Algebraic Topology

A solution manual by and for stupid student

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Contents

1 Topological spaces 5

2 Fundamental groups 7

3 Covering spaces 9

4 Elementary homotopy theory 11

4.1 The mapping cylinder 11

5 Cofibrations and fibrations 13

6 Homotopy groups 15

7 Stable homotopy. Daulity 17

8 Cell complexes 19

9 Singular homology 21

10 Homology 23

10.1 The Axioms of Eilenberg and Steenrod 23

11 Homological algebra 25

11.1 Diagrams 25

11.2 Exact sequences 25

11.3 Chain complex 25
11.4 Cochain complex 25
11.5 Natural chain maps and homotopies 25
11.6 Linear algebra of chain complexes 25
11.7 Tor and Ext 26
11.8 Universal coeffcients 26
11.9 The Künneth Formula 26

12 Cellular homology 27

13 Partition of unity in homotopy 29

I More

Bibliography 33
Articles 33
Books 33

1. Topological spaces

2. Fundamental groups

3. Covering spaces

4. Elementary homotopy theory

4.1 The mapping cylinder

Definition 4.1.1 Given a continuous map $f : X \longrightarrow Y$ of topological spaces, one can define its **mapping cylinder** as a pushout (fibered coproduct)

$$\begin{tikzcd}
B & & & \\
& Z(f) \arrow[ul, dashed, "\exists!"'] & & X \times I \arrow[lll, ""] \arrow[ll] \arrow[dl, "f\times id"'] \arrow[lllu] \\
& & Y \times I \arrow[ul, "r"] & \\
& Y \arrow[u, ""] \arrow[uu] \arrow[ur, "i_0"] \arrow[uuul] & & X, \arrow[uu, "i_0"'] \arrow[ll, "f"]
\end{tikzcd}$$

Set-theoretically, the mapping cylinder is usually represented as the quotient space $(X \times I \coprod Y)/\sim$, where $f(x) \sim (x,0)$. We use Mf to denote it. (other notations are used including Mf, M_f and $\mathrm{Cyl}(f)$.)

Notice that it is Mf rather than $Y \times I$ that plays the role of pushout because the map r is not unique. Our only restriction on r is $r \circ j = id$, where $j : Mf \longrightarrow Y \times I$ is the map that restricts to $f \times id$ on $X \times I$ and restricts to i_0 on Y.

R Another equivalent definition is used in tom Dieck.

In the following, we consider $X \coprod Y$ as subspace of $Z(f)$ via the map $J : J(x) = [(x,0)]$ and $J(y) = [y]$. Then we consider a homotopy commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\alpha \big\downarrow & & \big\downarrow \beta \\
X' & \xrightarrow{f'} & Y',
\end{array}$$

where the diagram commutes up to a homotopy $\Psi : f' \circ \alpha \simeq \beta \circ f$.

5. Cofibrations and fibrations

6. Homotopy groups

7. Stable homotopy. Daulity

8. Cell complexes

9. Singular homology

10. Homology

10.1 The Axioms of Eilenberg and Steenrod

11. Homological algebra

11.1 Diagrams

11.2 Exact sequences

11.3 Chain complex

11.4 Cochain complex

11.5 Natural chain maps and homotopies

11.6 Linear algebra of chain complexes

Exercise?? 11.6.A Tensor product is compatible with chain homotopy. Let $s : f \simeq g : C \longrightarrow C'$ be a chain homotopy. Then $s \otimes id : f \otimes id \simeq g \otimes id : C \otimes D \longrightarrow C' \otimes D$ is a chain homotopy. ■

Proof. Know: $s\partial_C + \partial_{C'} s = f - g$

Want: $(s \otimes id_D)\partial_{C\otimes D} + \partial_{C'\otimes D}(s \otimes id_D) = f \otimes id_D - g \otimes id_D$.

$C \otimes D$ is generated by pure tensors like $c'_n \otimes d_m$, therefore we can check the formula on element $c_n \otimes d_m \in C_n \otimes D_m$

$$\begin{aligned}
&(s \otimes id_D)\partial_{C\otimes D}(c_n \otimes d_m)\\
&= (s \otimes id_D)\left(\partial_C c_n \otimes d_m + (-1)^n c_n \otimes \partial_D d_m\right)\\
&= s \circ \partial_C c_n \otimes d_m + (-1)^n s c_n \otimes \partial_D d_m
\end{aligned}$$

and

$$\begin{aligned}
&\partial_{C'\otimes D}(s \otimes id_D)(c_n \otimes d_m)\\
&= \partial_{C'\otimes D}(s c_n \otimes d_m)\\
&= \partial_{C'} s c_n \otimes d_m + (-1)^{\deg(s c_n)} s c_n \otimes \partial_D d_m,
\end{aligned}$$

where $\deg(sc_n) = n-1$. Then we have

$$\begin{aligned}
&(\partial_{C'\otimes D}(s\otimes id_D) + (s\otimes id_D)\partial_{C\otimes D})\,(c_n\otimes d_m) \\
&= (s\partial_C + \partial_{C'}s)c_n\otimes d_m + 0 \\
&= (f\otimes id_D - g\otimes id_D)(c_n\otimes d_m)
\end{aligned}$$

We are done. Also we can generalize this statemnt to

Let $s: f\simeq g: C\longrightarrow C'$ and $t: p\simeq q: D\longrightarrow D'$ be chain homotopies. Then $s\otimes t: f\otimes p\simeq g\otimes q: C\otimes D\longrightarrow C'\otimes D'$ is a chain homotopy. We easily conclude by $s\otimes id$ and $id\otimes s$ are chain homotopy and composition of chain homotopies is a chain homotopy. ■

11.7 Tor and Ext

11.8 Universal coeffcients

11.9 The Künneth Formula

12. Cellular homology

13. Partition of unity in homotopy

More

Bibliography **33**
Articles
Books

Bibliography

Articles
Books