Periods and the conjectures of Grothendieck and Kontsevich–Zagier

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This paper concerns a class of complex numbers, called periods, *that appear naturally when comparing two cohomology theories for algebraic varieties (the first defined topologically and the second algebraically). Our goal is to explain the fundamental conjectures of Grothendieck and Kontsevich–Zagier that give very precise information about the transcendence properties of periods. The notion of* motive *(due to Grothendieck) plays an important conceptual role. Finally, we explain a geometric version of these conjectures. In contrast with the original conjectures whose solution seems to lie in a very distant future, if at all it exists, a solution for the geometric conjectures is within reach of the actual motivic technology.*

1 Introduction

1.1 Integration and cohomology

Let M be a real C^∞-manifold. Let $\mathcal{A}^n(M,\mathbb{C})$ be the $\mathbb{C}$-vector space of C^∞-differential forms of degree n on M. These vector spaces are the components of the de Rham complex $\mathcal{A}^\bullet(M;\mathbb{C})$ whose cohomology (i.e., the quotient of the space of closed differential forms by its subspace of exact differential forms) is the *de Rham cohomology* of M denoted by $H^\bullet_{\mathrm{dR}}(M;\mathbb{C})$. In practice, for instance if M is compact, the $H^n_{\mathrm{dR}}(M;\mathbb{C})$'s are finite dimensional vector spaces; in any case, they vanish unless $0 \leq n \leq \dim(M)$.

On the other hand, one has the singular chain complex of M, denoted by $C_\bullet(M;\mathbb{Q})$. For $n \in \mathbb{N}$, $C_n(M;\mathbb{Q})$ is the $\mathbb{Q}$-vector space with basis consisting of C^∞-maps from the n-th simplex

$$\Delta^n = \left\{ (x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{c} x_0 \geq 0,\ldots,x_n \geq 0, \\ x_0+\cdots+x_n = 1 \end{array} \right\}$$

to M. The *singular homology* of M, denoted by $H_\bullet(M;\mathbb{Q})$, is the homology of this complex.

Integration of forms yields a well-defined pairing

$$\langle -,- \rangle : H_n(M;\mathbb{Q}) \times H^n_{\mathrm{dR}}(M;\mathbb{C}) \to \mathbb{C}. \qquad (1)$$

If $\gamma = \sum_{s=1}^r a_s \cdot [f_s : \Delta^n \to M] \in C_n(M;\mathbb{Q})$ is a closed chain and $\omega \in \mathcal{A}^n(M,\mathbb{C})$ is a closed differential form, then

$$\langle \overline{\gamma}, \overline{\omega} \rangle = \sum_{s=1}^r a_s \cdot \int_{\Delta^n} f_s^* \omega. \qquad (2)$$

Stokes formula insures that the right hand side depends only on the classes $\overline{\gamma} \in H_n(M;\mathbb{Q})$ and $\overline{\omega} \in H^n_{\mathrm{dR}}(X;\mathbb{C})$ of γ and ω.

The classical theorem of de Rham asserts that the pairing (1) is perfect, i.e., that it identifies $H^n_{\mathrm{dR}}(M;\mathbb{C})$ with the space of linear maps from $H_n(M;\mathbb{Q})$ to $\mathbb{C}$.

1.2 Periods

Periods are complex numbers obtained by evaluating the pairing (1) on special de Rham cohomology classes with interesting arithmetic properties.

Assume that M is an algebraic manifold defined by equations with rational coefficients. In the language of algebraic geometry, we are fixing a smooth $\mathbb{Q}$-variety X such that M is the manifold of complex points in X, which we write simply as $M = X(\mathbb{C})$. In this case, it makes sense to speak of algebraic differential forms over M (or X). In general, there are few of these globally, and one is forced to work locally for the Zariski topology on X, i.e., to consider the sheaf Ω^n_X of algebraic differential forms of degree n on X. Locally for the Zariski topology, a section of Ω^n_X is an element of $\mathcal{A}^n(M;\mathbb{C})$ that can be written as a linear combination of $f_0 \cdot \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_n$ where the f_i's are *regular functions* over X. (For example, if $M = \mathbb{C}^n$, the f_i's can be written as fractions P_i/Q_i with P_i's and Q_i's polynomials in n variables with rational coefficients.) Varying the degree, one gets the algebraic de Rham complex $\Omega^\bullet_X$; this is a complex of sheaves for the Zariski topology on X. Following Grothendieck, we define the *algebraic de Rham cohomology* of X (or M) to be the Zariski hyper-cohomology of $\Omega^\bullet_X$:

$$H^\bullet_{\mathrm{AdR}}(X) = \mathbb{H}^\bullet_{\mathrm{Zar}}(X;\Omega^\bullet_X).$$

The elements of $H^\bullet_{\mathrm{AdR}}(X)$ are those special de Rham cohomology classes that produce periods when paired with singular homology classes. More precisely, there is a canonical isomorphism

$$H^\bullet_{\mathrm{AdR}}(X) \otimes \mathbb{C} \simeq H^\bullet_{\mathrm{dR}}(M;\mathbb{C})$$

and, in particular, $H^\bullet_{\mathrm{AdR}}(X)$ is a sub-$\mathbb{Q}$-vector space of $H^\bullet_{\mathrm{dR}}(M;\mathbb{C})$. (Roughly speaking, this inclusion is obtained by considering an algebraic differential form on X as an ordinary differential form on M.) Now, restricting the pairing (1), we get a pairing

$$H^{\mathrm{Sing}}_\bullet(X) \otimes H^\bullet_{\mathrm{AdR}}(X) \to \mathbb{C} \qquad (3)$$

where $H^{\mathrm{Sing}}_\bullet(X)$ denotes $H_\bullet(M;\mathbb{Q})$. (As the notation suggests, the above pairing is canonically associated to the algebraic variety X.)

Definition 1. A *period* of the algebraic variety X is a complex number which is in the image of the pairing (3).

