' = Ker(f)$, we get an isomorphism

$$M/Ker(f) \xrightarrow{\bar{f}} Im(f).$$

If M is an $\mathcal{A}$-module and $(M_i)_{i\in I}$ a family of submodules then $\cap_{i\in I}M_i$ is a submodule. If $X \subset M$ be a subset then the intersection of all submodules containing X is a submodule containing X, called the submodule generated by X, denote it by $\langle X\rangle$. One checks that

$$\begin{aligned}\langle X\rangle &= \{\text{linear combination of elements of } X\}\\ &= \left\{\sum_i a_i x_i | K \geq 0\mathbb{Z}, a_i \in \mathcal{A}, x_i \in X\right\}\end{aligned}$$

We write

$$\sum_{i\in I} M_i = \langle \cup_{i\in I} M_i\rangle$$

Definition 2.9. *If M satisfies $M = \langle X\rangle$ with X finite, then M is called finitely generated.*

Warning: A submodule of a finitely generated module is not necessarily finitely generated.

Example 2.10.

$$A = \mathbb{C}[X_1, ..., X_n, ...].$$

A is finitely generated by 1 however, the ideal $I = (X_1, .., X_n, ...)$ is note finitely generated

Lemma 2.11.

1. *$L \supset M \supset N$ are $\mathcal{A}$-modules, then there is an isomorphism*

$$(L/N)/(M/N) \to L/M$$

$$(x + N) + M/N \mapsto x + M$$

Rigorously: $\pi : L \longrightarrow L/M$ is surjective
$\Longrightarrow \bar{\pi} : L/N \to L/M$ is surjective
and $Ker(\bar{\pi}) = M/N$ so

$$(L/N)/(M/N) \cong Im(\bar{\pi}),$$

by Remark 2.8.

2. *$(M_1 + M_2)/M_2 \cong M_1/(M_1 \cap M_2)$*

Definition 2.12. *$I \subset A$ ideal; M module $IM = \langle \{ax | a \in I, x \in M\} \rangle \subset M$ as a submodule.*

M/IM is naturally an $\mathcal{A}/I$-module.

Definition 2.13. *$(M_i)_{i \in I}$ is a family of $\mathcal{A}$-modules*

1. *$\prod_{i \in I} M_i$ is an $\mathcal{A}$-module with $a(x_i) = (ax_i)$.*
2. *$\oplus_{i \in I} M_i \subset \prod_{i \in I} M_i$ is the submodule of $(x_i)_{i \in I}$ s.t. $x_i = 0$ for all but finitely many $i \in I$.*

Cartesian product and direct product are the same when there only finitely many summand. If $M_i = M, \forall i \in I$,we denote $M^{(I)} := \oplus_i M_i$. When I is finite, we denote it by M^I.

Definition 2.14. *An $\mathcal{A}$-module M is called **free** if there exists a set I s.t. M is isomorphic to $\mathcal{A}^{(I)}$.*

Example 2.15.

1. *if $\mathcal{A}$ is a field, then every $\mathcal{A}$-module is free.*
2. *$\mathcal{A} = \mathbb{Z}$: $\mathbb{Z}/2\mathbb{Z}$ is not free.*
3. ***Warning!** A submodule of a free module is not necessarily free.(e.g. ideals in $\mathcal{A}$)*
4. *If $\mathcal{A} \neq \{0\}$, $n, m \geq 0$ are integer and $\mathcal{A}^n \cong \mathcal{A}^m$ then $n = m$. $I \subset \mathcal{A}$ maximal ideal, then we get an isomorphism of $\mathcal{A}/I$-vector spaces,*

$$(\mathcal{A}/I)^n \cong (\mathcal{A}/I)^m \Longrightarrow n = m.$$

*This is called the **invariant basis number property**, all nontrivial commutative ring has the property.*

Proposition 2.16. *(Nakayama's lemma)*
M finitely generated $\mathcal{A}$-module, $I \subset$ Jacobson radical of A, which is just $\cap_{\mathfrak{m} \subset \mathcal{A}} \mathfrak{m}$, where $\mathfrak{m}$ are maximal ideals in $\mathcal{A}$. If $IM = M$, then $M = \{0\}$. e.g. $\mathcal{A}$ being a local ring and $I = \mathfrak{m}$ the only maximal in $\mathcal{A}$.

Proof. Suppose $M \neq 0$, and let $\{x_1, ..., x_n\}$ be a generating set with $n \geq 1$ minimal. Since $IM = M$, we have $x_n \in IM$, so

$$x_n = \sum_{i=1}^{k} a_i y_i, y_i \in M, a_i \in I$$

where $y_i = \sum_j b_{ij} x_j$. Then we have

$$x_n = \sum_{j=1}^{n} c_j x_j$$

$$c_j = \sum_{i}^{k} a_i b_{ij} \in I$$

$$\implies (1 - c_n) x_n = \sum_{j=1}^{n-1} c_j x_j$$

and $(1 - c_n) \equiv 1 \mod I \implies c_n \in$ the Jacobson radical, then $1 - c_n$ is invertible by Proposition 1.17.

$$x_n = (1 - c_n)^{-1} \sum_{j=1}^{n-1} c_j x_j,$$

which contradict the minimality of the generating set. □

Corollary 2.17. *M fin. gen. $\mathcal{A}$-module, $I \subset$ Jacobson radical , $N \subset M$. If $M = IM + N$, then $M = N$.*

Proof. $I(M/N) = (IM + N)/N = (M/N)$, then by Nakayama's lemma we know

$$M/N = 0.$$

□

Corollary 2.18. *$\mathcal{A}$ local ring, $\mathfrak{m} \subset \mathcal{A}$ the maximal ideal. M fin. gen. Then if $(x_1, ..., x_n) \in M$ are such that their classes modulo $\mathfrak{m}$ form a basis of $M/\mathfrak{m}M$ as $\mathcal{A}/\mathfrak{m}$-vector space, then they generate M.*

Proof. $N = \langle x_1, ..., x_n \rangle$ and apply Nakayama's lemma. □

Exact sequence

Definition 2.19. *(1)*$M' \to (f)M \to (g)M''$ *is* ***exact*** *if* $Im(f) = ker(g)$
(2) $M' \to (f_1)M \to (f_2)M'' \to ...$ *is* ***exact*** *if it is exact at each node.*

Example 2.20.

