4. Mixed Tate motives

The goal of this chapter is to give a precise meaning to the statement that the diagram D_U^H of Definition 3.220 has motivic origin. The theory of motives has been a very active area of research in the last decades. This is a rather abstract theory and it is remarkable that, up until today, the only proof that we have for the upper bound of the dimension of the space of multiple zeta values of a given weight uses the theory of motives. A proper treatment of the theory of motives falls outside the scope of this notes. We will use the theory of motives as a black-box and we will limit ourselves to give an idea of its origin and the properties that we will use. The interested reader is referred to the book [**And04**] and the references therein.

4.1. Tannakian formalism. The link between mixed Tate motives and multiple zeta values is made through the group of symmetries of mixed Tate motives. To make this idea precise we need the formalism of Tannakian categories that we summarize in this section.

The Galois group of a field extension is one of the basic tools in arithmetic and one of its more studied objects. In topology, the fundamental group of a topological space is the analogueue of the absolute Galois group of a field and is one of the basic invariants of a topological space. Fueled by the utility of the Galois and fundamental groups it is natural to seek for analogues in other situations. The Tannakian formalism is the basic tool to define analogueues of the Galois group in many algebro-geometric situations. The origin of this formalism is Pontryagin duality, according to which a locally compact abelian group is characterized by its character group, and the Tannaka-Krein duality that states that we can recover a compact Lie group from the category of its continuous finite-dimensional real representations. Grothendieck extended the Tannaka-Krein duality to affine algebraic groups. Saavedra-Rivano [**SR72**] encoded the properties of the category of linear representations of an algebraic group in the concept of a Tannakian category. Conversely, every Tannakian category is isomorphic to the category of linear representations of an algebraic group.

Note that the formalism of Tannakian categories is tailored to the study of affine group schemes. Thus we will not recover the "true" fundamental group of a topological space nor the Galois group of a field extension with this formalism, but only its so called pro-algebraic envelope.

We will follow the exposition in [**DM82**] to which the reader is referred for further details. Another nice reference is Chapter 6 of [**Sza09**]. Through this section we fix a field k (of any characteristic), that will play the role of field of coefficients.

4.1.1. *Tensor categories.* The definition of Tannakian category gathers together the properties of finite-dimensional representations of affine group schemes. In what follows, we shall identify them. First of all, since morphisms between k-linear representations form a vector space, we need the concept of a k-linear category.

Definition 4.1. A *k-linear category* $\mathcal{C}$ is an additive category such that, for each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the group $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a k-vector space and the composition maps are bilinear.

The tensor product of two representations is again a representation. Therefore, our category should have a tensor product, which is a bilinear functor with some additional properties.

Definition 4.2. Let $\mathcal{C}$ be a k-linear category, together with a bilinear functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

(a) An *associativity constraint* for $(\mathcal{C}, \otimes)$ is a natural transformation

$$\phi = \phi_{\cdot,\cdot,\cdot}\colon \; \cdot \otimes (\cdot \otimes \cdot) \longrightarrow (\cdot \otimes \cdot) \otimes \cdot$$

such that the following two conditions hold:

(1) For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the map $\phi_{X,Y,Z}$ is an isomorphism.

(2) (*Pentagon axiom*) For all $X, Y, Z, T \in \mathrm{Ob}(\mathcal{C})$, the following diagram commutes:

$$\begin{tikzcd}[column sep=small]
 & & X \otimes (Y \otimes (Z \otimes T)) \arrow[dll, "\mathrm{Id} \otimes \phi_{Y,Z,T}"'] \arrow[drr, "\phi_{X,Y,Z\otimes T}"] & & \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[dr, "\phi_{X,Y\otimes Z,T}"'] & & & & (X \otimes Y) \otimes (Z \otimes T). \arrow[dl, "\phi_{X\otimes Y,Z,T}"] \\
 & (X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\phi_{X,Y,Z} \otimes \mathrm{Id}"'] & & ((X \otimes Y) \otimes Z) \otimes T &
\end{tikzcd}$$

(b) A *commutativity constraint* is a natural transformation

$$\psi = \psi_{\cdot,*}\colon \; \cdot \otimes * \longrightarrow * \otimes \cdot$$

such that, for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, the map $\psi_{X,Y}$ is an isomorphism, and the following composition is the identity:

$$\psi_{Y,X} \circ \psi_{X,Y} \colon X \otimes Y \longrightarrow X \otimes Y.$$

(c) An associativity and a commutativity constrain are said to be *compatible* if, for all objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the following diagram commutes (*hexagon axiom*):

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[r, "\phi_{X,Y,Z}"] \arrow[dl, "\operatorname{Id} \otimes \psi_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[dr, "\psi_{X \otimes Y, Z}"] & \\
X \otimes (Z \otimes Y) \arrow[dr, "\phi_{X,Z,Y}"'] & & & Z \otimes (X \otimes Y). \arrow[dl, "\phi_{Z,X,Y}"] \\
 & (X \otimes Z) \otimes Y \arrow[r, "\psi_{X,Z} \otimes \operatorname{Id}"'] & (Z \otimes X) \otimes Y &
\end{tikzcd}$$

(d) Finally, we say that a pair (U, u) consisting of an object U of $\mathcal{C}$ and an isomorphism $u \colon U \to U \otimes U$ is an *identity object* if the functor $X \mapsto U \otimes X$ is an equivalence of categories.

We now have all the ingredients to define one of the underlying structures of Tannakian categories.

Definition 4.3. A *k-linear tensor category* is a tuple $(\mathcal{C}, \otimes, \phi, \psi)$ consisting of a k-linear category $\mathcal{C}$, a bilinear functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and compatible associativity and commutativity constraints ϕ and ψ, such that $\mathcal{C}$ contains an identity object.

The constraints ϕ and ψ are usually omitted from the notation and one simply denotes a k-linear tensor category by $(\mathcal{C}, \otimes)$.

Remark 4.4. Two identity objects are canonically isomorphic. From now on, we will fix one and denote by $(\mathbf{1}, e)$.

Definition 4.5. An object L in $\mathcal{C}$ is called *invertible* if the functor $X \mapsto L \otimes X$ is an equivalence of categories.

One easily shows that an object L is invertible if and only if there exists an object L' such that $L \otimes L' \cong \mathbf{1}$. Then L' is also invertible.

4.1.2. *Rigid categories.* The set of k-linear maps between two representations is again a representation and, in particular, a representation on a vector space induces a representation on the dual vector space. Thus our candidate to be the category of representations of a group should contain internal Hom's and duals.

Let $(\mathcal{C}, \otimes)$ be a tensor category and let $X, Y \in \operatorname{Ob}(\mathcal{C})$. We say that the functor $T \mapsto \operatorname{Hom}(T \otimes X, Y)$ is *representable* if there exist an object $Z \in \operatorname{Ob}(\mathcal{C})$ such that there are functorial isomorphisms

$$\operatorname{Hom}(T, Z) \longrightarrow \operatorname{Hom}(T \otimes X, Y) \tag{4.6}$$

for all $T \in \operatorname{Ob}(\mathcal{C})$. If this is the case, we denote Z by $\underline{\operatorname{Hom}}(X, Y)$ and we call it *internal Hom* between the objects X and Y. Note that any two such objects Z are related by a unique compatible isomorphism.

Taking $T = \underline{\mathrm{Hom}}(X,Y)$ in (4.6), the image of the identity $\mathrm{Id}_{\underline{\mathrm{Hom}}(X,Y)}$ is a morphism which will be denoted by

$$\mathrm{ev}_{X,Y}\colon \underline{\mathrm{Hom}}(X,Y)\otimes X \to Y.$$

The *dual* of an object X is defined as $X^\vee = \underline{\mathrm{Hom}}(X,\mathbf{1})$. If $X^\vee$ and $(X^\vee)^\vee$ exist, there is a natural morphism $X \mapsto (X^\vee)^\vee$. We say that X is *reflexive* if this morphism is an isomorphism.

Example 4.7. In the category of groups, $\mathbb{Z}/2$ is not reflexive since its dual is 0. In the category of vector spaces, finite dimensional vector spaces are reflexive, whereas infinite dimensional ones are not.

Definition 4.8. A k-linear tensor category is said to be *rigid* if

(1) $\underline{\mathrm{Hom}}(X,Y)$ exists for all $X,Y\in \mathrm{Ob}(\mathcal{C})$;
(2) for all $X_1,X_2,Y_1,Y_2\in\mathrm{Ob}(\mathcal{C})$, the natural morphism
$$\underline{\mathrm{Hom}}(X_1,Y_1)\otimes\underline{\mathrm{Hom}}(X_2,Y_2)\to\underline{\mathrm{Hom}}(X_1\otimes X_2,Y_1\otimes Y_2)$$
is an isomorphism;
(3) all objects of $\mathcal{C}$ are reflexive.

4.1.3. *Neutral Tannakian categories.* The category of finite-dimensional k-linear representations $\mathbf{Rep}_k(G)$ of an algebraic group G over k has other relevant properties. First, it is an abelian category. Second, the one-dimensional representation given by the vector space k with trivial G-action is an identity object $\mathbf{1}$ that satisfies $\mathrm{End}(\mathbf{1}) = k$. Finally, the forgetful functor from $\mathbf{Rep}_k(G)$ to the category of finite-dimensional vector spaces $\mathbf{Vec}_k$ that consists in forgetting the action of G is exact, faithful and compatible with the tensor structure on both categories. These will turn out to be all the necessary ingredients to identify the categories of finite-dimensional representations of algebraic groups.

Definition 4.9. A *neutral Tannakian category* over k is a rigid k-linear abelian tensor category $\mathcal{C}$ such that $\mathrm{End}(\mathbf{1}) = k$ and that there exists an exact faithful k-linear tensor functor $\omega\colon \mathcal{C}\to\mathbf{Vec}_k$. Any such functor is called a *fibre functor.*

Since we shall never consider non-neutral Tannakian categories in the sequel, we will just refer to them as "Tannakian categories".

Examples 4.10.

(1) The category $\mathbf{Vec}_k$ of finite-dimensional vector spaces over k, together with the identity functor, is a Tannakian category.
(2) Let $\mathbf{GrVec}_k$ be the category of graded vector spaces. The objects are finite-dimensional k-vector spaces V together with a direct sum decomposition $V=\bigoplus_{n\in\mathbb{Z}}V_n$, and the morphisms are homogeneous

k-linear maps. The tensor structure comes from the tensor product of vector spaces, graded by

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j.$$

The forgetful functor $\omega\colon \mathbf{GrVec}_k \to \mathbf{Vec}_k$ sending $(V, (V_n)_{n\in\mathbb{Z}})$ to V makes $\mathbf{GrVec}_k$ into a Tannakian category.

(3) Let G be any abstract group and $\mathbf{Rep}_k(G)$ the category of finite-dimensional k-linear representations of G. Let

$$\omega\colon\ \mathbf{Rep}_k(G) \to \mathbf{Vec}_k$$

be the functor that forgets the action of G. Then $\mathbf{Rep}_k(G)$ is a Tannakian category over k and ω is a fibre functor.

(4) Let $\mathbf{MHS}(\mathbb{Q})$ be the category of mixed Hodge structures over $\mathbb{Q}$ and let ω_{B} and ω_{dR} the forgetful functors of Definition 2.93. Then $\mathbf{MHS}(\mathbb{Q})$ is a Tannakian category over $\mathbb{Q}$ and both of the functors, ω_{B} and ω_{dR} are fibre functors.

(5) Let X be a path connected, locally path connected and locally simply connected topological space. The category $\mathbf{Loc}_k(X)$ of local systems (*i.e.* locally constant sheaves) of finite-dimensional k-vector spaces is a Tannakian category. For each point $x \in X$, the functor

$$\begin{array}{rccc} \omega_x\colon & \mathbf{Loc}_k(X) & \longrightarrow & \mathbf{Vec}_k \\ & V & \longmapsto & V_x \end{array}$$

that sends a local system V to its fibre at x is a fibre functor.

4.1.4. *The fundamental group of a Tannakian category.* Fix a Tannakian category $\mathcal{C}$ over k and a fibre functor ω.

Definition 4.11. For every k-algebra R, let $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ denote the set of families $(\lambda_X)_{X\in\mathrm{Ob}(\mathcal{C})}$ of R-linear automorphisms

$$\lambda_X\colon \omega(X)\otimes R \longrightarrow \omega(X)\otimes R$$

such that the following diagrams are commutative:

(1)

$$\begin{tikzcd} \omega(X_1\otimes X_2)\otimes R \arrow[r, "\lambda_{X_1\otimes X_2}"] \arrow[d] & \omega(X_1\otimes X_2)\otimes R \arrow[d] \\ \omega(X_1)\otimes\omega(X_2)\otimes R \arrow[d] & \omega(X_1)\otimes\omega(X_2)\otimes R \arrow[d] \\ (\omega(X_1)\otimes R)\otimes_R(\omega(X_2)\otimes R) \arrow[r, "\lambda_{X_1}\otimes_R\lambda_{X_2}"] & (\omega(X_1)\otimes R)\otimes_R(\omega(X_2)\otimes R), \end{tikzcd}$$

(2)

$$\begin{tikzcd} \omega(\mathbf{1})\otimes R \arrow[r, "\lambda_{\mathbf{1}}"] \arrow[d] & \omega(\mathbf{1})\otimes R \arrow[d] \\ R \arrow[r, "\mathrm{Id}"] & R \end{tikzcd}$$

(3) for every morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, the diagram

$$\begin{tikzcd} \omega(X)\otimes R \arrow[r, "\lambda_X"] \arrow[d, "\omega(\alpha)\otimes\mathrm{Id}"'] & \omega(X)\otimes R \arrow[d, "\omega(\alpha)\otimes\mathrm{Id}"] \\ \omega(Y)\otimes R \arrow[r, "\lambda_Y"] & \omega(Y)\otimes R \end{tikzcd}$$

In the above diagrams, all unlabeled tensor products of vector spaces are over k and the unlabeled arrows are the obvious isomorphisms.

In particular, we will define $\mathrm{Aut}^{\otimes}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega)(k)$. This is the group of k-linear automorphisms of the functor ω.

The main theorem of the theory of Tannakian categories is

THEOREM 4.12. [DM82, Theorem 2.11] *Let $\mathcal{C}$ be a Tannakian category, together with a fibre functor ω. Then*

(1) the functor $R \mapsto \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ is representable by an affine group scheme G;

(2) for every $X \in \mathrm{Ob}(\mathcal{C})$, the group G acts naturally on $\omega(X)$ and the functor $\mathcal{C} \mapsto \mathbf{Rep}_k(G)$ sending X to the vector space $\omega(X)$ with this action of G is an equivalence of categories.

DEFINITION 4.13. The affine group scheme $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is called the *Tannaka group* of $(\mathcal{C},\omega)$. Whenever we want to stress the category we are considering, we will write $\underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega)$.

Given a second fibre functor ω', the functor from k-algebras to sets

$$R \longmapsto \underline{\mathrm{Isom}}^{\otimes}(\omega,\omega')$$

is representable by an affine scheme which is a torsor under $\underline{\mathrm{Aut}}^{\otimes}(\omega)$, see [DM82, Theorem 3.2].

4.1.5. *Matrix coefficients.* Instead of proving Theorem 4.12, we will content ourselves with a description of the Hopf algebra of the Tannaka group using the notion of matrix coefficients from[8] [Del90, §4.7].

DEFINITION 4.14. Let $(\mathcal{C},\omega)$ be a neutral Tannakian category over k, together with a fibre functor. A *matrix coefficient* in $(\mathcal{C},\omega)$ is the data

$$(X, v, f)$$

of an object X of $\mathcal{C}$, and elements $v \in \omega(X)$ and $f \in \omega(X)^{\vee} = \mathrm{Hom}(\omega(X), k)$.

[8] See also [Bro17] and compare with the notion of framed objects from [BGSV90].

Let H be the k-vector space generated by all matrix coefficients, and $V \subseteq H$ the subspace spanned by

(1) (*bilinearity relations*) for every pair of matrix coefficients (X, v_1, f) and (X, v_2, f), and elements $\lambda, \mu \in k$, the relation
$$(X, \lambda v_1 + \mu v_2, f) - \lambda(X, v_1, f) - \mu(X, v_2, f) \in V.$$
Similarly, for every pair of matrix coefficients of the form (X, v, f_1) and (X, v, f_2), and elements $\lambda, \mu \in k$, the relation
$$(X, v, \lambda f_1 + \mu f_2) - \lambda(X, v, f_1) - \mu(X, v, f_2) \in V;$$

(2) (*compatibility relations*) for every pair of objects X, X', every morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, X')$, and $v \in \omega(X)$ and elements $f' \in \omega(X')^\vee$, the relation
$$(X, v, \omega(\phi)^\vee f') - (X', \omega(\phi)v, f') \in V.$$

We set $A = H/V$ and write $[X, v, f]$ for the class in A of a matrix coefficient (X, v, f). The vector space A comes with the following structures:

(1) *Product:* The tensor structure of $\mathcal{C}$ induces the product
$$[X, v, f] \cdot [X', v', f'] = [X \otimes X', v \otimes v', f \otimes f'].$$
The associativity and commutativity constraints together with the compatibility relation imply that this product is associative and commutative.

(2) *Unit:* Let $\mathbf{1}$ be an identity object. Then $\omega(\mathbf{1}) \simeq k$. Choose any $v \in \omega(\mathbf{1}) \setminus \{0\}$ and let $f \in \omega(\mathbf{1})^\vee$ be its dual, so that $f^*(v) = 1$. Then $[\mathbf{1}, v, f]$ is a unit for the product. By the bilinearity relations, this class does not depend on the choice of v.

(3) *Counit:* The counit is the map $A \to k$ given by $[X, v, f] \mapsto f(v)$.

(4) *Coproduct:* The coproduct is modeled on the Hopf algebra of GL_n (see Example 3.50). Given an object $X \in \mathcal{C}$, we choose a basis $(e_1, \dots, e_n)$ of $\omega(X)$. If $(e_1^*, \dots, e_n^*)$ is the dual basis, then
$$\Delta[X, v, f] = \sum_{j=1}^{n} [X, v, e_j^*] \otimes [X, e_j, f]. \tag{4.15}$$
One checks that (4.15) does not depend on the choice of the basis.

(5) *Antipode:* Finally, the rigidity of $\mathcal{C}$ allows us to define an antipode. If we identify $\omega(X^\vee)$ with $\omega(X)^\vee$, then
$$S([X, v, f]) = [X^\vee, f, v].$$

It is an easy verification to prove the following:

Proposition 4.16. *Together with the above structures, A is a commutative Hopf k-algebra.*

Taking Theorem 4.12 for granted, we can show that A is the Hopf algebra of the Tannaka group $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$. More precisely,

PROPOSITION 4.17. *The map $\varphi\colon A \to \mathcal{O}(G)$ given by*

$$\varphi([X, v, f])(\lambda) = f(\lambda_X(v))$$

is an isomorphism of Hopf algebras.

