0.1 The structure sheaf of an affine scheme

Definition 0.1.1 Define $\mathscr{O}_{\mathrm{Spec}A}(D(f))$ to be localization of A at the multiplicative set S, where

$$S := \{\text{All functions that do not vanish outside } V(f) \text{ (Do not vanish on } D(f))\}.$$

(i.e., those $g \in A$ such that $V(g) \subset V(f)$ or equivalently $D(f) \subset D(g)$)

In particular, $\mathscr{O}_{\mathrm{Spec}A}(\emptyset) = \{0\}$, where localize at the multiplicative set of functions g such that $V(g) \subset \mathrm{Spec}A$. This multiplicative set includes 0, hence the localization is $\{0\}$ ring.

Exercise 0.1.A Show that the natural map $A_f \longrightarrow \mathscr{O}_{\mathrm{Spec}A}(D(f))$ is an isomorphism. ■

Proof. In particular, $S_f := \{1, f, f^2, \ldots\}$ is a multiplicative subset of the multiplicative set T in the definition of $\mathscr{O}_{\mathrm{Spec}A}(D(f))$, where

$$T := \{\text{All functions that do not vanish outside } V(f) \text{ (Do not vanish on } D(f))\}.$$

$S_f \subset T$. There is a natural homomorphism

$$A \xrightarrow{S_f^{-1}} A_f \xrightarrow{\tilde{T}^{-1}} \mathscr{O}_{\mathrm{Spec}A}(D(f)),$$

where we have denoted the image of T in A_f by $\tilde{T}$. $g \in S \iff D(f) \subset D(g)$
$\iff T^{-1}g$ is invertible in A_f by Exercise **??**.
$\iff \tilde{T} \subseteq A_f^{\times}$
$\iff \tilde{T}^{-1}$ is an isomorphism. $A_f \cong \mathscr{O}_{\mathrm{Spec}A}(D(f))$. ■

Exercise 0.1.B Prove the base identity axiom for any distinguished open $D(f)$. ■

Proof. Consider the $D(f) = \cup_{i \in I} D(f_i)$. We already showed that $\mathrm{Spec}A_f \cong D(f)$ as topological spaces **??**. If $D(f) = \cup_{i \in I} D(f_i) = \cup_{i \in I} D(f_i) \cap D(f) = \cup_{i \in I} D(f_i f)$.

$D(f_i f) \cong \mathrm{Spec}A_{ff_i}$ A_{ff_i} is the localization of A_f at the image of f_i. $D(f_i f)$ corresponds to the point $[\mathfrak{q}] \in \mathrm{Spec}A_f$ such that $\mathfrak{q} \not\ni \frac{f_i}{1}$.

Then $D(f) = \cup_{i \in I} D(f_i) \subset \mathrm{Spec}A \iff \mathrm{Spec}A_f = \cup_{i \in I} D(f_i/1)$

$\mathscr{O}_{\mathrm{Spec}A}(D(f)) \cong A_f = \mathscr{O}_{\mathrm{Spec}A_f}(\mathrm{Spec}A_f)$. The function restricts to 0 on each $D(f_i)$ iff its restriction to $D(f_i/1)$ vanishes.

Then the problem reduces to the proved case $D(f) = \mathrm{Spec}A$. ■

Exercise 0.1.C Alter this argument appropriately to show base gluability for any distinguished open $D(f)$. ■

Proof. Again, we regard $D(f) \cong \mathrm{Spec}A_f$.
Then $D(f) = \cup_{i \in I} D(f_i) \subset \mathrm{Spec}A \iff \mathrm{Spec}A_f = \cup_{i \in I} D(f_i/1)$.
$\mathscr{O}_{\mathrm{Spec}A}(D(f)) \cong A_f = \mathscr{O}_{\mathrm{Spec}A_f}(\mathrm{Spec}A_f)$.
The base gluability follows from the special case we have proved for $\mathrm{Spec}A = D(f)$. ■

Exercise 0.1.D Suppose M is an A-module. Show that the following construction describes a sheaf M on the distinguished base. Define $\tilde{M}(D(f))$ to be the localization of M at the multiplicative set of all functions that do not vanish outside of $V(f)$. Define restriction maps $res_{D(f),D(g)}$ in the analogous way to $\mathscr{O}_{\mathrm{Spec}A}$. Show that this defines a sheaf on the distinguished base, and hence a sheaf on $\mathrm{Spec}A$. Then show that this is an $\mathscr{O}_{\mathrm{Spec}A}$-module. ∎

Proof. Define $\tilde{M}_{\mathrm{Spec}A}(D(f))$ to be localization of M at the multiplicative set S, where

$$S := \{\text{All functions that do not vanish outside } V(f) \text{ (Do not vanish on } D(f)\}.$$

Claim: $\tilde{M}(D(f)) \cong M_f$.
In particular, $S_f := \{1, f, f^2, \ldots\}$ is a multiplicative subset of the multiplicative set S.
There is a natural homomorphism

$$M \xrightarrow{S_f^{-1}} M_f \xrightarrow{\tilde{S}^{-1}} \tilde{M}(D(f)),$$

where we have denoted the image of S in A_f by $\tilde{S}$. $g \in S \iff D(f) \subset D(g)$
$\iff T^{-1}g$ is invertible in A_f by Exercise **??**.
$\iff \tilde{S} \subseteq A_f^\times$
$\iff \tilde{S}^{-1}$ is an isomorphism. $M_f \cong \tilde{M}(D(f))$. ∎

Exercise 0.1.E The disjoint union of schemes is defined as you would expect: it is the disjoint union of sets, with the expected topology, with the expected sheaf.

(a) Show that the disjoint union of a finite number of affine schemes is also an affine scheme.
(b) (a first example of a non-affine scheme) Show that an infinite disjoint union of (nonempty) affine schemes is not an affine scheme.

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Proof. (a) In Exercise **??**, we see that for finite index set I:

$$\coprod_{i \in I} \mathrm{Spec}A_i \cong \mathrm{Spec}\prod_i A_i$$

and we only need to describe the structure sheaf and verify that

$$\mathscr{O}_{\coprod_i \mathrm{Spec}A_i} \cong \mathscr{O}_{\mathrm{Spec}\prod_i A_i}$$

Consider the inclusion map $\iota_i : \mathrm{Spec}A_i \hookrightarrow \mathrm{Spec}\coprod A_i$

$$\mathscr{O}_{\mathrm{Spec}\prod_i A_i} := \coprod_i (\iota_i)_* \mathscr{O}_{\mathrm{Spec}A_i}$$

For $U = \coprod_i U_i \subset \coprod \mathrm{Spec}A_i$,

$$\left(\coprod_i (\iota_i)_* \mathscr{O}_{\mathrm{Spec}A_i}\right)(U) = \coprod_i \mathscr{O}_{\mathrm{Spec}A_i}(\iota_i^{-1}U) = \coprod_i \mathscr{O}_{\mathrm{Spec}A_i}(U_i).$$

The later coproduct means disjoint union in *Set*-value valued case and means tensor product in the case of *Mod* and so on.

(b)

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