Lecture Notes for Algebraic Geomtry I

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Recall: $V(I) \subset \mathbb{A}^n = \{x | \forall f \in I, f(x) = 0\}$.

Theorem 1.1. *Let $Y_1 \subset \mathbb{A}^n, X_1, ..., X_n$, $Y_2 \subset \mathbb{A}^m, T_1, , ..., T_m$ affine algebraic sets. There are bijections*

$$\begin{aligned}
&Hom_{K-alg}(\mathcal{O}(Y_2), \mathcal{O}(Y_1)) \\
&\xleftrightarrow{(*)} \{(f_1, ..., f_m) \in K[X]^m | \forall x \in Y_1, (f_1(x), ..., f_m(x)) \in Y_2)\} \\
&\xleftrightarrow{(**)} \{f : Y_1 \longrightarrow Y_2 | \forall \varphi \in \mathcal{O}(Y_2), \varphi \circ f \ \textit{is in}\ \mathcal{O}(Y_1)\}
\end{aligned}$$

Proof. Key observation

To give $(f_1, ..., f_m) \in K[X]^m$ is "the same " as giving a ring morphism $g_0 : K[T] \longrightarrow K[X] : T_i \mapsto f_i$, which gives by composition $g_1 = \pi_1 \circ g_0$, where $\pi_1 : K[X] \longrightarrow \mathcal{O}(Y_1)$ is the canonical projection.

$$g_1 : K[T] \longrightarrow \mathcal{O}(Y_1)$$

which has a factorization

$$\begin{tikzcd}
K[T] \arrow[r, "g_1"] \arrow[d, "\pi_2"] & \mathcal{O}(Y_1) \\
\mathcal{O}(Y_2) \arrow[ur, "g"] &
\end{tikzcd}$$

iff $g_1(I(Y_2)) = 0$, which means iff

$$g_1(\varphi) = \text{ “replace } T_i \text{ by } f_i \text{ in } \varphi\text{”}$$

belongs to $I(Y_1)$ if $\varphi \in I(Y_2)$, which means if $x \in Y_1$, then $g_1(\varphi)(x) = 0$. That means $\varphi(f_1(x), ..., f_m(x)) = 0$ for $\varphi \in I(Y_2)$, i.e., $(f_1(x), ..., f_m(x)) \in Y_2$. If $x \in Y_1$. In the statement, this gives the $(*)$ bijection. Any k-algebra morphism $\mathcal{O}(Y_1) \longrightarrow \mathcal{O}(Y_2)$ comes from $K[T] \longrightarrow \mathcal{O}(Y_1)$ s.t. it vanishes on $I(Y_2)$.

For the bijection $(**)$, suppose

$$g : Y_1 \xrightarrow{g} Y_2 \xrightarrow{\varphi} K$$

sends $\varphi(Y_2)$ to $\varphi \circ g \in \mathcal{O}(Y_1)$. Then we get

$$\begin{aligned}
&\mathcal{O}(Y_2) \longrightarrow \mathcal{O}(Y_1) \\
&\varphi \longmapsto \varphi \circ g,
\end{aligned}$$

which is a K-algebra morphism.

As for the reverse direction, given g. From $\mathcal{O}(Y_2) \longrightarrow \mathcal{O}(Y_1)$ to get a $g : Y_1 \longrightarrow Y_2$. We get a $\tilde{g} : Y_1 \longrightarrow Y_2$ in the second set

$$\tilde{g}(x) = (f_1(x), ..., f_m(x))$$

then we have $\varphi \circ g \in \mathcal{O}(Y_1)$ for $\varphi \in \mathcal{O}(Y_2)$. One checks that this shows that the first and third sets are the same. $\square$

Define morphism $Y_1 \longrightarrow Y_2$ by the second(and third) set. Composition in the obvious way and identity is a morphism. $\Longrightarrow$ get a category (Alg_K) of affine algebraic sets over K.

Corollary 1.2. $Y \mapsto \mathcal{O}(Y)$, $g \mapsto [\varphi \mapsto \varphi \circ g]$ *is a functor:* $(Alg_K) \longrightarrow (K - Alg)^{opp}$.

Facts: The "image" of this functor is the category of finitely generated K-algebras which are reduced.

Proof. A finitely generated reduced K-algebra. ($\exists n \geq 1$, so that $K[X_1, ..., X_n]/I \cong A$). Then "$A$ is reduced"$\Longleftrightarrow$ I is radical ideal. $\Longrightarrow A = \mathcal{O}(V(I))$, where $V(I) \subset \mathbb{A}^n$. $\square$

Corollary 1.3. *There is a equivalence of categories between*

$$(Alg_K) \longleftrightarrow (\text{fin. gen. reduced.} K - Algs.)$$

Example 1.4.

(1) $\mathbb{A}^1 \longrightarrow V(Y^2 - X^3 - X^2) \subset \mathbb{A}^2$, $t \mapsto (t^2 - 1, t(t^2 - 1))$

(2) $\mathbb{A}^1 \longrightarrow V(Y^2 - X^3) \subset \mathbb{A}^2$*:* $t \longmapsto (t^2, t^3)$ *is a bijection but Not an isomorphism.*

(3) Assume K *with characteristic* $p > 0$, $K \supset \mathbb{F}_p$. $Y = V(f_1, ..., f_m)$ *where* $f_i \in \mathbb{F}_p[X] \subset K[X]$. *Consider the morphism:*

$$Y \longrightarrow Y$$
$$(x_1, .., x_n) \longmapsto (x_1^p, ..., x_n^p).$$

It is bijective and homeomorphism but not an isomorphism if $dim(Y) \geq 1$.