PROOF. We leave it to the reader to check that φ is a morphism of Hopf algebras. By Theorem 4.12, $\mathcal{C}$ is equivalent to the category $\mathbf{Rep}_k(G)$ of finite-dimensional k-representations of G, and we can identify ω with the forgetful functor $\mathbf{Rep}_k(G) \to \mathbf{Vec}_k$.

We first prove that φ is surjective. Note that there is a left group action of G on $\mathcal{O}(G)$ given by

$$(\lambda h)(\mu) = h(\mu\lambda).$$

By Lemma 3.85, $\mathcal{O}(G)$ is the union of its finite-dimensional subrepresentations. In other words, given $h \in \mathcal{O}(G)$, there exists a finite-dimensional subrepresentation (V, ρ) of $\mathcal{O}(G)$ containing h. It determines an object X of $\mathcal{C}$ such that h belongs to $\omega(X) = V$. Let $f \in V^{\vee}$ be the element given by $f(u) = u(e)$, where e is the unit of G and $u \in V \subseteq \mathcal{O}(G)$. Then, for each element $\lambda \in G$, we have

$$[X, h, f](\lambda) = f(\lambda h) = (\lambda h)(e) = h(e\lambda) = h(\lambda).$$

Therefore, $\varphi([X, h, f]) = h$ and φ is surjective.

We next prove the injectivity. Assume that $\varphi([X, f, u]) = 0$. We identify X with a finite-dimensional representation (V, ρ) of G such that $u \in V$. Let V' be the simple subrepresentation of V containing u. Then V' is generated by elements of the form λu for $\lambda \in \mathcal{O}(G)$. Since $\varphi([X, f, u]) = 0$, we deduce that $f|_{V'} = 0$. Let X' be the object of $\mathcal{C}$ corresponding to (V', ρ). By the compatibility relation

$$[X, f, u] = [X', f|_{V'}, u] = [X', 0, u] = 0.$$

This concludes the proof. $\square$

EXAMPLE 4.18. Let $\mathbf{GrVec}_k$ be the category of finite-dimensional vector spaces from Example 4.10. Equipped with the forgetful functor ω, it forms a Tannakian category which is equivalent to semisimple category generated by objects k_n, $n \in \mathbb{Z}$ with

$$\mathrm{Hom}(k_n, k_m) = \begin{cases} k, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases} \qquad k_n \otimes k_m = k_{n+m}, \qquad \omega(k_n) \simeq k.$$

For each n choose a non-zero element $u_n \in \omega(k_n)$ and let $u_n^{\vee} \in \omega(k_n)^{\vee}$ be the element defined by $u_n^{\vee}(u_n) = 1$. Then every matrix coefficient in $\mathcal{T}$ can be written as a linear combination of the elements

$$[k_n, u_n, u_n^{\vee}], \quad n \in \mathbb{Z}.$$

Moreover,

$$[k_n, u_n, u_n^\vee] \cdot [k_m, u_m, u_m^\vee] = [k_{n+m}, u_{n+m}, u_{n+m}^\vee].$$

Thus, if we write $t = [k_1, u_1, u_1^\vee]$, there is an isomorphism of algebras

$$\mathcal{O}(\underline{\mathrm{Aut}}^{\otimes}(\omega)) = k[t, t^{-1}].$$

Moreover, the coproduct, the counit and the antipode are given by

$$\Delta t = t \otimes t, \quad \epsilon(t) = \epsilon(t^{-1}) = 1, \quad S(t) = t^{-1}.$$

From part (2) of Example 3.50, we deduce that $\underline{\mathrm{Aut}}^{\otimes}(\omega) = \mathbb{G}_m$, the multiplicative group. It is a general fact that the presence of a grading is related to an action of $\mathbb{G}_m$.

EXAMPLE 4.19. Consider the subgroup of $\mathrm{GL}_2(\mathbb{R})$ given by:

$$\{ \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix} \right) \in \mathrm{GL}_2(\mathbb{R}) \mid x^2 + y^2 > 0 \}.$$

These are the real points of an affine algebraic group $\mathbb{S}$ over $\mathbb{R}$ called the *Deligne torus*. Alternatively, one can define it as

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m),$$

where $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ is the Weil restriction functor. This means that, if A is an $\mathbb{R}$-algebra, then $\mathbb{S}(A) = \mathbb{G}_m(A \otimes_{\mathbb{R}} \mathbb{C}) = (A \otimes_{\mathbb{R}} \mathbb{C})^{\times}$. The category of representations of $\mathbb{S}$ is equivalent to the category of split $\mathbb{R}$-mixed Hodge structures.

4.1.6. *Tannakian subcategories.* Let Y be an objects in a neutral Tannakian category $\mathcal{C}$, we denote by $\langle Y \rangle$ the full subcategory of $\mathcal{C}$ that contains Y and is stable by sums, tensor products, dual and subquotients. Then $\langle Y \rangle$, together with the restriction of any fibre functor ω on $\mathcal{C}$ is again a neutral Tannakian category. The action of $G = \underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega)$ on the vector space $\omega(Y)$ induces a map $G \to \mathrm{GL}(\omega(Y))$. The following is shown in the proof of [**DM82**, Proposition 2.8]

LEMMA 4.20. *The image $G^Y \subset \mathrm{GL}(\omega(Y))$ of G by the above map is a closed subgroup of $\mathrm{GL}(\omega(Y))$ which agrees with the Tannaka group $\underline{\mathrm{Aut}}^{\otimes}_{\langle Y \rangle}(\omega)$ of the subcategory $\langle Y \rangle$.*

We can order the subcategories of the form $\langle Y \rangle$ for Y an object of $\mathcal{C}$ by inclusion. With this order they form a directed system. The following lemma exhibits the pro-algebraic nature of G.

LEMMA 4.21. *Let $(\mathcal{C}, \omega)$ be a neutral Tannakian category. Then:*

$$\underline{\mathrm{Aut}}^{\otimes}_{\mathcal{C}}(\omega) = \varprojlim_{\langle Y \rangle} \underline{\mathrm{Aut}}^{\otimes}_{\langle Y \rangle}(\omega) = \varprojlim_{\langle Y \rangle} G^Y.$$

Proof. By Lemma 4.20, there is a surjection $G \to G^Y$ for every object Y of $\mathcal{C}$. These surjections are compatible with the maps $G^Z \to G^Y$ induced by an inclusion $\langle Y\rangle \subset \langle Z\rangle$. Therefore, there is a surjection

$$G \longrightarrow \varprojlim_{\langle Y\rangle} G^Y.$$

This map is also injective, because if an element of G is sent to unit, then it acts trivially on $\omega(Y)$ for every object Y. Therefore it is the unit of G. $\square$

4.1.7. *Tannakian categories and the fundamental group.* We next explore what can be recovered from the classical fundamental group of a topological space using the Tannakian formalism. This includes the pro-unipotent completion.

Let X be a path connected, locally path connected and locally simply connected topological space. Let x_0 be a point of X and $\pi_1(X, x_0)$ be the fundamental group of X with base point x_0. By part (5) of Example 4.10, the category $\mathbf{Loc}_k(X)$ of local systems of finite dimensional k-vector spaces over X is a Tannakian category with fibre functor ω_{x_0}. Given a local system V, the fibre at x_0 is a k-vector space with an action of $\pi_1(X, x_0)$. This yields the so-called *monodromy representation*

$$\rho_V\colon \pi_1(X, x_0) \to \mathrm{GL}(\omega_{x_0}(V)).$$

It follows that $\mathbf{Loc}_k(X)$ is equivalent to the category of finite-dimensional k-linear representations of $\pi_1(X, x_0)$. However, since this is not an affine group scheme, it cannot be the Tannaka group of the category $\mathbf{Loc}_k(X)$. In fact, the Tannaka group $\underline{\mathrm{Aut}}^{\otimes}(\omega_{x_0})$ will be its *pro-algebraic completion.*

Following Lemma 4.21 we can give a description of the pro-algebraic completion of Γ. Let $Y = (V, \rho)$ be a k-linear finite-dimensional representation ρ of Γ. The group G^Y from Lemma 4.21 is the Zariski closure $\overline{\rho(\Gamma)}^{\mathrm{Zar}}$ of the image of $\rho\colon \Gamma \to \mathrm{GL}(V)$. Given another representation $Y' = (V', \rho')$, an inclusion $\langle Y'\rangle \subset \langle Y\rangle$ means that the representation Y' is a subquotient of a tensor of Y (a representation obtained from Y using dual, tensor products and direct sums). Then

$$\Gamma^{\mathrm{alg}} = \varprojlim_{\langle (V,\rho)\rangle} \overline{\rho(\Gamma)}^{\mathrm{Zar}},$$

where the limit is taken with respect to the subcategories $\langle (V, \rho)\rangle$ ordered by inclusion.

Analogously we can recover the pro-unipotent completion of Γ using the Tannakian formalism. A local system $\mathbf{V}$ is called *unipotent* if its monodromy representation is unipotent (Definition 3.90). The category of unipotent local systems $\mathbf{ULoc}_k(X)$ on X is again a Tannakian category and ω_{x_0} is again a fibre functor. In this case the Tannaka group $\underline{\mathrm{Aut}}^{\otimes}(\omega_{x_0})$ is the pro-unipotent completion of Γ. It admits a similar description as the pro-algebraic completion but restricting to finite-dimensional unipotent representations.

$$\Gamma^{\text{un}} = \varprojlim_{\langle (V,\rho)\rangle \text{ unip.}} \overline{\rho(\Gamma)}^{\text{Zar}},$$

where the limit again is taken with respect to the subcategories $\langle (V,\rho)\rangle$ ordered by inclusion.

$\star\star\star$

EXERCISE 4.22. Prove that

$$[X \oplus Y, u \oplus v, f \oplus g] = [X, u, f] + [Y, v, g].$$

EXERCISE 4.23. Consider the unit circle S^1 as a topological space. Its fundamental group is $\pi_1(S^1, 1) \simeq \mathbb{Z}$. Prove that the pro-algebraic completion $\mathbb{Z}^{\text{pro}-\text{alg}}$ is infinite-dimensional, while

$$\mathbb{Z}^{\text{un}} \simeq \mathbb{G}_a,$$

the additive group. For the second part use that giving a unipotent representation of $\mathbb{Z}$ is equivalent to giving a finite-dimensional vector space V together with a unipotent endomorphism of V and the fibre functor is just the forgetful functor. Then use the explicit description of the Hopf algebra of the Tannaka group.

EXERCISE 4.24. Consider the Tannakian category $\mathbf{Vec}_k$ with the identity as the fibre functor ω. Prove that $\underline{\text{Aut}}^{\otimes}(\omega) = \text{Spec}(k)$, the trivial group.

EXERCISE 4.25 (The pro-algebraic completion of a group). Let k be a field and Γ an abstract group. In this exercise, we present three equivalent constructions of the *pro-algebraic completion* of Γ, which is an affine group scheme $G = \Gamma^{\text{alg}}$ over k together with a group morphism $\Gamma \to G(k)$.

(a) Let $\mathcal{C}$ be the category of finite-dimensional k-linear representations of Γ. Equipped with the forgetful functor, it is a Tannakian category, and one defines G as its fundamental group. A k-point of G is thus a collection $(\lambda_V)_{V\in \text{Ob}(\mathcal{C})}$ of automorphisms $\lambda_V \colon V \to V$ satisfying the constraints of Definition 4.11. To each element $\gamma \in \Gamma$ one associates the collection of automorphisms $\lambda^\gamma = (\lambda^\gamma_V)_V$ defined as $\lambda^\gamma_V(v) = \lambda \cdot \gamma$. This yields the map $\Gamma \to G(k)$.

(b) Consider the collection of pairs (H, φ_H) consisting of an affine group scheme H over k and a group morphism $\varphi_H \colon \Gamma \to H(k)$ with Zariski dense image. We define a partial order by setting $(H, \varphi_H) \leq (H', \varphi_{H'})$ whenever there exists a morphism $f \colon H \to H'$ such that the induced map of k-points commutes with φ_H and $\varphi_{H'}$ and we define the pro-algebraic completion G as the limit:

$$G = \lim H.$$

(c) The pro-algebraic completion G is an affine group scheme over k with a group morphism $\varphi\colon \Gamma \to G(k)$ such that, for any affine group scheme H over k and any group morphism $\varphi_H\colon \Gamma \to H(k)$, there exists a unique morphism $f\colon G \to H$ such that $f \circ \varphi = \varphi_H$.

4.2. Triangulated categories and t-structures.

4.2.1. *Triangulated categories.*

Definition 4.26 (Verdier). A *triangulated category* $\mathcal{T}$ is an additive category, together with

a) a self-equivalence of categories

$$\begin{aligned}[1]\colon \mathcal{T} &\longrightarrow \mathcal{T}\\ X &\longmapsto X[1].\end{aligned}$$

Once the self equivalence [1] is given, we shall call *triangles* all diagrams of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

A morphism of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[r, "w"] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]. \end{tikzcd}$$

We will use the convention that an arrow decorated with [1] like $A \xrightarrow{[1]} B$ means a map $A \to B[1]$.

b) A class of triangles called *distinguished triangles*.

These data are required to satisfy the following axioms:

(T1) a) For any $X \in \mathrm{Ob}(\mathcal{T})$, the triangle

$$X \xrightarrow{\mathrm{Id}} X \to 0 \to X[1]$$

is distinguished.

b) Any triangle isomorphic to a distinguished one is distinguished.

c) Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

(T2) The triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished.

(T3) Given two distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1], \qquad X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1],$$

and morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$ such that $g \circ u = u' \circ f$, there exists $h\colon Z \to Z'$ (not necessarily unique) such that

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & X[1] \arrow[d, "{f[1]}"'] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z \arrow[r, "w'"] & X'[1].
\end{tikzcd}$$

is a morphism of triangles.

(T4) Given a diagram of solid arrows

$$\begin{tikzcd}
& Y' \arrow[dr, dashed, "g"] \arrow[ddl, "{[1]}"', "n"] & \\
Z' \arrow[ur, dashed, "f"] \arrow[d, "k", "{[1]}"'] & & X' \arrow[ll, "{j[1]\circ i}", "{[1]}"'] \arrow[ddl, "i", "{[1]}"'] \\
X \arrow[rr, "v\circ u"] \arrow[dr, "u"'] & & Z \arrow[u, "\ell"'] \arrow[uul, "m"] \\
& Y, \arrow[uul, "j"] \arrow[ur, "v"'] &
\end{tikzcd}$$

if the three triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{k} X[1]$$

$$Y \xrightarrow{v} Z \xrightarrow{\ell} X' \xrightarrow{i} Y[1]$$

$$X \xrightarrow{v\circ u} Z \xrightarrow{m} Y \xrightarrow{n} X[1]$$

are distinguished, then there exist dashed arrows f and g as in the diagram such that the triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{j[1]\circ i} Z'[1]$$

is distinguished and the following commutation relations hold:

$$k = n \circ f, \qquad \ell = g \circ m,$$
$$m \circ v = f \circ j, \qquad u[1] \circ n = i \circ g.$$

EXAMPLE 4.27. The main example of a triangulated category is the (bounded) derived category of an abelian category. We quickly recall the main structures associated to the definition of derived categories. Let $\mathcal{A}$ be an abelian category. The category of bounded cochain complexes $\mathcal{C}^b(\mathcal{A})$ consists of sequences of maps

$$\dots \to A^k \xrightarrow{d^k} A^{k+1} \xrightarrow{d^{k+1}} A^{k+2} \xrightarrow{d^{k+2}} A^{k+3} \to \dots$$

with $d^{k+1} \circ d^k = 0$ and such that $A^n = 0$ for $n \ll 0$ and $n \gg 0$. Morphisms of complexes are commutative diagrams

$$\begin{tikzcd} \dots \arrow[r] & A^k \arrow[r, "d^k"] \arrow[d, "f^k"] & A^{k+1} \arrow[r, "d^{k+1}"] \arrow[d, "f^{k+1}"] & A^{k+2} \arrow[r] \arrow[d, "f^{k+2}"] & \dots \\ \dots \arrow[r] & B^k \arrow[r, "d^k"] & B^{k+1} \arrow[r, "d^{k+1}"] & B^{k+2} \arrow[r] & \dots \end{tikzcd}$$

Complexes are denoted by A^* or, if we want to emphasize the differential, by (A^*, d). Given a complex A^*, its *cohomology groups* are

$$H^n(A^*) = \frac{\operatorname{Ker}(d^n \colon A^n \to A^{n+1})}{\operatorname{Im}(d^{n-1} \colon A^{n-1} \to A^n}).$$

Given a complex (A^*, d), its *shift* $(A[1]^*, d[1])$ is defined as

$$A[1]^n = A^{n+1}, \text{ with } d[1] = -d.$$

A morphism of complexes $f \colon A^* \to B^*$ induces a morphism of cohomology groups

$$H(f) \colon H^*(A^*) \to H^*(B^*).$$

A morphism of complexes is called a *quasi-isomorphism* if it induces an isomorphism of cohomology groups.

Another important construction in the category $\mathcal{C}^b(\mathcal{A})$ is the *cone* of a morphism of complexes. Let f be a morphism of complexes as before, then the cone of f is the complex

$$\operatorname{cone}(f)^n = A^{n+1} \oplus B^n, \text{ with } d(a,b) = (-da, db + f(a)).$$

The cone is provided with two morphisms of complexes

$$\begin{aligned} B &\to \operatorname{cone}(f), & b &\mapsto (0,b) \\ \operatorname{cone}(f) &\to A[1], & (a,b) &\mapsto a, \end{aligned}$$

that induce a long exact sequence of cohomology groups

$$H^n(A^*) \xrightarrow{H(f)} H^n(B^*) \to H^n(\operatorname{cone}(f)) \to H^{n+1}(A^*).$$

Given two morphisms of complexes $f, g \colon A^* \to B^*$, a *homotopy* between them is a collection of maps $s^n \colon A^n \to B^{n-1}$ such that

$$f^n - g^n = d^{n-1} \circ s^n + s^{n+1} \circ d^n.$$

two morphisms linked by a homotopy are said to be homotopicaly equivalent. Two homotopicaly equivalent morphisms induce the same morphism in cohomology groups.

