Appendix

This appendix is comprised of three parts. In §A.1, we will review some ideas from classical category theory, such as monoidal structures, enriched categories, and Quillen's small object argument. We give a brief overview of the theory of model categories in §A.2. The main result here is Proposition A.2.6.13, which will allow us to establish the existence of model category structures in a variety of situations with a minimal amount of effort. In §A.3, we will use this result to make a detailed study of the theory of simplicial categories. Our exposition is rather dense; for a more leisurely account of the theory of model categories, we refer the reader to one of the standard texts (such as [40]).

A.1 CATEGORY THEORY

Familiarity with classical category theory is the main prerequisite for reading this book. In this section, we will fix some of the notation that we use when discussing categories and summarize (generally without proofs) some of the concepts employed in the body of the text.

If $\mathcal{C}$ is a category, we let $\mathrm{Ob}(\mathcal{C})$ denote the set of objects of $\mathcal{C}$. We will write $X \in \mathcal{C}$ to mean that X is an object of $\mathcal{C}$. For $X, Y \in \mathcal{C}$, we write $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for the set of morphisms from X to Y in $\mathcal{C}$. We also write id_X for the identity automorphism of $X \in \mathcal{C}$ (regarded as an element of $\mathrm{Hom}_{\mathcal{C}}(X, X)$).

If Z is an object in a category $\mathcal{C}$, then the *overcategory* $\mathcal{C}_{/Z}$ of *objects over* Z is defined as follows: the objects of $\mathcal{C}_{/Z}$ are diagrams $X \to Z$ in $\mathcal{C}$. A morphism from $f : X \to Z$ to $g : Y \to Z$ is a commutative triangle

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z. & \end{tikzcd}$$

Dually, we have an *undercategory* $\mathcal{C}_{Z/} = ((\mathcal{C}^{op})_{/Z})^{op}$ of *objects under* Z.

If $f : X \to Z$ and $g : Y \to Z$ are objects in $\mathcal{C}_{/Z}$, then we will often write $\mathrm{Hom}_Z(X, Y)$ rather than $\mathrm{Hom}_{\mathcal{C}_{/Z}}(f, g)$.

We let $\mathcal{S}\mathrm{et}$ denote the category of sets and $\mathcal{C}\mathrm{at}$ the category of (small) categories (where the morphisms are given by functors).

If κ is a regular cardinal, we will say that a set S is *κ-small* if it has cardinality less than κ. We will also use this terminology when discussing mathematical objects other than sets, which are built out of sets. For example, we will say that a category $\mathcal{C}$ is *κ-small* if the set of all objects of $\mathcal{C}$ is κ-small and the set of all morphisms in $\mathcal{C}$ is likewise κ-small.

We will need to discuss categories which are not small. In order to minimize the effort spent dealing with set-theoretic complications, we will adopt the usual device of *Grothendieck universes*. We fix a strongly inaccessible cardinal κ and refer to a mathematical object (such as a set or category) as *small* if it is κ-small, and *large* otherwise. It should be emphasized that this is primarily a linguistic device and that none of our results depend in an essential way on the existence of a strongly inaccessible cardinal κ.

Throughout this book, the word "topos" will always mean *Grothendieck topos*. Strictly speaking, a knowledge of classical topos theory is not required to read this book: all of the relevant classical concepts will be introduced (though sometimes in a hurried fashion) in the course of our search for suitable ∞-categorical analogues.

A.1.1 Compactness and Presentability

Let κ be a regular cardinal.

Definition A.1.1.1. A partially ordered set $\mathcal{I}$ is *κ-filtered* if, for any subset $\mathcal{I}_0 \subseteq \mathcal{I}$ having cardinality $< \kappa$, there exists an upper bound for $\mathcal{I}_0$ in $\mathcal{I}$.

Let $\mathcal{C}$ be a category which admits (small) colimits and let X be an object of $\mathcal{C}$. Suppose we are given a κ-filtered partially ordered set $\mathcal{I}$ and a diagram $\{Y_\alpha\}_{\alpha \in \mathcal{I}}$ in $\mathcal{C}$ indexed by $\mathcal{I}$. Let Y denote a colimit of this diagram. Then there is an associated map of sets

$$\psi : \varinjlim \mathrm{Hom}_{\mathcal{C}}(X, Y_\alpha) \to \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

We say that X is *κ-compact* if ψ is bijective for *every* κ-filtered partially ordered set $\mathcal{I}$ and *every* diagram $\{Y_\alpha\}$ indexed by $\mathcal{I}$. We say that X is *small* if it is κ-compact for some (small) regular cardinal κ. In this case, X is κ-compact for all sufficiently large regular cardinals κ.

Definition A.1.1.2. A category $\mathcal{C}$ is *presentable* if it satisfies the following conditions:

(1) The category $\mathcal{C}$ admits all (small) colimits.

(2) There exists a (small) set S of objects of $\mathcal{C}$ which generates $\mathcal{C}$ under colimits; in other words, every object of $\mathcal{C}$ may be obtained as the colimit of a (small) diagram taking values in S.

(3) Every object in $\mathcal{C}$ is small. (Assuming (2), this is equivalent to the assertion that every object which belongs to S is small.)

(4) For any pair of objects $X, Y \in \mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is small.

Remark A.1.1.3. In §5.5, we describe an ∞-categorical generalization of Definition A.1.1.2.

Remark A.1.1.4. For more details of the theory of presentable categories, we refer the reader to [1]. Note that our terminology differs slightly from that of [1], in which our presentable categories are called *locally presentable* categories.

A.1.2 Lifting Problems and the Small Object Argument

Let $\mathcal{C}$ be a category and let $p : A \to B$ and $q : X \to Y$ be morphisms in $\mathcal{C}$. Recall that p is said to have the *left lifting property* with respect to q, and q the *right lifting property* with respect to p, if given any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "p"] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a dotted arrow as indicated, rendering the diagram commutative.

Remark A.1.2.1. In the case where Y is a final object of $\mathcal{C}$, we will instead say that X has the *extension property* with respect to $p : A \to B$.

Let S be any collection of morphisms in $\mathcal{C}$. We define ${}_{\perp}S$ to be the class of all morphisms which have the right lifting property with respect to all morphisms in S, and $S_{\perp}$ to be the class of all morphisms which have the left lifting property with respect to all morphisms in S. We observe that

$$S \subseteq ({}_{\perp}S)_{\perp}.$$

The class of morphisms $({}_{\perp}S)_{\perp}$ enjoys several stability properties which we axiomatize in the following definition.

Definition A.1.2.2. Let $\mathcal{C}$ be a category with all (small) colimits and let S be a class of morphisms of $\mathcal{C}$. We will say that S is *weakly saturated* if it has the following properties:

(1) (Closure under the formation of pushouts) Given a pushout diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & D \arrow[d] \\ C' \arrow[r, "f'"] & D' \end{tikzcd}$$

such that f belongs to S, the morphism f' also belongs to S.

(2) (Closure under transfinite composition) Let $C \in \mathcal{C}$ be an object, let α be an ordinal, and let $\{D_\beta\}_{\beta<\alpha}$ be a system of objects of $\mathcal{C}_{C/}$ indexed by α: in other words, for each $\beta < \alpha$, we are supplied with a morphism $C \to D_\beta$ which belongs to S, and for each $\gamma \leq \beta < \alpha$ a commutative diagram

$$\begin{tikzcd} & D_\gamma \arrow[dd, "\phi_{\gamma,\beta}"] \\ C \arrow[ur] \arrow[dr] & \\ & D_\beta \end{tikzcd}$$

satisfying $\phi_{\gamma,\delta} \circ \phi_{\beta,\gamma} = \phi_{\beta,\delta}$. For $\beta \leq \alpha$, we let $D_{<\beta}$ be a colimit of the system $\{D_\gamma\}_{\gamma<\beta}$ taken in the category $\mathcal{C}_{C/}$.

Suppose that, for each $\beta < \alpha$, the natural map $D_{<\beta} \to D_\beta$ belongs to S. Then the induced map $C \to D_{<\alpha}$ belongs to S.

(3) (Closure under the formation of retracts) Given a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"] & C' \arrow[r] \arrow[d, "g"] & C \arrow[d, "f"] \\ D \arrow[r] & D' \arrow[r] & D \end{tikzcd}$$

in which both horizontal compositions are the identity, if g belongs to S, then so does f.

It is worth noting that saturation has the following consequences:

Proposition A.1.2.3. *Let $\mathcal{C}$ be a category which admits all (small) colimits and let S be a weakly saturated class of morphism in $\mathcal{C}$. Then*

(1) *Every isomorphism belongs to S.*

(2) *The class S is stable under composition: if $f : X \to Y$ and $g : Y \to Z$ belong to S, then so does $g \circ f$.*

Proof. Assertion (1) is equivalent to the closure of S under transfinite composition in the special case where $\alpha = 0$; (2) is equivalent to the special case where $\alpha = 2$. □

Remark A.1.2.4. A reader who is ill at ease with the style of the preceding argument should feel free to take the asserted properties as part of the definition of a weakly saturated class of morphisms.

The intersection of any collection of weakly saturated classes of morphisms is itself weakly saturated. Consequently, for any category $\mathcal{C}$ which admits small colimits, and any collection A of morphisms of $\mathcal{C}$, there exists a *smallest* weakly saturated class of morphisms containing A: we will call this the weakly saturated class of morphisms *generated* by A. We note that $({}_\perp A)_\perp$ is weakly saturated. Under appropriate set-theoretic assumptions, Quillen's "small object argument" can be used to establish that $({}_\perp A)_\perp$ is the weakly saturated class generated by A:

Proposition A.1.2.5 (Small Object Argument)**.** *Let $\mathcal{C}$ be a presentable category and $A_0 = \{\phi_i : C_i \to D_i\}_{i \in I}$ a collection of morphisms in $\mathcal{C}$ indexed by a (small) set I. For each $n \geq 0$, let $\mathcal{C}^{[n]}$ denote the category of functors from the linearly ordered set $[n] = \{0, \dots, n\}$ into $\mathcal{C}$. There exists a functor $T : \mathcal{C}^{[1]} \to \mathcal{C}^{[2]}$ with the following properties:*

(1) *The functor T carries a morphism $f : X \to Z$ to a diagram*

$$\begin{tikzcd} & Y \arrow[dr, "f''"] & \\ X \arrow[ur, "f'"] \arrow[rr, "f"] & & Z \end{tikzcd}$$

where f' belongs to the weakly saturated class of morphisms generated by A_0 and f'' has the right lifting property with respect to each morphism in A_0.

(2) *If κ is a regular cardinal such that each of the objects C_i, D_i is κ-compact, then T commutes with κ-filtered colimits.*

Proof. Fix a regular cardinal κ as in (2) and fix a morphism $f : X \to Z$ in $\mathcal{C}$. We will give a functorial construction of the desired diagram

$$\begin{tikzcd} & Y \arrow[dr, "f''"] & \\ X \arrow[ur, "f'"] \arrow[rr, "f"] & & Z. \end{tikzcd}$$

We define a transfinite sequence of objects

$$Y_0 \to Y_1 \to \cdots$$

in $\mathcal{C}_{/Z}$ indexed by ordinals smaller than κ. Let $Y_0 = X$ and let $Y_\lambda = \varinjlim_{\alpha<\lambda} Y_\alpha$ when λ is a nonzero limit ordinal. For $i \in I$, let $F_i : \mathcal{C}_{/Z} \to \mathcal{S}\mathrm{et}$ be the functor

$$(W \to Z) \mapsto \mathrm{Hom}_{\mathcal{C}}(D_i, Z) \times_{\mathrm{Hom}_{\mathcal{C}}(C_i, Z)} \mathrm{Hom}_{\mathcal{C}}(C_i, W).$$

Supposing that Y_α has been defined, we define $Y_{\alpha+1}$ by the following pushout diagram

$$\begin{tikzcd} \coprod_{i \in I, \eta \in F_i(Y_\alpha)} C_i \arrow[r] \arrow[d] & Y_\alpha \arrow[d] \\ \coprod_{i \in I, \eta \in F_i(Y_\alpha)} D_i \arrow[r] & Y_{\alpha+1}. \end{tikzcd}$$

We conclude by defining Y to be $\varinjlim_{\alpha<\kappa} Y_\alpha$. It is easy to check that this construction has the desired properties. □

Remark A.1.2.6. If $\mathcal{C}$ is enriched, tensored, and cotensored over another presentable monoidal category $\mathbf{S}$ (see §A.1.4), then a similar construction shows that we can choose T to be an $\mathbf{S}$-enriched functor.

Corollary A.1.2.7. *Let $\mathcal{C}$ be a presentable category and let A be a* set *of morphisms of $\mathcal{C}$. Then $({}_{\perp}A)_{\perp}$ is the smallest weakly saturated class of morphisms containing A.*

Proof. Let $\overline{A}$ be the smallest weakly saturated class of morphisms containing A, so that $\overline{A} \subseteq ({}_{\perp}A)_{\perp}$. For the reverse inclusion, let us suppose that $f : X \rightarrow Z$ belongs to $({}_{\perp}A)_{\perp}$. Proposition A.1.2.5 implies the existence of a factorization

$$X \stackrel{f'}{\rightarrow} Y \stackrel{f''}{\rightarrow} Z,$$

where $f' \in \overline{A}$ and f'' belongs to ${}_{\perp}A$. It follows that f has the left lifting property with respect to f'', so that f is a retract of f' and therefore belongs to $\overline{A}$. $\square$

Remark A.1.2.8. Let $\mathcal{C}$ be a presentable category, let S be a (small) set of morphisms in $\mathcal{C}$, and suppose that $f : X \rightarrow Y$ belongs to the weakly saturated class of morphisms generated by S. The proofs of Proposition A.1.2.5 and Corollary A.1.2.7 show that there exists a transfinite sequence

$$Y_0 \rightarrow Y_1 \rightarrow \cdots$$

of objects of $\mathcal{C}_{X/}$, indexed by a set of ordinals $\{\beta | \beta < \alpha\}$, with the following properties:

(i) For each $\beta < \alpha$, there is a pushout diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d] & D \arrow[d] \\ \varinjlim_{\gamma < \beta} Y_\gamma \arrow[r] & Y_\beta, \end{tikzcd}$$

where the colimit is formed in $\mathcal{C}_{X/}$ and $g \in S$.

(ii) The object Y is a retract of $\varinjlim_{\gamma < \alpha} Y_\gamma$ in the category $\mathcal{C}_{X/}$.

A.1.3 Monoidal Categories

A *monoidal category* is a category $\mathcal{C}$ equipped with a (coherently) associative "product" functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and a unit object $\mathbf{1}$. The associativity is expressed by demanding isomorphisms

$$\eta_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C),$$

and the requirement that $\mathbf{1}$ be unital is expressed by demanding isomorphisms

$$\alpha_A : A \otimes \mathbf{1} \rightarrow A$$

$$\beta_A : \mathbf{1} \otimes A \rightarrow A.$$

We do not merely require the existence of these isomorphisms: they are part of the structure of a monoidal category. Moreover, these isomorphisms are required to satisfy the following conditions:

- The isomorphism $\eta_{A,B,C}$ depends *functorially* on the triple (A, B, C); in other words, η may be regarded as a natural isomorphism between the functors
$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
$$(A, B, C) \mapsto (A \otimes B) \otimes C$$
$$(A, B, C) \mapsto A \otimes (B \otimes C).$$
Similarly, α_A and β_A depend functorially on A.

- Given any quadruple (A, B, C, D) of objects of $\mathcal{C}$, the *MacLane pentagon*
$$\begin{tikzcd} & ((A \otimes B) \otimes C) \otimes D \arrow[dl, "\eta_{A,B,C} \otimes \mathrm{id}_D"'] \arrow[dr, "\eta_{A \otimes B, C, D}"] & \\ (A \otimes (B \otimes C)) \otimes D \arrow[d, "\eta_{A, B \otimes C, D}"] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\eta_{A,B,C \otimes D}"] \\ A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\mathrm{id}_A \otimes \eta_{B,C,D}"] & & A \otimes (B \otimes (C \otimes D)) \end{tikzcd}$$
is commutative.

- For any pair (A, B) of objects of $\mathcal{C}$, the triangle
$$\begin{tikzcd} (A \otimes \mathbf{1}) \otimes B \arrow[rr, "\eta_{A,\mathbf{1},B}"] \arrow[dr, "\alpha_A \otimes \mathrm{id}_B"] & & A \otimes (\mathbf{1} \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \beta_B"] \\ & A \otimes B & \end{tikzcd}$$
is commutative.

MacLane's coherence theorem asserts that the commutativity of this pair of diagrams implies the commutativity of *all* diagrams that can be written using only the isomorphisms $\eta_{A,B,C}$, α_A, and β_A. More precisely, any monoidal category is equivalent (as a monoidal category) to a *strict* monoidal category: that is, a monoidal category in which $\otimes$ is literally associative, $\mathbf{1}$ is literally a unit with respect to $\otimes$, and the isomorphisms $\eta_{A,B,C}$, α_A, β_A are the identity maps.

Example A.1.3.1. Let $\mathcal{C}$ be a category which admits finite products. Then $\mathcal{C}$ admits the structure of a monoidal category where the operation $\otimes$ is given by Cartesian product
$$A \otimes B \simeq A \times B$$
and the isomorphisms $\eta_{A,B,C}$ are induced from the evident associativity of the Cartesian product. The identity $\mathbf{1}$ is defined to be the final object of $\mathcal{C}$,

and the isomorphisms α_A and β_A are determined in the obvious way. We refer to this monoidal structure on $\mathcal{C}$ as the *Cartesian monoidal structure*.

We remark that the Cartesian product $A \times B$ is well-defined only up to (unique) isomorphism (as is the final object $\mathbf{1}$), so that strictly speaking the Cartesian monoidal structure on $\mathcal{C}$ depends on various choices; however, all such choices lead to (canonically) equivalent monoidal categories.

Remark A.1.3.2. Let $(\mathcal{C}, \otimes, \mathbf{1}, \eta, \alpha, \beta)$ be a monoidal category. We will generally abuse notation by simply saying that $\mathcal{C}$ is a monoidal category, that $(\mathcal{C}, \otimes)$ is a monoidal category, or that $\otimes$ is a *monoidal structure* on $\mathcal{C}$; the other structure is implicitly understood to be present as well.

Remark A.1.3.3. Let $\mathcal{C}$ be a category equipped with a monoidal structure $\otimes$. Then we may define a new monoidal structure on $\mathcal{C}$ by setting $A \otimes^{op} B = B \otimes A$. We refer to this monoidal structure $\otimes^{op}$ as the *opposite* of the monoidal structure $\otimes$.

Definition A.1.3.4. A monoidal category $(\mathcal{C}, \otimes)$ is said to be *left-closed* if, for each $A \in \mathcal{C}$, the functor

$$N \mapsto A \otimes N$$

admits a right adjoint

$$Y \mapsto {}^{A}Y.$$

We say that $(\mathcal{C}, \otimes)$ is *right-closed* if the opposite monoidal structure $(\mathcal{C}, \otimes^{op})$ is left-closed; in other words, if every functor

$$N \mapsto N \otimes A$$

has a right adjoint

$$Y \mapsto Y^{A}.$$

Finally, we say that $(\mathcal{C}, \otimes)$ is *closed* if it is both right-closed and left-closed.

In the setting of monoidal categories, it is appropriate to consider only those functors which are compatible with the monoidal structures in the following sense:

Definition A.1.3.5. Let $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \otimes)$ be monoidal categories. A *right-lax monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

- A functor $G : \mathcal{C} \to \mathcal{D}$.
- A natural transformation $\gamma_{A,B} : G(A) \otimes G(B) \to G(A \otimes B)$ rendering commutative the diagram

$$\begin{tikzcd}
(G(A) \otimes G(B)) \otimes G(C) \arrow[r] \arrow[d, "\gamma_{A,B}"] & G(A) \otimes (G(B) \otimes G(C)) \arrow[d, "\gamma_{B,C}"] \\
G(A \otimes B) \otimes G(C) \arrow[d, "\gamma_{A \otimes B, C}"] & G(A) \otimes G(B \otimes C) \arrow[d, "\gamma_{A, B \otimes C}"] \\
G((A \otimes B) \otimes C) \arrow[r, "G(\eta_{A,B,C})"] & G(A \otimes (B \otimes C)).
\end{tikzcd}$$

- A map $e : \mathbf{1}_{\mathcal{D}} \to G(\mathbf{1}_{\mathcal{C}})$ rendering commutative the diagrams

$$\begin{tikzcd}
G(A) \otimes \mathbf{1}_{\mathcal{D}} \arrow[r, "\mathrm{id} \otimes e"] \arrow[dr, "\alpha_{G(A)}"'] & G(A) \otimes G(\mathbf{1}_{\mathcal{C}}) \arrow[r, "\gamma_{A,\mathbf{1}_{\mathcal{C}}}"] & G(A \otimes \mathbf{1}_{\mathcal{C}}) \arrow[dl, "G(\alpha_A)"] \\
 & G(A) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{1}_{\mathcal{D}} \otimes G(B) \arrow[r, "e \otimes \mathrm{id}"] \arrow[dr, "\beta_{G(B)}"'] & G(\mathbf{1}_{\mathcal{C}}) \otimes G(B) \arrow[r, "\gamma_{\mathbf{1}_{\mathcal{C}},A}"] & G(\mathbf{1}_{\mathcal{C}} \otimes B) \arrow[dl, "G(\beta_B)"] \\
 & G(B) &
\end{tikzcd} .$$

A natural transformation between right-lax monoidal functors is *monoidal* if it commutes with the maps $\gamma_{A,B}$, e.

Dually, a *left-lax monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ consists of a right-lax monoidal functor from $\mathcal{C}^{op}$ to $\mathcal{D}^{op}$; it is determined by giving a functor $F : \mathcal{C} \to \mathcal{D}$ together with a map $e' : F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$ and a natural transformation

$$\gamma'_{A,B} : F(A \otimes B) \to F(A) \otimes F(B)$$

satisfying the appropriate analogues of the conditions listed above.

If F is a right-lax monoidal functor via *isomorphisms*

$$e : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$$

$$\gamma_{A,B} : F(A) \otimes F(B) \to F(A \otimes B),$$

then F may be regarded as a left-lax monoidal functor by setting $e' = e^{-1}$, $\gamma'_{A,B} = \gamma^{-1}_{A,B}$. In this case, we simply say that F is a *monoidal* functor.

Remark A.1.3.6. Let

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"]
\end{tikzcd}$$

be an adjunction between categories $\mathcal{C}$ and $\mathcal{D}$. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are equipped with monoidal structures. Then endowing G with the structure of a right-lax monoidal functor is equivalent to endowing F with the structure of a left-lax monoidal functor.

Example A.1.3.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories which admit finite products and let $F : \mathcal{C} \to \mathcal{D}$ be a functor between them. If we regard $\mathcal{C}$ and $\mathcal{D}$ as endowed with the Cartesian monoidal structure, then F acquires the structure of a left-lax monoidal functor in a canonical way via the maps $F(A \times B) \to F(A) \times F(B)$ induced from the functoriality of F. In this case, F is a monoidal functor if and only if it commutes with finite products.

A.1.4 Enriched Category Theory

One frequently encounters categories $\mathcal{D}$ in which the collections of morphisms $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ between two objects $X, Y \in \mathcal{D}$ have additional structure: for example, a topology, a group structure, or the structure of a vector space. These situations may all be efficiently described using the language of *enriched category theory*, which we now introduce.

Let $(\mathcal{C}, \otimes)$ be a monoidal category. A $\mathcal{C}$-*enriched category* $\mathcal{D}$ consists of the following data:

(1) A collection of objects.

(2) For every pair of objects $X, Y \in \mathcal{D}$, a mapping object $\mathrm{Map}_{\mathcal{D}}(X, Y)$ of $\mathcal{C}$.

(3) For every triple of objects $X, Y, Z \in \mathcal{D}$, a composition map

$$\mathrm{Map}_{\mathcal{D}}(Y, Z) \otimes \mathrm{Map}_{\mathcal{D}}(X, Y) \to \mathrm{Map}_{\mathcal{D}}(X, Z).$$

Composition is required to be associative in the sense that for any $W, X, Y, Z \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
& \mathrm{Map}_{\mathcal{D}}(Y, Z) \otimes \mathrm{Map}_{\mathcal{D}}(X, Y) \otimes \mathrm{Map}_{\mathcal{D}}(W, X) \arrow[dl] \arrow[dr] & \\
\mathrm{Map}_{\mathcal{D}}(Y, Z) \otimes \mathrm{Map}_{\mathcal{D}}(W, Y) \arrow[dr] & & \mathrm{Map}_{\mathcal{D}}(X, Z) \otimes \mathrm{Map}_{\mathcal{D}}(W, X) \arrow[dl] \\
& \mathrm{Map}_{\mathcal{D}}(W, X) &
\end{tikzcd}$$

is commutative.

(4) For every object $X \in \mathcal{D}$, a unit map $\mathbf{1} \to \mathrm{Map}_{\mathcal{D}}(X, X)$ rendering commutative the diagrams

$$\begin{tikzcd}
\mathbf{1} \otimes \mathrm{Map}_{\mathcal{D}}(Y, X) \arrow[rr] \arrow[dr] & & \mathrm{Map}_{\mathcal{D}}(X, X) \otimes \mathrm{Map}_{\mathcal{D}}(Y, X) \arrow[dl] \\
& \mathrm{Map}_{\mathcal{D}}(Y, X) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{D}}(X, Y) \otimes \mathbf{1} \arrow[rr] \arrow[dr] & & \mathrm{Map}_{\mathcal{D}}(X, Y) \otimes \mathrm{Map}_{\mathcal{D}}(X, X) \arrow[dl] \\
& \mathrm{Map}_{\mathcal{D}}(X, Y). &
\end{tikzcd}$$

Example A.1.4.1. Suppose that $(\mathcal{C}, \otimes)$ is a *right-closed* monoidal category. Then $\mathcal{C}$ is enriched over itself in a natural way if one defines $\mathrm{Map}_{\mathcal{C}}(X, Y) = Y^X$.

Example A.1.4.2. Let $\mathcal{C}$ be the category of sets endowed with the Cartesian monoidal structure. Then a $\mathcal{C}$-enriched category is simply a category in the usual sense.

Remark A.1.4.3. Let $G : \mathcal{C} \to \mathcal{C}'$ be a right-lax monoidal functor between monoidal categories. Suppose that $\mathcal{D}$ is a category enriched over $\mathcal{C}$. We may define a category $G(\mathcal{D})$ enriched over $\mathcal{C}'$ as follows:

(1) The objects of $G(\mathcal{D})$ are the objects of $\mathcal{D}$.

(2) Given objects $X, Y \in \mathcal{D}$, we set

$$\mathrm{Map}_{G(\mathcal{D})}(X,Y) = G(\mathrm{Map}_{\mathcal{D}}(X,Y)).$$

(3) The composition in $G(\mathcal{D})$ is given by the composite map

$$\begin{aligned} G(\mathrm{Map}_{\mathcal{D}}(Y,Z)) \otimes G(\mathrm{Map}_{\mathcal{D}}(X,Y)) &\to G(\mathrm{Map}_{\mathcal{D}}(Y,Z) \otimes \mathrm{Map}_{\mathcal{D}}(X,Y)) \\ &\to G(\mathrm{Map}_{\mathcal{D}}(X,Z)). \end{aligned}$$

Here the first map is determined by the right-lax monoidal structure on the functor G, and the second is obtained by applying G to the composition law in the category $\mathcal{D}$.

(4) For every object $X \in \mathcal{D}$, the associated unit $G(\mathcal{D})$ is given by the composition

$$\mathbf{1}_{\mathcal{C}'} \to G(\mathbf{1}_{\mathcal{C}}) \to G(\mathrm{Map}_{\mathcal{D}}(X,X)).$$

Suppose that $\mathcal{C}$ is any monoidal category. Consider the functor $\mathcal{C} \to \mathcal{S}\mathrm{et}$ given by

$$X \mapsto \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X).$$

This is a right-lax monoidal functor from $(\mathcal{C}, \otimes)$ to $\mathcal{S}\mathrm{et}$, where the latter is equipped with the Cartesian monoidal structure. By the above remarks, we see that we may equip any $\mathcal{C}$-enriched category $\mathcal{D}$ with the structure of an ordinary category by setting

$$\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathrm{Map}_{\mathcal{D}}(X,Y)).$$

We will generally not distinguish notationally between $\mathcal{D}$ as a $\mathcal{C}$-enriched category and this (underlying) category having the same objects. However, to avoid confusion, we use different notations for the morphisms: $\mathrm{Map}_{\mathcal{D}}(X,Y)$ is an object of $\mathcal{C}$, while $\mathrm{Hom}_{\mathcal{D}}(X,Y)$ is a set.

Remark A.1.4.4. If $\mathcal{D}$ and $\mathcal{D}'$ are categories which are enriched over the same monoidal category $\mathcal{C}$, then one can define a category of $\mathcal{C}$-*enriched* functors from $\mathcal{D}$ to $\mathcal{D}'$ in the evident way. Namely, an enriched functor $F : \mathcal{D} \to \mathcal{D}'$ consists of a map from the objects of $\mathcal{D}$ to the objects of $\mathcal{D}'$ and a collection of morphisms

$$\eta_{X,Y} : \mathrm{Map}_{\mathcal{D}}(X,Y) \to \mathrm{Map}_{\mathcal{D}'}(FX, FY)$$

with the following properties:

(i) For each object $X \in \mathcal{D}$, the composition

$$\mathbf{1}_{\mathcal{C}} \to \mathrm{Map}_{\mathcal{D}}(X,X) \overset{\eta_{X,X}}{\to} \mathrm{Map}_{\mathcal{D}'}(FX,FX)$$

coincides with the unit map for $FX \in \mathcal{D}'$.

(ii) For every triple of objects $X, Y, Z \in \mathcal{D}$, the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{D}}(Y,Z) \otimes \mathrm{Map}_{\mathcal{D}}(X,Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(X,Z) \arrow[d] \\ \mathrm{Map}_{\mathcal{D}'}(FY,FZ) \otimes \mathrm{Map}_{\mathcal{D}}(FX,FY) \arrow[r] & \mathrm{Map}_{\mathcal{D}}(FX,FZ) \end{tikzcd}$$

is commutative.

If F and F' are enriched functors, an *enriched natural transformation* α from F to F' consists of specifying, for each object $X \in \mathcal{D}$, a morphism $\alpha_X \in \mathrm{Hom}_{\mathcal{D}'}(FX, F'X)$ which renders commutative the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{D}}(X,Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}'}(FX,FY) \arrow[d, "\alpha_Y"] \\ \mathrm{Map}_{\mathcal{D}'}(F'X,F'Y) \arrow[r, "\alpha_X"] & \mathrm{Map}_{\mathcal{D}'}(FX,F'Y). \end{tikzcd}$$

Let $\mathcal{C}$ be a right-closed monoidal category and let $\mathcal{D}$ be a category enriched over $\mathcal{C}$. Fix objects $C \in \mathcal{C}$, $X \in \mathcal{D}$, and consider the functor

$$\mathcal{D} \to \mathcal{C}$$

$$Y \mapsto \mathrm{Map}_{\mathcal{D}}(X,Y)^C.$$

This functor may or may not be *corepresentable* in the sense that there exists an object $Z \in \mathcal{D}$ and an isomorphism of functors

$$\eta : \mathrm{Map}_{\mathcal{D}}(X,\bullet)^C \simeq \mathrm{Map}_{\mathcal{D}}(Z,\bullet).$$

If such an object Z exists, we will denote it by $X \otimes C$. The natural isomorphism η is determined by specifying a single map

$$\eta(X) : C \to \mathrm{Map}_{\mathcal{D}}(X, X \otimes C).$$

By general nonsense, the map $\eta(X)$ determines $X \otimes C$ up to (unique) isomorphism provided that $X \otimes C$ exists. If the object $X \otimes C$ exists for every $C \in \mathcal{C}$, $X \in \mathcal{D}$, then we say that $\mathcal{D}$ is *tensored over* $\mathcal{C}$. In this case, the construction

$$(X, C) \mapsto X \otimes C$$

determines a functor $\mathcal{D} \times \mathcal{C} \to \mathcal{D}$. Moreover, one has canonical isomorphisms

$$X \otimes (C \otimes D) \simeq (X \otimes C) \otimes D,$$

which express the idea that $\mathcal{D}$ may be regarded as equipped with an "action" of $\mathcal{C}$. Here we imagine $\mathcal{C}$ as a kind of generalized monoid (via its monoidal structure).

Dually, if $\mathcal{C}$ is left-closed, then an object of $\mathcal{D}$ which represents the functor

$$Y \mapsto^C \mathrm{Map}_{\mathcal{D}}(Y, X)$$

will be denoted by ${}^C X$; the object ${}^C X$ (if it exists) is determined up to (unique) isomorphism by a map $C \to \mathrm{Map}_{\mathcal{D}}({}^C X, X)$. If this object exists for all $C \in \mathcal{C}$, $X \in \mathcal{D}$, then we say that $\mathcal{D}$ is *cotensored over* $\mathcal{C}$.

Example A.1.4.5. Let $\mathcal{C}$ be a right-closed monoidal category. Then $\mathcal{C}$ may be regarded as enriched over itself in a natural way. It is automatically tensored over itself; it is cotensored over itself if and only if it is left-closed.

A.1.5 Trees

Let $\mathcal{C}$ be a presentable category and S a small collection of morphisms in $\mathcal{C}$. According to Remark A.1.2.8, the smallest weakly saturated class of morphisms $\overline{S}$ containing S can be obtained from S using pushouts, retracts, and transfinite composition. It is natural to ask if the formation of retracts is necessary: that is, does the weakly saturated class of morphisms generated by S coincide with the class of morphisms which generated by transfinite compositions of pushouts of morphisms of S? Our goal for the remainder of this section is to give an affirmative answer, at least after S has been suitably enlarged (Proposition A.1.5.12). This result is of a somewhat technical nature and will be needed only during our discussion of combinatorial model categories in §A.2.6.

We begin by introducing a generalization of the notion of a transfinite chain of morphisms.

Definition A.1.5.1. Let $\mathcal{C}$ be a presentable category and let S be a collection of morphisms in $\mathcal{C}$. An S-*tree* in $\mathcal{C}$ consists of the following data:

(1) An object $X \in \mathcal{C}$ called the *root* of the S-tree.

(2) A partially ordered set A which is *well-founded* (so that every nonempty subset of A has a minimal element).

(3) A diagram $A \to \mathcal{C}_{X/}$, which we will denote by $\alpha \mapsto Y_\alpha$.

(4) For each $\alpha \in A$, a pushout diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & D \arrow[d] \\ \varinjlim_{\beta<\alpha} Y_\beta \arrow[r] & Y_\alpha, \end{tikzcd}$$

where $f \in S$.

Let κ be a regular cardinal. We will say that an S-tree in $\mathcal{C}$ is *κ-good* if each of the objects C and D appearing above is κ-compact, and if for each $\alpha \in A$, the set $\{\beta \in A : \beta < \alpha\}$ is κ-small.

Notation A.1.5.2. Let $\mathcal{C}$ be a presentable category and S a collection of morphisms in $\mathcal{C}$. We will indicate an S-tree by writing $\{Y_\alpha\}_{\alpha \in A}$. Here the root $X \in \mathcal{C}$ and the relevant pushout diagrams are understood implicitly to be part of the data.

Suppose we are given an S-tree $\{Y_\alpha\}_{\alpha \in A}$ and a subset $B \subseteq A$ which is downward-closed in the following sense: if $\alpha \in B$ and $\beta \leq \alpha$, then $\beta \in B$. Then $\{Y_\alpha\}_{\alpha \in B}$ is an S-tree. We let Y_B denote the colimit $\varinjlim_{\alpha \in B} Y_\alpha$ formed in the category $\mathcal{C}_{X/}$. In particular, we have a canonical isomorphism $Y_\emptyset \simeq X$. If $B = \{\alpha \in A | \alpha \leq \beta\}$, then $Y_B \simeq Y_\beta$.

Remark A.1.5.3. Let $\mathcal{C}$ be a presentable category, S a collection of morphisms in $\mathcal{C}$, and $\{Y_\alpha\}_{\alpha \in A}$ an S-tree in $\mathcal{C}$ with root X. Given a map $f : X \to X'$, we can form an *associated S-tree* $\{Y_\alpha \coprod_X X'\}_{\alpha \in A}$ having root X'.

Example A.1.5.4. Let $\mathcal{C}$ be a presentable category, S a collection of morphisms in $\mathcal{C}$, and $\{Y_\alpha\}_{\alpha \in A}$ an S-tree in $\mathcal{C}$ with root X. If A is linearly ordered, then we may identify $\{Y_\alpha\}_{\alpha \in A}$ with a (possibly transfinite) sequence of morphisms belonging to S,

$$X \to Y_0 \to Y_1 \to \cdots ,$$

as in the statement of (2) in Definition A.1.2.2.

Remark A.1.5.5. Let $\mathcal{C}$ be a presentable category, S a collection of morphisms in $\mathcal{C}$, and $\{Y_\alpha\}_{\alpha \in A}$ an S-tree in $\mathcal{C}$. Let $B \subseteq A$ be downward-closed. For $\alpha \in A - B$, let $B_\alpha = B \cup \{\beta \in A : \beta \leq \alpha\}$ and let $Z_\alpha = Y_{B_\alpha}$. Then $\{Z_\alpha\}_{\alpha \in A - B}$ is an S-tree in $\mathcal{C}$ with root Y_B.

Lemma A.1.5.6. *Let $\mathcal{C}$ be a presentable category and let S be a collection of morphisms in $\mathcal{C}$. Let $\{Y_\alpha\}_{\alpha \in A}$ be an S-tree in $\mathcal{C}$ and let $A'' \subseteq A' \subseteq A$ be subsets which are downward-closed in A. Then the induced map $Y_{A''} \to Y_{A'}$ belongs to the weakly saturated class of morphisms generated by S. In particular, the canonical map $Y_\emptyset \to Y_A$ belongs to the weakly saturated class of morphisms generated by S.*

Proof. Using Remarks A.1.5.5 and A.1.5.3, we can assume without loss of generality that $A'' = \emptyset$ and $A' = A$. Using the assumption that A is well-founded, we can write A as the union of a transfinite sequence (downward-closed) subsets $\{B(\gamma) \subseteq A\}_{\gamma < \beta}$ with the following property:

(∗) For each $\gamma < \beta$, the set $B(\gamma)$ is obtained from $B'(\gamma) = \bigcup_{\gamma' < \gamma} B(\gamma')$ by adjoining a minimal element α_γ of $A - B'(\gamma)$.

For $\gamma < \beta$, let $Z_\gamma = Y_{B(\gamma)}$. We now observe that $Y_A \simeq \varinjlim_{\gamma < \beta} Z_\gamma$ and that for each $\gamma < \beta$ there is a pushout diagram

$$\begin{tikzcd} \varinjlim_{\alpha<\alpha_\gamma} Y_\alpha \arrow[r] \arrow[d] & Y_{\alpha_\gamma} \arrow[d] \\ \varinjlim_{\gamma'<\gamma} Z_{\gamma'} \arrow[r, "f"] & Z_\gamma, \end{tikzcd}$$

so that f is the pushout of a morphism belonging to S. □

Lemma A.1.5.7. *Let $\mathcal{C}$ be a presentable category, let κ be a regular cardinal, and let $S = \{f_s : C_s \to D_s\}$ be a collection of morphisms in $\mathcal{C}$, where each of the objects C_s and D_s is κ-compact. Suppose that $\{Y_\alpha\}_{\alpha\in A}$ is an S-tree in $\mathcal{C}$ indexed by a partially ordered set $(A, \leq)$. Then there exists the following:*

(1) *A new ordering $\preceq$ on A which refines $\leq$ in the following sense: if $\alpha \preceq \beta$, then $\alpha \leq \beta$. Let A' denote the partially ordered set A with this new partial ordering.*

(2) *A κ-good S-tree $\{Y'_\alpha\}_{\alpha\in A'}$ having the same root X as $\{Y_\alpha\}_{\alpha\in A}$.*

(3) *A collection of maps $f_\alpha : Y'_\alpha \to Y_\alpha$ which form a commutative diagram*

$$\begin{tikzcd} Y'_{\alpha'} \arrow[r] \arrow[d, "f_{\alpha'}"] & Y'_\alpha \arrow[d, "f_\alpha"] \\ Y_{\alpha'} \arrow[r] & Y_\alpha \end{tikzcd}$$

when $\alpha' \preceq \alpha$.

(4) *For every subset $B \subseteq A$ which is downward-closed with respect to $\preceq$, the induced map $f_B : Y'_B \to Y_B$ is an isomorphism.*

Proof. Choose a transfinite sequence of downward-closed subsets $\{A(\gamma) \subseteq A\}_{\gamma\leq\beta}$ so that the following conditions are satisfied:

(i) If $\gamma' \leq \gamma \leq \beta$, then $A(\gamma') \subseteq A(\gamma)$.

(ii) If $\lambda \leq \beta$ is a limit ordinal (possibly zero), then $A(\lambda) = \bigcup_{\gamma<\lambda} A(\gamma)$.

(iii) If $\gamma + 1 \leq \beta$, then $A(\gamma + 1) = A(\gamma) \cup \{\alpha_\gamma\}$, where α_γ is a minimal element of $A - A(\gamma)$.

(iv) The subset $A(\beta)$ coincides with A.

We will construct a compatible family of orderings $A'(\gamma) = (A(\gamma), \preceq)$, S-trees $\{Y'_\alpha\}_{\alpha\in A'(\gamma)}$, and collections of morphisms $\{Y'_\alpha \to Y_\alpha\}_{\alpha\in A(\gamma)}$ by induction on γ, so that the analogues of conditions (1) through (4) are satisfied. If γ is a limit ordinal, there is nothing to do; let us assume therefore that

$\gamma < \beta$ and that the data $(A'(\gamma), \{Y'_\alpha\}_{\alpha \in A'(\gamma)}, \{f_\alpha\}_{\alpha \in A(\gamma)})$ has already been constructed. Let $B = \{\alpha \in A : \alpha < \alpha_\gamma\}$, so that we have a pushout diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "i"] & D \arrow[d] \\ Y_B \arrow[r] & Y_\alpha \end{tikzcd}$$

where $f \in S$. By the inductive hypothesis, we may identify Y_B with Y'_B. Since C is κ-compact, the map i admits a factorization

$$C \xrightarrow{i'} Y'_{B'} \xrightarrow{i''} Y'_B,$$

where B' is κ-small. Enlarging B' if necessary, we may suppose that B' is downward-closed under $\preceq$. We now extend the partial ordering $\preceq$ to $A'(\gamma + 1) = A'(\gamma) \cup \{\alpha_\gamma\}$ by declaring that $\alpha \preceq \alpha_\gamma$ if and only if $\alpha \in B'$. We define Y'_{α_γ} by forming a pushout diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "i'"] & D \arrow[d] \\ Y'_{B'} \arrow[r] & Y'_{\alpha_\gamma}, \end{tikzcd}$$

and we define $f_{\alpha_\gamma} : Y'_{\alpha_\gamma} \to Y_{\alpha_\gamma}$ to be the map induced by i''. It is readily verified that these data satisfy the desired conditions. $\square$

Lemma A.1.5.8. *Let $\mathcal{C}$ be a presentable category, κ an uncountable regular cardinal, and S a collection of morphisms in $\mathcal{C}$. Let $\{Y_\alpha\}_{\alpha \in A}$ be a κ-good S-tree with root X and let $T_A : Y_A \to Y_A$ be an idempotent endomorphism of Y_A in the category $\mathcal{C}_{X/}$. Let B_0 be an arbitrary κ-small subset of A. Then there exists a κ-small subset $B \subseteq A$ which is downward-closed and contains B_0 and an idempotent endomorphism $T_B : Y_B \to Y_B$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "="] & Y_B \arrow[r] \arrow[d, "T_B"] & Y_A \arrow[d, "T_A"] \\ X \arrow[r] & Y_B \arrow[r] & Y_A. \end{tikzcd}$$

Proof. Enlarging B_0 if necessary, we may assume that B_0 is downward-closed. For every pair of downward-closed subsets $A'' \subseteq A' \subseteq A$, let $i_{A'',A'}$ denote the canonical map from $Y_{A''}$ to $Y_{A'}$. Note that because $\{Y_\alpha\}_{\alpha \in A}$ is a κ-good S-tree, if $A' \subseteq A$ is downward-closed and κ-small, $Y_{A'}$ is κ-compact when viewed as an object of $\mathcal{C}_{X/}$. In particular, Y_{B_0} is a κ-compact object of $\mathcal{C}_{X/}$. It follows that the composition

$$Y_{B_0} \xrightarrow{i_{B_0,A}} Y_A \xrightarrow{T_A} Y_A$$

can also be factored as a composition

$$Y_{B_0} \xrightarrow{T_0} Y_{B_1} \xrightarrow{i_{B_1,A}} Y_A,$$

where $B_1 \subseteq A$ is downward-closed and κ-small. Enlarging B_1 if necessary, we may suppose that B_1 contains B_0.

We now proceed to define a sequence of κ-small downward-closed subsets

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$$

of A and maps $T_i : Y_{B_i} \to Y_{B_{i+1}}$. Suppose that $i > 0$ and that B_i and T_{i-1} have already been constructed. By compactness again, we conclude that the composite map

$$Y_{B_i} \xrightarrow{i_{B_i,A}} Y_A \xrightarrow{T_A} Y_A$$

can be factored as

$$Y_{B_i} \xrightarrow{T_i} Y_{B_{i+1}} \xrightarrow{i_{B_{i+1},A}} Y_A,$$

where B_{i+1} is κ-small. Enlarging B_{i+1} if necessary, we may assume that B_{i+1} contains B_i and that the following diagrams commute:

$$\begin{tikzcd} Y_{B_{i-1}} \arrow[r, "T_{i-1}"] \arrow[d, "i_{B_{i-1},B_i}"] & Y_{B_i} \arrow[d, "i_{B_i,B_{i+1}}"] \\ Y_{B_i} \arrow[r, "T_i"] & Y_{B_{i+1}} \end{tikzcd} \qquad \begin{tikzcd} Y_{B_{i-1}} \arrow[r, "T_{i-1}"] \arrow[d, "T_{i-1}"] & Y_{B_i} \arrow[d, "T_i"] \\ Y_{B_i} \arrow[r, "i_{B_i,B_{i+1}}"] & Y_{B_{i+1}}. \end{tikzcd}$$

Let $B = \bigcup B_i$; then B is κ-small by virtue of our assumption that κ is uncountable. The collection of maps $\{T_i\}$ assemble to a map $T_B : Y_B \to Y_B$ with the desired properties. $\square$

Lemma A.1.5.9. *Let $\mathcal{C}$ be a presentable category, κ an uncountable regular cardinal, and S a collection of morphisms in $\mathcal{C}$. Let $\{Y_\alpha\}_{\alpha \in A}$ be a κ-good S-tree with root X, let $B \subseteq A$ be downward-closed, and suppose we are given a commutative diagram*

$$\begin{tikzcd} Y_B \arrow[r] \arrow[d, "T_B"] & Y_A \arrow[d, "T_A"] \\ Y_B \arrow[r] & Y_A \end{tikzcd}$$

in $\mathcal{C}_{X/}$, where T_A and T_B are idempotent. Let $C_0 \subseteq A$ be a κ-small subset. Then there exists a downward-closed κ-small subset $C \subseteq A$ containing C_0 and a pair of idempotent maps

$$T_C : Y_C \to Y_C$$

$$T_{B \cap C} : Y_{B \cap C} \to Y_{B \cap C}$$

such that the following diagram commutes (in $\mathcal{C}_{X/}$):

$$\begin{tikzcd} Y_B \arrow[d, "T_B"] & Y_{B \cap C} \arrow[l] \arrow[r] \arrow[d, "T_{B \cap C}"] & Y_C \arrow[r] \arrow[d, "T_C"] & Y_A \arrow[d, "T_A"] \\ Y_B & Y_{B \cap C} \arrow[l] \arrow[r] & Y_C \arrow[r] & Y_A. \end{tikzcd}$$

Proof. Enlarging C_0 if necessary, we may suppose that C_0 is downward-closed. We will define sequences of κ-small downward-closed subsets

$$C_0 \subseteq C_1 \subseteq \cdots \subseteq A$$

$$D_1 \subseteq D_2 \subseteq \cdots \subseteq B$$

and idempotent maps $\{T_{C_i} : Y_{C_i} \rightarrow Y_{C_i}\}_{i \geq 1}$, $\{T_{D_i} : Y_{D_i} \rightarrow Y_{D_i}\}_{i \geq 1}$. Moreover, we will guarantee that the following conditions are satisfied:

(i) For each $i > 0$, the set D_i contains the intersection $B \cap C_{i-1}$.

(ii) For each $i > 0$, the set C_i contains D_i.

(iii) For each $i > 0$, the diagrams

$$\begin{tikzcd} Y_{D_i} \arrow[r] \arrow[d, "T_{D_i}"] & Y_B \arrow[d, "T_B"] \\ Y_{D_i} \arrow[r] & Y_B \end{tikzcd} \qquad \begin{tikzcd} Y_{C_i} \arrow[r] \arrow[d, "T_{C_i}"] & Y_A \arrow[d, "T_A"] \\ Y_{C_i} \arrow[r] & Y_A \end{tikzcd}$$

are commutative.

(iv) For each $i > 2$, the diagrams

$$\begin{tikzcd} Y_{D_{i-2}} \arrow[r] \arrow[d, "T_{D_{i-2}}"] & Y_{D_{i-1}} \arrow[d, "T_{D_{i-1}}"] \\ Y_{D_{i-2}} \arrow[d] & Y_{D_{i-1}} \arrow[d] \\ Y_{D_i} \arrow[r, "="] & Y_{D_i} \end{tikzcd} \qquad \begin{tikzcd} Y_{C_{i-2}} \arrow[r] \arrow[d, "T_{C_{i-2}}"] & Y_{C_{i-1}} \arrow[d, "T_{C_{i-1}}"] \\ Y_{C_{i-2}} \arrow[d] & Y_{C_{i-1}} \arrow[d] \\ Y_{C_i} \arrow[r, "="] & Y_{C_i} \end{tikzcd}$$

commute.

(v) For each $i > 1$, the diagram

$$\begin{tikzcd} Y_{D_{i-1}} \arrow[r] \arrow[d, "T_{D_{i-1}}"] & Y_{C_{i-1}} \arrow[d, "T_{C_{i-1}}"] \\ Y_{D_{i-1}} \arrow[d] & Y_{C_{i-1}} \arrow[d] \\ Y_{D_i} \arrow[r] & Y_{C_i} \end{tikzcd}$$

is commutative.

