Cosmic Galois Group: a reading project under supervision of P. Jossen

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1 Preliminary knowledge about Algebraic Group and Tannakian Category

2 Statement of main results

Marcolli and Connes defined it

Theorem 2.1. *For any Feynman graph G with generic kinematics q, m there is a canonical way to associate to a* ***convergent integral***

- *an object mot_G in $\mathcal{H}(S)$, where S is a Zarikski open in a space of kinematics*
- *.....*
- *....*

3 Feynman graph and graph polynomials

A Feynman graph is a graph G defined by (V_G, E_G, E_G^{ext}), where V_G is the set of vertices of G, E_G is the set of internal edges of G, and E_G^{ext} is a set of external legs. Their endpoints are encoded by the maps $\partial : E_G \longrightarrow Sym^2 V_G$ and $\partial : E_G^{ext} \longrightarrow V_G$. We shall assume that the vertices with external legs lie in a single connected component of G. A Feynman graph additionally comes with kinematic data:

- a particle mass $m_e \in \mathbb{R}$ for every internal edge $e \in E_G$.
- a momentum $q_i \in \mathbb{R}^d$ for every external leg $i \in E_G^{ext}$,

where $d \geq 0$ is the dimension of space-time. All the external legs will be oriented inwards, so all momenta are incoming and are subject to momentum conservation.

In this paper, a subgraph H of G will be graph defined by a triple (V_H, E_H, E_H^{ext}) where $V_H \subset V_G$, $E_H \subset E_G$ and either $E_H^{ext} = E_G^{ext}$ or $E_H^{ext} = \emptyset$.

A tadpole is a subgraph of the the form $\{\{v\}, \{v, v\}, \emptyset\}$. We shall use the following notation for the basic combinatorial invariants of G:

- $h_G = dim(H^1(G))$ the loop number of G

- $\kappa_G = dim(H^0(G))$ the number of connected components of G
- $N_G = |E_G|$ the number of connected components of G.

They do not depends on the external legs of G. Euler's formula states that

$$N_G - V_G = h_G - \kappa_G.$$

We define that If a vertex $v \in V_G$ has several incoming momenta $q_1, ..., q_n$ we can replace it with a single incoming momentum $q_1 + ... + q_n$. Our notion of Feynman subgraph respects this equivalence relation. Then the graph polynomial defined latter would only depend on the equivalence classes.

We say that a Feynman graph is **of type** (Q, M) is it is equivalent to a graph with at most Q external kinematic parameters and at most M nonzero particle mass. We shall call a graph one-particle irreducible, or 1PI, if every connected component is 2-edge connected (i.e. deleting any edge causes the loop number to drop).

3.1 Graph polynomials

Let G be a Feynman graph. Recall that a tree is a connected graph T with $h_T = 0$. A forest is any graph with $h_T = 0$.

Definition 3.1. *Let G be a connected Feynman graph. The **Kirchhoff polynomial** (or first Symanzik polynomal) is the polynomial in $\mathbb{Z}[\alpha_e, e \in E_G]$ defined by*

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin T} \alpha_e, \tag{1}$$

where the sum is over all spanning trees T of G. If G has several connected components $G_1, ..., G_n$, we shall defined

$$\Psi_G = \prod_1^n \Psi_{G_i}$$

The secondSymanzikpolynomial is defined for connected G by

$$\Phi_G(q) = \sum_{T_1 \cup T_2 \subset G} (q^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e, \tag{2}$$

where the sum is over all spanning 2-trees $T = T_1 \cup T_2$ of G, and $q^{T_1} := \sum_{i \in E_{T_1}^{ext}} q_i$ is the total momentum entering T_1.

Remark 3.2. *α_e are just the Schwinger parameters*

Definition 3.3. *Let G be a Feynman graph. Define*

$$\Xi_G(q,m) = \Phi_G(q) + \left(\sum_{e \in E_G} m_e^2 \alpha_e \right) \Psi_G.$$

It is the denominator of Feynman integral, and it is homogeneous in α_e of degree $h_G + 1$

Since the graph polynomials only depend on the total momentum flow, they are well-defined on equivalence classes of graphs.