1. $0 \longrightarrow M \xrightarrow{g} M''$ *is exact, is equivalent to say that g is injective*

2. $M' \xrightarrow{f} M \longrightarrow 0$ *is exact, it is equivalent to say that f is surjective.*

3. *"Short exact sequence"* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *For instance,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{f} & M' \oplus M'' & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & x & \longmapsto & (x,0) & & & & \\ & & & & (x,y) & \longmapsto & y & & \end{array}$$

   the splitting sequence is exact. In fact short exact sequence of free modules always splits.

4. $\mathcal{A} = \mathbb{Z}$, *for non-free modules, for example*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{Z}/w\mathbb{Z} & \longrightarrow & 0 \\ & & x & \longmapsto & 2x & & & & \\ & & & & x & \longmapsto & x \ mod\ 2 & & \end{array}$$

   the exact sequence does not split.

2.2 Lecture 4. Snake Lemma, Tensor Product by Professor Kowalski

Proposition 2.21. *(Snake Lemma) Suppose we have such a commutative diagram, each row is exact,*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0 \end{tikzcd}$$

then we have a map $\delta : Ker(f'') \longrightarrow Coker(f')$ *s.t.*

$$0 \longrightarrow Ker(f') \longrightarrow Ker(f) \to Ker(f'') \xrightarrow{\delta} Coker(f') \longrightarrow Coker(f) \longrightarrow Coker(f'') \longrightarrow 0$$

is exact.

Proof. Consider the kernels and cokernels with the induced map between them. For notion consideration, we write $Ker(f')$ as K' and $Coker(f')$ as C' and so on. We have the extended commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, "\hat{u}"] \arrow[d, hook, "k'"] & K \arrow[r, "\hat{v}"] \arrow[d, hook, "k"] & K'' \arrow[d, hook, "k''"] & \\
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'"] \arrow[d, two heads, "q'"] & N \arrow[r, "v'"] \arrow[d, two heads, "q"] & N'' \arrow[r] \arrow[d, two heads, "q''"] & 0 \\
 & C' \arrow[r, "\bar{u}"] & C \arrow[r, "\bar{v}"] & C'' \arrow[r] & 0,
\end{tikzcd}$$

where the maps k', k, k'' are inclusion of the kernels as submodules and q', q, q'' are canonical projections, hence each column become exact now. $\bar{u}, \bar{v}$ are the morphism induced on quotient modules while $\hat{u}, \hat{v}$ are restrictions of u, v on submodules. One can check the induced maps on Cokernels are well defined, for example, for $\bar{v}$ to be well defined, because $q'' \circ v' \circ f = q'' \circ f'' \circ v = 0$, thus $Im(f) \subset Ker(q'' \circ v')$. One can also check that the above diagram is commutative. For example $x \in K'$, we have $f(\hat{u}(x)) = f(u(x)) = u'(f'(x)) = 0 \Longrightarrow \hat{u}(x) \in K$, then we have $u \circ k' = k \circ \hat{u}$.

1. Exactness at K'
   We already know $\hat{u} = u|_{Ker(f')}$, u injective implies that $\hat{u}$ is injective.

2. Exactness at K
   We easily check that $Im(\hat{u}) \subset Ker(\hat{v})$, because $k'' \circ \hat{v} \circ \hat{u} = v \circ u \circ k' = 0$, by the fact k'' is injective, we know $\hat{v} \circ \hat{u} = 0$. For the converse inclusion, if $x \in Ker(\hat{v}) = Ker(v) \cap Ker(f)$, then $x \in Im(u) \cap Ker(f)$. $\exists y \in M'$ s.t. $u(y) = x \Longrightarrow f(u(y)) = 0 \Longrightarrow u'(f'(y)) = 0$. Then because u' is injective, $f'(y) = 0 \Longrightarrow y \in K' \Longrightarrow x = \hat{u}(y)$. Then we conclude $Ker(\hat{v}) \subset Im(\hat{u})$, thus $Ker(\hat{v}) = Im(\hat{u})$.

3. Exactness at C''
   $q'' \circ v' = \bar{v} \circ q$, q'', v', q are all surjective, then we conclude that $\bar{v}$ has to be surjective.

4. Exactness at C
   We easily verify that $\bar{v} \circ \bar{u} = 0$, i.e. $\bar{v} \circ \bar{u} \circ q' = q'' \circ v' \circ u' = 0$ and

q' is surjective $\Longrightarrow \bar{v} \circ \bar{u} = 0$. For the converse inclusion, we choose $x + Im(f) \in Ker(\bar{v})$, where $x \in N$. $\bar{v}(x + Im(f)) = 0 = q'' \circ v'(x)$. $v'(x) \in Ker(q'') = Im(f'')$. $\exists y \in M''$ s.t. $f''(y) = v'(x)$, On the other hand, v is surjective ,$\Longrightarrow \exists z \in M$ s.t. $v(z) = y$. Then, we have $f''(v(z)) = v'(x) = v'(f(z))$. Then we choose $\tilde{x} = x - f(z)$, $\Longrightarrow x + Im(f) = \tilde{x} + Im(f)$ & $v'(\tilde{x}) = 0$. Then there exists $w \in N'$ s.t. $u'(w) = \tilde{x}$. Then, we check that $q \circ u'(w) = q(\tilde{x}) = \tilde{x} + Im(f)$, thus $\bar{u}(q(w)) = \tilde{x} + Im(f) \Longrightarrow \bar{u}(w + Im(f')) = x + Im(f)$. Then we conclude $Ker(\bar{v}) \subset Im(\bar{u})$.