Remark 2. Although $H_{\bullet}^{\mathrm{Sing}}(X)$ and $H_{\mathrm{AdR}}^{\bullet}(X)$ are finite dimensional $\mathbb{Q}$-vector spaces, the values of the paring (3) are not rational numbers in general. Indeed, *periods* are often (expected to be) transcendental numbers.

Remark 3. When X is smooth and proper, its periods are called *pure*. The previous construction, in the case where X is proper, yields all the pure periods.

If X is not necessarily proper, its periods are called *mixed*. In contrast with the pure case, the previous construction is not expected to give all the *mixed periods*. Indeed, more mixed periods are obtained by considering relative cohomology of pairs (see §2.1).

1.3 Transcendance

It is an important and fascinating problem to understand the arithmetic properties of periods. In the case of *abelian periods* (i.e., those obtained by taking $n = 1$ in (3) with X possibly singular or, equivalently, those arising from abelian varieties and more generally 1-motives) much is known thanks to the *analytic subgroup theorem* of Wüstholz [16] generalizing results of Baker. The result of Wüstholz can be interpreted as follows: *every $\overline{\mathbb{Q}}$-linear relation between abelian periods is of motivic origin.* This statement appears explicitly, for example, in [17]. We also refer to [8] for a similar interpretation of the result of Wüstholz in the context of another (albeit related) conjecture of Grothendieck (in the style of the Hodge and Tate conjectures).

Beside the case of abelian periods, very little is known (but see [15] for a comprehensive catalogue of what transcendence theory knows about periods and [9] for some spectacular recent advances concerning periods of Tate motives, aka., multiple zeta values). Nonetheless, the conjectural picture is very satisfactory and conjectures of Grothendieck and Kontsevich–Zagier yield a precise understanding of the ring of periods. Unfortunately, these conjectures seem so desperately out of reach of the present mathematics and – this is to be taken as my personal opinion – I can't think of any other conjecture that looks as intractable !

A typical question of interest is the following:

Question. *Let X be a smooth $\mathbb{Q}$-variety and let $\mathcal{P}er(X)$ be the subfield of $\mathbb{C}$ generated by the image of the pairing* (3)*. What is the transcendence degree of the finitely generated extension $\mathcal{P}er(X)/\mathbb{Q}$?*

Grothendieck's conjecture gives an answer to this question: *the transcendence degree of $\mathcal{P}er(X)/\mathbb{Q}$ is equal to the dimension of the motivic Galois group of X*. Of course, the motivic Galois group of X is quite a complicated object and its dimension can be very hard to compute. Nonetheless, it is easy to convey that the dimension of a not-so-explicit algebraic group is much easier to compute than the transcendence degree of an explicit subfield of $\mathbb{C}$. This is already the case for $X = \mathbf{P}^1$ (the projective line): the motivic Galois group is given by $\mathbf{G}_m$ whereas the subfield generated by periods is $\mathbb{Q}(\pi)$.

The conjecture of Kontsevich–Zagier is more ambitious and goes beyond the above question: it aims at describing all the algebraic relations among periods. Roughly speaking, it says that two periods are equal if and only if there is a *geometric reason* (see Definition 6 for the list of geometric reasons.)

Remark 4. The conjecture of Kontsevich–Zagier is stronger than the conjecture of Grothendieck. This is not obvious from their statements and will be explained in §4. However, one can argue that both conjectures are essentially, or morally, equivalent.

Remark 5. The conjecture of Kontsevich–Zagier is remarkable in its simplicity as it can be stated in elementary terms. However, in practice, the conjecture of Grothendieck is better suited for deducing algebraic independence of periods.

1.4 The geometric version

As it is the case with many deep and difficult problems on numbers, the conjectures of Grothendieck and Kontsevich–Zagier admit geometric (aka., functional) analogues that are accessible. Some ideas of the proof of the geometric versions will be discussed in §5.

2 The Kontsevich–Zagier conjecture

2.1 The ring of abstract periods

The Kontsevich–Zagier conjecture is best stated by introducing the ring of *abstract periods*. Roughly speaking, the ring of abstract periods is the free $\mathbb{Q}$-vector space generated by formal symbols, one for each pairing of a homology class with an algebraic de Rham cohomology class, modulo the relations that come from geometry. If the Kontsevich–Zagier conjecture was true, the ring of abstract periods would be identical to the subring of $\mathbb{C}$ generated by periods.

More precisely, consider 5-tuples $(X, Z, n, \gamma, \omega)$ where X is a $\mathbb{Q}$-variety (possibly singular), $Z \subset X$ a closed subvariety, $n \in \mathbb{N}$, $\gamma \in H_n^{\mathrm{Sing}}(X, Z)$ a relative homology class of the pair $(X(\mathbb{C}), Z(\mathbb{C}))$ and $\omega \in H_{\mathrm{AdR}}^n(X, Z)$ a relative algebraic de Rham cohomology class. To such a 5-tuple, one associates a period

$$\mathrm{Ev}(X, Z, n, \gamma, \omega) := \int_\gamma \omega \in \mathbb{C}. \tag{4}$$

Following Kontsevich–Zagier [12], we make the following:

Definition 6. The ring of *abstract (effective) periods*, denoted by $\mathcal{P}^{\mathrm{eff}}_{\mathrm{KZ}}$, is the free $\mathbb{Q}$-vector space generated by symbols $[X, Z, n, \gamma, \omega]$ modulo the following relations:

(a) (Additivity) the map $(\gamma, \omega) \mapsto [X, Z, n, \gamma, \omega]$ is bilinear on $H_n^{\mathrm{Sing}}(X, Z) \times H^n_{\mathrm{AdR}}(X, Z)$.

