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# Čech and de Rham cohomology of integral forms

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ABSTRACT

We present a study on the integral forms and their Čech and de Rham cohomology. We analyze the problem from a general perspective of sheaf theory and we explore examples in superprojective manifolds. Integral forms are fundamental in the theory of integration in a supermanifold. One can define the integral forms introducing a new sheaf containing, among other objects, the new basic forms $\delta(d\theta)$ where the symbol $\delta$ has the usual formal properties of Dirac's delta distribution and acts on functions and forms as a Dirac measure. They satisfy in addition some new relations on the sheaf. It turns out that the enlarged sheaf of integral and "ordinary" superforms contains also forms of "negative degree" and, moreover, due to the additional relations introduced it is, in a non trivial way, different from the usual superform cohomology.



## 1. Introduction

Supermanifolds are rather well-known in supersymmetric theories and in string theory. They provide a very natural ground to understand the supersymmetry and supergravity from a geometric point of view. Indeed, a supermanifold contains the anticommuting coordinates which are needed to construct the superfields whose natural environment is the graded algebras [1,2].

Before explaining the content of the present work, we stress the relevance of this analysis observing that recently the construction of a formulation of superstrings [3] requires the introduction of the superforms described here. In addition, the physics behind that formalism is encoded into the BRST cohomology which, in mathematical terms, is translated into the Čech and de Rham cohomology objects of our study.

The best way to understand supermanifold theory is using the theory of sheaves [2,4]. In the present notes we review this approach and we use the results of our previous paper [5]. In the first section, we recall some definitions and some auxiliary material. We point out that in order to formulate the theory of integration for superforms, one needs some additional ingredients such as integral forms. Enlarging the space of superforms to take into account those new quantities results in bigger complexes of superforms. These new mathematical objects should be understood in the language of the sheaves in order that the previous considerations about the morphisms are applicable. In particular, we study the behaviour of integral forms under morphisms and we show that they can be globally defined.

By a hand-waving argument, we can describe as follows the need of the integral forms for the theory of integration in the supermanifold. In the theory of integration of conventional forms for a manifold $\mathcal{M}$, we consider a $\omega \in \Omega^\bullet(\mathcal{M})$. We can introduce a supermanifold [6] $\widehat{\mathcal{M}}$ whose anticommuting coordinates are generated by the fibers $T^*\mathcal{M}$. Therefore, a function on $\widehat{\mathcal{M}}$ is the same of a differential form of $\Omega^\bullet(\mathcal{M})$, $\mathcal{F}(\widehat{\mathcal{M}}) \equiv \mathcal{C}^\infty(\widehat{\mathcal{M}}) \cong \Omega^\bullet(\mathcal{M})$. The correspondence is simply $dx^i \leftrightarrow \psi^i$. For the manifold $\mathcal{M}$ we can integrate differential forms of the top degree $\Omega^{(n)}(\mathcal{M})$, but in general we cannot integrate functions

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since $\mathcal{M}$ has no natural measure. On the other hand in $\widehat{\mathcal{M}}$ we can indeed write $\widehat{\mu} = dx^1 \wedge \cdots \wedge dx^n \wedge d\psi^1 \wedge \cdots \wedge d\psi^n$ where the integral on the variables $\psi^i$ is the Berezin integral ($\int d\psi f(\psi) = \partial_\psi f(\psi)$). If $\hat{\omega}$ is a function of $\mathcal{F}(\widehat{\mathcal{M}})$, we have $\int_{\widehat{\mathcal{M}}} \widehat{\mu}\,\widehat{\omega} = \int_{\mathcal{M}} \omega$ where the superspace integration is the integration of forms. We have to notice that being the integral on the anticommuting variables a Berezin integral, it selects automatically the degree of the form.

Now, the same construction can be performed in the case of a supermanifold $\mathcal{N}$ with only fermionic coordinates $\theta^a$. In that case its cotangent space $T^*\mathcal{N}$ is not finite dimensional. Therefore, mimicking the above construction, we define the form integration by considering the measure $\widehat{\mu}$ for the manifold $\mathcal{N} \oplus T^*\mathcal{N}$ where the commuting superforms $d\theta^a$ are replaced by commuting coordinates $\lambda^a$ and the measure is given $\widehat{\mu} = d\theta^a \wedge \cdots d\theta^b \wedge d\lambda^a \wedge \cdots d\lambda^b$. Thus, in contrast to the commuting case the integral over the coordinates $\theta^a$ is a Berezin integral, while the integration over the 1-forms $\lambda^a$ is an ordinary $n$-dimensional integral. In order that the latter has finite answer for a given superform, we introduce the integration forms $\omega_{a_1 \cdots a_n} \delta(\lambda^{a_1}) \wedge \cdots \wedge \delta(\lambda^{a_n})$ where the Dirac delta functions $\delta(\lambda^a)$ localize the integral at the point $\lambda^a = 0$. These new quantities behave as "distributions", and therefore they satisfy new relations that we will describe in Section 4. We show that the set of relations they ought to obey is preserved in passing from one patch to another and therefore that they are global properties. This implies that the sheaf of integral forms is well defined. Finally, we derive a Čech–de Rham theorem for these new superforms. The interesting aspect is that the distributional relations (here translated into an algebraic language) modifies the cohomology and therefore we find non-trivial results.

In Section 2, we review briefly the construction of the supermanifolds, the underlying structure using ringed spaces, their morphisms and the local charts on them. We specify the constructions to the case of superprojective manifolds. In Section 5 and in Section 6 we compute some examples of Čech and de Rham cohomology groups for superprojective spaces. We also prove a generalization of usual Čech–de Rham and Künneth theorems.

## 2. Supermanifolds

We collect here some definitions and considerations about supermanifolds.

### 2.1. Definitions

A *super-commutative* ring is a $\mathbb{Z}_2$-graded ring $A = A_0 \oplus A_1$ such that if $i, j \in \mathbb{Z}_2$, then $a_i a_j \in A_{i+j}$ and $a_i a_j = (-1)^{i+j} a_j a_i$, where $a_k \in A_k$. Elements in $A_0$ (resp. $A_1$) are called *even* (resp. *odd*).

A *super-space* is a super-ringed space such that the stalks are local super-commutative rings (Manin–Varadarajan). Since the odd elements are nilpotent, this reduces to require that the even component reduces to a local commutative ring.

A *super-domain* $U^{p|q}$ is the super-ringed space $\left(U^p, \mathcal{C}^{\infty p|q}\right)$, where $U^p \subseteq \mathbb{R}^p$ is open and $\mathcal{C}^{\infty p|q}$ is the sheaf of super-commutative rings given by:

$$V \mapsto \mathcal{C}^\infty(V)\left[\theta^1, \theta^2, \ldots, \theta^q\right], \tag{1}$$

where $V \subseteq U^p$ and $\theta^1, \theta^2, \ldots, \theta^q$ are generators of a Grassmann algebra. The grading is the natural grading in even and odd elements. The notation is taken from [7] and from the notes [8].

Every element of $\mathcal{C}^{\infty p|q}(V)$ may be written as $\sum_I f_I \theta^I$, where $I$ is a multi-index. A *supermanifold* of dimension $p|q$ is a super-ringed space locally isomorphic, as a ringed space, to $\mathbb{R}^{p|q}$. The coordinates $x_i$ of $\mathbb{R}^p$ are usually called the even coordinates (or bosonic), while the coordinates $\theta^j$ are called the odd coordinates (or fermionic). We will denote by $(M, \mathcal{O}_M)$ the supermanifold whose underlying topological space is $M$ and whose sheaf of super-commutative rings is $\mathcal{O}_M$.

To a section $s$ of $\mathcal{O}_M$ on an open set containing $x$ one may associate the *value* of $s$ in $x$ as the unique real number $\tilde{s}(x)$ such that $s - \tilde{s}(x)$ is not invertible on every neighborhood of $x$. The sheaf of algebras $\tilde{\mathcal{O}}$, whose sections are the functions $\tilde{s}$, defines the structure of a differentiable manifold on $M$, called the *reduced manifold* and denoted $\tilde{M}$.