5. Construct δ

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, "\hat{u}"] \arrow[d, hook, "k'"] & K \arrow[r, "\hat{v}"] \arrow[d, hook, "k"] & K'' \arrow[d, hook, "k''", "x \mapsto x"'] \arrow[dddll, snake, "\delta"] & \\
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v", "\exists y \mapsto x"'] \arrow[d, "f", "y \mapsto f(y)"'] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "u'", "\exists z \mapsto f(y)"'] \arrow[d, two heads, "q'", "z \mapsto {[z]}"'] & N \arrow[r, "v'"] \arrow[d, two heads, "q"] & N'' \arrow[r] \arrow[d, two heads, "q''"] & 0 \\
 & C' \arrow[r, "\bar{u}"] & C \arrow[r, "\bar{v}"] & C'' \arrow[r] & 0,
\end{tikzcd}$$

For an element $x \in K''$, $k''(x) = x \in M''$ and $f''(x) = 0$. $\because v$ is surjective, $\therefore \exists y \in M$ s.t. $v(y) = x$. Then $f''(x) = f''(v(y)) = v'(f(y)) = 0 \Longrightarrow f(y) \in Ker(v') = Im(u')$. Therefore, there exists $z \in N'$ s.t. $u'(z) = f(y)$. The choice of z is unique once we fix y, because u' is injective. **We define** $\delta : K'' \longrightarrow C', x \mapsto [z] = z + Im(f')$. For δ to be well defined, it can not depend on the choice of y and z. Choose another $\tilde{y} \in M$ and corresponding $\tilde{z} \in N'$ s.t. $v(\tilde{y}) = x$ and $u'(\tilde{z}) = f(\tilde{y})$. We have $v(\tilde{y} - y) = 0$, $\exists w \in M'$ s.t. $u(w) = \tilde{y} - y$. Then $f(u(w)) = u'(f'(w)) = f(\tilde{y} - y) = f(\tilde{y}) - f(y)$. Then we have $u'(\tilde{z}) - u'(z) = u'(f'(w))$. Since u' is injective, we have $\tilde{z} = z + f'(w)$, thus $\tilde{z} + Im(f') = z + Im(f')$. Then we conclude that δ is well defined.

6. Exactness at K''

For $x \in K$, we formally write

$$\begin{aligned}
\delta(\hat{v}(x)) &= u'^{-1}(f(v^{-1}(k''(\hat{v}(x))))) + Im(f') \\
&= u'^{-1}(f(v^{-1}(v(k(x))))) + Im(f') \\
&= u'^{-1}(f(k(x))) + Im(f') \\
&= 0 \text{ because } f \circ k = 0. \\
&\Longrightarrow Im(\hat{v}) \subset Ker(\delta)
\end{aligned}$$

For the converse inclusion. $\forall x \in Ker(\delta)$, we trace back to the construction of δ, and select the corresponding $y \in M,\ z \in N'$, where $v(y) = x$ and $u'(z) = f(y)$. $\because x \in Ker(\delta), \therefore z \in Im(f')$. $\Longrightarrow \exists w \in M'$ s.t. $f'(w) = z$. Then we choose another $\tilde{y} = y - u(w)$, one verifies that $v(\tilde{y}) = v(y) - v(u(w)) = v(y) = x$. (this is legal, because we know δ does not depend on the choice of y) Also, we know $f(\tilde{y}) = f(y) - f(u(w)) = f(y) - u'(f'(w)) = f(y) - u'(z) = 0$. Then we know $\tilde{y} \in Ker(f) = K$, we conclude that $\hat{v}(\tilde{y}) = x$, thus $Ker(\delta) \subset Im(\hat{v})$.

7. Exactness at C'
For $x \in K''$, we formally write

$$\begin{aligned}
\bar{u}(\delta(x)) &= \bar{u}\left(u'^{-1}(f(v^{-1}(k''(x)))) + Im(f')\right) \\
&= (q \circ u')\left(u'^{-1}(f(v^{-1}(k''(x))))\right) \\
&= q(0 + f(v^{-1}(k''(x)))) \\
&= 0 \\
&\Longrightarrow Im(\delta) \subset Ker(\bar{u})
\end{aligned}$$

For the converse inclusion, we choose an element $z + Im(f') \in Ker(\bar{u})$. Then $\bar{u}(z + Im(f')) = q \circ u'(z) = 0$, then we have $\exists y \in M$ s.t. $u'(z) = f(y)$. Also we have $v'(u'(z)) = v'(f(y)) = 0, \Longrightarrow f''(v(y)) = 0$. $v(y) \in Ker(f'') = K''$. We can check that $\delta(v(y)) = z + Im(f')$. Hence, we conclude that $Ker(\bar{u}) \subset Im(\delta)$.

$\square$

Example 2.22. *(Application of snake lemma) We have such a commutative diagram, each row is exact. Suppose the middle map is isomorphism.*

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

then we have a map $\delta : Ker(f'') \longrightarrow Coker(f')$ *s.t.*

$$0 \longrightarrow Ker(f') \longrightarrow \{0\} \to Ker(f'') \xrightarrow{\delta} Coker(f') \longrightarrow \{0\} \longrightarrow Coker(f'') \longrightarrow 0$$

is exact. Thus we get $\delta : Ker(f'') \longrightarrow Coker(f')$ *is an isomorphism.*

Proposition 2.23.

If $0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ is exact, then for any $\mathcal{A}$-module N,

$$\begin{array}{ccccccc} 0 & \longrightarrow & Hom_{\mathcal{A}}(M'', N) & \xrightarrow{v^*} & Hom_{\mathcal{A}}(M, N) & \xrightarrow{u^*} & Hom_{\mathcal{A}}(M', N) \\ & & f & \longmapsto & f \circ v & & \\ & & & & g & \longmapsto & g \circ u \end{array} \tag{*}$$

is exact, in general u^* is not surjective. Also,

$$\begin{array}{ccccccc} Hom_{\mathcal{A}}(N, M'') & \xrightarrow{u_*} & Hom_{\mathcal{A}}(N, M) & \xrightarrow{v_*} & Hom_{\mathcal{A}}(N, M') & \longrightarrow & 0 \\ f & \longmapsto & u \circ f & & & & \\ & & g & \longmapsto & v \circ g & & \end{array} \tag{**}$$

is exact but u_* is in general not always injective.
More precisely, we have **right exactness of functor** $Hom(\_, N)$:

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0 \text{ is exact} \Longleftrightarrow (*) \text{ is exact for all } N$$

and **Left exactness of functor** $Hom(N, \_)$:

$$0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \text{ is exact } \Longleftrightarrow (**) \text{ is exact for all } N.$$

Proof. For "$\Longrightarrow$" part of the first statement, we assume $M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ is exact. Let N be $\mathcal{A}$-module, then we check that:

1. $u^* \circ v^* = 0$
   Let $f : M'' \longrightarrow N$, $(u^* \circ v^*)(f) = f \circ v \circ u = f \circ (v \circ u) = 0$

2. v^* is injective
   Let $f : M'' \longrightarrow N$ be such that $v^*(f) = f \circ v = 0 \Longrightarrow f(Im(v)) = 0$
   $\Longrightarrow f = 0$ because v is surjective.