(b) (Base-change) Given a morphism of $\mathbb{Q}$-varieties $f : X' \to X$ such that $f(Z') \subset Z$, a relative singular homology class $\gamma' \in H_n^{\mathrm{Sing}}(X', Z')$ and a relative algebraic de Rham cohomology class $\omega \in H^n_{\mathrm{AdR}}(X, Z)$, we have the relation

$$[X, Z, n, f_*\gamma', \omega] = [X', Z', n, \gamma', f^*\omega].$$

(c) (Stokes formula) Given a $\mathbb{Q}$-variety X, closed subvarieties $Z \subset Y$ of X, a relative singular homology class $\gamma \in H_n^{\mathrm{Sing}}(X, Y)$ and a relative algebraic de Rham cohomology class $\omega \in H^{n-1}_{\mathrm{AdR}}(Y, Z)$, we have the relation

$$[X, Y, n, \gamma, \mathrm{d}\omega] = [Y, Z, n-1, \partial\gamma, \omega].$$

We denote by

$$\underline{2\pi i} := \left[\mathbf{G}_m, \emptyset, 1, t \in [0,1] \mapsto \exp(2\pi i \cdot t), \frac{\mathrm{d}t}{t}\right]$$

and set $\mathcal{P}_{\mathrm{KZ}} := \mathcal{P}^{\mathrm{eff}}_{\mathrm{KZ}}[\underline{2\pi i}^{-1}]$. This is the ring of *abstract periods*.

It is clear that the function Ev of (4) induces a morphism of $\mathbb{Q}$-algebras

$$\mathrm{Ev} : \mathcal{P}_{\mathrm{KZ}} \to \mathbb{C}. \tag{5}$$

(Note that $\mathrm{Ev}(\underline{2\pi i}) = 2\pi i$.) We can now state:

Conjecture 7 (Kontsevich–Zagier)**.** *The evaluation homomorphism* (5) *is injective.*

Remark 8. The above conjecture is widely open and desperately out of reach. However, as said in the introduction, the *analytic subgroup theorem* of Wüstholz [16] gives small (although highly non-trivial) evidence for this conjecture. Roughly speaking, Wüstholz result solves the Kontsevich–Zagier conjecture for abelian periods (aka., periods of level ≤ 1 in the jargon of Hodge theory).

2.2 A compact presentation of the ring of abstract periods

In this paragraph, we will give another presentation of the ring $\mathcal{P}_{\mathrm{KZ}}$ that was obtained rather accidentally by the author.

In retrospect, this presentation uses less generators than the presentation of Kontsevich–Zagier. With the notation of Definition 6, one restricts to:

- $X = \mathrm{Spec}(A)$ for A running among étale sub-$\mathbb{Q}[z_1, \ldots, z_n]$-algebras of the ring of convergent power series with radius strictly larger than 1.
- $Z \subset X$ is the normal crossing divisor given by the equation $\prod_{i=1}^n z_i(z_i - 1) = 0$.
- $\gamma : [0,1]^n \to X(\mathbb{C})$ the canonical lift of the obvious incluson $[0,1]^n \hookrightarrow \mathbb{C}^n$.
- $\omega = f \cdot \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$ with $f \in A$, a top degree differential form.

In return, one needs much less relations: a special case of the Stokes formula suffices to realize all the geometric relations described in Definition 6.

To write down precisely the above sketch, we introduce some notations. For an integer $n \in \mathbb{N}$, we denote by $\overline{\mathbb{D}}^n$ the closed unit polydisc in $\mathbb{C}^n$. Let $\mathcal{O}(\overline{\mathbb{D}}^n)$ be the ring of convergent power series in the system of n variables $(z_1, \ldots, z_n)$ with radius of convergence strictly larger than 1.

Definition 9. We denote by $\mathcal{O}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ the sub-$\mathbb{Q}$-vector space of $\mathcal{O}(\overline{\mathbb{D}}^n)$ consisting of those power series $f = f(z_1, \ldots, z_n)$ which are algebraic over the field $\mathbb{Q}(z_1, \ldots, z_n)$ of rational functions. We also set $\mathcal{O}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty) = \bigcup_{n \in \mathbb{N}} \mathcal{O}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$.

Definition 10. Let $\mathcal{P}^{\mathrm{eff}}$ be the quotient of $\mathcal{O}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ by the sub-$\mathbb{Q}$-vector space spanned by elements of the form

$$\frac{\partial f}{\partial z_i} - f|_{z_i=1} + f|_{z_i=0}$$

for $f \in \mathcal{O}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ and $i \in \mathbb{N} \setminus \{0\}$. We also define $\mathcal{P}$ to be $\mathcal{P}^{\mathrm{eff}}[\underline{2\pi i}^{-1}]$ with $\underline{2\pi i}$ the class of a well-chosen element of $\mathcal{O}_{\mathrm{alg}}(\overline{\mathbb{D}}^1)$ whose integral on $[0,1]$ is $2\pi i$.

Proposition 11. *There is an isomorphism*

$$\mathcal{P} \simeq \mathcal{P}_{\mathrm{KZ}}. \tag{6}$$

The image of $f \in \mathcal{O}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ can be described as follows. Let $A \subset \mathcal{O}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ be an étale $\mathbb{Q}[z_1, \ldots, z_n]$-algebra containing f. Let $Z \subset X = \mathrm{Spec}(A)$ be the divisor given by the equation $\prod_{i=1}^n z_i(z_i - 1) = 0$. Then the image of $[f]$ by (6) *is given by*

$$[X, Z, n, \tau_n, f \cdot \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n]$$

with τ_n the tautological relative homology class given by the composition of $[0,1]^n \hookrightarrow \overline{\mathbb{D}}^n \to X(\mathbb{C})$.

Remark 12. The actual proof of Proposition 11 is very indirect. It relies on the comparison of two constructions of motivic Galois groups: the one by M. Nori [13] and the one by the author [3]. The details of this comparison will appear in [10]. See Remark 34 for more details.

It would be interesting to find a direct proof of the Proposition 11 avoiding motives. It is certainly easy to construct the morphism $\mathcal{P} \to \mathcal{P}_{\mathrm{KZ}}$ that realizes the isomorphism (6). It is also easy to prove that this morphism is surjective. However, injectivity seems to be the interesting and difficult part.