The construction proceeds by induction on i. Using a compactness argument, we see that conditions (iv) and (v) are satisfied provided that we choose C_i and D_i to be sufficiently large. The existence of the desired idempotent maps satisfying (iii) then follows from Lemma A.1.5.8 applied to the roots $\{Y_\alpha\}_{\alpha \in A}$ and $\{Y_\alpha\}_{\alpha \in B}$. We now take $C = \bigcup C_i$. Conditions (i) and (ii) guarantee that $B \cap C = \bigcup D_i$. Using (iv), it follows that the maps $\{T_{C_i}\}$ and $\{T_{D_i}\}$ glue to give idempotent endomorphisms $T_C : Y_C \to Y_C$, $T_{B\cap C} : Y_{B \cap C} \to Y_{B\cap C}$. Using (iii) and (v), we deduce that all of the desired diagrams are commutative. □

Lemma A.1.5.10. *Let $\mathcal{C}$ be a presentable category, let κ be a regular cardinal, and suppose that $\mathcal{C}$ is κ-accessible: that is, $\mathcal{C}$ is generated under κ-filtered colimits by κ-compact objects (Definition 5.4.2.1). Let $f : C \to D$ be a morphism between κ-compact objects of $\mathcal{C}$, let $g : X \to Y$ be a pushout of f (so that $Y \simeq X \coprod_C D$), and let $g' : X' \to Y'$ be a retract of g in the category of morphisms of $\mathcal{C}$. Then there exists a morphism $f' : C' \to D'$ with the following properties:*

(1) *The objects $C', D' \in \mathcal{C}$ are κ-compact.*

(2) *The morphism g' is a pushout of f'.*

(3) *The morphism f' belongs to the weakly saturated class of morphisms generated by f.*

Proof. Since g' is a retract of g, there exists a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "g'"] & X \arrow[r] \arrow[d, "g"] & X' \arrow[d, "g'"] \\ Y' \arrow[r] & Y \arrow[r] & Y'. \end{tikzcd}$$

Replacing g by the induced map $X' \to X' \coprod_X Y$, we can reduce to the case where $X = X'$ and Y' is a retract of Y in $\mathcal{C}_{X/}$. Then Y' can be identified with the image of some idempotent $i : Y \to Y$.

Since $\mathcal{C}$ is κ-accessible, we can write X as the colimit of a κ-filtered diagram $\{X_\lambda\}$. The object C is κ-compact by assumption. Refining our diagram if necessary, we may assume that it takes values in $\mathcal{C}_{C/}$ and that Y is given as the colimit of the κ-filtered diagram $\{X_\lambda \coprod_C D\}$.

Because D is κ-compact, the composition $D \to Y \xrightarrow{i} Y$ admits a factorization

$$D \xrightarrow{j} X_\lambda \coprod_C D \to Y.$$

The κ-compactness of C implies that, after enlarging λ if necessary, we may suppose that the composition $j \circ f$ coincides with the canonical map from C to $X_\lambda \coprod_C D$. Consequently, j and the id_{X_λ} determine a map i' from $Y_\lambda = X_\lambda \coprod_C D$ to itself. Enlarging λ once more, we may suppose that i' is idempotent and that the diagram

$$\begin{tikzcd} Y_\lambda \arrow[r] \arrow[d, "i'"] & Y \arrow[d, "i"] \\ Y_\lambda \arrow[r] & Y \end{tikzcd}$$

is commutative. Let Y'_λ be the image of the idempotent i' and let $f' : X_\lambda \to Y'_\lambda$ be the canonical map. Then f' is a retract of the map $X_\lambda \to Y_\lambda$, which is a pushout of f. This proves (3). The objects X_λ and Y'_λ are κ-compact by construction, so that (1) is satisfied. We now observe that the diagram

$$\begin{tikzcd} X_\lambda \arrow[r] \arrow[d] & Y'_\lambda \arrow[d] \\ X \arrow[r] & Y' \end{tikzcd}$$

is a retract of the pushout diagram

$$\begin{tikzcd} X_\lambda \arrow[r] \arrow[d] & Y_\lambda \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

and therefore itself a pushout diagram. This proves (2) and completes the proof. $\square$

Lemma A.1.5.11. *Let $\mathcal{C}$ be a presentable category, κ a regular cardinal such that $\mathcal{C}$ is κ-accessible, and $S = \{f_s : C_s \to D_s\}$ a collection of morphisms $\mathcal{C}$ such that each C_s is κ-compact. Let $\{Y_\alpha\}_{\alpha \in A}$ be an S-tree in $\mathcal{C}$ with root X and suppose that A is κ-small. Then there exists a map $X' \to X$, where X is κ-compact, an S-tree $\{Y'_\alpha\}_{\alpha\in A}$ with root X', and an isomorphism of S-trees*

$$\{Y'_\alpha \coprod_{X'} X\}_{\alpha \in A} \simeq \{Y_\alpha\}_{\alpha\in A}$$

(*see Remark A.1.5.3*).

Proof. Since $\mathcal{C}$ is κ-accessible, we can write X as the colimit of diagram $\{X_i\}_{i\in I}$ indexed by a κ-filtered partially ordered set I, where each X_i is κ-compact. Choose a transfinite sequence of downward-closed subsets $\{A(\gamma) \subseteq A\}_{\gamma \leq \beta}$ so that the following conditions are satisfied:

(i) If $\gamma' \leq \gamma \leq \beta$, then $A(\gamma') \subseteq A(\gamma)$.

(ii) If $\lambda \leq \beta$ is a limit ordinal (possibly zero), then $A(\lambda) = \bigcup_{\gamma<\lambda} A(\gamma)$.

(iii) If $\gamma + 1 \leq \beta$, then $A(\gamma+1) = A(\gamma) \cup \{\alpha_\gamma\}$, where α_γ is a minimal element of $A - A(\gamma)$.

(iv) The subset $A(\beta)$ coincides with A.

Note that, since A is κ-small, we have $\beta < \kappa$.

We will construct:

(a) A transfinite sequence of elements $\{i_\gamma \in I\}_{\gamma\leq\beta}$ such that $i_\gamma \leq i_{\gamma'}$ for $\gamma \leq \gamma'$.

(b) A sequence of S-trees $\{Y^{\gamma}_{\alpha}\}_{\alpha \in A(\gamma)}\}$ having roots X_{i_γ}.

(c) A collection of isomorphisms of S-trees

$$\{Y^{\gamma}_{\alpha} \coprod_{X_{i_\gamma}} X_{i_{\gamma'}}\}_{\alpha \in A(\gamma)} \simeq \{Y^{\gamma'}_{\alpha}\}_{\alpha \in A(\gamma)}$$

$$\{Y^{\gamma}_{\alpha} \coprod_{X_{i_\gamma}} X\}_{\alpha \in A(\gamma)} \simeq \{Y_{\alpha}\}_{\alpha \in A(\gamma)}$$

which are compatible with one another in the obvious sense.

If γ is a limit ordinal (or zero), we simply choose i_γ to be any upper bound for $\{i_{\gamma'}\}_{\gamma' < \gamma}$ in I. The rest of the data is uniquely determined. The existence of such an upper bound is guaranteed by our assumption that I is κ-filtered since $\gamma \leq \beta < \kappa$. Let us therefore suppose that the above data has been constructed for all ordinals $\leq \gamma$, and proceed to define $i_{\gamma+1}$. Let $i = i_\gamma$, let $\alpha = \alpha_\gamma$, and let $B = \{\beta \in A : \beta < \alpha\}$. Then we have canonical isomorphisms

$$Y_B \simeq Y^{\gamma}_{B} \coprod_{X_i} X \simeq \varinjlim \{Y^{\gamma}_{B} \coprod_{X_i} X_j\}_{j \geq i}$$

and a pushout diagram

$$\begin{tikzcd}
C_s \arrow[r, "f_s"] \arrow[d, "g"] & D_s \arrow[d] \\
Y_B \arrow[r] & Y_\alpha.
\end{tikzcd}$$

The κ-compactness of C_s implies that g factors as a composition

$$C_s \xrightarrow{g'} Y^{\gamma}_{B} \coprod_{X_i} X_j$$

for some $j \geq i$. We now define $i_{\gamma+1} = j$, and $Y^{\gamma+1}_{\alpha}$ by forming a pushout diagram

$$\begin{tikzcd}
C_s \arrow[r] \arrow[d, "g_s"] & D_s \arrow[d] \\
Y^{\gamma}_{B} \coprod_{X_i} X_j \arrow[r] & Y^{\gamma+1}_{\alpha}.
\end{tikzcd}$$

□

Proposition A.1.5.12. *Let $\mathcal{C}$ be a presentable ∞-category, κ a regular cardinal, and $\overline{S}$ a weakly saturated class of morphisms in $\mathcal{C}$. Let $S \subseteq \overline{S}$ be the subset consisting of those morphisms $f : X \to Y$ in $\overline{S}$ such that X and Y are κ-compact. Assume that*

(i) *The regular cardinal κ is uncountable, and $\mathcal{C}$ is κ-accessible.*

(*ii*) *The set* S *generates* $\overline{S}$ *as a weakly saturated class of morphisms.*

Then, for every morphism $f : X \to Y$ *belonging to* $\overline{S}$*, there exists a transfinite sequence of objects* $\{Y_\gamma\}_{\gamma<\beta}$ *of* $\mathcal{C}_{X/}$ *with the following properties:*

(1) *For every ordinal* $\gamma < \beta$*, the natural map* $\varinjlim_{\gamma'<\gamma} Z_{\gamma'} \to Z_\gamma$ *is the pushout of a morphism in* S*.*

(2) *The colimit* $\varinjlim_{\gamma<\beta} Z_\gamma$ *is isomorphic to* Y *(as objects of* $\mathcal{C}_{X/}$*).*

Proof. Remark A.1.2.8 implies the existence of a transfinite sequence of objects

$$Y_0 \to Y_1 \to \cdots$$

in $\mathcal{C}_{X/}$ indexed by a set of ordinals $A = \{\alpha | \alpha < \lambda\}$, satisfying condition (1), such that Y is a *retract* of $\varinjlim_{\alpha<\lambda} Y_\alpha$ in $\mathcal{C}_{X/}$. We may view the sequence $\{Y_\alpha\}_{\alpha\in A}$ as an S-tree in $\mathcal{C}$ having root X. According to Lemma A.1.5.7, we can choose a new S-tree $\{Y'_\alpha\}_{\alpha\in A'}$ which is κ-good, where $Y'_{A'} \simeq Y_A$, so that Y is a retract of $Y'_{A'}$. Choose an idempotent map $T_{A'} : Y'_{A'} \to Y'_{A'}$ in $\mathcal{C}_{X/}$ whose image is isomorphic to Y.

We now define a transfinite sequence

$$B(0) \subseteq B(1) \subseteq B(2) \subseteq \cdots,$$

indexed by ordinals $\gamma < \beta$, and a compatible system of idempotent maps $T_{B(\gamma)} : Y'_{B_\gamma} \to Y'_{B_\gamma}$. Fix an ordinal γ and suppose that $B(\gamma')$ and $T_{B(\gamma')}$ have been defined for $\gamma' < \gamma$. Let $B'(\gamma) = \bigcup_{\gamma'<\gamma} B(\gamma')$ and let $T_{B'(\gamma)}$ be the result of amalgamating the maps $\{T_{B(\gamma')}\}_{\gamma'<\gamma}$. If $B'(\gamma) = A'$, we set $\beta = \gamma$ and conclude the construction; otherwise, we choose a minimal element $a \in A' - B'(\gamma)$. Applying Lemma A.1.5.9, we deduce the existence of a downward-closed subset $C(\gamma) \subseteq A'$ and a compatible collection of idempotent maps

$$T_{C(\gamma)} : Y'_{C(\gamma)} \to Y'_{C(\gamma)}$$

$$T_{C(\gamma)\cap B'(\gamma)} : Y'_{C(\gamma)\cap B'(\gamma)} \to Y'_{C(\gamma)\cap B'(\gamma)}.$$

We then define $B(\gamma) = B'(\gamma) \cup C(\gamma)$ and define $T_{B(\gamma)}$ to be the result of amalgamating $T_{B'(\gamma)}$ and $T_{C(\gamma)}$.

For every ordinal γ, there is a κ-good S-tree $\{Y''_\alpha\}_{\alpha\in B(\gamma)-B'(\gamma)}$ with root $Y'_{B(\gamma)}$ such that $Y''_{B(\gamma)-B'(\gamma)} \simeq Y'_{B(\gamma)}$ (Remark A.1.5.5). Combining Lemma A.1.5.11 with the observation that $B(\gamma) - B'(\gamma)$ is κ-small, we deduce that the map

$$Y'_{B'(\gamma)} \to Y'_{B(\gamma)}$$

is the pushout of a morphism in S.

For each ordinal $\gamma < \beta$, let Z_γ denote the image of the idempotent map $T_{B(\gamma)}$. Then $\varinjlim_{\gamma<\beta} Z_\gamma \simeq Y$, so that (2) is satisfied. Condition (1) follows from Lemma A.1.5.10. □

Corollary A.1.5.13. *Under the hypotheses of Proposition A.1.5.12, there exists a* κ*-good* S*-tree* $\{Y_\alpha\}_{\alpha\in A}$ *such that* $Y_A \simeq Y$ *in* $\mathcal{C}_{X/}$*.*

Proof. Combine Proposition A.1.5.12 with Lemma A.1.5.7. □

A.2 MODEL CATEGORIES

One of the oldest and most successful approaches to the study of higher-categorical phenomena is Quillen's theory of model categories. In this book, Quillen's theory will play two (related) roles:

(1) The structures that we use to describe higher categories are naturally organized into model categories. For example, ∞-categories are precisely those simplicial sets which are fibrant with respect to the Joyal model structure (Theorem 2.4.6.1). The theory of model categories provides a convenient framework for phrasing certain results and for comparing different models of higher category theory (see, for example, §2.2.5).

(2) The theory of model categories can itself be regarded as an approach to higher category theory. If $\mathbf{A}$ is a simplicial model category, then the subcategory $\mathbf{A}^{\circ} \subseteq \mathbf{A}$ of fibrant-cofibrant objects forms a fibrant simplicial category. Proposition 1.1.5.10 implies that the simplicial nerve $\mathrm{N}(\mathbf{A}^{\circ})$ is an ∞-category. We will refer to $\mathrm{N}(\mathbf{A}^{\circ})$ as the *underlying ∞-category* of $\mathbf{A}$. Of course, not every ∞-category arises in this way, even up to equivalence: for example, the existence of homotopy limits and homotopy colimits in $\mathbf{A}$ implies the existence of various limits and colimits in $\mathrm{N}(\mathbf{A}^{\circ})$ (Corollary 4.2.4.8). Nevertheless, we can often use the theory of model categories to prove theorems about general ∞-categories by reducing to the situation of ∞-categories which arise via the above construction (every ∞-category $\mathcal{C}$ admits a fully faithful embedding into $\mathrm{N}(\mathbf{A}^{\circ})$ for an appropriately chosen simplicial model category $\mathbf{A}$). For example, our proof of the ∞-categorical Yoneda lemma (Proposition 5.1.3.1) uses this strategy.

The purpose of this section is to review the theory of model categories with an eye toward the sort of applications described above. Our exposition is somewhat terse, and we will omit many proofs. For a more detailed account, we refer the reader to [40] (or any other text on the theory of model categories).

A.2.1 The Model Category Axioms

Definition A.2.1.1. A *model category* is a category $\mathcal{C}$ which is equipped with three distinguished classes of morphisms in $\mathcal{C}$, called *cofibrations, fibrations,* and *weak equivalences*, in which the following axioms are satisfied:

(1) The category $\mathcal{C}$ admits (small) limits and colimits.

(2) Given a composable pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, if any two of $g \circ f$, f, and g are weak equivalences, then so is the third.

(3) Suppose $f : X \to Y$ is a retract of $g : X' \to Y'$: that is, suppose there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & X' \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "i'"] & Y' \arrow[r, "r'"] & Y, \end{tikzcd}$$

where $r \circ i = \mathrm{id}_X$ and $r' \circ i' = \mathrm{id}_Y$. Then

(i) If g is a fibration, so is f.

(ii) If g is a cofibration, then so is f.

(iii) If g is a weak equivalence, then so is f.

(4) Given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

a dotted arrow can be found rendering the diagram commutative if either

(i) The map i is a cofibration, and the map p is both a fibration and a weak equivalence.

(ii) The map i is both a cofibration and a weak equivalence, and the map p is a fibration.

(5) Any map $X \to Z$ in $\mathcal{C}$ admits factorizations

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$X \xrightarrow{f'} Y' \xrightarrow{g'} Z,$$

where f is a cofibration, g is a fibration and a weak equivalence, f' is a cofibration and a weak equivalence, and g' is a fibration.

A map f in a model category $\mathcal{C}$ is called a *trivial cofibration* if it is both a cofibration and a weak equivalence; similarly, f is called a *trivial fibration* if it is both a fibration and a weak equivalence. By axiom (1), any model category $\mathcal{C}$ has an initial object $\emptyset$ and a final object $*$. An object $X \in \mathcal{C}$ is said to be *fibrant* if the unique map $X \to *$ is a fibration and *cofibrant* if the unique map $\emptyset \to X$ is a cofibration.

Example A.2.1.2. Let $\mathcal{C}$ be any category which admits small limits and colimits. Then $\mathcal{C}$ can be endowed with the *trivial* model structure:

(W) The weak equivalences in $\mathcal{C}$ are the isomorphisms.

(C) Every morphism in $\mathcal{C}$ is a cofibration.

(F) Every morphism in $\mathcal{C}$ is a fibration.

A.2.2 The Homotopy Category of a Model Category

Let $\mathcal{C}$ be a model category containing an object X. A *cylinder object* for X is an object C together with a diagram $X \coprod X \xrightarrow{i} C \xrightarrow{j} X$ where i is a cofibration, j is a weak equivalence, and the composition $j \circ i$ is the "fold map" $X \coprod X \to X$. Dually, a *path object* for $Y \in \mathcal{C}$ is an object P together with a diagram

$$Y \xrightarrow{q} P \xrightarrow{p} Y \times Y$$

such that q is a weak equivalence, p is a fibration, and $p \circ q$ is the diagonal map $Y \to Y \times Y$. The existence of cylinder and path objects follows from the factorization axiom (5) of Definition A.2.1.1 (factor the fold map $X \coprod X \to X$ as a cofibration followed by a trivial fibration and the diagonal map $Y \to Y \times Y$ as a trivial cofibration followed by a fibration).

Proposition A.2.2.1. *Let $\mathcal{C}$ be a model category. Let X be a cofibrant object of $\mathcal{C}$, Y a fibrant object of $\mathcal{C}$, and $f, g : X \to Y$ two maps. The following conditions are equivalent:*

(1) *For every cylinder object $X \coprod X \xrightarrow{j} C$, there exists a commutative diagram*

$$\begin{tikzcd} X \coprod X \arrow[rr, "j"] \arrow[dr, "(f,g)"'] & & C \arrow[dl] \\ & Y. & \end{tikzcd}$$

(2) *There exists a cylinder object $X \coprod X \xrightarrow{j} C$ and a commutative diagram*

$$\begin{tikzcd} X \coprod X \arrow[rr, "j"] \arrow[dr, "(f,g)"'] & & C \arrow[dl] \\ & Y. & \end{tikzcd}$$

(3) *For every path object $P \xrightarrow{p} Y \times Y$, there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "(f,g)"'] & & P \arrow[dl, "p"] \\ & Y \times Y. & \end{tikzcd}$$

(4) *There exists a path object $P \xrightarrow{p} Y \times Y$ and a commutative diagram*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "(f,g)"'] & & P \arrow[dl, "p"] \\ & Y \times Y. & \end{tikzcd}$$

If $\mathcal{C}$ is a model category containing a cofibrant object X and a fibrant object Y, we say two maps $f, g : X \to Y$ are *homotopic* if the hypotheses of Proposition A.2.2.1 are satisfied and write $f \simeq g$. The relation $\simeq$ is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. The *homotopy category* $\mathrm{h}\mathcal{C}$ may be defined as follows:

- The objects of $\mathrm{h}\mathcal{C}$ are the fibrant-cofibrant objects of $\mathcal{C}$.

- For $X, Y \in \mathrm{h}\mathcal{C}$, the set $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ is the set of $\simeq$-equivalence classes of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Composition is well-defined in $\mathrm{h}\mathcal{C}$ by virtue of the fact that if $f \simeq g$, then $f \circ h \simeq g \circ h$ (this is clear from characterization (2) of Proposition A.2.2.1) and $h' \circ f \simeq h' \circ g$ (this is clear from characterization (4) of Proposition A.2.2.1) for any maps h, h' such that the compositions are defined in $\mathcal{C}$.

There is another way of defining $\mathrm{h}\mathcal{C}$ (or, at least, a category equivalent to $\mathrm{h}\mathcal{C}$): one begins with all of $\mathcal{C}$ and formally adjoins inverses to all weak equivalences. Let $H(\mathcal{C})$ denote the category so obtained. If $X \in \mathcal{C}$ is cofibrant and $Y \in \mathcal{C}$ is fibrant, then homotopic maps $f, g : X \to Y$ have the same image in $H(\mathcal{C})$; consequently, we obtain a functor $\mathrm{h}\mathcal{C} \to H(\mathcal{C})$ which can be shown to be an equivalence. We will generally ignore the distinction between these two categories, employing whichever description is more useful for the problem at hand.

Remark A.2.2.2. Since $\mathcal{C}$ is (generally) not a small category, it is not immediately clear that $H(\mathcal{C})$ has small morphism sets; however, this follows from the equivalence between $H(\mathcal{C})$ and $\mathrm{h}\mathcal{C}$.

A.2.3 A Lifting Criterion

The following basic principle will be used many times throughout this book:

Proposition A.2.3.1. *Let $\mathcal{C}$ be a model category containing cofibrant objects A and B and a fibrant object X. Suppose we are given a cofibration $i : A \to B$ and any map $f : A \to X$. Suppose moreover that there exists a commutative diagram*

$$\begin{tikzcd} A \arrow[dr, "{[f]}"] \arrow[dd, "{[i]}"'] & \\ & X \\ B \arrow[ur, "\overline{g}"'] & \end{tikzcd}$$

in the homotopy category $h\,\mathcal{C}$*. Then there exists a commutative diagram*

$$\begin{tikzcd} A \arrow[d, "i"] \arrow[dr, "f"] & \\ B \arrow[r, "g"'] & X \end{tikzcd}$$

in $\mathcal{C}$*, with* $[g] = \overline{g}$*. (Here we let* $[p]$ *denote the homotopy class in* $\mathrm{h}\mathcal{C}$ *of a morphism* p *in* $\mathcal{C}$*.)*

Proof. Choose a map $g' : B \to X$ representing the homotopy class $\overline{g}$. Choose a cylinder object

$$A \coprod A \to C(A) \to A$$

and a factorization

$$C(A) \coprod_{A \coprod A} (B \coprod B) \to C(B) \to B,$$

where the first map is a cofibration and the second is a trivial fibration. We observe that $C(B)$ is a cylinder object for B.

Since $g' \circ i$ is homotopic to f, there exists a map $h_0 : C(A) \coprod_A B \to X$ with $h_0|B = g'$ and $h_0|A = f$. The inclusion $C(A) \coprod_A B \to C(B)$ is a trivial cofibration, so h_0 extends to a map $h : C(B) \to X$. We may regard h as a homotopy from g' to g, where $g \circ i = f$. □

Proposition A.2.3.1 will often be applied in the following way. Suppose we are given a diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, "i"] \arrow[dr, "f"] & \\ B' \arrow[r] & B \arrow[r, dashed] & X \end{tikzcd}$$

which we would like to extend as indicated by the dotted arrow. If X is fibrant, i is a cofibration between cofibrant objects, and the horizontal arrows are weak equivalences, then it suffices to solve the (frequently easier) problem of constructing the dotted arrow in the diagram

$$\begin{tikzcd} A' \arrow[d] \arrow[dr] & \\ B' \arrow[r, dashed] & X \end{tikzcd}.$$

A.2.4 Left Properness and Homotopy Pushout Squares

Definition A.2.4.1. A model category $\mathcal{C}$ is *left proper* if, for any pushout square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d, "j'"] \\ A' \arrow[r, "i'"] & B' \end{tikzcd}$$

in which i is a cofibration and j is a weak equivalence, the map j' is also a weak equivalence. Dually, $\mathcal{C}$ is *right proper* if, for any pullback square

$$\begin{tikzcd} X' \arrow[r, "p'"] \arrow[d, "q'"] & Y' \arrow[d, "q"] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

in which p is a fibration and q is a weak equivalence, the map q' is also a weak equivalence.

In this book, we will deal almost exclusively with left proper model categories. The following provides a useful criterion for establishing left properness.

Proposition A.2.4.2. *Let $\mathcal{C}$ be a model category in which every object is cofibrant. Then $\mathcal{C}$ is left proper.*

Proposition A.2.4.2 is an immediate consequence of the following basic lemma:

Lemma A.2.4.3. *Let $\mathcal{C}$ be a model category containing a pushout diagram*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d, "j'"] \\ A' \arrow[r, "i'"] & B'. \end{tikzcd}$$

Suppose that A and A' are cofibrant, i is a cofibration, and j is a weak equivalence. Then j' is a weak equivalence.

Proof. We wish to show that j' is an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}$. In other words, we need to show that for every fibrant object Z of $\mathcal{C}$, composition with j' induces a bijection $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(B', Z) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(B, Z)$.

We first show that composition with j' is surjective on homotopy classes. Suppose we are given a map $f : B \to Z$. Since j is a weak equivalence, the composition $f \circ i$ is homotopic to $g \circ j$ for some $g : A' \to Z$. According to Proposition A.2.3.1, there is a map $f' : B \to Z$ such that $f' \circ i = g \circ j$ and such that f' is homotopic to f. The amalgamation of f' and g determines a map $B' \to Z$ which lifts f'.

We now show that j' is injective on homotopy classes. Suppose we are given a pair of maps $s, s' : B' \to Z$. Let P be a path object for Z. If $s \circ j'$ and $s' \circ j'$ are homotopic, then there exists a commutative diagram

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "j'"] & P \arrow[d] \\ B' \arrow[r, "s \times s'"] & Z \times Z. \end{tikzcd}$$

We now replace $\mathcal{C}$ by $\mathcal{C}_{/Z \times Z}$ and apply the surjectivity statement above to deduce that there is a map $h' : B' \to P$ such that h is homotopic to $h' \circ j'$. The existence of h' shows that s and s' are homotopic, as desired. $\square$

Suppose we are given a diagram

$$A_0 \leftarrow A \rightarrow A_1$$

in a model category $\mathcal{C}$. In general, the pushout $A_0 \coprod_A A_1$ is poorly behaved in the sense that a map of diagrams

$$\begin{tikzcd} A_0 \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & A_1 \arrow[d] \\ B_0 & B \arrow[l] \arrow[r] & B_1 \end{tikzcd}$$

need not induce a weak equivalence $A_0 \coprod_A A_1 \to B_0 \coprod_B B_1$, even if each of the vertical arrows in the diagram is individually a weak equivalence. To correct this difficulty, it is convenient to introduce the left derived functor of "pushout". The *homotopy pushout* of the diagram

$$A_0 \longleftarrow A \longrightarrow A_1$$

is defined to be the pushout $A'_0 \coprod_{A'} A'_1$, where we have chosen a commutative diagram

$$\begin{tikzcd} A'_0 \arrow[d] & A' \arrow[l, "j"'] \arrow[r, "i"] \arrow[d] & A'_1 \arrow[d] \\ A_0 & A \arrow[l] \arrow[r] & A_1 \end{tikzcd}$$

where the vertical maps are weak equivalences, and the top row is *cofibrant* diagram in the sense that A' is cofibrant and the maps i and j are both cofibrations. One can show that such a diagram exists and that the pushout $A'_0 \coprod_{A'} A'_1$ depends on the choice of diagram only up to weak equivalence. (For a more systematic approach which includes a definition of "cofibrant" for more complicated diagrams, we refer the reader to §A.3.3.)

More generally, we will say that a diagram

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ A_0 \arrow[dr] & & A_1 \arrow[dl] \\ & M & \end{tikzcd}$$

is a *homotopy pushout square* if the composite map

$$A'_0 \coprod_{A'} A'_1 \to A_0 \coprod_{A} A_1 \to M$$

is a weak equivalence. In this case we will also say that M is a *homotopy pushout* of A_0 and A_1 over A. One can show that this condition is independent of the choice of a "cofibrant resolution"

$$A'_0 \longleftarrow A' \longrightarrow A'_1$$

of the original diagram. In particular, we note that if the diagram

$$A_0 \longleftarrow A \longrightarrow A_1$$

is *already* cofibrant, then the ordinary pushout $A_0 \coprod_A A_1$ is a homotopy pushout. However, the condition that the diagram be cofibrant is quite strong; in good situations we can get away with quite a bit less:

Proposition A.2.4.4. *Let $\mathcal{C}$ be a model category and let*

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ A_0 \arrow[dr] & & A_1 \arrow[dl] \\ & A_0 \coprod_A A_1 & \end{tikzcd}$$

be a pushout square in $\mathcal{C}$. This diagram is also a homotopy pushout square if either of the following conditions is satisfied:

(*i*) *The objects A and A_0 are cofibrant, and j is a cofibration.*

(*ii*) *The map j is a cofibration, and $\mathcal{C}$ is left proper.*

Remark A.2.4.5. The above discussion of homotopy pushouts can be dualized; one obtains the notion of *homotopy pullbacks*, and the analogue of Proposition A.2.4.4 requires either that $\mathcal{C}$ be a *right* proper model category or that the objects in the diagram be fibrant.

A.2.5 Quillen Adjunctions and Quillen Equivalences

Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and suppose we are given a pair of adjoint functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

(here F is the left adjoint and G is the right adjoint). The following conditions are equivalent:

(1) The functor F preserves cofibrations and trivial cofibrations.

(2) The functor G preserves fibrations and trivial fibrations.

(3) The functor F preserves cofibrations, and the functor G preserves fibrations.

(4) The functor F preserves trivial cofibrations, and the functor G preserves trivial fibrations.

If any of these equivalent conditions is satisfied, then we say that the pair (F, G) is a *Quillen adjunction* between $\mathcal{C}$ and $\mathcal{D}$. We also say that F is a *left Quillen functor* and that G is a *right Quillen functor*. In this case, one can show that F preserves weak equivalences between cofibrant objects and G preserves weak equivalences between fibrant objects.

Suppose that $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ is a Quillen adjunction. We may view the homotopy category $\mathrm{h}\mathcal{C}$ as obtained from $\mathcal{C}$ by first passing to the full subcategory consisting of cofibrant objects and then inverting all weak equivalences. Applying a similar procedure with $\mathcal{D}$, we see that because F preserves weak equivalence between cofibrant objects, it induces a functor $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$; this functor is called the *left derived functor of* F and denoted LF. Similarly, one may define the *right derived functor* RG of G. One can show that LF and RG determine an adjunction between the homotopy categories $\mathrm{h}\mathcal{C}$ and $\mathrm{h}\mathcal{D}$.

Proposition A.2.5.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and let*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be a Quillen adjunction. The following are equivalent:

(1) *The left derived functor $LF : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an equivalence of categories.*

(2) *The right derived functor $RG : \mathrm{h}\mathcal{D} \to \mathrm{h}\mathcal{C}$ is an equivalence of categories.*

(3) *For every cofibrant object $C \in \mathcal{C}$ and every fibrant object $D \in \mathcal{D}$, a map $C \to G(D)$ is a weak equivalence in $\mathcal{C}$ if and only if the adjoint map $F(C) \to D$ is a weak equivalence in $\mathcal{D}$.*

Proof. Since the derived functors LF and RG are adjoint to one another, it is clear that (1) is equivalent to (2). Moreover, (1) and (2) are equivalent to the assertion that the unit and counit of the adjunction

$$u : \mathrm{id}_{\mathcal{C}} \to RG \circ LF$$

$$v : LF \circ RG \to \mathrm{id}_{\mathcal{D}}$$

are weak equivalences. Let us consider the unit u. Choose a fibrant object C of $\mathcal{C}$. The composite functor $(RG \circ LF)(C)$ is defined to be $G(D)$, where $F(C) \to D$ is a weak equivalence in $\mathcal{D}$ and D is a fibrant object of $\mathcal{D}$. Thus u is a weak equivalence when evaluated on C if and only if for any weak equivalence $F(C) \to D$, the adjoint map $C \to G(D)$ is a weak equivalence. Similarly, the counit v is a weak equivalence if and only if the converse holds. Thus (1) and (2) are equivalent to (3). □

If the equivalent conditions of Proposition A.2.5.1 are satisfied, then we say that the adjunction (F, G) gives a *Quillen equivalence* between the model categories $\mathcal{C}$ and $\mathcal{D}$.

A.2.6 Combinatorial Model Categories

In this section, we give an overview of Jeff Smith's theory of *combinatorial model categories*. Our main goal is to prove Proposition A.2.6.13, which allows us to construct model structures on a category $\mathcal{C}$ by specifying the class of weak equivalences together with a small amount of additional data.

Definition A.2.6.1 (Smith)**.** Let $\mathbf{A}$ be a model category. We say that $\mathbf{A}$ is *combinatorial* if the following conditions are satisfied:

(1) The category $\mathbf{A}$ is presentable.

(2) There exists a set I of *generating cofibrations* such that the collection of all cofibrations in $\mathbf{A}$ is the smallest weakly saturated class of morphisms containing I (see Definition A.1.2.2).

(3) There exists a set J of *generating trivial cofibrations* such that the collection of all trivial cofibrations in $\mathbf{A}$ is the smallest weakly saturated class of morphisms containing J.

If $\mathcal{C}$ is a combinatorial model category, then the model structure on $\mathcal{C}$ is uniquely determined by the generating cofibrations and generating trivial cofibrations. However, in practice these generators might be difficult to find. Our goal in this section is to reformulate Definition A.2.6.1 in a manner which puts more emphasis on the collection of weak equivalences in $\mathbf{A}$.

In practice, it is often easier to describe the class of *all* weak equivalences than it is to describe a class of generating trivial cofibrations.

Definition A.2.6.2. Let $\mathcal{C}$ be a presentable category and κ a regular cardinal. We will say that a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is a *κ-accessible subcategory* of $\mathcal{C}$ if the following conditions are satisfied:

(1) The full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is stable under κ-filtered colimits.

(2) There exists a (small) set of objects of $\mathcal{C}_0$ which generates $\mathcal{C}_0$ under κ-filtered colimits.

We will say that $\mathcal{C}_0 \subseteq \mathcal{C}$ is an *accessible subcategory* if $\mathcal{C}_0$ is a κ-accessible subcategory of $\mathcal{C}$ for some regular cardinal κ.

Condition (2) of Definition A.2.6.2 admits the following reformulation:

Proposition A.2.6.3. *Let κ be a regular cardinal, $\mathcal{C}$ a presentable category, and $\mathcal{C}_0 \subseteq \mathcal{C}$ a full subcategory which is stable under κ-filtered colimits. Then $\mathcal{C}_0$ satisfies condition* (2) *of Definition A.2.6.2 if and only if the following condition is satisfied for all sufficiently large regular cardinals $\tau \gg \kappa$:*

($2'_\tau$) *Let A be a τ-filtered partially ordered set and $\{X_\alpha\}_{\alpha \in A}$ a diagram of τ-compact objects of $\mathcal{C}$ indexed by A. For every κ-filtered subset $B \subseteq A$, we let X_B denote the* (κ*-filtered*) *colimit of the diagram $\{X_\alpha\}_{\alpha \in B}$. Suppose that X_A belongs to $\mathcal{C}_0$. Then for every τ-small subset $C \subseteq A$, there exists a τ-small κ-filtered subset $B \subseteq A$ which contains C, such that X_B belongs to $\mathcal{C}_0$.*

First, we need the following preliminary result:

Lemma A.2.6.4. *Let $\tau \gg \kappa$ be regular cardinals such that $\tau > \kappa$, let $\mathcal{D}$ be a presentable ∞-category, and let $\{C_a\}_{a \in A}$ and $\{D_b\}_{b \in B}$ be families of τ-compact objects in $\mathcal{D}$ indexed by τ-filtered partially ordered sets A and B, such that*

$$\varinjlim_{a \in A} C_a \simeq \varinjlim_{b \in B} D_b.$$

Then, for every pair of τ-small subsets $A_0 \subseteq A$, $B_0 \subseteq B$, there exist τ-small κ-filtered subsets $A' \subseteq A$, $B' \subseteq B$ such that $A_0 \subseteq A'$, $B_0 \subseteq B'$, and $\varinjlim_{a \in A'} C_a \simeq \varinjlim_{b \in B'} D_b$.

Proof. Let $\mathcal{A}$ be the partially ordered set of all τ-small κ-filtered subsets of A which contain A_0, let $\mathcal{B}$ be the partially ordered set of all τ-small κ-filtered subsets of B which contain B_0, let $X \in \mathcal{D}$ be the common colimit $\varinjlim_{a \in A} C_a \simeq \varinjlim_{b \in B} D_b$, and let $\mathcal{C}$ be the full subcategory of $\mathcal{D}_{/X}$ spanned by those morphisms $Y \to X$, where Y is a τ-compact object of $\mathcal{D}$. Let $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ be the functors described by the formulas

$$f(A') = (\varinjlim_{a \in A'} C_a \to \varinjlim_{a \in A} C_a)$$

$$g(B') = (\varinjlim_{b \in B'} D_b \to \varinjlim_{b \in B} D_b).$$

The desired result now follows by applying Lemma 5.4.6.3 to the associated diagram

$$\mathrm{N}(\mathcal{A}) \to \mathrm{N}(\mathcal{C}) \leftarrow \mathrm{N}(\mathcal{B}).$$

$\square$

Proof of Proposition A.2.6.3. First suppose that $(2'_\tau)$ is satisfied for all sufficiently large $\tau \gg \kappa$. Choose $\tau \gg \kappa$ large enough that $\mathcal{C}$ is generated under colimits by its full subcategory $\mathcal{C}^\tau$ of τ-compact objects and such that $(2'_\tau)$ is satisfied. Let $\mathcal{D} = \mathcal{C}^\tau \cap \mathcal{C}_0$, so that $\mathcal{D}$ is essentially small. We will show that $\mathcal{D}$ generates $\mathcal{C}_0$ under τ-filtered colimits. By assumption, every object $X \in \mathcal{C}$ can be obtained as a τ-filtered colimit of τ-compact objects $\{X_\alpha\}_{\alpha \in A}$. Let A' denote the collection of all τ-small κ-filtered subsets $B \subseteq A$ such that $X_B \in \mathcal{C}_0$. We regard A' as partially ordered via inclusions. Invoking condition $(2'_\tau)$, we deduce that X_A is the colimit of the τ-filtered collection of objects $\{X_{A'}\}_{A' \in B}$. We now observe that each $X_{A'}$ belongs to $\mathcal{D}$.

Now suppose that condition (2) is satisfied, so that $\mathcal{C}_0$ is generated under κ-filtered colimits by a small subcategory $\mathcal{D} \subseteq \mathcal{C}_0$. Choose $\tau \gg \kappa$ large enough that every object of $\mathcal{D}$ is τ-compact. Enlarging τ if necessary, we may suppose that $\tau > \kappa$. We claim that $(2'_\tau)$ is satisfied. To prove this, we consider any system of morphisms $\{X\}_{\alpha \in A}$ satisfying the hypotheses of $(2'_\tau)$. In particular, X_A belongs to $\mathcal{C}_0$, so that X_A may be obtained in some *other* way as a κ-filtered colimit of a system $\{Y_\beta\}_{\beta \in B}$, where each of the objects Y_β belongs to $\mathcal{D}$ and is therefore τ-compact. Let C' denote the family of all τ-small κ-filtered subsets $B_0 \subseteq B$. Replacing B by B' and the family $\{Y_\beta\}_{\beta \in B}$ by $\{Y_{B_0}\}_{B_0 \in B'}$, we may assume that B is τ-filtered.

Let $A_0 \subseteq A$ be a τ-small subset. Applying Lemma A.2.6.4 to the diagram category $\mathcal{C}$, we deduce that $A_0 \subseteq A'$, where A' is a τ-small, κ-filtered subset of A and there is an isomorphism $X_{A'} \simeq Y_{B'}$; here B' is a κ-filtered subset of B, so that $Y_{B'} \in \mathcal{C}_0$ by virtue of our assumption that $\mathcal{C}_0$ is stable under κ-filtered colimits. $\square$

Corollary A.2.6.5. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between presentable categories which preserves κ-filtered colimits and let $\mathcal{D}_0 \subseteq \mathcal{D}$ be a κ-accessible subcategory. Then $f^{-1}\,\mathcal{D}_0 \subseteq \mathcal{C}$ is a κ-accessible subcategory.*

Corollary A.2.6.6 (Smith)**.** *Let $\mathbf{A}$ be a combinatorial model category, let $\mathbf{A}^{[1]}$ be the category of morphisms in $\mathbf{A}$, let $W \subseteq \mathbf{A}^{[1]}$ be the full subcategory spanned by the weak equivalences, and let $F \subseteq \mathbf{A}^{[1]}$ be the full subcategory spanned by the fibrations. Then F, W, and $F \cap W$ are accessible subcategories of $\mathbf{A}^{[1]}$.*

Proof. For every morphism $i : A \to B$, let $F_i : \mathbf{A}^{[1]} \to \mathcal{S}\mathrm{et}^{[1]}$ be the functor which carries a morphism $f : X \to Y$ to the induced map of sets

$$\mathrm{Hom}_{\mathbf{A}}(B, X) \to \mathrm{Hom}_{\mathbf{A}}(B, Y) \times_{\mathrm{Hom}_{\mathbf{A}}(A,Y)} \mathrm{Hom}_{\mathbf{A}}(A, X).$$

We observe that if A and B are κ-compact objects of $\mathbf{A}$, then F_i preserves κ-filtered colimits.

Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{S}\mathrm{et}^{[1]}$ spanned by the collection of *surjective* maps between sets. It is easy to see that $\mathcal{C}_0$ is an accessible category of $\mathcal{S}\mathrm{et}^{[1]}$. It follows that the full subcategories $R(i) = F_i^{-1}\,\mathcal{C}_0 \subseteq \mathbf{A}^{[1]}$ are accessible subcategories of $\mathbf{A}^{[1]}$ (Corollary A.2.6.5).

Let I be a set of generating cofibrations for $\mathbf{A}$ and let J be a set of generating trivial cofibrations. Then Proposition 5.4.7.10 implies that the subcategories

$$F = \bigcap_{j \in J} R(j)$$

$$W \cap F = \bigcap_{i \in I} R(i)$$

are accessible subcategories of $\mathbf{A}^{[1]}$.

Applying Proposition A.1.2.5, we deduce that there exists a pair of functors $T', T'' : \mathbf{A}^{[1]} \to \mathbf{A}^{[1]}$, which carry an arbitrary morphism $f : X \to Z$ to a factorization

$$X \stackrel{T'(f)}{\to} Y \stackrel{T''(f)}{\to} Z,$$

where $T'(f)$ is a trivial cofibration and $T''(f)$ is a fibration. Moreover, the functor T'' can be chosen to commute with κ-filtered colimits for a sufficiently large regular cardinal κ. We now observe that W is the inverse image of $F \cap W$ under the functor $T'' : \mathbf{A}^{[1]} \to \mathbf{A}^{[1]}$ and is therefore an accessible subcategory of $\mathbf{A}^{[1]}$ by Corollary A.2.6.5. □

Our next goal is to prove a converse to Corollary A.2.6.6, which will allow us to construct examples of combinatorial model categories. First, we need the following preliminary result.

Lemma A.2.6.7. *Let $\mathbf{A}$ be a presentable category. Suppose W and C are collections of morphisms of $\mathbf{A}$ with the following properties:*

1. *The collection C is a weakly saturated class of morphisms of $\mathbf{A}$, and there exists a (small) subset $C_0 \subseteq C$ which generates C as a weakly saturated class of morphisms.*
2. *The intersection $C \cap W$ is a weakly saturated class of morphisms of $\mathbf{A}$.*
3. *The full subcategory $W \subseteq \mathbf{A}^{[1]}$ is an accessible subcategory of $\mathbf{A}^{[1]}$.*
4. *The class W has the two-out-of-three property.*

Then $C \cap W$ is generated, as a weakly saturated class of morphisms, by a (small) subset $S \subseteq C \cap W$.

Proof. Let κ be a regular cardinal such that W is κ-accessible. Choose a regular cardinal $\tau \gg \kappa$ such that W satisfies condition $(2'_\tau)$ of Proposition A.2.6.3. Enlarging τ if necessary, we may assume that $\tau > \kappa$ (so that τ is uncountable), that $\mathcal{C}$ is τ-accessible, and that the source and target of every morphism in C_0 is τ-compact. Enlarging C_0 if necessary, we may suppose that C_0 consists of *all* morphisms $f : X \to Y$ in C such that X and Y are τ-compact. Let $S = C_0 \cap W$. We will show that S generates $C \cap W$ as a weakly saturated class of morphisms.

Let $\overline{S}$ be the weakly saturated class of morphisms generated by S and let $f : X \to Y$ be a morphism which belongs to $C \cap W$. We wish to show that $f \in \overline{S}$. Corollary A.1.5.13 implies that there exists a τ-good C_0-tree $\{Y_\alpha\}_{\alpha \in A}$ with root X, such that Y is isomorphic to Y_A as objects of $\mathcal{C}_{X/}$. Let us say that a subset $B \subseteq A$ is *good* if it is downward-closed and the canonical map $i : X \to Y_B$ belongs to W (we note that i automatically belongs to C, by virtue of Lemma A.1.5.6).

We now make the following observations:

(i) Given an increasing transfinite sequence of good subsets $\{A_\gamma\}_{\gamma<\beta}$, the union $\bigcup A_\gamma$ is good. This follows from the assumption that $C \cap W$ is weakly saturated.

(ii) Let $B \subseteq A$ be good and let $B_0 \subseteq B$ be τ-small. Then there exists a τ-small subset $B' \subseteq B$ containing B_0. This follows from our assumption that W satisfies condition $(2'_\tau)$ of Proposition A.2.6.3.

(iii) Suppose that $B, B' \subseteq A$ are such that B, B', and $B \cap B'$ are good. Then $B \cup B'$ is good. To prove this, we consider the pushout diagram

$$\begin{tikzcd} Y_{B\cap B'} \arrow[r] \arrow[d] & Y_B \arrow[d] \\ Y_{B'} \arrow[r] & Y_{B\cup B'}. \end{tikzcd}$$

Every morphism in this diagram belongs to C (Lemma A.1.5.6), and the upper horizontal map belongs to W by virtue of assumption (4). Since $C \cap W$ is stable under pushouts, we conclude that the lower vertical map belongs to W. Assumption (4) now implies that the composite map $X \to Y_{B'} \to Y_{B\cup B'}$ belongs to W, as desired.

The next step is to prove the following claim:

$(*)$ Let A' be a good subset of A and let $B_0 \subseteq A$ be τ-small. Then there exists a τ-small subset $B \subseteq A$ such that $B_0 \subseteq B$, B is good and $B \cap A'$ is good.

To prove $(*)$, we begin by setting $B'_0 = A' \cap B_0$. We now define sequences of τ-small subsets

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$$

$$B'_0 \subseteq B'_1 \subseteq B'_2 \subseteq \cdots$$

as follows. Suppose that B_i and B'_i have been defined. Applying (ii), we choose B_{i+1} to be any τ-small good subset of A which contains $B_i \cup B'_i$. Applying (ii) again, we select B'_{i+1} to be any τ-small good subset of A' which contains $A' \cap B_{i+1}$. Let $B = \bigcup B_i$. It follows from (i) that B and $A' \cap B = \bigcup_i B'_i$ are both good.

We now choose a transfinite sequence of good subsets $\{A(\gamma) \subseteq A\}_{\gamma<\beta}$. Suppose that $A(\gamma')$ has been defined for $\gamma' < \gamma$ and let $A'(\gamma) = \bigcup_{\gamma'<\gamma} A(\gamma')$.

It follows from (i) that $A'(\gamma)$ is good. If $A'(\gamma) = A$, we set $\beta = \gamma$ and conclude the construction. Otherwise, we choose a minimal element $a \in A - A'(\gamma)$. Applying $(*)$, we deduce that there exists a τ-small good subset $B(\gamma) \subseteq A$ containing a, such that $A'(\gamma) \cap B(\gamma)$ is good. Let $A(\gamma) = A'(\gamma) \cup B(\gamma)$. It follows from (iii) that $A(\gamma)$ is good.

We observe that $\{Y_{A(\gamma)}\}_{\gamma<\beta}$ is a transfinite sequence of objects of $\mathcal{C}_{X/}$ having colimit Y. To prove that $f : X \to Y$ belongs to $\overline{S}$, it will suffice to show that for each $\gamma < \beta$, the map $g : Y_{A'(\gamma)} \to Y_{A(\gamma)}$ belongs to $\overline{S}$. Remark A.1.5.5 implies the existence of a C_0-tree $\{Z_\alpha\}_{\alpha \in A(\gamma)-A'(\gamma)}$ with root $Y_{A'(\gamma)}$ and colimit $Y_{A(\gamma)}$. Since $A(\gamma) - A'(\gamma)$ is τ-small, Lemma A.1.5.11 implies the existence of a pushout diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & N \arrow[d] \\
Y_{A'(\gamma)} \arrow[r] & Y_{A(\gamma)}
\end{tikzcd}$$

where $g \in C_0$.

Since $\mathcal{C}$ is τ-accessible, we can write $Y_{A'(\gamma)}$ as the colimit of a family of τ-compact objects $\{Z_\lambda\}_{\lambda \in P}$ indexed by a τ-filtered partially ordered set P. Since M is τ-compact, we can assume (reindexing the colimit if necessary) that we have a compatible family of maps $\{M \to Z_\lambda\}$. For each λ, let $g_\lambda : Z_\lambda \to Z_\lambda \coprod_M N$ be the induced map. Then g is the filtered colimit of the family $\{g_\lambda\}_{\lambda \in P}$. Since W satisfies condition $(2'_\tau)$ of Proposition A.2.6.3, we conclude that there exists a τ-small κ-filtered subset $P_0 \subseteq P$, such that $g' = \varinjlim_{\lambda \in P_0} g_\lambda$ belongs to W. We now observe that $g' \in S$ and that g is a pushout of g', so that $g \in \overline{S}$, as desired. $\square$

Proposition A.2.6.8. *Let* $\mathbf{A}$ *be a presentable category and let* W *and* C *be classes of morphisms in* $\mathbf{A}$ *with the following properties:*

(1) *The collection* C *is a weakly saturated class of morphisms of* $\mathbf{A}$*, and there exists a (small) subset* $C_0 \subseteq C$ *which generates* C *as a weakly saturated class of morphisms.*

(2) *The intersection* $C \cap W$ *is a weakly saturated class of morphisms of* $\mathbf{A}$*.*

(3) *The full subcategory* $W \subseteq \mathbf{A}^{[1]}$ *is an accessible subcategory of* $\mathbf{A}^{[1]}$*.*

(4) *The class* W *has the two-out-of-three property.*

(5) *If* f *is a morphism in* $\mathbf{A}$ *which has the right lifting property with respect to each element of* C*, then* $f \in W$*.*

Then $\mathbf{A}$ *admits a combinatorial model structure, which may be described as follows:*

(C) *The cofibrations in* $\mathbf{A}$ *are the elements of* C*.*

(W) *The weak equivalences in* $\mathbf{A}$ *are the elements of* W*.*

(*F*) *A morphism in* $\mathbf{A}$ *is a fibration if it has the right lifting property with respect to every morphism in* $C \cap W$*.*

Proof. The category $\mathbf{A}$ has all (small) limits and colimits since it is presentable. The two-out-three property for W is among our assumptions, and the stability of W under retracts follows from the accessibility of $W \subseteq \mathbf{A}^{[1]}$ (Corollary 4.4.5.16). The class of cofibrations is stable under retracts by (1), and the class of fibrations is stable under retracts by definition. The classes of fibrations and cofibrations are stable under retracts by definition.