### 2.2. Morphisms

In order to understand the structure of supermanifolds it is useful to study their morphisms. Here we describe how a morphism of supermanifolds looks like locally. A *morphism* $\psi$ from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is given by a smooth map $\tilde{\psi}$ from $\tilde{X}$ to $\tilde{Y}$ together with a sheaf map:

$$\psi_V^* : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\psi^{-1}(V)), \tag{2}$$

where $V$ is open in $Y$. The homomorphisms $\psi_V^*$ must commute with the restrictions and they must be compatible with the super-ring structure. Moreover they satisfy

$$\psi_V^*(s)^\sim = \tilde{s} \circ \tilde{\psi}. \tag{3}$$

Let us recall some fundamental local properties of morphisms. A morphism $\psi$ between two super-domains $U^{p|q}$ and $V^{r|s}$ is given by a smooth map $\tilde{\psi} : U \to V$ and a homomorphism of super-algebras

$$\psi^* : \mathcal{C}^{\infty r|s}(V) \to \mathcal{C}^{\infty p|q}(U). \tag{4}$$

It must satisfy the following properties:

- If $t = (t_1, \dots, t_r)$ are coordinates on $V^r$, each component $t_j$ can also be interpreted as a section of $\mathcal{C}^{\infty r|s}(V)$. If $f_i = \psi^*(t_i)$, then $f_i$ is an even element of the algebra $\mathcal{C}^{\infty,p|q}(U)$.
- The smooth map $\tilde{\psi} : U \to V$ must be $\tilde{\psi} = (\tilde{f}_1, \dots, \tilde{f}_r)$, where the $\tilde{f}_r$ are the values of the even elements above.
- If $\theta_j$ is a generator of $\mathcal{C}^{\infty r|s}(V)$, then $g_j = \psi^*(\theta_j)$ is an odd element of the algebra $\mathcal{C}^{\infty p|q}(U)$.

The following fundamental theorem (see for example [8]) gives a local characterization of morphisms:

**Theorem 1** (*Structure of Morphisms*)**.** *Suppose $\phi : U \to V$ is a smooth map and $f_i, g_j$, with $i = 1, \dots, r, j = 1, \dots, s$, are given elements of $\mathcal{C}^{\infty p|q}(U)$, with $f_i$ even, $g_j$ odd and satisfying $\phi = (\tilde{f}_1, \dots, \tilde{f}_r)$. Then there exists a unique morphism $\psi : U^{p|q} \to V^{r|s}$ with $\tilde{\psi} = \phi$ and $\psi^*(t_i) = f_i$ and $\psi^*(\theta_j) = g_j$.*

### 2.3. Local charts on supermanifolds

We describe now how supermanifolds can be constructed by patching local charts. Let $X = \bigcup_i X_i$ be a topological space, with $\{X_i\}$ open, and let $\mathcal{O}_i$ be a sheaf of rings on $X_i$, for each $i$. We write (see [7]) $X_{ij} = X_i \cap X_j$, $X_{ijk} = X_i \cap X_j \cap X_k$, and so on. We now introduce isomorphisms of sheaves which represent the "coordinate changes" on our supermanifold. They allow us to glue the single pieces to get the final supermanifold. Let

$$f_{ij} : \left(X_{ji}, \mathcal{O}_j|_{X_{ji}}\right) \longrightarrow \left(X_{ij}, \mathcal{O}_i|_{X_{ij}}\right) \tag{5}$$

be an isomorphisms of sheaves with

$$\tilde{f}_{ij} = \mathrm{Id}. \tag{6}$$

This means that these maps represent differentiable coordinate changes on the underlying manifold.

To say that we glue the ringed spaces $(X_i, \mathcal{O}_i)$ through the $f_{ij}$ means that we are constructing a sheaf of rings $\mathcal{O}$ on $X$ and for each $i$ a sheaf isomorphism

$$f_i : (X_i, \mathcal{O}|_{X_i}) \longrightarrow (X_i, \mathcal{O}_i), \tag{7}$$

$$\tilde{f}_i = \mathrm{Id}_{X_i} \tag{8}$$

such that

$$f_{ij} = f_i f_j^{-1}, \tag{9}$$

for all $i$ and $j$.

The following usual cocycle conditions are necessary and sufficient for the existence of the sheaf $\mathcal{O}$:

i. $f_{ii} = \mathrm{Id}$ on $\mathcal{O}_i$;
ii. $f_{ij} f_{ji} = \mathrm{Id}$ on $\mathcal{O}_i|_{X_i}$;
iii. $f_{ij} f_{jk} f_{ki} = \mathrm{Id}$ on $\mathcal{O}_i|_{X_{ijk}}$.

## 3. Projective superspaces

Due to their importance in mathematical and physical applications we now give a description of projective superspaces (see also [5]). One can work either on $\mathbb{R}$ or on $\mathbb{C}$, but we choose to stay on $\mathbb{C}$. Let $X$ be the complex projective space of dimension $n$. The super-projective space will be called $Y$. The homogeneous coordinates are $\{z_i\}$. Let us consider the underlying topological space as $X$, and let us construct the sheaf of super-commutative rings on it. For any open subset $V \subseteq X$ we denote by $V'$ its preimage in $\mathbb{C}^{n+1} \setminus \{0\}$. Then, let us define $A\left(V'\right) = H\left(V'\right)\left[\theta^1, \theta^2, \dots, \theta^q\right]$, where $H\left(V'\right)$ is the algebra of holomorphic functions on $V'$ and $\left\{\theta^1, \theta^2, \dots, \theta^q\right\}$ are the odd generators of a Grassmann algebra. $\mathbb{C}^*$ acts on this super-algebra by:

$$t : \sum_I f_I(z)\theta^I \longrightarrow \sum_I t^{-|I|} f_I\left(t^{-1}z\right)\theta^I. \tag{10}$$

The super-projective space has a ring over $V$ given by:

$$\mathcal{O}_Y(V) = A\left(V'\right)^{\mathbb{C}^*} \tag{11}$$

which is the subalgebra of elements invariant by this action. This is the formal definition of a projective superspace (see for example [8]), however we would like to construct the same space more explicitly from gluing different superdomains as in Section 2.3.

Let $X_i$ be the open set where the coordinate $z_i$ does not vanish. Then the super-commutative ring $\mathcal{O}_Y(X_i)$ is generated by elements of the type

$$f_0\left(\frac{z_0}{z_i},\ldots,\frac{z_{i-1}}{z_i},\frac{z_{i+1}}{z_i},\ldots,\frac{z_n}{z_i}\right), \tag{12}$$

$$f_r\left(\frac{z_0}{z_i},\ldots,\frac{z_{i-1}}{z_i},\frac{z_{i+1}}{z_i},\ldots,\frac{z_n}{z_i}\right)\frac{\theta^r}{z_i}, \quad r=1,\ldots,q. \tag{13}$$

In fact, to be invariant with respect to the action of $\mathbb{C}^*$, the functions $f_I$ in Eq. (10) must be homogeneous of degree $-|I|$. Then, it is obvious that the only coordinate we can divide by, on $X_i$, is $z_i$: all functions $f_I$ are of degree $-|I|$ and holomorphic on $X_i$. If we put, on $X_i$, for $l \neq i$, $\Xi_l^{(i)} = \frac{z_l}{z_i}$ and $\Theta_r^{(i)} = \frac{\theta^r}{z_i}$, then $\mathcal{O}_Y(X_i)$ is generated, as a super-commutative ring, by the objects of the form

$$F_0^{(i)}\left(\Xi_0^{(i)},\Xi_1^{(i)},\ldots,\Xi_{i-1}^{(i)},\Xi_{i+1}^{(i)},\ldots,\Xi_n^{(i)}\right), \tag{14}$$

$$F_a^{(i)}\left(\Xi_0^{(i)},\Xi_1^{(i)},\ldots,\Xi_{i-1}^{(i)},\Xi_{i+1}^{(i)},\ldots,\Xi_n^{(i)}\right)\Theta_a^{(i)}, \tag{15}$$

where $F_0^{(i)}$ and the $F_a^{(i)}$'s are analytic functions on $\mathbb{C}^n$. In order to avoid confusion we have put the index $i$ in parenthesis; it just denotes the fact that we are defining objects over the local chart $X_i$. In the following, for convenience of notation, we also adopt the convention that $\Xi_i^{(i)} = 1$ for all $i$.