3.2 Feynman integral in projective space

After omitting certain pre-factors, we define the Feynman integral

$$I_G(q,m) = \int_\sigma \omega_G(q,m),$$

where

$$\omega_G(q,m) = \frac{1}{\Psi_G^{d/2}} \left(\frac{\Psi_G}{\Xi_G(q,m)} \right)^{N_G - h_G d/2} \Omega_G$$

and

$$\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge ... \wedge \widehat{d\alpha_i} \wedge ... \wedge d\alpha_{N_G}$$

Following form the fact that $deg(\Psi_G) = h_G$ and $deg(\Xi_G) = h_G + 1$, we know that ω_G is homogeneous of degree 0.

Finally, let $\sigma \subset \mathbb{P}^{N_G - 1}(\mathbb{R})$ be the coordinate simplex defined in projective coordinates by

$$\sigma = \{(\alpha_1 : ... : \alpha_{N_G}) \in \mathbb{P}^{N_G - 1}(\mathbb{R}) : \alpha_i \geq 0\}.$$

3.3 Edge subgraphs and their quotients

Let $G = (V_G, E_G, E_G^{ext})$ be a Feynman graph. A set of internal edges $\gamma \subset E_G$ defines a subgraph of G as follows. Write $E_\gamma = \gamma$ and let V_γ be the set of endpoints of elements of E_γ.

Definition 3.4. *A set of edges $\gamma \subset E_G$ is* ***momentum-spanning*** *if $\partial E_G^{ext} \subset V_\gamma$, and the vertices E_G^{ext} lie in a single connected component of the graph (V_γ, E_γ).*

we define the subgraph associated to $\gamma \subset E_G$ by

$$(V_\gamma, E_\gamma, E_\gamma^{ext}),$$

where $E_\gamma^{ext} = E_G^{ext}$ if γ is momentum-spanning and $E_\gamma^{ext} = \emptyset$ otherwise. We all $(V_\gamma, E_\gamma, E_\gamma^{ext})$ the edge-subgraph associated to γ and denote it also by γ when no confusion arises.

The quotient of G by an edge-subgraph γ is defined by

$$G/\gamma = (V_G/\sim, (E_Q \backslash \gamma)/\sim, {E_G}^{ext}/\sim),$$

where $\sim$ is the equivalence relation on vertices of G where two vertives are equivalent if and only if they are vertices of the same connected component of γ, and the induced equivalence relation on edges (unordered pairs of vertices). It is a Feynman graph. Every connected components of γ corresponds to a unique vertex in G/γ. Note that γ is momentum-spanning if and only if G/γ is equivalent to a graph with no external momenta. (which amounts to compress each component of γ to a single vertex.).

In this way, exactly one of the two Feynman graphs γ and G/γ is equivalent to a Feynman graph with non-zero external momenta: if is momentum spanning it is γ, otherwise it is G/γ.

3.4 Contraction-deletion

Let $G = (V_G, E_G, E_G^{ext})$ be a Feynman graph. The deletion fo an edge e in G is the graph G/e defined by deleting the edge e but retaining its endpoints:

$$G/e = (V_G, E_G \backslash \{e\}, E_G^{ext}).$$

In general, it is not a union of Feynman graphs since momentum conservation may not hold on each of its connected components.

One sometimes encounters the following variant of the previous notion of graph-quotient. It will be denoted by a double slash to distinguish it from the ordinary quotient. For an edge-subgraph γ, let $G//\gamma$ be the empty graph if $h_\gamma > 0$ and

$$G//\gamma = G/\gamma$$

if γ is a forest. In the case of a single edge e, G/e is empty whenever e is a tadpole.

It follows from Euler's formula that $h_G = h_\gamma + h_{G/\gamma}$ for any edge-subgraph $\gamma \subset G$ (which is not necessarily connected).