Remark 13. There is an evaluation homomorphism

$$\mathrm{Ev} : \mathcal{P} \to \mathbb{C} \tag{7}$$

that takes the class of $f \in O_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ to $\int_{[0,1]^n} f$. This evaluation homomorphism coincides with (5) modulo the isomorphism (6). Therefore, we may restate the Kontsevich–Zagier conjecture in more elementary terms (i.e., without speaking of algebraic varieties and their cohomologies) as follows: *the evaluation homomorphism* (7) *is injective*.

2.3 Over more general base-fields

Fix a base field k and a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then most of the previous discussion extends to varieties over k.

Indeed, given a pair (X, Z) consisting of a k-variety X and closed subvariety $Z \subset X$, one can still define $H^{\mathrm{Sing}}_\bullet(X, Z)$ and $H^\bullet_{\mathrm{AdR}}(X, Z)$ and the canonical pairing $H^{\mathrm{Sing}}_\bullet(X, Z) \otimes H^\bullet_{\mathrm{AdR}}(X, Z) \to \mathbb{C}$.

Remark 14. In contrast with $H^{\mathrm{Sing}}_\bullet(X, Z)$ which is still a finite dimensional $\mathbb{Q}$-vector space independently of k and σ, $H^\bullet_{\mathrm{AdR}}(X, Z)$ is naturally a k-vector space. Also, note that the canonical pairing is now perfect in a slightly twisted manner: it induces an isomorphism $H^\bullet_{\mathrm{AdR}}(X, Z) \otimes_{k,\sigma} \mathbb{C} \simeq \hom(H^{\mathrm{Sing}}_\bullet(X, Z), \mathbb{C})$.

One can also define the ring of abstract periods over k, that we denote by $\mathcal{P}_{\mathrm{KZ}}(k, \sigma)$, together with an evaluation homomorphism $\mathrm{Ev} : \mathcal{P}_{\mathrm{KZ}}(k, \sigma) \to \mathbb{C}$. One can wonder to which extent the Kontsevich–Zagier conjecture is reasonable for general fields. We discuss this in the following:

Remark 15. When $k/\mathbb{Q}$ is algebraic (e.g., k is a number field), it is easy to see that $\mathcal{P}_{\mathrm{KZ}}(k, \sigma) = \mathcal{P}_{\mathrm{KZ}}(\mathbb{Q})$. This shows that the Kontsevich–Zagier conjecture holds for k if and only if it holds for $\mathbb{Q}$.

On the other hand, the Kontsevich–Zagier conjecture extended to fields of higher transcendence degrees is not reasonable unless σ is "general". Indeed, if $\sigma(k)$ contains a transcendental period of a $\mathbb{Q}$-variety (e.g., $\pi \in \sigma(k)$), then $\mathrm{Ev} : \mathcal{P}_{\mathrm{KZ}}(k, \sigma) \to \mathbb{C}$ cannot be injective. (However, see Remark 24 for what is expected without any condition on σ.)

3 Motives and the Grothendieck conjecture

It is not our aim to give an overview of the theory of motives. We will simply list some facts that are necessary for stating and appreciating the Grothendieck conjecture. (For the reader who wants to learn more about motives, we recommend [1].)

3.1 Abelian category of motives

Let k be a base field. According to Grothendieck, there should exist a $\mathbb{Q}$-linear abelian category $\mathbf{MM}(k)$ whose objects are called *mixed motives*. Given an embedding $\sigma : k \hookrightarrow \mathbb{C}$, one has a realization functor

$$R_\sigma : \mathbf{MM}(k) \to \mathbf{MHS}$$

to the category of ($\mathbb{Q}$-linear) mixed Hodge structure. (This functor is beleived to be fully faithful as a consequence of the Hodge conjecture, but this will be irrelevant for us.) Also, given an algebraic closure $\overline{k}/k$ and a prime number ℓ invertible in k, there is a realization functor

$$R_\ell : \mathbf{MM}(k) \to \mathbf{Rep}(\mathrm{Gal}(\overline{k}/k); \mathbb{Q}_\ell)$$

to the category of ℓ-adic Galois representations. (After tensoring the source category by $\mathbb{Q}_\ell$ and when k is finitely generated over its prime field, this functor is also believed to be fully faithful as a consequence of the Tate conjecture.)

Given a k-variety X, there are objets $H^i_{\mathcal{M}}(X)$ of $\mathbf{MM}(k)$, called the *motives* of X, that play the role of the universal cohomological invariants attached to X. Every classical cohomological invariant of X is then obtained from one of the $H^i_{\mathcal{M}}(X)$'s by applying a suitable realization functor. For instance,

- $R_\sigma(H^i_{\mathcal{M}}(X))$ is the singular cohomology group $H^i_{\mathrm{Sing}}(X)$, endowed with its mixed Hodge structure;
- $R_\ell(H^i_{\mathcal{M}}(X))$ is the ℓ-adic cohomology group $H^i_\ell(X)$ endowed with the natural action of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$.

Remark 16. When k has characteristic zero, M. Nori [13] has constructed a candidate for the category of mixed motives. While his construction is not known to satisfy all the expected properties (for instance, the ext-groups between Nori's motives are poorly related to Quillen K-groups), its enough for the purpose of the article.

3.2 The absolute motivic Galois group of a field

The category $\mathbf{MM}(k)$ is expected to share the formal properties of $\mathbf{MHS}$ and $\mathbf{Rep}(\mathrm{Gal}(\overline{k}/k); \mathbb{Q}_\ell)$. For instance, $\mathbf{MM}(k)$ has an exact tensor product $\otimes$ and every motive M has a strong dual $M^\vee$. Moreover, given an embedding $\sigma : k \hookrightarrow \mathbb{C}$, singular cohomology yields an exact faithful monoidal functor

$$F_{\mathrm{Sing}} : \mathbf{MM}(k) \to \mathbf{Mod}(\mathbb{Q})$$

sending $H^i_{\mathcal{M}}(X)$ to the $\mathbb{Q}$-vector space $H^i_{\mathrm{Sing}}(X)$. (In fact, F_{Sing} is just R_σ composed with the forgetful functor from **MHS** to $\mathbf{Mod}(\mathbb{Q})$.) This makes $\mathbf{MM}(k)$ into a *neutralized Tannakian category* with fiber functor F_{Sing}.