We have the two sheaves $\mathcal{O}_Y(X_i)|_{X_j}$ and $\mathcal{O}_Y(X_j)|_{X_i}$. In the same way as before, we have the morphisms given by the "coordinate changes". So, on $X_i \cap X_j$, the isomorphism simply affirms the equivalence between the objects of the super-commutative ring expressed either by the first system of affine coordinates, or by the second one. So for instance we have that $\Xi_l^{(j)} = \frac{z_l}{z_j}$ and $\Theta_r^{(j)} = \frac{\theta^r}{z_j}$ can be also expressed as

$$\Xi_l^{(j)} = \frac{\Xi_l^{(i)}}{\Xi_j^{(i)}}, \qquad \Theta_r^{(j)} = \frac{\Theta_r^{(i)}}{\Xi_j^{(i)}}. \tag{16}$$

Which, in the language used in the previous section, means that the morphism $\psi_{ji}$ gluing $(X_i \cap X_j, \mathcal{O}_Y(X_i)|_{X_j})$ and $(X_j \cap X_i, \mathcal{O}_Y(X_j)|_{X_i})$ is such that $\tilde{\psi}_{ji}$ is the usual change of coordinates map on projective space and

$$\psi_{ji}^*(\Xi_l^{(j)}) = \frac{\Xi_l^{(i)}}{\Xi_j^{(i)}}, \qquad \psi_{ji}^*(\Theta_r^{(j)}) = \frac{\Theta_r^{(i)}}{\Xi_j^{(i)}}. \tag{17}$$

The supermanifold is obtained by observing that the coordinate changes satisfy the cocycle conditions of the previous section.

## 4. Integral forms and integration

Most supergeometry can be obtained straightforwardly by extending the commuting geometry by means of the rule of signs, but this is not the case in the theory of differential forms on supermanifolds. Indeed the naive notion of "superforms" obtainable just by adding a $\mathbb{Z}_2$ grading to the exterior algebra turns out not to be suitable for Berezin integration. In this section we briefly recall the definition of "integral forms" and their main properties referring mainly to [9] for a detailed exposition. The theory of superforms and their integration theory has been widely studied in the literature and it is based on the notion of the integral superforms (see for example [2,10]). The problem is that we can build the space $\Omega^k$ of $k$-superforms out of basic 1-superforms $d\theta^i$ and their wedge products, however these products are necessarily commutative, since the $\theta_i$'s are odd variables. Therefore, together with a differential operator $d$, the spaces $\Omega^k$ form a differential complex

$$0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \cdots \xrightarrow{d} \Omega^n \xrightarrow{d} \cdots \tag{18}$$

which is bounded from below, but not from above. In particular there is no notion of a top form to be integrated on the supermanifold $\mathbb{C}^{p+1|q}$.

The space of "integral forms" is obtained by adding to the usual space of superforms a new set of basic forms $\delta(d\theta)$, together with its "derivatives" $\delta^n(d\theta)$, and defining a product which satisfies certain formal properties. These properties are motivated and can be deduced from the following heuristic approach. In analogy with usual distributions acting on the space of smooth functions, we think of $\delta(d\theta)$ as an operator acting on the space of superforms as the usual Dirac's delta measure. We write this as

$$\langle f(d\theta), \delta(d\theta)\rangle = f(0),$$

where $f$ is a superform. This means that $\delta(d\theta)$ kills all monomials in the superform $f$ which contain the term $d\theta$. The derivatives $\delta^{(n)}(d\theta)$ satisfy

$$\left\langle f(d\theta), \delta^{(n)}(d\theta)\right\rangle = -\left\langle f'(d\theta), \delta^{(n-1)}(d\theta)\right\rangle = (-1)^n f^{(n)}(0),$$

like the derivatives of the usual Dirac $\delta$ measure. Moreover we can consider objects such as $g(d\theta)\delta(d\theta)$, which act by first multiplying by $g$ then applying $\delta(d\theta)$ (in analogy with a measure of type $g(x)\delta(x)$), and so on. The wedge products among these objects satisfy some simple relations such as (we will always omit the symbol $\wedge$ of the wedge product):

$$\begin{aligned} &dx^I \wedge dx^J = -dx^J \wedge dx^I, \qquad dx^I \wedge d\theta^j = d\theta^j \wedge dx^I, \\ &d\theta^i \wedge d\theta^j = d\theta^j \wedge d\theta^i, \qquad \delta(d\theta) \wedge \delta(d\theta') = -\delta(d\theta') \wedge \delta(d\theta), \\ &d\theta\delta(d\theta) = 0, \qquad d\theta\delta'(d\theta) = -\delta(d\theta). \end{aligned} \tag{19}$$

The second and third property can be easily deduced from the above approach. To prove these formulas we observe the usual transformation property of the usual Dirac's delta function

$$\delta(ax+by)\delta(cx+dy) = \frac{1}{\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\delta(x)\delta(y) \tag{20}$$

for $x, y \in \mathbb{R}$. By setting $a = 0, b = 1, c = 1$ and $d = 1$, the anticommutation property of Dirac's delta function of $d\theta$'s of (19) follows.

We do not wish here to give an exhaustive and rigorous treatment of integral forms. As we will see later, it is sufficient for our purposes that these simple rules give a well defined construction in the case of superprojective spaces. A systematic exposition of these rules can be found in [11] and they can be put in a more mathematical framework using the results of [4] and [12]. An interesting consequence of this procedure is the existence of "negative degree" forms, which are those which reduce the degree of forms (e.g. $\delta'(d\theta)$ has degree $-1$). The integral forms could be also called "pseudodifferential forms".

We introduce also the *picture number* by counting the number of delta functions (and their derivatives) and we denote by $\Omega^{r|s}$ the $r$-forms with picture $s$. For example, in the case of $\mathbb{C}^{p+1|q}$, the integral form

$$dx^{[K_1} \cdots dx^{K_l]} d\theta^{(i_{l+1}} \cdots d\theta^{i_r)} \delta(d\theta^{[i_{r+1}}) \cdots \delta(d\theta^{i_{r+s}]}) \tag{21}$$

is an $r$-form with picture $s$. All indices $K_i$ are antisymmetrized among themselves, while the first $r-l$ indices are symmetric and the last $s+1$ are antisymmetrized. We denote by $[I_1 \cdots I_s]$ the antysimmetrization of the indices and by $(i_1 \cdots i_n)$ the symmetrization. Indeed, by also adding derivatives of delta forms $\delta^{(n)}(d\theta)$, even negative form-degree can be considered, e.g. a form of the type:

$$\delta^{(n_1)}(d\theta^{i_1}) \cdots \delta^{(n_s)}(d\theta^{i_s}) \tag{22}$$

is a $-(n_1 + \cdots n_s)$-form with picture $s$. Clearly $\Omega^{k|0}$ is just the set $\Omega^k$ of superforms, for $k \geq 0$.

We now briefly discuss how these forms behave under change of coordinates, i.e. under sheaf morphisms. For a very general type of morphisms it is necessary to work with infinite formal sums in $\Omega^{r|s}$ as the following example clearly shows.

Suppose $\Phi^*(\tilde{\theta}^1) = \theta^1 + \theta^2$, $\Phi^*(\tilde{\theta}^2) = \theta^2$ is the odd part of a morphism. We want to compute

$$\Phi^*(\delta\left(d\tilde{\theta}^1\right)) = \delta\left(d\theta^1 + d\theta^2\right) \tag{23}$$

in terms of the above forms. We can formally expand in series about, for example, $d\theta^1$:

$$\delta\left(d\theta^1 + d\theta^2\right) = \sum_j \frac{\left(d\theta^2\right)^j}{j!}\delta^{(j)}(d\theta^1). \tag{24}$$

Recall that any usual superform is a polynomial in the $d\theta$, therefore only a finite number of terms really matter in the above sum, when we apply it to a superform. In fact, applying the formulae above, we have for example,

$$\left\langle (d\theta^1)^k, \sum_j \frac{\left(d\theta^2\right)^j}{j!}\delta^{(j)}(d\theta^1)\right\rangle = (-1)^k (d\theta^2)^k. \tag{25}$$

Notice that this is equivalent to the effect of replacing $d\theta^1$ with $-d\theta^2$. We could have also interchanged the role of $\theta^1$ and $\theta^2$ and the result would be to replace $d\theta^2$ with $-d\theta^1$. Both procedures correspond precisely to the action we expect when we apply the $\delta\left(d\theta^1 + d\theta^2\right)$ Dirac measure. We will not enter into more detailed treatment of other types of morphisms, as this simple example will suffice. In the case of super-projective spaces the change of coordinate rule is simple and will be discussed in the next section. In the rest of the paper we will ignore the action $\langle , \rangle$ and do the computations following the above rules.