3. $Ker(u^*) \subset Im(v^*)$
   Let $f : M \longrightarrow N$ be such that $u^*(f) = f \circ u = 0$. Then $f(Im(u)) = 0$ so $f(Ker(v)) = 0$, so there is $\bar{f} : M/Ker(v) \longrightarrow N$ s.t. $\bar{f} \circ p = f$.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "p"] & N \\
M/Ker(v) \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

   We know that v induces an isomorphism

$$\begin{tikzcd}
Im(v) = M'' \arrow[dr, bend right, "\bar{v}^{-1}"'] & M \arrow[l, "v"'] \arrow[r, "f"] \arrow[d, "p"] & N \\
 & M/Ker(v) \arrow[ul, "\bar{v}"] \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

   Let $f' = \bar{f} \circ \bar{v}^{-1} \in Hom(M'', N)$, we compute $v^*(f') = f' \circ v = \bar{f} \circ \bar{v}^{-1} \circ v = \bar{f} \circ p = f$ thus $f \in Im(v^*)$

We then give an example where the surjectivity of u^* fails
Consider $\mathcal{A} = \mathbb{Z}$, $0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ is exact.

$$\begin{aligned} v^* : & Hom(\mathbb{Z}, N) \to Hom(\mathbb{Z}, N) \\ & f \longmapsto f \circ (\times 2) \end{aligned}$$

is not surjective if $N = \mathbb{Z}$, because $f = Id_{\mathbb{Z}}$, we want to find a map g such that the following diagram commutes,

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "Id"] & \mathbb{Z} \arrow[dl, dashed, "?g"] \\
 & \mathbb{Z} &
\end{tikzcd}$$

but there is no g such that $g \circ (\times 2) = Id_{\mathbb{Z}}$ because every morphism in $Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is of the form $\times q$, where $q \in \mathbb{Z}$.

Conversely, for the "$\Longleftarrow$" part of the first statement, assume (*) is always exact. We want to show that $M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ is exact,

$$\begin{array}{ccccccc} 0 & \longrightarrow & Hom_{\mathcal{A}}(M'', N) & \xrightarrow{v^*} & Hom_{\mathcal{A}}(M, N) & \xrightarrow{u^*} & Hom_{\mathcal{A}}(M', N) \\ & & f & \longmapsto & f \circ v & & \\ & & & & g & \longmapsto & g \circ u \end{array}$$

1. Let $N = Coker(v)$ and $[p : M'' \longrightarrow Coker(v)] \in Hom(M'', N)$, then $v^*(p = p \circ v = 0)$. Since v^* is injective, we have $p = 0$, in other words $M'' = Ker(p) = Im(v)$ so v is surjective.

2. Take $N = M''$ and $f = Id_{M''}$, $(u^* \circ v^*)(f) = 0$ means $Id_{M''} \circ v \circ u = 0$ $\implies v \circ u = 0$, hence $Im(u) \subset Ker(v)$.

3. Take $N = M/Im(u)$, and $p : M \longrightarrow N$ projection, we have $u^*(p) = p \circ u = 0$. So $p \in Ker(u^*)$, so there exists $f \in Hom(M'', N)$ s.t. $v^*(f) = f \circ v = p$.

$$\begin{tikzcd} M' \arrow[r, "f"] & N = M/Im(u) \\ M \arrow[u, "v"] \arrow[ur, "p"] & \end{tikzcd}$$

   Hence $Ker(v) \subset Ker(p)$ and $Ker(v) \subset Im(u)$, then we can conclude that $Ker(v) = Im(u)$.

The above steps proves the first statement and proof of the second statement is similar. $\square$

Tensor Product

Definition 2.24. *M, N, P are $\mathcal{A}$-modules, A map $f : M \times N \longrightarrow P$ is called $\mathcal{A}$-**bilinear** if*

$$f(ax + by, z) = af(x, z) + bf(y, z)$$

$$f(x, ay + bz) = af(x, y) + bf(x, z)$$

$$Bil_{\mathcal{A}}(M, N, P) = \{ \text{ all } \mathcal{A}\text{-bilinear maps form } M \times N \text{ to } P\}.$$

$Bil_{\mathcal{A}}(M, N, P)$ is an $\mathcal{A}$-module.

Definition 2.25. *M, N are $\mathcal{A}$-modules and the **tensor product** gives an $\mathcal{A}$-module $M \otimes_{\mathcal{A}} N$ such that $Bil_{\mathcal{A}}(M, N; P) = Hom_{\mathcal{A}}(M \otimes_{\mathcal{A}} N, P)$. $Bil_{\mathcal{A}}(M, N; P)$ is obviously an $\mathcal{A}$-module, with sum and scalar multiplication performed valuewise.*

Theorem 2.26. *M, N are $\mathcal{A}$-modules. There exists a pair (T, β) where T is an $\mathcal{A}$-module and $\beta : M \times N \longrightarrow T$ s.t. any $\mathcal{A}$-bilinear map $b : M \times N \longrightarrow P$*

factors through (T, β), i.e. there exists a unique $f : T \longrightarrow P$ s.t. the following diagram commutes.

$$\begin{tikzcd} M \times N \arrow[r, "b"] \arrow[d, "\beta"'] & P \\ T \arrow[ur, "\exists! f"'] & \end{tikzcd}$$

This is what we call ***universal property****. One can check that if it exists, it is unique.*

2.3 Lecture 5. Properties of Tensor Product

The motivation of tensor product is to "classify" bilinear/multilinear maps between modules over some ring $\mathcal{A}$.