A *multiplicative operation* $\gamma = (\gamma_M)_M$ on F_{Sing} is a family of automorphisms $\gamma_M \in \mathbf{GL}(F_{\mathrm{Sing}}(M))$, one for each $M \in \mathbf{MM}(k)$, such that

- for every morphism of motives $a : M \to N$, one has $\gamma_N \circ F_{\mathrm{Sing}}(a) = F_{\mathrm{Sing}}(a) \circ \gamma_M$;
- for motives M and N, one has $\gamma_{M\otimes N} = \gamma_M \otimes \gamma_N$ modulo the identification $F_{\mathrm{Sing}}(M \otimes N) \simeq F_{\mathrm{Sing}}(M) \otimes F_{\mathrm{Sing}}(N)$.

Definition 17. The multiplicative operations of F_{Sing} are the $\mathbb{Q}$-rational points of a pro-$\mathbb{Q}$-algebraic group $\underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}})$ called the *motivic Galois group* (of k) and denoted by $\mathbf{G}_{mot}(k, \sigma)$. (Note that this depends on the choice of the complex embedding σ.)

By the Tannaka reconstruction theorem [14], the functor F_{Sing} induces an equivalence of categories

$$\widetilde{F}_{\mathrm{Sing}} : \mathbf{MM}(k) \xrightarrow{\sim} \mathbf{Rep}(\mathbf{G}_{mot}(k, \sigma))$$

between motives and algebraic representations of the motivic Galois group.

Remark 18. One may think about $\mathbf{G}_{mot}(k, \sigma)$ as a linearization of the absolute Galois group of k. For instance, there is a continuous morphism

$$\mathrm{Gal}(\bar{k}/k) \to \mathbf{G}_{mot}(k, \sigma)(\mathbb{Q}_\ell)$$

which induces the realization functor R_ℓ.

3.3 The motivic Galois group of a motive

In the previous subsection, we introduced the absolute motivic Galois group of a field k endowed with an embedding σ; this was the analogue of the absolute Galois group of a field endowed with a choice of an algebraic closure. In order to formulate the Grothendieck conjecture, we need the motivic Galois group of a motive; this is the analogue of the Galois group of a finite Galois extension.

Definition 19. Let $M \in \mathbf{MM}(k)$ be a mixed motive. The *motivic Galois group* of M, denoted by $\mathbf{G}(M)$, is the image of the morphism

$$\mathbf{G}_{mot}(k, \sigma) \to \mathbf{GL}(F_{\mathrm{Sing}}(M))$$

given by the natural action of $\mathbf{G}_{mot}(k, \sigma)$ on the $\mathbb{Q}$-vector space $F_{\mathrm{Sing}}(M)$, i.e., sending a multiplicative operation γ to γ_M.

Remark 20. By construction $\mathbf{G}(M)$ is an algebraic linear group. Moreover, $\mathbf{G}_{mot}(k, \sigma)$ is the inverse limit of the $\mathbf{G}(M)$'s when M runs over larger and larger motives.

3.4 Statement of Grothendieck conjecture

From now on, we assume that k has characteristic zero. Algebraic de Rham cohomology yields a functor

$$F_{\mathrm{AdR}} : \mathbf{MM}(k) \to \mathbf{Mod}(k)$$

sending $H^i_{\mathcal{M}}(X)$ to the k-vector space $H^i_{\mathrm{AdR}}(X)$. Fixing an embedding $\sigma : k \hookrightarrow \mathbb{C}$, the pairing (3) can be extended to any motive M yielding a pairing

$$F_{\mathrm{Sing}}(M)^\vee \otimes F_{\mathrm{AdR}}(M) \to \mathbb{C} \qquad (8)$$

(This is truly and extension of (3): for $M = H^i_{\mathcal{M}}(X)$, $F_{\mathrm{Sing}}(M)^\vee$ and $F_{\mathrm{AdR}}(M)$ are indeed canonically isomorphic to $H_i^{\mathrm{Sing}}(X)$ and $H^i_{\mathrm{AdR}}(X)$.)

Conjecture 21 (Grothendieck)**.** *Assume that $k = \mathbb{Q}$. Let M be a motive and let $\mathcal{P}er(M)$ be the subfield of $\mathbb{C}$ generated by the image of the pairing* (8)*. Then, one has the equality:*

$$\mathrm{degtr}(\mathcal{P}er(M)/\mathbb{Q}) = \dim(\mathbf{G}(M)).$$

Remark 22. If one is interested in the transcendence degree of the field $\mathcal{P}er(X)$ generated by the periods of a $\mathbb{Q}$-variety X, one should take

$$M = \bigoplus_{i=0}^{2\dim(X)} H^i_{\mathcal{M}}(X)$$

in the previous conjecture.

Remark 23. It is not difficult to show that

$$\mathrm{degtr}(\mathcal{P}er(M)/\mathbb{Q}) \leq \dim(\mathbf{G}(M)).$$

This is not very surprising: it is much harder to prove algebraic independence than constructing algebraic relations.

Remark 24. In [1, §23.4.1], Y. André proposes an extension of Grothendieck conjecture for base fields of non-zero transcendence degree. This extension states that the inequality

$$\mathrm{degtr}(\mathcal{P}er(M)/\mathbb{Q}) \geq \dim(\mathbf{G}(M))$$

holds for every $M \in \mathbf{MM}(k)$. By the previous remark, this is indeed an extension of Grothendieck conjecture. Note also that the above inequality is expected to be strict if the complex embedding σ is "general" (see Remark 15). Indeed, in this case, the equality

$$\mathrm{degtr}(\mathcal{P}er(M)/k) = \dim(\mathbf{G}(M)),$$

which can be restated as

$$\mathrm{degtr}(\mathcal{P}er(M)/\mathbb{Q}) = \dim(\mathbf{G}(M)) + \mathrm{degtr}(k/\mathbb{Q}),$$

is expected to hold.