We next establish the factorization axioms. By the small object argument, any morphism $X \to Z$ admits a factorization

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

where $f \in C$ and g has the right lifting property with respect to every morphism in C. In particular, g has the right lifting property with respect to every morphism in $C \cap W$, so that g is a fibration; assumption (5) then implies that g is a trivial fibration. Similarly, using Lemma A.2.6.7, we may choose a factorization as above, where $f \in C \cap W$ and g has the right lifting property with respect to $C \cap W$; g is then a fibration by definition.

To complete the proof, it suffices to show that cofibrations have the left lifting property with respect to trivial fibrations, and trivial cofibrations have the left lifting property with respect to fibrations. The second of these statements is clear (it is the definition of a fibration). For the first statement, let us consider an arbitrary trivial fibration $p : X \to Z$. By the small object argument, there exists a factorization of p

$$X \xrightarrow{q} Y \xrightarrow{r} Z,$$

where q is a cofibration and r has the right lifting property with respect to all cofibrations. Then r is a weak equivalence by (3), so that q is a weak equivalence by the two-out-of-three property. Considering the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "q"] & X \arrow[d, "p"] \\ Y \arrow[r, "r"] \arrow[ur, dashed] & Z, \end{tikzcd}$$

we deduce the existence of the dotted arrow from the fact that p is a fibration and q is a trivial cofibration. It follows that p is a retract of r, and therefore p also has the right lifting property with respect to all cofibrations. This completes the proof that $\mathbf{A}$ is a model category. The assertion that $\mathbf{A}$ is combinatorial follows immediately from (1) and from Lemma A.2.6.7. □

Corollary A.2.6.9. *Let* $\mathbf{A}$ *be a presentable category equipped with a model structure. Suppose that there exists a* (*small*) *set which generates the collection of cofibrations in* $\mathbf{A}$ (*as a weakly saturated class of morphisms*)*. Then the following are equivalent:*

(1) *The model category* $\mathbf{A}$ *is combinatorial; in other words, there exists a* (*small*) *set which generates the collection of trivial cofibrations in* $\mathbf{A}$ (*as a weakly saturated class of morphisms*).

(2) *The collection of weak equivalences in* $\mathbf{A}$ *determines an accessible subcategory of* $\mathbf{A}^{[1]}$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Corollary A.2.6.6, and the reverse implication follows from Proposition A.2.6.8. □

Our next goal is to prove a weaker version of Proposition A.2.6.8 which is somewhat easier to apply in practice.

Definition A.2.6.10. Let $\mathbf{A}$ be a presentable category. A class W of morphisms in $\mathcal{C}$ is *perfect* if it satisfies the following conditions:

(1) Every isomorphism belongs to W.

(2) Given a pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, if any two of the morphisms f, g, and $g \circ f$ belong to W, then so does the third.

(3) The class W is stable under filtered colimits. More precisely, suppose we are given a family of morphisms $\{f_\alpha : X_\alpha \to Y_\alpha\}$ which is indexed by a filtered partially ordered set. Let X denote a colimit of $\{X_\alpha\}$, Y a colimit of $\{Y_\alpha\}$, and $f : X \to Y$ the induced map. If each f_α belongs to W, then so does f.

(4) There exists a (small) subset $W_0 \subseteq W$ such that every morphism belonging to W can be obtained as a filtered colimit of morphisms belonging to W_0.

Example A.2.6.11. If $\mathcal{C}$ is a presentable category, then the class W consisting of all isomorphisms in $\mathcal{C}$ is perfect.

The following is an immediate consequence of Corollary A.2.6.5:

Corollary A.2.6.12. *Let* $F : \mathcal{C} \to \mathcal{C}'$ *be a functor between presentable categories which preserves filtered colimits and let* $W_{\mathcal{C}'}$ *be a perfect class of morphisms in* $\mathcal{C}'$. *Then* $W_{\mathcal{C}} = F^{-1} W_{\mathcal{C}'}$ *is a perfect class of morphisms in* $\mathcal{C}$.

Proposition A.2.6.13. *Let* $\mathbf{A}$ *be a presentable category. Suppose we are given a class* W *of morphisms of* $\mathbf{A}$*, which we will call weak equivalences, and a* (*small*) set C_0 *of morphisms of* $\mathbf{A}$*, which we will call generating cofibrations. Suppose furthermore that the following assumptions are satisfied:*

(1) *The class* W *of weak equivalences is perfect* (*Definition A.2.6.10*).

(2) *For any diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[d, "g"] & Y' \arrow[d, "g'"] \\ X'' \arrow[r] & Y'' \end{tikzcd}$$

in which both squares are coCartesian, f belongs to C_0, and g belongs to W, the map g' also belongs to W.

(3) *If $g : X \to Y$ is a morphism in $\mathbf{A}$ which has the right lifting property with respect to every morphism in C_0, then g belongs to W.*

Then there exists a left proper combinatorial model structure on $\mathbf{A}$ which may be described as follows:

(C) *A morphism $f : X \to Y$ in $\mathbf{A}$ is a cofibration if it belongs to the weakly saturated class of morphisms generated by C_0.*

(W) *A morphism $f : X \to Y$ in $\mathbf{A}$ is a weak equivalence if it belongs to W.*

(F) *A morphism $f : X \to Y$ in $\mathbf{A}$ is a fibration if it has the right lifting property with respect to every map which is both a cofibration and a weak equivalence.*

Proof. We first show that the class of weak equivalences is stable under pushouts by cofibrations. Let P denote the collection of all morphisms f in $\mathbf{A}$ with the following property: for coCartesian diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[d, "g"] & Y' \arrow[d, "g'"] \\ X'' \arrow[r] & Y'', \end{tikzcd}$$

where g belongs to W, the map g' also belongs to W. By assumption, $C_0 \subseteq P$. It is easy to see that P is weakly saturated (using the stability of W under filtered colimits), so that every cofibration belongs to P.

It remains only to show that $\mathbf{A}$ is a model category. In view of Proposition A.2.6.8, it will suffice to show that $C \cap W$ is a weakly saturated class of morphisms. It is clear that $C \cap W$ is stable under retracts. It will therefore suffice to verify the stability of $C \cap W$ under pushouts and transfinite composition. The case of transfinite composition is easy: C is stable under transfinite composition because C is weakly saturated, and W is stable under transfinite composition because it is stable under finite composition and filtered colimits.

It remains to show that $C \cap W$ is stable under pushouts. Suppose we are given a coCartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X'' \arrow[d, "f''"] \\ Y \arrow[r] & Y'' \end{tikzcd}$$

in which f belongs to $C \cap W$; we wish to show that f'' also belongs to $C \cap W$. Since C is weakly saturated, it will suffice to show that f'' belongs to W. Using the small object argument, we can factor the top horizontal map to produce a coCartesian rectangle

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & X' \arrow[r, "h"] \arrow[d, "f'"] & X'' \arrow[d, "f''"] \\ Y \arrow[r] & Y' \arrow[r, "h'"] & Y'' \end{tikzcd}$$

in which g is a cofibration and h has the right lifting property with respect to all the morphisms in C_0. Since W is stable under the formation of pushouts by cofibrations, we deduce that f' belongs to W. Moreover, by assumption (3), h belongs to W. Since h' is a pushout of h by the cofibration f', we deduce that h' belongs to W as well. Applying the two-out-of-three property (twice), we deduce that f'' belongs to W. $\square$

Remark A.2.6.14. Let $\mathbf{A}$ be a model category. Then $\mathbf{A}$ arises via the construction of Proposition A.2.6.13 if and only if it is combinatorial and left proper and the collection of weak equivalences in $\mathbf{A}$ is stable under filtered colimits.

A.2.7 Simplicial Sets

The formalism of simplicial sets plays a prominent role throughout this book. In this section, we will review the definition of a simplicial set and establish some notation.

For each $n \geq 0$, we let $[n]$ denote the linearly ordered set $\{0, \ldots, n\}$. We let $\mathbf{\Delta}$ denote the category of *combinatorial simplices*: the objects of $\mathbf{\Delta}$ are the linearly ordered sets $[n]$, and morphisms in $\mathbf{\Delta}$ are given by (nonstrictly) order-preserving maps.

If $\mathcal{C}$ is any category, a *simplicial object* of $\mathcal{C}$ is a functor $\mathbf{\Delta}^{op} \to \mathcal{C}$. Dually, a *cosimplicial object* of $\mathcal{C}$ is a functor $\mathbf{\Delta} \to \mathcal{C}$. A *simplicial set* is a simplicial object in the category of sets. More explicitly, a simplicial set S is determined by the following data:

- A set S_n for each $n \geq 0$ (the value of S on the object $[n] \in \mathbf{\Delta}$).

- A map $p^* : S_n \to S_m$ for each order-preserving map $[m] \to [n]$, the formation of which is compatible with composition (including empty composition, so that $(\mathrm{id}_{[n]})^* = \mathrm{id}_{S_n}$).

Let us recall a bit of standard notation for working with a simplicial set S. For each $0 \leq j \leq n$, the *face map* $d_j : S_n \to S_{n-1}$ is defined to be the pullback p^*, where $p : [n-1] \to [n]$ is given by

$$p(i) = \begin{cases} i & \text{if } i < j \\ i+1 & \text{if } i \geq j. \end{cases}$$

Similarly, the *degeneracy map* $s_j : S_n \to S_{n+1}$ is defined to be the pullback q^*, where $q : [n+1] \to [n]$ is defined by the formula

$$q(i) = \begin{cases} i & \text{if } i \leq j \\ i-1 & \text{if } i > j. \end{cases}$$

Because every order-preserving map from $[n]$ to $[m]$ can be factored as a composition of face and degeneracy maps, the structure of a simplicial set S is completely determined by the sets S_n for $n \geq 0$ together with the face and degeneracy operations defined above. These operations are required to satisfy certain identities, which we will not make explicit here.

Remark A.2.7.1. The category $\mathbf{\Delta}$ is equivalent to the (larger) category of all finite nonempty linearly ordered sets. We will sometimes abuse notation by identifying $\mathbf{\Delta}$ with this larger subcategory and by regarding simplicial sets (or more general simplicial objects) as functors which are defined on all nonempty linearly ordered sets.

Notation A.2.7.2. The category of simplicial sets will be denoted by $\mathcal{S}\mathrm{et}_\Delta$. If J is a linearly ordered set, we let $\Delta^J \in \mathcal{S}\mathrm{et}_\Delta$ denote the representable functor $[n] \mapsto \mathrm{Hom}([n], J)$, where the morphisms are taken in the category of linearly ordered sets. For each $n \geq 0$, we will write Δ^n in place of $\Delta^{[n]}$. We observe that, for any simplicial set S, there is a natural identification of sets $S_n \simeq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, S)$.

Example A.2.7.3. For $0 \leq j \leq n$, we let $\Lambda^n_j \subset \Delta^n$ denote the "jth horn." It is determined by the following property: an element of $(\Lambda^n_j)_m$ is given by an order-preserving map $p : [m] \to [n]$ satisfying the condition that $\{j\} \cup p([m]) \neq [n]$. Geometrically, Λ^n_j corresponds to the subset of an n-simplex Δ^n in which the jth face and the interior have been removed.

More generally, if J is any finite linearly ordered set containing an element j, we let Λ^J_j denote the simplicial subset of Δ^J obtained by removing the interior and the face opposite the vertex j.

The category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets has a (combinatorial, left proper, and right proper) model structure, which we will refer to as the *Kan model structure*. It may be described as follows:

- A map of simplicial sets $f : X \to Y$ is a *cofibration* if it is a monomorphism; that is, if the induced map $X_n \to Y_n$ is injective for all $n \geq 0$.

- A map of simplicial sets $f : X \to Y$ is a *fibration* if it is a Kan fibration: that is, if for any diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

it is possible to supply the dotted arrow rendering the diagram commutative.

- A map of simplicial sets $f : X \to Y$ is a *weak equivalence* if the induced map of geometric realizations $|X| \to |Y|$ is a homotopy equivalence of topological spaces.

To prove this, we observe that the class of all cofibrations is generated by the collection of all inclusions $\partial \Delta^n \subseteq \Delta^n$; it is then easy to see that the conditions of Proposition A.2.6.13 are satisfied. The nontrivial point is to verify that the fibrations for the resulting model structure are precisely the Kan fibrations and that $\mathcal{S}\mathrm{et}_\Delta$ is right proper; these facts ultimately rely on a delicate analysis due to Quillen (see [32]).

Remark A.2.7.4. In §2.2.5, we introduce another model structure on $\mathcal{S}\mathrm{et}_\Delta$, the *Joyal model structure*. This model structure has the same class of cofibrations, but the fibrations and the weak equivalences differ from those defined in this section. To avoid confusion, we will refer to the fibrations and weak equivalences for the usual model structure on simplicial sets as *Kan fibrations* and *weak homotopy equivalences*, respectively.

A.2.8 Diagram Categories and Homotopy (Co)limits

Let $\mathbf{A}$ be a combinatorial model category and $\mathcal{C}$ a small category. We let $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ denote the category of all functors from $\mathcal{C}$ to $\mathbf{A}$. In this section, we will see that $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ again admits the structure of a combinatorial model category: in fact, it admits two such structures. Moreover, by considering the functoriality of this construction in the category $\mathcal{C}$, we will obtain the theory of *homotopy limits* and *homotopy colimits*.

Definition A.2.8.1. Let $\mathcal{C}$ be a small category and let $\mathbf{A}$ be a model category. We will say that a natural transformation $\alpha : F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is:

- an *injective cofibration* if the induced map $F(C) \to G(C)$ is a cofibration in $\mathbf{A}$ for each $C \in \mathcal{C}$.

- a *projective fibration* if the induced map $F(C) \to G(C)$ is a fibration in $\mathbf{A}$ for each $C \in \mathcal{C}$.

- a *weak equivalence* if the induced map $F(C) \to G(C)$ is a weak equivalence in $\mathbf{A}$ for each $C \in \mathcal{C}$.
- an *injective fibration* if it has the right lifting property with respect to every morphism β in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ which is simultaneously a weak equivalence and an injective cofibration.
- a *projective cofibration* if it has the left lifting property with respect to every morphism β in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ which is simultaneously a weak equivalence and a projective fibration.

Proposition A.2.8.2. *Let $\mathbf{A}$ be a combinatorial model category and let $\mathcal{C}$ be a small category. Then there exist two combinatorial model structures on* $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$*:*

- *The projective model structure determined by the projective cofibrations, weak equivalences, and projective fibrations.*
- *The injective model structure determined by the injective cofibrations, weak equivalences, and injective fibrations.*

The following is the key step in the proof of Proposition A.2.8.2:

Lemma A.2.8.3. *Let $\mathbf{A}$ be a presentable category and let $\mathcal{C}$ be a small category. Let S_0 be a (small) set of morphisms of $\mathbf{A}$ and let $\overline{S}_0$ be the weakly saturated class of morphisms generated by S_0. Let $\widetilde{S}$ be the collection of all morphisms $F \to G$ in* $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ *with the following property: for every $C \in \mathcal{C}$, the map $F(C) \to G(C)$ belongs to $\overline{S}_0$. Then there exists a (small) set of morphisms S of* $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ *which generates $\widetilde{S}$ as a weakly saturated class of morphisms.*

We prove a generalization of Lemma A.2.8.3 in §A.3.3 (Lemma A.3.3.3).

Proof of Proposition A.2.8.2. We first treat the case of the projective model structure. For each object $C \in \mathcal{C}$ and each $A \in \mathbf{A}$, we define

$$\mathcal{F}_A^C : \mathcal{C} \to \mathbf{A}$$

by the formula

$$\mathcal{F}_A^C(C') = \coprod_{\alpha \in \mathrm{Map}_{\mathcal{C}}(C, C')} A.$$

We note that if $i : A \to A'$ is a (trivial) cofibration in $\mathbf{A}$, then the induced map $\mathcal{F}_A^C \to \mathcal{F}_{A'}^C$ is a projective (trivial) cofibration in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$.

Let I_0 be a set of generating cofibrations $i : A \to B$ for $\mathbf{A}$ and let I be the set of all induced maps $\mathcal{F}_A^C \to \mathcal{F}_B^C$ (where C ranges over $\mathcal{C}$). Let J_0 be a set of generating trivial cofibrations for $\mathbf{A}$ and define J likewise. It follows immediately from the definitions that a morphism in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is a projective fibration if and only if it has the right lifting property with respect to every morphism in J, and a projective trivial fibration if and only if it has the right lifting property with respect to every morphism in I. Let $\overline{I}$ and $\overline{J}$ be the weakly saturated classes of morphisms of $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ generated by I and J, respectively. Using the small object argument, we deduce the following:

(i) Every morphism $f : X \to Z$ in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ admits a factorization

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z,$$

where $f' \in \overline{I}$ and f'' is a projective trivial fibration.

(ii) Every morphism $f : X \to Z$ in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ admits a factorization

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z,$$

where $f' \in \overline{J}$ and f'' is a projective fibration.

(iii) The class $\overline{I}$ coincides with the class of projective cofibrations in $\mathbf{A}$.

Furthermore, since the class of trivial projective cofibrations in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is weakly saturated and contains J, it contains $\overline{J}$. This proves that $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ satisfies the factorization axioms. The only other nontrivial point to check is that $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ satisfies the lifting axioms. Consider a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ C \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$, where i is a projective cofibration and p is a projective fibration. We wish to show that there exists a dotted arrow as indicated provided that either i or p is a weak equivalence. If p is a weak equivalence, then this follows immediately from the definition of a projective fibration. Suppose instead that i is a trivial projective cofibration. We wish to show that i has the left lifting property with respect to every projective fibration. It will suffice to show that every trivial projective fibration belongs to $\overline{J}$ (this will also imply that J is a set of generating trivial projective cofibrations for $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$, which shows that the projective model structure on $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is combinatorial). Suppose then that i is a trivial projective cofibration and choose a factorization

$$A \xrightarrow{i'} B \xrightarrow{i''} C,$$

where $i' \in \overline{J}$ and i'' is a projective fibration. Then i' is a weak equivalence, so that i'' is a weak equivalence by the two-out-of-three property. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "i'"] \arrow[d, "i"'] & B \arrow[d, "i''"] \\ C \arrow[r, "="] \arrow[ur, dashed] & C. \end{tikzcd}$$

Since i is a cofibration, there exists a dotted arrow as indicated. This proves that i is a retract of i' and therefore belongs to $\overline{J}$, as desired.

We now prove the existence of the injective model structure on $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$. Here it is difficult to proceed directly, so we will instead apply Proposition A.2.6.8. It will suffice to check each of the hypotheses in turn:

(1) The collection of injective cofibrations in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is generated (as a weakly saturated class) by some small set of morphisms. This follows from Lemma A.3.3.3.

(2) The collection of trivial injective cofibrations in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is weakly saturated: this follows immediately from the fact that the class of injective cofibrations in $\mathbf{A}$ is weakly saturated.

(3) The collection of weak equivalences in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is an accessible subcategory of $\mathrm{Fun}(\mathcal{C}, \mathbf{A})^{[1]}$: this follows from the proof of Proposition 5.4.4.3 since the collection of weak equivalences in $\mathbf{A}$ form an accessible subcategory of $\mathbf{A}^{[1]}$.

(4) The collection of weak equivalences in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ satisfy the two-out-of-three property: this follows immediately from the fact that the weak equivalences in $\mathbf{A}$ satisfy the two-out-of-three property.

(5) Let $f : X \to Y$ be a morphism in $\mathbf{A}$ which has the right lifting property with respect to every injective cofibration. In particular, f has the right lifting property with respect to each of the morphisms in the class I defined above, so that f is a trivial projective fibration and, in particular, a weak equivalence.

□

Remark A.2.8.4. In the situation of Proposition A.2.8.2, if $\mathbf{A}$ is assumed to be right or left proper, then $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is likewise right or left proper (with respect to either the projective or the injective model structures).

Remark A.2.8.5. It follows from the proof of Proposition A.2.8.2 that the class of projective cofibrations is generated (as a weakly saturated class of morphisms) by the maps $j : \mathcal{F}^C_A \to \mathcal{F}^C_{A'}$, where $C \in \mathcal{C}$ and $A \to A'$ is a cofibration in $\mathbf{A}$. We observe that j is an injective cofibration. It follows that every projective cofibration is an injective cofibration; dually, every injective fibration is a projective fibration.

Remark A.2.8.6. The construction of Proposition A.2.8.2 is functorial in the following sense: given a Quillen adjunction of combinatorial model categories $\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B}$ and a small category $\mathcal{C}$, composition with F and G determines a Quillen adjunction

$$\mathrm{Fun}(\mathcal{C}, \mathbf{A}) \underset{G^{\mathcal{C}}}{\overset{F^{\mathcal{C}}}{\rightleftarrows}} \mathrm{Fun}(\mathcal{C}, \mathbf{B})$$

(with respect to either the injective or the projective model structures). Moreover, if (F, G) is a Quillen equivalence, then so is $(F^{\mathcal{C}}, G^{\mathcal{C}})$.

Because the projective and injective model structures on $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ have the same weak equivalences, the identity functor $\mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathbf{A})}$ is a Quillen equivalence between them. However, it is important to distinguish between these two model structures because they have different variance properties, as we now explain.

Let $f : \mathcal{C} \to \mathcal{C}'$ be a functor between small categories. Then composition with f yields a pullback functor $f^* : \mathrm{Fun}(\mathcal{C}', \mathbf{A}) \to \mathrm{Fun}(\mathcal{C}, \mathbf{A})$. Since $\mathbf{A}$ admits small limits and colimits, f^* has a right adjoint, which we will denote by f_*, and a left adjoint, which we shall denote by $f_!$.

Proposition A.2.8.7. *Let $\mathbf{A}$ be a combinatorial model category and let $f : \mathcal{C} \to \mathcal{C}'$ be a functor between small categories. Then*

(1) *The pair $(f_!, f^*)$ determines a Quillen adjunction between the* projective *model structures on* $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ *and* $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$.

(2) *The pair (f^*, f_*) determines a Quillen adjunction between the* injective *model structures on* $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ *and* $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$.

Proof. This follows immediately from the simple observation that f^* preserves weak equivalences, projective fibrations, and weak cofibrations. □

We now review the theory of homotopy limits and colimits in a combinatorial model category $\mathbf{A}$. For simplicity, we will discuss homotopy limits and leave the analogous theory of homotopy colimits to the reader. Let $\mathbf{A}$ be a combinatorial model category and let $f : \mathcal{C} \to \mathcal{C}'$ be a functor betweeen (small) categories. We wish to consider the right derived functor Rf_* of the right Kan extension $f_* : \mathrm{Fun}(\mathcal{C}, \mathbf{A}) \to \mathrm{Fun}(\mathcal{C}', \mathbf{A})$. This derived functor is called the *homotopy right Kan extension* functor. The usual way of defining it involves choosing a fibrant replacement functor $Q : \mathrm{Fun}(\mathcal{C}, \mathbf{A}) \to \mathrm{Fun}(\mathcal{C}, \mathbf{A})$ and setting $Rf_* = f_* \circ Q$. The assumption that $\mathbf{A}$ is combinatorial guarantees that such a fibrant replacement functor exists. However, for our purposes it is more convenient to address the indeterminacy in the definition of Rf_* in another way.

Let $F \in \mathrm{Fun}(\mathcal{C}, \mathbf{A})$, let $G \in \mathrm{Fun}(\mathcal{C}', \mathbf{A})$, and let $\eta : G \to f_* F$ be a map in $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$. We will say that η *exhibits G as the homotopy right Kan extension of F* if, for some weak equivalence $F \to F'$ where F' is injectively fibrant in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$, the composite map $G \to f_* F \to f_* F'$ is a weak equivalence in $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$. Since f_* preserves weak equivalences between injectively fibrant objects, this condition is independent of the choice of F'.

Remark A.2.8.8. Given an object $F \in \mathrm{Fun}(\mathcal{C}, \mathbf{A})$, it is not necessarily the case that there exists a map $\eta : G \to f_* F$ which exhibits G as a homotopy right Kan extension of F. However, such a map can always be found after replacing F by a weakly equivalent object; for example, if F is injectively fibrant, we may take $G = f_* F$ and η to be the identity map.

Let $[0]$ denote the final object of $\mathcal{C}\mathrm{at}$: that is, the category with one object and only the identity morphism. For *any* category $\mathcal{C}$, there is a unique functor $f : \mathcal{C} \to [0]$. If $\mathbf{A}$ is a combinatorial model category, $F : \mathcal{C} \to \mathbf{A}$ is a functor, and $A \in \mathbf{A} \simeq \mathrm{Fun}([0], \mathbf{A})$ is an object, then we will say that a natural transformation $\alpha : f^* A \to F$ *exhibits A as a homotopy limit of F* if it exhibits A as a homotopy right Kan extension of F. Note that we can identify α with a map $A \to \lim_{C \in \mathcal{C}} F(C)$ in the model category $\mathbf{A}$.

The theory of homotopy right Kan extensions in general can be reduced to the theory of homotopy limits in view of the following result:

Proposition A.2.8.9. *Let $\mathbf{A}$ be a combinatorial model category, let $f : \mathcal{C} \to \mathcal{D}$ be a functor between small categories, and let $F : \mathcal{C} \to \mathbf{A}$ and $G : \mathcal{D} \to \mathbf{A}$ be diagrams. A natural transformation $\alpha : f^* G \to F$ exhibits G as a homotopy right Kan extension of F if and only if for each object $D \in \mathcal{D}$, α exhibits $G(D)$ as a homotopy limit of the composite diagram*

$$F_{D/} : \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/} \to \mathcal{C} \xrightarrow{F} \mathbf{A}.$$

To prove Proposition A.2.8.9, we can immediately reduce to the case where F is an injectively fibrant diagram. In this case, α exhibits G as a homotopy right Kan extension of F if and only if it induces a weak homotopy equivalence $G(D) \to \lim F_{D/}$ for each $D \in \mathcal{D}$. It will therefore suffice to prove the following result (in the case $\mathcal{C}' = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$):

Lemma A.2.8.10. *Let $\mathbf{A}$ be a combinatorial model category and let $g : \mathcal{C}' \to \mathcal{C}$ be a functor which exhibits $\mathcal{C}'$ as cofibered in sets over $\mathcal{C}$. Then the pullback functor $g^* : \mathrm{Fun}(\mathcal{C}, \mathbf{A}) \to \mathrm{Fun}(\mathcal{C}', \mathbf{A})$ preserves injective fibrations.*

Proof. It will suffice to show that the left adjoint $g_!$ preserves injective trivial cofibrations. Let $\alpha : F \to F'$ be a map in $\mathrm{Fun}(\mathcal{C}', \mathbf{A})$. We observe that for each object $C \in \mathcal{C}$, the map $(q_! \alpha)(C) : (q_! F)(C) \to (q_! F')(C)$ can be identified with the coproduct of the maps $\{\alpha(C') : F(C') \to F'(C')\}_{C' \in g^{-1}\{C\}}$. If α is an injective trivial cofibration, then each of these maps is a trivial cofibration in $\mathbf{A}$, so that $q_! \alpha$ is again an injective trivial cofibration, as desired. $\square$

Remark A.2.8.11. In the preceding discussion, we considered injective model structures, Rf_*, and homotopy limits. An entirely dual discussion may be carried out with projective model structures and $Lf_!$; one obtains a notion of *homotopy colimit* which is the dual of the notion of homotopy limit.

Example A.2.8.12. Let $\mathbf{A}$ be a combinatorial model category and consider a diagram

$$X' \xleftarrow{f} X \xrightarrow{g} X''.$$

This diagram is projectively cofibrant if and only if the object X is cofibrant and the maps f and g are both cofibrations. Consequently, the definition of homotopy colimits given above recovers, as a special case, the theory of homotopy pushouts presented in §A.2.4.

A.2.9 Reedy Model Structures

Let $\mathbf{A}$ be a combinatorial model category and $\mathcal{J}$ a small category. In §A.2.8, we saw that the diagram category $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ can again be regarded as a combinatorial model category via either the projective or the injective model structure of Proposition A.2.8.2. In the special case where $\mathcal{J}$ is a *Reedy category* (see Definition A.2.9.1), it is often useful to consider still another model structure on $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$: the *Reedy model structure*. We will sketch the definition and some of the basic properties of Reedy model categories below; we refer the reader to [38] for a more detailed treatment.

Definition A.2.9.1. A *Reedy category* is a small category $\mathcal{J}$ equipped with a factorization system $\mathcal{J}^L, \mathcal{J}^R \subseteq \mathcal{J}$ satisfying the following conditions:

(1) Every isomorphism in $\mathcal{J}$ is an identity map.

(2) Given a pair of objects $X, Y \in \mathcal{J}$, let us write $X \preceq_0 Y$ if either there exists a morphism $f : X \to Y$ belonging to $\mathcal{J}^R$ or there exists a morphism $g : Y \to X$ belonging to $\mathcal{J}^L$. We will write $X \prec_0 Y$ if $X \preceq_0 Y$ and $X \neq Y$. Then there are no infinite descending chains

$$\cdots \prec_0 X_2 \prec_0 X_1 \prec_0 X_0.$$

Remark A.2.9.2. Let $\mathcal{J}$ be a category equipped with a factorization system $(\mathcal{J}^L, \mathcal{J}^R)$ and let $\preceq_0$ be the relation described in Definition A.2.9.1. This relation is generally not transitive. We will denote its transitive closure by $\preceq$. Then condition (2) of Definition A.2.9.1 guarantees that $\preceq$ is a well-founded partial ordering on the set of objects of $\mathcal{J}$. In other words, every nonempty set S of objects of $\mathcal{J}$ contains a $\preceq$-minimal element.

Remark A.2.9.3. In the situation of Definition A.2.9.1, we will often abuse terminology and simply refer to $\mathcal{J}$ as a Reedy category, implicitly assuming that a factorization system on $\mathcal{J}$ has been specified as well.

Warning A.2.9.4. Condition (1) of Definition A.2.9.1 is not stable under equivalence of categories. Suppose that $\mathcal{J}$ is equivalent to a Reedy category. Then $\mathcal{J}$ can itself be regarded as a Reedy category if and only if every isomorphism class of objects in $\mathcal{J}$ contains a unique representative. (Definition A.2.9.1 can easily be modified so as to be invariant under equivalence, but it is slightly more convenient not to do so.)

Example A.2.9.5. The category $\mathbf{\Delta}$ of combinatorial simplices is a Reedy category with respect to the factorization system $(\mathbf{\Delta}^L, \mathbf{\Delta}^R)$; here a morphism $f : [m] \to [n]$ belongs to $\mathbf{\Delta}^L$ if and only if f is surjective, and to $\mathbf{\Delta}^R$ if and only if f is injective.

Example A.2.9.6. Let $\mathcal{J}$ be a Reedy category with respect to the factorization system $(\mathcal{J}^L, \mathcal{J}^R)$. Then $\mathcal{J}^{op}$ is a Reedy category with respect to the factorization system $((\mathcal{J}^R)^{op}, (\mathcal{J}^L)^{op})$.

Notation A.2.9.7. Let $\mathcal{J}$ be a Reedy category, $\mathcal{C}$ a category which admits small limits and colimits, and $X : \mathcal{J} \to \mathcal{C}$ a functor. For every object $J \in \mathcal{J}$, we define the *latching object* $L_J(X)$ to be the colimit

$$\varinjlim_{J' \in \mathcal{J}^R_{/J}, J' \neq J} X(J').$$

Similarly, we define the *matching object* to be the limit

$$\varprojlim_{J' \in \mathcal{J}^L_{J/}, J' \neq J} X(J').$$

We then have canonical maps $L_J(X) \to X(J) \to M_J(X)$.

Example A.2.9.8. Let $X : \mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ be a simplicial set and regard $\mathbf{\Delta}^{op}$ as a Reedy category using Examples A.2.9.5 and A.2.9.6. For every nonnegative integer n, the latching object $L_{[n]}X$ can be identified with the collection of all degenerate simplices of X. In particular, the map $L_{[n]}(X) \to X([n])$ is always a monomorphism.

More generally, we observe that a map of simplicial sets $f : X \to Y$ is a monomorphism if and only if, for every $n \geq 0$, the map

$$L_{[n]}(Y) \coprod_{L_{[n]}(X)} X([n]) \to Y([n])$$

is a monomorphism of sets. The "if" direction is obvious. For the converse, let us suppose that f is a monomorphism; we must show that if σ is an n-simplex of X such that $f(\sigma)$ is degenerate, then σ is already degenerate. If $f(\sigma)$ is degenerate, then $f(\sigma) = \alpha^* f(\sigma) = f(\alpha^* \sigma)$, where $\alpha : [n] \to [n]$ is a map of linearly ordered sets other than the identity. Since f is a monomorphism, we deduce that $\sigma = \alpha^* \sigma$, so that σ is degenerate, as desired.

Remark A.2.9.9. Let $X : \mathcal{J} \to \mathcal{C}$ be as in Notation A.2.9.7. Then the Jth matching object $M_J(X)$ can be identified with the Jth latching object of the induced functor $X^{op} : \mathcal{J}^{op} \to \mathcal{C}^{op}$.

Remark A.2.9.10. Let $X : \mathcal{J} \to \mathcal{C}$ be as in Notation A.2.9.7. Then the Jth matching object $M_J(X)$ can also be identified with the colimit

$$\varinjlim_{(f: J' \to J) \in S} X(J'),$$

where S is any full subcategory of $\mathcal{J}_{/J}$ with the following properties:

(1) Every morphism $f : J' \to J$ which belongs to $\mathcal{J}^R$ and is not an isomorphism also belongs to S.

(2) If $f : J' \to J$ belongs to S, then $J \not\leq J'$.

This follows from a cofinality argument since every morphism $f : J' \to J$ in S admits a canonical factorization

$$J' \xrightarrow{f'} J'' \xrightarrow{f''} J,$$

where f' belongs to $\mathcal{J}^L$ and f'' belongs to $\mathcal{J}^R$. Assumption (2) guarantees that the map f'' is not an isomorphism.

Similarly, when convenient, we can replace the limit $\varprojlim_{f:J\to J'} X(J')$ defining the matching object $M_J(X)$ by a limit over a slightly larger category.

Notation A.2.9.11. Let $\mathcal{J}$ be a Reedy category. A *good filtration* of $\mathcal{J}$ is a transfinite sequence

$$\{\mathcal{J}_\beta\}_{\beta<\alpha}$$

of full subcategories of $\mathcal{J}$ with the following properties:

(a) The filtration is exhaustive in the following sense: every object of $\mathcal{J}$ belongs to $\mathcal{J}_\beta$ for sufficiently large $\beta < \alpha$.

(b) For each ordinal $\beta < \alpha$, the category $\mathcal{J}_\beta$ is obtained from the subcategory $\mathcal{J}_{<\beta} = \bigcup_{\gamma<\beta} \mathcal{J}_\gamma$ by adjoining a single new object J_β satisfying the following condition: if $J \in \mathcal{J}$ satisfies $J \prec J_\beta$, then $J \in \mathcal{J}_{<\beta}$.

Remark A.2.9.12. Let $\mathcal{J}$ be a Reedy category. Then there exists a good filtration of $\mathcal{J}$. In fact, the existence of a good filtration is *equivalent* to the second assumption of Definition A.2.9.1.

Remark A.2.9.13. Let $\mathcal{J}$ be a Reedy category with respect to the factorization system $(\mathcal{J}^L, \mathcal{J}^R)$ and let $\{\mathcal{J}_\beta\}_{\beta<\alpha}$ be a good filtration of $\mathcal{J}$. Then each $\mathcal{J}_\beta$ admits a factorization system $(\mathcal{J}^L \cap \mathcal{J}_\beta, \mathcal{J}^R \cap \mathcal{J}_\beta)$. In other words, if $f : I \to K$ is a morphism in $\mathcal{J}_\beta$ which admits a factorization

$$I \xrightarrow{f'} J \xrightarrow{f''} K,$$

where f' belongs to $\mathcal{J}^L$ and f'' belongs to $\mathcal{J}^R$, then the object J also belongs to $\mathcal{J}_\beta$. This is clear: either f'' is an isomorphism, in which case $J = K \in \mathcal{J}_\beta$, or f'' is not an isomorphism, so that $J \prec K$ implies that $J \in \mathcal{J}_{<\beta}$.

The following result summarizes the essential features of a good filtration:

Proposition A.2.9.14. *Let $\mathcal{J}$ be a Reedy category with a good filtration $\{\mathcal{J}_\beta\}_{\beta<\alpha}$ and let $\beta < \alpha$ be an ordinal, so that $\mathcal{J}_\beta$ is obtained from $\mathcal{J}_{<\beta}$ by adjoining a single new object J. Then we have a homotopy pushout square (with respect to the Joyal model structure):*

$$\begin{tikzcd}
\mathrm{N}(\mathcal{J}_{<\beta})_{/J} \star \mathrm{N}(\mathcal{J}_{<\beta})_{J/} \arrow[r] \arrow[d, hook] & \mathrm{N}(\mathcal{J}_{<\beta}) \arrow[d, hook] \\
\mathrm{N}(\mathcal{J}_{<\beta})_{/J} \star \{J\} \star \mathrm{N}(\mathcal{J}_{<\beta})_{J/} \arrow[r] & \mathrm{N}(\mathcal{J}_\beta).
\end{tikzcd}$$

Corollary A.2.9.15. *Let $\mathcal{J}$ be a Reedy category equipped with a good filtration $\{\mathcal{J}_\beta\}_{\beta<\alpha}$ and let let $\beta < \alpha$ be an ordinal, so that $\mathcal{J}_\beta$ is obtained from $\mathcal{J}_{<\beta}$ by adjoining a single new object J. Let $\mathcal{C}$ be a category which admits small limits and colimits, let $X : \mathcal{J}_{<\beta} \to \mathcal{C}$ be a functor, and let the latching* *and matching objects $L_J(X)$ and $M_J(X)$ be defined as in Notation A.2.9.7 (note that this does not require that the functor X be defined on the object J), so that we have a canonical map $\alpha : L_J(X) \to M_J(X)$. The following data are equivalent:*

1. *A functor $\overline{X} : \mathcal{J}_\beta \to \mathcal{C}$ extending X.*
2. *A commutative diagram*

$$\begin{tikzcd} & C \arrow[dr] & \\ L_J(X) \arrow[ur] \arrow[rr, "\alpha"] & & M_J(X) \end{tikzcd}$$

in the category $\mathcal{C}$.

The equivalence carries a functor $\overline{X}$ to the evident diagram with $C = \overline{X}(J)$.

Proof. Using Proposition A.2.9.14, we see that giving an extension $\overline{X} : \mathcal{J}_\beta \to \mathcal{C}$ of X is equivalent to giving an extension $\overline{Y} : (\mathcal{J}_{<\beta})_{/J} \star \{J\} \star (\mathcal{J}_{<\beta})_{J/} \to \mathcal{C}$ of the composite functor

$$Y : (\mathcal{J}_{<\beta})_{/J} \star (\mathcal{J}_{<\beta})_{J/} \to \mathcal{J}_{<\beta} \stackrel{X}{\to} \mathcal{C}.$$

This, in turn, is equivalent to giving a commutative diagram

$$\begin{tikzcd} & C \arrow[dr] & \\ \varinjlim Y|(\mathcal{J}_{<\beta})_{/J} \arrow[ur] \arrow[rr, "\alpha'"] & & \varprojlim Y|(\mathcal{J}_{<\beta})_{J/}), \end{tikzcd}$$

where α' is the map induced by the diagram Y. The equivalence of this with the data (2) follows immediately from Remark A.2.9.10. $\square$

Remark A.2.9.16. The proof of Corollary A.2.9.15 carries over without essential change to the case where $\mathcal{C}$ is an ∞-category which admits small limits and colimits. In this case, to extend a functor $X : \mathrm{N}(\mathcal{J}_{<\beta}) \to \mathcal{C}$ to a functor $\overline{X}$ defined on the whole of $\mathcal{J}_\beta$, it will suffice to specify the object

$$\overline{X}(J) \in \mathcal{C}_{X|(\mathcal{J}_{<\beta})_{/J}/\, X|(\mathcal{J}_{<\beta})_{J/}} \simeq \mathcal{C}_{L_J(X)/\,/M_J(X)},$$

where the latching and matching objects $L_J(X), M_J(X) \in \mathcal{C}$ are defined in the obvious way.

The proof of Proposition A.2.9.14 will require a few preliminaries.

Lemma A.2.9.17. *Let $\mathcal{J}$ be a Reedy category equipped with a good filtration $\{\mathcal{J}_\beta\}_{\beta<\alpha}$. Fix $\beta < \alpha$ and let $\mathcal{J}_\beta$ be obtained from $\mathcal{J}_{<\beta}$ by adjoining the object J. Let $f : J \to J$ be a map which is not the identity, let $\mathcal{I}$ denote the category $(\mathcal{J}_{J/})_{/f} \simeq (\mathcal{J}_{/J})_{f/}$ of factorizations of the morphism f, and let $\mathcal{I}_0 = \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{<\beta}$. Then the nerve $\mathrm{N}\,\mathcal{I}_0$ is weakly contractible.*

Proof. Let $\mathcal{I}_1$ denote the full subcategory of $\mathcal{I}_0$ spanned by those diagrams

$$\begin{tikzcd} & I \arrow[dr, "f''"] & \\ J \arrow[ur, "f'"] \arrow[rr, "f"] & & J, \end{tikzcd}$$

where $I \in \mathcal{J}_{<\beta}$ and f'' is a morphism in $\mathcal{J}^R$. The inclusion $\mathcal{I}_1 \subseteq \mathcal{I}_0$ admits a left adjoint, so that $\mathrm{N}\,\mathcal{I}_1$ is a deformation retract of $\mathrm{N}\,\mathcal{I}_0$. It will therefore suffice to show that $\mathrm{N}\,\mathcal{I}_1$ is weakly contractible. Let $\mathcal{I}_2$ denote the full subcategory of $\mathcal{I}_1$ spanned by those diagrams as above where, in addition, the morphism f' belongs to $\mathcal{J}^L$. Then the inclusion $\mathcal{I}_2 \subseteq \mathcal{I}_1$ admits a right adjoint, so that $\mathrm{N}\,\mathcal{I}_2$ is a deformation retract of $\mathrm{N}\,\mathcal{I}_1$. It will therefore suffice to show that $\mathrm{N}\,\mathcal{I}_2$ is weakly contractible. This is clear since the category $\mathcal{I}_2$ consists of a single object (with no nontrivial endomorphisms). □

Lemma A.2.9.18. *Let $n \geq 1$ and suppose we are given a sequence of weakly contractible simplicial sets $\{A_i\}_{1 \leq i \leq n}$. Let L denote the iterated join*

$$\{J_0\} \star A_1 \star \{J_1\} \star A_2 \star \cdots \star A_n \star \{J_n\}$$

and let K denote the simplicial subset of L spanned by those simplices which do not contain all of the vertices $\{J_i\}_{0 \leq i \leq n}$. Then the inclusion $K \subseteq L$ is a categorical equivalence of simplicial sets.

Proof. If $n = 1$, then this follows immediately from Lemma 5.4.5.10. Suppose that $n > 1$. Let X denote the iterated join $A_1 \star \{J_1\} \star A_2 \star \cdots \star \{J_{n-1}\} \star A_n$. For every subset $S \subseteq \{1, \ldots, n-1\}$, let $X(S)$ denote the simplicial subset of X spanned by those simplices which do not contain any vertex J_i for $i \in S$. Let $X' = \bigcup_{S \neq \emptyset} X(S) \subseteq X(\emptyset) = X$. Then X' is the homotopy colimit of the diagram of simplicial sets $\{X(S)\}_{S \neq \emptyset}$. Each $X(S)$ is a join of weakly contractible simplicial sets, and is therefore weakly contractible. Since $n > 1$, the partially ordered set $\{S \subseteq \{1, \ldots, n-1\} : S \neq \emptyset\}$ has a largest element and is therefore weakly contractible. It follows that the simplicial set X' is weakly contractible.

The assertion that the inclusion $K \subseteq L$ is a categorical equivalence is equivalent to the assertion that the diagram

$$\begin{tikzcd} (\{J_0\} \star X') \coprod_{X'} (X' \star \{J_n\}) \arrow[r, hook] \arrow[d] & (\{J_0\} \star X) \coprod_X (X \star \{J_0\}) \arrow[d] \\ \{J_0\} \star X' \star \{J_n\} \arrow[r, hook] & \{J_0\} \star X \star \{J_n\} \end{tikzcd}$$

is a homotopy pushout square (with respect to the Joyal model structure). To prove this, it suffices to observe that the vertical maps are both categorical equivalences (Lemma 5.4.5.10). □

Proof of Proposition A.2.9.14. Let S denote the collection of all composable chains of morphisms

$$\overline{f} : J \xrightarrow{f_1} J \xrightarrow{f_2} \cdots \xrightarrow{f_n} J.$$

where $n \geq 1$ and each $f_i \neq \mathrm{id}_J$. For every subset $S' \subseteq S$, let $X(S')$ denote the simplicial subset of $\mathrm{N}(\mathcal{J}_\beta)$ spanned by those simplices σ satisfying the following condition:

$(*)$ For every nondegenerate face τ of σ of positive dimension, if every vertex of τ coincides with J, then τ belongs to S'.

Note that $X(S)$ coincides with $\mathrm{N}(\mathcal{J}_\beta)$, while $X(\emptyset)$ coincides with the pushout

$$(\mathrm{N}(\mathcal{J}_{<\beta})_{/J} \star \{J\} \star \mathrm{N}(\mathcal{J}_{<\beta})_{J/}) \coprod_{\mathrm{N}(\mathcal{J}_{<\beta})_{/J} \star \mathrm{N}(\mathcal{J}_{<\beta})_{J/}} \mathrm{N}(\mathcal{J}_{<\beta}).$$

It will therefore suffice to show that the inclusion $X(\emptyset) \subseteq X(S)$ is a categorical equivalence of simplicial sets.

Choose a well-ordering

$$S = \{\overline{f}_0 < \overline{f}_1 < \overline{f}_2 < \cdots\}$$

with the following property: if $\overline{f}$ has length shorter than $\overline{g}$ (when regarded as a chain of morphisms), then $\overline{f} < \overline{g}$. For every ordinal α, let $S_\alpha = \{\overline{f}_\beta\}_{\beta < \alpha}$. We will prove that for every ordinal α, the inclusion $X(\emptyset) \subseteq X(S_\alpha)$ is a categorical equivalence. The proof proceeds by induction on α. If $\alpha = 0$, there is nothing to prove, and if α is a limit ordinal, then the desired result follows from the inductive hypothesis and the fact that the class of categorical equivalences is stable under filtered colimits. We may therefore assume that $\alpha = \beta + 1$ is a successor ordinal. The inductive hypothesis guarantees that $X(\emptyset) \subseteq X(S_\beta)$ is a categorical equivalence. It will therefore suffice to show that the inclusion $j : X(S_\beta) \subseteq X(S_\alpha)$ is a categorical equivalence. We may also suppose that β is smaller than the order type of S, so that $\overline{f}_\beta$ is well-defined (otherwise, the inclusion j is an isomorphism and the result is obvious).

Let $\overline{f} = \overline{f}_\beta$ be the composable chain of morphisms

$$\overline{f} : J \xrightarrow{f_1} J \xrightarrow{f_2} \cdots \xrightarrow{f_n} J.$$

For $1 \leq i \leq n$, let A_i denote the nerve of the category

$$\mathcal{J}_{<\beta} \times_{\mathcal{J}} (\mathcal{J}_{J/})_{/f_i} \simeq \mathcal{J}_{<\beta} \times_{\mathcal{J}} (\mathcal{J}_{/J})_{f_i/}.$$

Let K denote the simplicial subset of

$$\{J_0\} \star A_1 \star \{J_1\} \star A_2 \star \cdots \star A_n \star \{J_n\}$$

spanned by those simplices which do not contain every vertex J_n. We then have a homotopy pushout diagram

$$\begin{tikzcd}
\mathrm{N}(\mathcal{J}_{<\beta})_{/J} \star K \star \mathrm{N}(\mathcal{J}_{<\beta})_{J/} \arrow[r] \arrow[d, hook] & X(S_\beta) \arrow[d] \\
\mathrm{N}(\mathcal{J}_{<\beta})_{/J} \star \{J_0\} \star A_1 \star \cdots \star A_n \star \{J_n\} \star \mathrm{N}(\mathcal{J}_{<\beta})_{J/} \arrow[r] & X(S_\alpha).
\end{tikzcd}$$

It will therefore suffice to prove that the left vertical map is a categorical equivalence. In view of Corollary 4.2.1.3, it will suffice to show that the inclusion

$$K \subseteq \{J_0\} \star A_1 \star \{J_1\} \star A_2 \star \cdots \star A_n \star \{J_n\}$$

is a categorical equivalence. Since each A_i is weakly contractible (Lemma A.2.9.17), this follows immediately from Lemma A.2.9.18. $\square$

Proposition A.2.9.19. *Let $\mathcal{J}$ be a Reedy category and let $\mathbf{A}$ be a model category. Then there exists a model structure on the category of functors $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ with the following properties:*

- (C) *A morphism $X \to Y$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ is a Reedy cofibration if and only if, for every object $J \in \mathcal{J}$, the induced map $X(J) \coprod_{L_J(X)} L_J(Y) \to Y(J)$ is a cofibration in $\mathbf{A}$.*
- (F) *A morphism $X \to Y$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ is a Reedy fibration if and only if, for every object $J \in \mathcal{J}$, the induced map $X(J) \to Y(J) \times_{M_J(Y)} M_J(X)$ is a fibration in $\mathbf{A}$.*
- (W) *A morphism $X \to Y$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ is a weak equivalence if and only if, for every $J \in \mathcal{J}$, the map $X(J) \to Y(J)$ is a weak equivalence.*

Moreover, a morphism $f : X \to Y$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ is a trivial cofibration if and only if the following condition is satisfied:

- (WC) *For every object $J \in \mathcal{J}$, the map $X(J) \coprod_{L_J(X)} L_J(Y) \to Y(J)$ is a trivial cofibration in $\mathbf{A}$.*

Similarly, f is a fibration if and only if it satisfies the dual condition:

- (WF) *For every object $J \in \mathcal{J}$, the map $X(J) \to Y(J) \times_{M_J(Y)} M_J(X)$ is a trivial fibration in $\mathbf{A}$.*

The model structure of Proposition A.2.9.19 is called the *Reedy model structure* on $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$. Note that Proposition A.2.9.19 does not require the model category $\mathbf{A}$ to be combinatorial.