Definition/Theorem 2.27. *M and N are $\mathcal{A}$-modules,* ***there exists a best possible bilinear map*** *$M \times N \to M \otimes N$. That is to say : there exists a module T (denoted $M \otimes N$ or $M \otimes_{\mathcal{A}} N$) and a bilinear map $f : M \times N \longrightarrow T$. By "best possible", we mean: For all module P and all bilinear map $b : M \times N \to P$, here exists a unique $\tilde{b} : T \longrightarrow P$ s.t. the following diagram commutes.*

$$\begin{tikzcd} M \times N \arrow[r, "b"] \arrow[d, "f"'] & P \\ T \arrow[ur, "\exists! \tilde{b}"'] & \end{tikzcd}$$

What's more (T, f) is ***strongly unique*** *which means* ***it is unique up to unique isomorphism***

$$\begin{tikzcd} M \times N \arrow[r, "f'"] \arrow[d, "f"'] & T' \arrow[dl, "\exists! k"'] \\ T \arrow[ur, bend right, "\exists! j"'] & \end{tikzcd}$$

Proof. **Uniqueness**
The uniqueness is just the direct result of universal property. By definition, f is bilinear. Apply the universal property with $P = T'$, $b = f'$, then we know $j := \tilde{b} : T \to T'$. Similarly, we can construct k by swapping T, T'.

Consider $k \circ j : T \to T$, apply the universal property with $P := T$, $b := f$

$$\begin{tikzcd}
M \times N \arrow[r, "f"] \arrow[d, "f"] & T \\
T \arrow[ur, "\exists! \tilde{b}"'] &
\end{tikzcd}$$

We know $\exists! \tilde{b}$ s.t. the diagram commutes. Then we have $\tilde{b} \circ f = f$, but another obvious map having this property is just id_T. Then, we get to the conclusion $k \circ j = id_T$ by the uniqueness of $\tilde{b}$. Similarly, we get $j \circ k = id_{T'}$. Altogether, we conclude that (T, f) is unique up to unique isomorphism.
Existence
Form the free module $C := \mathcal{A}^{M \times N}$, where

$$\mathcal{A}^{(M\times N)} = \left\{ \sum_{(x,y)\in M\times N} a_{(x,y)}(x,y) \middle| a_{(x,y)} \in \mathcal{A}, \text{almost all } a_{(x,y)} = 0 \right\}.$$

We'd better mention the universal property of the free module $\mathcal{A}^{(M\times N)}$, every map $q : M \times N \longrightarrow P$ can be extended to $\tilde{q} : \mathcal{A}^{(M\times N)} \longrightarrow P$
Let submodule $D \subseteq C$, then there is an induced map $\bar{g} : M \times N \longrightarrow C/D$ for defining map $g : M \times N \longrightarrow C$ of the free module. Then we consider a certain submodule D with the following two equivalent definitions

- D is the smallest submodule for which all the induced map $\bar{g} : M \times N \longrightarrow C/D$ is bilinear.

- D it the submodule generated by the following elements

$$\left\{ \begin{array}{l} (x+x', y) - (x,y) - (x',y) \\ (x, y+y') - (x,y) - (x,y') \\ a(x,y) - (ax,y) \\ a(x,y) - (x,ay) \end{array} \middle| \forall a \in \mathcal{A}, \forall x, x' \in M, \forall y, y' \in N \right\}$$

The equivalence of two definition can be explained by the definition of "bilinear maps".
We want to show that C/D is what we are looking for. First, we claim, for all bilinear mop $b : M \times N \to P$, $Ker(\tilde{b}) \supseteq D$.

The proof is to just check it by hand, e.g.

$$\begin{aligned}
&\tilde{b}((x+x',y)-(x,y)-(x',y)) \\
&=\tilde{b}((x+x',y))-\tilde{b}((x,y))-\tilde{b}((x',y)) \\
&=b(x+x',y)-b(x,y)-b(x',y) \\
&=0(\text{by } b \text{ is bilinear})
\end{aligned}$$

The characterization of $\tilde{b}$ determines its restriction of $g(M \times N) \subseteq T$. Clear by construction that $g(M \times N)$ generates T. We get the conclusion that $\bar{g} : M \times N \to C/D = T$. □

Also note that, in general

$$S := \{m \otimes n | (m,n) \in M \times N\} \neq M \otimes N$$

, e.g. $\mathbb{Z}^n \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ but S generates $M \otimes N$ as we saw in the proof.

Example 2.28. *Natural isomorphisms, $\exists!$ isomorphisms*

1. $M \otimes N \cong N \otimes M$
2. $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$
3. $M \otimes (N_1 \oplus N_2) \cong (M \otimes N_1) \oplus (M \otimes N_2)$
4. $\mathcal{A} \otimes M \cong M$

Proof. we prove part 3. Consider a map:

$$\begin{aligned}
b : M \times (N_1 \oplus N_2) &\to M \otimes N_1 \oplus M \otimes N_2 \\
(m,(n_1,n_2)) &\mapsto (m \otimes n_1, m \otimes n_2).
\end{aligned}$$

We can check that b is bilinear, for example

$$\begin{aligned}
&b(m+m',(n_1,n_2)) \\
&=((m+m') \otimes n_1, (m+m') \otimes n_2) \\
&=(m \otimes n_1 + m' \otimes n_1, m \otimes n_2 + m' \otimes n_2) \\
&=(m \otimes n_1, m \otimes n_2) + (m' \otimes n_1, m' \otimes n_2) \\
&=b(m,(n_1,n_2)) + b(m',(n_1,n_2)).
\end{aligned}$$

As a result the bilinear map b must factor through $M \otimes (N_1 \oplus N_2)$, and we denote the corresponding map $f : M \otimes (N_1 \oplus N_2) \to M \otimes N_1 \oplus M \otimes N_2$.

$$f(m \otimes (n_1,n_2)) = (m \otimes n_1, m \otimes n_2).$$

We use the terminology **pure tensor** to name the tensors like $x \otimes y \in M \otimes N$, obviously, $M \otimes N$ is linearly generated by pure tensors. We want to show that f is an isomorphism. Need to find the inverse map g of f.