Remark 25. Grothendieck conjecture is the basis for a (conjectural) Galois theory for periods. We refer the interested reader to [2].

3.5 Reformulation of Grothendieck conjecture

We reformulate Grothendieck conjecture in terms of the absolute motivic Galois group and the so-called torsor of periods. We start with the following basic fact from the general theory of Tannakian categories.

Proposition 26. *Let F be a field of characteristic zero and E/F an extension.*

Let $\mathcal{T}$ be an F-linear Tannakian category neutralized by a fiber functor $\omega : \mathcal{T} \to \mathbf{Mod}(F)$ and let $\delta : \mathcal{T} \to \mathbf{Mod}(E)$ be another fiber functor. Then, the multiplicative operations $\delta \xrightarrow{\sim} \omega \otimes_F E$ are the E-points of a pro-algebraic E-variety $\underline{\mathrm{Iso}}^{\otimes}(\delta, \omega)$ which is naturally a pro-E-torsor (on the right) over the pro-F-algebraic group $\underline{\mathrm{Aut}}^{\otimes}(\omega)$.

Let M be a motive and let $\langle M \rangle$ be the Tannakian subcategory of $\mathbf{MM}(k)$ generated by M; this is the smallest abelian subcategory of $\mathbf{MM}(k)$ closed under tensor products and duals and containing M. We have the following lemma.

Lemma 27. *$\underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M \rangle})$ identifies with $\mathbf{G}(M)$. Moreover, $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}|_{\langle M \rangle}, F_{\mathrm{Sing}}|_{\langle M \rangle})$ has a canonical complex valued point, denoted by* comp*, whose residue field is exactly the subfield $\mathcal{P}er(M) \subset \mathbb{C}$.*

Proof. The pairing (8) yields an isomorphism of $\mathbb{C}$-vector spaces $F_{\mathrm{AdR}}(M) \otimes_k \mathbb{C} \xrightarrow{\sim} F_{\mathrm{Sing}}(M) \otimes \mathbb{C}$. Replacing M by motives in $\langle M \rangle$ yields a multiplicative operation

$$\mathrm{comp} : F_{\mathrm{AdR}}|_{\langle M \rangle} \otimes_k \mathbb{C} \xrightarrow{\sim} F_{\mathrm{Sing}}|_{\langle M \rangle} \otimes \mathbb{C}$$

and hence a complex-valued point

$$\mathrm{comp} \in \underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}|_{\langle M \rangle}, F_{\mathrm{Sing}}|_{\langle M \rangle})(\mathbb{C}).$$

Formal manipulations show that the residue field of this point is generated by the image of the pairing (8). □

Corollary 28. *Grothendieck conjecture is equivalent to the following statement.* Let $M \in \mathbf{MM}(\mathbb{Q})$ be a motive over $\mathbb{Q}$. Then, comp is a generic point of the $\mathbb{Q}$-variety $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}|_{\langle M \rangle}, F_{\mathrm{Sing}}|_{\langle M \rangle})$.

Proof. If $\xi \in W(\mathbb{C})$ is a complex point of an equidimensional $\mathbb{Q}$-variety W, the following conditions are equivalent:

- ξ is a generic point;
- $\dim(W) = \mathrm{degtr}(\mathbb{Q}(\xi))$.

Now, the $\mathbb{Q}$-variety $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}|_{\langle M \rangle}, F_{\mathrm{Sing}}|_{\langle M \rangle})$ is a torsor over $\mathbf{G}(M) = \underline{\mathrm{Aut}}^{\otimes}(R_{\mathrm{Sing}}|_{\langle M \rangle})$. Hence, it is equidimensional and

$$\dim(\underline{\mathrm{Iso}}^{\otimes}(R_{\mathrm{AdR}}|_{\langle M \rangle}, R_{\mathrm{Sing}}|_{\langle M \rangle})) = \dim(\mathbf{G}(M)).$$

This proves the claim as $\mathbb{Q}(\mathrm{comp}) = \mathcal{P}er(M)$. □

Passing to the limit, we get a complex point comp of the pro-k-variety $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})$. We also obtain the the following:

Proposition 29. *Grothendieck conjecture is equivalent to the following statement.* If $k = \mathbb{Q}$, then comp is a generic point of $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})$.

Definition 30. $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})$ is called the *torsor of periods*.

4 The relation between the two conjectures

Here we explain why the Grothendieck conjecture is only slightly weaker than the Kontsevich–Zagier conjecture.

In fact, one has the following theorem (due to Kontsevich and proven in detail in [11]).

Theorem 31. *There is a canonical isomorphism of k-algebras*

$$\mathcal{O}(\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})) \simeq \mathcal{P}_{\mathrm{KZ}}(k, \sigma).$$

Moreover, modulo this isomorphism, the evaluation homomorphism (5) *corresponds to evaluating a regular function on $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})$ at the complex point* comp*:*

$$\begin{array}{ccc} \mathcal{O}(\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})) & \to & \mathbb{C} \\ f & \mapsto & f(\mathrm{comp}). \end{array}$$

Corollary 32. *The following assertions are equivalent.*

(a) The Kontsevich–Zagier conjecture holds.

(b) The Grothendieck conjecture holds and the ring $\mathcal{P}_{\mathrm{KZ}}$ is an integral domain.

Proof. Indeed, by the previous theorem, the injectivity of the evaluation homomorphism (5) is equivalent to the fact that the complex-valued point comp is generic and that $\mathcal{P}_{\mathrm{KZ}}$ is an integral domain. We conclude using Proposition 29. □

Remark 33. Some authors, for instance Y. André in [2], refer to the Grothendieck conjecture as the combination of the statement in Conjecture 21 and the property that $\mathcal{P}_{\mathrm{KZ}}$ is an integral domain.

