# Personal Notes All the information I need

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### 1 Affine Group Scheme, Hopf Algebra

**Definition 1.1.** Let A be a commutative K-algebra. The corresponding affine K-scheme G = Spec(A) is said to be a group scheme if it is endowed with algebraic operations

$$\mu: G \times G \longrightarrow G \ (product),$$
  
 $e: Spec(K) \longrightarrow G(unit),$   
 $\iota: G \longrightarrow G(inverse),$ 

satisfying the usual axioms of a group, which are expressed by the commutativity of the following three diagrams

(1) Associativity:

$$G \times G \times G \xrightarrow{\mu \times Id} G \times G$$

$$\downarrow^{Id \times \mu} \qquad \qquad \downarrow^{\mu}$$

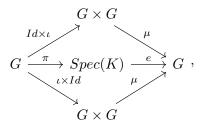
$$G \times G \xrightarrow{\mu} G$$

(2) Unit:

$$G \times Spec(K) \xrightarrow{Id \times e} G \times G \xleftarrow{e \times Id} Spec(K) \times G$$

$$\downarrow \mu \qquad \qquad pr_1 \qquad \downarrow \mu \qquad pr_2$$

(3) Inverse:



where  $\pi$  denotes the structural map of G as a K-scheme. If the algebra A is finitely generated, we say that G is algebraic. We will see below that every affine group scheme is in fact a projective limit of affine group schemes.

Equivalent definition: A group scheme over K is a functor between the categories of commutative K-algebras and abstract groups. Namely, given  $G = \operatorname{Spec}(A) A, P, Q$  are K-algebras, one considers the functor:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow^G & \downarrow^G & \downarrow^G \\ Hom_{K-alg}(A,P) = G(P) & \xrightarrow{f} G(Q) = Hom_{K-alg}(A,Q) \end{array},$$

#### 1.1 Hopf Algebra

The category of affine schemes over K is equivalent to the category of the commutative K-algebras through the contravariant functors

$$A \mapsto \operatorname{Spec}(A)$$
  
 $G \mapsto \mathcal{O}(G),$ 

where  $\mathcal{O}(G)$  is the ring of regular functions on G. Thus the defining properties of a group scheme can be transferred to the corresponding algebra, yielding the concept of Hopf algebra.

**Definition 1.2.** Let H be an associative (not necessarily commutative) K-algebra. Let  $\nabla: H \otimes H \longrightarrow H$  be the product of H and  $\eta: k \longrightarrow H$  the unit.

(1) We say that H is a **bialgebra** if it is provided with two morphisms of algebras

$$\Delta: H \longrightarrow H \otimes H(coproduct),$$
  
 $\epsilon: H \longrightarrow k(counit)$ 

such that the following diagrams commute:

- (a) Coassociativity:
- *(b)*

## 2 Tannakian Categories

K is a field of characteristic 0,  $\mathcal{A}$  rigid tensor  $\mathbb{K}$ -linear abelian category, L is an extension of K.

**Definition 2.1.** An L-valued fibre functor is a tensor functor  $\omega : \mathcal{A} \longrightarrow Vec_L$  which is faithful and exact.

#### Definition 2.2. A is

- Neutralized Tannakian if one can endow it with a K-valued fibre functor.
- Neutral Tannakian if  $\exists K$ -valued fibre functor.
- Tannakian if  $\exists L$ -valued fibre functor for some L.

**Example 2.3.** G is an affine K-group scheme,  $\mathcal{A} = Rep_K(G), \omega : \mathcal{A} \longrightarrow Vec_K$  the forgetful functor.