define

$$\begin{aligned} g_1 : M \otimes N_1 &\longrightarrow M \otimes (N_1 \oplus N_2) \\ (m \otimes n_1) &\longmapsto m \otimes (n_1, 0) \end{aligned}$$

similarly, we can construct

$$\begin{aligned} g_2 : M \otimes N_2 &\longrightarrow M \otimes (N_1 \oplus N_2) \\ (m \otimes n_2) &\longmapsto m \otimes (0, n_2) \end{aligned}$$

Then, we define $g = g_1 \oplus g_2$. We want to show $f \circ g = id, g \circ f = id$.

$$\begin{aligned} & f \circ g(m \otimes n, m' \otimes n_2) \\ & = f(m \otimes (n_1, 0) + m' \otimes (0, n_2)) \\ & = (m \otimes n_1, 0) + (0, m' \otimes n_2) \\ & = (m \otimes n_1, m' \otimes n_2) \end{aligned}$$

Then $f \circ g = id$ on pure tensors, hence it is identity on all tensors, because $f \circ g$ is linear, and pure tensor generates the whole tensor product module. $\square$

Consider $\mathcal{A}^m = \mathcal{A} \oplus \mathcal{A} \oplus ... \oplus \mathcal{A}$ (finite free module), by the isomorphism 4 in the above example

$$\begin{aligned} \mathcal{A} \otimes \mathcal{A} &\cong \mathcal{A} \\ x \otimes y &\mapsto xy \end{aligned}$$

also by iterating (3) and (4), we get

$$\mathcal{A}^m \otimes \mathcal{A}^n \cong \mathcal{A}^{mn},$$

compared to the known result

$$\mathcal{A}^m \oplus \mathcal{A}^n \cong \mathcal{A}^{m+n}.$$

More directly, if $e_1^{(1)}, ..., e_m^{(1)}$ standard basis for $\mathcal{A}^m$, $e_1^{(2)}, ..., e_n^{(2)}$ standard basis for $\mathcal{A}^n$, then

$$\left\{ e_i^{(1)} \otimes e_j^{(2)} \middle| , m \geq i \geq 1, n \geq j \geq 1 \right\}$$

form a basis of $\mathcal{A}^m \otimes \mathcal{A}^n$ and induces $\cong \mathcal{A}^{mn}$

To see this directly, consider a bilinear map $f : \mathcal{A}^m \times \mathcal{A}^n \longrightarrow P$, where P is some module.

$$\mathcal{A}^m \ni x = x_1 e_1^{(1)} + ... + x_m e_m^{(1)},\ x_i \in \mathcal{A}$$

$$\mathcal{A}^n \ni y = y_1 e_1^{(1)} + ... + y_m e_m^{(1)},\ y_i \in \mathcal{A}$$

Then

$$f(x,y) = \sum_{\substack{i=1...m \\ j=1...n}} x_i y_j f(e_i^{(1)} \otimes e_j^{(2)}),$$

where we can define $f(e_i^{(1)} \otimes e_j^{(2)}) =: a_{ij} \in P$ Generally, given an mn-tuple (a_{ij}) in P we may define a bilinear $f : \mathcal{A}^m \times \mathcal{A}^n \longrightarrow P$ by the above formula.

$$\begin{tikzcd}
\mathcal{A}^m \times \mathcal{A}^n \arrow[r, "(e_i^{(1)}, e_j^{(2)}) \longmapsto e_i^{(1)} \otimes e_j^{(2)}"] \arrow[d, "f"] & \mathcal{A}^{\oplus\{e_i^{(1)} \otimes e_j^{(2)}\}} \arrow[dl, "\exists! \tilde{f} \ s.t. \ \tilde{f}(e_{ij}) = a_{ij}"] \\
P &
\end{tikzcd}$$

Remark 2.29. *More generally, we may define the n-fold tensor products $M_1 \otimes ... \otimes M_n$.*

$\{$*multilinear maps* $: M_1 \times ... \times M_n \longrightarrow P\} \leftrightarrow \{$*linear maps* $: M_1 \otimes ... \otimes M_n \longrightarrow P\}$

Let $V = \mathbb{R}^n$, then

$$\{\textit{inner products on } V\} \leftrightarrow \{\textit{linear functions on } V \otimes V\}$$

Remark 2.30. ***Extension of scalars*** *Consider a ring morphism $f : \mathcal{A} \to \mathcal{B}$ and an $\mathcal{A}$-module M, we can construct a $\mathcal{B}$-module*

$$M_{\mathcal{B}} := M \otimes_{\mathcal{A}} \mathcal{B},$$

where $\mathcal{B}$ is regarded as an $\mathcal{A}$-module via f, i.e. $a \cdot b = f(a)b$. And the $\mathcal{B}$ action on $M_{\mathcal{B}}$ is like $b \cdot (m \otimes z) := m \otimes bz$

Example 2.31.

- $M = \mathcal{A}^m \Longrightarrow M_{\mathcal{B}} = \mathcal{B}^m$
- $\mathcal{A} = \mathbb{R}, \mathcal{B} = \mathbb{C} \Longrightarrow (\mathbb{R}^n)_{\mathbb{C}} := (\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^n$

2.4 Lecture 6. Flatness

The meaning of $x \otimes y$ depends on the modules to which we regard x and y are belonging. In fact, one can have $x \in M' \subseteq M$ and $y \in N' \subset N$ but

$$M' \otimes N' \ni x \otimes y \neq x \otimes y \in M \otimes N$$

Example 2.32. *$\mathcal{A} = \mathbb{Z}$, $M' = 2\mathbb{Z} \subseteq M = \mathbb{Z}$, $N' = \mathbb{Z}/2 = N$, then $2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \ni 2 \otimes 1 \neq 0$, but $\mathbb{Z} \otimes \mathbb{Z}/w\mathbb{Z} \ni 2 \otimes 1 = 0$*

In summary, we no $M' \subset M, N' \subset N$ does not indicate that $M' \otimes N' \subset M \otimes N$, which means the simple inclusion is not an injective morphism.

But $\otimes$ is indeed a **bifunctor**. Given module morphisms

$$\begin{aligned} f : M' &\longrightarrow M \\ g : N' &\longrightarrow N \\ \exists! f \otimes g : M' \otimes N' &:\longrightarrow M \otimes N \\ x \otimes y &\longmapsto f(x) \otimes g(y) \end{aligned}$$

and

$$(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g')$$

For example, we alway consider the case $g = 1_N$ with N $\mathcal{A}$-module, then each morphism $f : M' \longrightarrow M$ is mapped to $f \otimes 1_N : M' \otimes N \longrightarrow M \otimes N$.