The construction of the derived category is done in two steps. First one defines the homotopy category $\mathcal{K}^b(\mathcal{A})$ whose objects are the same as $\mathcal{C}^b(\mathcal{A})$ but whose morphisms are equivalence classes with respect the homotopy equivalence of morphisms of $\mathcal{C}^b(\mathcal{A})$. In the second step one constructs $\mathcal{D}^b(\mathcal{A})$ by inverting the quasi-isomorphisms. That is, the objects of $\mathcal{D}^b(\mathcal{A})$ are the same as the objects of $\mathcal{K}^b(\mathcal{A})$ (which are the same as the ones of $\mathcal{C}^b(\mathcal{A})$), while the morphism on $\mathcal{D}^b(\mathcal{A})$ between two objects A^* and B^* are equivalence classes of diagrams of the form

$$A^* \xleftarrow{\simeq} C^* \to B^*$$

where the arrow to the left is a quasi-isomorphism. The diagrams

$$A^* \xleftarrow{\simeq} C_1^* \to B^* \text{ and } A^* \xleftarrow{\simeq} C_2^* \to B^*$$

are equivalent if there is a third diagram of the same type such that

$$\begin{tikzcd} & C_1^* \arrow[dl, "\simeq"'] \arrow[dr] & \\ A^* & C_3^* \arrow[l, "\simeq"'] \arrow[u] \arrow[r] \arrow[d] & B^* \\ & C_2^* \arrow[ul, "\simeq"] \arrow[ur] & \end{tikzcd} \tag{4.28}$$

commutes in $\mathcal{K}^b(\mathcal{A})$.

This means that all the triangles in diagram (4.28) are commutative up to homotopy but they are not necessarily commutative. To have such a simple description of the morphisms is the main reason to define the derived category in to steps. One can invert directly the quasi-isomorphisms in $\mathcal{C}^b(\mathcal{A})$, but then morphisms will be chains of the form

$$A^* \xleftarrow{\simeq} C_1^* \to C_2^* \dots C_{k-1}^* \xleftarrow{\simeq} C_k^* \to B^*$$

where all the arrows in the left direction are quasi-isomorphisms.

The category $\mathcal{D}^b(\mathcal{A})$ is a triangulated category, where the self equivalence [1] is defined by the shift, while the class of distinguished triangles are those triangles that are isomorphic (in $\mathcal{D}^b(\mathcal{A})$) to one of the form

$$A^* \xrightarrow{f} B^* \to \operatorname{cone}(f) \to A[1]^{ast}.$$

4.2.2. *t-structures.* There are many natural situations where one is able to construct a triangulated category but would like to obtain an abelian category instead. In their work on perverse sheaves [**BBD82**], Beilinson, Bernstein and Deligne introduced the notion of *t-structure* as a way of extracting an abelian category from a triangulated category. This is how mixed Tate motives over a number field will be constructed in the sections to follow.

DEFINITION 4.29 (Beilinson-Bernstein-Deligne). Let $\mathcal{T}$ be a triangulated category. A *t-structure* on $\mathcal{T}$ is a pair of strictly full[9] subcategories

$$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$$

such that, defining for each integer n

$$\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n], \quad \mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n],$$

the following three conditions are satisfied: m

(1) One has $\mathcal{T}^{\leq -1} \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 1} \subseteq \mathcal{T}^{\geq 0}$.
(2) (*Orthogonality*) If $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$, then $\operatorname{Hom}_{\mathcal{T}}(X, Y) = 0$.
(3) Each object X of $\mathcal{T}$ fits into a distinguished triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow Y[-1] \tag{4.30}$$

with $Y \in \mathcal{T}^{\leq 0}$ and $Z \in \mathcal{T}^{\geq 1}$.

We say that the t-structure is *non-degenerate* if, moreover, the intersections $\cap_{n\in\mathbb{Z}}\mathcal{T}^{\leq n}$ and $\cap_{n\in\mathbb{Z}}\mathcal{T}^{\geq n}$ are reduced to zero.

DEFINITION 4.31. The *heart* of a t-structure on $\mathcal{T}$ is the full subcategory

$$\mathcal{T}^{0} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}.$$

A functor $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ between triangulated categories equipped with t-structures is said to be *t-exact* whenever $F(\mathcal{T}_1^{\leq 0}) \subseteq \mathcal{T}_2^{\leq 0}$ and $F(\mathcal{T}_1^{\geq 0}) \subseteq \mathcal{T}_2^{\geq 0}$. It restricts thus to a functor between the hearts.

Note that the objects Y and Z in the triangle (4.30) are not a priori required to be unique. However, this follows from the other axioms:

LEMMA 4.32.
(1) The inclusion of $\mathcal{T}^{\leq n}$ into $\mathcal{T}$ admits a right adjoint $t_{\leq n}\colon \mathcal{T} \to \mathcal{T}^{\leq n}$ and the inclusion $\mathcal{T}^{\geq n}$ into $\mathcal{T}$ admits a left adjoint $t_{\geq n}\colon \mathcal{T} \to \mathcal{T}^{\geq n}$.
(2) For each object X in $\mathcal{T}$, there exists a unique morphism

$$w \in \operatorname{Hom}_{\mathcal{T}}(t_{\geq 1}X, t_{\leq 0}X)$$

such that the following is a distinguished triangle:

$$t_{\leq 0}X \longrightarrow X \longrightarrow t_{\geq 1}X \xrightarrow{w} t_{\leq 0}X[-1].$$

Up to unique isomorphism, it is only the only one satisfying (3).

[9] By this we wean full and closed under isomorphism.

Moreover, if $a \leq b$, there is a unique isomorphism

$$t_{\geq a} t_{\leq b} X \xrightarrow{\sim} t_{\leq b} t_{\geq a} X. \tag{4.33}$$

The standard example of t-structure is the following:

EXAMPLE 4.34. Let $\mathcal{A}$ be an abelian category. Recall from Example 4.27 that the bounded derived category $D^b(\mathcal{A})$ is a triangulated category. It comes together with a canonical t-structure which measures how far a complex is from having cohomology concentrated in degree zero. Precisely, for each integer n, one considers the full subcategories

$$\mathcal{T}^{\leq n} = \{C^\bullet \in D^b(\mathcal{A}) \mid H^m(C^\bullet) = 0 \text{ for all } m > n\},$$
$$\mathcal{T}^{\geq n} = \{C^\bullet \in D^b(\mathcal{A}) \mid H^m(C^\bullet) = 0 \text{ for all } m < n\}.$$

It is easy to check that the pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\leq 0})$ satisfies the axioms (1)-(3) from Definition 4.29. Moreover, this t-structure is non-degenerate.

The functors $t_{\leq n}$ and $t_{\geq n}$ are given by the *canonical truncations*

$$(t_{\leq n} C)^m = \begin{cases} C^m, & \text{if } m < n, \\ \operatorname{Ker} d, & \text{if } m = n \\ 0, & \text{if } m > n. \end{cases} \qquad (t_{\geq n} C)^m = \begin{cases} C^m, & \text{if } m > n, \\ C^n / dC^{n-1}, & \text{if } m = n \\ 0, & \text{if } m < n. \end{cases}$$

It follows that

$$t_{\leq n} t_{\geq n} C^\bullet = H^n(C^\bullet).$$

Viewing an object of $\mathcal{A}$ as a complex concentrated in degree zero, one gets an equivalence between $\mathcal{A}$ and the heart of $D^b(\mathcal{A})$.

For more general triangulated categories, the following theorem makes it possible to extract an abelian category [**BBD82**, Thm 1.3.6]. Recall that an abelian subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ is said to be *admissible* whenever short exact sequences in $\mathcal{A}$ are exactly those sequences

$$0 \longrightarrow B \xrightarrow{u} C \xrightarrow{v} A \to 0$$

such that there exists a distinguished triangle

$$B \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B[1]. \tag{4.35}$$

REMARK 4.36. The extension to a distinguished triangle is not unique, unless $\mathcal{A}$ is a full subcategory, that is, $\operatorname{Hom}_{\mathcal{A}}(X, Y) = \operatorname{Hom}_{\mathcal{T}}(X, Y)$ for all objects $X, Y \in \mathcal{A}$. Indeed, it follows from axiom (T3) in the definition of triangulated categories that, given two extensions as in (4.35), the identity maps $B \to B$ and $C \to C$ can be completed to a morphism of triangles

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[d, equal] & C \arrow[r, "v"] \arrow[d, equal] & A \arrow[r, "w"] \arrow[d, "h"'] & B[1] \arrow[d, equal] \\ B \arrow[r, "u"] & C \arrow[r, "v"] & A \arrow[r, "w'"] & B[1]. \end{tikzcd}$$

in $\mathcal{T}$. In particular, $w = w' \circ h$ and uniqueness amounts to proving that h is the identity. Since $\mathcal{A}$ is a full subcategory, $h\colon A \to A$ is a morphism in $\mathcal{A}$ such that $v \circ h = h$, and the surjectivity of v implies $h = \mathrm{Id}_A$.

The following theorem is proved in [**BBD82**, Thm. 1.3.6]:

THEOREM 4.37 (Beilinson-Bernstein-Deligne). *The heart of a t-structure on a triangulated category is a full admissible abelian subcategory.*

REMARK 4.38. It is not true, however, that $\mathcal{T}$ is equivalent, as triangulated category, to the derived category of the heart of a t-structure. Usually, one does not even have a functor $D^b(\mathcal{T}^0) \to \mathcal{T}$ (see Exercice 4.45).

DEFINITION 4.39. Let n be an integer. The n-th cohomology of $X \in \mathcal{T}$ with respect to the t-structure is the following object of the heart:

$$h^n(X) = t_{\leq n} t_{\geq n} X \in \mathcal{T}^0. \tag{4.40}$$

This yields a *cohomological* functor $h^n\colon \mathcal{T} \to \mathcal{T}^0$, in the sense that it maps distinguished triangles $X \to Y \to Z \to X[1]$ to long exact sequences

$$\cdots \to h^n(X) \longrightarrow h^n(Y) \longrightarrow h^n(Z) \longrightarrow h^{n+1}(X) \to \ldots$$

4.2.3. *Extensions.* We now explain the relation between Hom groups in a triangulated category and extensions in its abelian subcategories. First recall the basic definitions:

DEFINITION 4.41. Let $\mathcal{A}$ be an abelian category. Given two objects A and B, a degree n extension of A by B is an exact sequence

$$E\colon \quad 0 \longrightarrow B \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow A \longrightarrow 0.$$

Two extensions of the same degree E and E' are said to be equivalent if there exists a commutative diagram

$$\begin{tikzcd}
E\colon & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
E'\colon & 0 \arrow[r] & B \arrow[r] & C'_{n-1} \arrow[r] & \cdots \arrow[r] & C'_0 \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

The set of equivalent classes of degree n extensions of A by B forms a group $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ with respect to the *Baer sum.*

Consider a full admissible abelian subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$. Let $0 \to B \to C \to A \to 0$ be an extension in $\mathcal{A}$. By Remark 4.36, it extends to a *unique* distinguished triangle $B \to C \to A \to B[1]$, yielding a map $w\colon A \to B[1]$. Moreover, the same argument shows that two equivalent extensions give rise to the same w. We thus obtain a homomorphism

$$\varphi_1\colon \mathrm{Ext}^1_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(A, B[1]).$$

More generally, breaking a degree n extension

$$0 \to B \to C_{n-1} \to \cdots \to C_0 \to A \to 0$$

into several short exact sequences gives a morphism $A \to B[n]$ which only depends on the equivalence class of the extension. For instance, if $n = 2$, one associates to $0 \to B \to C_1 \xrightarrow{a} C_0 \to A \to 0$ the short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & C_1 \arrow[r] & \mathrm{Im}(a) \arrow[r] \arrow[d, equal] & 0 & & \\
& & 0 \arrow[r] & \mathrm{Ker}(a) \arrow[r] & C_0 \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Setting $D = \mathrm{Im}(a) = \mathrm{Ker}(a)$ and applying φ_1 to the rows of the above diagram, we get maps $\alpha\colon D \to B[1]$ and $\beta\colon A \to D[1]$. Then we form

$$\alpha[1] \circ \beta\colon A \to B[2].$$

Proposition 4.42. *Let $\mathcal{A}$ be a full admissible abelian subcategory of a triangulated category $\mathcal{T}$. Assume that $\mathcal{A}$ is stable under extension. Then*

$$\varphi_n\colon \mathrm{Ext}^n_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(A, B[n]).$$

is an isomorphism for $n = 1$ and an injection for $n = 2$.

Proof. See [**Lev93**, Prop. 1.6]. □

⋆ ⋆ ⋆

Exercise 4.43. Show that the distinguished triangle (4.30) in the definition of t-structure is uniquely determined by X up to a unique isomorphism. Thus, it makes sense to write $Y = X^{\leq 0}$ and $Z = X^{>0}$. Moreover, the assignments $X \mapsto X^{\leq 0}$ and $X \mapsto X^{>0}$ determine functors

Exercise 4.44 (Weight structures). Let $\mathcal{T}$ be a triangulated category. Following Bondarko [**Bon10**], we call *weight structure* on $\mathcal{T}$ a pair

$$(\mathcal{T}_{w\leq 0}, \mathcal{T}_{w\geq 0})$$

of strictly full subcategories such that, setting

$$\mathcal{T}_{w\leq n} = \mathcal{T}_{w\leq 0}[n], \quad \mathcal{T}_{w\geq n} = \mathcal{T}_{w\geq 0}[n],$$

for each integer n, the following conditions hold:

(1) One has $\mathcal{T}_{w\leq 0} \subseteq \mathcal{T}_{w\leq 1}$ and $\mathcal{T}_{w\geq 1} \subseteq \mathcal{T}_{w\geq 0}$.
(2) If $X \in \mathcal{T}_{w\leq 0}$ and $Y \in \mathcal{T}_{w\geq 1}$, then $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$.
(3) For each object X in $\mathcal{T}$, there exists a distinguished triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow Y[1]$$

such that $Y \in \mathcal{T}_{\leq 0}$ and $Z \in \mathcal{T}_{\geq 1}$.

Notice that the order of the inclusions in (1) is reversed with respect to the definition of t-structure.

Exercise 4.45 (A t-structure such that the derived category of the heart is not equivalent to the original triangulated category). Let X be a connected finite CW-complex and let $\mathrm{Sh}(X)$ be the abelian category of sheaves of $\mathbb{Q}$-vector spaces on X. Consider the full subcategory

$$\mathcal{T} \subseteq D^b(\mathrm{Sh}(X))$$

consisting of complexes of sheaves C such that all the cohomology sheaves $\mathcal{H}^i(C)$ are constant. Then $\mathcal{T}$ inherits a structure of triangulated category. We define

$$\mathcal{T}^{\leq 0} = \{C \mid \mathcal{H}^i(C) = 0 \text{ for } i > 0\},$$
$$\mathcal{T}^{\geq 0} = \{C \mid \mathcal{H}^i(C) = 0 \text{ for } i < 0\}.$$

(1) Show that the pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ forms a t-structure on $\mathcal{T}$, whose heart is equivalent to the category $\mathbf{Vec}_{\mathbb{Q}}$ of finite-dimensional $\mathbb{Q}$-vector spaces.
(2) Let $\mathbb{Q}_X$ be the constant sheaf on X. Show that

$$\mathrm{Hom}_{\mathcal{T}}(\mathbb{Q}_X, \mathbb{Q}_X[2]) = H^2(X, \mathbb{Q}).$$

However, using the fact that $D^b(\mathcal{T}^0)$ is equivalent to the category $D^b(\mathbf{Vec}_{\mathbb{Q}})$, we have

$$\mathrm{Hom}_{D^b(\mathcal{T}^2)}(\mathbb{Q}_X, \mathbb{Q}_X[2]) = 0.$$

Deduce that, as long as $H^2(X, \mathbb{Q}) \neq 0$, the triangulated category $\mathcal{T}$ is not equivalent to the derived category of the heart.

4.3. Voevodsky's category of motives.

4.3.1. *A universal cohomology.* Different cohomology theories have been proved useful in the study of algebraic varieties. For instance, as we saw in Chapter 2, to any variety X over a subfield k of $\mathbb{C}$, it is attached the Betti cohomology

$$H^*_{\mathrm{B}}(X) = H^*(X(\mathbb{C}), \mathbb{Q}),$$

which is a finite-dimensional graded $\mathbb{Q}$-vector space. If, in addition, X is smooth, one has also at disposal the de Rham cohomology

$$H^*_{\mathrm{dR}}(X) = \mathbb{H}^*(X, \Omega^*_X),$$

which is now a finite-dimensional graded k-vector space. Recall from Theorem 2.55 that both cohomologies are related, after complexification, by the period isomorphism

$$H^*_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} H^*_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.46}$$

Another important example is ℓ-adic cohomology, defined for varieties over a field k of arbitrary characteristic p, a choice of a separable closure $\bar{k}$ of k, and a prime number ℓ different from p by

$$H_\ell^*(X) = \varprojlim H_{\text{ét}}^*(X_{\bar{k}}, \mathbb{Z}/\ell^n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

When $\bar{k}$ is embeddable into $\mathbb{C}$, Artin proved that there exists a canonical isomorphism

$$H_\ell(X) \simeq H_B(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell. \tag{4.47}$$

All the cohomology theories we have mentioned satisfy similar properties, such as homotopy invariance, Poincaré duality, Künneth formulas, Mayer-Vietoris exact sequences etc. A fundamental feature is that the corresponding vector spaces usually come together with extra structures. We have already seen that Betti cohomology can be provided with a mixed Hodge structure, and ℓ-adic cohomology carries a continuous $\mathbb{Q}_\ell$-linear action of the Galois group $\mathrm{Gal}(\bar{k}/k)$.

The similarities between different cohomology theories, as well as the existence of comparison isomorphisms such as (4.46) or (4.47), led Grothendieck to postulate the existence of a universal cohomology theory which factors all the others: this should be the *motive* of the variety. Since its introduction by Grothendieck, the theory of motives has inspired a wealth of research but, although we have advanced a lot in our understanding, many fundamental questions remain still unanswered.

Restricting to the case of smooth projective varieties, Grothendieck constructed a category of *pure* motives over a field k with some of the desired properties. However, in order to prove that it has all of them, he stated a set of conjectures, the *standard conjectures*, that have proven to be very difficult. The terminology "pure" comes from the fact that for any smooth projective variety, its n-th cohomology group always has certain properties that are encoded in the statement "$H^n(X)$ is of pure weight n". For instance, if X is a smooth projective complex variety, the group $H^n_{\mathrm{B}}(X, \mathbb{C})$ has a Hodge decomposition

$$H^n_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{p+q=n} H^{p,q}(X).$$

The fact that only factors with $p+q=n$ appear means that its Hodge structure is pure of weight n. For varieties over a finite field, the corresponding purity is reflected by the fact that the eigenvalues of the action of Frobenius on tale cohomology have absolute value $q^{\frac{n}{2}}$.

Using resolution of singularities, we can express the cohomology of a singular quasi-projective variety in terms of the cohomology of smooth projective varieties, but in this expression cohomologies of different degrees get mixed. As we have seen in Section 2.5.2 this gives rise to a mixed Hodge structure in the cohomology of X. Thus, the motive of a smooth projective variety should be *pure* while the motive of a singular or quasi-projective variety should be *mixed*. Since Grothendieck, there has been a great effort to develop a theory of mixed motives.

