Personal Notes All the information I need

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October 13, 2017

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1 Affine Group Scheme, Hopf Algebra

Definition 1.1. *Let A be a commutative K-algebra. The corresponding affine K-scheme $G = Spec(A)$ is said to be a group scheme if it is endowed with algebraic operations*

$$\begin{aligned}
&\mu : G \times G \longrightarrow G \ (product),\\
&e : Spec(K) \longrightarrow G(unit),\\
&\iota : G \longrightarrow G(inverse),
\end{aligned}$$

satisfying the usual axioms of a group, which are expressed by the commutativity of the following three diagrams

(1) Associativity:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mu\times Id"] \arrow[d, "Id\times\mu"] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"'] & G
\end{tikzcd}$$

(2) Unit:

$$\begin{tikzcd}
G \times Spec(K) \arrow[r, "Id\times e"] \arrow[dr, "pr_1"] & G \times G \arrow[d, "\mu"] & Spec(K) \times G \arrow[l, "e\times Id"'] \arrow[dl, "pr_2"] \\
& G &
\end{tikzcd}$$

(3) Inverse:

$$\begin{tikzcd}
& G \times G \arrow[dr, "\mu"] & \\
G \arrow[ur, "Id\times\iota"] \arrow[r, "\pi"] \arrow[dr, "\iota\times Id"'] & Spec(K) \arrow[r, "e"] & G \\
& G \times G \arrow[ur, "\mu"'] &
\end{tikzcd} ,$$

where π denotes the structural map of G as a K-scheme. If the algebra A is finitely generated, we say that G is algebraic. We will see below that every affine group scheme is in fact a projective limit of affine group schemes.

Equivalent definition: A group scheme over K is a functor between the categories of commutative K-algebras and abstract groups. Namely, given $G = \mathrm{Spec}(A)$ A, P, Q are K-algebras, one considers the functor:

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "G"] & Q \arrow[d, "G"] \\
Hom_{K-alg}(A,P) = G(P) \arrow[r, "g \mapsto G(g) = f \circ g"] & G(Q) = Hom_{K-alg}(A,Q)
\end{tikzcd},$$

1.1 Hopf Algebra

The category of affine schemes over K is equivalent to the category of the commutative K-algebras through the contravariant functors

$$A \mapsto \mathrm{Spec}(A)$$

$$G \mapsto \mathcal{O}(G),$$

where $\mathcal{O}(G)$ is the ring of regular functions on G. Thus the defining properties of a group scheme can be transfered to the corresponding algebra, yielding the concept of Hopf algebra.

Definition 1.2. *Let H be an associative (not necessarily commutative) K-algebra. Let $\nabla : H \otimes H \longrightarrow H$ be the product of H and $\eta : k \longrightarrow H$ the unit.*

(1) *We say that H is a **bialgebra** if it is provided with two morphisms of algebras*

$$\Delta : H \longrightarrow H \otimes H (\textit{coproduct}),$$
$$\epsilon : H \longrightarrow k (\textit{counit})$$

such that the following diagrams commute:

(a) *Coassociativity:*

$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[d, "\Delta"] & H \otimes H \arrow[d, "Id \otimes \Delta"] \\
H \otimes H \arrow[r, "\Delta \otimes Id"] & H \otimes H \otimes H,
\end{tikzcd}$$

(b) *Counit.*

$$\begin{tikzcd} H \otimes K & H \otimes H \arrow[l, "Id\otimes\epsilon"'] \arrow[r, "\epsilon\otimes Id"] & K \otimes H \\ & H \arrow[ul] \arrow[u, "\Delta"] \arrow[ur] & \end{tikzcd}$$

(2) *A bialgebra H is called a* ***Hopf algebra*** *if it is further equipped witha morphism of algebras*

$$S : H \longrightarrow H \ (antipode)$$

such that the following diagram commutes:

(c) *Antipode.*

$$\begin{tikzcd} & H \otimes H \arrow[rr, "S\otimes Id"] & & H \otimes H \arrow[dr, "\nabla"] & \\ H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "\eta"] & & H \\ & H \otimes H \arrow[rr, "Id\otimes S"] & & H \otimes H \arrow[ur, "\nabla"'] & \end{tikzcd}$$

(3) *A bialgebra H is called commutative if the product is commutative, and cocommutative if the coproduct satisfies $\Delta = \tau \circ \Delta$, where $\tau : H \otimes H \longrightarrow H \otimes H$ is the flip of the factors.*

2 Tannakian Categories

K is a field of characteristic 0, $\mathcal{A}$ rigid tensor $\mathbb{K}$-linear abelian category, L is an extension of K.

Definition 2.1. *An L-valued* ***fibre functor*** *is a tensor functor $\omega : \mathcal{A} \longrightarrow Vec_L$ which is faithful and exact.*

Definition 2.2. *$\mathcal{A}$ is*

- ***Neutralized Tannakian*** *if one can endow it with a K-valued fibre functor.*
- ***Neutral Tannakian*** *if $\exists$ K-valued fibre functor.*
- ***Tannakian*** *if $\exists L$-valued fibre functor for some L.*

Example 2.3. *G is an affine K-group scheme, $\mathcal{A} = Rep_K(G), \omega : \mathcal{A} \longrightarrow Vec_K$ the forgetful functor.*