We will see later, in Section 6, that integral forms form a new complex as follows

$$\cdots \xrightarrow{d} \Omega^{(r|q)} \xrightarrow{d} \Omega^{(r+1|q)} \cdots \xrightarrow{d} \Omega^{(p+1|q)} \xrightarrow{d} 0 \tag{26}$$

where $\Omega^{(p+1|q)}$ is the top "form" $dx^{[K_1} \cdots dx^{K_{p+1}]}\delta(d\theta^{[i_1}) \cdots \delta(d\theta^{i_q]})$ which can be integrated on the supermanifold. As in the usual commuting geometry, there is an isomorphism between the cohomologies $H^{(0|0)}$ and $H^{(p+1|q)}$ on a supermanifold of dimension $(p+1|q)$. In addition, one can define two operations acting on the cohomology groups $H^{(r|s)}$ which change the picture number $s$ (see [11]).

Given a function $f(x,\theta)$ on the superspace $\mathbb{C}^{(p+1|q)}$, we define its integral by the super top-form $\omega^{(p+1|q)} = f(x,\theta)dx^1 \cdots dx^{p+1}\delta(d\theta^1)\cdots\delta(d\theta^q)$ belonging to $\Omega^{(p+1|q)}$ as follows

$$\int_{\mathbb{C}^{(p+1|q)}} \omega^{(p+1|q)} = \epsilon^{i_1\cdots i_q}\partial_{\theta^{i_1}}\cdots\partial_{\theta^{i_q}}\int_{\mathbb{C}^{p+1}} f(x,\theta) \tag{27}$$

where the last equalities is obtained by integrating on the delta functions and selecting the bosonic top form. The remaining integrals are the usual integral of densities and the Berezin integral. The latter can be understood in terms of the Berezinian sheaf [12]. It is easy to show that indeed the measure is invariant under general coordinate changes and the density transform as a Berezinian with the superdeterminant.

## 5. Čech cohomology of $\mathbb{P}^{1|1}$

We describe now Čech cohomology on super-projective spaces, with respect to this particular sheaf of "integral 1-forms". We will begin by considering $\mathbb{P}^{1|1}$. $\mathbb{P}^1$ has a natural covering with two charts, $U_0$ and $U_1$, where

$$U_0 = \{[z_0; z_1] \in \mathbb{P}^1 : z_0 \neq 0\}, \tag{28}$$

$$U_1 = \{[z_0; z_1] \in \mathbb{P}^1 : z_1 \neq 0\}. \tag{29}$$

The affine coordinates are $\gamma = \frac{z_1}{z_0}$ on $U_0$ and $\widetilde{\gamma} = \frac{z_0}{z_1}$ on $U_1$. The odd generators are $\psi$ on $U_0$ and $\widetilde{\psi}$ on $U_1$. The gluing morphism of sheaves on the intersection $U_0 \cap U_1$ has pull-back given by:

$$\Phi^* : \mathcal{O}(U_0\cap U_1)[\psi] \longmapsto \mathcal{O}(U_0\cap U_1)[\widetilde{\psi}] \tag{30}$$

with the requirement that:

$$\Phi^*(\gamma) = \frac{1}{\widetilde{\gamma}}, \qquad \Phi^*(\psi) = \frac{\widetilde{\psi}}{\widetilde{\gamma}}. \tag{31}$$

We now consider a sheaf of differential on $\mathbb{P}^{1|1}$. As we already said in the previous section, we must add objects of the type "$d\gamma$" and of the type "$d\psi$" on $U_0$. But $d\psi$ is an even generator, because $\psi$ is odd, so we are not able to find a differential form of maximal degree. We introduce then the generator $\delta(d\psi)$, which allows us to perform integration in the "variable" $d\psi$. It satisfies the rule $d\psi\,\delta(d\psi) = 0$. This means that $\delta(d\psi)$ is like a Dirac measure on the space of the analytic functions in $d\psi$ which gives back the evaluation at zero. We must also introduce the derivatives $\delta^{(n)}(d\psi)$, where $d\psi\,\delta'(d\psi) = -\delta(d\psi)$, and, in general, $d\psi\,\delta^{(n)}(d\psi) = -\delta^{(n-1)}(d\psi)$. In this way, the derivatives of the delta represent anticommuting differential forms of negative degree.

Let's define the following sheaves of modules:

$$\Omega^{0|0}(U_0) = \mathcal{O}(U_0)[\psi]; \tag{32}$$

$$\Omega^{1|0}(U_0) = \mathcal{O}(U_0)[\psi]d\gamma \oplus \mathcal{O}(U_0)[\psi]d\psi; \tag{33}$$

and similarly un $U_1$. The general sheaf $\Omega^{n|0}$ is locally made up by objects of the form

$$\mathcal{O}(U_0)[\psi](d\gamma)^i(d\psi)^j, \tag{34}$$

where $i = 0; 1$ and $i + j = n$. The definitions on $U_1$ are similar, the only difference is that we will use the corresponding coordinates on $U_1$. Note that $\Omega^{n|0}$ is non zero for all integers $n \geq 0$.

We also define the sheaves of modules $\Omega^{l|1}$, which, on $U_0$, contain elements of the form:

$$\mathcal{O}(U_0)[\psi](d\gamma)^i\delta^{(j)}(d\psi), \tag{35}$$

with $i - j = l$. The elements containing "$d\psi$" cannot appear, since they cancel with the delta forms. On $U_1$, the sections of this sheaf assume a similar structure with respect to the coordinates on $U_1$.

Notice that $\Omega^{l|1}$ is non zero for all integers $l$ with $l \leq 1$, in particular for all negative integers. We still have to describe coordinate change in the intersection $U_0 \cap U_1$ of the objects $\{d\gamma, d\psi, \delta(d\psi)\}$. They are given by:

$$\Phi^* d\widetilde{\gamma} = -\frac{1}{\gamma^2}d\gamma, \tag{36}$$

and

$$\Phi^* d\widetilde{\psi} = \frac{d\psi}{\gamma} - \frac{d\gamma\ \psi}{\gamma^2}. \tag{37}$$

More generally, for any integer $n > 0$, we have the formula

$$\Phi^*(d\widetilde{\psi})^n = \left(\frac{d\psi}{\gamma}\right)^n - \frac{d\gamma\ \psi}{\gamma^2}\left(\frac{d\psi}{\gamma}\right)^{n-1}. \tag{38}$$

It only remains to compute how $\delta(d\psi)$ transforms in a coordinate change. We can proceed as outlined in the previous section.

In this case, we write:

$$\Phi^*\delta(d\widetilde{\psi}) = \delta\left(\frac{d\psi}{\gamma} - \frac{d\gamma\ \psi}{\gamma^2}\right). \tag{39}$$

Then:

$$\Phi^*\delta(d\widetilde{\psi}) = \gamma\delta\left(d\psi - \frac{d\gamma\,\psi}{\gamma}\right) = \gamma\delta\,(d\psi) - \gamma\frac{d\gamma\ \psi}{\gamma}\delta(d\psi) = \gamma\delta\,(d\psi) - \psi\, d\gamma\,\delta'(d\psi). \tag{40}$$

Notice that the latter equation, together with (37), implies that

$$\Phi^*(d\tilde{\psi}\delta(d\tilde{\psi})) = 0$$

as expected.

Hence the generator $\delta(d\tilde{\psi})$ and its properties are well defined. Similarly, one can compute that the derivatives $\delta^n(d\tilde{\psi})$ satisfy the following change of coordinates formula

$$\Phi^*\delta^n(d\widetilde{\psi}) = \gamma^{n+1}\delta^n\,(d\psi) - \gamma^n\psi\ d\gamma\ \delta^{n+1}(d\psi). \tag{41}$$

Now, we can proceed in calculating sheaf cohomology for each of the sheaves $\Omega^{i|j}$ with respect to the covering $\{U_0; U_1\}$.

**Theorem 2.** *The covering $\{U_0; U_1\}$ is acyclic with respect to each of the sheaves $\Omega^{i|j}$.*

**Proof.** We know that $U_0$ and $U_1$ are both isomorphic to $\mathbb{C}$, while $U_0 \cap U_1$ is isomorphic to $\mathbb{C}^*$. Moreover, we know that, classically, $H^q(\mathbb{C}; \mathcal{O}) = \{0\}$, and that $H^q(\mathbb{C}^*; \mathcal{O}) = 0$. We note that the restriction to each open set of the sheaf $\Omega^{i|j}$ is simply the direct sum of the sheaf $\mathcal{O}$ a certain finite number of times.