Lemma A.2.9.20. *Let $\mathcal{J}$ be a Reedy category containing an object J, let $\mathbf{A}$ be a model category, and let $f : F \to G$ be a natural transformation in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$. Let $\mathcal{I} \subseteq \mathcal{J}^R_{/J}$ be a sieve: that is, $\mathcal{I}$ is a full subcategory of $\mathcal{J}^R_{/J}$ with the property that if $I \to I'$ is a morphism in $\mathcal{J}^R_{/J}$ such that $I' \in \mathcal{I}$, then $I \in \mathcal{I}$. Let $\mathcal{I}' \subseteq \mathcal{I}$ be another sieve. Then*

- (a) *If the map f satisfies condition (C) of Proposition A.2.9.19 for every object $I \in \mathcal{I}$, then the induced map*

$$\chi_{\mathcal{I}',\mathcal{I}} : \varinjlim(F|\mathcal{I}) \coprod_{\varinjlim(F|\mathcal{I}')} \varinjlim(G|\mathcal{I}') \to \varinjlim(G|\mathcal{I})$$

is a cofibration in $\mathbf{A}$.

(*b*) *If the map f satisfies condition (WC) of Proposition A.2.9.19 for every object $I \in \mathcal{I}$, then the map $\chi_{\mathcal{I}',\mathcal{I}}$ is a trivial cofibration in $\mathbf{A}$.*

Proof. We will prove (a); the proof of (b) is identical. Choose a transfinite sequence of sieves $\{\mathcal{I}_\beta \subseteq \mathcal{I}\}_{\beta<\alpha}$ with the following properties:

(i) The union $\bigcup_{\beta<\alpha} \mathcal{I}_\beta$ coincides with $\mathcal{I}$.

(ii) For each $\beta < \alpha$, the sieve $\mathcal{I}_\beta$ is obtained from $\mathcal{I}_{<\beta} = \mathcal{I}' \cup (\bigcup_{\gamma<\beta} \mathcal{I}_\gamma)$ by adjoining a single new object $J_\beta \in \mathcal{J}^R_{/J}$.

For every triple $\delta \leq \gamma \leq \beta \leq \alpha$, let $\chi_{\delta,\gamma,\beta}$ denote the induced map

$$\varinjlim(F|\,\mathcal{I}_{<\beta}) \coprod_{\varinjlim(F|\,\mathcal{I}_{<\delta})} \varinjlim(G|\,\mathcal{I}_{<\delta}) \to \varinjlim(F|\,\mathcal{I}_{<\beta}) \coprod_{\varinjlim(F|\,\mathcal{I}_{<\gamma})} \varinjlim(G|\,\mathcal{I}_{<\gamma}).$$

We wish to prove that $\chi_{0,\alpha,\alpha}$ is a cofibration. We will prove more generally that $\chi_{\delta,\gamma,\beta}$ is an equivalence for every $\delta \leq \gamma \leq \beta \leq \alpha$. The proof uses induction on γ. If γ is a limit ordinal, then we can write $\chi_{\delta,\gamma,\beta}$ as the transfinite composition of the maps $\{\chi_{\epsilon,\epsilon+1,\beta}\}_{\delta\leq\epsilon<\gamma}$, which are cofibrations by the inductive hypothesis. We may therefore assume that $\gamma = \gamma_0 + 1$ is a successor ordinal. If $\delta = \gamma$, then $\chi_{\delta,\gamma,\beta}$ is an isomorphism; otherwise, we have $\delta \leq \gamma_0$. In this case, we have

$$\chi_{\delta,\gamma,\beta} = \chi_{\gamma_0,\gamma,\beta} \circ \chi_{\delta,\gamma_0,\beta}.$$

Using the inductive hypothesis, we can reduce to the case $\delta = \gamma_0$. The map $\chi_{\gamma_0,\gamma,\beta}$ is a pushout of the map $\chi_{\gamma_0,\gamma,\gamma}$. We are therefore reduced to proving that $\chi_{\gamma_0,\gamma,\gamma}$ is a cofibration. But $\chi_{\gamma_0,\gamma,\gamma}$ is a pushout of the map $L_I(G) \coprod_{L_I(F)} F(I) \to G(I)$ for $I = J_{\gamma_0}$. This map is a cofibration by virtue of our assumption that f satisfies (C). $\square$

Proof of Proposition A.2.9.19. Let $f : X \to Z$ be a morphism in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$. We will prove that f admits a factorization

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z,$$

where

(i) The map f'' is a fibration, and f' satisfies (WC).

(ii) The map f' is a cofibration, and f'' satisfies (WF).

By symmetry, it will suffice to consider case (i). Choose a good filtration $\{\mathcal{J}_\beta\}_{\beta<\alpha}$ of $\mathcal{J}$. For $\beta < \alpha$, let $X_\beta = X|\,\mathcal{J}_\beta$, let $Z_\beta = Z|\,\mathcal{J}_\beta$, and let $f_\beta : X_\beta \to Z_\beta$ be the restriction of f. We will construct a compatible family of factorizations of f_β as a composition

$$X_\beta \xrightarrow{f'_\beta} Y_\beta \xrightarrow{f''_\beta} Z_\beta.$$

Suppose that $\mathcal{J}_\beta$ is obtained from $\mathcal{J}_{<\beta}$ by adjoining a single new object J. Assuming that (f'_γ, f''_γ) has been constructed for all $\gamma < \beta$, we note that constructing (f'_β, f''_β) is equivalent (by virtue of Corollary A.2.9.15) to giving a commutative diagram

$$\begin{tikzcd}
L_J(X) \arrow[r] \arrow[d] & L_J(Y_{<\beta}) \arrow[d] & \\
X(J) \arrow[r] & Y_\beta(J) \arrow[r] \arrow[d] & Z(J) \arrow[d] \\
 & M_J(Y_{<\beta}) \arrow[r] & M_J(Z).
\end{tikzcd}$$

In other words, we must factor a certain map

$$g : L_J(Y_{<\beta}) \coprod_{L_J(X)} X(J) \to M_J(Y_{<\beta}) \times_{M_J(Z)} Z(J)$$

as a composition

$$L_J(Y_{<\beta}) \coprod_{L_J(X)} X(J) \xrightarrow{g'} Y_\beta(J) \xrightarrow{g''} M_J(Y_{<\beta}) \times_{M_J(Z)} Z(J).$$

Using the fact that $\mathbf{A}$ is a model category, we can choose a factorization where g' is a trivial cofibration and g'' a fibration. It is readily verified that this construction has the desired properties.

We now prove the following:

- (i') A morphism $f : X \to Y$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ satisfies (WC) if and only if f is both a fibration and a weak equivalence.
- (ii') A morphism $f : X \to Y$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ satisfies (WF) if and only if f is both a cofibration and a weak equivalence.

By symmetry, it will suffice to prove (i'). The "only if" direction follows from Lemma A.2.9.20. For the "if" direction, it will suffice to show that for each $\beta < \alpha$, the induced transformation $f_\beta : X_\beta \to Y_\beta$ satisfies (WC) when regarded as a morphism of $\mathrm{Fun}(\mathcal{J}_\beta, \mathbf{A})$. Suppose that $\mathcal{J}_\beta$ is obtained from $\mathcal{J}_{<\beta}$ by adjoining a single new element J. We have a commutative diagram

$$\begin{tikzcd}
 & L_J(Y) \coprod_{L_J(X)} X(J) \arrow[dr, "q"] & \\
X(J) \arrow[ur, "p"] \arrow[rr, "r"] & & Y(J).
\end{tikzcd}$$

We wish to prove that q is a trivial cofibration in $\mathbf{A}$. Since f is a cofibration in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$, the map q is a cofibration in $\mathbf{A}$. It will therefore suffice to show that q is a weak equivalence. By the two-out-of-three property, it will suffice to show that p and r are weak equivalences. For r, this follows from our assumption that f is a weak equivalence in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$. The map p is a pushout of the map of latching objects $L_J(X) \to L_J(Y)$, which is a cofibration in $\mathbf{A}$ by virtue of the inductive hypothesis and Lemma A.2.9.20.

Combining (i) and (i') (and the analogous assertions (ii) and (ii')), we deduce that $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ satisfies the factorization axioms for a model category. To complete the proof, it will suffice to verify the lifting axioms:

(i'') Every fibration in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ has the right lifting property with respect to morphisms in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ which satisfy (WC).

(ii'') Every cofibration in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ has the left lifting property with respect to morphisms in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ which satisfy (WF).

Again, by symmetry it will suffice to prove (i''). Consider a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y, \end{tikzcd}$$

where f satisfies (WC) and g satisfies (F); we wish to prove that there exists a dotted arrow h as indicated, rendering the diagram commutative. To prove this, we will construct a compatible family of natural transformations $\{h_\beta : B|\mathcal{J}_\beta \to X|\mathcal{J}_\beta\}_{\beta<\alpha}$ which render the diagram

$$\begin{tikzcd} A|\mathcal{J}_\beta \arrow[r] \arrow[d] & X|\mathcal{J}_\beta \arrow[d, "g"] \\ B|\mathcal{J}_\beta \arrow[r] \arrow[ur, "h_\beta"] & Y|\mathcal{J}_\beta \end{tikzcd}$$

commutative. Suppose that $\mathcal{J}_\beta$ is obtained from $\mathcal{J}_{<\beta}$ by adjoining a single new object J. Assume that the maps $\{h_\gamma\}_{\gamma<\beta}$ have already been constructed and can be amalgamated into a single natural transformation $h_{<\beta} : B|\mathcal{J}_{<\beta} \to X|\mathcal{J}_{<\beta}$. Using Corollary A.2.9.15, we see that extending $h_{<\beta}$ to a map h_β with the desired properties is equivalent to solving a lifting problem of the kind depicted in the following diagram:

$$\begin{tikzcd} L_J(B)\coprod_{L_J(A)} A(J) \arrow[r] \arrow[d, "f'"] & X(J) \arrow[d, "g'"] \\ B(J) \arrow[r] \arrow[ur, dashed] & Y(J) \times_{M_J(Y)} M_J(X). \end{tikzcd}$$

Since our assumptions guarantee that f' is a trivial cofibration and that g' is a fibration, this lifting problem has a solution, as desired. $\square$

Example A.2.9.21. Let $\mathbf{A}$ be the category of *bisimplicial* sets, which we will identify with $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S}\mathrm{et}_\Delta)$ and endow with the Reedy model structure. It follows from Example A.2.9.8 that a morphism $f : X \to Y$ of bisimplicial sets is a Reedy cofibration if and only if it is a monomorphism. Consequently, the Reedy model structure on $\mathbf{A}$ coincides with the injective model structure on $\mathbf{A}$.

Example A.2.9.22. Let $\mathcal{J}$ be a Reedy category with $\mathcal{J}^L = \mathcal{J}$ and let $\mathbf{A}$ be a model category. Then the weak equivalences and cofibrations of the Reedy model structure (Proposition A.2.9.19) are the injective cofibrations and the weak equivalences appearing in Definition A.2.8.1. It follows that the Reedy model structure on $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ coincides with the injective model structure of Proposition A.2.8.2 (in particular, the injective model structure is well-defined in this case even without the assumption that $\mathbf{A}$ is combinatorial). Similarly, if $\mathcal{J}^R = \mathcal{J}$, then we can identify the Reedy model structure on $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ with the projective model structure of Proposition A.2.8.2.

In the general case, we can regard the Reedy model structure on $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ as a mixture of the projective and injective model structures. More precisely, we have the following:

(i) A natural transformation $F \to G$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ satisfies condition (C) of Proposition A.2.9.19 if and only if the induced transformation $F|\mathcal{J}^R \to G|\mathcal{J}^R$ is a projective cofibration in $\mathrm{Fun}(\mathcal{J}^R, \mathbf{A})$.

(ii) A natural transformation $F \to G$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ satisfies condition (F) of Proposition A.2.9.19 if and only if the induced transformation $F|\mathcal{J}^L \to G|\mathcal{J}^L$ is an injective fibration in $\mathrm{Fun}(\mathcal{J}^L, \mathbf{A})$.

(iii) A natural transformation $F \to G$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ satisfies condition (WC) of Proposition A.2.9.19 if and only if the induced transformation $F|\mathcal{J}^R \to G|\mathcal{J}^R$ is a trivial projective cofibration in $\mathrm{Fun}(\mathcal{J}^R, \mathbf{A})$.

(iv) A natural transformation $F \to G$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ satisfies condition (WF) of Proposition A.2.9.19 if and only if the induced transformation $F|\mathcal{J}^L \to G|\mathcal{J}^L$ is a trivial injective fibration in $\mathrm{Fun}(\mathcal{J}^L, \mathbf{A})$.

Remark A.2.9.23. Let $\mathcal{J}$ be a Reedy category and $\mathbf{A}$ a combinatorial model category, so that the injective and projective model structures on $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ are well-defined. The identity functor from $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ to itself can be regarded as a left Quillen equivalence from the projective model structure to the Reedy model structure and from the Reedy model structure to the injective model structure.

Corollary A.2.9.24. *Let $\mathcal{C}$ be a small category. Suppose that there exists a well-ordering $\leq$ on the collection of objects of $\mathcal{C}$ satisfying the following condition: for every pair of objects $X, Y \in \mathcal{C}$, we have*

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) = \begin{cases} \emptyset & \text{if } X \text{ ¿ } Y \\ \{\mathrm{id}_X\} & \text{if } X = Y. \end{cases}$$

Let $\mathbf{A}$ be a model category. Then

(i) *A natural transformation $F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$ is a (trivial) projective cofibration if and only if, for every object $C \in \mathcal{C}$, the induced map*

$$F(C) \coprod_{\varinjlim_{D \to C, D \neq C} F(D)} \varinjlim_{D \to C, D \neq C} G(D) \to G(C)$$

is a (trivial) cofibration in $\mathbf{A}$.

(*ii*) *A natural transformation $F \to G$ in* $\mathrm{Fun}(\mathcal{C}^{op}, \mathbf{A})$ *is a (trivial) injective fibration if and only if, for every object $C \in \mathcal{C}$, the induced map*

$$F(C) \to G(C) \times_{\varprojlim_{D \to C, D \neq C} G(D)} \varprojlim_{D \to C, D \neq C} F(D)$$

is a (trivial) fibration in $\mathbf{A}$.

Proof. Combine Example A.2.9.22 with Proposition A.2.9.19. $\square$

Corollary A.2.9.25. *Let $\mathbf{A}$ be a model category, let α be an ordinal, and let (α) denote the linearly ordered set $\{\beta < \alpha\}$ regarded as a category. Then*

(1) *Let $F \to F'$ be a natural transformation of diagrams $(\alpha) \to \mathbf{A}$. Suppose that, for each $\beta < \alpha$, the maps*

$$\varinjlim_{\gamma < \beta} F(\gamma) \to F(\beta)$$

$$\varinjlim_{\gamma < \beta} F'(\gamma) \to F'(\beta)$$

are cofibrations, while the map $F(\beta) \to F'(\beta)$ is a weak equivalence. Then the induced map

$$\varinjlim_{\gamma < \alpha} F(\gamma) \to \varinjlim_{\gamma < \alpha} F'(\gamma)$$

is a weak equivalence.

(2) *Let $G \to G'$ be a natural transformation of diagrams $(\alpha)^{op} \to \mathbf{A}$. Suppose that, for each $\beta < \alpha$, the maps*

$$G(\beta) \to \lim_{\gamma < \beta} G(\gamma)$$

$$G'(\beta) \to \lim_{\gamma < \beta} G'(\gamma)$$

are fibrations, while the map $G(\beta) \to G'(\beta)$ is a weak equivalence. Then the induced map

$$\lim_{\gamma < \alpha} G(\gamma) \to \lim_{\gamma < \alpha} G'(\gamma)$$

is a weak equivalence.

Proof. We will prove (1); (2) follows by the same argument. Let $p : (\alpha) \to *$ be the unique map, let $p^* : \mathbf{A} \to \mathbf{A}^{(\alpha)}$ be the diagonal map, and let $p_! : \mathbf{A}^{(\alpha)} \to \mathbf{A}$ be a left adjoint to p^*. Then $p_!$ can be identified with the functor $F \mapsto \varinjlim_{\gamma < \alpha} F(\gamma)$. We observe that $(p_!, p^*)$ is a Quillen adjunction (where $\mathbf{A}^{(\alpha)}$ is endowed with the projective model structure) so that $p_!$ preserves weak equivalence between projectively cofibrant objects. The desired result now follows from Corollary A.2.9.24. $\square$

Suppose that we are given a bifunctor

$$\otimes : \mathbf{A} \times \mathbf{B} \to \mathbf{C},$$

where $\mathbf{C}$ is a category which admits small limits. For any small category $\mathcal{J}$, we define the *coend* functor $\int_{\mathcal{J}} : \mathrm{Fun}(\mathcal{J}, \mathbf{A}) \times \mathrm{Fun}(\mathcal{J}^{op}, \mathbf{B}) \to \mathbf{C}$ so that the integral $\int_{\mathcal{J}}(F, G)$ is defined to be the coequalizer of the diagram

$$\coprod_{J \to J'} F(J) \otimes G(J') \rightrightarrows \coprod_J F(J) \otimes G(J) .$$

We then have the following result:

Proposition A.2.9.26. *Let $\otimes : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ be a left Quillen bifunctor (see Proposition A.3.1.1) and let $\mathcal{J}$ be a Reedy category. Then the coend functor*

$$\int_{\mathcal{J}} : \mathrm{Fun}(\mathcal{J}, \mathbf{A}) \times \mathrm{Fun}(\mathcal{J}^{op}, \mathbf{B}) \to \mathbf{C}$$

is also a left Quillen bifunctor, where we regard $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ *and* $\mathrm{Fun}(\mathcal{J}^{op}, \mathbf{B})$ *as endowed with the Reedy model structure.*

Proof. Let $f : F \to F'$ be a Reedy cofibration in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ and $g : G \to G'$ a Reedy cofibration in $\mathrm{Fun}(\mathcal{J}^{op}, \mathbf{B})$. Set $C = \int_{\mathcal{J}}(F, G') \coprod_{\int_{\mathcal{J}}(F,G)} \int_{\mathcal{J}}(F', G) \in \mathbf{C}$ and $C' = \int_{\mathcal{J}}(F', G')$. We wish to show that the induced map $C \to \int_{\mathcal{J}}(F', G')$ is a cofibration, which is trivial if either f or g is trivial.

Choose a good filtration $\{\mathcal{J}_\beta\}_{\beta < \alpha}$ of $\mathcal{J}$. For $\beta \leq \alpha$, we define

$$C_\beta = \int_{\mathcal{J}_\beta} (F|\mathcal{J}_{<\beta}, G'|\mathcal{J}_{<\beta}) \coprod_{\int \mathcal{J}_\beta (F|\mathcal{J}_{<\beta}, G|\mathcal{J}_{<beta})} \int_{\mathcal{J}_\beta} (F'|\mathcal{J}_{<\beta}, G|\mathcal{J}_{<\beta})$$

$$C'_\beta = \int_{\mathcal{J}_\beta} (F'|\mathcal{J}_{<\beta}, G'|\mathcal{J}_{<\beta}).$$

We wish to show that the map

$$C_\alpha \simeq C_\alpha \coprod_{C_0} C'_0 \to C_\alpha \coprod_{C_\alpha} C'_\alpha$$

is a cofibration (which is trivial if either f or g is trivial). We will prove more generally that for $\delta \leq \gamma \leq \beta \leq \alpha$, the map

$$\eta_{\delta,\gamma,\beta} : C_\beta \coprod_{C_\delta} C'_\delta \to C_\beta \coprod_{C_\gamma} C'_\gamma$$

is a cofibration (trivial if either f or g is trivial). The proof proceeds by induction on γ. If γ is a limit ordinal, then $\eta_{\delta,\gamma,\beta}$ can be obtained as a transfinite composition of the maps $\{\eta_{\epsilon,\epsilon+1,\beta}\}_{\delta \leq \epsilon < \gamma}$, and the result follows from the inductive hypothesis. We may therefore assume that $\gamma = \gamma_0 + 1$ is a successor ordinal. Since $\eta_{\delta,\gamma,\beta} = \eta_{\gamma_0,\gamma,\beta} \circ \eta_{\delta,\gamma_0,\beta}$, we can use the inductive hypothesis to reduce to the case where $\delta = \gamma_0$. Since $\eta_{\delta,\gamma,\beta}$ is a pushout of

$\eta_{\delta,\gamma,\gamma}$, we can also assume that $\beta = \gamma$. In other words, we are reduced to proving that the map

$$h : C_{\gamma_0+1} \coprod_{C_{\gamma_0}} C'_{\gamma_0} \to C'_{\gamma_0}$$

is a cofibration, which is trivial if either f or g is trivial. Let J be the object of $\mathcal{J}_{\gamma_0}$ which does not belong to $\mathcal{J}_{<\gamma_0}$. Form a pushout diagram

$$\begin{tikzcd}[column sep=-4em] & (F(J) \coprod_{L_J(F)} L_J(F')) \otimes (G(J) \coprod_{L_J(G)} L_J(G')) \arrow[dl] \arrow[dr] & \\ (F(J) \coprod_{L_J(F)} L_J(F')) \otimes G'(J) \arrow[dr] & & (F'(J) \otimes G(J) \coprod_{L_J(G)} L_J(G')) \arrow[dl] \\ & X. & \end{tikzcd}$$

We have an evident map $h' : X \to F'(J) \otimes G'(J)$ which is a cofibration (trivial if either f or g is trivial) by virtue of our assumptions on f and g (together with the fact that $\otimes$ is a left Quillen bifunctor). We conclude by observing that h is a pushout of h'. $\square$

Remark A.2.9.27. Proposition A.2.9.26 has an analogue for the model structures introduced in Proposition A.2.8.2. That is, suppose that $\mathbf{A}$ and $\mathbf{B}$ are *combinatorial* model categories and let $\mathcal{J}$ be an arbitrary small category. Then any left Quillen bifunctor $\otimes : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ induces a left Quillen bifunctor

$$\int_{\mathcal{J}} : \mathrm{Fun}(\mathcal{J}, \mathbf{A}) \times \mathrm{Fun}(\mathcal{J}^{op}, \mathbf{B}) \to \mathbf{C},$$

where we regard $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ as endowed with the projective model structure and $\mathrm{Fun}(\mathcal{J}^{op}, \mathbf{B})$ with the injective model structure. To see this, we must show that for any projective cofibration $f : F \to F'$ in $\mathrm{Fun}(\mathcal{J}, \mathbf{A})$ and any injective cofibration $g : G \to G'$ in $\mathrm{Fun}(\mathcal{J}^{op}, \mathbf{B})$, the induced map

$$h : \int_{\mathcal{J}} (F, G') \coprod_{\int_{\mathcal{J}} (F,G)} \int_{\mathcal{J}} (F', G) \to \int_{\mathcal{J}} (F', G')$$

is a cofibration in $\mathbf{C}$ which is trivial if either f or g is trivial. Without loss of generality, we may suppose that f is a generating projective cofibration of the form $\mathcal{F}^J_A \to \mathcal{F}^J_{A'}$ associated to an object $J \in \mathcal{J}$ and a cofibration $i : A \to A'$ in $\mathbf{A}$, which is trivial if f is trivial (see the proof of Proposition A.2.8.2 for an explanation of this notation). Unwinding the definitions, we can identify h with the map

$$(A \otimes G'(J)) \coprod_{A \otimes G(J)} (A' \otimes G(J)) \to A' \otimes G'(J).$$

Since i is a cofibration in $\mathbf{A}$ and the map $G(J) \to G'(J)$ is a cofibration in $\mathbf{B}$, we deduce that h is a cofibration in $\mathbf{C}$ (since $\otimes$ is a left Quillen bifunctor) which is trivial if either i or h is trivial.

Example A.2.9.28. Let $\mathbf{A}$ be a simplicial model category, so that we have a left Quillen bifunctor

$$\otimes : \mathbf{A} \times \mathcal{S}\mathrm{et}_{\Delta} \to \mathbf{A}.$$

The coend construction determines a left Quillen bifunctor

$$\int_{\mathbf{\Delta}} : \mathrm{Fun}(\mathbf{\Delta}, \mathbf{A}) \times \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S}\mathrm{et}_{\Delta}) \to \mathbf{A}.$$

where $\mathrm{Fun}(\mathbf{\Delta}, \mathbf{A})$ and $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S}\mathrm{et}_{\Delta})$ are both endowed with the Reedy model structure. In particular, if we fix a cosimplicial object $X^{\bullet} \in \mathrm{Fun}(\mathbf{\Delta}, \mathbf{A})$ which is Reedy cofibrant, then forming the coend against $X^{\bullet}$ determines a left Quillen functor from the category of bisimplicial sets (with the Reedy model structure, which coincides with the injective model structure by Example A.2.9.21) to $\mathbf{A}$.

Example A.2.9.29. Let $\mathbf{A}$ be a simplicial model category, so that we have a left Quillen bifunctor

$$\otimes : \mathbf{A} \times \mathcal{S}\mathrm{et}_{\Delta} \to \mathbf{A},$$

and consider the coend functor

$$\int_{\mathbf{\Delta}^{op}} \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathbf{A}) \times \mathrm{Fun}(\mathbf{\Delta}, \mathcal{S}\mathrm{et}_{\Delta}) \to \mathbf{A}.$$

Let $\Delta^{\bullet} \in \mathrm{Fun}(\mathbf{\Delta}, \mathcal{S}\mathrm{et}_{\Delta})$ denote the standard simplex (that is, the functor $[n] \mapsto \Delta^n$) and let $\mathbf{1}$ denote the final object of $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{S}\mathrm{et}_{\Delta})$ (that is, the constant functor given by $[n] \mapsto \Delta^0$). The unique map $\Delta^{\bullet} \to \mathbf{1}$ is a weak equivalence, and $\Delta^{\bullet}$ is Reedy cofibrant: we may therefore regard $\Delta^{\bullet}$ as a cofibrant replacement for the constant functor $\mathbf{1}$.

The functor $X_{\bullet} \mapsto \int_{\mathbf{\Delta}^{op}}(X_{\bullet}, \mathbf{1})$ can be identified with the colimit functor $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathbf{A}) \to \mathbf{A}$. This is a left Quillen functor if $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathbf{A})$ is endowed with the projective model structure but not the Reedy model structure. However, the *geometric realization* functor $X_{\bullet} \mapsto |X_{\bullet}| = \int_{\mathbf{\Delta}^{op}}(X_{\bullet}, \Delta^{\bullet})$ is a left Quillen functor with respect to the Reedy model structure.

Corollary A.2.9.30. *Let $\mathbf{A}$ be a combinatorial simplicial model category and let $X_{\bullet}$ be a simplicial object of $\mathbf{A}$. There is a canonical map*

$$\gamma : \mathrm{hocolim}\, X_{\bullet} \to |X_{\bullet}|$$

in the homotopy category of $\mathbf{A}$. This map is an equivalence if $X_{\bullet}$ is Reedy cofibrant.

Proof. Let $\Delta^{\bullet}$ and $*$ be the cosimplicial objects of $\mathcal{S}\mathrm{et}_{\Delta}$ described in Example A.2.9.29. Choose a weak equivalence of simplicial objects $X'_{\bullet} \to X_{\bullet}$, where $X'_{\bullet}$ is projectively cofibrant. We then have a diagram

$$\mathrm{hocolim}\, X_{\bullet} \simeq \varinjlim X'_{\bullet} \simeq \int_{\mathbf{\Delta}^{op}} (X'_{\bullet}, *) \xleftarrow{\alpha} \int_{\mathbf{\Delta}^{op}} (X'_{\bullet}, \Delta^{\bullet}) \xrightarrow{\beta} \int_{\mathbf{\Delta}^{op}} (X_{\bullet}, \Delta^{\bullet}).$$

Since $X'_{\bullet}$ is projectively cofibrant, Remark A.2.9.27 implies that the coend functor $\int_{\mathbf{\Delta}^{op}}(X'_{\bullet}, \bullet)$ preserves weak equivalences between injectively cofibrant cosimplicial objects of $\mathcal{S}\mathrm{et}_{\Delta}$; in particular, α is a weak equivalence in $\mathbf{A}$.

This gives the desired map γ. Proposition A.2.9.26 implies that $\int_{\mathbf{\Delta}^{op}}(\bullet, \Delta^\bullet)$ preserves weak equivalences between Reedy cofibrant simplicial objects of $\mathbf{A}$, which proves that γ is an isomorphism if $X_\bullet$ is Reedy cofibrant. $\square$

Example A.2.9.31. If $\mathbf{A}$ is the category of simplicial sets, then the map γ of Corollary A.2.9.30 is always an isomorphism; this follows from Example A.2.9.21. In other words, if $X_{\bullet,\bullet}$ is a bisimplicial set, then we can identify the diagonal simplicial set $[n] \mapsto X_{n,n}$ with the homotopy colimit of corresponding diagram $\mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}_\Delta$.

A.3 SIMPLICIAL CATEGORIES

Among the many different models for higher category theory, the theory of simplicial categories is perhaps the most rigid. This can be either a curse or a blessing, depending on the situation. For the most part, we have chosen to use the less rigid theory of ∞-categories (see §1.1.2) throughout this book. However, some arguments are substantially easier to carry out in the setting of simplicial categories. For this reason, we have devoted the final section of this appendix to a review of the theory of simplicial categories.

There exists a model structure on the category $\mathcal{C}\mathrm{at}_\Delta$ of (small) simplicial categories, which was constructed by Bergner ([7]). In §A.3.2, we will describe an analogous model structure on the category $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ of $\mathbf{S}$-enriched categories, where $\mathbf{S}$ is a suitable model category. To formulate this generalization, we will need to employ the language of monoidal model categories, which we review in §A.3.1. Under mild assumptions on $\mathbf{S}$, one can show that an $\mathbf{S}$-enriched category $\mathcal{C}$ is fibrant if and only if each of the mapping objects $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a fibrant object of $\mathbf{S}$.

In §A.3.3, we will study the category $\mathbf{A}^{\mathcal{C}}$ of diagrams $\mathcal{C} \to \mathbf{A}$, where $\mathcal{C}$ is a small category and $\mathbf{A}$ is a model category, both enriched over some fixed model category $\mathbf{S}$. In the enriched setting we can again endow $\mathbf{A}^{\mathcal{C}}$ with projective and injective model structures, which can be used to define homotopy limits and colimits.

Putting aside set-theoretic technicalities, every $\mathbf{S}$-enriched model category $\mathbf{A}$ gives rise to a fibrant object of $\mathcal{C}\mathrm{at}_{\mathbf{S}}$: namely, the full subcategory $\mathbf{A}^\circ \subseteq \mathbf{A}$ spanned by the fibrant-cofibrant objects. In §A.3.4, we will introduce a path object for $\mathbf{A}^\circ$, which will enable us to perform some calculations in the homotopy category of $\mathcal{C}\mathrm{at}_{\mathbf{S}}$.

In §A.3.5, we will consider the problem of constructing homotopy colimits in the category $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ of $\mathbf{S}$-enriched categories. Our main result, Theorem A.3.5.15, asserts that the formation of homotopy colimits in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is compatible with the formation of (tensor) products in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$. We will apply this result in §A.3.6 to study the homotopy theory of internal mapping objects in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$.

We conclude this section with §A.3.7, where we discuss localizations of (simplicial) model categories.

A.3.1 Enriched and Monoidal Model Categories

Many of the model categories which arise naturally are *enriched* over the category of simplicial sets. Our goal in this section to study enrichments of one model category over another.

Definition A.3.1.1. Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be model categories. We will say that a functor $F : \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ is a *left Quillen bifunctor* if the following conditions are satisfied:

(a) Let $i : A \to A'$ and $j : B \to B'$ be cofibrations in $\mathbf{A}$ and $\mathbf{B}$, respectively. Then the induced map

$$i \wedge j : F(A', B) \coprod_{F(A,B)} F(A, B') \to F(A', B')$$

is a cofibration in $\mathbf{C}$. Moreover, if either i or j is a trivial cofibration, then $i \wedge j$ is also a trivial cofibration.

(b) The functor F preserves small colimits separately in each variable.

Definition A.3.1.2. A *monoidal model category* is a monoidal category $\mathbf{S}$ equipped with a model structure, which satisfies the following conditions:

(i) The tensor product functor $\otimes : \mathbf{S} \times \mathbf{S} \to \mathbf{S}$ is a left Quillen bifunctor.

(ii) The unit object $\mathbf{1} \in \mathbf{S}$ is cofibrant.

(iii) The monoidal structure on $\mathbf{S}$ is closed.

Remark A.3.1.3. Some authors demand only a weakened form of axiom (ii) in the preceding definition.

Example A.3.1.4. The category of simplicial sets $\mathcal{S}\mathrm{et}_\Delta$ is a monoidal model category with respect to the Cartesian product and the Kan model structure defined in §A.2.7.

Definition A.3.1.5. Let $\mathbf{S}$ be a monoidal model category. A $\mathbf{S}$-*enriched model category* is an $\mathbf{S}$-enriched category $\mathbf{A}$ equipped with a model structure satisfying the following conditions:

(1) The category $\mathbf{A}$ is tensored and cotensored over $\mathbf{S}$.

(2) The tensor product $\otimes : \mathbf{A} \times \mathbf{S} \to \mathbf{A}$ is a left Quillen bifunctor.

In the special case where $\mathbf{S}$ is the category of simplicial sets (regarded as a monoidal model category as in Example A.3.1.4), we will simply refer to $\mathbf{A}$ as a *simplicial model category*.

Remark A.3.1.6. An easy formal argument shows that condition (2) is equivalent to either of the following statements:

(2′) Given any cofibration $i : D \to D'$ in $\mathbf{A}$ and any fibration $j : X \to Y$ in $\mathbf{A}$, the induced map

$$k : \mathrm{Map}_{\mathbf{A}}(D', X) \to \mathrm{Map}_{\mathbf{A}}(D, X) \times_{\mathrm{Map}_{\mathbf{A}}(D,Y)} \mathrm{Map}_{\mathbf{A}}(D', Y)$$

is a fibration in $\mathbf{S}$, which is trivial if either i or j is a weak equivalence.

(2″) Given any cofibration $i : C \to C'$ in $\mathbf{S}$ and any fibration $j : X \to Y$ in $\mathbf{A}$, the induced map

$$k : X^{C'} \to X^C \times_{Y^C} Y^{C'}$$

is a fibration in $\mathbf{A}$, which is trivial if either i or j is trivial.

The following provides a criterion for detecting simplicial model structures:

Proposition A.3.1.7. *Let $\mathcal{C}$ be a simplicial category that is equipped with a model structure (not assumed to be compatible with the simplicial structure on $\mathcal{C}$). Suppose that every object of $\mathcal{C}$ is cofibrant and that the collection of weak equivalences in $\mathcal{C}$ is stable under filtered colimits. Then $\mathcal{C}$ is a simplicial model category if and only if the following conditions are satisfied:*

(1) *As a simplicial category, $\mathcal{C}$ is both tensored and cotensored over $\mathcal{S}\mathrm{et}_\Delta$.*

(2) *Given a cofibration of simplicial sets $i : K \to L$ and a cofibration $j : C \to D$ in $\mathcal{C}$, the induced map*

$$(C \otimes L) \coprod_{C \otimes K} D \otimes K \to D \otimes L$$

is a cofibration in $\mathcal{C}$.

(3) *For every $n \geq 0$ and every object C in $\mathcal{C}$, the natural map*

$$C \otimes \Delta^n \to C \otimes \Delta^0 \simeq C$$

is a weak equivalence in $\mathcal{C}$.

Proof. Suppose first that $\mathcal{C}$ is a simplicial model category. It is clear that (1) and (2) are satisfied. To prove (3), we note that the projection $\Delta^n \to \Delta^0$ admits a section $s : \Delta^0 \to \Delta^n$ which is a trivial cofibration. If $\mathcal{C}$ is a simplicial model category, then since C is cofibrant, it follows that $C \otimes \Delta^0 \to C \otimes \Delta^n$ is a trivial cofibration and, in particular, a weak equivalence. Thus the projection $C \otimes \Delta^n \to C \otimes \Delta^0$ is a weak equivalence by the two-out-of-three property.

Now suppose that (1), (2), and (3) are satisfied. We wish to show that $\mathcal{C}$ is a simplicial model category. We first show that the bifunctor

$$(C, K) \mapsto C \otimes K$$

preserves weak equivalences separately in each variable.

Fix the object $C \in \mathcal{C}$ and suppose that $f : K \to K'$ is a weak homotopy equivalence of simplicial sets. Choose a cofibration $K \to K''$, where K'' is a contractible Kan complex. Then we may factor f as a composition

$$K \xrightarrow{f'} K \times K'' \xrightarrow{f''} K.$$

To prove that $\mathrm{id}_C \otimes f$ is a weak equivalence, it suffices to prove that $\mathrm{id}_C \otimes f'$ and $\mathrm{id}_C \otimes f''$ are weak equivalences. Note that the map f'' has a section s which is a trivial cofibration. Thus, to prove that $\mathrm{id}_C \otimes f''$ is a weak equivalence, it suffices to show that $\mathrm{id}_C \otimes s$ is a weak equivalence. In other words, we may reduce to the case where f is itself a trivial cofibration of simplicial sets.

Consider the collection A of all monomorphisms $f : K \to K'$ of simplicial sets having the property that $\mathrm{id}_C \otimes f$ is a weak equivalence in $\mathcal{C}$. It is easy to see that this collection of morphisms is weakly saturated. Thus, to prove that it contains all trivial cofibrations of simplicial sets, it suffices to show that every horn inclusion $\Lambda^n_i \to \Delta^n$ belongs to A. We prove this by induction on $n > 0$. Choose a vertex v belonging to Λ^n_i. We note that the inclusion $\{v\} \to \Lambda^n_i$ is a pushout of horn inclusions in dimensions $< n$; by the inductive hypothesis, this inclusion belongs to A. Thus it suffices to show that $\{v\} \to \Delta^n$ belongs to A, which is equivalent to assumption (3).

Now let us show that for each simplicial set K, the functor

$$C \mapsto C \otimes K$$

preserves weak equivalences. We will prove this by induction on the (possibly infinite) dimension of K. Choose a weak equivalence $g : C \to C'$ in $\mathcal{C}$. Let S denote the collection of all simplicial subsets $L \subseteq K$ such that $g \otimes \mathrm{id}_L$ is a weak equivalence. We regard S as a partially ordered set with respect to inclusions of simplicial subsets. Clearly, $\emptyset \in S$. Since weak equivalences in $\mathcal{C}$ are stable under filtered colimits, the supremum of every chain in S belongs to S. By Zorn's lemma, S has a maximal element L. We wish to show that $L = K$. If not, we may choose some nondegenerate simplex σ of K which does not belong to L. Choose σ of the smallest possible dimension, so that all of the faces of σ belong to L. Thus there is an inclusion $L' = L \coprod_{\partial \sigma} \sigma \subseteq K$. Since $\mathcal{C}$ is left proper, assumption (2) implies that the diagram

$$\begin{tikzcd}
D \otimes \partial \sigma \arrow[r] \arrow[d] & D \otimes \sigma \arrow[d] \\
D \otimes L \arrow[r] & D \otimes L'
\end{tikzcd}$$

is a homotopy pushout for every object $D \in \mathcal{C}$. We observe that $g \otimes \mathrm{id}_L$ is a weak equivalence by assumption, $g \otimes \mathrm{id}_{\partial \sigma}$ is a weak equivalence by the inductive hypothesis (since $\partial \sigma$ has dimension smaller than the dimension of K), and $g \otimes \mathrm{id}_\sigma$ is a weak equivalence by virtue of assumption (3) and the fact that g is a weak equivalence. It follows that $g \otimes \mathrm{id}_{L'}$ is a weak equivalence, which contradicts the maximality of L. This completes the proof that the bifunctor $\otimes : \mathcal{C} \times \mathcal{S}\mathrm{et}_\Delta \to \mathcal{C}$ preserves weak equivalences separately in each variable.

Now suppose we are given a cofibration $i : C \to C'$ in $\mathcal{C}$ and another cofibration $j : S \to S'$ in $\mathcal{S}\mathrm{et}_\Delta$. We wish to prove that the induced map

$$i \wedge j : (C \otimes S') \coprod_{C \otimes S} (C' \otimes S) \to C' \otimes S'$$

is a cofibration in $\mathcal{C}$, which is trivial if either i or j is trivial. The first point follows immediately from (2). For the triviality, we will assume that i is a weak equivalence (the case where j is a weak equivalence follows using the same argument). Consider the diagram

$$\begin{tikzcd} C \otimes S \arrow[r, "i \otimes \mathrm{id}_S"] \arrow[d] & C' \otimes S \arrow[d] & \\ C \otimes S' \arrow[r, "f"] & (C' \otimes S) \coprod_{C \otimes S} (C \otimes S') \arrow[r] & C' \otimes S'. \end{tikzcd}$$

The arguments above show that $i \otimes \mathrm{id}_S$ and $i \otimes \mathrm{id}_{S'}$ are weak equivalences. The square in the diagram is a homotopy pushout, so Proposition A.2.4.2 implies that f is a weak equivalence as well. Thus $i \wedge j$ is a weak equivalence by the two-out-of-three property. □

If $\mathcal{C}$ is a simplicial model category, then there is automatically a strong relationship between the homotopy theory of the underlying model category and the homotopy theory of the simplicial sets $\mathrm{Map}_{\mathcal{C}}(\bullet, \bullet)$. For example, we have the following:

Remark A.3.1.8. Let $\mathcal{C}$ be a simplicial model category, let X be a cofibrant object of $\mathcal{C}$, and let Y be a fibrant object of $\mathcal{C}$. The simplicial set $K = \mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex; moreover, there is a canonical bijection

$$\pi_0 K \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y).$$

We conclude this section by studying a situation which arises in Chapter 3. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories enriched over another model category $\mathbf{S}$, and suppose we are given a Quillen adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

between the underlying model categories. We wish to study the situation where G (but not F) has the structure of an $\mathbf{S}$-enriched functor. Thus, for every triple of objects $X \in \mathcal{C}$, $Y \in \mathcal{D}$, $S \in \mathbf{S}$, we have a canonical map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(S \otimes X, GY) &\simeq \mathrm{Hom}_{\mathbf{S}}(S, \mathrm{Map}_{\mathcal{C}}(X, GY)) \\ &\to \mathrm{Hom}_{\mathbf{S}}(S, \mathrm{Map}_{\mathcal{D}}(FX, FGY)) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(S \otimes FX, FGY) \\ &\to \mathrm{Hom}_{\mathcal{D}}(S \otimes FX, Y). \end{aligned}$$

Taking $Y = F(S \otimes X)$ and applying this map to the unit of the adjunction between F and G, we obtain a map $S \otimes FX \to F(S \otimes X)$, which we will denote by $\beta_{X,S}$. The collection of maps $\beta_{X,S}$ is simply another way of encoding the data of G as an $\mathbf{S}$-enriched functor. If the maps $\beta_{X,S}$ are isomorphisms, then F is again an $\mathbf{S}$-enriched functor and (F, G) is an adjunction between $\mathbf{S}$-enriched categories. We wish to study an analogous situation where the maps $\beta_{X,S}$ are only assumed to be weak equivalences.

Remark A.3.1.9. Suppose that $\mathbf{S}$ is the category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets with its usual model structure. Then the map $\beta_{X,S}$ is automatically a weak equivalence for every cofibrant object $X \in \mathcal{C}$. To prove this, we consider the collection $\mathcal{K}$ of all simplicial sets S such that $\beta_{S,X}$ is an equivalence. It is not difficult to show that $\mathcal{K}$ is closed under weak equivalences, homotopy pushout squares, and coproducts. Since $\Delta^0 \in \mathcal{K}$, we conclude that $\mathcal{K} = \mathcal{S}\mathrm{et}_\Delta$.

Proposition A.3.1.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbf{S}$-enriched model categories. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a Quillen adjunction between the underlying model categories. Assume that every object of $\mathcal{C}$ is cofibrant, and that the map $\beta_{X,S} : S \otimes F(X) \to F(S \otimes X)$ is a weak equivalence for every pair of cofibrant objects $X \in \mathcal{C}$, $S \in \mathbf{S}$. The following are equivalent:*

1. *The adjunction (F, G) is a Quillen equivalence.*
2. *The restriction of G determines a weak equivalence of $\mathbf{S}$-enriched categories $\mathcal{D}^\circ \to \mathcal{C}^\circ$ (see §A.3.2).*

Remark A.3.1.11. Strictly speaking, in §A.3.2, we define only weak equivalences between *small* $\mathbf{S}$-enriched categories; however, the definition extends to large categories in an obvious way.

Proof. Since G preserves fibrant objects and every object of $\mathcal{C}$ is cofibrant, it is clear that G carries $\mathcal{D}^\circ$ into $\mathcal{C}^\circ$. Condition (1) is equivalent to the assertion that for every pair of fibrant-cofibrant objects $C \in \mathcal{C}$, $D \in \mathcal{D}$, a map $g : C \to GD$ is a weak equivalence in $\mathcal{C}$ if and only if the adjoint map $f : FC \to D$ is a weak equivalence in $\mathcal{D}$. Choose a factorization of f' as a composition $FC \xrightarrow{f'} D' \xrightarrow{f''} D$, where f' is a trivial cofibration and f'' is a fibration. By the two-out-of-three property, f is a weak equivalence if and only if f'' is a weak equivalence. We note that g admits an analogous factorization as

$$C \xrightarrow{g'} GD' \xrightarrow{g''} GD.$$

Using (2), we deduce that f'' is an equivalence in $\mathcal{D}^\circ$ if and only if g'' is an equivalence in $\mathcal{C}^\circ$. It will therefore suffice to show that g' is an equivalence in $\mathcal{C}^\circ$. For this, it will suffice to show that C and GD' corepresent the same functor on the homotopy category $\mathrm{h}\mathcal{C}$. Invoking (2) again, it will suffice to show that for every fibrant-cofibrant object $D'' \in \mathcal{D}$, the induced map

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(GD', GD'') \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, GD'') \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{D}}(FC, D'')$$

is bijective. Using (2), we deduce that the map

$$\mathrm{Hom}_{\mathrm{h}\mathcal{D}}(D', D'') \to \mathrm{Hom}_{\mathrm{h}\mathcal{D}}(GD', GD'')$$

is bijective. The desired result now follows from the fact that f' is a weak equivalence in $\mathcal{D}$.

We now show that (1) $\Rightarrow$ (2). The $\mathbf{S}$-enriched functor $G^\circ : \mathcal{D}^\circ \to \mathcal{C}^\circ$ is essentially surjective since the right derived functor RG is essentially surjective on homotopy categories. It suffices to show that G° is fully faithful: in other words, that for every pair of fibrant-cofibrant objects $X, Y \in \mathcal{D}$, the induced map

$$i : \mathrm{Map}_{\mathcal{D}}(X,Y) \to \mathrm{Map}_{\mathcal{C}}(G(X), G(Y))$$

is a weak equivalence in $\mathbf{S}$.

Since the left derived functor LF is essentially surjective, there exists an object $X' \in \mathcal{C}$ and a weak equivalence $FX' \to X$. We may regard X as a fibrant replacement for FX' in $\mathcal{D}$; it follows that the adjoint map $X' \to GX$ may be identified with the adjunction $X' \to (RG \circ LF)X'$ and is therefore a weak equivalence by (1). Thus we have a diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{D}}(X,Y) \arrow[r, "i"] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(G(X), G(Y)) \arrow[d] \\ \mathrm{Map}_{\mathcal{D}}(F(X'),Y) \arrow[r, "i'"] & \mathrm{Map}_{\mathcal{C}}(X', G(Y)) \end{tikzcd}$$

in which the vertical arrows are homotopy equivalences; thus, to show that i is a weak equivalence, it suffices to show that i' is a weak equivalence. For this, it suffices to show that i' induces a bijection from $[S, \mathrm{Map}_{\mathcal{D}}(F(X'), Y)]$ to $[S, \mathrm{Map}_{\mathcal{C}}(X', G(Y))]$ for every cofibrant object $S \in \mathbf{S}$; here $[S, K]$ denotes the set of homotopy classes of maps from S into K in the homotopy category h$\mathbf{S}$. But we may rewrite this map of sets as

$$i'_S : \mathrm{Map}_{\mathrm{h}\mathcal{D}}(F(X') \otimes S, Y) \to \mathrm{Map}_{\mathrm{h}\mathcal{C}}(X' \otimes S, G(Y)) = \mathrm{Map}_{\mathrm{h}\mathcal{D}}(F(X' \otimes S), Y),$$

and it is given by composition with $\beta_{X',S}$. (Here h$\mathcal{C}$ and h$\mathcal{D}$ denote the homotopy categories of $\mathcal{C}$ and $\mathcal{D}$ as *model categories*; these are equivalent to the corresponding homotopy categories of $\mathcal{C}^\circ$ and $\mathcal{D}^\circ$ as $\mathbf{S}$-enriched categories). Since $\beta_{X',S}$ is an isomorphism in the homotopy category h$\mathcal{D}$, the map i'_S is bijective and (2) holds, as desired. $\square$

Corollary A.3.1.12. *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be a Quillen equivalence between simplicial model categories, where every object of* $\mathcal{C}$ *is cofibrant. Suppose that* G *is a simplicial functor. Then* G *induces an equivalence of* ∞*-categories* $\mathrm{N}(\mathcal{D}^\circ) \to \mathrm{N}(\mathcal{C}^\circ)$.

A.3.2 The Model Structure on S-Enriched Categories

Throughout this section, we will fix a symmetric monoidal model category $\mathbf{S}$ and study the category of $\mathbf{S}$-enriched categories. The main case of interest to us is that in which $\mathbf{S}$ is the category of simplicial sets (with its usual model structure and the Cartesian monoidal structure). However, the treatment of the general case requires little additional effort, and there are a number of other examples which arise naturally in other contexts:

(i) The category of simplicial sets equipped with the Cartesian monoidal structure and the *Joyal* model structure defined in §2.2.5.

(*ii*) The category of complexes

$$\cdots \to M_n \to M_n \to M_{n-1} \to \cdots,$$

of vector spaces over a field k with its usual model structure (in which weak equivalences are quasi-isomorphisms, fibrations are epimorphisms, and cofibrations are monomorphisms) and monoidal structure given by the formation of tensor products of complexes.