Definition 2.33. *N is **flat** if $\forall f : M' \longrightarrow M$ s.t.*

$$f : \textit{injective} \implies f \otimes 1_N \textit{ is injective}$$

In other words,

$$M' \subset M \implies \text{“}M' \otimes N \subset M \otimes N\text{”}$$

Example 2.34.

- *$\{0\}$ is a flat $\mathcal{A}$-module*
- *$\mathcal{A}$ is a flat $\mathcal{A}$-module, because $M \otimes_{\mathcal{A}} \mathcal{A} = M$ and $f = f \otimes 1_{\mathcal{A}}$*

Lemma 2.35. *Let $(N_i)_{i \in I}$ be a family of modules over $\mathcal{A}$, then $\oplus_{i \in I} N_i$ is flat iff each N_i is flat.*

Proof. Suppose each N_i is flat. Let $M' \xrightarrow{f} M$ be injective. Suppose,

$$M' \otimes (\oplus_i N_i) \xrightarrow{f \otimes 1} M \otimes (\oplus_i N_i)$$

is not injective, i.e. $z \in Ker(f \otimes 1_N) \neq 0$. Let N denote $\oplus_i N_i$ and the i-th projection $\pi_i : N \longrightarrow N_i$.

$$\begin{tikzcd}
0 \neq z \quad \in & \oplus_i(M' \otimes N_i) \arrow[r, "\rho_i'"] \arrow[d, equal] & M' \otimes N_i \arrow[d, equal] \\
& M' \otimes (\oplus_i N_i) \arrow[r, "1_{M'} \otimes \pi_i"] \arrow[d, "f \otimes 1_N"] & M' \otimes N_i \arrow[d, "f \otimes 1_{N_i}"] \\
& M \otimes (\oplus_i N_i) \arrow[r, "1_M \otimes \pi_i"] \arrow[d, equal] & M \otimes N_i \arrow[d, equal] \\
& \oplus_i(M \otimes N_i) \arrow[r, "\rho_i"] & M \otimes N_i
\end{tikzcd}$$

$z \neq 0 \Longrightarrow \exists i \in I$ s.t. $\rho_i'(z) \neq 0 \Longrightarrow (f \otimes 1_{N_i})(\rho_i'(z)) \neq 0 \in M \otimes N_i$. But $(f \otimes 1_{N_i})(\rho_i'(z)) = \rho_i(f \otimes 1_N(z))$ is the i-th component of $(f \otimes 1_N)(z) = 0$ by assumption, which gives the contradiction. The converse is simpler. $\square$

Corollary 2.36. *If M is a free $\mathcal{A}$-module, then it is a flat module.*

Proof. We already know $\mathcal{A}$ is flat, then by the previous lemma, we know $\oplus_{i \in I} \mathcal{A}$ is flat. $\square$

Example 2.37. *Consider a system of linear equations*

$$S : f_1(x_1, ..., x_n) = ... = f_m(x_1, ..., x_n) = 0,$$

where these f_i's has coefficients in $\mathbb{R}$. Then S has solution over $\mathbb{R}$ iff S has solution over $\mathbb{C}$ (This claim works for any field extension L/K instead of $\mathbb{C}/\mathbb{R}$) A simple proof goes like: "$\Longrightarrow$" is trivial, for the converse, we take the real or the imaginary part of a complex solution.

For a second proof:

$$M' = \mathbb{R}^n \xrightarrow{f} M = \mathbb{R}^m,$$

where $f = (f_1, ..., f_m)$. $\mathcal{A} = \mathbb{R}$, $N = \mathbb{C} \cong \mathbb{R} \oplus \mathbb{R}i$ is free, then by the above corollary, we know N is flat. Then S has a solution over $\mathbb{R}$ iff $Ker(f) \neq 0$, *and S has a solution over $\mathbb{C}$ iff $Ker(f \otimes 1_{\mathbb{C}}) \neq 0$. If $f \otimes 1$ is not injective, by the definition of flat module, we know f is not injective, which conclude the proof. This second proof works for arbitrary field extension, because the field extensions are always free modules over the initial field.*

Proposition 2.38. *(Right exactness of $\otimes N$)*
Consider an exact sequence of $\mathcal{A}$-modules

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

Then we have

$$M' \otimes N \xrightarrow{f\otimes 1} M \otimes N \xrightarrow{g\otimes 1} M'' \otimes N \longrightarrow 0$$

is exact for arbitrary $\mathcal{A}$-module N.

Proof. Obviously $g \otimes 1$ is surjective. We only need to prove the exactness at $M \otimes N$. As for the easier inclusion, $Im(f \otimes 1) \subseteq Ker(g \otimes 1)$ because $(g \otimes 1) \circ (f \otimes 1) = (g \circ f) \otimes 1 = 0$. Then it remains to show

$$\frac{M \otimes N}{Im(f \otimes 1)} \xrightarrow{\psi} M'' \otimes N$$

is an isomorphism. ψ is induced by $g \otimes 1$, well defined because $Im(f \otimes 1) \subseteq Ker(g \otimes 1)$.

Now, we construct a two-sided inverse φ of ψ.

$$\begin{tikzcd}
M'' \otimes N \arrow[r, dashed, "\exists\varphi"] & \frac{M\otimes N}{Im(f\otimes 1)} \\
M'' \times N \arrow[u] \arrow[ur, dashed, "\exists\varphi_0"] & \\
M \times N \arrow[u, "g\times 1"] \arrow[uur, "\varphi_1"'] &
\end{tikzcd}$$

Consider the map φ_1, it is the composition of the canonical projection and the defining map of tensor product. $\varphi_1(x, y) \mapsto x \otimes y + Im(f \otimes 1)$. Consider $(x'', y) \in M'' \times N$, which is the image of (x, y) under $g \times 1$. Then we can define $\varphi_0(x'', y) := \varphi_1(x, y)$. It is well-defined, because if there is another (x_1, y) also map to (x'', y), the difference

$$x - x_1 \in Ker(g) = Im(f),$$

hence $\exists z \in M'$ $x - x_1 = f(z)$. $\implies (x - x_1) \otimes y = (f \otimes 1)(z \otimes y)$ Then

$$\varphi_1(x, y) - \varphi(x_1, y) = (x - x_1) \otimes y + Im(f \otimes 1) = 0.$$

Then it remains to check φ_0 is bilinear so that φ_0 lifts to a φ on $M'' \otimes N$. Also we need to check the φ is indeed the two-sided inverse of ψ.