Remark 34. We take the opportunity to give some hints concerning the compact presentation of the ring of abstract periods.

It is possible to construct a motivic Galois group and a torsor of periods starting from Voevodsky's triangulated category of motives. This is the approach pursued in [3, 4]. Using some flexibility pertaining to the theory of motives à la Voevodsky one is able to "compute" more efficiently the ring of regular functions on the torsor of periods arriving eventually at the ring $\mathcal{P}$ of Definition 10.

Now, it turns out that both approaches yield isomorphic motivic Galois groups (see [10]). As the canonical map between $\mathrm{Spec}(\mathcal{P})$ and $\mathrm{Spec}(\mathcal{P}_{\mathrm{KZ}})$ is equivariant and the latters are torsors over isomorphic pro-$\mathbb{Q}$-algebraic groups, this gives Proposition 11.

5 The geometric version of the Grothendieck and the Kontsevich–Zagier conjectures

We now turn to the geometric version of the conjectures of Grothendieck and Kontsevich–Zagier for which a proof is available.

5.1 The relative motivic Galois group

Definition 35. Let k be a field. Given an extension K/k and an embedding $\sigma : K \hookrightarrow \mathbb{C}$, one has an induced morphism of motivic Galois groups

$$\mathbf{G}_{mot}(K,\sigma) \to \mathbf{G}_{mot}(k,\sigma).$$

The *relative motivic Galois group* $\mathbf{G}_{rel}(K/k,\sigma)$ is the kernel of this morphism.

Proposition 36. *Let $l \subset K$ be the algebraic closure of k in K. One has an exact sequence (of groups and sets)*

$$\{1\} \to \mathbf{G}_{rel}(K/k,\sigma) \to \mathbf{G}_{mot}(K,\sigma) \to \mathbf{G}_{mot}(k,\sigma) \to \hom_k(l,\mathbb{C}) \to \star.$$

In particular, if k is algebraically closed in K, then $\mathbf{G}_{mot}(K,\sigma) \to \mathbf{G}_{mot}(k,\sigma)$ is surjective.

Proof. It is shown in [4, Théorème 2.34] that the morphism

$$\mathbf{G}_{mot}(K,\sigma) \to \mathbf{G}_{mot}(l,\sigma)$$

is surjective. Therefore, it remains to show that $\mathbf{G}_{mot}(l,\sigma)$ identifies with the stabilizer in $\mathbf{G}_{mot}(k,\sigma)$ of the point in $\sigma|_l \in \hom_k(l,\mathbb{C})$. This follows easily from the exact sequence

$$\{1\} \to \mathbf{G}_{mot}(\bar{k},\sigma) \to \mathbf{G}_{mot}(k,\sigma) \to \mathrm{Gal}(\bar{k}/k) \to \{1\}$$

which is a consequence of [4, Corollaire 2.31]. □

We also note the following easy consequence of [4, Théorème 2.34].

Proposition 37. *Assume that k is algebraically closed. Then, the exact sequence*

$$\{1\} \to \mathbf{G}_{rel}(K/k) \to \mathbf{G}_{mot}(K) \to \mathbf{G}_{mot}(k) \to \{1\}$$

splits (non canonically). In particular, there is an isomorphism

$$\mathbf{G}_{mot}(K,\sigma) \simeq \mathbf{G}_{mot}(k,\sigma) \ltimes \mathbf{G}_{rel}(K/k,\sigma).$$

An important fact about the relative motivic Galois group is that it is "controlled" by a group of topological origin. In order to explain this, we need some notation.

Definition 38. Assume that k is algebraically closed in K and denote by $\mathrm{Mod}(K/k)$ the pro-k-variety of smooth models of K. More precisely, the objects of the indexing category of $\mathrm{Mod}(K/k)$ are pairs (X,i) where X is a smooth k-variety and $i : k(X) \simeq K$ an isomorphism. The pro-objet $\mathrm{Mod}(K/k)$ is the functor $(X,i) \mapsto X$.

Remark 39. Consider the pro-manifold

$$(K/k)^{an} := \mathrm{Mod}(K/k)(\mathbb{C})$$

obtained by taking $\mathbb{C}$-points of each k-variety appearing in $\mathrm{Mod}(K/k)$. The complex embedding σ makes $(K/k)^{an}$ into a pointed pro-manifold and we may consider the associated pro-system of fundamental groups $\pi_1((K/k)^{an},\sigma)$. This is a pro-discrete group.

We can now state the following crucial fact. This theorem was obtained independently by M. Nori (unpublished) and the author [4, Théorème 2.57].

Theorem 40. *There is a canonical morphism*

$$\pi_1((K/k)^{an},\sigma) \to \mathbf{G}_{rel}(K/k,\sigma)$$

with Zariski dense image.

Remark 41. Let X be a geometrically irreducible algebraic k-variety and $M \in \mathbf{MM}(X)$ a *motivic local system*. (On can think about M as an object of $\mathbf{MM}(k(X))$ which is unramified over X.) Given the complex embedding σ, M realizes to a topological local system on $X(\mathbb{C})$. If this local system is trivial, then M is the pull-back of a motive $M_0 \in \mathbf{MM}(k)$; such a motive is called *constant* (relative to k). This is a direct consequence of Theorem 40.

For later use, we give a reformulation of Theorem 40.

Proposition 42. *Assume that k is algebraically closed in K. Let $M \in \mathbf{MM}(K)$ and denote by*

$$\langle M\rangle_0 \subset \langle M\rangle$$

the largest Tannakian subcategory consisting of constant motives (i.e., in the image of the pull-back $\mathbf{MM}(k) \to \mathbf{MM}(K)$). Then, there is an exact sequence

$$\pi_1^{\mathrm{alg}}((K/k)^{an},\sigma) \to \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M\rangle}) \to \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M\rangle_0}) \to \{1\}$$

where $\pi_1^{\mathrm{alg}}((K/k)^{an},\sigma)$ is the pro-algebraic completion of $\pi_1((K/k)^{an},\sigma)$.