Abstractly we can think of a cohomology theory in the following way. Fix a field k, denote by $\mathbf{Var}_k$ the category of varieties over k, and let $\mathcal{A}$ be an abelian category (or more precisely a Tannakian category). Denote by $\mathcal{D}\mathcal{A}$ the derived category of $\mathcal{A}$. Then $\mathcal{D}\mathcal{A}$ is a triangulated category provided with a t-structure (see Section 4.2.2 for a definition) that allows us to recover $\mathcal{A}$ from $\mathcal{D}\mathcal{A}$. A cohomology theory (with values in $\mathcal{A}$) is a contravariant functor

$$H\colon \mathbf{Var}_k \longrightarrow \mathcal{D}\mathcal{A}$$

satisfying certain properties. We can recover the "cohomology groups" of X from $H(X)$ using the t-structure:

$$H^n(X) = t_{\leq n} t_{\geq n} H(X) \in \mathcal{A}.$$

Voevodsky was able to define a triangulated category $\mathbf{DM}_{gm}(k)$, which is a candidate for the derived category of mixed motives over k. The main missing piece is a suitable "motivic" t-structure. Recently, Beilinson [**Bei12**] showed that, when k has characteristic zero, its existence implies the standard conjectures[10].

4.3.2. *The derived category of mixed motives.* Let k be a field. In what follows, we give a sketch of Voevodsky's construction of a derived category of mixed motives over k with rational coefficients, which will be denoted by

$$\mathbf{DM}(k) = \mathbf{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}.$$

For more details we refer the reader to the original paper [**Voe00**], the lecture notes [**MVW06**] or part II of the introductory book [**And04**].

We start with the category $\mathbf{Sm}(k)$ of smooth varieties over k. This category is not additive, for it does not make sense to "sum" two morphisms of schemes. The first step of the construction will be to enlarge the set of morphisms through the notion of finite correspondence.

4.3.3. *First step: the category of finite correspondences.*

Definition 4.48. Let X and Y be objects of $\mathbf{Sm}(k)$. A *finite correspondence* from X to Y is a $\mathbb{Z}$-linear combination of integral closed subschemes $W \subseteq X \times Y$ such that the projection $W \to X$ is finite and surjective over a connected component of X.

Finite correspondences form an abelian subgroup of the algebraic cycles $\mathcal{Z}^{\dim Y}(X \times Y)$ which will be denoted by $c(X, Y)$.

[10] Conversely, Hanamura proved in [**Han99**] that, over any field k, the conjunction of the standard conjectures and conjectures by Murre and Beilinson-Soulé implies the existence of the motivic t-structure.

EXAMPLE 4.49. Given any morphism of schemes $f\colon X \to Y$, the graph $\Gamma_f \subseteq X \times Y$ is a finite correspondence. In general, we can think of finite correspondences as multivalued maps on a connected component of X.

Recall that, when dealing with pure motives, one needs to pass to an equivalence relation in order to compose morphisms; this leads to difficult questions on algebraic cycles (like the standard conjectures). In contrast, there is no need of equivalence relations if we work with finite correspondences.

Given $X, Y, Z \in \mathbf{Sm}(k)$, we will denote by p_{XY}, p_{XZ} and p_{YZ} the projections from $X \times Y \times Z$ to $X \times Y$, $X \times Z$ and $Y \times Z$ respectively.

LEMMA 4.50. *Let X, Y, Z be objects in $\mathbf{Sm}(k)$. Consider finite correspondences $W \in c(X,Y)$ and $W' \in c(Y,Z)$. Then the cycles $p_{XY}^*(W)$ and $p_{YZ}^*(W')$ intersect properly on $X \times Y \times Z$. Moreover, the projection of the cycle $p_{XZ}(p_{XY}^*\alpha \cdot p_{YZ}^*\beta)$ is finite over X and surjective over a connected component.*

Thanks to the above lemma, one defines the composition

$$\begin{aligned} \circ : c(X,Y) \times c(Y,Z) &\to c(X,Z) \\ \alpha \circ \beta &= p_{XZ}(p_{XY}^*\alpha \cdot p_{YZ}^*\beta). \end{aligned} \tag{4.51}$$

The category $\mathbf{SmCor}(k)$ has the same objects as $\mathbf{Sm}(k)$ and morphisms given by finite correspondences with $\mathbb{Q}$-coefficients, that is

$$\mathrm{Hom}_{\mathbf{SmCor}(k)}(X,Y) = c(X,Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

There is a functor $\mathbf{Sm}(k) \to \mathbf{SmCor}(k)$ that is the identity on objects and sends a map $f\colon X \to Y$ to its graph Γ_f. By Exercise 4.72, the composition of maps is compatible with the composition (4.51) of finite correspondences. We denote by $[X]$ the image in $\mathbf{SmCor}(k)$ of a smooth variety X.

The direct sum in $\mathbf{SmCor}(k)$ is given by the disjoint union of subvarieties. This category is also equipped with the tensor product

$$[X] \otimes [Y] = [X \times_k Y].$$

4.3.4. *Second step: A triangulated category with homotopy invariance and Mayer Vietoris.* The second step is similar to the construction of the derived category of an abelian category. We start with the category

$$C^b(\mathbf{SmCor}(k))$$

of bounded chain complexes in $\mathbf{SmCor}(k)$. The objects are diagrams

$$\cdots \to [X_n] \xrightarrow{\partial_n} [X_{n-1}] \to \cdots,$$

where X_i is in $\mathbf{Sm}(k)$ and $\partial_n \in c(X_n, X_{n-1}) \otimes \mathbb{Q}$ are finite correspondences such that $\partial_{n-1} \circ \partial_n = 0$. Then we define the homotopy category

$K^b(\mathbf{SmCor}(k))$ as the one having the same objects as $C^b(\mathbf{SmCor}(k))$, and morphisms given by homotopy classes of morphisms of complexes.

Two examples of objects of $K^b(\mathbf{SmCor}(k))$ are:

(1) (homotopy complex) for any X in $\mathbf{Sm}(k)$, the complex

$$[X \times \mathbb{A}^1] \xrightarrow{\mathrm{pr}} [X]$$

placed in degrees 1 and 0.

(2) (Mayer-Vietoris complex) for any X in $\mathbf{Sm}(k)$ and any open cover $X = U \cup V$, the complex

$$[U \cap V] \xrightarrow{i_{U\cap V,U}+i_{U\cap V,V}} [U] \oplus [V] \xrightarrow{i_{U,X}-i_{V,X}} [X],$$

where $[U \cap V]$ sits in degree 2, and the arrows $i_{U,X}, i_{V,X}, i_{U\cap V,U}$ and $i_{U\cap V,V}$ are the obvious inclusions.

We want to force the homotopy invariance and the Mayer-Vietoris property, which mean that the above two complexes are acyclic. To this end, we take the quotient of $K^b(\mathbf{SmCor}(k))$ by the thick triangulated subcategory generated by all homotopy and Mayer-Vietoris complexes. It has the structure of a triangulated category.

4.3.5. *Third step: The pseudo-abelian envelope.* The next step is to take the pseudo-abelian envelope of the quotient obtained in the previous step. The resulting category is denoted by $DM_{gm}^{\mathrm{eff}}(k)$.

Recall the construction of the pseudo-abelian envelope

Definition 4.52. Let $\mathcal{C}$ be an additive category. The *pseudo-abelian envelope* of $\mathcal{C}$ is the category with

- *objects*: (X, p) where X is an object of $\mathcal{C}$ and $p \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ is an idempotent, that is, $p^2 = p$.
- *morphisms*: $\mathrm{Hom}((X, p), (Y, q)) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the subgroup of those f such that $f = q \circ f \circ p$.

There is a fully faithful functor $\mathcal{C} \to \mathcal{C}_{pa}$ sending X to (X, id). Passing to the pseudo-abelian envelope allows us to consider the kernel of each idempotent $p\colon X \to X$ as a subobject of X. This will be crucial when we want to talk about "pieces of the cohomology".

Remark 4.53. By a result of Balmer and Schlichting [BS01], the pseudo-abelian envelope of a triangulated category remains triangulated. Thus, $DM_{gm}^{\mathrm{eff}}(k)$ is still a triangulated category.

We have a functor $M\colon \mathbf{Sm}(k) \to DM_{gm}^{\mathrm{eff}}(k)$ sending X to $[X]$, regarded as a complex concentrated in degree zero. The category $DM_{gm}^{\mathrm{eff}}(k)$ is also equipped with a tensor product that is characterized by the property

$$M(X) \otimes M(Y) = M(X \times Y).$$

The unit object is the motive of the base field, which will be denoted by

$$\mathbb{Q}(0) = M(\mathrm{Spec}(k)).$$

Note also that there is a functor

$$C^b(\mathbf{SmCor}(k)_{\mathrm{pa}}) \longrightarrow DM_{gm}^{\mathrm{eff}}(k) \tag{4.54}$$

from the category of bounded complexes in the pseudo-abelian envelope of $\mathbf{SmCor}(k)$ to the category of effective motives $DM_{gm}^{\mathrm{eff}}(k)$.

4.3.6. *Fourth step: inversion of the Tate motive.* Given X in $\mathbf{Sm}(k)$, let $X \to \mathrm{Spec}(k)$ denote the structural morphism. We can think of it as a complex sitting in degrees 0 and -1:

$$[X] \longrightarrow [\mathrm{Spec}\, k]. \tag{4.55}$$

DEFINITION 4.56. The *reduced motive* of X is the object $\widetilde{M}(X)$ of $DM_{gm}^{\mathrm{eff}}(k)$ determined by the complex (4.55).

When X has a k-rational point, there is a direct sum decomposition (see exercise 4.73)

$$M(X) = \mathbb{Q}(0) \oplus \widetilde{M}(X).$$

DEFINITION 4.57. The *Tate motive* $\mathbb{Q}(1)$ is $\widetilde{M}(\mathbb{P}^1_k)[-2]$. For $n \geq 0$, one defines $\mathbb{Q}(n)$ as $\mathbb{Q}(1)^{\otimes n}$.

The last step of the construction of $\mathbf{DM}(k)$, necessary to obtain a rigid tensor category, is to formally invert the motive $\mathbb{Q}(1)$. By this we mean the following: an object of the new category $\mathbf{DM}(k)$ is a pair (M, m), where M is an object of $DM_{gm}^{\mathrm{eff}}(k)$ and $m \in \mathbb{Z}$. Morphisms are given by

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{DM}(k)}((M,m),(N,n)) \\ &\qquad = \varinjlim_{r \geq -m,-n} \mathrm{Hom}_{DM_{gm}^{\mathrm{eff}}(k)}(M \otimes \mathbb{Q}(m+r), N \otimes \mathbb{Q}(n+r)).\end{aligned}$$

The resulting category has the following property:

THEOREM 4.58 (Voevodsky). *The category $\mathbf{DM}(k)$ is a rigid tensor $\mathbb{Q}$-linear triangulated category.*

PROOF. See [**MVW06**, Theorem 20.17]. □

4.3.7. *Properties of $\mathbf{DM}(k)$.* All the usual machinery to compute the homology of algebraic varieties is still available in the derived category of motives:

(1) (Künneth): $M(X \times Y) = M(X) \otimes M(Y)$.
(2) ($\mathbb{A}^1$-homotopy invariance): $M(X \times \mathbb{A}^1) = M(X)$.

(3) (Mayer-Vietoris): For $X = U \cup V$ as before, there is a distinguished triangle

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1].$$

(4) (Gysin) If $Z \subset X$ is a smooth closed subscheme of codimension c of a smooth scheme X, then there is a distinguished triangle

$$M(X \setminus Z) \to M(X) \to M(Z)(c)[2c] \to M(X \setminus Z)[1].$$

(5) (Blow-ups) Let $Z \subseteq X$ be a smooth closed subscheme of a smooth scheme, $\mathrm{Bl}_Z X$ the blow-up of X along Z, and E the exceptional divisor. Then there is a distinguished triangle

$$M(E) \to M(\mathrm{Bl}_Z X) \oplus M(Z) \to M(X) \to M(E)[1].$$

Moreover, if Z has codimension c in Z, the triangle yields a canonical isomorphism

$$M(\mathrm{Bl}_Z X) = M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)(i)[2i].$$

(6) (Duality) There is a duality $A \mapsto A^\vee$ that, for X smooth and projective of dimension d, satisfies

$$M(X)^\vee = M(X)(-d)[-2d].$$

(7) (Adjunction) The duality and tensor product are related by the adjunction formulas

$$\mathrm{Hom}(A \otimes B^\vee, C) = \mathrm{Hom}(A, C \otimes B),$$
$$\mathrm{Hom}(A \otimes B, C) = \mathrm{Hom}(B, A^\vee \otimes C).$$

Remark 4.59. We observe that the functor from $\mathbf{Sm}(k)$ to $\mathbf{DM}(k)$ is covariant, thus is a "homological" functor in contrast to the contravariant functor chosen by Grothendieck for pure motives that was cohomological.

Example 4.60. Let us use some of these properties to show that

$$M(\mathbb{P}^n) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2] \oplus \cdots \oplus \mathbb{Q}(n)[2n].$$

This should be compared with Example 2.86. We will proceed by induction on n, the case $n = 1$ being reduced to the definition of $\mathbb{Q}(1)$. For $n \geq 2$, the standard closed immersion $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ satisfies $\mathbb{P}^n \setminus \mathbb{P}^{n-1} = \mathbb{A}^n$. By the Gysin property, we have the distinguished triangle

$$M(\mathbb{A}^n) \to M(\mathbb{P}^n) \to M(\mathbb{P}^{n-1})(1)[2] \to M(\mathbb{A}^n)[1]. \tag{4.61}$$

Note that $M(\mathbb{A}^n) = \mathbb{Q}(0)$, as one can prove by repeatedly applying the $\mathbb{A}^1$-homotopy property. Moreover, the composition

$$M(\mathbb{A}^n) \to M(\mathbb{P}^n) \to M(\mathrm{Spec}(k))$$

is the identity $\mathbb{Q}(0) \to \mathbb{Q}(0)$. Thus, the triangle (4.61) is split and $M(\mathbb{P}^n) = \mathbb{Q}(0) \oplus M(\mathbb{P}^{n-1})(1)[2]$. The result follows by induction hypothesis.

Remark 4.62. To understand the different roles of the twist and the shift, it is instructive to compare the reduced motives of $\mathbb{P}^1$ and $\mathbb{G}_m$. In the first case, we have $\widetilde{M}(\mathbb{P}^1) = \mathbb{Q}(1)[2]$. For the second case, one can use the Mayer-Vietoris triangle for the open covering $\mathbb{P}^1 = U \cup V$, with $U = \mathbb{P}^1 \setminus \{0\}$ and $V = \mathbb{P}^1 \setminus \{\infty\}$. One gets an exact triangle

$$M(\mathbb{G}_m) \longrightarrow \mathbb{Q}(0) \oplus \mathbb{Q}(0) \longrightarrow \mathbb{Q}(1)[2] \longrightarrow M(\mathbb{G}_m)[1],$$

from which it follows that $M(\mathbb{G}_m) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[1]$, thus $\widetilde{M}(\mathbb{G}_m) = \mathbb{Q}(1)[1]$. This should be compared with the fact that, for any cohomology theory, $H^1(\mathbb{G}_m)$ and $H^2(\mathbb{P}^1)$ are isomorphic, but they lie in different degree. In particular, the Hodge structure $H^2(\mathbb{P}^1)$ is pure of weight 2 and Hodge type $(1,1)$. The same is true for $H^1(\mathbb{G}_m)$, but, since this last group lies in degree one, we consider it as a mixed Hodge structure.

4.3.8. *Motivic cohomology.* Voevodsky also computed some morphism groups in the category $\mathbf{DM}(k)$. In particular, he defined:

Definition 4.63. The *motivic cohomology* of X is

$$H^n_{\mathcal{M}}(X, \mathbb{Q}(p)) = \operatorname{Hom}_{\mathbf{DM}(k)}(M(X), \mathbb{Q}(p)[n]).$$

Using Bloch's formula relating higher Chow groups and K-theory he proves ([Voe02], [Blo86], [Lev94])

Theorem 4.64. *Given a smooth variety X, there is an isomorphism*

$$H^n_{\mathcal{M}}(X, \mathbb{Q}(p)) = (K_{2p-n}(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(p)},$$

where $K_\bullet(X)$ denote Quillen's K-theory groups of X and the index (p) means the eigenspace for the Adams operations.

4.3.9. *The normalization of a cosimplicial scheme.* To every variety X, not necessarily smooth, it is attached a motive $M(X)$ in Voevodsky's category. Using tools from homological algebra, one can construct more general motives, for instance starting from a cosimplicial variety.

Recall that in Section 3.5.3 we defined the normalized complex associated to a cosimplicial object in an abelian category. It turns out that it is enough to work in a pseudo-abelian category.

Lemma 4.65. *Let $X^\bullet$ be a cosimplicial object in $\mathbf{Sm}(k)$. Given integers $m > n \geq 0$, the following endomorphism in $\mathbf{SmCor}(k)$ is idempotent:*

$$p_n = (1 - \delta^0\sigma^0)(1 - \delta^1\sigma^1) \cdots (1 - \delta^n\sigma^n) : [X^m] \to [X^m].$$

Proof. We argue by induction on n. For $n = 0$, note that the relation $\sigma^0\delta^0 = \mathrm{Id}$ implies that $\delta^0\sigma^0$ is an idempotent, hence the same holds for $1 - \delta^0\sigma^0$. Let us now assume that p_{n-1} is idempotent. We next observe that for $i = 0, \dots, n-1$, the face σ^n commutes with $\delta^i\sigma^i$. Indeed, by relations (c) and (b) in (3.148),

$$\sigma^n(\delta^i\sigma^i) = \delta^i\sigma^{n-1}\sigma^i = (\delta^i\sigma^i)\sigma^n.$$

Moreover, relation (d) in (3.148) implies $\sigma^n(1-\delta^n\sigma^n) = 0$. These two equations together imply

$$\sigma^n(1-\delta^0\sigma^0)\cdots(1-\delta^n\sigma^n) = 0. \tag{4.66}$$

We now compute, using equation (4.66), and the induction hypothesis,

$$\begin{aligned} p_n^2 = \underbrace{(1-\delta^0\sigma^0)\cdots(1-\delta^{n-1}\sigma^{n-1})}_{p_{n-1}}(1-\delta^n\sigma^n) \\ \underbrace{(1-\delta^0\sigma^0)\cdots(1-\delta^{n-1}\sigma^{n-1})}_{p_{n-1}}(1-\delta^n\sigma^n) \\ = p_{n-1}^2(1-\delta^n\sigma^n) = p_{n-1}(1-\delta^n\sigma^n) = p_n, \end{aligned}$$

as we wanted to show. □

Since p_n is idempotent, $\operatorname{Im}(p_n)$ is an object of the pseudo-abelian enveloppe of $\mathbf{SmCor}(k)$. By convention, we write $p_{-1} = \mathrm{Id}$.