For example,

$$\Omega^{1|1}(U_0 \cap U_1) = \mathcal{O}(\mathbb{C}^*)d\gamma\delta(d\psi) + \mathcal{O}(\mathbb{C}^*)\psi d\gamma\delta(d\psi). \tag{42}$$

Note that the symbols $d\gamma\delta(d\psi)$ and $\psi d\gamma\delta(d\psi)$ represent the generators of a vector space, then, each of the direct summands can be treated separately. So, we see that a chain of $\Omega^{i|j}$ (on $\mathbb{C}$ or $\mathbb{C}^*$) is a cocycle if and only if each of the summands is a cocycle, and it is a coboundary if and only if every summand is a coboundary. □

We now begin the computation of the main cohomology groups on $\mathbb{P}^{1|1}$. For $\check{H}^0$ we have the following result:

**Theorem 3.** *For integers $n \geq 0$, the following isomorphisms hold*

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{n|0}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1}) \cong \mathbb{C}^{4n+4},$$

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong 0.$$

**Proof.**

- Let's begin from $\check{H}^0(\mathbb{P}^{1|1}, \Omega^{0|0})$. On $U_1$, the sections of the sheaf have the structure:

  $$f(\widetilde{\gamma}) + f_1(\widetilde{\gamma})\widetilde{\psi}. \tag{43}$$

  On the intersection $U_0 \cap U_1$ they transform in the following way:

  $$f\left(\frac{1}{\gamma}\right) + \frac{\psi}{\gamma}f_1\left(\frac{1}{\gamma}\right). \tag{44}$$

So, the only globally defined sections (i.e. which can be extended also on $\mathbb{P}^{1|1}$) are the constants:

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{0|0}) \cong \mathbb{C}. \tag{45}$$

- Let's consider $\check{H}^0(\mathbb{P}^{1|1}, \Omega^{n|0})$, with $n > 0$. On $U_1$, the sections of the sheaf have the structure:

$$\left(f_0(\widetilde{\gamma}) + f_1(\widetilde{\gamma})\widetilde{\psi}\right) \, d\widetilde{\gamma} \, (d\widetilde{\psi})^{n-1} + \left(f_2(\widetilde{\gamma}) + f_3(\widetilde{\gamma})\widetilde{\psi}\right) \, (d\widetilde{\psi})^n. \tag{46}$$

Since both $d\widetilde{\gamma}$ and $d\widetilde{\psi}$ transform, by coordinate change, producing a term $1/\gamma^2$, none of these sections can be extended on the whole $\mathbb{P}^{1|1}$, except the zero section. So,

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{n|0}) \cong 0. \tag{47}$$

- Let us now compute $\check{H}^0(\mathbb{P}^{1|1}; \Omega^{-n|1})$ for every integer $n \geq 0$. On $U_1$, the sections of the sheaf have the form:

$$\left(f_0(\widetilde{\gamma}) + f_1(\widetilde{\gamma})\widetilde{\psi}\right) \, \delta^n(d\widetilde{\psi}) + \left(f_2(\widetilde{\gamma}) + f_3(\widetilde{\gamma})\widetilde{\psi}\right) \, d\widetilde{\gamma} \, \delta^{n+1}(d\widetilde{\psi}). \tag{48}$$

Using the change of coordinates formula (41) one can verify that on the intersection $U_0 \cap U_1$ they transform in the following way:

$$\begin{aligned}
&\left(f_0\left(\frac{1}{\gamma}\right) + f_1\left(\frac{1}{\gamma}\right)\frac{\psi}{\gamma}\right)\left(\gamma^{n+1}\delta^n\,(d\psi) - \gamma^n\psi\, d\gamma\, \delta^{n+1}(d\psi)\right) \\
&\quad - \left(f_2\left(\frac{1}{\gamma}\right) + f_3\left(\frac{1}{\gamma}\right)\frac{\psi}{\gamma}\right)\frac{d\gamma}{\gamma^2}\left(\gamma^{n+2}\delta^{n+1}\,(d\psi) - \gamma^{n+1}\psi\, d\gamma\, \delta^{n+2}(d\psi)\right) \\
&= \left(f_0\left(\frac{1}{\gamma}\right)\gamma^{n+1} + f_1\left(\frac{1}{\gamma}\right)\gamma^n\psi\right)\delta^n\,(d\psi) \\
&\quad - \left(f_2\left(\frac{1}{\gamma}\right)\gamma^n\left(f_0\left(\frac{1}{\gamma}\right)\gamma^n + f_3\left(\frac{1}{\gamma}\right)\gamma^{n-1}\right)\psi\right) d\gamma\, \delta^{n+1}\,(d\psi)\,.
\end{aligned} \tag{49}$$

Therefore this expression extends to a global section if and only if the following conditions hold. The coefficient $f_0$ is a polynomial of degree $n+1$, while $f_1, f_2$ and $f_3$ are polynomials of degree $n$. Moreover, if $a_{n+1}$ and $b_n$ are the coefficients of maximal degree in $f_0$ and $f_3$ respectively, then $a_{n+1} = -b_n$. This establishes that $\check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1})$ has dimension $4n+4$.

- Let's consider $\check{H}^0(\mathbb{P}^{1|1}; \Omega^{1|1})$. On $U_1$, the sections of the sheaf have the structure:

$$\left(f_0(\widetilde{\gamma}) + f_1(\widetilde{\gamma})\widetilde{\psi}\right) d\widetilde{\gamma}\,\delta(d\widetilde{\psi}). \tag{50}$$

These sections cannot be defined on the whole $\mathbb{P}^1$, since they transform as:

$$-\left(f_0\left(\frac{1}{\gamma}\right) + f_1\left(\frac{1}{\gamma}\right)\frac{\psi}{\gamma}\right)\frac{d\gamma}{\gamma^2}\left(\gamma\,\delta\,(d\psi) - \psi\, d\gamma\,\delta'(d\psi)\right) = -\left(f_0\left(\frac{1}{\gamma}\right)\frac{1}{\gamma} + f_1\left(\frac{1}{\gamma}\right)\frac{\psi}{\gamma^2}\right) d\gamma\,\delta\,(d\psi)\,.$$

So,

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{1|1}) = 0. \quad \square \tag{51}$$

A similar computation can be done to obtain the groups $\check{H}^1(\mathbb{P}^{1|1}; \Omega^{i|j})$. The elements of the Čech cohomology are sections $\sigma_{01}$ of $\Omega^{i|j}_{|U_0 \cap U_1}$ which cannot be written as differences $\sigma_0 - \sigma_1$, with $\sigma_0$ defined on $U_0$ and $\sigma_1$ defined on $U_1$. We have the following result:

**Theorem 4.** *For integers $n \geq 0$, the following isomorphisms hold*

$$\begin{aligned}
&\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n|0}) \cong \mathbb{C}^{4n} \\
&\check{H}^1(\mathbb{P}^{1|1}, \Omega^{-n|1}) \cong 0, \\
&\check{H}^1(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong \mathbb{C}.
\end{aligned}$$

**Proof.**

- $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{0|0}) = \{0\}$, since for every section on $U_1 \cap U_0$ we have the structure:

$$f(\widetilde{\gamma}) + f_1(\widetilde{\gamma})\widetilde{\psi}, \tag{52}$$

we can decompose the Laurent series of $f$ and $f_1$ in a singular part and in a holomorphic component. The singular part is defined on $U_0$, while the holomorphic part is defined on $U_1$. So, it's easy to write every section of $\Omega^{0|0}$ on $U_0 \cap U_1$ as a difference of sections on $U_0$ and $U_1$.

- We now compute $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n|0})$ for $n > 0$. A section on $U_0$ is of the type

$$(f_0(\gamma) + f_1(\gamma)\psi)\; d\gamma\; (d\psi)^{n-1} + (f_2(\gamma) + f_3(\gamma)\psi)\; (d\psi)^n$$

while a section on $U_1$ is of the type

$$\big(g_0(\widetilde{\gamma}) + g_1(\widetilde{\gamma})\widetilde{\psi}\big)\; d\widetilde{\gamma}\; (d\widetilde{\psi})^{n-1} + \big(g_2(\widetilde{\gamma}) + g_3(\widetilde{\gamma})\widetilde{\psi}\big)\; (d\widetilde{\psi})^n.$$

All functions here are regular. A computation shows that, taking the difference of the two on $U_0 \cap U_1$ and expressing everything in the coordinates $\gamma$ and $\psi$, gives us an expression of the type

$$\begin{aligned}&\big(f_0(\gamma) + g_0(\gamma^{-1})\gamma^{-(n+1)}\big)\; d\gamma\; (d\psi)^{n-1} + \big(f_1(\gamma) + g_1(\gamma^{-1})\gamma^{-(n+2)} + g_2(\gamma^{-1})\gamma^{-(n+1)}\big)\; \psi d\gamma\; (d\psi)^{n-1}\\ &\quad + \big(f_2(\gamma) - g_2(\gamma^{-1})\gamma^{-n}\big)\; (d\psi)^n + \big(f_3(\gamma) - g_3(\gamma^{-1})\gamma^{-(n+1)}\big)\; \psi(d\psi)^n.\end{aligned}$$

It is clear that in the first row there are no terms of the type $a_k\gamma^{-k}$ with $1 \le k \le n$, so this gives us $n$ parameters for an element of $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n|0})$. Similarly, the second row gives us $n$ parameters, the third gives us $n-1$ and the fourth $n$. This gives a total of $4n-1$. Notice now that in the above expression the coefficient of $\gamma^{-(n+1)}$ in the second row must be equal to the coefficient of $\gamma^{-n}$ in the third row. This constraint on the terms of the above type gives us room for an extra parameter in the elements of $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n|0})$. We therefore have a total of $4n$ parameters.