Consider $\varphi_0(x'', ay + bv)$ and $\varphi_0(ax'' + bw'', y)$. Chose x and w in the preimages $g^{-1}(x'')$ and $g^{-1}(w'')$. By the linearity of g, we can safely choose $ax + bw$ in the pre-image of $ax'' + bw''$ Knowing that φ_1 is bilinear (because the defining map of tensor product is bilinear and canonical projection is linear), we have

$$\begin{aligned}&\varphi_0(x'', ay + bv) = \varphi_1(x, ay + bv)\\&= a\varphi_1(x, y) + b\varphi_1(x, v) = a\varphi_0(x'', y) + b\varphi_0(x'', v)\end{aligned}$$

and

$$\begin{aligned}&\varphi_0(ax'' + bw'', y) = \varphi_1(ax + bw, y)\\&= a\varphi_1(x, y) + b\varphi(w, y) = a\varphi_0(x'', y) + b\varphi_0(w'', y).\end{aligned}$$

Explicitly, with $x \in g^{-1}(x'')$,

$$\varphi(x'' \otimes y) = x \otimes y + Im(f \otimes 1)$$

and

$$\psi(x \otimes y + Im(f \otimes 1)) = g(x) \otimes y$$

$$\implies$$

$$\begin{aligned}&\psi \circ \varphi(x'' \otimes y) = g(x) \otimes y = x'' \otimes y\\&\varphi \circ \psi(x \otimes y + Im(f \otimes 1)) = x_1 \otimes y + Im(f \otimes 1) = x \otimes y + Im(f \otimes 1),\end{aligned}$$

where in the last line x_1 is another representative in $g^{-1}(x'')$. □

Corollary 2.39. *N is flat iff $\otimes N$ preserves the exactness of any sequence of modules*

Proof. Any exact sequence can be split up into short exact sequence, and the flatness does indicate it preserve the exactness of short exact sequence. □

Example 2.40. *An ideal $\mathfrak{a} \subset \mathcal{A}$, and M is an $\mathcal{A}$-module,*

$$M \otimes_{\mathcal{A}} \mathcal{A}/\mathfrak{a} \cong M/\mathfrak{a}M,$$

where $\mathfrak{a}M := \{\sum x_i m_i | x_i \in \mathfrak{a}, m_i \in M\}$. $\mathfrak{a}M$ is a submodule of M.

Proof.

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathfrak{a} \longrightarrow 0$$

is an exact sequence (of A-modules). Tensorring it with M, we have

$$\mathfrak{a} \otimes M \xrightarrow{\psi} M \longrightarrow M \otimes \mathcal{A}/\mathfrak{a} \longrightarrow 0$$

is exact, where ψ is induced by the inclusion $\mathfrak{a} \hookrightarrow \mathcal{A}$, $\psi : x \otimes m \mapsto xm$. $Im(\psi) = \mathfrak{a}M$ Then by the exactness, we have

$$M \otimes \mathcal{A}/\mathfrak{a} \cong M/Im(\psi) = M/\mathfrak{a}M.$$

□

Example 2.41.

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/gcd(m,n)\mathbb{Z}.$$

Pf. Take $M = \mathbb{Z}/m\mathbb{Z}$, $\mathcal{A} = \mathbb{Z}$, $\mathfrak{a} = n\mathbb{Z}$. *Then* $\mathfrak{a}M = (n\mathbb{Z} + m\mathbb{Z})/m\mathbb{Z} = gcd(m,n)\mathbb{Z}/m\mathbb{Z}$. $\mathcal{A}/\mathfrak{a} = \mathbb{Z}/n\mathbb{Z}$

Then by the result of Example 2.40, we have

$$M \otimes \mathcal{A}/\mathfrak{a} = \frac{\mathbb{Z}}{m\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}/m\mathbb{Z}}{gcd(m,n)\mathbb{Z})/m\mathbb{Z}} = \frac{\mathbb{Z}}{gcd(m,n)\mathbb{Z}} = M/\mathfrak{a}M.$$

Let $n \in \mathbb{Z}$. Then $\mathbb{Z}/n\mathbb{Z}$ is flat iff $n = \pm 1, 0$, i.e. $\mathbb{Z}/n\mathbb{Z} = \{0\}$ or $\mathbb{Z}$. This is easy to prove, consider the following short exact sequence for $|n| \geq 2$,

$$0 \longrightarrow n\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

Suppose $\mathbb{Z}/n\mathbb{Z}$ is flat. Tensorring it with the above exact sequence, we get

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

which gives the contradiction.

Fact
Any finitely generated $\mathbb{Z}$-module is of the form

$$M = \mathbb{Z}^r \left(\oplus_{i \in I}(\mathbb{Z}/n_i\mathbb{Z})\right)$$

, the second part of M is denoted M_{tors}, then we get the corollary that a finitely generated $\mathbb{Z}$-module is flat iff M_{tors} vanishes.

Definition 2.42. *$\mathcal{A}$ a ring, M an $\mathcal{A}$-module, we call M **torsion free** if $\forall a \in \mathcal{A}$ non-zerodivisor. $m \in M$ $am = 0 \Longrightarrow m = 0$*

Theorem 2.43.

1. *M if flat $\implies M$ is torsion free*

2. *If $\mathcal{A}$ is PID, M is torsion free $\implies M$ is flat.*

Proof. Bosch section 4.2 □

Some other facts about tensor product

Example 2.44. *For $\mathcal{A} = \mathbb{F}$ being a field, V, W finite dimensional vector space over $\mathbb{F}$*

$$V^* \otimes W \cong Hom_{\mathbb{F}}(V, W)$$
$$l \otimes w \mapsto [v \mapsto l(v)w]$$