Definition 4.67. Let $X^\bullet$ be a cosimplicial object in $\mathbf{Sm}(k)$. The *normalization* of $X^\bullet$ is the complex in $\mathbf{SmCor}(k)_{\mathrm{pa}}$ given by

$$\mathcal{N}(X^\bullet)^n = \operatorname{Im}(p_{n-1}\colon [X^n] \to [X^n]),$$

together with the differential

$$d = \sum_{i=0}^{n+1}(-1)^i\delta^i\colon \mathcal{N}(X^\bullet)^n \to \mathcal{N}(X^\bullet)^{n+1}.$$

In general, the complex $\mathcal{N}(X^\bullet)$ is not bounded. To obtain a bounded complex, we consider the *bête* truncation $\sigma_{\le N}\mathcal{N}(X^\bullet)$, that is,

$$\sigma_{\le N}\mathcal{N}(X^\bullet)^n = \begin{cases} \mathcal{N}(X^\bullet)^n & n \le N, \\ 0 & n > N. \end{cases}$$

This is now an element of $C^b(\mathbf{SmCor}(k)_{\mathrm{pa}})$. For each $N \ge 0$, applying the functor (4.54), we obtain a motive

$$[\sigma_{\le N}\mathcal{N}(X^\bullet)].$$

Clearly, given integers $M \ge N \ge 0$, there is a morphism of complexes

$$\sigma_{\le M}\mathcal{N}(X^\bullet) \to \sigma_{\le N}\mathcal{N}(X^\bullet).$$

The system $([\sigma_{\le N}\mathcal{N}(X^\bullet)])_{N\ge 0}$ is a pro-object in $\mathbf{DM}(k)$.

Remark 4.68. The advantage of using Lemma 4.65 is that it provides us with an explicit idempotent cutting out the normalized complex from the cochain complex. However, we could have also constructed it directly by abstract means, as we now explain[11]. Recall that a category is said to be *preadditive* if the morphism sets are abelian groups and the composition of maps is bilinear. Given a preadditive category $\mathcal{A}$, let $\mathbf{Ab}(\mathcal{A})$ denote the category of presheaves of abelian groups on $\mathcal{A}$, by which we simply mean additive contravariant functors from $\mathcal{A}$ to $\mathbf{Ab}$. Then $\mathbf{Ab}(\mathcal{A})$ is an abelian category, and the Yoneda lemma ensures that the natural functor

$$h\colon \mathcal{A} \longrightarrow \mathbf{Ab}(\mathcal{A})$$

which sends X to $\mathrm{Hom}(-, X)$ is fully faithful. Assume now that $\mathcal{A}$ is pseudo-abelian. If Y' is a direct factor of an object of the form $h(X)$, then projecting to the complement one gets an idempotent p of $h(X)$ such that $Y' = \mathrm{Ker}(p)$. By fully-faithfulness, we can see p as an idempotent of X, and the object $Y = \mathrm{Ker}(p)$ in $\mathcal{A}$, determined up to unique isomorphism, satisfies $h(Y) = Y'$. If $X^\bullet$ is a cosimplicial object in $\mathcal{A}$, the associated cochain complex CX^* is a complex in $\mathcal{A}$ whose formation commutes with the functor h, in the sense that $h(CX^*) = C^*(h(X^\bullet))$. Since $\mathbf{Ab}(\mathcal{A})$ is abelian, the normalized complex $\mathcal{N}^*(h(X^\bullet))$, as introduced in Section 3.5.3, is a direct factor of $C^*(h(X^\bullet))$. Proceeding as above, one gets a complex (up to unique isomorphism) $\mathcal{N}X^*$ such that $h(\mathcal{N}X^*) = \mathcal{N}^*(h(X^\bullet))$.

4.3.10. *Hodge realization.* From now on, we assume that k has characteristic zero and comes with an embedding $k \hookrightarrow \mathbb{C}$. We end this section recalling the existence of the Hodge realization functor.

Theorem 4.69. *There is a covariant functor of $\mathbb{Q}$-linear rigid tensor triangulated categories*

$$R^{\mathrm{H}}\colon\ \mathbf{DM}(k) \longrightarrow D^b(\mathbf{MHS}(k)).$$

The proof of this theorem is sketched in [DG05, §1.5]. The main difficulty is the covariance of the de Rham complex for finite correspondences. A more detailed version of the argument is exposed in [Bou09].

We now give a sketch of the construction of the Hodge realization functor in the case of the motive $[\sigma_{\leq N}\mathcal{N}(X^\bullet)]$ from the previous section. Let $X^\bullet$ be a cosimplicial object in $\mathbf{Sm}(k)$. Assume that there is an embedding of cosimplicial smooth varieties over k,

$$j_\bullet\colon X^\bullet \to \overline{X}^\bullet,$$

such that all the $\overline{X}^n$ are smooth projective varieties and $D^n = \overline{X}^n \setminus X^n$ is a simple normal crossing divisor. The Hodge realization of $[\sigma_{\leq N}\mathcal{N}(X^\bullet)]$ is constructed as follows.

[11] We thank J. Ayoub for pointing this argument to us.

(1) *Betti part* R^{B}. For each n, let $\mathcal{C}^*(X^n(\mathbb{C}), \underline{\mathbb{Q}})$ be the Godement canonical flasque resolution of the locally constant sheaf $\mathbb{Q}$ of the complex manifold $X^n(\mathbb{C})$ and let $j_{n,*}\mathcal{C}^*(X^n(\mathbb{C}), \underline{\mathbb{Q}})$ be the complex of sheaves on $\overline{X}^n$ obtained by direct image by the inclusion $j_n \colon X_n \to \overline{X}_n$. On this complex of sheaves we put the canonical increasing filtration

$$W_m j_{n,*}\mathcal{C}^k(X^n(\mathbb{C}), \underline{\mathbb{Q}}) = \begin{cases} j_{n,*}\mathcal{C}^k(X^n(\mathbb{C}), \underline{\mathbb{Q}}), & \text{if } k < m, \\ \ker d, & \text{if } k = m, \\ 0, & \text{if } k > m. \end{cases}$$

we construct filtered acyclic resolutions $(K^*_{\mathrm{B},n}, W)$ of the complex $(j_{n,*}\mathcal{C}(X^n(\mathbb{C}), \underline{\mathbb{Q}}), W)$ in a functorial way. For instance using again the Godement canonical flasque resolution, this time on $\overline{X}^n$. Taking now global sections we obtain a filtered simplicial complex $(\Gamma(\overline{X}^\bullet, K^*_{\mathrm{B},n}), W)$. Finally, taking the normalization, the truncation and the total complex of the resulting double complex we obtain a filtered complex

$$(\operatorname{Tot} \sigma_{\leq N} \mathcal{N}\Gamma(\overline{X}^\bullet, K^*_{\mathrm{B},n}), W).$$

Finally, since we want the realization functor to be covariant, so we write

$$(R^{\mathrm{B}}(\sigma_{\leq N}\mathcal{N}X^\bullet), W) = (\operatorname{Tot} \sigma_{\leq N} \mathcal{N}\Gamma(\overline{X}^\bullet, K^*_{\mathrm{B},n}), W)^\vee. \tag{4.70}$$

Here it is important to note that the normalization of simplicial and cosimplicial objects are dual of each other.

(2) *de Rham part* R^{dR}. For each n let $\Omega^*_{\overline{X}^n}(\log D^n)$ be the de Rham complex of algebraic forms on $\overline{X}^n$ with logarithmic poles along D^n. This complex has a decreasing Hodge filtration F that counts the number of differentials and an increasing weight filtration W that counts the number of poles of a differential form. We construct an acyclic bifiltered resolution (K_{dR}, F, W) again in a functorial way. We now repeat the process done in the Betti case: we take global sections, the normalization and truncation on the simplicial direction, the total complex and the dual to obtain a bifiltered complex

$$(R^{\mathrm{dR}}(\sigma_{\leq N}\mathcal{N}X^\bullet), F, W) = ((\operatorname{Tot} \sigma_{\leq N} \mathcal{N}\Gamma(X^\bullet, K_{\mathrm{dR}})^\vee, F, W). \tag{4.71}$$

(3) *The comparison isomorphism.* Going to the cosimplicial complex manifold $\overline{X}^\bullet(\mathbb{C})$, we can construct a bifiltered complex

$$(R^{\mathrm{dR}}(\sigma_{\leq N}\mathcal{N}X^\bullet(\mathbb{C})), F, W),$$

that is the analogueue of the complex $(R^{\mathrm{dR}}(\sigma_{\leq N}\mathcal{N}X^\bullet), F, W)$ but using holomorphic forms. Then the maps

$$\begin{tikzcd}
(R^{\mathrm{B}}(\sigma_{\leq N}\mathcal{N}X^\bullet), W) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r] & (R^{\mathrm{dR}}(\sigma_{\leq N}\mathcal{N}X^\bullet(\mathbb{C})), W) \\
& (R^{\mathrm{dR}}(\sigma_{\leq N}\mathcal{N}X^\bullet), W) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[u]
\end{tikzcd}$$

are filtered quasi-isomorphisms giving the comparison isomorphism.

⋆ ⋆ ⋆

Exercise 4.72. Prove that the composition of the finite correspondences given by the graphs of two morphisms of algebraic varieties $f\colon X \to Y$ and $g\colon Y \to Z$, as defined in (4.51), is the graph of $g \circ f\colon X \to Z$.

Exercise 4.73. Let X be a smooth variety over k, together with a rational point $x\colon \operatorname{Spec}(k) \to X$. Consider the composition

$$p\colon X \longrightarrow \operatorname{Spec}(k) \xrightarrow{x} X.$$

(1) Show that p is a projector and that the class of $(X, 1-p)$ agrees with the reduced motive $\widetilde{M}(X)$ from Definition 4.56. Thus there is a decomposition

$$M(X) = \mathbb{Q}(0) \oplus \widetilde{M}(X).$$

(2) Show that the reduced motive $\widetilde{M}(X)$ is independent of the choice of the rational point x.

4.4. Mixed Tate motives. As was mentioned in the previous section, it is not known how to construct a motivic t-structure yielding the desired abelian category of mixed motives. However, when k is number field, one can extract from $\mathbf{DM}(k)$ an abelian category of mixed Tate motives with similar properties to mixed Hodge Tate structures. The keystone is Borel's computation of the K-theory of number fields.

4.4.1. *The derived category of mixed Tate motives.* The motives $\mathbb{Q}(n)$ are the simplest non-trivial objects of the category $\mathbf{DM}(k)$. It is thus reasonable to figure out what can be built starting from them.

Definition 4.74. The *derived category of mixed Tate motives over k* is the smallest triangulated subcategory $\mathbf{DMT}(k)$ of $\mathbf{DM}(k)$ containing the objects $\mathbb{Q}(n)$, for all $n \in \mathbb{Z}$, and stable under extensions.

Recall that the latter condition means that if $A \to B \to C \to A[1]$ is a distinguished triangle in $\mathbf{DM}(k)$ and two objects among A, B, C belong to $\mathbf{DMT}(k)$, then so does the third.

Thanks to the comparison between motivic cohomology and K-theory (Theorem 4.64), the extension groups of simple objects in the category $\mathbf{DMT}(k)$ are given by

$$\begin{aligned}\mathrm{Ext}^i(\mathbb{Q}(l),\mathbb{Q}(m)) &= \mathrm{Ext}^i(\mathbb{Q}(0),\mathbb{Q}(m-l))\\ &= \mathrm{Hom}_{\mathbf{DM}(k)}(M(\mathrm{Spec}(k)),\mathbb{Q}(m-l)[i])\\ &= (K_{2(m-l)-i}(k)\otimes\mathbb{Q})^{(m-l)}.\end{aligned}$$

The K-theory groups of general fields are still largely unknown, but, when k is a number field, Borel computed their ranks:

THEOREM 4.75 (Borel, [Bor74]). *Let k be a number field with r_1 (resp. $2r_2$) real (resp. complex) embeddings. Then:*

$$(K_{2(m-l)-i}(k)\otimes\mathbb{Q})^{(m-l)} = \begin{cases}\mathbb{Q}, & \textit{if } i=0,\ m-l=0,\\ k^\times\otimes_{\mathbb{Z}}\mathbb{Q}, & \textit{if } i=1,\ m-l=1,\\ \mathbb{Q}^{r_1+r_2}, & \textit{if } i=1,\ m-l\geq 3 \textit{ odd},\\ \mathbb{Q}^{r_2}, & \textit{if } i=1,\ m-l\geq 2 \textit{ even},\\ 0, & \textit{otherwise}.\end{cases}$$

The important information we should get from this is

(1) the only non-zero groups Ext^i occur for $i=0,1$;
(2) $\mathrm{Ext}^0(\mathbb{Q}(l),\mathbb{Q}(m))=\mathrm{Hom}(\mathbb{Q}(l),\mathbb{Q}(m))=0$ unless $m=l$, for which it is equal to $\mathbb{Q}$;
(3) if $\mathrm{Ext}^1(\mathbb{Q}(l),\mathbb{Q}(m))\neq 0$, then $m>l$;
(4) the only infinite-dimensional group is $\mathrm{Ext}^1(\mathbb{Q}(l),\mathbb{Q}(l+1))$.

In particular, when $k=\mathbb{Q}$, we have $r_1=1$ and $r_2=0$, so

$$\mathrm{Ext}^1_{\mathbf{DMT}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) = \begin{cases}\mathbb{Q}^\times\otimes_{\mathbb{Z}}\mathbb{Q} & \text{if } n=1,\\ \mathbb{Q} & \text{if } n\geq 3 \text{ odd}\\ 0 & \text{otherwise}.\end{cases}$$

This and the fact that $\mathrm{Ext}^i_{\mathbf{DMT}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n))=0$ for $i\geq 2$ will determine the structure of the category of mixed Tate motives over $\mathbb{Q}$.

EXAMPLE 4.76 (Kummer motives). Since

$$\mathrm{Ext}^1_{\mathbf{DMT}(k)}(\mathbb{Q}(0),\mathbb{Q}(1)) = k^\times\otimes_{\mathbb{Z}}\mathbb{Q},$$

there are plenty of non-trivial extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. They are all rational linear combinations of Kummer motives. For each $t\in k^\times\setminus\{1\}$, consider the complex K_t in $\mathbf{SmCor}(k)$ given by

$$\mathrm{Spec}(k)\oplus\mathrm{Spec}(k)\xrightarrow{f_t}\mathbb{P}^1_k\setminus\{0,\infty\},$$

where $\mathrm{Spec}(k)\oplus\mathrm{Spec}(k)=\{*_1,*_2\}$ sits in degree 0 and the finite correspondence f_t is defined by the cycle $[(*_1,t)]-[(*_2,1)]$.

The class of K_t in $\mathbf{DM}(k)$ belongs to $\mathbf{DMT}(k)$ and the degrees are chosen so that it belongs to $\mathbf{MT}(k)$. The Kummer motive K_t^{Mot} is the class of K_t in $\mathbf{MT}(k)$. For $t = 1$ we write K_1^{Mot} for the trivial extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$. The Hodge realization of the Kummer motive is the Kummer mixed Hodge structure of Example 2.134.

Another well understood case is the K-theory of finite fields, which was completely computed by Quillen in [**Qui72**, Thm. 8], shortly after he introduced the definition of higher algebraic K-theory:

Theorem 4.77 (Quillen, [**Qui72**]). *Let $\mathbb{F}_q$ be the finite field with q elements. Then:*

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}/(q^n - 1) & i = 2n - 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Conjecture 4.78 (Beilinson-Soulé). *If k is a field, then $K_n(k)_{\mathbb{Q}}^{(r)}$ vanishes for all $n > 2r$.*

An immediate corollary of Borel and Quillen's theorems is:

Corollary 4.79. *The Beilinson-Soulé conjecture holds when k is either a number field or a finite field.*

4.4.2. *A t-structure on* $\mathbf{DMT}(k)$ *after Levine* [**Lev93**]. For each pair of integers a and b, let us denote by $\mathcal{T}_{[a,b]}$ the strictly full triangulated subcategory of $\mathbf{DMT}(k)$ generated by the objects $\mathbb{Q}(n)$ for $a \leq -2n \leq b$. We denote $\mathcal{T}_{[a,a]}$ simply by $\mathcal{T}_a$, and we extend the definition to cover the cases $a = -\infty$ or $b = \infty$ as well. In particular, $\mathcal{T}_{(-\infty,\infty)} = \mathbf{DMT}(k)$.

Lemma 4.80. *Let $a \leq b \leq c$ be integers (the cases $a = -\infty$ and $c = \infty$ are also allowed). Then $(\mathcal{T}_{[a,b-1]}, \mathcal{T}_{[b,c]})$ is a t-structure on $\mathcal{T}_{[a,c]}$.*

In particular, for each integer b, the pair $(\mathcal{T}_{(-\infty,b]}, \mathcal{T}_{[b+1,\infty)})$ provides a t-structure on $\mathbf{DMT}(k)$. Let us emphasize that this is not the t-structure we are looking for, since its heart is reduced to zero. However, it will allow us to define a *weight structure*.

The truncation functors for the t-structure $(\mathcal{T}_{(-\infty,b]}, \mathcal{T}_{[b+1,\infty)})$ on $\mathbf{DMT}(k)$ will be denoted by

$$W_{\leq b} \colon \mathbf{DMT}(k) \longrightarrow \mathcal{T}_{(-\infty,b]}$$
$$W^{>b} \colon \mathbf{DMT}(k) \longrightarrow \mathcal{T}_{[b+1,\infty)}.$$

The reason for the subindex or superindex is that one will give an increasing filtration whereas the other will give a decreasing filtration.

Let $W^{\geq b}$ denote $W^{>b-1}$ and define

$$\mathrm{Gr}_b^W(M) = W^{\geq b} W_{\leq b}(M).$$

For each even integer a, let $\mathcal{T}_a^{\leq 0}$ (resp. $\mathcal{T}_a^{\geq 0}$) be the full subcategory of $\mathcal{T}_a$ generated by $\mathbb{Q}(-a/2)[n]$ for $n \leq 0$ (resp. $n \geq 0$). Finally, let $\mathcal{T}_{[a,b]}^{\leq 0}$ (resp. $\mathcal{T}_{[a,b]}^{\geq 0}$) be the full subcategory of $\mathcal{T}_{[a,b]}$ generated by the objects M such that $\mathrm{Gr}_c^W(M)$ belongs to $T_c^{\leq 0}$ (resp. $\mathcal{T}_c^{\geq 0}$) for all $a \leq c \leq b$.