- We compute in a similar way $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{-n|1})$ for $n \ge 0$. A computation shows that a difference between a section on $U_0$ and a section on $U_1$ is of the type

$$\begin{aligned}&\big(f_0(\gamma) - g_0(\gamma^{-1})\gamma^{n+1}\big)\; \delta^n(d\psi) + \big(f_1(\gamma) - g_1(\gamma^{-1})\gamma^n\big)\; \psi\delta^n(d\psi) + \big(f_2(\gamma) + g_2(\gamma^{-1})\gamma^n\big)\; d\gamma\; \delta^{n+1}(d\psi)\\ &\quad + \big(f_3(\gamma) + g_0(\gamma^{-1})\gamma^n + g_3(\gamma^{-1})\gamma^{n-1}\big)\; \psi d\gamma\; \delta^{n+1}(d\psi).\end{aligned}$$

It is clear that every section on $U_0 \cap U_1$ is represented in such an expression. Therefore we have $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{-n|1}) = 0$.

- We see in a similar way that $\check{H}^1(\mathbb{P}^{1|1}; \Omega^{1|1}) = \mathbb{C}$, in fact the section on $U_0 \cap U_1$ which are not differences are all generated by

$$\frac{\psi d\gamma\delta(d\psi)}{\gamma}. \tag{53}$$

This completes the proof. □

Notice that $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n+1|0})$ and $\check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1})$ have the same dimension. There is an interesting explanation of this fact, in fact we can construct a pairing

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n+1|0}) \times \check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1}) \to \check{H}^1(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong \mathbb{C}$$

as follows. As explained above, an element of $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n+1|0})$ is of the type

$$\big(f_0(\gamma^{-1}) + f_1(\gamma^{-1})\psi\big)\; d\gamma\; (d\psi)^n + \big(f_2(\gamma^{-1}) + f_3(\gamma^{-1})\psi\big)\; (d\psi)^{n+1} \tag{54}$$

where $f_0$ and $f_1$ are polynomials of degree at most $n+1$, while $f_1$ and $f_2$ can be chosen to be respectively of degree at most $n+2$ and $n$ or both of degree at most $n+1$. An element of $\check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1})$ is of the type

$$(g_0(\gamma) + g_1(\gamma)\psi)\; \delta^n(d\psi) + (g_2(\gamma) + g_3(\gamma)\psi)\; d\gamma\; \delta^{n+1}(d\psi), \tag{55}$$

where $g_0$ is a polynomial of degree $n+1$, $g_1, \ldots, g_3$ are polynomials of degree $n$ and the coefficients of maximal degree in $g_0$ and $g_3$ are opposite to each other. Now recall that we have a pairing

$$\Omega^{n+1|0} \times \Omega^{-n|1} \to \Omega^{1|1}$$

obeying the rules explained in Section 4. For instance

$$\begin{aligned}&\langle d\gamma\; (d\psi)^n, \delta^n(d\psi)\rangle = (-1)^n n!\, d\gamma\; \delta(d\psi),\\ &\langle (d\psi)^{n+1}, d\gamma\; \delta^{n+1}(d\psi)\rangle = -(-1)^n(n+1)!\, d\gamma\; \delta(d\psi),\\ &\langle d\gamma\; (d\psi)^n, d\gamma\; \delta^{n+1}(d\psi)\rangle = \langle (d\psi)^{n+1}, \delta^n(d\psi)\rangle = 0.\end{aligned}$$

It can be checked that this product descends to a pairing in cohomology. We have the following

**Lemma 5.** *On $\mathbb{P}^{1|1}$ the above product in cohomology is non-degenerate.*

**Proof.** The product between (54) and (55) is cohomologous to the expression

$$(-1)^n n!\,((f_0g_1+f_1g_0)-(n+1)(f_2g_3+f_3g_2))\,\psi\,d\gamma\,\delta(d\psi). \tag{56}$$

We have to prove that if (55) is arbitrary and non zero, then we can chose $f_0,\ldots,f_3$ so that the above expression is cohomologous to (53). We can assume one of the $g_0,\ldots,g_3$ to be non zero. If $g_0\neq 0$, let $a_k$ be the coefficient of highest degree in $g_0$, hence $k\le n+1$. Define

$$f_1 = C\gamma^{-k+1},$$

and $f_0, f_2, f_3$ to be zero. Then, for suitably chosen $C\neq 0$ we can easily see that (56) is cohomologous to (53). Notice also that $k+1\le n+2$, so the choice of $f_0,\ldots,f_3$ gives a well defined element of $\check{H}^1(\mathbb{P}^{1|1},\Omega^{n+1|0})$. Similar arguments hold when $g_1, g_2$ or $g_3$ are not zero. □

A consequence of this lemma is that $\check{H}^1(\mathbb{P}^{1|1},\Omega^{n+1|0})$ and $\check{H}^0(\mathbb{P}^{1|1},\Omega^{-n|1})$ are dual to each other. This explains why they have the same dimension.

## 6. Super de Rham cohomology

We now briefly describe smooth and holomorphic de Rham cohomology with respect to the $d$ differential on superforms.

On a fixed complex supermanifold $M^{n|m}$ we denote by $\mathcal{A}^{i|j}$ and $\Omega^{i|j}$ respectively the sheaf of smooth and holomorphic superforms of degree $i$ with picture number $j$ and by $\mathbf{A}^{i|j}$ and $\mathbf{\Omega}^{i|j}$ the global sections of these sheaves. As usual for superforms, $i$ can also have negative values. On $\mathbf{A}^{*|j}$ (or locally on $\mathcal{A}^{*|j}$) we can define the exterior differential operator $d:\mathbf{A}^{i|j}\to\mathbf{A}^{i+1|j}$ which satisfies the following rules:

1. $d$ behaves as a differential on functions;
2. $d^2=0$;
3. $d$ commutes with $\delta$ and its derivatives, and so $d(\delta^{(k)}(d\psi))=0$.

Similarly, the same operator $d$ is defined on $\mathbf{\Omega}^{*|j}$, and behaves as the $\partial$ operator on holomorphic functions (since $\overline{\partial}$ always vanishes).

It is easy to verify that, on the intersection of 2 charts, $d$ commutes with the pull-back map $\Phi^*$ expressing the "coordinate changes". This is due to the particular definition of the pull-back of the differentials, and it implies that $d$ is well defined and it does not depend on coordinate systems.

As an example, we prove it on $\mathbb{P}^{1|1}$ in the holomorphic case, leaving to the reader the easy generalization to every other super-projective space.

- We know that $\Phi^*(\widetilde{\gamma})=\frac{1}{\gamma}$, so it's easy to see that $d\left(\frac{1}{\gamma}\right)=\Phi^*d(\widetilde{\gamma})=-\frac{1}{\gamma^2}d\gamma$.
- We know that $\Phi^*(\widetilde{\psi})=\frac{\psi}{\gamma}$, so it's easy to see that $d\left(\frac{\psi}{\gamma}\right)=\Phi^*d(\widetilde{\psi})=-\frac{1}{\gamma^2}d\gamma\,\psi+\frac{d\psi}{\gamma}$.
- We know that $\Phi^*\delta(d\widetilde{\psi})=\gamma\delta\,(d\psi)-d\gamma\,\psi\delta'(d\psi)$. Then, $\Phi^*d(\delta(d\widetilde{\psi}))=0$.
  But, $d(\Phi^*\delta(d\widetilde{\psi}))=d(\gamma\delta\,(d\psi)-d\gamma\,\psi\delta'(d\psi))=d\gamma\,\delta\,(d\psi)+d\gamma\,d\psi\,\delta'(d\psi)=0$.