Let $\mathbf{S}$ be an monoidal model category and let $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ denote the category of (small) $\mathbf{S}$-enriched categories in which morphisms are given by $\mathbf{S}$-enriched functors. The goal of this section is to describe a model structure on $\mathcal{C}\mathrm{at}_{\mathbf{S}}$. We first note that the monoidal structure on $\mathbf{S}$ induces a monoidal structure on its homotopy category $\mathrm{h}\mathbf{S}$, which is determined up to (unique) isomorphism by the requirement that there exist a monoidal structure on the functor

$$\mathbf{S} \to \mathrm{h}\mathbf{S}$$

given by inverting all weak equivalences. Consequently, we note that any $\mathbf{S}$-enriched category $\mathcal{C}$ gives rise to an $\mathrm{h}\mathbf{S}$-enriched category $\mathrm{h}\mathcal{C}$ having the same objects as $\mathcal{C}$ and where mapping spaces are given by

$$\mathrm{Map}_{\mathrm{h}\mathcal{C}}(X, Y) = [\mathrm{Map}_{\mathcal{C}}(X, Y)].$$

Here we let $[K]$ denote the image in $\mathrm{h}\mathbf{S}$ of an object $K \in \mathbf{S}$. We will refer to $\mathrm{h}\mathcal{C}$ as the *homotopy category* of $\mathcal{C}$; the passage from $\mathcal{C}$ to $\mathrm{h}\mathcal{C}$ is a special case of Remark A.1.4.3.

Definition A.3.2.1. Let $\mathbf{S}$ be an monoidal model category. We say that a functor $F : \mathcal{C} \to \mathcal{C}'$ in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is a *weak equivalence* if the induced functor $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{C}'$ is an equivalence of $\mathrm{h}\mathbf{S}$-enriched categories. In other words, F is a weak equivalence if and only if:

(1) For every pair of objects $X, Y \in \mathcal{C}$, the induced map

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{C}'}(F(X), F(Y))$$

is a weak equivalence in $\mathbf{S}$.

(2) Every object $Y \in \mathcal{C}'$ is equivalent to $F(X)$ in the homotopy category $\mathrm{h}\mathcal{C}'$ for some $X \in \mathcal{C}$.

Remark A.3.2.2. If $\mathbf{S}$ is the category $\mathcal{S}\mathrm{et}_{\Delta}$ (endowed with the Kan model structure), then Definition A.3.2.1 reduces to the definition given in §1.1.3.

Remark A.3.2.3. Suppose that the collection of weak equivalences in $\mathbf{S}$ is stable under filtered colimits. Then it is easy to see that the collection of weak equivalences in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is also stable under filtered colimits. If $\mathbf{S}$ is also a combinatorial model category, then a bit more effort shows that the class of weak equivalences in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is perfect in the sense of Definition A.2.6.10.

We now introduce a bit of notation for working with $\mathbf{S}$-enriched categories. If A is an object of $\mathbf{S}$, we will let $[1]_A$ denote the $\mathbf{S}$-enriched category having two objects X and Y, with

$$\mathrm{Map}_{[1]_A}(Z,Z') = \begin{cases} \mathbf{1_S} & \text{if } Z = Z' = X \\ \mathbf{1_S} & \text{if } Z = Z' = Y \\ A & \text{if } Z = X, Z' = Y \\ \emptyset & \text{if } Z = Y, Z' = X. \end{cases}$$

Here $\emptyset$ denotes the initial object of $\mathbf{S}$, and $\mathbf{1_S}$ denotes the unit object with respect to the monoidal structure on $\mathbf{S}$. We will denote $[1]_{\mathbf{1_S}}$ simply by $[1]_{\mathbf{S}}$. We let $[0]_{\mathbf{S}}$ denote the $\mathbf{S}$-enriched category having only a single object X, with $\mathrm{Map}_*(X,X) = \mathbf{1_S}$.

We let C_0 denote the collection of all morphisms in $\mathbf{S}$ of the following types:

(i) The inclusion $\emptyset \hookrightarrow [0]_{\mathbf{S}}$.

(ii) The induced maps $[1]_S \to [1]_{S'}$, where $S \to S'$ ranges over a set of generators for the weakly saturated class of cofibrations in $\mathbf{S}$.

Proposition A.3.2.4. *Let* $\mathbf{S}$ *be a combinatorial monoidal model category. Assume that every object of* $\mathbf{S}$ *is cofibrant and that the collection of weak equivalences in* $\mathbf{S}$ *is stable under filtered colimits. Then there exists a left proper combinatorial model structure on* $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ *characterized by the following conditions:*

(C) *The class of cofibrations in* $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ *is the smallest weakly saturated class of morphisms containing the set of morphisms* C_0 *appearing above.*

(W) *The weak equivalences in* $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ *are defined as in §A.3.2.1.*

Proof. It suffices to verify the hypotheses of Proposition A.2.6.13. Condition (1) follows from Remark A.3.2.3. For condition (3), we must show that any functor $F : \mathcal{C} \to \mathcal{C}'$ having the right lifting property with respect to all morphisms in C_0 is a weak equivalence. Since F has the right lifting property with respect to $i : \emptyset \to [0]_{\mathbf{S}}$, it is surjective on objects and therefore essentially surjective. The assumption that F has the right lifting property with respect to the remaining morphisms of C_0 guarantees that for every $X, Y \in \mathcal{C}$, the induced map

$$\mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{C}'}(F(X), F(Y))$$

is a trivial fibration in $\mathbf{S}$ and therefore a weak equivalence.

It remains to verify condition (2): namely, that the class of weak equivalences is stable under pushout by the elements of C_0. We must show that given any pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \to \mathcal{C}'$ with F a weak equivalence and G a pushout of some morphism in C_0, the induced map $F' : \mathcal{C}' \to \mathcal{D}' = \mathcal{D} \coprod_{\mathcal{C}} \mathcal{C}'$ is a weak equivalence. There are two cases to consider.

First, suppose that G is a pushout of the generating cofibration $i : \emptyset \to *$. In other words, the category $\mathcal{C}'$ is obtained from $\mathcal{C}$ by adjoining a new object X, which admits no morphisms to or from the objects of $\mathcal{C}$ (and no endomorphisms other than the identity). The category $\mathcal{D}'$ is obtained from $\mathcal{D}$ by adjoining X in the same fashion. It is easy to see that if F is a weak equivalence, then F' is also a weak equivalence.

The other basic case to consider is one in which G is a pushout of one of the generating cofibrations $[1]_S \to [1]_T$, where $S \to T$ is a cofibration in $\mathbf{S}$. Let $H : [1]_S \to \mathcal{C}$ denote the "attaching map," so that H is determined by a pair of objects $x = H(X)$ and $y = H(Y)$ and a map of $h : S \to \mathrm{Map}_{\mathcal{C}}(x, y)$. By definition, $\mathcal{C}'$ is universal with respect to the property that it receives a map from $\mathcal{C}$, and the map h extends to a map $\widetilde{h} : T \to \mathrm{Map}_{\mathcal{C}'}(x, y)$. To carry out the proof, we will give an explicit construction of an $\mathbf{S}$-enriched category $\mathcal{C}'$ which has this universal property.

For the remainder of the proof, we will assume that $\mathbf{S}$ is the category of simplicial sets. This is purely for notational convenience; the same arguments can be employed without change in the general case.

We begin by declaring that the objects of $\mathcal{C}'$ are the objects of $\mathcal{C}$. The definition of the morphisms in $\mathcal{C}'$ is a bit more complicated. Let w and z be objects of $\mathcal{C}$. We define a sequence of simplicial sets $M^k_{\mathcal{C}'}$ as follows:

$$M^0_{\mathcal{C}} = \mathrm{Map}_{\mathcal{C}}(w, z)$$

$$M^1_{\mathcal{C}} = \mathrm{Map}_{\mathcal{C}}(y, z) \times T \times \mathrm{Map}_{\mathcal{C}}(w, x)$$

$$M^2_{\mathcal{C}} = \mathrm{Map}_{\mathcal{C}}(y, z) \times T \times \mathrm{Map}_{\mathcal{C}}(y, x) \times T \times \mathrm{Map}_{\mathcal{C}}(w, x),$$

and so forth. More specifically, for $k \geq 1$, the m-simplices of $M^k_{\mathcal{C}}$ are finite sequences

$$(\sigma_0, \tau_1, \sigma_1, \tau_2, \ldots, \tau_k, \sigma_k),$$

where $\sigma_0 \in \mathrm{Map}_{\mathcal{C}}(y, z)_m$, $\sigma_k \in \mathrm{Map}_{\mathcal{C}}(w, x)_m$, $\sigma_i \in \mathrm{Map}_{\mathcal{C}}(y, x)_m$ for $0 < i < k$, and $\tau_i \in T_m$ for $1 \leq i \leq k$.

We define $\mathrm{Map}_{\mathcal{C}'}(w, z)$ to be the quotient of the disjoint union $\coprod_k M^k_{\mathcal{C}}$ by the equivalence relation which is generated by making the identification

$$(\sigma_0, \tau_1, \ldots, \sigma_k) \simeq (\sigma_0, \tau_1, \ldots, \tau_{j-1}, \sigma_{j-1} \circ h(\tau_j) \circ \sigma_j, \tau_{j+1}, \ldots, \sigma_k)$$

whenever the simplex τ_j belongs to $S_m \subseteq T_m$.

We equip $\mathcal{C}'$ with an associative composition law, which is given on the level of simplices by

$$(\sigma_0, \tau_1, \ldots, \sigma_k) \circ (\sigma'_0, \tau'_1, \ldots, \sigma'_l) = (\sigma_0, \tau_1, \ldots, \tau_k, \sigma_k \circ \sigma'_0, \tau'_1, \ldots, \sigma'_l).$$

It is easy to verify that this composition law is well-defined (that is, compatible with the equivalence relation introduced above) and associative and that the identification $M^0_{\mathcal{C}} = \mathrm{Map}_{\mathcal{C}}(w, z)$ gives rise to an inclusion of categories $\mathcal{C} \subseteq \mathcal{C}'$. Moreover, the map $h : S \to \mathrm{Map}_{\mathcal{C}}(x, y)$ extends to $\widetilde{h} : T \to \mathrm{Map}_{\mathcal{C}'}(x, y)$ given by the composition

$$T \simeq \{\mathrm{id}_y\} \times T \times \{\mathrm{id}_x\} \subseteq \mathrm{Map}_{\mathcal{C}}(y, y) \times T \times \mathrm{Map}_{\mathcal{C}}(x, x) = M^1_{\mathcal{C}} \to \mathrm{Map}_{\mathcal{C}'}(x, y).$$

Moreover, it is not difficult to see that $\mathcal{C}'$ has the desired universal property.

We observe that, by construction, the simplicial sets $\mathrm{Map}_{\mathcal{C}'}(w,z)$ come equipped with a natural filtration. Namely, define $\mathrm{Map}_{\mathcal{C}'}(w,z)^k$ to be the image of

$$\coprod_{0 \leq i \leq k} M^i_{\mathcal{C}}$$

in $\mathrm{Map}_{\mathcal{C}'}(w,z)$. Then we have

$$\mathrm{Map}_{\mathcal{C}}(w,z) = \mathrm{Map}_{\mathcal{C}'}(w,z)^0 \subseteq \mathrm{Map}_{\mathcal{C}'}(w,z)^1 \subseteq \cdots$$

and $\bigcup_k \mathrm{Map}_{\mathcal{C}'}(w,z)^k = \mathrm{Map}_{\mathcal{C}'}(w,z)$. Moreover, the inclusion

$$\mathrm{Map}_{\mathcal{C}'}(w,z)^k \subseteq \mathrm{Map}_{\mathcal{C}'}(w,z)^{k+1}$$

is a pushout of the inclusion $N^{k+1}_{\mathcal{C}} \subseteq M^{k+1}_{\mathcal{C}}$, where N^{k+1} is the simplicial subset of $M^{k+1}_{\mathcal{C}}$ whose m-simplices consist of those $(2m+1)$-tuples $(\sigma_0, \tau_1, \ldots, \sigma_m)$ such that $\tau_i \in S_m$ for at least one value of i.

Let us now return to the problem at hand: namely, we wish to prove that $F' : \mathcal{C}' \to \mathcal{D}'$ is an equivalence. We note that the construction outlined above may also be employed to produce a model for $\mathcal{D}'$ and an analogous filtration on its morphism spaces.

Since $G' : \mathcal{D} \to \mathcal{D}'$ and $F : \mathcal{C} \to \mathcal{D}$ are essentially surjective, we deduce that F' is essentially surjective. Hence it will suffice to show that, for any objects $w, z \in \mathcal{C}'$, the induced map

$$\phi : \mathrm{Map}_{\mathcal{C}'}(w,z) \to \mathrm{Map}_{\mathcal{D}'}(w,z)$$

is a weak homotopy equivalence. For this, it will suffice to show that for each $i \geq 0$, the induced map $\phi_i : \mathrm{Map}_{\mathcal{C}'}(w,z)^i \to \mathrm{Map}_{\mathcal{D}'}(w,z)^i$ is a weak homotopy equivalence; then ϕ, being a filtered colimit of weak homotopy equivalences ϕ_i, will itself be a weak homotopy equivalence.

The proof now proceeds by induction on i. When $i = 0$, ϕ_i is a weak homotopy equivalence by assumption (since F is an equivalence of simplicial categories). For the inductive step, we note that ϕ_{i+1} is obtained as a pushout

$$\mathrm{Map}_{\mathcal{C}'}(w,z)^i \coprod_{N^{i+1}_{\mathcal{C}}} M^{i+1}_{\mathcal{C}} \to \mathrm{Map}_{\mathcal{D}'}(w,z)^i \coprod_{N^{i+1}_{\mathcal{D}}} M^{i+1}_{\mathcal{D}}.$$

Since $\mathbf{S}$ is left proper, both of these pushouts are homotopy pushouts. Consequently, to show that ϕ_{i+1} is a weak equivalence, it suffices to show that ϕ_i is a weak equivalence and that both of the maps

$$N^{i+1}_{\mathcal{C}} \to N^{i+1}_{\mathcal{D}}$$

$$M^{i+1}_{\mathcal{C}} \to M^{i+1}_{\mathcal{D}}$$

are weak equivalences. These statements follow easily from the compatibility of the monoidal structure of $\mathbf{S}$ with the model structure and the assumption that every object of $\mathbf{S}$ is cofibrant. $\square$

Remark A.3.2.5. It follows from the proof of Proposition A.3.2.4 that if $f : \mathcal{C} \to \mathcal{C}'$ is a cofibration of $\mathbf{S}$-enriched categories, then the induced map $\mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{C}'}(fX, fY)$ is a cofibration for every pair of objects $X, Y \in \mathcal{C}$.

Remark A.3.2.6. The model structure of Proposition A.3.2.4 enjoys the following functoriality: suppose that $f : \mathbf{S} \to \mathbf{S}'$ is a monoidal left Quillen functor between model categories satisfying the hypotheses of Proposition A.3.2.4, with right adjoint $g : \mathbf{S}' \to \mathbf{S}$. Then f and g induce a Quillen adjunction

$$\mathcal{C}\mathrm{at}_{\mathbf{S}} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}\mathrm{at}_{\mathbf{S}'},$$

where F and G are as in Remark A.1.4.3. Moreover, if (f, g) is a Quillen equivalence, then (F, G) is likewise a Quillen equivalence.

In order for Proposition A.3.2.4 to be useful in practice, we need to understand the fibrations in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$. For this, we first introduce a few definitions.

Definition A.3.2.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. We will say that F is a *quasi-fibration* if, for every object $X \in \mathcal{C}$ and every isomorphism $f : F(X) \to Y$ in $\mathcal{D}$, there exists an isomorphism $\overline{f} : X \to \overline{Y}$ in $\mathcal{C}$ such that $F(\overline{f}) = f$.

Remark A.3.2.8. The relevance of Definition A.3.2.7 is as follows: the category $\mathcal{C}\mathrm{at}$ admits a model structure in which the weak equivalences are the equivalences of categories and the fibrations are the quasi-fibrations. This is a special case of Theorem A.3.2.24, which we will prove below (namely, the special case where we take $\mathbf{S} = \mathcal{S}\mathrm{et}$ endowed with the trivial model structure of Example A.2.1.2).

Definition A.3.2.9. Let $\mathbf{S}$ be a monoidal model category and let $\mathcal{C}$ be an $\mathbf{S}$-enriched category. We will say that a morphism f in $\mathcal{C}$ is an *equivalence* if the homotopy class $[f]$ of f is an isomorphism in $\mathrm{h}\mathcal{C}$.

We will say that $\mathcal{C}$ is *locally fibrant* if, for every pair of objects $X, Y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(X,Y)$ is a fibrant object of $\mathbf{S}$.

We will say that an $\mathbf{S}$-enriched functor $F : \mathcal{C} \to \mathcal{C}'$ is a *local fibration* if the following conditions are satisfied:

(*i*) For every pair of objects $X, Y \in \mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{C}'}(FX, FY)$ is a fibration in $\mathbf{S}$.

(*ii*) The induced map $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{C}'$ is a quasi-fibration of categories.

Remark A.3.2.10. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between $\mathbf{S}$-enriched categories which satisfies condition (*i*) of Definition A.3.2.9. Let $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$ be objects. If $\mathcal{C}'$ is locally fibrant, then every isomorphism $[f] : F(X) \to Y$ in $\mathrm{h}\mathcal{C}'$ can be represented by an equivalence $f : F(X) \to Y$

in $\mathcal{C}'$. Let $\overline{Y}$ be an object of $\mathcal{C}$ such that $F(\overline{Y}) = Y$. Since $\mathbf{1_S}$ is a cofibrant object of $\mathbf{S}$ and the map $\mathrm{Map}_{\mathcal{C}}(X, \overline{Y}) \to \mathrm{Map}_{\mathcal{C}'}(F(X), Y)$ is a fibration, Proposition A.2.3.1 implies that $[f]$ can be lifted to an isomorphism $[\overline{f}] : X \to \overline{Y}$ in $\mathrm{h}\mathcal{C}$ if and only if f can be lifted to an equivalence $\overline{f} : X \to \overline{Y}$ in $\mathcal{C}$. Consequently, when $\mathcal{C}'$ is locally fibrant, condition (ii) is equivalent to the following analogous assertion:

(ii') For every equivalence $f : F(X) \to Y$ in $\mathcal{C}'$, there exists an equivalence $\overline{f} : X \to \overline{Y}$ in $\mathcal{C}$ such that $F(\overline{f}) = f$.

Notation A.3.2.11. We let $[1]^{\sim}_{\mathbf{S}}$ denote the $\mathbf{S}$-enriched category containing a pair of objects X, Y, with

$$\mathrm{Map}_{[1]^{\sim}_{\mathbf{S}}}(Z, Z') = \mathbf{1_S}$$

for all $Z, Z' \in \{X, Y\}$.

Definition A.3.2.12 (Invertibility Hypothesis)**.** Let $\mathbf{S}$ be a monoidal model category satisfying the hypotheses of Proposition A.3.2.4. We will say that $\mathbf{S}$ *satisfies the invertibility hypothesis* if the following condition is satisfied:

$(*)$ Let $i : [1]_{\mathbf{S}} \to \mathcal{C}$ be a cofibration of $\mathbf{S}$-enriched categories, classifying a morphism f in $\mathcal{C}$ which is invertible in the homotopy category $\mathrm{h}\mathcal{C}$, and form a pushout diagram

$$\begin{tikzcd} {[1]_{\mathbf{S}}} \arrow[r, "i"] \arrow[d] & \mathcal{C} \arrow[d, "j"] \\ {[1]^{\sim}_{\mathbf{S}}} \arrow[r] & \mathcal{C}\langle f^{-1} \rangle. \end{tikzcd}$$

Then j is an equivalence of $\mathbf{S}$-enriched categories.

In other words, the invertibility hypothesis is the assertion that inverting a morphism f in an $\mathbf{S}$-enriched category $\mathcal{C}$ does not change the homotopy type of $\mathcal{C}$ when f is already invertible up to homotopy.

Remark A.3.2.13. Let $\mathbf{S}$, f, and $\mathcal{C}$ be as in Definition A.3.2.12 and choose a trivial cofibration $F : \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}'$ is a fibrant $\mathbf{S}$-enriched category. Since $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is left proper, the induced map $\mathcal{C}\langle f^{-1} \rangle \to \mathcal{C}'\langle F(f)^{-1} \rangle$ is an equivalence of $\mathbf{S}$-enriched categories. Consequently, assertion $(*)$ holds for $(\mathcal{C}, f)$ if and only if it holds for $(\mathcal{C}', F(f))$. In other words, to test whether $\mathbf{S}$ satisfies the invertibility hypothesis, we need only check $(*)$ in the case where $\mathcal{C}$ is fibrant.

Remark A.3.2.14. In Definition A.3.2.12, the condition that i be a cofibration guarantees that the construction $\mathcal{C} \mapsto \mathcal{C}\langle f^{-1} \rangle$ is homotopy invariant. Alternatively, we can guarantee this by choosing a cofibrant replacement for the map $j : [1]_{\mathbf{S}} \to [1]^{\sim}_{\mathbf{S}}$. Namely, choose a factorization for j as a composition

$$[1]_{\mathbf{S}} \xrightarrow{j'} \mathcal{E} \xrightarrow{j''} [1]^{\sim}_{\mathbf{S}},$$

where j'' is a weak equivalence and j' is a cofibration. For every **S**-enriched category containing a morphism f, define $\mathcal{C}[f^{-1}] = \mathcal{C} \coprod_{[1]_{\mathbf{S}}} \mathcal{E}$. Then we have a canonical map $\mathcal{C}[f^{-1}] \to \mathcal{C}\langle f^{-1}\rangle$, which is an equivalence whenever the map $[1]_{\mathbf{S}} \to \mathcal{C}$ classifying f is a cofibration. Moreover, the construction $\mathcal{C} \mapsto \mathcal{C}[f^{-1}]$ preserves weak equivalences in $\mathcal{C}$. Consequently, we may reformulate the invertibility hypothesis as follows:

($*'$) For every **S**-enriched category $\mathcal{C}$ containing an equivalence f, the map $\mathcal{C} \to \mathcal{C}[f^{-1}]$ is a weak equivalence of **S**-enriched categories.

Remark A.3.2.15. Let $\mathcal{C}$ be a fibrant **S**-enriched category containing an equivalence $f : X \to Y$ and let $\mathcal{C}[f^{-1}]$ be defined as in Remark A.3.2.14. The canonical map $\mathcal{C} \to \mathcal{C}[f^{-1}]$ is a trivial cofibration and therefore admits a section. This section determines a map of **S**-enriched categories $h : \mathcal{E} \to \mathcal{C}$. We observe that $\mathcal{E}$ is a mapping cylinder for the object $[0]_{\mathcal{C}\mathrm{at}_{\mathbf{S}}} \in \mathcal{C}\mathrm{at}_{\mathbf{S}}$, so we can view h as a homotopy between the maps $[0]_{\mathcal{C}\mathrm{at}_{\mathbf{S}}} \to \mathcal{C}$ classifying the objects X and Y.

More generally, the same argument shows that if $F : \mathcal{C} \to \mathcal{D}$ is a fibration of **S**-enriched categories and $f : X \to Y$ is an equivalence in $\mathcal{C}$ such that $F(f) = \mathrm{id}_D$ for some object $D \in \mathcal{D}$, then the functors $[0]_{\mathbf{S}} \to \mathcal{C}$ classifying the objects X and Y are homotopic in the model category $(\mathcal{C}\mathrm{at}_{\mathbf{S}})_{/\mathcal{D}}$.

Definition A.3.2.16. We will say that a model category **S** is *excellent* if it is equipped with a symmetric monoidal structure and satisfies the following conditions:

($A1$) The model category **S** is combinatorial.

($A2$) Every monomorphism in **S** is a cofibration, and the collection of cofibrations is stable under products.

($A3$) The collection of weak equivalences in **S** is stable under filtered colimits.

($A4$) The monoidal structure on **S** is compatible with the model structure. In other words, the tensor product functor $\otimes : \mathbf{S} \times \mathbf{S} \to \mathbf{S}$ is a left Quillen bifunctor.

($A5$) The monoidal model category **S** satisfies the invertibility hypothesis.

Remark A.3.2.17. Axiom ($A2$) of Definition A.3.2.16 implies that every object of **S** is cofibrant. In particular, **S** is left proper.

Example A.3.2.18 (Dwyer, Kan)**.** The category of simplicial sets is an excellent model category when endowed with the Kan model structure and the Cartesian product. The only nontrivial point is to show that $\mathcal{S}\mathrm{et}_{\Delta}$ satisfies the invertibility hypothesis. This is one of the main theorems of [21].

Example A.3.2.19. Let **S** be a presentable category equipped with a closed symmetric monoidal structure. Then **S** is an excellent model category with respect to the trivial model structure of Example A.2.1.2.

The following lemma guarantees a good supply of examples of excellent model categories:

Lemma A.3.2.20. *Suppose we are given a monoidal left Quillen functor $T : \mathbf{S} \rightarrow \mathbf{S}'$ between model categories $\mathbf{S}$ and $\mathbf{S}'$ satisfying axioms* $(A1)$ *through* $(A4)$ *of Definition A.3.2.16. If $\mathbf{S}$ satisfies axiom* $(A5)$*, then so does $\mathbf{S}'$.*

Proof. As indicated in Remark A.3.2.6, the functor T determines a Quillen adjunction

$$\mathcal{C}\mathrm{at}_{\mathbf{S}} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}\mathrm{at}_{\mathbf{S}'}.$$

Let $\mathcal{C}$ be a $\mathbf{S}'$-enriched category and $i : [1]_{\mathbf{S}'} \rightarrow \mathcal{C}$ a cofibration classifying an equivalence f in $\mathcal{C}$. We wish to prove that the map $\mathcal{C} \rightarrow \mathcal{C}\langle f^{-1} \rangle$ is an equivalence of $\mathbf{S}'$-enriched categories. In view of Remark A.3.2.13, we may assume that $\mathcal{C}$ is fibrant.

Choose a factorization of the map $[1]_{\mathbf{S}} \rightarrow [1]^{\sim}_{\mathbf{S}}$ as a composition

$$[1]_{\mathbf{S}} \stackrel{j}{\rightarrow} \mathcal{E} \stackrel{j'}{\rightarrow} [1]^{\sim}_{\mathbf{S}}$$

as in Remark A.3.2.14, so that we have an analogous factorization

$$[1]_{\mathbf{S}'} \rightarrow F(\mathcal{E}) \rightarrow [1]^{\sim}_{\mathbf{S}'}$$

in $\mathcal{C}\mathrm{at}_{\mathbf{S}'}$. Using the latter factorization, we can define $\mathcal{C}[f^{-1}]$ as in Remark A.3.2.14; we wish to show that the map $h : \mathcal{C} \rightarrow \mathcal{C}[f^{-1}]$ is a trivial cofibration.

Let f_0 be the morphism in $G(\mathcal{C})$ classified by f, and let $G(\mathcal{C})[f_0^{-1}] \in \mathcal{C}\mathrm{at}_{\mathbf{S}}$ be defined as in Remark A.3.2.14. Using the fact that $\mathcal{C}$ is locally fibrant (see Theorem A.3.2.24 below), we conclude that f_0 is an equivalence in $G(\mathcal{C})$. Since $\mathbf{S}$ satisfies the invertibility hypothesis, the map $h_0 : G(\mathcal{C}) \rightarrow G(\mathcal{C})[f_0^{-1}]$ is a trivial cofibration. We now conclude by observing that h is a pushout of $F(h_0)$. □

Remark A.3.2.21. Using a similar argument, we can prove a converse to Lemma A.3.2.20 in the case where T is a Quillen equivalence.

Example A.3.2.22. Let $\mathbf{S}$ be the category $\mathcal{S}\mathrm{et}^{+}_{\Delta}$ of marked simplicial sets endowed with the Cartesian model structure defined in §3.1. Then the functor $X \mapsto X^{\sharp}$ is a monoidal left Quillen functor $\mathcal{S}\mathrm{et}_{\Delta} \rightarrow \mathbf{S}$. Combining Example A.3.2.18 with Lemma A.3.2.20, we conclude that $\mathbf{S}$ satisfies the invertibility hypothesis, so that $\mathbf{S}$ is an excellent model category (with respect to the Cartesian product).

Example A.3.2.23. Let $\mathbf{S}$ denote the category of simplicial sets, endowed with the Joyal model structure. The functor $X \mapsto X^{\flat}$ determines a monoidal left Quillen equivalence $\mathbf{S} \rightarrow \mathcal{S}\mathrm{et}^{+}_{\Delta}$. Using Remark A.3.2.21, we deduce that $\mathbf{S}$ satisfies the invertibility hypothesis, so that $\mathbf{S}$ is an excellent model category (with respect to the Cartesian product).

We are now ready to state our main result:

Theorem A.3.2.24. *Let* $\mathbf{S}$ *be an excellent model category. Then*

(1) *An* $\mathbf{S}$*-enriched category* $\mathcal{C}$ *is a fibrant object of* $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ *if and only if it is* locally fibrant*: that is, if and only if the mapping object* $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathbf{S}$ *is fibrant for every pair of objects* $X, Y \in \mathcal{C}$.

(2) *Let* $F : \mathcal{C} \to \mathcal{D}$ *be an* $\mathbf{S}$*-enriched functor, where* $\mathcal{D}$ *is a fibrant object of* $\mathcal{C}\mathrm{at}_{\mathbf{S}}$*. Then* F *is a fibration in* $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ *if and only if* F *is a local fibration.*

Remark A.3.2.25. In the case where $\mathbf{S}$ is the category of simplicial sets (with its usual model structure), Theorem A.3.2.24 is due to Bergner; see [7]. Moreover, Bergner proves a stronger result in this case: assertion (2) holds without the assumption that $\mathcal{D}$ is fibrant.

Before giving the proof of Theorem A.3.2.24, we need to establish some preliminaries. Fix an excellent model category $\mathbf{S}$. We observe that $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is naturally *cotensored* over $\mathbf{S}$. That is, for every $\mathbf{S}$-enriched category $\mathcal{C}$ and every object $K \in \mathbf{S}$, we can define a new $\mathbf{S}$-enriched category $\mathcal{C}^K$ as follows:

(i) The objects of $\mathcal{C}^K$ are the objects of $\mathcal{C}$.

(ii) Given a pair of objects $X, Y \in \mathcal{C}$, we have
$$\mathrm{Map}_{\mathcal{C}^K}(X, Y) = \mathrm{Map}_{\mathcal{C}}(X, Y)^K \in \mathbf{S}.$$

This construction does not endow $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ with the structure of an $\mathbf{S}$-enriched category because the construction $\mathcal{D} \mapsto \mathcal{D}^K$ is not compatible with colimits in K. However, we can remedy the situation as follows. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbf{S}$-enriched categories and let ϕ be a function from the set of objects of $\mathcal{C}$ to the set of objects of $\mathcal{D}$. Then there exists an object $\mathrm{Map}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathcal{D}) \in \mathbf{S}$ which is characterized by the following universal property: for every $K \in \mathbf{S}$, there is a natural bijection
$$\mathrm{Hom}_{\mathbf{S}}(K, \mathrm{Map}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Hom}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathcal{D}^K),$$
where $\mathrm{Hom}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathcal{D}^K)$ denotes the set of all functors from $\mathcal{C}$ to $\mathcal{D}^K$ which is given on objects by the function ϕ.

Lemma A.3.2.26. *Let* $\mathbf{S}$ *be an excellent model category. Fix a diagram of* $\mathbf{S}$*-enriched categories*
$$\begin{tikzcd} \mathcal{C} \arrow[r, "u"] \arrow[d, "F"] & \mathcal{C}' \arrow[d, "F'"] \\ \mathcal{D} \arrow[r, "u'"] & \mathcal{D}'. \end{tikzcd}$$

Assume that

(a) *For every pair of objects* $X, Y \in \mathcal{C}$*, the diagram*
$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{D}}(FX, FY) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}'}(uX, uY) \arrow[r] & \mathrm{Map}_{\mathcal{D}'}(u'FX, u'FY) \end{tikzcd}$$

is a homotopy pullback square involving fibrant objects of $\mathbf{S}$ *and the vertical arrows are fibrations.*

Let $G : \mathcal{A} \to \mathcal{B}$ *be a functor between* $\mathbf{S}$*-enriched categories which is a transfinite composition of pushouts of generating cofibrations of the form* $[1]_S \to [1]_{S'}$*, where* $S \to S'$ *is a cofibration in* $\mathbf{S}$ *and let* ϕ *be a function from the set of objects of* $\mathcal{B}$ *(which is isomorphic to the set of objects of* $\mathcal{A}$*) to* $\mathcal{C}$*. Then the diagram*

$$\begin{tikzcd}
\mathrm{Map}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Map}^{F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{D}) \times_{\mathrm{Map}^{F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{D})} \mathrm{Map}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{C}) \arrow[d] \\
\mathrm{Map}^{u\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{C}') \arrow[r] & \mathrm{Map}^{u'F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{D}') \times_{\mathrm{Map}^{u'F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{D}')} \mathrm{Map}^{u\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{C}')
\end{tikzcd}$$

is a homotopy pullback square between fibrant objects of $\mathbf{S}$*, and the vertical arrows are fibrations.*

Proof. It is easy to see that the collection of morphisms $G : \mathcal{A} \to \mathcal{B}$ which satisfy the conclusion of the lemma is weakly saturated. It will therefore suffice to show that G contains every morphism of the form $[1]_S \to [1]_{S'}$, where $S \to S'$ is a cofibration in $\mathbf{S}$. In this case, ϕ determines a pair of objects $X, Y \in \mathcal{C}$, and we can rewrite the diagram of interest as

$$\begin{tikzcd}
& \mathrm{Map}_{\mathcal{C}}(X,Y)^{S'} \arrow[dl] \arrow[dr] & \\
\mathrm{Map}_{\mathcal{C}'}(uX,uY)^{S'} \arrow[dr] & & \mathrm{Map}_{\mathcal{C}}(X,Y)^{S} \times_{\mathrm{Map}_{\mathcal{D}}(FX,FY)^S} \mathrm{Map}_{\mathcal{D}}(FX,FY)^{S'} \arrow[dl] \\
& \mathrm{Map}_{\mathcal{C}'}(uX,uY)^{S} \times_{\mathrm{Map}_{\mathcal{D}'}(u'FX,u'FY)^S} \mathrm{Map}_{\mathcal{D}'}(u'FX,u'FY)^{S'}. &
\end{tikzcd}$$

The desired result now follows from (a) since the map $S \to S'$ is a cofibration between cofibrant objects of $\mathbf{S}$. □

Proof of Theorem A.3.2.24. Assertion (1) is just a special case of (2) where we take $\mathcal{D}$ to be the final object of $\mathcal{C}\mathrm{at}_{\mathbf{S}}$. It will therefore suffice to prove (2).

We first prove the "only if" direction. If F is a fibration, then F has the right lifting property with respect to every trivial cofibration of the form $[1]_S \to [1]_{S'}$, where $S \to S'$ is a trivial cofibration in $\mathbf{S}$. It follows that for every pair of objects $X, Y \in \mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{D}}(FX,FY)$ is a fibration in $\mathbf{S}$. In particular, $\mathcal{C}$ is locally fibrant.

To complete the proof that F is a local fibration, we will show that F satisfies condition (ii') of Remark A.3.2.10. Suppose $X \in \mathcal{C}$ and that $f : FX \to Y$ is an equivalence in $\mathcal{D}$. We wish to show that we can lift f to an equivalence $\overline{f} : X \to \overline{Y}$. Let $\mathcal{E}$ and $\mathcal{D}[f^{-1}]$ be defined as in Remark A.3.2.14.

Since $\mathbf{S}$ satisfies the invertibility hypothesis, the map $h : \mathcal{D} \to \mathcal{D}[f^{-1}]$ is a trivial cofibration. Because we have assumed $\mathcal{D}$ to be fibrant, the map h admits a section. This section determines a map $s : \mathcal{E} \to \mathcal{D}$. We now consider the lifting problem

$$\begin{tikzcd} {[0]_{\mathbf{S}}} \arrow[r, "X"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{E} \arrow[r, "s"] \arrow[ur, dashed] & \mathcal{D}. \end{tikzcd}$$

Since F is a fibration and the left vertical map is a trivial cofibration, there exists a solution as indicated. This solution determines a morphism $\overline{f} : X \to \overline{Y}$ in $\mathcal{C}$ lifting f. Moreover, $\overline{f}$ is the image of a morphism in $\mathcal{E}$. Since every morphism in $\mathcal{E}$ is an equivalence, we deduce that $\overline{f}$ is an equivalence in $\mathcal{C}$.

Let us now suppose that F is a local fibration. We wish to show that F is a fibration. Choose a factorization of F as a composition

$$\mathcal{C} \xrightarrow{u} \mathcal{C}' \xrightarrow{F'} \mathcal{D},$$

where u is a weak equivalence and F' is a fibration. We will prove the following:

$(*)$ Suppose we are given a commutative diagram of $\mathbf{S}$-enriched categories

$$\begin{tikzcd} \mathcal{A} \arrow[r, "v"] \arrow[d, "G"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{B} \arrow[r, "v'"] & \mathcal{D}, \end{tikzcd}$$

where G is a cofibration. If there exists a functor $\alpha : \mathcal{B} \to \mathcal{C}'$ such that $\alpha G = uv$ and $F'\alpha = v'$, then there exists a functor $\beta : \mathcal{B} \to \mathcal{C}$ such that $\beta G = v$ and $F\beta = v'$.

Since the map F' has the right lifting property with respect to all trivial cofibrations, assertion $(*)$ implies that F also has the right lifting property with respect to all trivial cofibrations, so that F is a fibration as desired.

We now prove $(*)$. Using the small object argument, we deduce that the functor G is a retract of some functor $G' : \mathcal{A} \to \mathcal{B}'$, where G' is a transfinite composition of morphisms obtained as pushouts of generating cofibrations. It will therefore suffice to prove $(*)$ after replacing G by G'.

Reordering the transfinite composition if necessary, we may assume that G' factors as a composition

$$\mathcal{A} \xrightarrow{G'_0} \mathcal{B}'_0 \xrightarrow{G'_1} \mathcal{B}',$$

where $\mathcal{B}'_0$ is obtained from $\mathcal{A}$ by adjoining a collection of new objects, $\{B_i\}_{i \in I}$, and $\mathcal{B}'$ is obtained from $\mathcal{B}'_0$ by a transfinite sequence of pushouts by generating cofibrations of the form $[1]_S \to [1]_{S'}$, where $S \to S'$ is a cofibration in $\mathbf{S}$. Let $C'_i = \alpha(B_i)$ for each $i \in I$. Since u is an equivalence of

$\mathbf{S}$-enriched categories, there exists a collection of objects $\{C_i\}_{i\in I}$ and equivalences $f_i : uC_i \to C'_i$. Let g_i be the image of f_i in $\mathcal{D}$. Since F is a local fibration, we can lift each g_i to an equivalence $f'_i : C_i \to C''_i$ in $\mathcal{C}$. Since the maps $\mathrm{Map}_{\mathcal{C}'}(uC''_i, C'_i) \to \mathrm{Map}_{\mathcal{D}}(FC''_i, F'C'_i)$ are fibrations, we can choose morphisms $f''_i : uC''_i \to C'_i$ in $\mathcal{C}'$ such that $F'(f'_i)$ is the identity for each i, and the diagrams

$$\begin{tikzcd} & uC''_i \arrow[dr, "f''_i"] & \\ uC_i \arrow[ur, "f'_i"] \arrow[rr, "f_i"] & & C'_i \end{tikzcd}$$

commute up to homotopy. Replacing C_i by C''_i, we may assume that each of the maps f_i projects to the identity in $\mathcal{D}$.

Let $\alpha_0 = \alpha|\,\mathcal{B}'_0$ and let $\alpha'_0 : \mathcal{B}'_0 \to \mathcal{C}'$ be defined by the formula

$$\alpha'_0(A) = \begin{cases} \alpha_0(A) & \text{if } A \in \mathcal{A} \\ uC_i & \text{if } A = B_i, i \in I. \end{cases}$$

Remark A.3.2.15 implies that the maps α_0 and α'_0 are homotopic in the model category $(\mathcal{C}\mathrm{at}_{\mathbf{S}})_{\mathcal{A}\,/\,/\,\mathcal{D}}$. Applying Proposition A.2.3.1, we deduce the existence of a map $\alpha' : \mathcal{B}' \to \mathcal{C}$ which extends α_0 and satisfies $\alpha' G = uv$ and $F'\alpha' = v'$. We may therefore replace α by α', v by α'_0, and $\mathcal{A}$ by $\mathcal{B}'_0$ and thereby reduce to the case where the functor $G : \mathcal{A} \to \mathcal{B}$ is a transfinite composition of generating cofibrations of the form $[1]_S \to [1]_{S'}$, where $S \to S'$ is a cofibration in $\mathbf{S}$.

Let ϕ be the map from the objects of $\mathcal{B}$ to the objects of $\mathcal{C}$ determined by α. Applying Lemma A.3.2.26, we obtain a homotopy pullback diagram

$$\begin{tikzcd} \mathrm{Map}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Map}^{F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{D}) \times_{\mathrm{Map}^{F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{D})} \mathrm{Map}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{C}) \arrow[d] \\ \mathrm{Map}^{u\phi}\,\mathcal{C}\mathrm{at}_{\mathbf{S}}(\mathcal{B},\mathcal{C}') \arrow[r] & \mathrm{Map}^{F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{D}) \times_{\mathrm{Map}^{F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{D})} \mathrm{Map}^{u\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{C}'). \end{tikzcd}$$

in which the horizontal arrows are fibrations. We therefore have a weak equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{C}) \to M = \mathrm{Map}^{u\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{C}') \times_{\mathrm{Map}^{F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{D})} \mathrm{Map}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{C})$$

of fibrations over $N = \mathrm{Map}^{F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{D}) \times_{\mathrm{Map}^{F\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{D})} \mathrm{Map}^{u\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{A},\mathcal{C}')$. Moreover, the pair (α, v) determines a map $\mathbf{1_S} \to M$ lifting the map $(v', uv') : \mathbf{1_S} \to N$. Applying Proposition A.2.3.1, we deduce that $(v, uv') : \mathbf{1_S} \to N$ can be lifted to a map $\mathbf{1_S} \to \mathrm{Map}^{\phi}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{B},\mathcal{C})$, which is equivalent to the existence of the desired map β. $\square$

We conclude this section with a few easy results concerning homotopy limits in the model category $\mathcal{C}\mathrm{at}_{\mathbf{S}}$.

Proposition A.3.2.27. *Let* $\mathbf{S}$ *be an excellent model category,* $\mathcal{J}$ *a small category, and* $\{\mathcal{C}_J\}_{J\in\mathcal{J}}$ *a diagram of* $\mathbf{S}$*-enriched categories. Suppose we are given a compatible family of functors* $\{f_J : \mathcal{C} \to \mathcal{C}_J\}_{J\in\mathcal{J}}$ *which exhibits* $\mathcal{C}$ *as a homotopy limit of the diagram* $\{\mathcal{C}_J\}_{J\in\mathcal{J}}$ *in* $\mathcal{C}\mathrm{at}_{\mathbf{S}}$*. Then for every pair of objects* $X, Y \in \mathcal{C}$*, the maps* $\{\mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{C}_J}(f_J X, f_J Y)\}_{J\in\mathcal{J}}$ *exhibit* $\mathrm{Map}_{\mathcal{C}}(X,Y)$ *as a homotopy limit of the diagram* $\{\mathrm{Map}_{\mathcal{C}_J}(f_J X, f_J Y)\}_{J\in\mathcal{J}}$ *in* $\mathbf{S}$*.*

Proof. Without loss of generality, we may assume that the diagram $\{\mathcal{C}_J\}_{J\in\mathcal{J}}$ is injectively fibrant and that the maps f_J exhibit $\mathcal{C}$ as a limit of $\{\mathcal{C}_J\}_{J\in\mathcal{J}}$. It follows that $\mathrm{Map}_{\mathcal{C}}(X,Y)$ is a limit of the diagram $\{\mathrm{Map}_{\mathcal{C}_J}(f_J X, f_J Y)\}_{J\in\mathcal{J}}$. It will therefore suffice to show that the diagram $\{\mathrm{Map}_{\mathcal{C}_J}(f_J X, f_J Y)\}_{J\in\mathcal{J}}$ is injectively fibrant. For this, it will suffice to show that $\{\mathrm{Map}_{\mathcal{C}_J}(f_J X, f_J Y)\}_{J\in\mathcal{J}}$ has the right lifting property with respect to every injective trivial cofibration $\alpha : F \to F'$ of diagrams $F, F' : \mathcal{J} \to \mathbf{S}$. Let $G : \mathcal{J} \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ be defined by the formula $G(J) = [1]_{F(J)}$ and let $G' : \mathcal{J} \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ be defined likewise. The desired result now follows from the observation that α induces an injective trivial cofibration $G \to G'$ in $\mathrm{Fun}(\mathcal{J}, \mathcal{C}\mathrm{at}_{\mathbf{S}})$. □

Corollary A.3.2.28. *Let* $\mathbf{S}$ *be an excellent model category,* $\mathcal{J}$ *a small category, and* $\{\mathcal{C}_J\}_{J\in\mathcal{J}}$ *a diagram of* $\mathbf{S}$*-enriched categories. Suppose we are given* $\mathbf{S}$*-enriched functors*

$$\mathcal{D} \xrightarrow{\beta} \mathcal{C} \xrightarrow{\alpha} \lim\{\mathcal{C}_J\}_{J\in\mathcal{J}}$$

such that $\alpha\circ\beta$ *exhibits* $\mathcal{D}$ *as a homotopy limit of the diagram* $\{\mathcal{C}_J\}_{J\in\mathcal{J}}$*. Then the following conditions are equivalent:*

(1) *The functor* α *exhibits* $\mathcal{C}$ *as a homotopy limit of the diagram* $\{\mathcal{C}_J\}_{J\in\mathcal{J}}$*.*

(2) *For every pair of objects* $X, Y \in \mathcal{C}$*, the functor* α *exhibits* $\mathrm{Map}_{\mathcal{C}}(X,Y)$ *as a homotopy limit of the diagram* $\{\mathrm{Map}_{\mathcal{C}_J}(\alpha_J X, \alpha_J Y)\}_{J\in\mathcal{J}}$*.*

Proof. The implication (1) ⇒ (2) follows from Proposition A.3.2.27. To prove the converse, we may assume that the diagram $\{\mathcal{C}_J\}_{J\in\mathcal{J}}$ is injectively fibrant. In view of (2), Proposition A.3.2.27 implies that α induces a fully faithful functor between h$\mathbf{S}$-enriched homotopy categories. It will therefore suffice to show that α is essentially surjective on homotopy categories, which follows from our assumption that $\alpha\circ\beta$ is a weak equivalence. □

A.3.3 Model Structures on Diagram Categories

In this section, we consider enriched analogues of the constructions presented in §A.2.8. Namely, suppose that $\mathbf{S}$ is an excellent model category, $\mathbf{A}$ a combinatorial $\mathbf{S}$-enriched model category, and $\mathcal{C}$ a small $\mathbf{S}$-enriched category. Let $\mathbf{A}^{\mathcal{C}}$ denote the category of $\mathbf{S}$-enriched functors from $\mathcal{C}$ to $\mathbf{A}$. In this section, we will study the associated projective and injective model structures on $\mathbf{A}^{\mathcal{C}}$. The ideas described here will be used in §A.3.4 to construct certain mapping objects in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$.

We begin with the analogue of Definition A.2.8.1.

Definition A.3.3.1. Let $\mathcal{C}$ be a small $\mathbf{S}$-category and $\mathbf{A}$ a combinatorial $\mathbf{S}$-enriched model category. A natural transformation $\alpha : F \to G$ in $\mathbf{A}^{\mathcal{C}}$ is:

- an *injective cofibration* if the induced map $F(C) \to G(C)$ is a cofibration in $\mathbf{A}$ for each $C \in \mathcal{C}$.
- a *projective fibration* if the induced map $F(C) \to G(C)$ is a fibration in $\mathbf{A}$ for each $C \in \mathcal{C}$.
- a *weak equivalence* if the induced map $F(C) \to G(C)$ is a weak equivalence in $\mathbf{A}$ for each $C \in \mathcal{C}$.
- an *injective fibration* if it has the right lifting property with respect to every morphism β in $\mathbf{A}^{\mathcal{C}}$ which is simultaneously a weak equivalence and an injective cofibration.
- a *projective cofibration* if it has the left lifting property with respect to every morphism β in $\mathbf{A}^{\mathcal{C}}$ which is simultaneously a weak equivalence and a projective fibration.

Proposition A.3.3.2. *Let $\mathbf{S}$ be an excellent model category, let $\mathbf{A}$ be a combinatorial $\mathbf{S}$-enriched model category, and let $\mathcal{C}$ be a small $\mathbf{S}$-enriched category. Then there exist two combinatorial model structures on $\mathbf{A}^{\mathcal{C}}$:*

- *The projective model structure determined by the projective cofibrations, weak equivalences, and projective fibrations.*
- *The injective model structure determined by the injective cofibrations, weak equivalences, and injective fibrations.*

The proof of Proposition A.3.3.2 is identical to that of Proposition A.2.8.2, except that it requires the following more general form of Lemma A.2.8.3:

Lemma A.3.3.3. *Let $\mathbf{A}$ be a presentable category which is enriched, tensored, and cotensored over a presentable category $\mathbf{S}$. Let S_0 be a (small) set of morphisms of $\mathbf{A}$ and let $\overline{S}_0$ be the weakly saturated class of morphisms generated by S_0. Let $\mathcal{C}$ be a small $\mathbf{S}$-enriched category. Let $\widetilde{S}$ be the collection of all morphisms $F \to G$ in $\mathbf{A}^{\mathcal{C}}$ with the following property: for every $C \in \mathcal{C}$, the map $F(C) \to G(C)$ belongs to $\overline{S}_0$. Then there exists a (small) set of morphisms S of $\mathbf{A}^{\mathcal{C}}$ which generates $\widetilde{S}$ (as a weakly saturated class of morphisms).*

We will defer the proof until the end of this section.

Remark A.3.3.4. In the situation of Proposition A.3.3.2, the category $\mathbf{A}^{\mathcal{C}}$ is again enriched, tensored, and cotensored over $\mathbf{S}$. The tensor product with an object $K \in \mathbf{S}$ is computed pointwise; in other words, if $\mathcal{F} \in \mathbf{A}^{\mathcal{C}}$, then we have the formula

$$(K \otimes \mathcal{F})(A) = K \otimes \mathcal{F}(A).$$

Using criterion (2′) of Remark A.3.1.6, we deduce that $\mathbf{A}^{\mathcal{C}}$ is an **S**-enriched model category with respect to the injective model structure. A dual argument (using criterion (2″) of Remark A.3.1.6) shows that $\mathbf{A}^{\mathcal{C}}$ is also an **S**-enriched model category with respect to the projective model structure.