THEOREM 4.81 (Levine). *Assume that the field k satisfies the Beilinson-Soulé conjecture. Then the pair of strictly full subcategories*

$$(\mathcal{T}_{(-\infty,\infty)}^{\leq 0}, \mathcal{T}_{(-\infty,\infty)}^{\geq 0})$$

forms a non-degenerate t-structure on $\mathbf{DMT}(k)$.

DEFINITION 4.82. The category $\mathbf{MT}(k)$ of *mixed Tate motives over k* is the heart of the above t-structure.

The category $\mathbf{MT}(k)$ has the following properties:

(1) It is a neutral Tannakian category generated under extensions by the objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.
(2) Each object M of $\mathbf{MT}(k)$ has an increasing *weight filtration* $W_\bullet M$ such that
$$\mathrm{Gr}_{2n}^W M \simeq \mathbb{Q}(n)^{\oplus k_n}, \qquad \mathrm{Gr}_{2n+1}^W = 0$$
for some natural numbers k_n.
(3) A fibre functor is given by
$$\omega(M) = \bigoplus_n \mathrm{Hom}(\mathbb{Q}(n), \mathrm{Gr}_{2n}^W M). \tag{4.83}$$

Moreover, Wildeshaus [Wil09, Théorème 1.3] proved that there exists a canonical equivalence of categories

$$F\colon D^b(\mathbf{MT}(k)) \longrightarrow \mathbf{DMT}(k). \tag{4.84}$$

The functor F is t-exact, induces the identity on the heart $\mathbf{MT}(k)$, and has the property that the composition with the cohomology functor H^0 associated to the t-structure as in (4.40) coincides with the canonical cohomology functor $D^b(\mathbf{MT}(k)) \to \mathbf{MT}(k)$. In view of Remark 4.38, the main difficulty does not lie in proving that the two categories are equivalent but in constructing a functor between them.

4.4.3. *Examples.* If the motive of a variety X is of mixed Tate type, *i.e.* belongs to $\mathbf{DMT}(k)$, then decomposing $M(X)$ (or rather its dual) by means of Levine's t-stucture we obtain the cohomology motives

$$h^i(X) = t_{\leq 0} t_{\geq 0}(M(X)^\vee[i]) \in \mathbf{MT}(k).$$

Thus we can isolate the different cohomological degrees, something we do not know how to do for general motives.

EXAMPLE 4.85. By Example 4.60, the motive of the projective space $M(\mathbb{P}^n_k)$ is of mixed Tate type and one has

$$h^i(\mathbb{P}^n_k) = \begin{cases} \mathbb{Q}(-m) & i = 2m,\ 0 \le m \le n \\ 0 & \text{otherwise.} \end{cases}$$

Using properties of $\mathbf{DMT}(k)$ such as the homotopy invariance or the long exact sequence of a closed immersion, we can show that certain motives are mixed Tate. For instance, if a variety X possesses a stratification such that the motive of each locally closed stratum is mixed Tate, then the whole $M(X)$ is a mixed Tate motive.

EXAMPLE 4.86. Let $n \ge 3$ be an integer and consider the moduli space $M_{0,n}$ of distinct n-points in $\mathbb{P}^1$. It is a smooth variety of dimension $n-3$ which is defined over $\mathbb{Q}$. Since any three points can be sent to $0, 1, \infty$ by a projective transformation, one has $M_{0,3} = \mathrm{Spec}(\mathbb{Q})$ and $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. In general, $M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus$ diagonals. We will write elements of $M_{0,n}$ as tuples $(0, 1, \infty, x_4, \ldots, x_n)$.

Let us show by induction that $M(M_{0,n})$ belongs to $\mathbf{DMT}(\mathbb{Q})$. The result is clear for $n = 3$ and 4. For $n \ge 5$, we can decompose $M_{0,n}$ as follows:

$$M_{0,n} \simeq (M_{0,4} \times M_{0,n-1}) \setminus \bigsqcup_{i=5}^{n} \{x_i = x_4\}.$$

By the Künneth formula and the induction hypothesis, the motive of $X = M_{0,4} \times M_{0,n-1}$ belongs to $\mathbf{DMT}(\mathbb{Q})$. The same is true for the motive of $Z = \bigsqcup_{i=5}^{n} \{x_i = x_4\}$. Now the Gysin triangle reads

$$M(M_{0,n}) \to M(X) \to M(Z)(1)[2] \to M(M_{0,n})[1]$$

and since $M(X)$ and $M(Z)(1)[2]$ belong to $\mathbf{DMT}(\mathbb{Q})$, so does $M(M_{0,n})$.

EXAMPLE 4.87. Let $L = L_0 \cup \cdots \cup L_n$ and $M = M_0 \cup \cdots \cup M_n$ be hyperplanes in the projective space $\mathbb{P}^n$. Assume that they are in general position, meaning that the divisor $L \cup M$ has normal crossings. Then the following motive belongs to $\mathbf{MT}(k)$:

$$H^2(\mathbb{P}^n \setminus L, M \setminus (M \cap L)).$$

4.4.4. *Realizations.* Recall that in Definition 2.94 we introduced a category $\mathbf{MHTS}(\mathbb{Q})$ of mixed Hodge Tate structures over $\mathbb{Q}$. Then the functor R^{H} of Theorem 4.69 restricts to a functor

$$\mathbf{DMT}(\mathbb{Q}) \to D^b(\mathbf{MHTS}(\mathbb{Q})).$$

As explained in Example 4.34, the category appearing on the right-hand side has a canonical t-structure. We have also defined a t-structure on $\mathbf{DMT}(\mathbb{Q})$. Since it is motivic, any realization functor is t-exact in the sense of Definition 4.31, hence restricts to a functor on the hearts. Specializing to R^{H}, we obtain a functor from $\mathbf{MT}(\mathbb{Q})$ to $\mathbf{MHS}(\mathbb{Q})$. Taking into account that the Hodge realization of a mixed Tate motive is a mixed Hodge Tate structure, we actually get a functor

$$R^{\mathrm{H}}\colon \mathbf{MT}(\mathbb{Q}) \longrightarrow \mathbf{MHTS}(\mathbb{Q}) \tag{4.88}$$

which respects the weight filtrations.

It is important to note that the category $\mathbf{MHTS}(\mathbb{Q})$ is much bigger than $\mathbf{MT}(\mathbb{Q})$. For instance compare the set of extensions of $\mathbb{Q}(m)$ and $\mathbb{Q}(n)$ in the category $\mathbf{MHTS}(\mathbb{Q})$ given by Theorem 2.130, that is uncountable, with the set of extensions in $\mathbf{MT}(\mathbb{Q})$ given by Theorem 4.75, that is countable. Thus it is important to know which mixed Hodge structures come from geometry. This leads to the precise meaning to the word "motivic" when speaking about a mixed Hodge Tate structure:

Definition 4.89. We say that a mixed Hodge Tate structure over $\mathbb{Q}$ is *motivic* if it lies in the essential image of the functor R^{H}. The same definition applies to pro-mixed Hodge Tate structures. More generally, we say that a diagram of pro-mixed Hodge Tate structures is motivic if it is isomorphic to the image by the functor R^{H} of a diagram of pro-mixed Tate motives.

Even if $\mathbf{MHTS}(\mathbb{Q})$ is much bigger than $\mathbf{MT}(\mathbb{Q})$, the realization functor between them is fully faithful and stable by subobjects. This is a very useful result to prove that many mixed Hodge structures have motivic origin. We should mention that to determine whether the Hodge realization functor from the hypotetical category of mixed motives is fully faithful (*i.e.* bijective on Hom sets) would be a extremely difficult problem. For instance, if one restricts to the category of pure motives it amounts to the Hodge conjecture. That we can do it for $\mathbf{MT}(\mathbb{Q})$ relies again on Borel's results about the K-theory of number fields.

Proposition 4.90 (Deligne-Goncharov). *The realization functor* (4.88) *is fully faithful and its essential image is stable under subobjects.*

Proof. The key point of the argument is that the realization functor R^{H} determines injections

$$\operatorname{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) \longrightarrow \operatorname{Ext}^1_{\mathbf{MHS}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) \tag{4.91}$$

into the extension groups which were computed in Theorem 2.130. For $n=1$, this follows from the injectivity of

$$\log|\cdot|\colon \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{C}/2\pi i\mathbb{Q}.$$

For $n>1$, the injectivity follows by interpreting $\operatorname{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n))$ as a part of the motivic cohomology of $\operatorname{Spec}(\mathbb{Q})$, which can be computed using K-theory:

$$\operatorname{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) = H^1_{\mathcal{M}}(\operatorname{Spec}(\mathbb{Q}),\mathbb{Q}(n)) = K_{2n-1}(\mathbb{Q})\otimes\mathbb{Q},$$

then interpreting $\operatorname{Ext}^1_{\mathbf{MHS}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n))$ as Deligne cohomology groups:

$$\operatorname{Ext}^1_{\mathbf{MHS}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)) = H^1_{\mathcal{D}}(\operatorname{Spec}(\mathbb{Q}),\mathbb{Q}(n)).$$

Under this interpretation, the realization map (4.91) should correspond to the Borel regulator map mentioned in Digression 1.14, which is known to be injective by the work of Borel.

Consider now the fibre functors ω_{dR} on $\mathbf{MHS}(\mathbb{Q})$ (Definition 2.93) and ω on $\mathbf{MT}(\mathbb{Q})$ (4.83). These fibre functors are compatible and induce maps at the level of Tannaka groups

$$G^{\mathrm{H}}_{\omega_{\mathrm{dR}}} = \underline{\mathrm{Aut}}^{\otimes}_{\mathbf{MHTS}(\mathbb{Q})}(\omega_{\mathrm{dR}}) \to \underline{\mathrm{Aut}}^{\otimes}_{\mathbf{MT}(\mathbb{Q})}(\omega) = G_{\omega}. \tag{4.92}$$

By the Tannakian dictionary, the functor R^H is fully faithful if and only if the morphism (4.92) is surjective.

To show this we argue as follows: both $G^{\mathrm{H}}_{\omega_{\mathrm{dR}}}$ and G_ω can be written as the semidirect product of $\mathbb{G}_m$ and a pro-unipotent group.

$$G^{\mathrm{H}}_{\omega_{\mathrm{dR}}} = U^{\mathrm{H}}_{\omega_{\mathrm{dR}}} \rtimes \mathbb{G}_m, \quad G_\omega = U_\omega \rtimes \mathbb{G}_m.$$

Then the injectivity of (4.91) implies the surjectivity of (4.92) (see the proof of Theorem 4.124 for the precise relationship between the Ext groups and the Lie algebra of U_ω). □

EXAMPLE 4.93. Let $n > 0$ be an even integer and H a mixed Hodge structure over $\mathbb{Q}$ that is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$. If this extension is non-trivial then it is not motivic over $\mathbb{Q}$, in the sense that it can not be the Hodge realization of a motive over $\mathbb{Q}$. Indeed, assume that there is a mixed Tate motive over $\mathbb{Q}$ whose Hodge realization is H. Since the realization functor is fully faithful, from the exact sequence

$$0 \to \mathbb{Q}(n) \to H \to \mathbb{Q}(0) \to 0$$

corresponds an exact sequence of mixed Tate motives

$$0 \to \mathbb{Q}(n) \to M \to \mathbb{Q}(0) \to 0.$$

Since $\mathrm{Ext}^1_{\mathbf{DMT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ this extension is split. Hence the sequence of mixed Hodge structures is also split.

Of course, there exist motivic non-trivial extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ defined over non-totally real number fields.

⋆⋆⋆

EXERCISE 4.94. Prove that the pair of subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of Example 4.34 forms indeed a t-structure.

EXERCISE 4.95. Let $\mathrm{Gr}(d, n)$ be the Grassmanian scheme of d-planes in k^n. Show that the motive of $\mathrm{Gr}(d, n)$ belongs to $\mathbf{DMT}(k)$.

4.5. Mixed Tate motives over $\mathbb{Z}$. From now on, we specialize further to the case $k = \mathbb{Q}$. The category $\mathbf{MT}(\mathbb{Q})$ is still too big for our purposes since the extension group

$$\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(1)) \simeq \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \bigoplus_{p \text{ prime}} \mathbb{Q}$$

is infinite-dimensional. To remedy this, Goncharov [Gon01, §3] introduced a subcategory of "mixed Tate motives over $\mathbb{Z}$".

4.5.1. *Definition and basic properties.*

Definition 4.96. A motive M in $\mathbf{MT}(\mathbb{Q})$ is said to be *everywhere unramified* if, given any integer n, there is no subquotient E of M which fits into a non-split extension $0 \to \mathbb{Q}(n+1) \to E \to \mathbb{Q}(n) \to 0$. The full subcategory $\mathbf{MT}(\mathbb{Z})$ of $\mathbf{MT}(\mathbb{Q})$ consisting of everywhere unramified motives is called the category of *mixed Tate motives over* $\mathbb{Z}$.

To a motive M over $\mathbb{Q}$ and a prime number ℓ, we can associate the ℓ-adic realization of M. For instance, to the motive corresponding to a smooth variety X over $\mathbb{Q}$ we associate the dual of the ℓ-adic cohomology $H^*_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. The ℓ-adic realization is a $\mathbb{Q}_\ell$-vector space, together with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let p be a prime number distinct from ℓ. The choice of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and a field embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ allows one to see the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\overline{\mathbb{Q}}_p)$ as a subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By restriction, we obtain a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Recall that the Galois group of the maximal unramified extension $\mathbb{Q}_p \subset \mathbb{Q}_p^{\mathrm{ur}} \subset \overline{\mathbb{Q}}_p$ is isomorphic to $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. The *inertia subgroup* I_p is defined by

$$1 \to I_p \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1.$$

Definition 4.97. Let $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(V)$ be an ℓ-adic representation, and p a prime number distinct from ℓ. We say that ρ is *unramified* at p if its restriction to the inertia subgroup $I_p \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is trivial.

We have at our disposal the following criterion to decide whether a mixed Tate motive over $\mathbb{Q}$ belongs to $\mathbf{MT}(\mathbb{Z})$.

Proposition 4.98 (Deligne-Goncharov). *A mixed Tate motive M over $\mathbb{Q}$ belongs to $\mathbf{MT}(\mathbb{Z})$ if and only if, for each prime number p, there exists a prime $\ell \neq p$ such that the ℓ-adic realization $\omega_\ell(M)$ is unramified at p.*

Proof. See [DG05, Prop. 1.8]. □

Example 4.99. Let K_t^{Mot} be the Kummer motive associated to an element $t \in \mathbb{Q}^\times$ as in Example 4.76. For each prime ℓ, the ℓ-adic realization of K_t^{Mot} is the extension

$$0 \to \mathbb{Q}_\ell(1) \to K_t^\ell \xrightarrow{f} \mathbb{Q}(0) \to 0$$

corresponding to the $\mathbb{Q}_\ell(1)$-torsor given by the projective limit of ℓ^n-th roots of unity of t. This is unramified everywhere if and only if $t \in \mathbb{Z}^\times$. Thus, taking into account that $\mathbb{Z}^\times \otimes_\mathbb{Z} \mathbb{Q} = 0$, the only Kummer motive that belongs to $\mathbf{MT}(\mathbb{Z})$ is the trivial one K_1^{Mot}. This solves the problem of the extension groups being infinite-dimensional.

The main properties of the category $\mathbf{MT}(\mathbb{Z})$ are summarized in the following theorem

Theorem 4.100.

(1) $\mathbf{MT}(\mathbb{Z})$ *is a Tannakian category generated by the objects* $\mathbb{Q}(n)$ *for all integers* $n \in \mathbb{Z}$.

(2) *Each object* M *of* $\mathbf{MT}(\mathbb{Z})$ *has a canonical increasing weight filtration* W *indexed by even integers, and such that*

$$\mathrm{Gr}^W_{2n} M \cong \mathbb{Q}(-n)^{\oplus k_n}$$

for some integers $k_n \geq 0$.

(3) *The extension groups in the category* $\mathbf{MT}(\mathbb{Z})$ *are given by*

$$\mathrm{Ext}^i_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(l), \mathbb{Q}(m)) = \begin{cases} \mathbb{Q}, & \text{if } i = 0,\ m - l = 0, \\ \mathbb{Q}, & \text{if } i = 1,\ m - l \geq 3 \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Hence all of them are finitely dimensional.

Since $\mathbf{MT}(\mathbb{Z}) \subset \mathbf{MT}(\mathbb{Q})$ is stable under subobjects, we immediately deduce from Proposition 4.90:

Corollary 4.101. *The realization functor*

$$R\colon \mathbf{MT}(\mathbb{Z}) \to \mathbf{MHTS}(\mathbb{Q})$$

is fully faithful with essential image stable under subobjects.

4.5.2. *Fibre functors.* In this section, we introduce various fibre functors on the category $\mathbf{MT}(\mathbb{Z})$ and compute the corresponding Tannaka groups. The first one is defined using the weight structure on $\mathbf{MT}(\mathbb{Z})$ given by part (2) of Theorem 4.100. For each motive M in $\mathbf{MT}(\mathbb{Z})$ and each integer $n \in \mathbb{Z}$, we write

$$\omega_n(M) = \mathrm{Hom}_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(n), \mathrm{Gr}^W_{-2n}(M))$$

and define a fibre functor $\omega\colon \mathbf{MT}(\mathbb{Z}) \to \mathrm{Vec}_\mathbb{Q}$ by

$$\omega(M) = \bigoplus_n \omega_n(M). \tag{4.102}$$

Observe that ω factors through the category of graded $\mathbb{Q}$-vector spaces.

From the Hodge realization of a motive we obtain two fibre functors. The de Rham fibre functor, denoted by ω_{dR}, is the de Rham part of the Hodge structure. For a motive $M \in \mathbf{MT}(\mathbb{Z})$, the vector space $\omega_{\mathrm{dR}}(M)$

comes equipped with two filtrations, the decreasing Hodge filtration F, and the increasing weight filtration W. Since $(\omega_{\mathrm{dR}}(M), F, W)$ is part of a mixed Tate Hodge structure, these filtrations are opposed in the sense that, if we write

$$\omega_{\mathrm{dR}}(M)^n = F^{-n}\omega_{\mathrm{dR}}(M) \cap W_{-2n}\omega_{\mathrm{dR}}(M),$$

then

$$\begin{aligned} \omega_{\mathrm{dR}}(M) &= \bigoplus_n \omega_{\mathrm{dR}}(M)^n, \\ F^{-p}\omega_{\mathrm{dR}}(M) &= \bigoplus_{m \leq p} \omega_{\mathrm{dR}}(M)^m, \\ W_{-2n}\omega_{\mathrm{dR}}(M) &= \bigoplus_{m \geq n} \omega_{\mathrm{dR}}(M)^m. \end{aligned}$$

Thus the de Rham fibre functor ω_{dR} also factors through the category of graded vector spaces.