Now $(\mathbf{A}^{*|j}(M),d)$ and $(\mathbf{\Omega}^{*|j}(M),d)$ define complexes, whose cohomology groups we call respectively the smooth and holomorphic super de Rham cohomology groups:

**Definition 6.** If $Z^{i|j}$ is the set of the $d$-closed forms in $\mathbf{A}^{i|j}$, and $B^{i|j}=d\mathbf{A}^{i-1|j}$. Then, the $i|j$-th smooth de Rham cohomology group is the quotient of additive groups:

$$H^{i|j}_{\mathrm{DR}}(M^{n|m})=\frac{Z^{i|j}}{B^{i|j}}. \tag{57}$$

Similarly we define the holomorphic de Rham cohomology groups which we denote by $H^{i|j}_{\mathrm{DR}}(M^{n|m},\mathrm{hol})$.

We now calculate the holomorphic super de Rham cohomology of $\mathbb{C}^{m|n}$.

Let's call $\{\gamma_1,\gamma_2,\ldots,\gamma_m\}$ the even coordinates and $\{\psi_1,\psi_2,\ldots,\psi_n\}$ the odd coordinates of $\mathbb{C}^{m|n}$.

Clearly the following forms are closed:

(a) 1;
(b) $\{d\gamma_i\},\ i\in\{1;2;\ldots,m\}$;
(c) $\{d\psi_j\},\ j\in\{1;2;\ldots;n\}$;
(d) $\{d\gamma_h\cdot\psi_k+\gamma_h d\psi_k=d(\gamma_h\cdot\psi_k)\}, h\in\{1;2;\ldots,m\}, k\in\{1;2;\ldots,n\}$;
(e) $\{\delta^{(k)}(d\psi_a)\},\ a\in\{1;2;\ldots;n\}$ and $k\in\mathbb{N}$;
(f) $\{\psi_b\delta(d\psi_b)\},\ b\in\{1;2;\ldots,n\}$.

All other closed forms are products and linear combinations of these with coefficients some holomorphic functions in the even coordinates. Observe that $\{\psi_b\delta(d\psi_b)\}$, with $b \in \{1; 2; \dots, n\}$ are not exact. A calculation shows that the holomorphic super de Rham cohomology $H^{i|j}(\mathbb{C}^{m|n}, \text{hol})$ is zero whenever $i > 0$, it is generated by 1 when $i = j = 0$, by $\{\psi_b\delta(d\psi_b)\}$ when $i = 0$ and $j = 1$ and by their $j$-th exterior products when $i = 0$ and $j \geq 2$. Similarly we can compute the smooth de Rham cohomology of $\mathbb{R}^{m|n}$.

**Remark 7.** In particular, we see that the super-vector space $\mathbb{C}^{m|n}$ (or $\mathbb{R}^{m|n}$) does not satisfy the Poincarè lemma, since its de Rham cohomology is not trivial. The forms $\{\psi_i\delta(d\psi_i)\}$ can be seen as even generators of the "odd component" of the cohomology.

As an example we compute the holomorphic de Rham cohomology of $\mathbb{P}^{1|1}$. We have:

**Theorem 8.** *For $n \geq 0$, the holomorphic de Rham cohomology groups of $\mathbb{P}^{1|1}$ are as follows:*

$$H^{n|0}_{\text{DR}}(\mathbb{P}^{1|1}, \text{hol}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$H^{-n|1}_{\text{DR}}(\mathbb{P}^{1|1}, \text{hol}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$H^{1|1}_{\text{DR}}(\mathbb{P}^{1|1}, \text{hol}) \cong 0.$$

**Proof.** We have given explicit descriptions of global sections of the sheaves $\Omega^{i|j}$ in Theorem 3 and therefore it is a rather straightforward computation to determine which forms are closed and which are exact in terms of the coefficients describing the forms (see formulas (46) and (48)). We leave the details to the reader. Notice that $H^{0|1}_{\text{DR}}(\mathbb{P}^{1|1}, \text{hol})$ is generated by the closed form $\psi\delta(d\psi)$ which is globally defined on $\mathbb{P}^{1|1}$. $\square$

Now consider a general smooth super manifold $M^{n|m}$. On $M$ we can define the pre-sheaf which associates to every open subset $U \subset M$ the smooth super de Rham $i|j$-cohomology group of $U^{n|m}$ and we denote the corresponding sheaf by $\mathcal{H}^{i|j}$. If follows from the above remark that $\mathcal{H}^{i|j}$ is the constant $\mathbb{C}$-sheaf when $i, j = 0$, a non zero sheaf when $i = 0$ and $j > 0$ and the zero sheaf otherwise. It makes therefore sense to consider the Čech cohomology groups which we denote by $\check{H}^p(M^{n|m}, \mathcal{H}^{i|j})$ (which are zero when $i > 0$). Recall that a *good cover* is an open covering $U_\alpha$ of $M$ such that every non-empty finite intersection $U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_p}$ is diffeomorphic to $\mathbb{R}^n$. We can now prove a generalization of the classical equivalence of Čech and De Rham cohomology.

**Theorem 9.** *Given a supermanifold $M^{n|m}$, for $i \geq 0$ we have the following isomorphism*

$$H^{i|j}_{\text{DR}}(M^{n|m}) \cong \check{H}^i(M^{n|m}, \mathcal{H}^{0|j}). \tag{58}$$

**Proof.** For the proof we can use the same method used in [13] for the classical equivalence of Čech and De Rham cohomology. Let us fix a good cover $\underline{\mathcal{U}} = \{U_\alpha\}$ of $M$. For integers $p, q \geq 0$, let us set

$$K^{p,q} = \mathcal{C}^p(\mathcal{A}^{q|j}, \underline{\mathcal{U}}), \tag{59}$$

where the righthand side denotes the usual $p$-cochains of the sheaf $\mathcal{A}^{q|j}$, with respect to the covering $\underline{\mathcal{U}}$. Then we can form the double complex $(K, d, \delta)$, where $K = \oplus_{p,q\geq 0} K^{p,q}$ and the operators are the usual exterior differential operator $d$ and the Čech co-boundary operator $\delta$. From this double complex one can construct two spectral sequences $(E^{p,q}_r, d_r)$ and $(E'^{p,q}_r, d_r)$ both converging to the total cohomology $H_D(K)$ of the double complex (see [13]). We have that

$$E^{p,q}_2 = \check{H}^p(H^{q|j}_{\text{DR}}(\mathcal{A}^{q|j}), \underline{\mathcal{U}}) = \check{H}^p(M^{n|m}, \mathcal{H}^{q|j}). \tag{60}$$

In particular $E^{p,q}_2 = 0$ when $q > 0$, therefore $(E^{p,q}_r, d_r)$ stabilizes at $r = 2$. On the other hand we have

$$E'^{p,q}_2 = H^q_{\text{DR}}(\check{H}^p(\mathcal{A}^{q|j}, \underline{\mathcal{U}})). \tag{61}$$

We can easily see that the sheaves are fine i.e. that

$$\check{H}^0(\mathcal{A}^{q|j}, \underline{\mathcal{U}}) = \mathbf{A}^{q|j} \tag{62}$$

and

$$\check{H}^p(\mathcal{A}^{q|j}, \underline{\mathcal{U}}) = 0 \quad \text{when } p > 0. \tag{63}$$

The latter identity can be proved using standard partitions of unity relative to the covering $\underline{\mathcal{U}}$ of the underlying smooth manifold $M$. Therefore we conclude that $(E_r^{'p,q}, d_r)$ also stabilizes at $r = 2$ and $E_2^{'p,q} = 0$ when $p > 0$ and

$$E_2^{'0,q} = H_{\mathrm{DR}}^{q|j}(M^{n|m}). \tag{64}$$

The theorem is then proved by using the fact that the two spectral sequences must converge to the same thing and therefore

$$H_{\mathrm{DR}}^{q|j}(M^{n|m}) = E_2^{'0,q} \cong E^{q,0} = \check{H}^q(M^{n|m}, \mathcal{H}^{0|j}). \quad \square \tag{65}$$

It may happen the sheaf $\mathcal{H}^{0|j}$ is actually a constant sheaf, for instance on projective superspaces $\mathbb{P}^{n|m}$ the forms $\{\psi_i\delta(d\psi_i)\}$ are globally defined. In this case, as a corollary of the above result, we obtain a sort of "Kunneth formula" for the super de Rham cohomology on supermanifolds.