Remark A.3.3.5. For each object $C \in \mathcal{C}$ and each $A \in \mathbf{A}$, let $\mathcal{F}^C_A \in \mathbf{A}^{\mathcal{C}}$ be the functor given by

$$D \mapsto A \otimes \mathrm{Map}_{\mathcal{C}}(C, D).$$

As in the proof of Proposition A.2.8.2, we learn that the class of projective cofibrations in $\mathbf{A}^{\mathcal{C}}$ is generated by cofibrations of the form $j : \mathcal{F}^C_A \to \mathcal{F}^C_{A'}$, where $A \to A'$ is a cofibration in $\mathbf{A}$. It follows that every projective cofibration is an injective cofibration; dually, every injective fibration is a projective fibration.

As in §A.2.8, the construction $(\mathcal{C}, \mathbf{A}) \mapsto \mathbf{A}^{\mathcal{C}}$ is functorial in both $\mathcal{C}$ and $\mathbf{A}$. We summarize the situation in the following propositions, whose proofs are left to the reader:

Proposition A.3.3.6. *Let* **S** *be an excellent model category,* $\mathcal{C}$ *a small* **S***-enriched category, and* $\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B}$ *an* **S***-enriched Quillen adjunction between combinatorial* **S***-enriched model categories. The composition with* F *and* G *determines another* **S***-enriched Quillen adjunction*

$$\mathbf{A}^{\mathcal{C}} \underset{G^{\mathcal{C}}}{\overset{F^{\mathcal{C}}}{\rightleftarrows}} \mathbf{B}^{\mathcal{C}}$$

with respect to either the projective or the injective model structure. Moreover, if (F, G) *is a Quillen equivalence, then* $(F^{\mathcal{C}}, G^{\mathcal{C}})$ *is also a Quillen equivalence.*

Because the projective and injective model structures on $\mathbf{A}^{\mathcal{C}}$ have the same weak equivalences, the identity functor $\mathrm{id}_{\mathbf{A}^{\mathcal{C}}}$ is a Quillen equivalence between them. However, it is important to distinguish between these two model structures because they have different variance properties as we now explain.

Let $f : \mathcal{C} \to \mathcal{C}'$ be an **S**-enriched functor. Then composition with f yields a pullback functor $f^* : \mathbf{A}^{\mathcal{C}'} \to \mathbf{A}^{\mathcal{C}}$. Since $\mathbf{A}$ has all **S**-enriched limits and colimits, f^* has a right adjoint, which we will denote by f_*, and a left adjoint, which we will denote by $f_!$.

Proposition A.3.3.7. *Let* **S** *be an excellent model category,* **A** *a combinatorial* **S***-enriched model category, and* $f : \mathcal{C} \to \mathcal{C}'$ *an* **S***-enriched functor between small* **S***-enriched categories. Let* $f^* : \mathbf{A}^{\mathcal{C}'} \to \mathbf{A}^{\mathcal{C}}$ *be given by composition with* f*. Then* f^* *admits a right adjoint* f_* *and a left adjoint* $f_!$*. Moreover:*

(1) *The pair* $(f_!, f^*)$ *determines a Quillen adjunction between the* projective *model structures on* $\mathbf{A}^{\mathcal{C}}$ *and* $\mathbf{A}^{\mathcal{C}'}$*.*

(2) *The pair* (f^*, f_*) *determines a Quillen adjunction between the* injective *model structures on* $\mathbf{A}^{\mathcal{C}}$ *and* $\mathbf{A}^{\mathcal{C}'}$.

We now study some aspects of the theory which are unique to the enriched context.

Proposition A.3.3.8. *Let* $\mathbf{S}$ *be an excellent model category,* $\mathbf{A}$ *a combinatorial* $\mathbf{S}$*-enriched model category, and* $f : \mathcal{C} \to \mathcal{C}'$ *an equivalence of small* $\mathbf{S}$*-enriched categories. Then*

(1) *The Quillen adjunction* $(f_!, f^*)$ *determines a Quillen equivalence between the projective model structures on* $\mathbf{A}^{\mathcal{C}}$ *and* $\mathbf{A}^{\mathcal{C}'}$.

(2) *The Quillen adjunction* (f^*, f_*) *determines a Quillen equivalence between the injective model structures on* $\mathbf{A}^{\mathcal{C}}$ *and* $\mathbf{A}^{\mathcal{C}'}$.

Proof. We first note that (1) and (2) are equivalent: they are both equivalent to the assertion that f^* induces an equivalence on homotopy categories. It therefore suffices to prove (1). We first prove this under the following additional assumption:

($*$) For every pair of objects $C, D \in \mathcal{C}'$, the map
$$\operatorname{Map}_{\mathcal{C}'}(C, D) \to \operatorname{Map}_{\mathcal{C}}(f(C), f(D))$$
is a cofibration in $\mathbf{S}$.

Let $Lf_! : \mathbf{A}^{\mathcal{C}} \to \mathbf{A}^{\mathcal{C}'}$ denote the left derived functor of $f_!$. We must show that the unit and counit maps
$$h_F : F \mapsto f^* Lf_! F$$
$$k_G : Lf_! f^* G \to G$$
are isomorphisms for all $F \in \mathrm{h}\mathbf{A}^{\mathcal{C}}$, $G \in \mathrm{h}\mathbf{A}^{\mathcal{C}}$. Since f is essentially surjective on homotopy categories, a natural transformation $K \to K'$ of $\mathbf{S}$-enriched functors $K, K' : \mathcal{C}' \to \mathbf{A}$ is a weak equivalence if and only if $f^*K \to f^*K'$ is a weak equivalence. Consequently, to prove k_G is an isomorphism, it suffices to show that h_{f^*G} is an isomorphism.

Let us say that a map $F \to F'$ in $\mathbf{A}^{\mathcal{C}}$ is *good* if the induced map
$$f^* f_! F \coprod_F F' \to f^* f_! F'$$
is an injective trivial cofibration. To complete the proof, it will suffice to show that every projective cofibration is good. Since the collection of good transformations is weakly saturated, it will suffice to show that each of the generating cofibrations $\mathcal{F}^C_A \to \mathcal{F}^C_{A'}$ is good, where $C \in \mathcal{C}'$ and $j : A \to A'$ is a cofibration in $\mathbf{A}$. Unwinding the definitions, we must show that for each $D \in \mathcal{C}'$ the induced map
$$\begin{array}{c} A' \otimes \operatorname{Map}_{\mathcal{C}'}(C, D)) \coprod_{A \otimes \operatorname{Map}_{\mathcal{C}'}(C,D)} (A \otimes \operatorname{Map}_{\mathcal{C}}(f(C), f(D))) \\ \downarrow{\scriptstyle \theta} \\ A' \otimes \operatorname{Map}_{\mathcal{C}}(f(C), f(D)) \end{array}$$

is a trivial cofibration. This follows from the fact that j is a cofibration and our assumption $(*)$.

We now treat the general case. First, choose a trivial cofibration $g : \mathcal{C} \to \mathcal{C}''$, where $\mathcal{C}''$ is fibrant. Then g satisfies $(*)$, so $g_!$ is a Quillen equivalence. By a two-out-of-three argument, we see that $f_!$ is a Quillen equivalence if and only if $(g \circ f)_!$ is a Quillen equivalence. Replacing $\mathcal{C}$ by $\mathcal{C}''$, we may reduce to the case where $\mathcal{C}$ is itself fibrant.

Choose a cofibration $j : \mathcal{C} \coprod \mathcal{C}' \to \mathcal{D}$, where $\mathcal{D}$ is fibrant and equivalent to the final object of $\mathcal{C}\mathrm{at}_{\mathbf{S}}$. Then f factors as a composition

$$\mathcal{C}' \xrightarrow{f'} \mathcal{C} \times \mathcal{D} \xrightarrow{f''} \mathcal{C}.$$

Since $\mathcal{C}$ and $\mathcal{D}$ are fibrant, the product $\mathcal{C} \times \mathcal{D}$ is equivalent to $\mathcal{C}$. Moreover, the map f'' admits a section $s : \mathcal{C} \to \mathcal{C} \times \mathcal{D}$. Using another two-out-of-three argument, it will suffice to show that $f'_!$ and $s_!$ are Quillen equivalences. For this, it will suffice to show that f' and s satisfy $(*)$.

We first show that f' satisfies $(*)$. Fix a pair of objects $X, Y \in \mathcal{C}'$. Then f' induces the composite map

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}'}(X,Y) &\xrightarrow{u} \mathrm{Map}_{\mathcal{C}}(fX, fY) \times \mathrm{Map}_{\mathcal{C}'}(X,Y) \\ &\xrightarrow{u'} \mathrm{Map}_{\mathcal{C}}(fX, fY) \times \mathrm{Map}_{\mathcal{D}}(jX, jY) \\ &\simeq \mathrm{Map}_{\mathcal{C} \times \mathcal{D}}(f'X, f'Y). \end{aligned}$$

The map u is a monomorphism (since it admits a left inverse) and therefore a cofibration in view of axiom $(A2)$ of Definition A.3.2.16. The map u' is a product of cofibrations and therefore also a cofibration (again by axiom $(A2)$).

The proof that s satisfies $(*)$ is similar: for every pair of objects $U, V \in \mathcal{C}$, the map

$$\mathrm{Map}_{\mathcal{C}}(U,V) \to \mathrm{Map}_{\mathcal{C} \times \mathcal{D}}(sU, sV) \simeq \mathrm{Map}_{\mathcal{C}}(U,V) \times \mathrm{Map}_{\mathcal{D}}(jU, jV)$$

is a monomorphism since it admits a left inverse and is therefore a cofibration. $\square$

In the special case where $f : \mathcal{C} \to \mathcal{C}'$ is a *cofibration* between $\mathbf{S}$-enriched categories, we have some additional functoriality:

Proposition A.3.3.9. *Let $\mathbf{S}$ be an excellent model category and let $f : \mathcal{C} \to \mathcal{C}'$ be a cofibration of small $\mathbf{S}$-enriched categories. Then*

(1) *For every combinatorial $\mathbf{S}$-enriched model category $\mathbf{A}$, the pullback map $f^* : \mathbf{A}^{\mathcal{C}'} \to \mathbf{A}^{\mathcal{C}}$ preserves projective cofibrations.*

(2) *For every projectively cofibrant object $F \in \mathbf{S}^{\mathcal{C}}$, the unit map $F \to f^* f_! F$ is a projective cofibration.*

Lemma A.3.3.10. *Let* $\mathbf{S}$ *be an excellent model category and suppose we are given a pushout diagram*

$$\begin{tikzcd} {[1]_S} \arrow[r] \arrow[d, "i"] & {[1]_{S'}} \arrow[d] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{C}' \end{tikzcd}$$

of $\mathbf{S}$*-enriched categories, where* $j : S \to S'$ *is a cofibration in* $\mathbf{S}$*. Let* C *be an object of* $\mathcal{C}$ *and let* $F \in \mathbf{S}^{\mathcal{C}}$ *be the functor given by the formula* $D \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$*. Then the unit map* $F \to f^* f_! F$ *is a projective cofibration in* $\mathbf{S}^{\mathcal{C}}$*.*

Proof. The map i determines a pair of objects $X, Y \in \mathcal{C}$ and a map $S \to \mathrm{Map}_{\mathcal{C}}(X, Y)$. The proof of Proposition A.3.2.4 shows that the functor $f^* f_! F$ is the colimit of a sequence

$$F = F(0) \stackrel{h_1}{\to} F(1) \stackrel{h_2}{\to} F(2) \to \cdots,$$

where each h_k is a pushout of a map $\mathcal{F}^Y_A \to \mathcal{F}^Y_{A'}$ induced by a map $t : A \to A'$ in $\mathbf{S}$. Moreover, the map t can be identified with the tensor product

$$\mathrm{id}_{\mathrm{Map}_{\mathcal{C}}(C,X)} \otimes \mathrm{id}^{\otimes k-1}_{\mathrm{Map}_{\mathcal{C}}(Y,X)} \otimes \wedge^k (j),$$

where $\wedge^k(j)$ denotes the kth pushout power of j. It follows that t is a cofibration in $\mathbf{S}$, so that each h_k is a projective cofibration in $\mathbf{S}^{\mathcal{C}}$. $\square$

Proof of Proposition A.3.3.9. The collection of $\mathbf{S}$-enriched functors f which satisfy (1) and (2) is clearly closed under the formation of retracts. We may therefore assume without loss of generality that f is a transfinite composition of pushouts of generating cofibrations (see the discussion preceding Proposition A.3.2.4). Reordering these pushouts if necessary, we can factor f as a composition

$$\mathcal{C} \stackrel{f'}{\to} \overline{\mathcal{C}} \stackrel{f''}{\to} \mathcal{C}',$$

where $\overline{\mathcal{C}}$ is obtained from $\mathcal{C}$ by freely adjoining a collection of new objects and f'' is bijective on objects. Since f' clearly satisfies (1) and (2), it will suffice to prove that f'' satisfies (1) and (2). Replacing f by f'', we may assume that f is bijective on objects.

We now show that (2) $\Rightarrow$ (1). Since the functor f^* preserves colimits, the collection of morphisms α in $\mathbf{A}^{\mathcal{C}'}$ such that f^* is a projective cofibration in $\mathbf{A}^{\mathcal{C}}$ is weakly saturated. It will therefore suffice to show that for every object $X \in \mathcal{C}'$ and every cofibration $A \to A'$ in $\mathbf{A}$, if $\alpha : \mathcal{F}^X_A \to \mathcal{F}^X_{A'}$ denotes the corresponding generating projective cofibration, then $f^*(\alpha)$ is a projective cofibration in $\mathbf{S}$.

There is a canonical left Quillen bifunctor

$$\boxtimes : \mathbf{S}^{\mathcal{C}} \times \mathbf{A} \to \mathbf{A}^{\mathcal{C}}$$

described by the formula $(F \boxtimes A)(C) = F(C) \otimes A$. (Here we regard $\mathbf{S}^{\mathcal{C}}$ as endowed with the projective model structure.) We observe that $f^*(\alpha)$

is the induced map $(f^*F) \boxtimes A \to (f^*F) \boxtimes A'$, where $F \in \mathbf{S}^{\mathcal{C}'}$ is given by $F(C') = \mathrm{Map}_{\mathcal{C}'}(X, C')$. To prove (1), it will suffice to show that f^*F is projectively cofibrant.

Since F is bijective on objects, we can choose an object $X_0 \in \mathcal{C}$ such that $fX_0 = X$. We now observe that $F \simeq f_! F_0$, where $F_0 \in \mathbf{S}^{\mathcal{C}}$ is defined by the formula $F_0(C) = \mathrm{Map}_{\mathcal{C}}(X_0, C)$. If (2) is satisfied, then the unit map $F_0 \to f^*F$ is a projective cofibration in $\mathbf{S}^{\mathcal{C}}$. Since F_0 is projectively cofibrant, we conclude that f^*F is projectively cofibrant as well. This completes the proof that (2) $\Rightarrow$ (1).

To prove (2), let us write f as a transfinite composition of $\mathbf{S}$-enriched functors

$$\mathcal{C} = \mathcal{C}_0 \to \mathcal{C}_1 \to \cdots,$$

each of which is a pushout of a generating cofibration of the form $[1]_S \to [1]_{S'}$, where $S \to S'$ is a cofibration in $\mathbf{S}$. For each $\alpha \leq \beta$, let $f_\alpha^\beta : \mathcal{C}_\alpha \to \mathcal{C}_\beta$ be the corresponding cofibration. We will prove that the following statement holds for every pair of ordinals $\alpha \leq \beta$:

$(2_{\alpha,\beta})$ For every projectively cofibrant object $F \in \mathbf{S}^{\mathcal{C}_\alpha}$, the unit map $u : F \to (f_\alpha^\beta)^*(f_\alpha^\beta)_! F$ is a projective cofibration.

The proof proceeds by induction on β. We observe that u is a transfinite composition of maps of the form

$$u_\gamma : (f_\alpha^\gamma)^*(f_\alpha^\gamma)_! F \to (f_\alpha^\gamma)^*(f_\gamma^{\gamma+1})^*(f_\gamma^{\gamma+1})_!(f_\alpha^\gamma)_! F,$$

where $\gamma < \beta$. It will therefore suffice to show that each u_γ is a projective cofibration. Our inductive hypothesis therefore guarantees that $(2_{\alpha,\gamma})$ holds, so the first part of the proof shows that $(f_\alpha^\gamma)^*$ preserves trivial cofibrations. We are therefore reduced to proving assertion $(2_{\gamma,\gamma+1})$. In other words, to prove (2) in general, it will suffice to treat the case in which f is a pushout of a generating cofibration of the form $[1]_S \to [1]_{S'}$.

We will in fact prove the following stronger version of (2):

(3) For every projective cofibration $\phi : F' \to F$ in $\mathbf{S}^{\mathcal{C}}$, the induced map $\phi' : F \coprod_{F'} f^* f_! F' \to f^* f_! F$ is again a projective cofibration in $\mathbf{S}^{\mathcal{C}}$.

Consider the collection of *all* morphisms $\phi : F' \to F$ in $\mathbf{S}^{\mathcal{C}}$ such that the induced map $\phi' : F \coprod_{F'} f^* f_! F' \to f^* f_! F$ is a projective cofibration. It is easy to see that this collection is weakly saturated. Consequently, to prove (3) it suffices to treat the case where ϕ is a generating projective cofibration of the form $\mathcal{F}_A^C \to \mathcal{F}_{A'}^C$, where $A \to A'$ is a cofibration in $\mathbf{S}$. In this case, we can identify ϕ' with the map

$$(F_C \boxtimes A') \coprod_{F_C \boxtimes A} (f^* f_! F_C \boxtimes A) \to f^* f_! F_C \boxtimes A',$$

where $F_C \in \mathbf{S}^{\mathcal{C}}$ is the functor $D \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$. Since $\boxtimes$ is a left Quillen bifunctor, it will suffice to show that the unit map $f_C \to f^* f_! F_C$ is a projective cofibration in $\mathbf{S}^{\mathcal{C}}$. This is precisely the content of Lemma A.3.3.10. $\square$

In §A.2.8, we introduced the definitions of homotopy limits and colimits in an arbitrary combinatorial model category $\mathbf{A}$. We now discuss an analogous construction in the case where $\mathbf{A}$ is enriched over an excellent model category $\mathbf{S}$. To simplify the exposition, we will discuss only the case of homotopy limits; the case of homotopy colimits is entirely dual and is left to the reader.

Fix an excellent model category $\mathbf{S}$ and a combinatorial $\mathbf{S}$-enriched model category $\mathbf{A}$. Let $f : \mathcal{C} \to \mathcal{C}'$ be a functor between small $\mathbf{S}$-enriched categories, so that we have an induced Quillen adjunction

$$\mathbf{A}^{\mathcal{C}'} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{A}^{\mathcal{C}}.$$

We will refer to the right derived functor Rf_* as the *homotopy right Kan extension* functor. Suppose we are given a pair of functors $F \in \mathbf{A}^{\mathcal{C}}$, $G \in \mathbf{A}^{\mathcal{C}'}$ and a morphism $\eta : G \to f_* F$ in $\mathbf{A}^{\mathcal{C}'}$. We will say that η *exhibits G as the homotopy right Kan extension of F* if, for some weak equivalence $F \to F'$ where F' is injectively fibrant in $\mathbf{A}^{\mathcal{C}}$, the composite map $G \to f_* F \to f_* F'$ is a weak equivalence in $\mathbf{A}^{\mathcal{C}'}$. Since f_* preserves weak equivalences between injectively fibrant objects, this condition is independent of the choice of F'.

Remark A.3.3.11. In §A.2.8, we defined homotopy right Kan extensions in the setting of the diagram categories $\mathrm{Fun}(\mathcal{C}, \mathbf{A})$, where $\mathcal{C}$ is an ordinary category. In fact, this is a special case of the above construction. Namely, there is a unique colimit-preserving monoidal functor $F : \mathcal{S}\mathrm{et} \to \mathbf{S}$ given by $F(S) = \coprod_{s \in S} \mathbf{1_S}$. We can therefore define an $\mathbf{S}$-enriched category $\overline{\mathcal{C}}$ whose objects are the objects of $\mathcal{C}$, with $\mathrm{Map}_{\overline{\mathcal{C}}}(X, Y) = F \,\mathrm{Map}_{\mathcal{C}}(X, Y)$. We now observe that we have an identification $\mathrm{Fun}(\mathcal{C}, \mathbf{A}) \simeq \mathbf{A}^{\overline{\mathcal{C}}}$ which is functorial in both $\mathcal{C}$ and $\mathbf{A}$. This identification is compatible with the definition of the injective model structures on both sides, so that either point of view gives rise to the same theory of homotopy right Kan extensions.

We now discuss a special feature of the enriched theory of homotopy Kan extensions: they can be reduced to the theory of homotopy Kan extensions in the model category $\mathbf{S}$:

Proposition A.3.3.12. *Let $\mathbf{S}$ be an excellent model category, let $\mathbf{A}$ be a combinatorial model category enriched over $\mathbf{S}$, and let $f : \mathcal{C} \to \mathcal{C}'$ be a functor between small $\mathbf{S}$-enriched categories. Suppose given objects $F \in \mathbf{A}^{\mathcal{C}}$, $G \in \mathbf{A}^{\mathcal{C}'}$ and a map $\eta : G \to f_* F$. Assume that F and G are projectively fibrant. The following conditions are equivalent:*

(1) *The map η exhibits G as a homotopy right Kan extension of F.*

(2) *For each cofibrant object $A \in \mathbf{A}$, the induced map*

$$\eta_A : G_A \to f_* F_A$$

exhibits G_A as a homotopy right Kan extension of F_A. Here $F_A \in \mathbf{S}^{\mathcal{C}}$ and $G_A \in \mathbf{S}^{\mathcal{C}'}$ are defined by $F_A(C) = \mathrm{Map}_{\mathbf{A}}(A, F(C)), G_A(C) = \mathrm{Map}_{\mathbf{A}}(A, G(C))$.

(3) *For every fibrant-cofibrant object $A \in \mathbf{A}$, the induced map*

$$\eta_A : G_A \to f_* F_A$$

exhibits G_A as a homotopy right Kan extension of F_A.

Proof. Choose an equivalence $F \to F'$, where F' is injectively fibrant. We note that the induced maps $F_A \to F'_A$ are weak equivalences for any cofibrant $A \in \mathbf{A}$ since $\mathrm{Map}_{\mathbf{A}}(A, \bullet)$ preserves weak equivalences between fibrant objects. Consequently, we may without loss of generality replace F by F' and thereby assume that F is injectively fibrant.

Now suppose that A is any cofibrant object of $\mathbf{A}$; we claim that F_A is injectively fibrant. To show that F_A has the right lifting property with respect to a trivial weak cofibration $H \to H'$ of functors $\mathcal{C} \to \mathbf{S}$, one need only observe that F has the right lifting property with respect to trivial injective cofibration $A \otimes H \to A \otimes H'$ in $\mathbf{A}^{\mathcal{C}}$.

Now we note that (1) is equivalent to the assertion that η is a weak equivalence, (2) is equivalent to the assertion that η_A is a weak equivalence for any cofibrant object A, and (3) is equivalent to the assertion that η_A is a weak equivalence whenever A is fibrant-cofibrant. Because $\mathrm{Map}_{\mathbf{A}}(A, \bullet)$ preserves weak equivalences between fibrant objects, we deduce that (1) $\Rightarrow$ (2). It is clear that (2) $\Rightarrow$ (3). We will complete the proof by showing that (3) $\Rightarrow$ (1). Assume that (3) holds; we must show that $\eta(C') : G(C') \to f_* F(C')$ is an isomorphism in the homotopy category $\mathrm{h}\mathbf{A}$ for each $C' \in \mathcal{C}'$. For this, it suffices to show that $G(C')$ and $f_* F(C')$ represent the same $\mathcal{H}$-valued functors on the homotopy category $\mathrm{h}\mathbf{A}$, which is precisely the content of (3). $\square$

Remark A.3.3.13. It follows from Proposition A.3.3.12 that we can make sense of homotopy right Kan extensions for diagrams which do not take values in a model category. Let $f : \mathcal{C} \to \mathcal{C}'$ be an $\mathbf{S}$-enriched functor as in the discussion above and let $\mathcal{A}$ be an *arbitrary* locally fibrant $\mathbf{S}$-enriched category. Suppose we are given objects $F \in \mathcal{A}^{\mathcal{C}}$, $G \in \mathcal{A}^{\mathcal{C}'}$ and $\eta : f^* G \to F$; we say that η *exhibits G as a homotopy right Kan extension of F* if, for each object $A \in \mathcal{A}$, the induced map

$$\eta_A : G_A \to f_* F_A$$

exhibits $G_A \in \mathbf{S}^{\mathcal{C}'}$ as a homotopy right Kan extension of $F_A \in \mathbf{S}^{\mathcal{C}}$.

Suppose that the monoidal structure on $\mathbf{S}$ is given by the Cartesian product and take $\mathcal{C}'$ to be the final object of $\mathcal{C}\mathrm{at}_{\mathbf{S}}$, so that we can identify $\mathcal{A}^{\mathcal{C}'}$ with $\mathcal{A}$. In this case, we can identify G with a single object $B \in \mathcal{A}$ and the map η with a collection of maps $\{B \to F(C)\}_{C \in \mathcal{C}}$. We will say that η *exhibits B as a homotopy limit of F* if it identifies G with a homotopy right Kan extension of F. In other words, η exhibits B as a homotopy limit of F if, for every object $A \in \mathcal{A}$, the induced map

$$\mathrm{Map}_{\mathcal{A}}(A, B) \to \lim \{\mathrm{Map}_{\mathcal{A}}(A, F(C))\}_{C \in \mathcal{C}}$$

exhibits $\mathrm{Map}_{\mathcal{A}}(A, B)$ as a homotopy limit of the diagram

$$\{\mathrm{Map}_{\mathcal{A}}(A, F(C))\}_{C \in \mathcal{C}}$$

in the model category $\mathbf{S}$.

We also have a dual notion of *homotopy colimit* in an arbitrary fibrant $\mathbf{S}$-enriched category $\mathcal{A}$: a compatible family of maps $\{F(C) \to B\}_{C \in \mathcal{C}}$ *exhibits B as a homotopy colimit of F* if, for every object $A \in \mathcal{A}$, the induced maps $\{\mathrm{Map}_{\mathcal{A}}(B, A) \to \mathrm{Map}_{\mathcal{A}}(F(C), A)\}_{C \in \mathcal{C}}$ exhibit $\mathrm{Map}_{\mathcal{A}}(B, A)$ as a homotopy limit of the diagram $\{\mathrm{Map}_{\mathcal{A}}(F(C), A)\}_{C \in \mathcal{C}}$ in $\mathbf{S}$.

Remark A.3.3.14. In view of Proposition A.3.3.12, the terminology introduced in Remark A.3.3.13 for general $\mathcal{A}$ agrees with the terminology introduced for a combinatorial $\mathbf{S}$-enriched model category $\mathbf{A}$ if we set $\mathcal{A} = \mathbf{A}^{\circ}$. We remark that, in general, the two notions do *not* agree if we take $\mathcal{A} = \mathbf{A}$, so that our terminology is potentially ambiguous; however, we feel that there is little danger of confusion.

We conclude this section by giving the proof of Lemma A.3.3.3. Let $\mathbf{A}$ be a presentable category which is enriched, tensored, and cotensored over a presentable category $\mathbf{S}$. Let $\mathcal{C}$ be a small $\mathbf{S}$-enriched category and let $\overline{S}_0$ be a weakly saturated class of morphisms of $\mathbf{A}$ generated by a (small) set S_0. We regard this data as *fixed* for the remainder of this section.

Choose a regular cardinal κ satisfying the following conditions:

(i) The cardinal κ is uncountable.

(ii) The category $\mathcal{C}$ has fewer than κ-objects.

(iii) Let $X, Y \in \mathcal{C}$ and let $K = \mathrm{Map}_{\mathcal{C}}(X, Y)$. Then the functor from $\mathbf{A}$ to itself given by the formula $A \mapsto A^K$ preserves κ-filtered colimits. This implies, in particular, that the collection of κ-compact objects of $\mathbf{A}$ is stable with respect to the functors $\bullet \otimes K$.

(iv) The category $\mathbf{A}$ is κ-accessible. It follows also that $\mathbf{A}^{\mathcal{C}}$ is κ-accessible, and that an object $F \in \mathbf{A}^{\mathcal{C}}$ is κ-compact if and only if each $F(C) \in \mathbf{A}$ is κ-compact. We prove an ∞-category generalization of this statement as Proposition 5.4.4.3. The same proof also works in the setting of ordinary categories.

(v) The source and target of every morphism in S_0 is a κ-compact object of $\mathbf{A}$.

Enlarging S_0 if necessary, we may assume that S_0 consists of *all* morphisms in $f \in \overline{S}_0$ such that the source and target of f are κ-compact. Let S be the collection of all injective cofibrations between κ-compact objects of $\mathbf{A}$ (in view of (iv), we can equally well define S to be the set of morphisms $F \to G$ in $\mathbf{A}^{\mathcal{C}}$ such that each of the induced morphisms $F(C) \to G(C)$ belongs to S_0). Let $\overline{S}$ be the weakly saturated class of morphisms in $\mathbf{A}^{\mathcal{C}}$ generated by S and choose a map $f : F \to G$ in $\mathbf{A}^{\mathcal{C}}$ such that $f(C) \in \overline{S}_0$ for each $C \in \mathcal{C}$. We wish to show that $f \in \overline{S}$. Corollary A.1.5.13 implies that, for each $C \in \mathcal{C}$, there exists a κ-good S_0-tree $\{Y(C)_\alpha\}_{\alpha \in A(C)}$ with root $F(C)$ and colimit $G(C)$.

Let us define a *slice* to be the following data:

(a) For each object $C \in \mathcal{C}$, a downward-closed subset $B(C) \subseteq A(C)$.

(b) For every object $C \in \mathcal{C}$, a morphism

$$\eta_C : \coprod_{C' \in \mathcal{C}} Y(C')_{B(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \to Y(C)_{B(C)},$$

rendering the following diagrams commutative:

$$\begin{tikzcd}[column sep=tiny] & Y(C'')_{B(C'')} \otimes \mathrm{Map}_{\mathbf{A}}(C'', C') \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[dl, "\eta_{C'}"] \arrow[dr] & \\ Y(C')_{B(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[dr, "\eta_C"] & & Y(C'')_{B(C'')} \otimes \mathrm{Map}_{\mathbf{A}}(C'', C) \arrow[dl, "\eta_{C''}"] \\ & Y(C)_{B(C)} & \end{tikzcd}$$

$$\begin{tikzcd} F(C') \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[r] \arrow[d] & F(C) \arrow[d] \\ Y(C')_{B(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[r, "\eta_C"] \arrow[d] & Y(C)_{B(C)} \arrow[d] \\ G(C') \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[r] & G(C). \end{tikzcd}$$

We remark that (b) is precisely the data needed to endow $C \mapsto Y(C)_{B(C)}$ with the structure of an $\mathbf{S}$-enriched functor $\mathcal{C} \to \mathbf{A}$ lying between F and G in $\mathbf{A}^{\mathcal{C}}$.

Lemma A.3.3.15. *Suppose we are given a collection of κ-small subsets $\{B_0(C) \subseteq A(C)\}_{C \in \mathcal{C}}$. Then there exists a slice $\{(B(C), \eta_C\}_{C \in \mathcal{C}}$ such that each $B(C)$ is a κ-small subset of $A(C)$ containing $B_0(C)$.*

Proof. Enlarging each $B_0(C)$ if necessary, we may assume that each $B_0(C)$ is downward-closed. Note that because each $\{Y(C)_\alpha\}_{\alpha \in A(C)}$ is a κ-good S_0-tree, if $A' \subseteq A(C)$ is downward-closed and κ-small, $Y(C)_{A'}$ is κ-compact when viewed as an object of $\mathbf{A}_{F(C)/}$. It follows from (iii) that each tensor product $Y(C)_{B_0(C)} \otimes \mathrm{Map}_{\mathbf{A}}(C, C')$ is a κ-compact object of the category $\mathbf{A}_{(F(C) \otimes \mathrm{Map}_{\mathbf{A}}(C, C'))/}$. Consequently, each composition

$$\coprod_{C' \in \mathcal{C}} Y(C')_{B_0(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \to \coprod_{C' \in \mathcal{C}} G(C') \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \to G(C)$$

admits another factorization

$$\coprod_{C' \in \mathcal{C}} Y(C')_{B_0(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \xrightarrow{\eta_C^1} Y(C)_{B_1(C)} \to G(C),$$

where $B_1(C)$ is downward-closed and κ-small, and the diagram

$$\begin{tikzcd}
\coprod_{C' \in \mathcal{C}} F(C') \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[r] \arrow[d] & \coprod_{C' \in \mathcal{C}} Y(C')_{B_0(C')} \arrow[d, "\eta_C^1"] \\
F(C) \arrow[r] & Y(C)_{B_1(C)}
\end{tikzcd}$$

commutes. Enlarging $B_1(C)$ if necessary, we may suppose that each $B_1(C)$ contains $B_0(C)$.

We now continue the preceding construction by defining, for each $C \in \mathcal{C}$, a sequence of κ-small downward-closed subsets

$$B_0(C) \subseteq B_1(C) \subseteq B_2(C) \subseteq \cdots$$

of $A(C)$ and maps $\eta_C^i : \coprod_{C' \in \mathcal{C}} Y(C')_{B_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \to Y(C)_{B_i(C)}$. Suppose that $i > 1$ and that the sets $B_j(C)$ and maps η_C^j have been constructed for $j < i$. Using a compactness argument, we conclude that the composition

$$\coprod_{C' \in \mathcal{C}} Y(C')_{B_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \to \coprod_{C' \in \mathcal{C}} G(C') \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \to G(C)$$

coincides with

$$\coprod_{C' \in \mathcal{C}} Y(C')_{B_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \xrightarrow{\eta_C^i} Y(C)_{B_i(C)} \to G(C),$$

where $B_i(C)$ is κ-small and the diagram

$$\begin{tikzcd}
\coprod_{C' \in \mathcal{C}} F(C') \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[r] \arrow[d] & \coprod_{C' \in \mathcal{C}} Y(C')_{B_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[d, "\eta_C^i"] \\
F(C) \arrow[r] & Y(C)_{B_i(C)}
\end{tikzcd}$$

commutes. Enlarging $B_i(C)$ if necessary, we may suppose that $B_i(C)$ contains $B_{i-1}(C)$ and that the following diagrams commute as well (for all $C', C'' \in \mathcal{C}$):

$$\begin{tikzcd}
& Y(C'')_{B_{i-2}(C'')} \otimes \mathrm{Map}_{\mathbf{A}}(C'', C') \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[dl] \arrow[dr] & \\
Y(C')_{B_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[dr, "\eta_C^i"] & & Y(C'')_{B_{i-1}(C'')} \otimes \mathrm{Map}_{\mathbf{A}}(C'', C) \arrow[dl, "\eta_C^i"] \\
& Y(C)_{B_i(C)} &
\end{tikzcd}$$

$$\begin{tikzcd}
Y(C')_{B_{i-2}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[r] \arrow[d, "\eta_C^{i-1}"] & Y(C')_{B_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[d, "\eta_C^i"] \\
Y(C)_{B_{i-1}(C)} \arrow[r] & Y(C)_{B_i(C)}.
\end{tikzcd}$$

We now define $B(C) = \bigcup B_i(C)$, and η_C to be the amalgam of the compositions

$$\coprod_{C' \in \mathcal{C}} Y(C')_{B_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \stackrel{\eta_C^i}{\to} Y(C)_{B_i}(C) \to Y(C)_{B(C)}.$$

□

We now introduce a bit more terminology. Suppose we are given a pair of slices $M = \{(B(C), \eta_C)\}_{C \in \mathcal{C}}$, $M' = \{(B'(C), \eta'_C\}_{C \in \mathcal{C}}\}$. We will say that M is *κ-small* if each $B(C)$ is κ-small. We will say that M' *extends* M if $B(C) \subseteq B'(C)$ for each $C \in \mathcal{C}$ and each diagram

$$\begin{tikzcd}
Y(C')_{B(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[r] \arrow[d, "\eta_C"] & Y(C')_{B'(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[d, "\eta'_C"] \\
Y(C)_{B(C)} \arrow[r] & Y(C)_{B'(C)}
\end{tikzcd}$$

is commutative. Equivalently, M' extends M if $B(C) \subseteq B'(C)$ for each $C \in \mathcal{C}$, and the induced maps $Y(C)_{B(C)} \to Y(C)_{B(C')}$ constitute a natural transformation of simplicial functors from $\mathcal{C}$ to $\mathbf{A}$.

Lemma A.3.3.16. *Let $M' = \{(A'(C), \theta_C)\}_{C \in \mathcal{C}}$ be a slice and let $\{B_0(C) \subseteq A(C)\}_{C \in \mathcal{C}}$ be a collection of κ-small subsets of $A(C)$. Then there exists a pair of slices $N = \{(B(C), \eta_C)\}_{C \in \mathcal{C}}$, $N' = \{(B(C) \cap A'(C), \eta'_C)\}$, where $B(C)$ is κ-small and N' is compatible with both N and M'.*

Proof. Let $B'_0(C) = A'(C) \cap B_0(C)$. For every positive integer i, we will construct a pair of slices $N_i = \{(B_i(C), \eta(i)_C)\}$, $N'_i = \{(B'_i(C), \eta'(i)_C)\}$ so that the following conditions are satisfied:

(a) Each $B_i(C)$ is κ-small and contains $B_{i-1}(C)$.

(b) Each $B'_i(C)$ is κ-small, contains $B'_{i-1}(C)$ and the intersection $B_i(C) \cap A'(C)$, and is contained in $A'(C)$.

(c) Each N'_i is compatible with M'.

(d) If $i > 2$ and $C, C' \in \mathcal{C}$, then the diagram

$$\begin{tikzcd}
Y(C')_{B_{i-2}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[r] \arrow[d, "\eta(i-2)_C"] & Y(C')_{B_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[d, "\eta(i-1)_C"] \\
Y(C)_{B_{i-2}(C)} \arrow[d] & Y(C)_{B_{i-1}}(C) \arrow[d] \\
Y(C)_{B_i(C)} \arrow[r, equal] & Y(C)_{B_i(C)}
\end{tikzcd}$$

commutes.

(e) If $i > 2$ and $C, C' \in \mathcal{C}$, then the diagram

$$\begin{tikzcd}
Y(C')_{B'_{i-2}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[r] \arrow[d, "\eta'(i-2)_C"] & Y(C')_{B'_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[d, "\eta('i-1)_C"] \\
Y(C)_{B'_{i-2}(C)} \arrow[d] & Y(C)_{B'_{i-1}}(C) \arrow[d] \\
Y(C)_{B'_i(C)} \arrow[r, equal] & Y(C)_{B'_i(C)}
\end{tikzcd}$$

commutes.

(f) If $i > 1$ and $C, C' \in \mathcal{C}$, then the diagram

$$\begin{tikzcd}
Y(C')_{B'_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[r] \arrow[d, "\eta'(i-1)_C"] & Y(C')_{B_{i-1}(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \arrow[d, "\eta(i-1)_C"] \\
Y(C)_{B'_{i-1}(C)} \arrow[d] & Y(C)_{B_{i-1}(C)} \arrow[d] \\
Y(C)_{B'_i(C)} \arrow[r] & Y(C)_{B'_i(C)}
\end{tikzcd}$$

commutes.

The construction is by induction on i. The existence of N_i satisfying (a), (d), and (f) follows from Lemma A.3.3.15 (and a compactness argument). Similarly, the existence of N'_i satisfying (b), (c), and (e) follows by applying Lemma A.3.3.15 after replacing $G \in \mathbf{A}^{\mathcal{C}}$ by the functor G' given by $G'(C) = Y(C)_{A'(C)}$, and the S_0-trees $\{Y(C)_\alpha\}_{\alpha \in A(C)}$ by the smaller trees $\{Y(C)_\alpha\}_{\alpha \in A'(C)}$.

We now define $B(C) = \bigcup_i B_i(C)$. It follows from (d) that the $\eta(i)_C$ assemble to a map

$$\eta_C : \coprod_{C' \in \mathcal{C}} Y(C')_{B(C')} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \to Y(C)_{B(C)}.$$

Taken together, these maps determine a slice $N = \{(B(C), \eta_C)\}$. Similarly, (e) implies that the maps $\eta'(i)_C$ assemble to a slice $N' = \{(B(C) \cap A'(C), \eta'_C)\}$. The compatibility of N and N' follows from (f), while the compatibility of M' and N' follows from (c). $\square$

We now construct a transfinite sequence of compatible slices $\{M(\gamma) = \{(B(\gamma)(C), \eta(\gamma)_C)\}_{C \in \mathcal{C}}\}_{\gamma < \beta}$. The construction is by recursion. Assume that $M(\gamma')$ has been defined for $\gamma' < \gamma$ and let $M'(\gamma) = \{(B'(\gamma)(C), \eta'(\gamma)_C)\}_{C \in \mathcal{C}}$ denote the slice obtained by amalgamating the family of slices $\{M(\gamma')\}_{\gamma' < \gamma}$.

If $B'(\gamma)(C) = A(C)$ for all $C \in \mathcal{C}$, we set $\beta = \gamma$ and conclude the construction. Otherwise, we choose $C \in \mathcal{C}$ and $a \in A(C) - B'(\gamma)(C)$. According to Lemma A.3.3.16, there exists a pair of slices $N(\gamma) = \{(B''(C), \theta_C)\}_{C \in \mathcal{C}}$, $N'(\gamma) = \{(B''(C) \cap B'(\gamma)(C), \theta'_C\}_{C \in \mathcal{C}}$ such that $N'(\gamma)$ is compatible with both $N(\gamma)$ and $M'(\gamma)$. We now define $M(\gamma)$ to be the slice obtained by amalgamating $M'(\gamma)$ and $N(\gamma)$.

For $\gamma < \beta$, let $G(\gamma) : \mathcal{C} \to \mathbf{A}$ be the simplicial functor corresponding to the slice $M(\gamma)$. Then we have a transfinite sequence of composable morphisms

$$G(0) \to G(1) \to \cdots$$

in $(\mathbf{A}^{\mathcal{C}})_{F/}$ having colimit $G \simeq \varinjlim_{\gamma < \beta} G(\gamma)$. Since $\overline{S}$ is weakly saturated, to prove that the map $F \to G$ belongs to $\overline{S}$, it will suffice to show that for each $\gamma < \beta$, the map

$$f_\gamma : \varinjlim_{\gamma' < \gamma} G(\gamma') \to G(\gamma)$$

belongs to $\overline{S}$. We observe that for each $C \in \mathcal{C}$, the map $f_\gamma(C)$ can be identified with the map $Y(C)_{B'(\gamma)(C)} \to Y(C)_{B(\gamma)(C)}$. Since $B(\gamma)(C) - B'(\gamma)(C)$ is κ-small, Remark A.1.5.5, Lemma A.1.5.11, and Lemma A.1.5.6 imply that f_γ is the pushout of a morphism belonging to S_0. We now conclude by applying the following result (replacing G by $G(\gamma)$ and F by $\varinjlim_{\gamma' < \gamma} G(\gamma')$):

Lemma A.3.3.17. *Suppose that $f : F \to G$ has the property that, for each $C \in \mathcal{C}$, there exists a pushout diagram*

$$\begin{tikzcd} X_C \arrow[r, "g_C"] \arrow[d] & Y_C \arrow[d] \\ F(C) \arrow[r, "f(C)"] & G(C), \end{tikzcd}$$

where $g_C \in S_0$. Then f is the pushout of a morphism in S.

Proof. In view of (iv), we can write F as the colimit of a diagram $\{F_\lambda\}_{\lambda \in P}$ indexed by a κ-filtered partially ordered set P, where each F_λ is a κ-compact object of $\mathbf{A}^{\mathcal{C}}$ and is therefore a functor whose values are κ-compact objects of $\mathbf{A}$. Since each $X_C \in \mathbf{A}$ is κ-compact, the map $X_C \to F(C)$ factors through $F_{\lambda(C)}(C)$ for some sufficiently large $\lambda(C) \in P$. Since $\mathcal{C}$ has fewer than κ objects and P is κ-filtered we can choose a single $\lambda \in P$ which works for every object $C \in \mathcal{C}$.

Consider, for each $C \in \mathcal{C}$, the composite map

$$\begin{aligned} \coprod_{C' \in \mathcal{C}} Y_{C'} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) &\to \coprod_{C' \in \mathcal{C}} G(C') \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \\ &\to G(C) \\ &\simeq \varinjlim_{\lambda' \in P} F_{\lambda'}(C) \coprod_{X_C} Y_C. \end{aligned}$$

Using another compactness argument, we deduce that this map is equivalent to a composition

$$\coprod_{C' \in \mathcal{C}} Y_{C'} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) \to F_{\lambda'(C)}(C) \coprod_{X_C} Y_C$$

for some sufficiently large $\lambda'(C) \in P$. Once again, because P is κ-filtered we can choose a single $\lambda' \in P$ which works for all C. Enlarging λ and λ', we can assume $\lambda = \lambda'$. Using another compactness argument, we can (after enlarging λ if necessary) assume that each of the diagrams

$$\begin{array}{ccc} X_{C'} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) & \longrightarrow & F_\lambda(C) \\ \downarrow & & \downarrow \\ Y_{C'} \otimes \mathrm{Map}_{\mathbf{A}}(C', C) & \longrightarrow & F_\lambda(C) \coprod_{X_C} Y_C \end{array}$$

$$\begin{array}{ccc} Y_{C''} \otimes \mathrm{Map}_{\mathbf{A}}(C'', C') \otimes \mathrm{Map}_{\mathbf{A}}(C', C) & \longrightarrow & Y_{C''} \otimes \mathrm{Map}_{\mathbf{A}}(C'', C) \\ \downarrow & & \downarrow \\ (F_\lambda(C') \coprod_{X_{C'}} Y_{C'}) \otimes \mathrm{Map}_{\mathbf{A}}(C', C) & \longrightarrow & F_\lambda(C) \coprod_{X_C} Y_C \end{array}$$

is commutative. Then the above maps allow us to define an **S**-enriched functor $G_\lambda : \mathcal{C} \to \mathbf{A}$ by the formula $G_\lambda(C) = F_\lambda(C) \coprod_{X_C} Y_C$. We now observe that there is a pushout diagram

$$\begin{array}{ccc} F_\lambda & \xrightarrow{f_\lambda} & G_\lambda \\ \downarrow & & \downarrow \\ F & \xrightarrow{f} & G \end{array}$$

and that $f_\lambda \in S$. □

A.3.4 Path Spaces in S-Enriched Categories

Let **S** be a excellent model category. We have seen that there exists a model structure on the category $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ of **S**-enriched categories whose fibrant objects are precisely those categories which are enriched over the full subcategory $\mathbf{S}^\circ$ of fibrant objects of **S**.

The theory of model categories provides a plethora of examples: for every **S**-enriched model category **A**, the full subcategory $\mathbf{A}^\circ \subseteq \mathbf{A}$ of fibrant-cofibrant objects is a fibrant object of $\mathcal{C}\mathrm{at}_{\mathbf{S}}$. In other words, $\mathbf{A}^\circ$ is suitable to use for computing the homotopy set $[\mathcal{C}, \mathbf{A}^\circ] = \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathbf{A}^\circ)$: if $\mathcal{C}$ is cofibrant, then every map from $\mathcal{C}$ to $\mathbf{A}^\circ$ in the homotopy category of $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is represented by an actual **S**-enriched functor from $\mathcal{C}$ to $\mathbf{A}^\circ$. Moreover, two simplicial functors $F, F' : \mathcal{C} \to \mathbf{A}^\circ$ represent the same morphism in $\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}$ if and only if they are homotopic to one another. The relation of homotopy can be described in terms of either a cylinder object for $\mathcal{C}$ or a path object for $\mathbf{A}^\circ$. Unfortunately, it is rather difficult to construct a cylinder object for $\mathcal{C}$ explicitly since the cofibrations in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ are difficult to describe directly even when $\mathbf{S} = \mathcal{S}\mathrm{et}_\Delta$ (the class of cofibrations of simplicial categories is *not* stable under products, so the usual procedure of constructing mapping cylinders via "product with an interval" cannot be applied). On the other hand, Theorem A.3.2.24 gives a good understanding of the fibrations in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$, which will allow us to give a very explicit construction of a path object for $\mathbf{A}^{\circ}$.

Let $\mathbf{A}$ be an $\mathbf{S}$-enriched model category. Our goal in this section is to give a direct construction of a path space object for $\mathbf{A}^{\circ}$ in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$. In other words, we wish to supply a diagram of $\mathbf{S}$-enriched categories

$$\mathbf{A}^{\circ} \to P(\mathbf{A}) \to \mathbf{A}^{\circ} \times \mathbf{A}^{\circ},$$

where the composite map is the diagonal, the left map is a weak equivalence, and the right map is a fibration. For technical reasons, we will find it convenient to work not with the entire category $\mathbf{A}$ but with some (usually small) subcategory thereof. For this reason, we introduce the following definition:

Definition A.3.4.1. Let $\mathbf{S}$ be an excellent model category and let $\mathbf{A}$ be a combinatorial $\mathbf{S}$-enriched model category. A *chunk of* $\mathbf{A}$ is a full subcategory $\mathcal{U} \subseteq \mathbf{A}$ with the following properties:

(a) Let A be an object of $\mathcal{U}$ and let $\{\phi_i : A \to B_i\}_{i \in I}$ be a finite collection of morphisms in $\mathcal{U}$. Then there exists a factorization

$$A \xrightarrow{p} \overline{A} \xrightarrow{q} \prod_{i \in I} B_i$$

of the product map $\prod_{i \in I} \phi_i$, where p is a trivial cofibration, q is a fibration, and $\overline{A} \in \mathcal{U}$. Moreover, this factorization can be chosen to depend functorially on the collection $\{\phi_i\}$ via an $\mathbf{S}$-enriched functor.

(b) Let A be an object of $\mathcal{U}$ and let $\{\phi_i : B_i \to A\}_{i \in I}$ be a finite collection of morphisms in $\mathcal{U}$. Then there exists a factorization

$$\coprod_{i \in I} B_i \xrightarrow{p} \overline{A} \xrightarrow{q} A$$

of the coproduct map $\coprod_{i \in I} \phi_i$, where p is a cofibration, q is a trivial fibration, and $\overline{A} \in \mathcal{U}$. Moreover, this factorization can be chosen to depend functorially on the collection $\{\phi_i\}$ via an $\mathbf{S}$-enriched functor.

If $\mathcal{U}$ is a chunk of $\mathbf{A}$, we let $\mathcal{U}^{\circ}$ denote the full subcategory $\mathbf{A}^{\circ} \cap \mathcal{U} \subseteq \mathcal{U}$ consisting of fibrant-cofibrant objects of $\mathbf{A}$ which belong to $\mathcal{U}$.