Lemma 4.103. *The de Rham fibre functor ω_{dR} is canonically isomorphic to the fibre functor ω.*

There is also a Betti fibre functor ω_{B} given by the Betti part of the Hodge realization. The rational vector space ω_{B} is provided with a weight filtration W, but not a Hodge filtration. Note that ω_{B} does not factor canonically through the category of graded vector spaces.

Finally there is a comparison isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} \colon \omega_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \omega_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.104}$$

Example 4.105. In this example we compute explicitly the de Rham and Betti realizations of $\mathbb{Q}(1)$ and the comparison isomorphism. First we need a variety whose motive contains $\mathbb{Q}(1)$. Let

$$X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, \infty\} = \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \mathbb{G}_{m,\mathbb{Q}} = \mathrm{Spec}(\mathbb{Q}[x, x^{-1}]).$$

Recall from Remark 4.62 that $M(X) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[1]$, hence

$$t_0(M(X)[-i]) = \begin{cases} \mathbb{Q}(i), & \text{if } i = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

We already have a nice compactification $X \subset \mathbb{P}^1_{\mathbb{Q}}$. We can write down explicitly the complex of differential forms on $\mathbb{P}^1_{\mathbb{Q}}$ with logarithmic poles along $\{0, \infty\}$. The sheaf $\Omega^0_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0, \infty\})$ is $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}$, the sheaf of rational functions on $\mathbb{P}^1_{\mathbb{Q}}$. The sheaf $\Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0, \infty\})$ is the $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}$-module generated by the differential form $\frac{dx}{x} = -\frac{dx^{-1}}{x^{-1}}$. Thus, as a sheaf, is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}$. Since

$$H^i(\mathbb{P}^1_{\mathbb{Q}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}) = 0, \quad \text{for } i > 0,$$

there is no need to search for a resolution of the complex $\Omega^*_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0,\infty\})$ and we can use directly the complex of global sections to compute de Rham cohomology. We have

$$\Gamma(\mathbb{P}^1_{\mathbb{Q}}, \Omega^0_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0,\infty\})) = \mathbb{Q}[x, x^{-1}],$$

$$\Gamma(\mathbb{P}^1_{\mathbb{Q}}, \Omega^1_{\mathbb{P}^1_{\mathbb{Q}}}(\log\{0,\infty\})) = \mathbb{Q}[x, x^{-1}]\frac{dx}{x}.$$

The differential map is given by $dx^n = nx^{n-1}$. Hence

$$H^0_{\mathrm{dR}}(X) = \mathbb{Q}, \qquad H^1_{\mathrm{dR}}(X) = \mathbb{Q}\frac{dx}{x}.$$

Therefore

$$\omega_{\mathrm{dR}}(\mathbb{Q}(1)) = \left(\mathbb{Q}\frac{dx}{x}\right)^{\vee}.$$

Thus $\omega_{\mathrm{dR}}(\mathbb{Q}(1))$ is a one dimensional vector space and we have identified a canonical generator $(dx/x)^{\vee}$.

The Betti realization is given by the singular homology of the space of complex points. Thus

$$\omega_{\mathrm{B}}(\mathbb{Q}(1)) = H_1(\mathbb{C}\setminus\{0\}, \mathbb{Q})$$

This is again a rational vector space of dimension 1. A generator of it is given by the unit circle traveled in the counterclockwise direction, that we denote γ.

The comparison isomorphism is obtained from the integration of differential forms along singular chains. Since

$$\int_{\gamma} \frac{dx}{x} = 2\pi i$$

we deduce that $\mathrm{comp}_{\mathrm{dR,B}}(\gamma) = (dx/x)^{\vee} \otimes (2\pi i)$.

4.5.3. *Tannaka groups of* $\mathbf{MT}(\mathbb{Z})$. We now turn to the description of the affine group schemes associated to the various fibre functors on the category of mixed Tate motives over $\mathbb{Z}$.

Notation 4.106. The following notation will be used throughout:

$$G_{\mathrm{dR}} = \underline{\mathrm{Aut}}^{\otimes}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{dR}}), \tag{4.107}$$

$$G_{\mathrm{B}} = \underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{B}}), \tag{4.108}$$

$$P_{\mathrm{B,dR}} = \underline{\mathrm{Isom}}^{\otimes}(\omega_{dR}, \omega_{\mathrm{B}}), \tag{4.109}$$

$$P_{\mathrm{dR,B}} = \underline{\mathrm{Isom}}^{\otimes}(\omega_{\mathrm{B}}, \omega_{dR}). \tag{4.110}$$

Observe that both $P_{\mathrm{B,dR}}$ and $P_{\mathrm{dR,B}}$ are G_{dR}-torsors and $\mathrm{comp}_{\mathrm{B,dR}}$ (resp. $\mathrm{comp}_{\mathrm{dR,B}}$) is a complex point of $P_{\mathrm{B,dR}}$ (resp. $P_{\mathrm{dR,B}}$).

In what follows, we will use the subscript dR/B for properties which are common to G_{dR} and G_{B}.

Lemma 4.111. *The groups $G_{\mathrm{dR/B}}$ fit into an exact sequence*

$$1 \longrightarrow U_{\mathrm{dR/B}} \longrightarrow G_{\mathrm{dR/B}} \longrightarrow \mathbb{G}_m \longrightarrow 1, \tag{4.112}$$

where $U_{\mathrm{dR/B}}$ is a pro-unipotent group.

Proof. Recall that the category $\mathbf{MT}(\mathbb{Z})$ contains the object $\mathbb{Q}(1)$. Since $\omega_{\mathrm{dR/B}}(\mathbb{Q}(1))$ is a one-dimensional $\mathbb{Q}$-vector space, we obtain a morphism

$$t_{\mathrm{dR/B}}\colon G_{\mathrm{dR/B}} \to \mathrm{GL}(\omega_{\mathrm{dR/B}}(\mathbb{Q}(1))) = \mathbb{G}_m. \tag{4.113}$$

We define $U_{\mathrm{dR/B}}$ as the kernel of this morphism.

Since the action of $G_{\mathrm{dR/B}}$ is compatible with the tensor product, an element $g \in G_{\mathrm{dR/B}}$ acts on $\omega_{\mathrm{dR/B}}(\mathbb{Q}(n))$ as $t_{\mathrm{dR/B}}(g)^n$. Since the weight filtration is a filtration in the category of motives, $G_{\mathrm{dR/B}}$ respects the weight filtration. This means that, if $g \in G_{\mathrm{dR/B}}$ and $X \in \mathrm{Ob}(\mathbf{MT}(\mathbb{Z}))$, the action of g in $\omega_{\mathrm{dR/B}}(X)$ sends $W_n\omega_{\mathrm{dR/B}}(X) = \omega_{\mathrm{dR/B}}(W_nX)$ to $W_n\omega_{\mathrm{dR/B}}(X)$. Therefore, it acts on $\mathrm{Gr}_n^W \omega_{\mathrm{dR/B}}(X)$. Since $\mathrm{Gr}_n^W \omega_{\mathrm{dR/B}}(X)$ is a sum of copies of $\omega_{\mathrm{dR/B}}(\mathbb{Q}(n))$, g acts on $\mathrm{Gr}_n^W \omega_{\mathrm{dR/B}}(X)$ as $t_{\mathrm{dR/B}}(g)^n$ and the action of an element $u \in U_{\mathrm{dR/B}}$ on the same space is trivial. This implies that $U_{\mathrm{dR/B}}$ is a pro-unipotent group, that is, an inverse limit of unipotent affine algebraic groups. □

At this level, an advantage of using the de Rham fibre functor $\omega = \omega_{\mathrm{dR}}$ instead of the Betti one ω_{B} is that the exact sequence (4.112) admits a canonical splitting $\tau\colon \mathbb{G}_m \to G_{\mathrm{dR}}$. Indeed:

Lemma 4.114. *One has*

$$G_{\mathrm{dR}} = U_{\mathrm{dR}} \rtimes \mathbb{G}_m.$$

Proof. We use the fact that $\omega = \omega_{\mathrm{dR}}$ factors through the category of graded vector spaces. Given $t \in \mathbb{G}_m$, let $\tau(t) \in G_{\mathrm{dR}}$ denote the element that acts as multiplication by t^n on ω_n. This defines a section $\tau\colon \mathbb{G}_m \to G_{\mathrm{dR}}$ of t_{dR}. Hence G_{dR} is a semidirect product. □

Corollary 4.115. *Any G_{dR}-torsor is trivial.*

Proof. We assume that the reader is familiar with the vanishing of the Galois cohomology groups

$$H^1(\mathbb{Q}, \mathbb{G}_m) = H^1(\mathbb{Q}, \mathbb{G}_a) = 0$$

(see for instance [**Wat79**, 18.2] or [**Ser94**, Chap. II, §1.2, Prop. 1]). It follows that, for any unipotent group U or any group G that is an extension of $\mathbb{G}_m$ by U, the Galois cohomology groups are also trivial

$$H^1(\mathbb{Q}, U) = H^1(\mathbb{Q}, G) = 0.$$

Now, the group G_{dR} can be written as

$$G_{\mathrm{dR}} = \varprojlim_N G_{\mathrm{dR}}^N,$$

where each G_{dR}^N is an extension of $\mathbb{G}_m$ by a unipotent group and all the transition maps are surjective. By Mittag-Leffler we deduce that

$$H^1(\mathbb{Q}, G_{\mathrm{dR}}) = \varprojlim_N H^1(\mathbb{Q}, G_{\mathrm{dR}}^N) = 0,$$

which implies that any G_{dR}-torsor defined over $\mathbb{Q}$ is trivial. □

The corollary has the important following consequence, which will be exploited in the next chapter.

PROPOSITION 4.116. *There exists an element $a \in G_{\mathrm{dR}}(\mathbb{C})$ such that, for all motives M of $\mathbf{MT}(\mathbb{Z})$, one has*

$$\omega_B(M) = (\mathrm{comp}_{B,\mathrm{dR}} \circ a)(\omega_{dR}(M)). \tag{4.117}$$

Moreover, a can be chosen of the form $a = u_0 \cdot \tau(2\pi i)$ with $u_0 \in U_{\mathrm{dR}}(\mathbb{R})$.

PROOF. We follow [**Del89**, §8.10]. Recall from (4.109) that

$$P_{\mathrm{B,dR}} = \underline{\mathrm{Isom}}^{\otimes}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}})$$

is a G_{dR}-torsor with a complex point $\mathrm{comp}_{\mathrm{B,dR}} \in P_{\mathrm{B,dR}}(\mathbb{C})$. In particular, $P_{\mathrm{B,dR}}$ is non-empty. This implies that $P_{\mathrm{B,dR}}$ has a $\overline{\mathbb{Q}}$-rational point, hence it is a trivial torsor over $\overline{\mathbb{Q}}$. By Corollary 4.115, the torsor has to be trivial already over $\mathbb{Q}$, which implies the existence of a rational point, that is an isomorphism of fibre functors $\alpha\colon \omega_{\mathrm{dR}} \xrightarrow{\sim} \omega_{\mathrm{B}}$. Define

$$a = \mathrm{comp}_{\mathrm{dR,B}} \circ \alpha. \tag{4.118}$$

By construction, a is an element of $G_{\mathrm{dR}}(\mathbb{C})$ and $\mathrm{comp}_{\mathrm{B,dR}} \circ a = \alpha$, from which (4.117) follows. Note also that any other element of $G_{\mathrm{dR}}(\mathbb{C})$ satisfying this propery is of the form $a\gamma$ with $\gamma \in G_{\mathrm{dR}}(\mathbb{Q})$.

Let us now turn to the second assertion, that a can be chosen of the form $u_0 \cdot \tau(2\pi i)$ with $u_0 \in U_{\mathrm{dR}}(\mathbb{R})$. This uses in a crucial way the compatibility between the comparison isomorphism and complex conjugation explained in Proposition 2.68. Interpreted in our context, it says that the following diagram of fibre functors is commutative:

$$\begin{tikzcd}
\omega_{\mathrm{dR}} \arrow[r, "\alpha"] \arrow[rrr, bend left, "a"] & \omega_{\mathrm{B}} \arrow[r, hook] & \omega_{\mathrm{B}} \otimes \mathbb{C} \arrow[r, "\mathrm{comp}"] & \omega_{\mathrm{dR}} \otimes \mathbb{C} \\
\omega_{\mathrm{dR}} \arrow[r, "\alpha"'] \arrow[rrr, bend right, "a"'] & \omega_{\mathrm{B}} \arrow[r, hook] \arrow[u, "\rho"] & \omega_{\mathrm{B}} \otimes \mathbb{C} \arrow[r, "\mathrm{comp}"] \arrow[u, "\rho\otimes c"] & \omega_{\mathrm{dR}} \otimes \mathbb{C}, \arrow[u, "\mathrm{Id}\otimes c"']
\end{tikzcd}$$

where ρ is the map induced from complex conjugation on the topological space and c is complex conjugation on the coefficients. Note that ρ is a rational point of G_{B}. The complex conjugate of a is $\overline{a} = \mathrm{Id} \otimes \sigma \circ a$. Define

$x = a^{-1}\overline{a}$. By the commutativity of the diagram, $x = \alpha^{-1}\rho\alpha$. Thus $x \in G_{\mathrm{dR}}(\mathbb{Q})$ and has order two.

Let us apply (4.118) to the motive $\mathbb{Q}(1)$. Since

$$\mathrm{comp}_{\mathrm{dR,B}} \colon \omega_{\mathrm{B}}(\mathbb{Q}(1)) \longrightarrow \omega_{\mathrm{dR}}(\mathbb{Q}(1))$$

is multiplication by $2\pi i$ by Example 4.105 and $\alpha(\mathbb{Q}(1))$ is an invertible map of one-dimensional $\mathbb{Q}$-vector spaces, it follows that $t_{\mathrm{dR}}(a) \in \mathbb{G}_m$ lies in $2\pi i\mathbb{Q}^\times$. Thus, up to replacing a by $a\gamma$ with $\gamma \in G_{\mathrm{dR}}(\mathbb{Q})$, we can assume that

$$a^{-1}\overline{a} = \tau(-1). \tag{4.119}$$

Any other element satisfying both (4.117) and (4.119) is of the form $a\gamma$ for some $\gamma \in G_{\mathrm{dR}}(\mathbb{Q})$ such that $\gamma^{-1}\tau(-1)\gamma = \tau(-1)$. In particular, any $\gamma \in \tau(\mathbb{Q}^\times)$ works. Therefore, replacing a by $a\gamma$ with $\gamma \in \tau(\mathbb{Q}^\times)$, one can choose a such that $t_{\mathrm{dR}}(a) = 2\pi i$. This amounts to saying that $a = u_0 \cdot \tau(2\pi i)$ with $u_0 \in U_{\mathrm{dR}}(\mathbb{C})$.

It remains to show that $u_0 \in U_{\mathrm{dR}}(\mathbb{R})$. By (4.119),

$$\tau(2\pi i)^{-1}u_0^{-1}\overline{u_0}\tau(-2\pi i) = \tau(-1)$$

and writing $\tau(-1) = \tau(2\pi i)^{-1}\tau(-2\pi i)$ one gets $u_0 = \overline{u_0}$. $\square$

4.5.4. *The period map and the period conjecture.* Recall from the previous sections that $P_{\mathrm{dR,B}}$ denotes the scheme of tensor isomorphisms between ω_{B} and ω_{dR}, which has the structure of a pro-algebraic variety over $\mathbb{Q}$. The ring of regular functions $\mathcal{O}(P_{\mathrm{dR,B}})$ forms an ind-object in the category of $\mathbb{Q}$-algebras of finite type.

Definition 4.120. The *period map* is the ring morphism

$$\mathrm{per} \colon \mathcal{O}(P_{\mathrm{dR,B}}) \to \mathbb{C} \tag{4.121}$$

given by evaluation at the point $\mathrm{comp}_{\mathrm{dR,B}}$:

$$\mathrm{per}(f) = f(\mathrm{comp}_{\mathrm{dR,B}}).$$

Similarly, evaluation at the point $\mathrm{comp}_{\mathrm{B,dR}}$ yields a period map

$$\mathcal{O}(P_{\mathrm{B,dR}}) \to \mathbb{C}.$$

The following is a variant of Grothendieck's period conjecture for the category of mixed Tate motives over $\mathbb{Z}$ (cf. also [**And04**, 25.2]).

Conjecture 4.122 (Grothendieck). *The point* $\mathrm{comp}_{\mathrm{dR,B}}$ *is generic.*

To give a meaning to the word "generic", observe that, as in Lemma 4.21, $P_{\mathrm{B,dR}}$ can be written as the projective system of torsors $P^Y_{\mathrm{B,dR}}$ for mixed Tate motives Y. Then, by "generic" we mean that, for every quotient $P_{\mathrm{B,dR}} \to P^Y_{\mathrm{B,dR}}$ the image $\mathrm{comp}^Y_{\mathrm{B,dR}}$ of the point $\mathrm{comp}_{\mathrm{B,dR}}$ in $P^Y_{\mathrm{B,dR}}$ is not contained in any proper subvariety defined over $\mathbb{Q}$. Therefore $\mathrm{comp}_{\mathrm{B,dR}}$ is generic if and only if, for every mixed Tate motive, the period map

$$\mathrm{per} = \mathrm{ev}_{\mathrm{comp}^Y_{\mathrm{B,dR}}} : \mathcal{O}(P^Y_{\mathrm{B,dR}}) \longrightarrow \mathbb{C}$$

is injective. Moreover, if $\mathrm{comp}_{\mathrm{B,dR}}$ is generic, then the transcendence degree of the residue field of $\mathrm{comp}^Y_{\mathrm{B,dR}}$ is equal to the dimension of $P^Y_{\mathrm{B,dR}}$.

From the previous discussion, we see that Grothendieck's period conjecture for mixed Tate motives is equivalent to the following:

Conjecture 4.123. *The period map* (4.121) *is injective.*

4.5.5. *Lie algebras.* Let $\mathfrak{u}_{\mathrm{dR}}$ be the Lie algebra of U_{dR}. The decomposition $G_{\mathrm{dR}} = U_{\mathrm{dR}} \rtimes \mathbb{G}_m$ implies that $\mathbb{G}_m$ acts on $\mathfrak{u}_{\mathrm{dR}}$. Let $\mathfrak{u}^n_{\mathrm{dR}} \subseteq \mathfrak{u}_{\mathrm{dR}}$ be the subspace where $t \in \mathbb{G}_m$ acts as multiplication by t^n. By the compatibility of the action of $\mathbb{G}_m$ with the Lie algebra structure, one has

$$[\mathfrak{u}^n_{\mathrm{dR}}, \mathfrak{u}^m_{\mathrm{dR}}] \subseteq \mathfrak{u}^{n+m}_{\mathrm{dR}}$$

and therefore we get a graded Lie algebra

$$\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{u}^n_{\mathrm{dR}}.$$

The fibre functor ω_{dR} induces an equivalence of categories between finite-dimensional graded vector spaces together with an action of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ compatible with the gradings and the category $\mathbf{MT}(\mathbb{Z})$.