**Corollary 10.** *Let $M^{n|m}$ be a supermanifold, such that $\mathcal{H}^{0|j}$ is a constant sheaf, (e.g. when the locally defined forms $\{\psi_i\delta(d\psi_i)\}$ extend globally). Then the de Rham cohomology of $M^{n|m}$ is:*

$$H_{\mathrm{DR}}^{*|j}(M^{n|m}) = H_{\mathrm{DR}}^{*}(M) \otimes \mathcal{H}^{0|j}. \tag{66}$$

**Proof.** The map $\psi : H_{\mathrm{DR}}^*(M) \otimes \mathcal{H} \longrightarrow H_{\mathrm{DR}}^*(M^{n|m})$ given by multiplication is a map in cohomology. It is easy to show that, if $\gamma$ is an element of $H_{\mathrm{DR}}^*(M)$ and $\omega$ is an element of $\mathcal{H}$, then $\gamma\omega$ is an element of $H_{\mathrm{DR}}^*(M^{n|m})$. Moreover, if $\gamma$ and $\gamma'$ are cohomologous in $H_{\mathrm{DR}}^*(M)$, then $\gamma\omega$ and $\gamma'\omega$ are cohomologous in $H_{\mathrm{DR}}^*(M^{n|m})$: if $\gamma - \gamma' = df$, then $\gamma\omega - \gamma'\omega = d(f\omega)$, since $d\omega = 0$. Now, we proceed by induction on the number of open sets of the good cover of $M$. Obviously, if this number is equal to 1, then $M = \mathbb{R}^n$, and the thesis is true for the study we have performed above. We have to prove the truth of the thesis for an integer $s$, knowing that it is true for $s - 1$. So, let $M$ be covered by $s$ open sets forming a good cover. Then, we can call $U$ one of them, and $V$ the union of the remaining ones. We know that the thesis is true on $U$; $V$ and $U \cap V$. We will call $U^{m|n}$ and $V^{m|n}$ the open sets $U$ and $V$ endowed with the corresponding graded sheaves. Let $k$; $p$ be two integers; by the usual Mayer–Vietoris sequence,

$$\cdots \longrightarrow H^p(U \cup V) \longrightarrow H^p(U) \oplus H^p(V) \longrightarrow H^p(U \cap V) \longrightarrow \cdots . \tag{67}$$

If $\mathcal{H}^q$ are the elements of $\mathcal{H}$ of degree $\cdot|q$, we have the following exact sequence:

$$\cdots \longrightarrow H^p(U \cup V) \otimes \mathcal{H}^q \longrightarrow (H^p(U) \otimes \mathcal{H}^q) \oplus (H^p(V) \otimes \mathcal{H}^q) \longrightarrow (H^p(U \cap V) \otimes \mathcal{H}^q) \longrightarrow \cdots . \tag{68}$$

Summing up, we find that the following sequence is exact:

$$\begin{aligned}
&\cdots \longrightarrow \bigoplus_{p+q=k} H^p(U \cup V) \otimes \mathcal{H}^q \\
&\longrightarrow \bigoplus_{p+q=k} (H^p(U) \otimes \mathcal{H}^q) \oplus (H^p(V) \otimes \mathcal{H}^q) \\
&\longrightarrow \bigoplus_{p+q=k} (H^p(U \cap V) \otimes \mathcal{H}^q) \longrightarrow \cdots
\end{aligned}$$

where the sum is performed over $p, q$.

The following diagram is commutative:

$$\begin{array}{ccccc}
\bigoplus_{p+q=k} H^p(U \cup V) \otimes \mathcal{H}^q & \rightarrow & \bigoplus_{p+q=k} (H^p(U) \otimes \mathcal{H}^q) \oplus (H^p(V) \otimes \mathcal{H}^q) & \rightarrow & \bigoplus_{p+q=k} (H^p(U \cap V) \otimes \mathcal{H}^q) \\
\downarrow \psi & & \downarrow \psi & & \downarrow \psi \\
H^k(M^{n|m}) & \rightarrow & H^k(U^{n|m}) \oplus H^k(V^{n|m}) & \rightarrow & H^k((U \cap V)^{n|m})
\end{array}$$

The commutativity is clear except possibly for the square:

$$\begin{array}{ccc}
\oplus(H^p(U \cap V) \otimes \mathcal{H}^q) & \longrightarrow^{d^*} & \oplus H^{p+1}(U \cup V) \otimes \mathcal{H}^q \\
\downarrow \psi & & \downarrow \psi \\
H^k((U \cap V)^{n|m}) & \longrightarrow^{d^*} & H^{k+1}(M^{n|m})
\end{array}$$

Let $\omega \otimes \phi$ be in $(H^p(U \cap V) \otimes \mathcal{H}^q)$. Then, $\psi d^*(\omega \otimes \phi) = (d^*\omega) \cdot \phi$ and $d^*\psi(\omega \otimes \phi) = d^*(\omega\phi)$.

If $\{\rho_U; \rho_V\}$ is a partition of unity subordinate to $\{U; V\}$, then $d^*\omega = -d(\rho_V\omega)$ and $d^*(\omega\phi) = -d(\rho_V\omega\phi)$ on $U$, while $d^*\omega = d(\rho_U\omega)$ and $d^*(\omega\phi) = d(\rho_U\omega\phi)$ on $V$. Note that $-d(\rho_U\omega\phi) = d(\rho_V\omega\phi)$ on $U \cap V$, since both $\omega$ and $\phi$ are closed. So, $d^*(\omega\phi)$ is a global section of the sheaf of $M^{n|m}$.

By these relations, it's easy to see that the square is commutative:
$d^*\psi(\omega \otimes \phi) = d^*(\omega\phi) = d(\rho_U \omega\phi) = (d\rho_U \omega)\phi = (d^*\omega) \cdot \phi = \psi d^*(\omega \otimes \phi)$, since $\phi$ is closed.
By the Five Lemma, if the theorem is true for $U^{n|m}$, $V^{n|m}$ and $(U \cap V)^{n|m}$ then it holds also for $M^{n|m}$, by induction. □

## References


[1] J. Wess, J. Bagger, Supersymmetry and Supergravity, Univ. Pr., Princeton, USA, 1992, 259 p.
[2] Y.I. Manin, Gauge Field Theory and Complex Geometry, second ed., Springer, Berlin, 1997, Translated from the 1984 Russian original by N. Koblitz and J. R. King.
[3] N. Berkovits, Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring, J. High Energy Phys. 0409 (2004) 047. arXiv:hep-th/0406055.
[4] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, The Geometry of Supermanifolds, Kluwer Acad. Publ., Dordrecht, 1991.
[5] R. Catenacci, M. Debernardi, P.A. Grassi, D. Matessi, Balanced superprojective varieties, J. Geophys. Res. 59 (10) (2009) 1363–1378. arXiv:0707.4246.
[6] S. Cordes, G.W. Moore, S. Ramgoolam, Nuclear Phys. B Proc. Suppl. 41 (1995) 184. arXiv:hep-th/9411210.
[7] V.S. Varadarajan, Supersymmetry for Mathematicians: An Introduction, in: Courant Lectures Notes, American Mathematical Society, 2004.
[8] V.S. Varadarajan, Courant Lectures on Supersymmetry, http://www.math.ucla.edu/~vsv/susy.html.
[9] P.A. Grassi, M. Marescotti, hep-th/0712.2600v2 (in press).
[10] T. Voronov, A. Zorich, Integral transformations of pseudodifferential forms, Uspekhi Mat. Nauk 41 (1986) 167–168;
T. Voronov, A. Zorich, Complex of forms on a supermanifold, Funktsional. Anal. i Prilozhen. 20 (1986) 58–65;
T. Voronov, A. Zorich, Theory of bordisms and homotopy properties of supermanifolds, Funktsional. Anal. i Prilozhen. 21 (1987) 77–78;
T. Voronov, A. Zorich, Cohomology of supermanifolds, and integral geometry, Sov. Math. Dokl. 37 (1988) 96–101.
[11] A. Belopolsky, New geometrical approach to superstrings, arXiv:hep-th/9703183;
A. Belopolsky, Picture changing operators in supergeometry and superstring theory, arXiv:hep-th/9706033;
P.A. Grassi, G. Policastro, Super-Chern–Simons theory as superstring theory, arXiv:hep-th/0412272.
[12] M.J. Rothstein, Trans. Amer. Math. Soc. 299 (1987) 387–396;
Hernandez-Ruiperez, Muñoz-Masque, Differential Geometric Methods in Mathematical Physics, in: Lecture Notes in Mathematics, vol. 1251, 1987, pp. 137–149.
[13] R. Bott, L.W. Tu, Differential Forms in Algebraic Topology, Springer, New York, 1982.