Lemma 3.5. *(Contraction-deletion) Let G be connected and $e \in E_G$. Then*

$$\Psi_G = \Psi^0_{G\backslash e}\alpha_e + \Psi_{G//e},$$

$$\Phi_G(q) = \Phi^0_{G\backslash e}(q)\alpha_e + \Phi_{G//e}(q),$$

where $\Psi^0_{G\backslash e}$ is given by the right hand side of Eq 1: it is $\Psi_{G\backslash e}$ if $G\backslash e$ is connected and 0 otherwise. Likewise $\Phi^0_{G\backslash e}(q)$ is given by the right-hand side of of Eq 2: it equals to $\Phi_{G\backslash e}(q)$ if $G\backslash e$ is connected and equals to $\Psi_{G_1}\Psi_{G_2}(q^{G_1})^2) = \Psi_{G_1}\Psi_{G_2}(q^{G_2})^2)$ if $G\backslash e$ has two connected components G_1, G_2.

Proof. Let T be a spanning k-tree of G. The edge e is not an edge of T if and only if T is a spanning k-tree of $G\backslash e$. By the definition of graph polynomials, this gives rise to the first terms in the right-hand side of above equations in the lemma. Note that if e is a tadpole, this is the only case which can ovvur. Now suppose that e is not a tadpole. If e is an edge of T, the T/e is a spanning k-tree of $G\backslash e$. Conversely, if T' is a spanning k-tree of G/e, then there is a unique component of T' which meets the vertex in G/e. It follows that the inverse image of T' in G with the edge e, is a spanning k-tree in G. This establishes a bijection between the set of spanning k-trees in T which contain e and those G/e. The rest just follows from the definitoin of graph polynomials. $\square$

Goal, define and study a glaois grop of Feynman anmplitude

4 General construction on Tannakian category

5 Motivic Periods over $\mathbb{Q}$

Definition 5.1. ***Category $\mathcal{H}$ of Betti and de Rham realizations:*** *Consider a category $\mathcal{H}$ whose objects are triples (V_B, V_{dR}, c) consisting of the following data:*

1. *A finite dimensional $\mathbb{Q}$-vector space V_B with a finite increasing (Weight) filtration $W_\bullet V_B$.*

2. *A finite dimensional $\mathbb{Q}$-vector space V_{dR} with a finite increasing (Weight) filtration $W_\bullet V_{dR}$ and a finite decreasing (Hodge) filtration $F^\bullet V_{dR}$.*

3. *An isomorphism*

$$c : V_{dR} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} V_B \otimes_{\mathbb{Q}} \mathbb{C}$$

   which respect the $W_\bullet$ filtration on both sides.

4. *A linear involution $F_\infty : V_B \xrightarrow{\sim} V_B$ called the real Frobenius.*

The category $\mathcal{H}$ has two fibre functors:

$$\omega_\bullet : \mathcal{H} \longrightarrow Vec_{\mathbb{Q}}, \bullet = B, dR$$

Then we can regard $\mathcal{H}$ as a Tannakian category.

Definition 5.2. *The ring of $\mathcal{H}$-periods is $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$. It is equipped with a*

- *A period homomorphism*

$$per : \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \longrightarrow \mathbb{C}$$

  which sends $[(V_B, V_{dR}, c)\sigma, \omega]^{\mathfrak{m}}$

6 Connections with regular singularities

Let X be a smooth $\mathbb{C}$-variety, and in particular a smooth $\mathbb{C}$-manifold. An algebraic D-module on X whcih is $\mathcal{O}_X$ coherent is exactly an algebraic vector bundle E on X with a algebraic flat (Integrable) connection

7 $\mathcal{H}(S)$-realizations

Let $k = \mathbb{Q}$ and S be a smooth geometrically connected scheme over the field k. The category $\mathcal{H}(S)$ consists of triples $(\mathcal{V}_B, \mathcal{V}_{dR}, c)$, where

- $\mathcal{V}_B$ is a local system of finite dimensional $\mathbb{Q}$-vector spaces over $S(\mathbb{C})$, equipped with a finite increasing filtration $W_\bullet \mathcal{V}_B$ of local subsystems.

- $\mathcal{V}_{dR}$ is an algebraic vector bundle on S equipped with an integrable connection $\nabla : \mathcal{V}_{dR} \longrightarrow V_{dR} \otimes_{\mathcal{O}_S} \Omega^1_S$