Proposition 1.5. $Y = V(I) \subset \mathbb{A}^n$

(1) The points of Y are in bijection with maximal ideals $I \subset \mathcal{O}(Y)$ by

$$Y \ni x \longmapsto \{f \in \mathcal{O}(Y) | f(x) = 0\}$$

(2) We have a bijection

$$\mathcal{O}(Y) \longleftrightarrow \ Hom_{Alg_K}(Y, \mathbb{A}^1)$$

Proof. (1) $I_x := Ker(\mathcal{O}(Y) \longrightarrow K)$, $f \mapsto f(x)$, since the evaluation map is surjective $[1 \mapsto 1]$, we get an isomorphism

$$\mathcal{O}(Y)/I_x \xrightarrow{\sim} K,$$

so I_x is maximal in $\mathcal{O}(Y)$.

Conversely, if $I \subset \mathcal{O}(Y)$ is maximal, we get $I = I'/I(Y)$ for $I' \subset K[X]$ maximal.

Nullstellensatz says $\exists (x_1, ..., x_n) \in \mathbb{A}^n$ s.t., $I' = (X_1 - x_1, ..., X_n - x_n)$.

Since $I' \supset I(Y)$, we get $(x_1, .., x_n) \in Y$. Then we check that $\mathcal{O}(Y) \longrightarrow \mathcal{O}(Y)/I \cong K$ is just given by $f \mapsto f(x_1, ..., x_n)$. That means $I = I_x$.

(2) We saw in 1.1, that there is a bijection between sets

$$\text{Hom}_{Alg_k}(Y, \mathbb{A}^1) \longleftrightarrow \ \text{Hom}_{K-alg}(\mathcal{O}(\mathbb{A}^1), \mathcal{O}(Y)).$$

But $\text{Hom}_{K-alg}(\mathcal{O}(\mathbb{A}^1), \mathcal{O}(Y)) = \text{Hom}_{K-alg}(K[X], \mathcal{O}(Y)) \cong \mathcal{O}(Y)$ (by $g : \mathcal{O}(\mathbb{A}^1) \longrightarrow \mathcal{O}(Y)$, $g \mapsto g(X)$) □

Projective Algebraic sets

Projective sets can have a good notion of "compactness".

N.B. Any $Y \in (Alg_K)$ is **quasi-compact**(open cover have a finite subcover).

Definition 1.6. $\mathbb{P}^n_K = \mathbb{P}^n$ *can be either defined as*

"the set of lines in $\mathbb{A}^{n+1}$ that pass through the origin"

or

"the equivalence classes of points in $K^{n+1} \backslash \{0\}$ with the equivalence relation $x \sim y$ iff $x = \lambda y$ for some $\lambda \in K$" and we use the notion $[x_0 : ... : x_n]$ for the equivalence class of $(x_0, .., x_n)$

These two definitions are equivalent:

Given a line $l \in \mathbb{A}^1 \longleftrightarrow$ hyperplane in K^{n+1}, corresponds to a equation

$$a_0X_0 + ... + a_nX_n = 0$$

with at least one of a_i non-zero.

Conversely, from $[x_0 : .. : x_n]$, we we get the corresponding hyperplane/line trivially.

Notes the following fact:

$$\mathbb{P}^n = \cup_{0 \leq i \leq n} H_i,$$

where $H_i = \{[x_0, ..., x_n] | x_i \neq 0\}$ and there is a bijection

$$\begin{aligned}
&H_i \longrightarrow K^n \\
&[x_0 : ... : x_n] \longmapsto \left(\frac{x_0}{x_i}, ..., \frac{\widehat{x_i}}{x_i}, .., \frac{x_n}{x_i}\right) \\
&[y_1 : ... : y_{i-1} : 1 : y_i : ... : y_n] \hookleftarrow (y_1, ..., y_n)
\end{aligned}$$

We define from linear algebra some notions in $\mathbb{P}^n$ a line in $\mathbb{P}^n$ is the image by the projection $K^{n+1} \backslash \{0\} \longrightarrow \mathbb{P}^n$ of the two dimensional affine subspace.

Example 1.7. *$l_1, l_2 \subset \mathbb{P}^2$ lines $l_1 \cap l_2$ is a line if l_1 and l_2 are identical and would be a single point otherwise.*

Observation: If $f \in K[X_0, ..., X_{n+1}]$ is homogeneous, then for $x \in \mathbb{P}^n$, it makes no sense to speak of "$f(x) \in K$", but the zero-loci or the set where $f(x) \neq 0$ does make sense.

Definition 1.8. *A **projective algebraic set** $S \subset \mathbb{P}^n$ is*

$$S = \{x \in \mathbb{P}^n | f_1(x) = ... = f_m(x) = 0\},$$

where $f_1, ..., f_m$ are homogeneous of some degrees.

<u>Notation</u>: $V(f_1, .., f_n)$

Example 1.9. *$V(Y^2Z - X^3 - XZ^2) \subset \mathbb{P}^2$.*

Let $0 \leq i \leq n$, then $S \cap H_i = \{[x_0 : ... : x_n] \in S | x_i \neq 0\}$ is , via the bijection $H_i \longrightarrow K^n$, in bijection with an affine algebraic set $S_1 \subset \mathbb{A}^n$ given by $\tilde{f}_1(y) = ... = \tilde{f}_m(y) = 0$, where $\tilde{f}_i(y_1, .., y_n) = f_i(y_1, ..., y_{i-1}, 1, y_i, ..., y_n)$