We will say that two chunks $\mathcal{U}, \mathcal{U}' \subseteq \mathbf{A}$ are *equivalent* if they have the same essential image in the homotopy category $\mathrm{h}\mathbf{A}$.

Remark A.3.4.2. In particular, if $\mathcal{U}$ is a chunk of $\mathbf{A}$, then each object $A \in \mathcal{U}$ admits (functorial) fibrant and cofibrant replacements which also belong to $\mathcal{U}$ (take the set I to be empty in (a) and (b)).

Remark A.3.4.3. If $\mathcal{U} \subseteq \mathcal{U}' \subseteq \mathbf{A}$ are equivalent chunks of $\mathbf{A}$, then the inclusion $\mathcal{U}^{\circ} \subseteq \mathcal{U}'^{\circ}$ is a weak equivalence of $\mathbf{S}$-enriched categories.

Example A.3.4.4. Let $\mathbf{S}$ be an excellent model category and let $\mathbf{A}$ be a combinatorial $\mathbf{S}$-enriched model category. Then $\mathbf{A}$ is a chunk of itself; this follows from the small object argument.

Example A.3.4.5. Let $\mathcal{U} \subseteq \mathbf{A}$ be a chunk and let $\{X_\alpha\}$ be a collection of objects in $\mathbf{A}$. Let $\mathcal{V} \subseteq \mathcal{U}$ be the full subcategory spanned by those objects $X \in \mathcal{U}$ such that there exists an isomorphism $[X] \simeq [X_\alpha]$ in the homotopy category $\mathrm{h}\mathbf{A}$. Then $\mathcal{V}$ is also a chunk of $\mathbf{A}$.

We will prove a general existence theorem for chunks below (see Lemma A.3.4.15).

Lemma A.3.4.6. *Let* $\mathbf{S}$ *be an excellent model category and let* $\mathcal{C}$ *be a small* $\mathbf{S}$*-enriched category. Then there exists a weak equivalence of* $\mathbf{S}$*-enriched categories* $\mathcal{C} \to \mathcal{U}^\circ$*, where* $\mathcal{U}$ *is a chunk of a combinatorial* $\mathbf{S}$*-enriched category* $\mathbf{A}$*.*

Proof. Without loss of generality, we may suppose that $\mathcal{C}$ is fibrant. Let $\mathbf{A} = \mathbf{S}^{\mathcal{C}^{op}}$ endowed with the projective model structure. We can identify $\mathcal{C}$ with a full subcategory of $\mathbf{A}^\circ$ via the Yoneda embedding. Using Lemma A.3.4.15, we can enlarge $\mathcal{C}$ to a chunk in $\mathbf{A}$ having the same image in the homotopy category $\mathrm{h}\mathbf{A}$. □

Notation A.3.4.7. Let $\mathbf{S}$ be an excellent model category, let $\mathbf{A}$ be a combinatorial $\mathbf{S}$-enriched model category, and let $\mathcal{U}$ be a chunk of $\mathbf{A}$. We define a new category $P(\mathcal{U})$ as follows:

(i) The objects of $P(\mathcal{U})$ are fibrations $\phi : A \to B \times C$ in $\mathbf{A}$, where $A, B, C \in \mathcal{U}^\circ$ and the composite maps $A \to B$ and $A \to C$ are weak equivalences.

(ii) Morphisms in $P(\mathcal{U})$ are given by maps of diagrams

$$\begin{tikzcd} B \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & C \arrow[d] \\ B' & A' \arrow[l] \arrow[r] & C'. \end{tikzcd}$$

We let $\pi, \pi' : P(\mathcal{U}) \to \mathcal{U}^\circ$ be the functors described by the formulas

$$\pi(\phi : A \to B \times C) = B \qquad \pi'(\phi : A \to B \times C) = C.$$

We observe that both π and π' have the structure of $\mathbf{S}$-enriched functors. Invoking assumption (a) of Proposition A.3.4.1, we deduce the existence of another $\mathbf{S}$-enriched functor $\tau : \mathcal{U}^\circ \to P(\mathcal{U})$, which carries an object $A \in \mathbf{A}^\circ$ to the map q appearing in a functorial factorization

$$A \xrightarrow{p} \overline{A} \xrightarrow{q} A \times A$$

of the diagonal, where p is a trivial cofibration and q is a fibration.

Theorem A.3.4.8. *Let* $\mathbf{S}$ *be an excellent model category, let* $\mathbf{A}$ *be a combinatorial* $\mathbf{S}$*-enriched model category, and let* $\mathcal{U}$ *be a chunk of* $\mathbf{A}$*. Then the morphisms* $\pi, \pi' : P(\mathcal{U}) \to \mathcal{U}^\circ$ *and* $\tau : \mathcal{U}^\circ \to P(\mathcal{U})$ *furnish* $P(\mathcal{U})$ *with the structure of a path object for* $\mathcal{U}^\circ$ *in* $\mathcal{C}\mathrm{at}_{\mathbf{S}}$*.*

Proof. We first show that $\pi \times \pi'$ is a fibration of $\mathbf{S}$-enriched categories. In view of Theorem A.3.2.24, it will suffice to show that $\pi \times \pi'$ is a local fibration. Let $\phi : A \to B \times C$ and $\phi' : A' \to B' \times C'$ be objects of $P(\mathcal{U})$. We must show that the induced map

$$\mathrm{Map}_{P(\mathcal{U})}(\phi, \phi') \to \mathrm{Map}_{\mathbf{A}}(B, B') \times \mathrm{Map}_{\mathbf{A}}(C, C')$$

is a fibration in $\mathbf{S}$. This map is a base change of

$$\mathrm{Map}_{\mathbf{A}}(A, A') \to \mathrm{Map}_{\mathbf{A}}(A, B' \times C'),$$

which is a fibration by virtue of the assumption that ϕ' is a fibration (since A is assumed to be cofibrant).

To complete the proof that $\pi \times \pi'$ is a quasi-fibration, we must show that if $\phi : A \to B \times C$ is an object of $P(\mathcal{U})$ and we are given weak equivalences $f : B \to B'$, $g : C \to C'$, then we can lift f and g to an equivalence in $P(\mathcal{U})$. To do so, we factor the composite map $A \to B' \times C'$ as a trivial cofibration $A \to A'$ followed by a fibration $\phi' : A' \to B' \times C'$. Since $\mathcal{U}$ is a chunk of $\mathbf{A}$, we may assume that $A' \in \mathcal{U}$ so that $\phi' \in P(\mathcal{U})$. We have an evident natural transformation $\alpha : \phi \to \phi'$. We will show below that $\pi : P(\mathcal{U}) \to \mathcal{U}^\circ$ is an equivalence of $\mathbf{S}$-enriched categories; since $\pi(\alpha) = f$ is an isomorphism in $\mathrm{h}\mathcal{U}^\circ$, we conclude that α is an isomorphism in $\mathrm{h}P(\mathcal{U})$, as required.

To complete the proof, we must show that τ is a weak equivalence of $\mathbf{S}$-enriched categories. By the two-out-of-three property, it will suffice to show that π is a weak equivalence of $\mathbf{S}$-enriched categories. Since τ is a section of π, it is clear that π is essentially surjective. It remains only to prove that π is fully faithful. Let $\phi : A \to B \times C$ and $\phi' : A' \to B' \times C'$ be objects of $P(\mathcal{U})$; we wish to show that the induced map $p : \mathrm{Map}_{P(\mathcal{U})}(\phi, \phi') \to \mathrm{Map}_{\mathbf{A}}(B, B')$ is a weak equivalence in $\mathbf{S}$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{P(\mathcal{U})}(\phi, \phi') \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbf{A}}(A, A') \arrow[d, "u"] \\
\mathrm{Map}_{\mathbf{A}}(B, B') \times \mathrm{Map}_{\mathbf{A}}(C, C') \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbf{A}}(A, B' \times C') \arrow[d] \\
\mathrm{Map}_{\mathbf{A}}(B, B') \times \mathrm{Map}_{\mathbf{A}}(A, C') \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbf{A}}(A, B') \times \mathrm{Map}_{\mathbf{A}}(A, C') \arrow[d] \\
\mathrm{Map}_{\mathbf{A}}(B, B') \arrow[r] & \mathrm{Map}_{\mathbf{A}}(A, B').
\end{tikzcd}$$

We note that because the map $A \to B$ is a weak equivalence between cofibrant objects and B' is fibrant, the bottom horizontal map is a weak equivalence in $\mathbf{S}$. Consequently, to show that the top horizontal map is a weak equivalence in $\mathbf{S}$, it will suffice to show that each square in the diagram is homotopy Cartesian. The bottom square is Cartesian and fibrant, so there is nothing to prove. The middle square is homotopy Cartesian because both of the middle vertical maps are weak equivalences. The upper square is a pullback square between fibrant objects of $\mathbf{S}$, and the map u is a fibration; we now complete the proof by invoking Proposition A.2.4.4. $\square$

Fix an excellent model category $\mathbf{S}$. The symmetric monoidal structure on $\mathbf{S}$ induces a symmetric monoidal structure on $\mathcal{C}\mathrm{at}_{\mathbf{S}}$: if $\mathcal{C}$ and $\mathcal{D}$ are $\mathbf{S}$-enriched categories, then we can define a new $\mathbf{S}$-enriched category $\mathcal{C} \otimes \mathcal{D}$ as follows:

(i) The objects of $\mathcal{C} \otimes \mathcal{D}$ are pairs (C, D), where $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

(ii) Given a pair of objects $(C, D), (C', D') \in \mathcal{C} \otimes \mathcal{D}$, we have

$$\mathrm{Map}_{\mathcal{C} \otimes \mathcal{D}}((C, D), (C', D')) = \mathrm{Map}_{\mathcal{C}}(C, C') \otimes \mathrm{Map}_{\mathcal{D}}(D, D') \in \mathbf{S}.$$

In the case where the tensor product on $\mathbf{S}$ is the Cartesian product, this simply reduces to the usual product of $\mathbf{S}$-enriched categories.

Note that the operation $\otimes : \mathcal{C}\mathrm{at}_{\mathbf{S}} \times \mathcal{C}\mathrm{at}_{\mathbf{S}} \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ is *not* a left Quillen bifunctor even when $\mathbf{S} = \mathcal{S}\mathrm{et}_{\Delta}$: for example, a product of cofibrant simplicial categories is generally not cofibrant. Nevertheless, $\otimes$ behaves much like a left Quillen bifunctor at the level of homotopy categories. For example, the operation $\otimes$ respects weak equivalences in each argument and therefore induces a functor $\otimes : \mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}} \times \mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}} \to \mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}$, which is characterized by the existence of natural isomorphisms $[\mathcal{C} \otimes \mathcal{D}] \simeq [\mathcal{C}] \otimes [\mathcal{D}]$.

Our goal for the remainder of this section is to show that the monoidal structure $\otimes$ on $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is *closed*: that is, there exist internal mapping objects in $\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}$. This is not completely obvious. It is easy to see that the monoidal structure $\otimes$ on $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is closed: given a pair of $\mathbf{S}$-enriched categories $\mathcal{C}$ and $\mathcal{D}$, the category of $\mathbf{S}$-enriched functors $\mathcal{D}^{\mathcal{C}}$ is itself $\mathbf{S}$-enriched and possesses the appropriate universal property. However, this is not necessarily the "correct" mapping object in the sense that the homotopy equivalence class $[\mathcal{D}^{\mathcal{C}}]$ does not necessarily coincide with the internal mapping object $[\mathcal{D}]^{[\mathcal{C}]}$ in $\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}$. Roughly speaking, the problem is that $\mathcal{D}^{\mathcal{C}}$ consists of functors which are strictly compatible with composition; the correct mapping object should also incorporate functors which preserve composition only up to (coherent) weak equivalence. However, when $\mathcal{D}$ is the category of fibrant-cofibrant objects of an $\mathbf{S}$-enriched *model* category $\mathbf{A}$, then we can proceed more directly.

Definition A.3.4.9. Let $\mathbf{S}$ be an excellent model category, $\mathbf{A}$ a combinatorial $\mathbf{S}$-enriched model category, and $\mathcal{C}$ an $\mathbf{S}$-enriched category. We will say that a full subcategory $\mathcal{U} \subseteq \mathbf{A}$ is a $\mathcal{C}$-*chunk of* $\mathbf{A}$ if it is a chunk of $\mathbf{A}$ and the subcategory $\mathcal{U}^{\mathcal{C}}$ is a chunk of $\mathbf{A}^{\mathcal{C}}$. Here we regard $\mathbf{A}^{\mathcal{C}}$ as endowed with the *projective* model structure.

Lemma A.3.4.10. *Let* $\mathbf{S}$ *be an excellent model category,* $\mathbf{A}$ *a combinatorial* $\mathbf{S}$*-enriched model category,* $\mathcal{C}$ *a* (*small*) *cofibrant* $\mathbf{S}$*-enriched category, and* $\mathcal{U} \subseteq \mathbf{A}$ *a* $\mathcal{C}$*-chunk. Let* $f, f' : \mathcal{C} \to \mathcal{U}^{\circ}$ *be a pair of maps. The following conditions are equivalent:*

(1) *The homotopy classes* $[f]$ *and* $[f']$ *coincide in* $\mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathcal{U}^{\circ})$.

(2) *The maps f and f' are weakly equivalent when regarded as objects of* $\mathbf{A}^{\mathcal{C}}$.

Proof. Suppose first that (1) is satisfied. Using Theorem A.3.4.8, we deduce the existence of a homotopy $h : \mathcal{C} \to P(\mathcal{U})$ from $f = \pi \circ h$ to $f' = \pi' \circ h$. The map h determines another simplicial functor $f'' : \mathcal{C} \to \mathcal{U}$ equipped with weak equivalences $f'' \to f$, $f'' \to f'$. This proves that f and f' are isomorphic in the homotopy category of $\mathbf{A}^{\mathcal{C}}$, so that (2) is satisfied.

Now suppose that (2) is satisfied. Since $\mathcal{U}$ is a $\mathcal{C}$-chunk, we can find a projectively cofibrant $f'' : \mathcal{U} \to \mathcal{C}$ equipped with a weak equivalence $\alpha : f'' \to f$. Using (2), we deduce that there is also a weak equivalence $\beta : f'' \to f'$. Again using the assumption that $\mathcal{U}^{\mathcal{C}}$ is a chunk of $\mathbf{A}^{\mathcal{C}}$, we can choose a factorization of $\alpha \times \beta$ as a composition

$$f'' \xrightarrow{u} f''' \xrightarrow{v} f \times f'$$

where u is a trivial projective cofibration, v is a projective fibration, and $f''' \in \mathcal{U}^{\mathcal{C}}$. The map v can be viewed as an object of $\mathcal{P}(\mathcal{U})$, which determines a right homotopy from f to f'. □

Corollary A.3.4.11. *Let* $\mathbf{S}$ *be an excellent model category and let* $f : \mathcal{C} \to \mathcal{C}'$ *be an* $\mathbf{S}$*-enriched functor. Suppose that f is fully faithful in the sense that for every pair of objects $X, Y \in \mathcal{C}$, the induced map* $\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{C}'}(fX, fY)$ *is a weak equivalence in* $\mathbf{S}$. *Let $\mathcal{D}$ be an arbitrary* $\mathbf{S}$*-enriched category. Then*

(1) *Composition with f induces an injective map* $\phi : \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{D}, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{D}, \mathcal{C}')$.

(2) *The image of ϕ consists of those maps $g : \mathcal{D} \to \mathcal{C}'$ in* $\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}$ *such that the essential image of $[g]$ in* $\mathrm{h}\mathcal{C}'$ *is contained in the essential image of $[f]$ in* $\mathrm{h}\mathcal{C}'$.

Proof. Using Lemma A.3.4.6, we may assume without loss of generality that $\mathcal{C}' = \mathcal{U}^{\circ}$, where $\mathcal{U}$ is a chunk of an $\mathbf{S}$-enriched model category. Let $\mathcal{V} \subseteq \mathcal{U}$ be the full subcategory spanned by those objects which are weakly equivalent to an object lying in the image of f. Since f is fully faithful, the induced map $\mathcal{C} \to \mathcal{V}^{\circ}$ is a weak equivalence. We may therefore assume that $\mathcal{C} = \mathcal{V}^{\circ}$.

Without loss of generality, we may suppose that $\mathcal{D}$ is cofibrant. Enlarging $\mathcal{U}$ and $\mathcal{V}$ if necessary (using Lemma A.3.4.15), we may assume that $\mathcal{U}$ and $\mathcal{V}$ are $\mathcal{D}$-chunks. The desired results now follow immediately from Lemma A.3.4.10. □

Let $\pi_0 \mathbf{A}^{\mathcal{C}}$ denote the collection of weak equivalence classes of objects in $\mathbf{A}^{\mathcal{C}}$. Every equivalence class contains a fibrant-cofibrant representative which determines an $\mathbf{S}$-enriched functor $\mathcal{C} \to \mathbf{A}^{\circ}$.

Proposition A.3.4.12. *Let* $\mathbf{S}$ *be an excellent model category,* $\mathbf{A}$ *a combinatorial* $\mathbf{S}$*-enriched model category, and $\mathcal{C}$ a (small)* $\mathbf{S}$*-enriched category. Then the map*

$$\phi : \pi_0 \mathbf{A}^{\mathcal{C}} \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathbf{A}^{\circ})$$

described above is bijective.

Proof. In view of Proposition A.3.3.8, we may assume that $\mathcal{C}$ is cofibrant. Lemma A.3.4.10 shows that ϕ is well-defined and injective. We show that ϕ is surjective. Let $[f] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathcal{U}^\circ)$. Since $\mathcal{C}$ is cofibrant and $\mathbf{A}^\circ$ is fibrant in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$, we can find an $\mathbf{S}$-enriched functor $f : \mathcal{C} \to \mathbf{A}^\circ$ representing $[f]$. The simplicial functor f takes values in fibrant-cofibrant objects of $\mathbf{A}$ but is not necessarily fibrant-cofibrant *as an object of* $\mathbf{A}^{\mathcal{C}}$. However, we can choose a projective trivial fibration $f' \to f$, where f' is projectively cofibrant. Consequently, it will suffice to show that a weak equivalence $u : f' \to f$ of $\mathbf{S}$-enriched functors $\mathcal{C} \to \mathbf{A}^\circ$ guarantees that $[f] = [f'] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathbf{A}^\circ)$, which follows from Lemma A.3.4.10. $\square$

Proposition A.3.4.13. *Let* $\mathbf{S}$ *be an excellent model category,* $\mathbf{A}$ *a combinatorial* $\mathbf{S}$*-enriched model category, and* $\mathcal{C}$ *a small* $\mathbf{S}$*-enriched category. Then the evaluation map* $e : (\mathbf{A}^{\mathcal{C}})^\circ \otimes \mathcal{C} \to \mathbf{A}^\circ$ *has the following property: for every small* $\mathbf{S}$*-enriched category* $\mathcal{D}$*, composition with* e *induces a bijection*

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{D}, (\mathbf{A}^{\mathcal{C}})^\circ) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C} \otimes \mathcal{D}, \mathbf{A}^\circ).$$

Proof. Using Proposition A.3.4.12, we can identify both sides with $\pi_0 \mathbf{A}^{\mathcal{C} \otimes \mathcal{D}}$. $\square$

It is not clear that the conclusion of Proposition A.3.4.13 characterizes $(\mathbf{A}^{\mathcal{C}})^\circ$ up to equivalence since $(\mathbf{A}^{\mathcal{C}})^\circ$ is a *large* $\mathbf{S}$-enriched category, and the proof of the proposition applies only in the case where $\mathcal{D}$ is small. To remedy this defect, we establish a more refined version:

Corollary A.3.4.14. *Let* $\mathbf{S}$ *be an excellent model category,* $\mathbf{A}$ *a combinatorial* $\mathbf{S}$*-enriched model category, and* $\mathcal{C}$ *a small* $\mathbf{S}$*-enriched category. Let* $\mathcal{U}$ *be a* $\mathcal{C}$*-chunk of* $\mathbf{A}$*. Then the evaluation map* $e : (\mathcal{U}^{\mathcal{C}})^\circ \otimes \mathcal{C} \to \mathcal{U}^\circ$ *has the following property: for every small* $\mathbf{S}$*-enriched category* $\mathcal{D}$*, composition with* e *induces a bijection*

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{D}, (\mathcal{U}^{\mathcal{C}})^\circ) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{U}^\circ).$$

Proof. Combine Proposition A.3.4.13 with Corollary A.3.4.11. $\square$

We conclude this section with a technical result which ensures the existence of a good supply of chunks of combinatorial model categories.

Lemma A.3.4.15. *Let* $\mathbf{S}$ *be an excellent model category,* $\mathbf{A}$ *a combinatorial* $\mathbf{S}$*-enriched model category, and* $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ *a (small) collection of (small) cofibrant* $\mathbf{S}$*-enriched categories. Let* $\mathcal{U}$ *be a small full subcategory of* $\mathbf{A}$*. Then there exists a small subcategory* $\mathcal{V} \subseteq \mathbf{A}$ *containing* $\mathcal{U}$*, such that* $\mathcal{V}$ *is a* $\mathcal{C}_\alpha$*-chunk for each* $\alpha \in A$*. Moreover, we may arrange that* $\mathcal{U}$ *and* $\mathcal{V}$ *have the same essential image in the homotopy category* $\mathrm{h}\mathbf{A}$*.*

Proof. Enlarging A if necessary, we may suppose that the collection $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ includes the unit category $[0]_{\mathbf{S}}$. For each $\alpha \in A$, we can choose $\mathbf{S}$-enriched functors

$$F_\alpha : \mathbf{A}^{\mathcal{C}_\alpha \otimes [1]_{\mathbf{S}}} \to \mathbf{A}^{\mathcal{C}_\alpha \otimes [2]_{\mathbf{S}}} \qquad G_\alpha : \mathbf{A}^{\mathcal{C}_\alpha \otimes [1]_{\mathbf{S}}} \to \mathbf{A}^{\mathcal{C}_\alpha \otimes [2]_{\mathbf{S}}},$$

such that F carries each morphism $u : f \to g$ in $\mathbf{A}^{\mathcal{C}}$ to a factorization

$$f \xrightarrow{u'} f' \xrightarrow{u''} g,$$

where u' is a projective trivial cofibration and u'' is a projective fibration, and G carries u to a factorization

$$f \xrightarrow{v'} g' \xrightarrow{v''} g,$$

where v' is a projective cofibration and v'' is an injective trivial cofibration. For $C \in \mathcal{C}_\alpha$, let F^C_α be the functor $u \mapsto f'(C)$ and define G^C_α likewise.

Choose a regular cardinal κ such that each $\mathcal{C}_\alpha$ is κ-small. We define a sequence of full subcategories $\{\mathcal{U}_\alpha \subseteq \mathbf{A}\}_{\alpha < \kappa}$ as follows:

(i) If $\alpha = 0$, then $\mathcal{U}_\alpha = \mathcal{U}$.

(ii) If α is a nonzero limit ordinal, then $\mathcal{U}_\alpha = \bigcup_{\beta < \alpha} \mathcal{U}_\beta$.

(iii) If $\alpha = \beta + 1$, then $\mathcal{U}_\alpha$ is the full subcategory of $\mathbf{A}$ spanned by the following:

- (a) The objects which belong to $\mathcal{U}_\beta$.
- (b) The objects $F^C_\alpha(u) \in \mathbf{A}$, where $\alpha \in A$, $C \in \mathcal{C}_\alpha$, and $u : f \to g$ is a morphism from an object of $\mathcal{U}_\beta^{\mathcal{C}_\alpha}$ to a finite product of objects in $\mathcal{U}_\beta^{\mathcal{C}_\alpha}$.
- (c) The objects $G^C_\alpha(u) \in \mathbf{A}$, where $\alpha \in A$, $C \in \mathcal{C}_\alpha$, and $u : f \to g$ is a morphism from a finite coproduct of objects of $\mathcal{U}_\beta^{\mathcal{C}_\alpha}$ to an object in $\mathcal{U}_\beta^{\mathcal{C}_\alpha}$.

It is readily verified that the subcategory $\mathcal{V} = \bigcup_{\alpha < \kappa} \mathcal{U}_\alpha$ has the desired properties. $\square$

A.3.5 Homotopy Colimits of S-Enriched Categories

Our goal in this section is to give an explicit construction of (certain) homotopy colimits in the model category $\mathcal{C}\mathrm{at}_{\mathbf{S}}$, where $\mathbf{S}$ is an excellent model category. We begin with some general remarks concerning localization.

Notation A.3.5.1. Consider the canonical map $\bar{i} : [1]_{\mathbf{S}} \to [1]^{\sim}_{\mathbf{S}}$. We fix once and for all a factorization of $\bar{i}$ as a composition

$$[1]_{\mathbf{S}} \xrightarrow{i} \mathcal{E} \xrightarrow{i'} [1]^{\sim}_{\mathbf{S}},$$

where i is a cofibration and i' is a weak equivalence of **S**-enriched categories. For every **S**-enriched category $\mathcal{C}$ and every map $W \to \mathrm{Hom}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}([1]_{\mathbf{S}}, \mathcal{C})$, we define a new **S**-enriched category $\mathcal{C}[W^{-1}]$ by a pushout diagram

$$\begin{tikzcd}
\coprod_{w \in W} [1]_{\mathbf{S}} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\coprod_{w \in W} \mathcal{E} \arrow[r] & \mathcal{C}[W^{-1}].
\end{tikzcd}$$

Remark A.3.5.2. Since the model category $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is left proper, the construction $\mathcal{C} \mapsto \mathcal{C}[W^{-1}]$ preserves weak equivalences in $\mathcal{C}$.

We now characterize $\mathcal{C}[W^{-1}]$ by a universal property in $\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}$.

Lemma A.3.5.3. *Let $\mathcal{C}$ be a fibrant **S**-enriched category and let f be a morphism in $\mathcal{C}$ classified by a map $j_0 : [1]_{\mathbf{S}} \to \mathcal{C}$. The following conditions are equivalent:*

(1) *The map f is an equivalence in $\mathcal{C}$.*

(2) *The extension problem depicted in the diagram*

$$\begin{tikzcd}
{[1]_{\mathbf{S}}} \arrow[r, "j_0"] \arrow[d, "i"'] & \mathcal{C} \\
\mathcal{E} \arrow[ur, dashed, "j"] &
\end{tikzcd}$$

admits a solution.

Proof. The implication (2) $\Rightarrow$ (1) is clear since every morphism in $\mathcal{E}$ is an equivalence. For the converse, we observe that the desired lifting problem admits a solution if and only if the induced map $i' : \mathcal{C} \to \mathcal{C} \coprod_{[1]_{\mathbf{S}}} \mathcal{E}$ admits a left inverse. Since $\mathcal{C}$ is fibrant, it suffices to show that i' is a trivial cofibration. The map i' is a cofibration since it is a pushout of i, and a weak equivalence because of the invertibility hypothesis. $\square$

Lemma A.3.5.3 immediately implies the following apparently stronger claim:

Lemma A.3.5.4. *Let $f_0 : \mathcal{C} \to \mathcal{D}$ be an **S**-enriched functor, where $\mathcal{D}$ is a fibrant **S**-enriched category. Let $\psi : W \to \mathrm{Hom}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}([1]_{\mathbf{S}}, \mathcal{C})$ be a map of sets. The following conditions are equivalent:*

(1) *The map f_0 extends to a map $f : \mathcal{C}[W^{-1}] \to \mathcal{D}$.*

(2) *For each $w \in W$, f_0 carries the morphism $\psi(w)$ to an equivalence in $\mathcal{D}$.*

Proposition A.3.5.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be* **S***-enriched categories and let* $\psi : W \to \mathrm{Hom}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}([1]_{\mathbf{S}}, \mathcal{C})$ *be a map of sets. Then the induced map*

$$\phi : \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}[W^{-1}], \mathcal{D}) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathcal{D})$$

is injective, and its image is the subset $\mathrm{Hom}^{W}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, \mathcal{D})$ *consisting of those homotopy classes of maps which induce functors* $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ *carrying each element of W to an isomorphism in* $\mathrm{h}\mathcal{D}$*.*

Proof. Without loss of generality, we may suppose that $\mathcal{C}$ is cofibrant and $\mathcal{D}$ is fibrant. The description of the image of ϕ follows immediately from Lemma A.3.5.4. It will therefore suffice to show that ϕ is injective. Suppose we are given a pair of maps $[f], [g] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}[W^{-1}], \mathcal{D})$ such that $\phi([f]) = \phi([g])$. Since $\mathcal{C}[W^{-1}]$ is cofibrant, we may assume that $[f]$ and $[g]$ are represented by actual **S**-enriched functors $f, g : \mathcal{C}[W^{-1}] \to \mathcal{D}$. Moreover, the condition that $\phi([f]) = \phi([g])$ guarantees that the restrictions $f|\,\mathcal{C}$ and $g|\,\mathcal{C}$ are homotopic. We wish to show that f and g are homotopic.

Invoking Proposition A.2.3.1, we deduce that g is homotopic to a map $g' : \mathcal{C}[W^{-1}] \to \mathcal{D}$ such that $g'|\,\mathcal{C} = f|\,\mathcal{C}$. Replacing g by g' if necessary, we may assume that $g|\,\mathcal{C} = f|\,\mathcal{C}$. It will now suffice to show that f and g are homotopic in the model category $(\mathcal{C}\mathrm{at}_{\mathbf{S}})_{\mathcal{C}\,/}$. We observe that f and g determine a map

$$h : \mathcal{C}[(W \coprod W)^{-1}] \simeq \mathcal{C}[W^{-1}] \coprod_{\mathcal{C}} \mathcal{C}[W^{-1}] \to \mathcal{D}\,.$$

Using the invertibility hypothesis, we conclude that $\mathcal{C}[(W \coprod W)^{-1}]$ is a cylinder object for $\mathcal{C}[W^{-1}]$ in the model category $(\mathcal{C}\mathrm{at}_{\mathbf{S}})_{\mathcal{C}\,/}$, so that h is the desired homotopy from f to g. □

Lemma A.3.5.6. *Let $f : \mathcal{C} \to \mathcal{D}$ be an* **S***-enriched functor and let $\mathcal{M}$ be the categorical mapping cylinder of f defined as follows:*

(1) *An object of $\mathcal{M}$ is either an object of $\mathcal{C}$ or an object of $\mathcal{D}$.*

(2) *Given a pair of objects $X, Y \in \mathcal{M}$, we have*

$$\mathrm{Map}_{\mathcal{M}}(X, Y) = \begin{cases} \mathrm{Map}_{\mathcal{C}}(X, Y) & \text{if } X, Y \in \mathcal{C} \\ \mathrm{Map}_{\mathcal{D}}(X, Y) & \text{if } X, Y \in \mathcal{D} \\ \mathrm{Map}_{\mathcal{D}}(fX, Y) & \text{if } X \in \mathcal{C}, Y \in \mathcal{D} \\ \emptyset & \text{if } X \in \mathcal{D}, Y \in \mathcal{C}\,. \end{cases}$$

Here $\emptyset$ denotes the initial object of **S***.*

We observe that there is a canonical retraction j of $\mathcal{M}$ onto $\mathcal{D}$ described by the formula

$$j(X) = \begin{cases} fX & \text{if } X \in \mathcal{C} \\ X & \text{if } X \in \mathcal{D}\,. \end{cases}$$

Let W be a collection of morphisms in $\mathcal{M}$ with the following properties:

(*i*) *For each* $w \in W$, $j(w)$ *is an identity morphism in* $\mathcal{D}$.

(*ii*) *For every object* $C \in \mathcal{C}$, *the morphism* $C \to fC$ *in* $\mathcal{M}$ *classifying the identity map from* fC *to itself belongs to* W.

Assumption (*i*) *implies that the map* j *canonically extends to a map* $\overline{j} : \mathcal{M}[W^{-1}] \to \mathcal{D}$. *The map* $\overline{j}$ *is a weak equivalence of* $\mathbf{S}$*-enriched categories.*

Proof. It will suffice to show that composition with $\overline{j}$ induces a bijection

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{D}, \mathcal{A}) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{M}[W^{-1}], \mathcal{A})$$

for every $\mathbf{S}$-enriched category $\mathcal{A}$. Equivalently, we must show that the map

$$t : \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{D}, \mathcal{A}) \to \mathrm{Hom}^{W}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{M}, \mathcal{A})$$

is bijective, where $\mathrm{Hom}^{W}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{M}, \mathcal{A})$ is defined as in Proposition A.3.5.5. The map t has a section t' given by composition with the inclusion $\mathcal{D} \to \mathcal{M}$. It will therefore suffice to show that $t \circ t'$ is the identity on $\mathrm{Hom}^{W}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{M}, \mathcal{A})$.

Using Lemma A.3.4.6 and Corollary A.3.4.11, we can reduce to the case where $\mathcal{A} = \mathbf{A}^{\circ}$, where $\mathbf{A}$ is a combinatorial $\mathbf{S}$-enriched model category. Using Proposition A.3.4.12, we deduce that every element $[g] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{M}, \mathcal{A})$ can be represented by a diagram $g : \mathcal{M} \to \mathbf{A}^{\circ}$. We wish to prove that g and $g \circ i \circ j$ are homotopic. We observe that there is a canonical natural transformation $\alpha : g \to g \circ i \circ j$. Moreover, if g carries each element of W to an equivalence in $\mathbf{A}^{\circ}$, then assumption (*ii*) guarantees that α is a weak equivalence in the model category $\mathbf{A}^{\mathcal{M}}$. We now invoke Proposition A.3.4.12 to deduce that g and $g \circ i \circ j$ are homotopic, as desired. □

Definition A.3.5.7. Let A be a partially ordered set. An *A-filtered* $\mathbf{S}$*-enriched category* is an $\mathbf{S}$-enriched category $\mathcal{C}$ together with a map $r : \mathrm{Ob}(\mathcal{C}) \to A$ with the following property: if $C, D \in \mathcal{C}$ and $r(C) \not\leq r(D)$, then $\mathrm{Map}_{\mathcal{C}}(C, D) \simeq \emptyset$, where $\emptyset$ denotes an initial object of $\mathbf{S}$.

If $\mathcal{C}$ is an A-filtered $\mathbf{S}$-enriched category and $a \in A$, then we let $\mathcal{C}_{\leq a}$ denote the full subcategory of $\mathcal{C}$ spanned by those objects $C \in \mathcal{C}$ such that $r(C) \leq a$.

Remark A.3.5.8. Let $\mathcal{C}$ be an A-filtered $\mathbf{S}$-enriched category and let $\psi : W \to \mathrm{Hom}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}([1]_{\mathbf{S}}, \mathcal{C})$ be a map of sets. For each $a \in A$, we let $W_a \subseteq W$ be the subset consisting of those elements $w \in W$ such that the morphism $\psi(w)$ belongs to $\mathcal{C}_{\leq a}$. This data determines a diagram $\chi^{W} : A \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ described by the formula $a \mapsto \mathcal{C}_{\leq a}[W_a^{-1}]$. Moreover, we have a canonical isomorphism of $\mathbf{S}$-enriched categories $\mathcal{C}[W^{-1}] \simeq \varinjlim(\chi)$.

Using the small object argument, we easily deduce the following result:

Lemma A.3.5.9. *Let A be a partially ordered set and let $\mathcal{C}$ be an A-filtered* $\mathbf{S}$*-enriched category. Then there exists an* $\mathbf{S}$*-enriched functor* $f : \mathcal{C}' \to \mathcal{C}$ *with the following properties:*

(1) *The functor f is bijective on objects, and for every pair of objects* $C, D \in \mathcal{C}'$, *the map* $\mathrm{Map}_{\mathcal{C}'}(C, D) \to \mathrm{Map}_{\mathcal{C}}(fC, fD)$ *is a trivial fibration in* $\mathbf{S}$. *In particular, f is a weak equivalence of* $\mathbf{S}$*-enriched categories.*

(2) *The A-filtration on $\mathcal{C}$ induces an A-filtration on $\mathcal{C}'$. In other words, if $C, D \in \mathcal{C}'$ and $r(fC) \not\leq r(fD)$, then $\mathrm{Map}_{\mathcal{C}'}(C, D)$ is an initial object of $\mathbf{S}$.*

(3) *The diagram $A \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ described by the formula $a \mapsto \mathcal{C}'_{\leq a}$ is projectively cofibrant.*

Proposition A.3.5.10. *Let A be a partially ordered set, let $\mathcal{C}$ be an A-filtered $\mathbf{S}$-enriched category, and let $\psi : W \to \mathrm{Hom}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}([1]_{\mathbf{S}}, \mathcal{C})$ be a map of sets. Let $\chi : A \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ be defined as in Remark A.3.5.8. Then the isomorphism $\varinjlim \chi \simeq \mathcal{C}[W^{-1}]$ exhibits $\mathcal{C}$ as the homotopy colimit of the diagram χ.*

Proof. Choose a map $\mathcal{C}' \to \mathcal{C}$ as in Lemma A.3.5.9 and a map $\psi' : W \to \mathrm{Hom}_{\mathcal{C}\mathrm{at}_{\mathbf{S}}}([1]_{\mathbf{S}}, \mathcal{C}')$ lifting ψ, and let $\chi' : A \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ be defined as in Remark A.3.5.8. Then we have a canonical map $\chi' \to \chi$, which is a cofibrant replacement for χ in the model category $\mathrm{Fun}(A, \mathcal{C}\mathrm{at}_{\mathbf{S}})$. It will therefore suffice to show that the induced map $\mathcal{C}'[W^{-1}] \simeq \varinjlim \chi' \to \varinjlim \chi \simeq \mathcal{C}[W^{-1}]$ is a weak equivalence of $\mathbf{S}$-enriched categories, which follows immediately from Remark A.3.5.2. $\square$

Definition A.3.5.11. Let A be a partially ordered set and let $p : A \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ be an A-indexed diagram of $\mathbf{S}$-enriched categories. Let us denote the image of $a \in A$ under p by $\mathcal{C}_a$.

The *Grothendieck construction on p* is a $\mathbf{S}$-enriched category $\mathrm{Groth}(p)$ defined as follows:

(1) The objects of $\mathrm{Groth}(p)$ are pairs (a, C), where $a \in A$ and $C \in \mathcal{C}_a$.

(2) Given a pair of objects $(a, C), (a', C')$ in $\mathrm{Groth}(p)$, we set

$$\mathrm{Map}_{\mathrm{Groth}(p)}((a, C), (a', C')) = \begin{cases} \mathrm{Map}_{\mathcal{C}_{a'}}(p_a^{a'} C, C') & \text{if } a \leq a' \\ \emptyset & \text{otherwise.} \end{cases}$$

Here $p_a^{a'}$ denotes the functor $\mathcal{C}_a \to \mathcal{C}_{a'}$ determined by p, and $\emptyset$ denotes an initial object of $\mathbf{S}$.

(3) Composition in $\mathrm{Groth}(p)$ is defined in the obvious way.

We observe that $\mathrm{Groth}(p)$ is A-filtered via the map $r : \mathrm{Ob}(\mathrm{Groth}(p)) \to A$ given by the formula $r(a, C) = a$. We let $W(p)$ denote the collection of all morphisms in $\mathrm{Groth}(p)$ of the form $\alpha : (a, C) \to (a', p_a^{a'} C)$, where $a \leq a'$ and α corresponds to the identity in $\mathcal{C}_{a'}$.

For each $a \in A$, there is a canonical functor $\pi_a : \mathrm{Groth}(p)_{\leq a} \to \mathcal{C}_a$ given by the formula $(C, a') \mapsto p_{a'}^{a}(C)$. We note that π carries each element of $W(p)_a$ to an identity map in $\mathcal{C}_a$, so that π_a canonically extends to a map $\overline{\pi}_a : \mathrm{Groth}(p)_{\leq a}[W(p)_a^{-1}] \to \mathcal{C}_a$. The maps $\overline{\pi}_a$ are functorial in a and therefore determine a map of diagrams $\chi(p) \to p$, where $\chi(p)$ is defined as in Remark A.3.5.8.

Lemma A.3.5.12. *Let $p : A \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ be as in Definition A.3.5.11. Then for each $a \in A$, the map $\overline{\pi}_a : \mathrm{Groth}(p)_{\leq a}[W(p)_a^{-1}] \to \mathcal{C}_a$ is a weak equivalence of $\mathbf{S}$-enriched categories.*

Proof. This is a special case of Lemma A.3.5.6. □

Lemma A.3.5.13. *Let $p : A \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ be as in Definition A.3.5.11. Then there is a canonical isomorphism $\mathrm{Groth}(p)[W(p)^{-1}] \simeq \mathrm{hocolim}(p)$ in the homotopy category $\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}$.*

Proof. Combine Lemma A.3.5.12 with Proposition A.3.5.10. □

Lemma A.3.5.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be small $\mathbf{S}$-enriched categories. Let W be a collection of morphisms in $\mathcal{C}$ and let W' be the collection of all morphisms in $\mathcal{C} \otimes \mathcal{D}$ of the form $w \otimes \mathrm{id}_D$, where $w \in W$ and $D \in \mathcal{D}$. Then the canonical map*

$$(\mathcal{C} \otimes \mathcal{D})[W'^{-1}] \to \mathcal{C}[W^{-1}] \otimes \mathcal{D}$$

is a weak equivalence of $\mathbf{S}$-enriched categories.

Proof. It will suffice to show that for every $\mathbf{S}$-enriched category $\mathcal{A}$, the induced map

$$\phi : \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}[W^{-1}] \otimes \mathcal{D}, \mathcal{A}) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}((\mathcal{C} \otimes \mathcal{D})[W'^{-1}], \mathcal{A})$$

is bijective. Using Lemma A.3.4.6 and Corollary A.3.4.11, we can reduce to the case where $\mathcal{A} = \mathbf{A}^\circ$, where $\mathbf{A}$ is a combinatorial $\mathbf{S}$-enriched model category. We now invoke Propositions A.3.4.13 and A.3.5.5 to get a chain of bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}[W^{-1}] \otimes \mathcal{D}, \mathbf{A}^\circ) &\simeq \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}[W^{-1}], (\mathbf{A}^{\mathcal{D}})^\circ) \\
&\simeq \mathrm{Hom}^{W}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C}, (\mathbf{A}^{\mathcal{D}})^\circ) \\
&\simeq \mathrm{Hom}^{W'}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}(\mathcal{C} \otimes \mathcal{D}, \mathbf{A}^\circ) \\
&\simeq \mathrm{Hom}_{\mathrm{h}\mathcal{C}\mathrm{at}_{\mathbf{S}}}((\mathcal{C} \otimes \mathcal{D})[W'], \mathbf{A}^\circ)
\end{aligned}$$

whose composition is the map ϕ. □

Theorem A.3.5.15. *Let A be a partially ordered set and let $\mathcal{D}$ be an $\mathbf{S}$-enriched category. Then the functor $\mathcal{C} \mapsto \mathcal{C} \otimes \mathcal{D}$ commutes with A-indexed homotopy colimits. In other words, if $p : A \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ is a projectively cofibrant diagram and $p' : A \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ is defined by $p'(a) = p(a) \otimes \mathcal{D}$, then the canonical isomorphism $\varinjlim(p') \simeq \varinjlim(p) \otimes \mathcal{D}$ exhibits $\varinjlim(p) \otimes \mathcal{D}$ as a homotopy colimit of the diagram p'.*

Proof. In view of Lemma A.3.5.13, it will suffice to show that the canonical map $h : \mathrm{Groth}(p')[W(p')^{-1}] \to \mathrm{Groth}(p)[W(p)^{-1}] \otimes \mathcal{D}$ is a weak equivalence of $\mathbf{S}$-enriched categories. This is a special case of Lemma A.3.5.14. □

A.3.6 Exponentiation in Model Categories

Let $\mathcal{C}$ be a category which admits finite products, containing a pair of objects X and Y. An *exponential of* X *by* Y is an object $X^Y \in \mathcal{C}$ together with a map $e : X^Y \times Y \to X$, with the following universal property: for every object $W \in \mathcal{C}$, the composition

$$\mathrm{Hom}_{\mathcal{C}}(W, X^Y) \to \mathrm{Hom}_{\mathcal{C}}(W \times Y, X^Y \times Y) \stackrel{\circ e}{\to} \mathrm{Hom}_{\mathcal{C}}(W \times Y, Z)$$

is bijective.

Our goal in this section is to study the existence of exponentials in the homotopy category of a model category $\mathbf{A}$. Suppose we are given a pair of objects $X, Y \in \mathbf{A}$ such that there exists an exponential of $[X]$ by $[Y]$ in the homotopy category $\mathrm{h}\mathbf{A}$. We can then represent this exponential as $[Z]$ for some object $Z \in \mathbf{A}$. Without loss of generality, we may assume that X, Y, and Z are fibrant and cofibrant, so that we have a canonical identification $[Z] \times [Y] \simeq [Z \times Y]$. However, we encounter a technical difficulty: the evaluation map $[Z] \times [Y] \to [X]$ need not be representable by any morphism from $Z \times Y$ to X in the category $\mathbf{A}$ because $Z \times Y$ need not be cofibrant. We wish to work in certain contexts where this difficulty does arise (for example, where $\mathbf{A}$ is the category of simplicial categories). For this reason we are forced to work with the following somewhat cumbersome definition:

Definition A.3.6.1. Let $\mathbf{A}$ be a model category. We will say that a diagram

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr] & \\ Z \times Y & & X \end{tikzcd}$$

exhibits Z *as a weak exponential of* X *by* Y if the following conditions are satisfied:

(1) The map p exhibits P as a homotopy product of Z and Y; in other words, the induced map $[p] : [P] \to [Z] \times [Y]$ is an isomorphism in the homotopy category $\mathrm{h}\mathbf{A}$.

(2) The composition $[Z] \times [Y] \stackrel{[p]^{-1}}{\to} [P] \to [X]$ exhibits $[Z]$ as an exponential of $[X]$ by $[Y]$ in the homotopy category $\mathrm{h}\mathbf{A}$.

We will say that a map $Z \times Y \to X$ *exhibits* Z *as an exponential of* X *by* Y if the diagram

$$\begin{tikzcd} & Z \times Y \arrow[dl, "\mathrm{id}"'] \arrow[dr] & \\ Z \times Y & & X \end{tikzcd}$$

satisfies (1) and (2).

Remark A.3.6.2. Suppose we are given a diagram

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr] & \\ Z \times Y & & X \end{tikzcd}$$

as in Definition A.3.6.1. We will say that this diagram is *standard* if $X, Y, Z \in \mathbf{A}$ are fibrant, and the map p is a trivial fibration.

Suppose X and Y are fibrant objects of $\mathbf{A}$ such that there exists an exponential of $[X]$ by $[Y]$ in the homotopy category h$\mathbf{A}$. Without loss of generality, this exponential has the form $[Z]$, where Z is a fibrant object of $\mathbf{A}$. We can then choose a trivial fibration $P \to Z \times Y$, where P is cofibrant. The evaluation map $[Z \times Y] \simeq [Z] \times [Y] \to [X]$ is then representable by a map $P \to X$ in $\mathbf{A}$, so that we obtain a *standard* diagram which exhibits Z as a weak exponential of X by Y.

Remark A.3.6.3. Suppose we are given a diagram

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr] & \\ Z \times Y & & X \end{tikzcd}$$

in a model category $\mathbf{A}$. The condition that this diagram exhibits Z as a weak exponential of X by Y depends only on the image of this diagram in the homotopy category h$\mathbf{A}$. We may therefore replace the above diagram by a weakly equivalent diagram when testing whether or not the conditions of Definition A.3.6.1 are satisfied.

Remark A.3.6.4. Let $\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B}$ be a Quillen equivalence of model categories. Suppose we are given a standard diagram

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr] & \\ Z \times Y & & X \end{tikzcd}$$

in $\mathbf{B}$. Then this diagram exhibits Z as a weak exponential of X by Y in $\mathbf{B}$ if and only if the associated diagram

$$\begin{tikzcd} & GP \arrow[dl] \arrow[dr] & \\ GZ \times GY & & GX \end{tikzcd}$$

exhibits GZ as a weak exponential of GX by GY in $\mathbf{A}$.

To work effectively with weak exponentials, we need to introduce an additional assumption.

Definition A.3.6.5. Let $\mathbf{A}$ be a combinatorial model category containing a fibrant object Y. We will say that *multiplication by Y preserves homotopy colimits* if the following condition is satisfied:

($*$) Let A be a (small) partially ordered set, let $F : A \to \mathbf{A}$ be a projectively cofibrant diagram, and let $F' : A \to \mathbf{A}$ be another strongly cofibrant diagram equipped with a natural transformation $F'(a) \to F(a) \times Y$ which is weak equivalence for each $a \in A$. Then the induced map $\varinjlim F' \to (\varinjlim F) \times Y$ exhibits $\varinjlim F'$ as a homotopy product of Y with $\varinjlim F$ in $\mathbf{A}$.

We will say that *multiplication in* $\mathbf{A}$ *preserves homotopy colimits* if condition ($*$) is satisfied for every fibrant object $Y \in \mathbf{A}$.

Remark A.3.6.6. Definition A.3.6.5 refers only to homotopy colimits indexed by partially ordered sets. However, every diagram indexed by an arbitrary category can be replaced by a diagram indexed by a partially ordered set having the same homotopy colimit. We formulate and prove a precise statement to this effect (in the language of ∞-categories) in §4.2. However, we will not need (or use) any such result in this appendix.

Remark A.3.6.7. Let $\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B}$ be a Quillen equivalence between combinatorial model categories and let $Y \in \mathbf{B}$ be a fibrant object. Then multiplication by Y preserves homotopy colimits in $\mathbf{B}$ if and only if multiplication by $G(Y)$ preserves homotopy colimits in $\mathbf{A}$. Since the right derived functor RG is essentially surjective on homotopy categories, we see that multiplication in $\mathbf{B}$ preserves homotopy colimits if and only if multiplication in $\mathbf{A}$ preserves homotopy colimits.

Example A.3.6.8. Let $\mathbf{S}$ be an excellent model category with respect to the symmetric monoidal structure given by the Cartesian product in $\mathbf{S}$. Then multiplication in $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ preserves homotopy colimits. This is precisely the content of Theorem A.3.5.15.

Lemma A.3.6.9. *Let S be a collection of simplicial sets satisfying the following conditions:*

(i) *The simplicial set Δ^0 belongs to S.*

(ii) *If $f : X \to Y$ is a weak homotopy equivalence, then $X \in S$ if and only if $Y \in S$.*

(iii) *For every small partially ordered set A, if $F : A \to \mathcal{S}\mathrm{et}_\Delta$ is a projectively cofibrant diagram such that each $F(a) \in S$, then $\varinjlim F \in S$.*

Then every simplicial set belongs to S.