The main result of this section is the following.

Theorem 4.124. *The graded Lie algebra $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is free with one generator in each positive odd degree $n \geq 3$.*

The theorem will be a consequence of Lemma 4.127 below. Since we have not found a suitable reference, we include a proof of it.

The definition of nilpotent Lie algebras admits several generalizations to the infinite-dimensional case. The one that will be useful for us is the following.

Definition 4.125. A Lie algebra $\mathfrak{L}$ is called *quasi-nilpotent* if

$$\bigcap_n [\mathfrak{L}, [\mathfrak{L}, \overset{n}{\ldots}, [\mathfrak{L}, \mathfrak{L}] \ldots] = 0.$$

Examples 4.126. Any nilpotent Lie algebra is quasi-nilpotent. A pro-nilpotent Lie algebra is quasi-nilpotent. The graded Lie algebra associated to a pro-nilpotent graded Lie algebra is also quasi-nilpotent. Any subalgebra of a quasi-nilpotent Lie algebra is quasi-nilpotent.

Lemma 4.127. *Let $\mathfrak{L} = \bigoplus_n \mathfrak{L}_n$ be a quasi-nilpotent graded Lie algebra over $\mathbb{Q}$ with $H_1(\mathfrak{L}, \mathbb{Q})$ concentrated in positive degrees and $H_2(\mathfrak{L}, \mathbb{Q}) = 0$. Then $\mathfrak{L}$ is isomorphic to the free algebra generated by $H_1(\mathfrak{L}, \mathbb{Q})$.*

Proof. We use the Koszul complex of $\mathfrak{L}$ to compute its homology

$$\dots \longrightarrow \mathfrak{L}\wedge\mathfrak{L}\wedge\mathfrak{L} \longrightarrow \mathfrak{L}\wedge\mathfrak{L} \xrightarrow{[\ ,\ ]} \mathfrak{L} \xrightarrow{0} \mathbb{Q},$$

where the last map in the complex is the zero map and the previous to the last is given by the Lie bracket. From this complex we derive the well known identity

$$H_1(\mathfrak{L},\mathbb{Q}) = \mathfrak{L}/[\mathfrak{L},\mathfrak{L}].$$

The map $\mathfrak{L} \to H_1(\mathfrak{L},\mathbb{Q})$ is homogeneous and surjective, thus we can choose a homogeneous lifting $H_1(\mathfrak{L},\mathbb{Q}) \to \mathfrak{L}$. In general, this lifting is non-canonical. Let $\mathfrak{F}$ be the free Lie algebra generated by $H_1(\mathfrak{L},\mathbb{Q})$. It is a graded algebra. By the universal property of free Lie algebras, the chosen lifting defines a graded map $\mathfrak{F} \to \mathfrak{L}$. We want to show that this map is an isomorphism.

Let F_n denote the increasing filtration of $\mathfrak{L}$ and $\mathfrak{F}$ given by the degree:

$$F_n\mathfrak{L} = \bigoplus_{n'\le n} \mathfrak{L}_{n'}, \quad F_n\mathfrak{F} = \bigoplus_{n'\le n} \mathfrak{F}_{n'}.$$

We prove by induction on $n \ge 0$ that the map $F_n\mathfrak{F} \to F_n\mathfrak{L}$ is surjective. By construction, $F_0\mathfrak{F} = 0$. Since $\mathfrak{L}$ is graded, we deduce that $F_0\mathfrak{L}$ is a Lie subalgebra. Since $\mathfrak{L}$ is quasi-nilpotent, the same is true for $F_0\mathfrak{L}$. Since $H_1(\mathfrak{L},\mathbb{Q})$ is concentrated in positive degrees, $F_0\mathfrak{L}$ is also perfect: $F_0\mathfrak{L} = [F_0\mathfrak{L}, F_0\mathfrak{L}]$. This implies that $F_0\mathfrak{L} = \{0\}$ so we get the case $n = 0$ in the induction process.

We assume now that $F_{n'}\mathfrak{F} \to F_{n'}\mathfrak{L}$ is surjective for all $n' < n$. Since we can write

$$F_n\mathfrak{L}/F_{n-1}\mathfrak{L} = H_1(\mathfrak{L},\mathbb{Q})_n + [\mathfrak{L},\mathfrak{L}]_n$$

the definition of $\mathfrak{F}$, the fact that $F_0\mathfrak{L} = 0$ and the induction hypothesis imply that the map $F_n\mathfrak{F} \to F_n\mathfrak{L}$ is surjective. Since $\mathfrak{L}$ is graded,

$$\mathfrak{L} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{L}_n = \bigcup_{n\ge 0} F_n\mathfrak{L},$$

and we conclude the surjectivity of $\mathfrak{F} \to \mathfrak{L}$.

Let now $\mathfrak{k} \subset \mathfrak{F}$ denote the kernel of the map $\mathfrak{F} \to \mathfrak{L}$. We have a commutative diagram

$$\begin{tikzcd}
 & \mathfrak{F}\wedge\mathfrak{k} \arrow[r] \arrow[d, hook] & \mathfrak{k} \arrow[d, hook] & \\
\mathfrak{F}\wedge\mathfrak{F}\wedge\mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F}\wedge\mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{F} \arrow[r, two heads] \arrow[d] & H_1(\mathfrak{L},\mathbb{Q}) \arrow[d, equal] \\
\mathfrak{L}\wedge\mathfrak{L}\wedge\mathfrak{L} \arrow[r] & \mathfrak{L}\wedge\mathfrak{L} \arrow[r] & \mathfrak{L} \arrow[r, two heads] & H_1(\mathfrak{L},\mathbb{Q}),
\end{tikzcd}$$

where $\mathfrak{F}\wedge\mathfrak{k}$ is the image of $\mathfrak{F}\otimes\mathfrak{k}$ in $\mathfrak{F}\wedge\mathfrak{F}$. The long vertical sequences and the upper long horizontal sequence are exact by definition. The lower long sequence is exact because $H_2(\mathfrak{L}, \mathbb{Q}) = 0$. From this we deduce

$$\mathfrak{k} \subset [\mathfrak{k}, \mathfrak{F}].$$

Since $\mathfrak{F}$ is also quasi-nilpotent we conclude that $\mathfrak{k} = 0$, thus showing the injectivity of the map $\mathfrak{F} \to \mathfrak{L}$. $\square$

Proof of Theorem 4.124. We start by computing the Lie algebra cohomology of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$. To this end, let $\mathbf{Rep}^{\infty}_{\mathbb{Q}}(U_{\mathrm{dR}})$ (respectively $\mathbf{Rep}^{\infty}_{\mathbb{Q}}(G_{\mathrm{dR}})$) denote the category of continuous $\mathbb{Q}$-linear representations of U_{dR} (respectively G_{dR}), not necessarily of finite dimension. We have a fully faithful functor

$$\mathbf{MT}(\mathbb{Z}) = \mathbf{Rep}_{\mathbb{Q}}(G_{\mathrm{dR}}) \longrightarrow \mathbf{Rep}^{\infty}_{\mathbb{Q}}(G_{\mathrm{dR}}).$$

In particular, there are representations $\mathbb{Q}(n)$ of G_{dR} on which G_{dR} acts through its quotient $\mathbb{G}_m$. Then

$$H^i(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}, \mathbb{Q}) = \mathrm{Ext}^i_{\mathbf{Rep}^{\infty}_{\mathbb{Q}}(U_{\mathrm{dR}})}(\mathbb{Q}, \mathbb{Q}),$$

where $\mathbb{Q}$ is viewed as the trivial representation of U_{dR}.

In order to compute the groups $\mathrm{Ext}^i_{\mathbf{Rep}^{\infty}_{\mathbb{Q}}(U_{\mathrm{dR}})}(\mathbb{Q}, \mathbb{Q})$ we will use the theory of induction and restriction of representations. From the inclusion $U_{\mathrm{dR}} \to G_{\mathrm{dR}}$ we have a functor from the category of representations of G_{dR} to the category of representations of U_{dR} that consist simply in restricting the group that act. This functor is denoted $\mathrm{Res}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}$. This functor admits a left adjoint denoted $\mathrm{Ind}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}$.

The properties we need are the computations

$$\mathrm{Res}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}(\mathbb{Q}) = \mathbb{Q}, \quad \text{and} \quad \mathrm{Ind}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}(\mathbb{Q}) = \prod_{n \in \mathbb{Z}} \mathbb{Q}(n),$$

and the adjoint property. Then

$$\begin{aligned}
\mathrm{Ext}^i_{\mathbf{Rep}^{\infty}_{\mathbb{Q}}(U_{\mathrm{dR}})}(\mathbb{Q}, \mathbb{Q}) &= \mathrm{Ext}^i_{\mathbf{Rep}^{\infty}_{\mathbb{Q}}(U_{\mathrm{dR}})}(\mathbb{Q}, \mathrm{Res}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}(\mathbb{Q})) \\
&= \mathrm{Ext}^i_{\mathbf{Rep}^{\infty}_{\mathbb{Q}}(G_{\mathrm{dR}})}(\mathrm{Ind}^{G_{\mathrm{dR}}}_{U_{\mathrm{dR}}}(\mathbb{Q}), \mathbb{Q}) \\
&= \mathrm{Ext}^i_{\mathbf{Rep}^{\infty}_{\mathbb{Q}}(G_{\mathrm{dR}})}(\prod_{n \in \mathbb{Z}} \mathbb{Q}(n), \mathbb{Q}) \\
&= \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^i_{\mathbf{Rep}^{\infty}_{\mathbb{Q}}(G_{\mathrm{dR}})}(\mathbb{Q}(n), \mathbb{Q}).
\end{aligned}$$

It follows from part (3) of Theorem 4.100 that

$$\begin{aligned}
H^1(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}, \mathbb{Q}) &= \bigoplus_{\substack{n \leq -3 \\ n \text{ odd}}} \mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(n), \mathbb{Q}(0)), \\
H^2(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}, \mathbb{Q}) &= 0,
\end{aligned}$$

where each summand $\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(n),\mathbb{Q}(0))$ is one-dimensional and sits in odd degree $n \leq -3$. Going to homology we deduce that

$$H_1(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}},\mathbb{Q}) = \bigoplus_{\substack{n\geq 3\\ n \text{ odd}}} \mathbb{Q}, \tag{4.128}$$

$$H_2(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}},\mathbb{Q}) = 0. \tag{4.129}$$

To prove the theorem we only need to show that $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ satisfies the hypothesis of Lemma 4.127. By definition, it is a graded Lie algebra. Since U_{dR} is pro-unipotent, we deduce that $\mathfrak{u}_{\mathrm{dR}}$ is pro-nilpotent, hence $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is quasi-nilpotent. The other assumptions of the lemma are nothing but conditions (4.128) and (4.129) above. □

Remarks 4.130.

(1) Following [**DG05**] and [**Del13**], the grading on $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ that we consider is the one coming from the action of $\mathbb{G}_m$, where t acts as t on $\mathbb{Q}(1)$. This is why we obtain a positively graded Lie algebra in contrast with [**And04**] or [**Bro12a**] that have a negatively graded Lie algebra.

(2) Consider the abelianization

$$(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}} = \mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}/[\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}},\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}],$$

which is a graded vector space. The proof of Theorem 4.124 yields a canonical identification

$$(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}_n = (\mathrm{Ext}^1_{\mathbf{MT}(\mathbb{Z})}(\mathbb{Q}(0),\mathbb{Q}(n)))^\vee.$$

Moreover, $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is isomorphic to the free Lie algebra generated by $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}$. Nevertheless, there is no canonical lifting from $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}$ to $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$, hence no canonical isomorphism between $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ and the free Lie algebra generated by $(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})^{\mathrm{ab}}$.

(3) Note also that $\mathfrak{u}_{\mathrm{dR}}$ and $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ are not isomorphic. In fact, $\mathfrak{u}_{\mathrm{dR}}$ is the completion of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ with respect to the grading, which implies that $\mathfrak{u}_{\mathrm{dR}}$ is not a free Lie algebra.

4.5.6. *The Hilbert-Poincaré series.* From Theorem 4.124, we deduce that the universal enveloping algebra $\mathcal{U}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$ of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$ is the free associative graded algebra with one generator in each odd degree $n \geq 3$. The algebra of regular functions $\mathcal{O}(U_{\mathrm{dR}})$ is also graded and is the dual of the completed universal enveloping algebra $\hat{\mathcal{U}}(\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}})$ in the graded sense.

For simplicity we will consider the grading by the codegree in $\mathcal{O}(U_{\mathrm{dR}})$ that is the opposite of the one induced by the grading of $\mathfrak{u}^{\mathrm{gr}}_{\mathrm{dR}}$. Thus it is also positively graded. We can compute its Hilbert-Poincaré series

$$H_{\mathcal{O}(U_{\mathrm{dR}})}(t) = \frac{1}{1-t^3-t^5-t^7-\dots} = \frac{1-t^2}{1-t^2-t^3} \tag{4.131}$$

from the dimension of the graded pieces of $\mathcal{U}(\mathfrak{u}_{\mathrm{dR}}^{\mathrm{gr}})$.

Let us now, somehow artificially, introduce the algebra

$$\mathcal{H}^{\mathcal{MT}} = \mathcal{O}(U_{\mathrm{dR}}) \otimes_{\mathbb{Q}} \mathbb{Q}[f_2], \tag{4.132}$$

where f_2 is in degree 2. From (4.131) we immediately deduce:

LEMMA 4.133. *The Hilbert-Poincaré series of $\mathcal{H}^{\mathcal{MT}}$ is given by*

$$H_{\mathcal{H}^{\mathcal{MT}}}(t) = \frac{1}{1-t^2-t^3} = \sum_{k\geq 0} d_k t^k,$$

where the integers d_k are the same as in Zagier's Conjecture 1.68.

Following Deligne, Goncharov and Terasoma, in order to prove the upper bound $\dim \mathcal{Z}_k \leq d_k$ of Theorem 1.92, we will construct in Chapter 5 a $\mathbb{Q}$-algebra $\mathcal{H}$, which injects into $\mathcal{H}^{\mathcal{MT}}$, and comes together with a surjective graded map $\mathcal{H} \to \bigoplus \mathcal{Z}_k$. This will imply immediately the bound. The reason we have changed the grading of $\mathcal{O}(U_{\mathrm{dR}})$ is precisely to make this map compatible with the degree. We have already seen that multiple zeta values appear as periods of the pro-unipotent completion of the fundamental group of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$. The motivic interpretation will give the link between $\mathcal{H}$ and $\bigoplus \mathcal{Z}_k$.

⋆⋆⋆

EXERCISE 4.134. Find examples which show that all the hypothesis in Lemma 4.127 are needed.

4.6. The motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0,1,\infty\}$. We continue considering the algebraic variety

$$X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$$

over $\mathbb{Q}$ and the complex manifold

$$M = X(\mathbb{C}) = \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}.$$

As in Section 3.9, we set:

$\mathbf{0}$=the tangential base point $(0,1)$, *i.e.* the tangent vector 1 at 0,

$\mathbf{1}$=the tangential base point $(1,-1)$, *i.e.* the tangent vector -1 at 1.

Let $\boldsymbol{x}, \boldsymbol{y} \in X(\mathbb{Q}) \cup \{\mathbf{0},\mathbf{1}\}$ be rational or tangential base points. The aim of this section is to explain that the pro-unipotent completion of the torsor of paths from $\boldsymbol{x}$ to $\boldsymbol{y}$, as well as the extra structures given by composition of paths and local monodromy, are motivic in the sense of Definition 4.89. In fact, we want to add to Summary 3.216 a motivic side whose Betti and de Rham realizations give the Betti and de Rham sides of that summary. To exhibit the motivic nature of the group schemes and torsors in that summary, it seems necessary to use the language of algebraic geometry over a Tannakian category [**Del89**, §6]. In order to avoid this language, we will only consider the motivic analoguеues of ${}_\bullet U_\bullet^?$ and ${}_\bullet\mathcal{L}_\bullet^?$.

4.6.1. *The pro-mixed Tate motive* ${}_yU_x^{\mathrm{Mot}}$. We start with the case of two rational base points $x, y \in X(\mathbb{Q}) \subseteq M$. Recall the cosimplicial manifold ${}_yM_x^\bullet$ from Construction 3.152. As we already used in Section 3.6.1, when endowing the fundamental group with a mixed Hodge structure over $\mathbb{Q}$, all the maps involved in ${}_yM_x^\bullet$ are algebraic and, the points x, y being rational, defined over $\mathbb{Q}$. We will denote by ${}_yX_x^\bullet$ the corresponding cosimplicial object in the category $\mathbf{Sm}(\mathbb{Q})$.

As explained in Section 4.3.9, to ${}_yX_x^\bullet$ one associates a family of motives

$$\{[\sigma_{\leq N}\mathcal{N}{}_yX_x^\bullet]\}_{N\geq 0}.$$

By construction, given integers $M \geq N \geq 0$, there is a morphism

$$\sigma_{\leq M}\mathcal{N}{}_yX_x^\bullet \to \sigma_{\leq N}\mathcal{N}{}_yX_x^\bullet$$

making $\{[\sigma_{\leq N}\mathcal{N}{}_yX_x^\bullet]\}_{N\geq 0}$ into a projective system of motives.

Lemma 4.135. *The object* $[\sigma_{\leq N}\mathcal{N}{}_yX_x^\bullet]$ *belongs to* $\mathbf{DMT}(\mathbb{Q})$.

Proof. Exercise 4.163. □

We can therefore consider its cohomology with respect to the t-structure of $\mathbf{DMT}(\mathbb{Q})$.

Definition 4.136. For each $N \geq 0$, we define a mixed Tate motive

$${}_yU_x^{\mathrm{Mot},N} = H_0([\sigma_{\leq N}\mathcal{N}{}_yX_x^\bullet]) \in \mathbf{MT}(\mathbb{Q}).$$

As N varies, these motives fit into a pro-mixed Tate motive ${}_yU_x^{\mathrm{Mot}}$.

We also consider the constant cosimplicial variety $\mathrm{Spec}(\mathbb{Q})^\bullet$ given by $\mathrm{Spec}(\mathbb{Q})$ in all degrees, with coface and codegeneracy maps all equal to the identity. Applying the previous construction to $\mathrm{Spec}(\mathbb{Q})^\bullet$, one easily finds (Exercise 4.164) that, for all $N \geq 0$,

$$H_0([\sigma_{\leq N}\mathcal{N}\,\mathrm{Spec}(\mathbb{Q})^\bullet]) = \mathbb{Q}(0).$$