Proof. There is a commutative diagram

$$\begin{array}{ccccc} \pi_1^{\mathrm{alg}}((K/k)^{an}) & \longrightarrow & \mathbf{G}_{mot}(K,\sigma) & \twoheadrightarrow & \mathbf{G}_{mot}(k,\sigma) \\ \| & & \downarrow & & \downarrow \\ \pi_1^{\mathrm{alg}}((K/k)^{an}) & \to & \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M\rangle}) & \twoheadrightarrow & \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M\rangle_0}) \end{array}$$

The image of $\pi_1^{\mathrm{alg}}((K/k)^{an},\sigma)$ in $\underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M\rangle})$ is a normal subgroup $\mathbf{N}$ and an algebraic representation of $\underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M\rangle})/\mathbf{N}$ corresponds to a motive in $\langle M\rangle$ whose associated local system is trivial. By Theorem 40 (and Remark 41), this motive belongs to $\langle M\rangle_0$. □

Definition 43. Let $M \in \mathbf{MM}(K)$ be a motive. The kernel of the morphism

$$\underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M\rangle}) \to \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M\rangle_0})$$

is denoted by $\mathbf{G}_{rel}(M)$. This is the *relative motivic Galois group* of M. By construction, $\mathbf{G}_{rel}(M)$ is a closed subgroup of $\mathbf{G}(M)$ (cf., Lemma 27).

5.2 Relative motivic Galois groups and functional transcendance

Keep the situation as above. Given a motive M in $\mathbf{MM}(K)$, $F_{\mathrm{AdR}}(M)$ is naturally a holonomic $\mathcal{D}_{K/k}$-module with regular singularities. (If $M = H^i_{\mathcal{M}}(X)$, the $\mathcal{D}_{K/k}$-module is associated to the Gauss–Manin connexion on $H^i_{\mathrm{AdR}}(X)$.)

Theorem 44. *The Picard–Vessiot extension of K associated to the differential module $F_{\mathrm{AdR}}(M)$ has transcendance degree equal to the dimension of the algebraic group $\mathbf{G}_{rel}(M)$.*

Proof. Indeed, by differential Galois theory, the transcendence degree of the Picard–Vessiot extension associated to $F_{\mathrm{AdR}}(M)$ is equal to the dimension of its differential Galois group. By the Riemann–Hilbert correspondence, the latter group has the same dimension as the monodromy group of the local system associated to M. This monodromy group is by definition the Zariski closure of the image of

$$\pi_1((K/k)^{an},\sigma) \to \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M\rangle}) = \mathbf{G}(M)$$

which, by Proposition 42, is equal to $\mathbf{G}_{rel}(M)$. □

Remark 45. Theorem 44 is clearly a geometric analogue of the Grothendieck conjecture. Although it is a direct corollary of Theorem 40, its precise statement was obtained during an email exchange with Daniel Bertrand (and hence, did not appear before in the literature). It was also independently obtained by Peter Jossen and was probably known to Madhav Nori.

5.3 Geometric version of Kontsevich–Zagier

As for the Grothendieck conjecture, one can use Theorem 40 to obtain a geometric version of the Kontsevich–Zagier conjecture. Moreover, working in the realm of Voevodsky motives, one can give a very concrete statement in the style of the reformulation given in Remark 13. This was achieved in [6] and relies on previous work of the author (such as the theory of rigid analytic motives [5] and the construction of nearby motives [7, Chapitre 3]). We will not discuss the technical details here and we content ourself with stating the main result of [6]. We start by introducing some notation (compare with §2.2). Recall that $\overline{\mathbb{D}}^n$ denotes the closed unit polydisc in $\mathbb{C}^n$.

Definition 46. Let $\mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ be the sub-$\mathbb{C}$-vector space of $\mathcal{O}(\overline{\mathbb{D}}^n)[[\varpi]][\varpi^{-1}]$ consisting of those Laurent series

$$F = \sum_{i > -\infty} f_i(z_1,\dots,z_n)\cdot\varpi^i,$$

with coefficients in $\mathcal{O}(\overline{\mathbb{D}}^n)$, which are algebraic over the field $\mathbb{C}(\varpi, z_1,\dots,z_n)$. We also set $\mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty) = \bigcup_{n\in\mathbb{N}} \mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$.

Definition 47. Let $\mathcal{P}^{\dagger}$ be the quotient of $\mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ by the $\mathbb{C}$-vector space spanned by

- elements of the *first kind*:

  $$\frac{\partial F}{\partial z_i} - F|_{z_i=1} + F|_{z_i=0}$$

  for $F \in \mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ and $i \in \mathbb{N} \smallsetminus \{0\}$;
- and elements of the *second kind*:

  $$\left(g - \int_{[0,1]^\infty} g\right)\cdot F$$

  for $g,\ F \in \mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ such that g does not dependent on the variable ϖ (i.e., $\dfrac{\partial g}{\partial\varpi} = 0$) and

g and F do not depend simultaneously on the variable z_i (i.e., $\frac{\partial g}{\partial z_i} \cdot \frac{\partial F}{\partial z_i} = 0$) for every $i \in \mathbb{N} \smallsetminus \{0\}$.

There is an evaluation homomorphism

$$\mathrm{Ev} : \mathcal{P}^{\dagger} \to \mathbb{C}(\!(\varpi)\!) \tag{9}$$

sending the class of $F = \sum_{i > -\infty} f_i \cdot \varpi^i \in \mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ to

$$\sum_{i > -\infty} \left(\int_{[0,1]^n} f_i \right) \cdot \varpi^i.$$

The Laurent series belonging to the image of (9) are called *series of periods*. The main theorem of [6] is:

Theorem 48. *The evaluation homomorphism* (9) *is injective.*

Remark 49. An important difference between the original Kontsevich–Zagier conjecture and its geometric version is the presence, in the geometric case, of new obvious relations corresponding to elements of the second kind in Definition 47.

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