Proof. Using (ii) and (iii), we deduce that if $F : A \to \mathcal{S}\mathrm{et}_\Delta$ is *any* diagram such that each $F(a)$ belongs to S, then the homotopy colimit of F belongs to

S. In particular, S is closed under the formation of coproducts and homotopy pushouts.

We now prove by induction on n that every n-dimensional simplicial set X belongs to S. For this, we observe that there is a homotopy pushout diagram

$$\begin{tikzcd} B \times \partial \Delta^n \arrow[r] \arrow[d] & B \times \Delta^n \arrow[d] \\ \mathrm{sk}^{n-1} X \arrow[r] & X, \end{tikzcd}$$

where B denotes the set of n-simplices of X. The simplicial sets $B \times \partial \Delta^n$ and $\mathrm{sk}^{n-1} X$ belong to S by the inductive hypothesis. The simplicial set $B \times \Delta^n$ is weakly equivalent to the constant simplicial set B, which belongs to S in view of (i) and the fact that S is stable under coproducts. Since S is stable under homotopy pushouts, we conclude that $X \in S$, as desired.

An arbitrary simplicial set X can be written as the colimit of a projectively cofibrant diagram

$$\mathrm{sk}^0 X \subseteq \mathrm{sk}^1 X \subseteq \mathrm{sk}^2 X \subseteq \cdots$$

and therefore belongs to S by assumption (iii). $\square$

Proposition A.3.6.10. *Let $\mathbf{A}$ be a combinatorial simplicial model category containing a standard diagram*

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr] & \\ Z \times Y & & X. \end{tikzcd}$$

Assume further that multiplication by Y preserves homotopy colimits in $\mathbf{A}$. The following conditions are equivalent:

- (i) *The above diagram exhibits Z as a weak exponential of X by Y.*
- (ii) *Let W and W' be cofibrant objects of $\mathbf{A}$ and let $W' \to W \times Y$ be a map which exhibits W' as a homotopy product of W and Y. Then the induced map*

$$\mathrm{Map}_{\mathbf{A}}(W, Z) \times_{\mathrm{Map}_{\mathbf{A}}(W', Z \times Y)} \mathrm{Map}_{\mathbf{A}}(W', P) \to \mathrm{Map}_{\mathbf{A}}(W', X)$$

is a homotopy equivalence of Kan complexes.

Remark A.3.6.11. In the situation of part (ii) of Proposition A.3.6.10, the projection map $\mathrm{Map}_{\mathbf{A}}(W', P) \to \mathrm{Map}_{\mathbf{A}}(W', Z \times Y)$ is a trivial Kan fibration, so the fiber product $\mathrm{Map}_{\mathbf{A}}(W, Z) \times_{\mathrm{Map}_{\mathbf{A}}(W', Z \times Y)} \mathrm{Map}_{\mathbf{A}}(W', P)$ is automatically a Kan complex which is homotopy equivalent to $\mathrm{Map}_{\mathbf{A}}(W, Z)$.

Proof of Proposition A.3.6.10. First suppose that (ii) is satisfied. We wish to show that for every object $[W] \in \mathrm{h}\mathbf{A}$, the composition

$$\begin{aligned} \mathrm{Hom}_{\mathrm{h}\mathbf{A}}([W], [Z]) &\to \mathrm{Hom}_{\mathrm{h}\mathbf{A}}([W] \times [Y], [Z] \times [Y]) \\ &\simeq \mathrm{Hom}_{\mathrm{h}\mathbf{A}}([W] \times [Y], [P]) \\ &\to \mathrm{Hom}_{\mathrm{h}\mathbf{A}}([W] \times [Y], [P]) \end{aligned}$$

is bijective. Without loss of generality, we may assume that $[W]$ is the homotopy equivalence class of a fibrant-cofibrant object $W \in \mathbf{A}$. Choose a cofibrant replacement $W' \to W \times Y$. We observe that the map in question can be identified with the composition

$$\begin{aligned}\pi_0 \operatorname{Map}_{\mathbf{A}}(W, Z) &\to \pi_0 \operatorname{Map}_{\mathbf{A}}(W', Z \times Y)\\ &\simeq \pi_0 \operatorname{Map}_{\mathbf{A}}(W', P)\\ &\to \pi_0 \operatorname{Map}_{\mathbf{A}}(W', X),\end{aligned}$$

which is bijective in view of (ii) and Remark A.3.6.11.

We now assume (i) and prove (ii). It will suffice to show that for every simplicial set K, the induced map

$$\begin{array}{c}\operatorname{Hom}_{\mathrm{h}\mathcal{S}\mathrm{et}_{\Delta}}(K, \operatorname{Map}_{\mathbf{A}}(W, Z) \times_{\operatorname{Map}_{\mathbf{A}}(W', Z \times Y)} \operatorname{Map}_{\mathbf{A}}(W', P))\\ \downarrow\\ \operatorname{Hom}_{\mathrm{h}\mathcal{S}\mathrm{et}_{\Delta}}(K, \operatorname{Map}_{\mathbf{A}}(W', X))\end{array}$$

is a bijection. Using Remark A.3.6.11, we can identify the left side with the set $\operatorname{Hom}_{\mathrm{h}\mathcal{S}\mathrm{et}_{\Delta}}(K, \operatorname{Map}_{\mathbf{A}}(W, Z)) \simeq \operatorname{Hom}_{\mathrm{h}\mathbf{A}}(W \otimes K, Z)$. Similarly, the right side can be identified with $\operatorname{Hom}_{\mathrm{h}\mathbf{A}}(W' \otimes K, X)$. In view of assumption (i), it will suffice to show that the map $\beta_K : W' \otimes K \to Y \times (W \otimes K)$ exhibits $W' \otimes K$ as a homotopy product of Y and $W \otimes K$. The collection of simplicial sets K with this property clearly contains Δ^0 and is stable under weak homotopy equivalence. The assumption that multiplication by Y preserves homotopy colimits guarantees that the hypotheses of Lemma A.3.6.9 are satisfied, so that the desired conclusion holds for *every* simplicial set K. □

Lemma A.3.6.12. *Let $\mathbf{A}$ be a combinatorial model category and $i : B_0 \to B$ an inclusion of partially ordered sets. Suppose that there exists a retraction $r : B \to B_0$ such that $r(b) \leq b$ for each $b \in B$. Let $F : B \to \mathbf{A}$ be a diagram. Then a map $\alpha : X \to \lim(F)$ in $\mathbf{A}$ exhibits X as a homotopy limit of F if and only if α exhibits X as a homotopy limit of i^*F.*

Proof. Without loss of generality, we may assume that F is injectively fibrant. We have a canonical isomorphism $\lim(F) \simeq \lim(i^*F)$. It will therefore suffice to show that the functor i^* preserves injective fibrations. It now suffices to observe that i^* is right adjoint to r^* and that the functor r^* preserves injective trivial cofibrations. □

Lemma A.3.6.13. *Let $\mathbf{A}$ be a combinatorial model category, $\mathcal{C}$ a small category, $F : \mathcal{C} \to \mathbf{A}$ a diagram, and $\alpha : X \to \lim(F)$ a morphism in the category $\mathbf{A}$. Suppose that*

(i) *For each $C \in \mathcal{C}$, the induced map $X \to F(C)$ is a weak equivalence in $\mathbf{A}$.*

(ii) *The category $\mathcal{C}$ has a final object C_0.*

Then α exhibits X as a homotopy limit of the diagram F.

Proof. Without loss of generality, we may assume that the diagram F is projectively fibrant. Let $F' : \mathcal{C} \to \mathbf{A}$ be defined by the formula $F'(C) = F(C_0)$. We observe that, for every $G \in \mathrm{Fun}(\mathcal{C}, \mathbf{A})$, we have $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{A})}(G, F') = \mathrm{Hom}_{\mathbf{A}}(G(C_0), F(C_0))$. In particular, we have a canonical map $\beta : F \to F'$. Condition (i) guarantees that β is a weak equivalence. Since $F(C_0) \in \mathbf{A}$ is fibrant, the diagram F' is injectively fibrant. It therefore suffices to show that the induced map $X \to \lim(F') \simeq F(C_0)$ is a weak equivalence, which follows from (i). $\square$

Lemma A.3.6.14. *Let $\mathbf{A}$ be a combinatorial model category, let A be a partially ordered set, and set $B = \{(a, b) \in A^{op} \times A : a \geq b\}$, regarded as a partially ordered subset of $A^{op} \times A$. Let $\pi : B \to A^{op}$ denote the projection onto the first factor.*

Suppose we are given diagrams $F : B \to \mathbf{A}$, $G : A \to \mathbf{A}$, and a natural transformation $\alpha : \pi^*(G) \to F$, which induces weak equivalences $G(a) \to F(a, b)$ for each $(a, b) \in B$. Then α exhibits G as a homotopy right Kan extension of F.

Proof. In view of Proposition A.2.8.9, it will suffice to show that for each $a_0 \in A$, the transformation α exhibits $G(a_0)$ as a homotopy limit of the diagram $F|\{(a, b) \in B : a \leq a_0\}$. Let $B_0 = \{(a, b) \in B : a = a_0\}$. In view of Lemma A.3.6.12, it will suffice to show that α exhibits $G(a_0)$ as a limit of the diagram $F_0 = F|B_0$. This follows immediately from Lemma A.3.6.13. $\square$

Proposition A.3.6.15. *Let $\mathbf{A}$ be a combinatorial model category and $A = A_0 \cup \{\infty\}$ a partially ordered set with a largest element ∞. Let $B = \{(a, b) \in A^{op} \times A : a \geq b\}$, regarded as a partially ordered subset of $A^{op} \times A$.*

Suppose we are given an object $X \in \mathbf{A}$ together with functors $Y : A \to \mathbf{A}$, $Z : A^{op} \to \mathbf{A}$, $P : B \to \mathbf{A}$, and diagrams $\sigma_{a,b}$:

$$\begin{tikzcd} & P(a,b) \arrow[dl] \arrow[dr] & \\ Z(a) \times Y(b) & & X, \end{tikzcd}$$

which depend functorially on $(a, b) \in B$. Suppose further that

(i) *Each diagram $\sigma_{a,b}$ exhibits $P(a, b)$ as a homotopy product of $Z(a)$ and $Y(b)$ in $\mathbf{A}$.*

(ii) *The diagrams $\sigma_{a,a}$ exhibit $Z(a)$ as a weak exponential of X by $Y(a)$.*

(iii) *Multiplication in $\mathbf{A}$ preserves homotopy colimits.*

(iv) *The diagram Y exhibits $Y(\infty)$ as a homotopy colimit of $Y_0 = Y|A_0$.*

Then the diagram Z exhibits $Z(\infty)$ as the homotopy limit of the diagram $Z_0 = Z|A_0^{op}$.

Proof. Making fibrant replacements if necessary, we may assume that each diagram $\sigma_{a,b}$ is standard. According to the main result of [19], there exists a Quillen equivalence $\mathbf{A}' \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{A}$, where $\mathbf{A}'$ is a combinatorial *simplicial* model category. In view of Remark A.3.6.4, we may replace $\mathbf{A}$ by $\mathbf{A}'$ and thereby reduce to the case where $\mathbf{A}$ is a simplicial model category.

In view of Proposition A.3.3.12, it will suffice to prove the following: for every fibrant-cofibrant object $C \in \mathbf{A}$, if we define $G : A^{op} \to \mathcal{S}\mathrm{et}_\Delta$ by the formula $G(a) = \mathrm{Map}_{\mathbf{A}}(C, Z(a))$, then G exhibits $G(\infty)$ as a homotopy limit of the diagram $G|A_0^{op}$.

Let $W : A \to \mathbf{A}$ be a cofibrant replacement for the functor $a \mapsto C \times Y(a)$. Let $G' : A^{op} \to \mathcal{S}\mathrm{et}_\Delta$ be defined by the formula $G'(a) = \mathrm{Map}_{\mathbf{A}}(W(a), X)$.

Define $G'' : B \to \mathcal{S}\mathrm{et}_\Delta$ by the formula

$$G''(a,b) = \mathrm{Map}_{\mathbf{A}}(C, Z(a)) \times_{\mathrm{Map}_{\mathbf{A}}(W(a), Z(a) \times Y(b))} \mathrm{Map}_{\mathbf{A}}(W(a), P(a,b)).$$

Let $\pi : B \to A^{op}$ denote projection onto the first factor, so that we have natural transformations of diagrams

$$\pi^* G \overset{\alpha}{\leftarrow} G'' \overset{\beta}{\rightarrow} \pi^* G'.$$

We observe that β induces a trivial Kan fibration $G''(a,b) \to G'(a)$ for all $(a,b) \in B$. In particular, for $a \leq b$ the induced map $G''(a,a) \to G''(a,b)$ is a homotopy equivalence. Condition (ii) guarantees that α induces a homotopy equivalence $G''(a,b) \to G(a)$ if $a = b$ and therefore for all $(a,b) \in B$.

Using Lemma A.3.6.14, we conclude that α and β exhibit G and G' as homotopy right Kan extensions of G'' along π. In particular, G and G' are equivalent in the homotopy category $\mathrm{hFun}(A^{op}, \mathbf{A})$. Assumptions (iii) and (iv) guarantee that W exhibits $W(\infty)$ as the homotopy colimit of $W|A_0$. Using Proposition A.3.3.12, we deduce that G' exhibits $G'(\infty)$ as the homotopy limit of $G'|A_0^{op}$. It follows that G exhibits $G(\infty)$ as the homotopy limit of $G|A_0^{op}$, as desired. $\square$

We conclude this section with an application of Proposition A.3.6.15.

Proposition A.3.6.16. *Let* $\mathbf{S}$ *be an excellent model category in which the monoidal structure is given by the Cartesian product. Let* $\mathbf{A}$ *be a combinatorial* $\mathbf{S}$*-enriched model category,* $A = A_0 \cup \{\infty\}$ *a partially ordered set with a largest element* ∞*, and* $\{\mathcal{C}_a\}_{a \in A}$ *a diagram of small* $\mathbf{S}$*-enriched categories indexed by* A*. Let* $\mathcal{U} \subseteq \mathbf{A}$ *be a chunk. For each* $a \in A$*, let* $\mathcal{U}_f^{\mathcal{C}_a}$ *denote the full subcategory of* $\mathcal{U}^{\mathcal{C}_a} \subseteq \mathbf{A}^{\mathcal{C}_a}$ *spanned by the projectively fibrant diagrams and let* W_a *denote the collection of weak equivalences in* $\mathcal{V}_a$*. Assume that*

- (a) *For each* $a \in A$*, the* $\mathbf{S}$*-enriched category* $\mathcal{C}_a$ *is cofibrant and* $\mathcal{U}$ *is a* $\mathcal{C}_a$*-chunk of* $\mathbf{A}$*.*
- (b) *The diagram* $\{\mathcal{C}_a\}_{a \in A}$ *exhibits* $\mathcal{C}_\infty$ *as a homotopy colimit of the diagram* $\{\mathcal{C}_a\}_{a \in A_0}$*.*
- (c) *The chunk* $\mathcal{U}$ *is small.*

Then the induced diagram $\{\mathcal{U}_f^{\mathcal{C}_a}[W_a^{-1}]\}_{a \in A}$ *exhibits* $\mathcal{U}_f^{\mathcal{C}_\infty}[W_\infty^{-1}]$ *as a homotopy limit of the diagram* $\{\mathcal{U}_f^{\mathcal{C}_a}[W_a^{-1}]\}_{a \in A_0}$.

Before proving Proposition A.3.6.16, we need a simple lemma.

Lemma A.3.6.17. *Let* $\mathbf{S}$ *be an excellent model category,* $\mathbf{A}$ *a combinatorial* $\mathbf{S}$*-enriched model category, and* $\mathcal{U} \subseteq \mathbf{A}$ *a chunk. Let* $\mathcal{U}_f$ *denote the full subcategory of* $\mathcal{U}$ *spanned by those objects which are fibrant in* $\mathbf{A}$ *and let* W *denote the collection of weak equivalences in* $\mathcal{U}_f$*. Then the induced map* $\mathcal{U}^\circ \to \mathcal{U}_f[W^{-1}]$ *is a weak equivalence of* $\mathbf{S}$*-enriched categories.*

Proof. Let $W_0 = W \cap \mathcal{U}^\circ$. Since every weak equivalence in $\mathcal{U}^\circ$ is actually an equivalence, we conclude that the induced map $\mathcal{U}^\circ \to \mathcal{U}^\circ[W_0^{-1}]$ is a weak equivalence. It will therefore suffice to prove that the map $i : \mathcal{U}^\circ[W_0^{-1}] \to \mathcal{U}_f[W^{-1}]$ is a weak equivalence. Let F be an $\mathbf{S}$-enriched cofibrant replacement functor which carries $\mathcal{U}$ to itself, so that F induces a map $j : \mathcal{U}_f[W^{-1}]$ to $\mathcal{U}^\circ[W_0^{-1}]$. We claim that j is a homotopy inverse to i. To prove this, we observe that there is a natural transformation $\alpha : F \to \mathrm{id}$, which we can identify with a map

$$\overline{h} : \mathcal{U}_f \otimes [1]_{\mathbf{S}} \to \mathcal{U}_f \,.$$

Let W_0' be the collection of all morphisms in $\mathcal{U}_f \otimes [1]_{\mathbf{S}}$ of the form $e \otimes \mathrm{id}$, where e is an equivalence in $\mathcal{U}_f$, and let W_1' be the collection of all morphisms of $\mathcal{U}_f \otimes [1]_{\mathbf{S}}$ of the form $\mathrm{id} \otimes g$, where $g : 0 \to 1$ is the tautological morphism in $[1]_{\mathbf{S}}$. Let $W' = W_0' \cup W_1'$, so that $\overline{h}$ determines a map

$$h : (\mathcal{U}_f \otimes [1]_{\mathbf{S}})[W'^{-1}] \to \mathcal{U}_f[W^{-1}].$$

We will prove that h determines a homotopy from the identity to $j \circ i$, so that j is a left homotopy inverse to i. Applying the same argument to the restriction $\overline{h}| \,\mathcal{U}^\circ \otimes [1]_{\mathbf{S}}$ will show that j is a right homotopy inverse to i.

To prove that h gives the desired homotopy, it will suffice to show that the inclusions $\{0\}, \{1\} \hookrightarrow [1]_{\mathbf{S}}$ induce weak equivalences

$$\mathcal{U}_f[W^{-1}] \to (\mathcal{U}_f \otimes [1]_{\mathbf{S}})[W'^{-1}].$$

This follows immediately from Corollary A.3.4.11 and Lemma A.3.5.14. □

Proof of Proposition A.3.6.16. Let $B = \{(a, b) \in A^{op} \times A : a \geq b\}$. For each $(a, b) \in B$, we define $P(a, b) = (\mathcal{U}_f^{\mathcal{C}_a} \times \mathcal{C}_b)[V_{a,b}^{-1}]$, where $V_{a,b}$ is the collection of all morphisms of $\mathcal{U}_f^{\mathcal{C}_a} \times \mathcal{C}_b$ of the form $e \otimes \mathrm{id}_C$, where $e \in W_a$ and $C \in \mathcal{C}_b$. We have an evident family of diagrams $\sigma(a, b)$:

$$\begin{tikzcd} & P(a,b) \arrow[dl] \arrow[dr] & \\ \mathcal{U}_f^{\mathcal{C}_a}[W_a^{-1}] \times \mathcal{C}_b & & \mathcal{U}_f[W], \end{tikzcd}$$

where $\mathcal{U}_f$ denotes the full subcategory of $\mathcal{U}$ spanned by the fibrant objects and W is the collection of weak equivalences in $\mathcal{U}_f \subseteq \mathbf{A}$. To complete the proof, it will suffice to show that the hypotheses of Proposition A.3.6.15 are satisfied. Condition (i) follows from Lemma A.3.5.14, condition (iii) from Theorem A.3.5.15, and condition (iv) from (b). To prove (ii), we observe that for each $a \in A$, the diagram $\sigma(a,a)$ is weakly equivalent to the diagram

$$\begin{tikzcd} & (\mathcal{U}^{\mathcal{C}_a})^\circ \times \mathcal{C}_a \arrow[dl, "\sim"'] \arrow[dr] & \\ (\mathcal{U}^{\mathcal{C}_a})^\circ \times \mathcal{C}_a & & \mathcal{U}^\circ . \end{tikzcd}$$

This diagram exhibits $(\mathcal{U}^{\mathcal{C}_a})^\circ$ as a weak exponential of $\mathcal{U}^\circ$ by $\mathcal{C}_a$ by Corollary A.3.4.14. $\square$

Corollary A.3.6.18. *Let* $\mathbf{S}$ *be an excellent model category in which the monoidal structure is given by the Cartesian product. Let* $\mathbf{A}$ *be a combinatorial* $\mathbf{S}$*-enriched model category, let* $A = A_0 \cup \{\infty\}$ *be a partially ordered set with a largest element* ∞*, and let* $\{\mathcal{C}_a\}_{a\in A}$ *be a diagram of small* $\mathbf{S}$*-enriched categories indexed by* A*.*

For each $a \in A$*, let* $\mathbf{A}_f^{\mathcal{C}_a}$ *denote the collection of projectively fibrant objects of* $\mathbf{A}^{\mathcal{C}_a}$ *and let* W_a *denote the collection of weak equivalences in* $\mathbf{A}_f^{\mathcal{C}_a}$*. Assume that the diagram* $\{\mathcal{C}_a\}_{a\in A}$ *exhibits* $\mathcal{C}_\infty$ *as a homotopy colimit of the diagram* $\{\mathcal{C}_a\}_{a\in A_0}$*. Then the induced diagram* $\{\mathbf{A}_f^{\mathcal{C}_a}[W_a^{-1}]\}_{a\in A}$ *exhibits* $\mathbf{A}^{\mathcal{C}_\infty}[W_\infty^{-1}]$ *as a homotopy limit of the diagram* $\{\mathbf{A}_f^{\mathcal{C}_a}[W_a^{-1}]\}_{a\in A_0}$*.*

Proof. Without loss of generality, we may suppose that each $\mathcal{C}_a$ is cofibrant. The proof of Proposition A.2.8.2 shows that there exists a (small) regular cardinal κ such that the collection of homotopy limit diagrams in $\mathrm{Fun}(A, \mathcal{C}\mathrm{at}_{\mathbf{S}})$ is stable under κ-filtered colimits. This cardinal depends only on A and $\mathbf{S}$ and remains invariant if we enlarge the universe. Using Lemma A.3.4.15, we can write $\mathbf{A}$ as a κ-filtered union of full subcategories $\mathcal{U} \subseteq \mathbf{A}$, where $\mathcal{U}$ is a $\mathcal{C}_a$-chunk for each $a \in A$. We now conclude by applying Proposition A.3.6.16. $\square$

A.3.7 Localizations of Simplicial Model Categories

Let $\mathbf{A}$ and $\mathbf{A}'$ be two model categories with the same underlying category. We say that $\mathbf{A}'$ is a (*Bousfield*) *localization* of $\mathbf{A}$ if the following conditions are satisfied:

- (C) A morphism f of $\mathcal{C}$ is a cofibration in $\mathbf{A}$ if and only if f is a cofibration in $\mathbf{A}'$.
- (W) If a morphism f of $\mathcal{C}$ is a weak equivalence in $\mathbf{A}$, then f is a weak equivalence in $\mathbf{A}'$.

It then also follows that

- (F) If a morphism f of $\mathcal{C}$ is a fibration in $\mathbf{A}'$, then f is a fibration in $\mathbf{A}$.

Our goal in this section is to study the localizations of a fixed model category $\mathbf{A}$ and to relate this to our study of localizations of presentable ∞-categories (§5.5.4).

Let $\mathbf{A}$ be a simplicial model category. Let $\mathrm{h}\mathbf{A}$ be the homotopy category of $\mathbf{A}$ obtained from $\mathbf{A}$ by inverting all weak equivalences. Alternatively, we can obtain $\mathrm{h}\mathbf{A}$ by first passing to the full subcategory $\mathbf{A}^\circ \subseteq \mathbf{A}$ spanned by the fibrant-cofibrant objects and then passing to the homotopy category of the simplicial category $\mathbf{A}^\circ$. From the second point of view, we see that $\mathrm{h}\mathbf{A}$ has a natural enrichment over the homotopy category $\mathcal{H}$: if $X, Y \in \mathrm{h}\mathbf{A}$ are represented by fibrant-cofibrant objects $\overline{X}, \overline{Y} \in \mathbf{A}$, then we let

$$\mathrm{Map}_{\mathrm{h}\mathbf{A}}(X, Y) = [\mathrm{Map}_{\mathbf{A}}(\overline{X}, \overline{Y})].$$

Here $[K] \in \mathcal{H}$ denotes the object of $\mathcal{H}$ represented by a Kan complex K. In fact, this description is accurate if we assume only that $\overline{X}$ is cofibrant and $\overline{Y}$ fibrant.

Let S be a collection of morphisms in $\mathrm{h}\mathbf{A}$. Then

(i) We will say that an object $Z \in \mathrm{h}\mathbf{A}$ is *S-local* if, for every morphism $f : X \to Y$ in S, the induced map

$$\mathrm{Map}_{\mathrm{h}\mathbf{A}}(Y, Z) \to \mathrm{Map}_{\mathrm{h}\mathbf{A}}(X, Z)$$

is a homotopy equivalence. We say that an object $\overline{Z} \in \mathbf{A}$ is S-local if its image in $\mathrm{h}\mathbf{A}$ is S-local.

(ii) We will say that a morphism $f : X \to Y$ of $\mathrm{h}\mathbf{A}$ is an *S-equivalence* if, for every S-local object $Z \in \mathrm{h}\mathbf{A}$, the induced map

$$\mathrm{Map}_{\mathrm{h}\mathbf{A}}(Y, Z) \to \mathrm{Map}_{\mathrm{h}\mathbf{A}}(X, Z)$$

is a homotopy equivalence. We say that a morphism $\overline{f}$ in $\mathbf{A}$ is an *S-equivalence* if its image in $\mathrm{h}\mathbf{A}$ is an S-equivalence.

If $\overline{S}$ is a collection of morphisms in $\mathbf{A}$ with image S in $\mathrm{h}\mathbf{A}$, we will apply the same terminology: an object of $\mathbf{A}$ (or $\mathrm{h}\mathbf{A}$) is said to be *$\overline{S}$-local* if it is S-local, and a morphism of $\mathbf{A}$ (or $\mathrm{h}\mathbf{A}$) is said to be an *$\overline{S}$-equivalence* if it is an S-equivalence.

Lemma A.3.7.1. *Let $\mathbf{A}$ be a left proper simplicial model category, let S be a collection of morphisms in $\mathrm{h}\mathbf{A}$, and let $i : A \to B$ be a cofibration in $\mathbf{A}$. The following conditions are equivalent:*

(1) *The map i is an S-equivalence.*

(2) *For every fibrant object $X \in \mathbf{A}$ which is S-local, the map i induces a trivial Kan fibration $\mathrm{Map}_{\mathbf{A}}(B, X) \to \mathrm{Map}_{\mathbf{A}}(A, X)$.*

Proof. Choose a trivial fibration $f : A' \to A$, where A' is cofibrant, and choose a factorization

$$A' \xrightarrow{i'} B' \xrightarrow{f'} B$$

of $i \circ f$, where i' is a cofibration and f' is a trivial fibration. We have a commutative diagram

$$\begin{tikzcd} A' \arrow[r, "i'"] \arrow[d, "f"] & B' \arrow[d, "g"] \arrow[dr, "f'"] & \\ A \arrow[r, "i"] & A \coprod_{A'} B' \arrow[r, "j"] & B. \end{tikzcd}$$

Since f is a weak equivalence and i' is a cofibration, the left properness of $\mathbf{A}$ guarantees that g is a weak equivalence. It follows from the two-out-of-three property that j is also a weak equivalence.

Suppose first that (1) is satisfied. Let X be an S-local fibrant object of $\mathbf{A}$. The map $p : \mathrm{Map}_{\mathbf{A}}(B, X) \to \mathrm{Map}_{\mathbf{A}}(A, X)$ is a Kan fibration. We wish to show that p is a trivial Kan fibration. Our assumption that X is S-local guarantees that the map $q' : \mathrm{Map}_{\mathbf{A}}(B', X) \to \mathrm{Map}_{\mathbf{A}}(A', X)$ is a homotopy equivalence and therefore a trivial fibration (since i' is a cofibration). The map

$$q : \mathrm{Map}_{\mathbf{A}}(A \coprod_{A'} B', X) \to \mathrm{Map}_{\mathbf{A}}(A)$$

is a pullback of q' and therefore also a trivial fibration. To show that p is a trivial Kan fibration, it will suffice to show that for every $t : A \to X$, the fiber $p^{-1}\{t\}$ is a contractible Kan complex. Since the corresponding fiber $q^{-1}\{t\}$ is contractible, it will suffice to show that composition with j induces a homotopy equivalence

$$p^{-1}\{t\} \to q^{-1}\{t\}.$$

This is clear since j is a weak equivalence between cofibrant objects of the simplicial model category $\mathbf{A}_{A/}$.

Now assume that (2) holds. We wish to show that i is an S-equivalence. For this, it suffices to show that for every fibrant S-local object $X \in \mathbf{A}$, the map

$$q' : \mathrm{Map}_{\mathbf{A}}(B', X) \to \mathrm{Map}_{\mathbf{A}}(A', X)$$

is a trivial Kan fibration. The preceding argument shows that the fiber of q' over a morphism $t' : A' \to X$ is contractible, provided that t' factors as a composition

$$A' \xrightarrow{f} A \xrightarrow{t} X.$$

To complete the proof, it suffices to show that the same result holds for an arbitrary vertex t' of $\mathrm{Map}_{\mathbf{A}}(A', X)$. The map t' factors as a composition

$$A' \xrightarrow{u} Y \xrightarrow{v} X,$$

where u is a cofibration and v is a trivial fibration. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathbf{A}}(B', Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbf{A}}(A', Y) \arrow[d] \\ \mathrm{Map}_{\mathbf{A}}(B', X) \arrow[r] & \mathrm{Map}_{\mathbf{A}}(A', X) \end{tikzcd}$$

in which the vertical arrows are trivial Kan fibrations. It will therefore suffice to show that the fiber $\mathrm{Map}_{\mathbf{A}}(B', Y) \times_{\mathrm{Map}_{\mathbf{A}}(A',Y)} \{u\}$ is contractible. Choose a trivial cofibration $A' \coprod_A Y \to Z$, where Z is fibrant. We observe that the map $Y \to A' \coprod_A Y$ is the pushout of a weak equivalence by a cofibration and therefore a weak equivalence (since $\mathbf{A}$ is left proper). It follows that the map $Y \to Z$ is a weak equivalence between fibrant objects of $\mathbf{A}$. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathbf{A}}(B', Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbf{A}}(A', Y) \arrow[d] \\ \mathrm{Map}_{\mathbf{A}}(B', Z) \arrow[r, "q''"] & \mathrm{Map}_{\mathbf{A}}(A', Z) \end{tikzcd}$$

in which the vertical maps are homotopy equivalences and the horizontal maps are Kan fibrations. It will therefore suffice to show that the fiber of q'' is contractible when taken over the composite map $t'' : A' \stackrel{u}{\to} Y \to Z$. We now observe that t'' factors through A, so that the desired result follows from the first part of the proof. □

Corollary A.3.7.2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be simplicial model categories and suppose we are given a simplicial adjunction*

$$\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B}.$$

Assume that $\mathbf{B}$ *is left proper. The following conditions are equivalent:*

(1) *The adjunction between* F *and* G *is a Quillen adjunction.*

(2) *The functor* F *preserves cofibrations, and the functor* G *preserves fibrant objects.*

Proof. The implication (1) ⇒ (2) is obvious. Conversely, suppose that (2) is satisfied. We wish to prove that F is a left Quillen functor. Since F preserves cofibrations, it will suffice to show that for every trivial cofibration $u : A \to A'$ in $\mathbf{A}$, the image Fu is a weak equivalence in $\mathbf{B}$. Applying Lemma A.3.7.1 in the case $S = \emptyset$, it will suffice to prove the following: for every fibrant object $B \in \mathbf{B}$, the induced map

$$\mathrm{Map}_{\mathbf{B}}(FA', B) \to \mathrm{Map}_{\mathbf{B}}(FA, B)$$

is a trivial Kan fibration. Since F and G are adjoint simplicial functors, this is equivalent to the requirement that the map $\mathrm{Map}_{\mathbf{A}}(A', GB) \to \mathrm{Map}_{\mathbf{A}}(A, GB)$ be a trivial Kan fibration, which follows from our assumption that u is a trivial cofibration in $\mathbf{A}$ and that $GB \in \mathbf{A}$ is fibrant. □

Proposition A.3.7.3. *Let* $\mathbf{A}$ *be a left proper combinatorial simplicial model category and let* S *be a (small) set of cofibrations in* $\mathbf{A}$. *Let* $S^{-1}\mathbf{A}$ *denote the same category, with the following distinguished classes of morphisms:*

(C) *A morphism g in $S^{-1}\mathbf{A}$ is a cofibration if it is a cofibration when regarded as a morphism in $\mathbf{A}$.*

(W) *A morphism g in $S^{-1}\mathbf{A}$ is a weak equivalence if it is an S-equivalence.*

Then

(1) *The above definitions endow $S^{-1}\mathbf{A}$ with the structure of a combinatorial simplicial model category.*

(2) *The model category $S^{-1}\mathbf{A}$ is left proper.*

(3) *An object $X \in \mathbf{A}$ is fibrant in $S^{-1}\mathbf{A}$ if and only if X is S-local and fibrant in $\mathbf{A}$.*

Proof. Enlarging S if necessary, we may assume:

(a) For every morphism $f : A \to B$ in S and every $n \geq 0$, the induced map

$$(A \times \Delta^n) \coprod_{A \times \partial \Delta^n} (B \times \partial \Delta^n) \to B \times \Delta^n$$

belongs to S.

(b) The set S contains a collection of generating trivial cofibrations for $\mathbf{A}$.

It follows that an object $X \in \mathbf{A}$ is fibrant and S-local if and only if it has the extension property with respect to every morphism in S. Since $S \subseteq C \cap W$, we deduce that every fibrant object of $S^{-1}\mathbf{A}$ is S-local and fibrant in $\mathbf{A}$. The converse follows from Lemma A.3.7.1; this proves (3).

To prove (1), it will suffice to show that the classes C and W satisfy the hypotheses of Proposition A.2.6.8 (the compatibility of the simplicial structure on $S^{-1}\mathbf{A}$ with its model structure follows immediately from Proposition A.3.1.7). We observe that Lemma A.3.7.1 implies that $C \cap W$ is a weakly saturated class of morphisms in $\mathbf{A}$. The only other nontrivial point is to show that W is an accessible subcategory of $\mathbf{A}^{[1]}$.

Proposition A.1.2.5 implies the existence of a functor $T : \mathbf{A} \to \mathbf{A}$ and a natural transformation $\mathrm{id}_{\mathbf{A}} \to T$ having the following properties:

(i) For every $X \in \mathbf{A}$, the object $TX \in \mathbf{A}$ is fibrant and S-local.

(ii) For every $X \in \mathbf{A}$, the map $X \to TX$ belongs to the smallest weakly saturated class of morphisms containing S; in particular, it belongs to $W \cap C$ and is therefore an S-equivalence.

(iii) There exists a regular cardinal κ such that T commutes with κ-filtered colimits.

It follows that a morphism $f : X \to Y$ in $\mathbf{A}$ is an S-equivalence if and only if the induced map $Tf : TX \to TY$ is an S-equivalence. Since TX and TY are S-local, Yoneda's lemma (in the category h$\mathbf{A}$) implies that Tf is an S-equivalence if and only if Tf is a weak equivalence in $\mathbf{A}$. It follows that

W is the inverse image under T of the collection of weak equivalences in $\mathbf{A}$. Corollaries A.2.6.5 and A.2.6.6 imply that W is an accessible subcategory of $\mathbf{A}^{[1]}$, as desired. This completes the proof of (1).

We now prove (2). We need to show that the collection of S-equivalences in $\mathbf{A}$ is stable under pushouts by cofibrations. We observe that every morphism $f : X \to Z$ admits a factorization

$$X \stackrel{f'}{\to} Y \stackrel{f''}{\to} Z,$$

where f' is a cofibration and f'' is a weak equivalence in $\mathbf{A}$ (in fact, we can choose f'' to be a trivial fibration in $\mathbf{A}$). If f is an S-equivalence, then f' is an S-equivalence, so that $f' \in C \cap W$. It will therefore suffice to show that $C \cap W$ and the class of weak equivalences in $\mathbf{A}$ are stable under pushouts by cofibrations. The first follows from the assertion that $C \cap W$ is weakly saturated, and the second from the assumption that $\mathbf{A}$ is left proper. $\square$

Proposition A.3.7.4. *Let $\mathbf{A}$ be a left proper combinatorial simplicial model category. Then*

(1) *Every combinatorial localization of $\mathbf{A}$ has the form $S^{-1}\mathbf{A}$, where S is some (small) set of cofibrations in $\mathbf{A}$.*

(2) *Given two (small) sets of cofibrations S and T, the localizations $S^{-1}\mathbf{A}$ and $T^{-1}\mathbf{A}$ coincide if and only if the class of S-local objects of $\mathrm{h}\mathbf{A}$ coincides with the class of T-local objects of $\mathrm{h}\mathbf{A}$.*

Proof. The "if" direction of (2) is obvious, and the converse follows from the characterization of the fibrant objects of $S^{-1}\mathbf{A}$ given in Proposition A.3.7.3. We now prove (1). Let $\mathbf{B}$ be a combinatorial model category which is a localization of $\mathbf{A}$ and let S be a set of generating trivial cofibrations for $\mathbf{B}$. We claim that $\mathbf{B} = S^{-1}\mathbf{A}$. The cofibrations of $S^{-1}\mathbf{A}$ and $\mathbf{B}$ coincide. Moreover, the collection of trivial cofibrations in $S^{-1}\mathbf{A}$ is a weakly saturated class of morphisms which contains S and therefore contains every trivial cofibration in $\mathbf{B}$. To complete the proof, it will suffice to show that every trivial cofibration $f : X \to Y$ in $S^{-1}\mathbf{A}$ is a trivial cofibration in $\mathbf{B}$.

Choose a diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where X' is cofibrant, f' is a cofibration, and the vertical maps are weak equivalences in $\mathbf{A}$. Then f' is a trivial cofibration in $S^{-1}\mathbf{A}$, and it will suffice to show that f' is a trivial cofibration in $\mathbf{B}$. For this, it will suffice to show that for every fibrant object $Z \in \mathbf{B}$, the map

$$\mathrm{Map}_{\mathbf{B}}(Y', Z) \to \mathrm{Map}_{\mathbf{B}}(X', Z)$$

is a trivial fibration. In view of Lemma A.3.7.1, it will suffice to show that Z is S-local and fibrant as an object of $\mathbf{A}$. The second claim is obvious, and the first follows from the fact that S consists of trivial cofibrations in $\mathbf{B}$. $\square$

Remark A.3.7.5. In the situation of Proposition A.3.7.4, we may assume that for every cofibration $f : A \to B$ in S, the objects A and B are themselves cofibrant. To see this, choose for each cofibration $f : A \to B$ in S a diagram

$$\begin{tikzcd} A' \arrow[r, "g_f"] \arrow[d, "u"] & B' \arrow[d, "v"] \arrow[dr, "w"] & \\ A \arrow[r, "f'"] & A \coprod_{A'} B' \arrow[r, "f''"] & B \end{tikzcd}$$

as in the proof of Lemma A.3.7.1, so that u and w are trivial cofibrations, $f = f'' \circ f'$, and g_f is a cofibration between cofibrant objects. Then g_f is a trivial cofibration in $S^{-1}\mathbf{A}$. We claim that the localizations $S^{-1}\mathbf{A}$ and $T^{-1}\mathbf{A}$ coincide, where $T = \{g_f\}_{f \in S}$. To prove this, it will suffice to show that for each $f \in S$, every g_f-local fibrant object $X \in \mathbf{A}$ is also f-local.

Suppose that X is g_f-local. We wish to prove that the map

$$p : \mathrm{Map}_{\mathbf{A}}(B, X) \to \mathrm{Map}_{\mathbf{A}}(A, X)$$

is a trivial Kan fibration. Since p is automatically a Kan fibration, it will suffice to show that the fiber $p^{-1}\{t\}$ is contractible for every morphism $t : A \to X$. Since X is g_f-local, we deduce that the fiber $q^{-1}\{t\}$ is contractible, where q is the projection map $\mathrm{Map}_{\mathbf{A}}(A \coprod_{A'} B', X) \to \mathrm{Map}_{\mathbf{A}}(A, X)$. It will therefore suffice to show that f'' induces a homotopy equivalence of fibers

$$\mathrm{Map}_{\mathbf{A}_{A/}}(B, X) \to \mathrm{Map}_{\mathbf{A}_{A/}}(A \coprod_{A'} B', X).$$

This is clear because f'' is a weak equivalence between cofibrant objects of the simplicial model category $\mathbf{A}_{A/}$.

Proposition A.3.7.6. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is presentable.*

(2) *There exists a combinatorial simplicial model category $\mathbf{A}$ and an equivalence $\mathcal{C} \simeq \mathrm{N}(\mathbf{A}^\circ)$.*

Proof. According to Theorem 5.5.1.1 and Proposition 5.5.4.15, $\mathcal{C}$ is presentable if and only if there exists a small simplicial set K, a small set S of morphisms in $\mathcal{P}(K)$, and an equivalence $\mathcal{C} \simeq S^{-1}\,\mathcal{P}(K)$. Let $\mathcal{D}$ be the simplicial category $\mathfrak{C}[K]^{op}$ and let $\mathbf{B}$ be the category $\mathcal{S}\mathrm{et}_\Delta^{\mathcal{D}}$ of simplicial functors $\mathcal{D} \to \mathcal{S}\mathrm{et}_\Delta$ endowed with the injective model structure. Proposition 4.2.4.4 implies that there is an equivalence $\mathcal{P}(K) \simeq \mathrm{N}(\mathbf{B}^\circ)$. Moreover, Propositions A.3.7.3 and A.3.7.4 imply that there is a bijective correspondence between accessible localizations of $\mathcal{P}(K)$ (as a presentable ∞-category) and combinatorial localizations of $\mathbf{B}$ (as a model category). This proves the implication

(1) $\Rightarrow$ (2). Moreover, it also shows that (2) $\Rightarrow$ (1) in the special case where $\mathbf{A}$ is a localization of a category of simplicial presheaves.

We now complete the proof by invoking the following result, proven in [19]: for every combinatorial model category $\mathbf{A}$, there exists a small category $\mathcal{D}$, a set S of morphisms of $\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{D}^{op}}$, and a Quillen equivalence of $\mathbf{A}$ with $S^{-1}\,\mathcal{S}\mathrm{et}_{\Delta}^{\mathcal{D}}$. Moreover, the proof given in [19] shows that when $\mathbf{A}$ is a *simplicial* model category, then F and G can be chosen to be simplicial functors. $\square$

Remark A.3.7.7. Let $\mathbf{A}$ and $\mathbf{B}$ be combinatorial simplicial model categories. Then the underlying ∞-categories $\mathrm{N}(\mathbf{A}^{\circ})$ and $\mathrm{N}(\mathbf{B}^{\circ})$ are equivalent if and only if $\mathbf{A}$ and $\mathbf{B}$ can be joined by a chain of simplicial Quillen equivalences. The "only if" assertion follows from Corollary A.3.1.12, and the "if" direction can be proven using the methods described in [19].

Proposition A.3.7.8. *Let $\mathbf{A}$ be a left proper combinatorial simplicial model category and let $\mathcal{C} = \mathrm{N}(\mathbf{A}^{\circ})$ denote its underlying ∞-category. Suppose that $\mathcal{C}^0 \subseteq \mathcal{C}$ is an accessible localization of $\mathcal{C}$ and let $L : \mathcal{C} \to \mathcal{C}^0$ denote a left adjoint to the inclusion.*

Then there exists a localization $\mathbf{A}'$ of $\mathbf{A}$ satisfying the following conditions:

(1) *An object $X \in \mathbf{A}'$ is fibrant if and only if it is fibrant in $\mathbf{A}$ and the associated object of the homotopy category $\mathrm{h}\mathbf{A} \simeq \mathrm{h}\mathcal{C}$ belongs to the full subcategory $\mathrm{h}\mathcal{C}^0$.*

(2) *A morphism $f : X \to Y$ in $\mathbf{A}'$ is a weak equivalence if and only if the functor $L : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{C}^0$ carries f to an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}^0$.*

Proof. According to Proposition A.3.7.6, the ∞-category $\mathcal{C}$ is presentable. The results of §5.5.4 imply that we can write $\mathcal{C}^0 = S^{-1}\,\mathcal{C}$ for some small collection of morphisms S in $\mathcal{C}$. We then take $\widetilde{S}$ to be a collection of representatives for the elements of S as cofibrations between cofibrant objects of $\mathbf{A}$ and let $\mathbf{A}'$ denote the localization $\widetilde{S}^{-1}\mathbf{A}$. $\square$

We conclude this section by establishing a universal property enjoyed by the localization of a combinatorial simplicial model category.

Proposition A.3.7.9. *Suppose we are given a simplicial Quillen adjunction*

$$\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B}$$

between left proper combinatorial simplicial model categories and let $\mathbf{A}'$ be a Bousfield localization of $\mathbf{A}$. The following conditions are equivalent:

(1) *The adjoint functors F and G determine a Quillen adjunction between $\mathbf{A}'$ and $\mathbf{B}$.*

(2) *Let α be a morphism in $\mathbf{A}$ which is a weak equivalence in $\mathbf{A}'$. Then the left derived functor $LF : \mathrm{h}\mathbf{A} \to \mathrm{h}\mathbf{B}$ carries α to an isomorphism in the homotopy category $\mathrm{h}\mathbf{B}$.*

(3) *For every fibrant object $X \in \mathbf{B}$, the image GX is a fibrant object of $\mathbf{A}'$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, and the implication (3) $\Rightarrow$ (1) follows from Corollary A.3.7.2. We will complete the proof by showing that (2) $\Rightarrow$ (3). According to Proposition A.3.7.4 and Remark A.3.7.5, we may suppose that $\mathbf{A}' = S^{-1}\mathbf{A}$, where S is a small collection of cofibrations between cofibrant objects of $\mathbf{A}$. Let X be a fibrant object of $\mathbf{B}$; we wish to show that GX is a fibrant object of $\mathbf{A}'$. Since GX is fibrant in $\mathbf{A}$, it will suffice to show that GX is S-local (Proposition A.3.7.3). In other words, we must show that if $\alpha : A \to B$ belongs to S, then the induced map $p : \mathrm{Map}_{\mathbf{A}}(B, GX) \to \mathrm{Map}_{\mathbf{A}}(A, GX)$ is a weak homotopy equivalence. Since F and G are simplicial functors, we can identify p with the map $\mathrm{Map}_{\mathbf{B}}(FB, X) \to \mathrm{Map}_{\mathbf{B}}(FA, X)$. To prove that p is a weak homotopy equivalence, it will suffice to show that $F(\alpha)$ is a weak equivalence between cofibrant objects of $\mathbf{B}$. This follows immediately from assumption (2) (because α is a cofibration between cofibrant objects of $\mathbf{A}$, we can identify $F(\alpha)$ with the left derived functor $LF(\alpha)$). $\square$

Corollary A.3.7.10. *Let $\mathbf{A}$ and $\mathbf{B}$ be left proper combinatorial simplicial model categories and suppose we are given a simplicial Quillen adjunction*

$$\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B}.$$

Then

(1) *There exists a new left proper combinatorial simplicial model structure $\mathbf{A}'$ on the category $\mathbf{A}$ with the following properties:*

  (C) *A morphism α in $\mathbf{A}'$ is a cofibration if and only if it is a cofibration in $\mathbf{A}$.*

  (W) *A morphism α in $\mathbf{A}'$ is a weak equivalence if and only if the left derived functor LF carries α to an isomorphism in the homotopy category* h$\mathbf{B}$.

  (F) *A morphism α in $\mathbf{A}'$ is a fibration if and only if it has the right lifting property with respect to every morphism in $\mathbf{A}'$ satisfying (C) and (W).*

(2) *The functors F and G determine a new simplicial Quillen adjunction*

$$\mathbf{A}' \underset{G'}{\overset{F'}{\rightleftarrows}} \mathbf{B}.$$

(3) *Suppose that the right derived functor RG is fully faithful. Then the adjoint pair (F', G') is a Quillen equivalence.*

Proof. The functors F and G determine a pair of adjoint functors

$$\mathrm{N}\,\mathbf{A}^\circ \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{N}\,\mathbf{B}^\circ$$

between the underlying ∞-categories (see Proposition 5.2.4.6), which are themselves presentable (Proposition A.3.7.6). Let $\overline{S}$ be the collection of all morphisms u in $\mathrm{N}\,\mathbf{A}^\circ$ such that $f(u)$ is an equivalence in $\mathrm{N}\,\mathbf{B}^\circ$. Proposition 5.5.4.16 implies that $\overline{S}$ is generated (as a strongly saturated class of morphisms) by a small subset $S \subseteq \overline{S}$. Without loss of generality, we may suppose that the morphisms of S are represented by some (small) collection T of cofibrations between cofibrant objects of $\mathbf{A}$. Let $\mathbf{A}' = T^{-1}\mathbf{A}$. We claim that $\mathbf{A}'$ satisfies the description given in (1). In other words, we claim that a morphism α in $\mathbf{A}$ is a T-equivalence if and only if the left derived functor LF carries α to an isomorphism in $\mathrm{h}\mathbf{B}$. Without loss of generality, we may suppose that α is a morphism between fibrant-cofibrant objects of $\mathbf{A}$, so that we can view α as a morphism in the ∞-category $\mathrm{N}\,\mathbf{A}^\circ$. In this case, both conditions on α are equivalent to the requirement that α belong to $\overline{S}$. This completes the proof of (1). Assertion (2) follows immediately from Proposition A.3.7.9.

We now prove (3). Note that the homotopy category $\mathrm{h}\mathbf{A}'$ can be identified with a full subcategory of the homotopy category $\mathrm{h}\mathbf{A}$ and that under this identification the left derived functor LF restricts to the left derived functor LF'. It follows that for every fibrant object $X \in \mathbf{B}$, the counit map

$$(LF')(RG')X \simeq (LF')(GX) \simeq (LF)(GX) \simeq (LF)(RG)X \simeq X$$

is an isomorphism in $\mathrm{h}\mathbf{B}$ (where the last equivalence follows from our assumption that RG is fully faithful). It follows that the functor RG' is fully faithful. To complete the proof, it will suffice to show that the left derived functor LF' is conservative. In other words, we must show that if $\alpha : X \to Y$ is a morphism in $\mathbf{A}'$, then α is a weak equivalence if and only if $LF(\alpha)$ is an isomorphism in $\mathbf{B}$; this follows immediately from (1). $\square$

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General Index

Index of Notation