Algebraic Topology

An exercise-oriented notes

Vector_Cat

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1. Topological spaces

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4. Elementary homotopy theory

4.1 The mapping cylinder

Definition 4.1.1 Given a continuous map $f : X \longrightarrow Y$ of topological spaces, one can define its **mapping cylinder** as a pushout (fibered coproduct)

$$\begin{tikzcd}
B & & & \\
& Z(f) \arrow[ul, dashed, "\exists!"'] & & X\times I \arrow[ll] \arrow[dl, "f\times id"'] \arrow[ulll] \\
& & Y\times I \arrow[ul, "r"] & \\
& Y \arrow[uu] \arrow[ur, "i_0"] \arrow[uuul] & & X, \arrow[ll, "f"] \arrow[uu, "i_0"']
\end{tikzcd}$$

Set-theoretically, the mapping cylinder is usually represented as the quotient space $(X \times I \coprod Y)/\sim$, where $f(x) \sim (x,0)$. We use Mf to denote it. (other notations are used including Mf, M_f and $\mathrm{Cyl}(f)$.)

Notice that it is Mf rather than $Y \times I$ that plays the role of pushout because the map r is not unique. Our only restriction on r is $r \circ j = id$, where $j : Mf \longrightarrow Y \times I$ is the map that restricts to $f \times id$ on $X \times I$ and restricts to i_0 on Y.

R Another equivalent definition is used in tom Dieck.

In the following, we consider $X \coprod Y$ as subspace of $Z(f)$ via the map $J : J(x) = [(x,0)]$ and $J(y) = [y]$. Then we consider a homotopy commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\alpha \big\downarrow & & \big\downarrow \beta \\
X' & \xrightarrow{f'} & Y',
\end{array}$$

where the diagram commutes up to a homotopy $\Psi : f' \circ \alpha \simeq \beta \circ f$.

5. Cofibrations and fibrations

6. Homotopy groups

7. Stable homotopy. Daulity

8. Cell complexes

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10. Homology

10.1 The Axioms of Eilenberg and Steenrod

11. Homological algebra

11.1 Diagrams

11.2 Exact sequences

11.3 Chain complex

11.4 Cochain complex

11.5 Natural chain maps and homotopies

11.6 Linear algebra of chain complexes

Definition 11.6.1 Suppose $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ are two non-negative chain complexes. We define the **tensor complex** $(C_\bullet \otimes C'_\bullet, \Delta)$, where

$$(C_\bullet \otimes C'_\bullet)_n = \oplus_{i+j=n} C_i \otimes C'_j$$

and the differential Δ is defined by

$$\Delta(c_i \otimes c'_j) = \partial c_i \otimes c'_j + (-1)^i c_i \otimes \partial' c_j$$

Definition 11.6.2 Suppose $f_\bullet : C_\bullet \longrightarrow D_\bullet$ and $g_\bullet : C'_\bullet \longrightarrow D'_\bullet$ are two morphism of chain complexes. Then we can define a chain map

$$f \otimes g : C_\bullet \otimes C'_\bullet \longrightarrow D_\bullet \otimes D'_\bullet$$

by

$$(f \otimes g)_n = \sum_{i+j=n} f_i \otimes g_j$$

It is easy to check this is indeed a chain map.

Exercise?? 11.6.A Tensor product is compatible with chain homotopy. Let $s : f \simeq g : C_\bullet \longrightarrow C'_\bullet$ be a chain homotopy. Then $s \otimes id : f \otimes id \simeq g \otimes id : C_\bullet \otimes D_\bullet \longrightarrow C'_\bullet \otimes D_\bullet$ is a chain homotopy.

Proof. Know: $s\partial_C + \partial_{C'} s = f - g$

Want: $(s \otimes id_D)\partial_{C\otimes D} + \partial_{C'\otimes D}(s \otimes id_D) = f \otimes id_D - g \otimes id_D$.

$C \otimes D$ is generated by pure tensors like $c'_n \otimes d_m$, therefore we can check the formula on element $c_n \otimes d_m \in C_n \otimes D_m$

$$\begin{aligned}
&(s \otimes id_D)\partial_{C\otimes D}(c_n \otimes d_m)\\
&= (s \otimes id_D)\left(\partial_C c_n \otimes d_m + (-1)^n c_n \otimes \partial_D d_m\right)\\
&= s \circ \partial_C c_n \otimes d_m + (-1)^n s c_n \otimes \partial_D d_m
\end{aligned}$$

and

$$\begin{aligned}
&\partial_{C'\otimes D}(s \otimes id_D)(c_n \otimes d_m)\\
&= \partial_{C'\otimes D}(s c_n \otimes d_m)\\
&= \partial_{C'} s c_n \otimes d_m + (-1)^{\deg(s c_n)} s c_n \otimes \partial_D d_m,
\end{aligned}$$

where $\deg(sc_n) = n-1$. Then we have

$$\begin{aligned}
&\left(\partial_{C'\otimes D}(s \otimes id_D) + (s \otimes id_D)\partial_{C\otimes D}\right)(c_n \otimes d_m)\\
&= (s\partial_C + \partial_{C'} s)c_n \otimes d_m + 0\\
&= (f \otimes id_D - g \otimes id_D)(c_n \otimes d_m)
\end{aligned}$$

We are done. Also we can generalize this statement to

Let $s : f \simeq g : C \longrightarrow C'$ and $t : p \simeq q : D \longrightarrow D'$ be chain homotopies. Then $s \otimes t : f \otimes p \simeq g \otimes q : C \otimes D \longrightarrow C' \otimes D'$ is a chain homotopy. We easily conclude by $s \otimes id$ and $id \otimes t$ are chain homotopy and composition of chain homotopies is a chain homotopy. ■

Exercise?? 11.6.B Let $(C_\bullet, \partial)$ be a free chain complex. Then $C_\bullet$ is acyclic iff it has contracting chain homotopy

Proof. A contracting homotopy means $Q : C_n \longrightarrow C_{n+1}$ s.t. $Q\partial + \partial Q = id$.

If such Q exists then $H_n(C_\bullet) = 0 \forall n$. That direction doesn't require $C_\bullet$ to be free.

As for the reverse direction, consider

$$B_n \subseteq Z_n \subseteq C_n$$

If we assume $C_\bullet$ is acyclic then

$$B_n = Z_n, \forall n$$

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} Z_{n-1} \longrightarrow 0$$

Since Z_{n-1} is free abelian the sequence splits $\exists r_n : Z_{n-1} \longrightarrow C_n$ s.t. $\partial \circ r_n = id$. Note that $id - r_{n-1} \circ \partial$ has image in Z_{n-1}, $c \in C_n$. $\partial(c - r_n\partial c) = \partial c - \partial c = 0$

Now define $Q_n : C_n \longrightarrow C_{n+1}$ by $Q_n = r_n(id - r_{n-1} \circ \partial)$. This works.

$$\begin{aligned}
\partial Q_n + Q_{n-1}\partial &= \partial r_n(id - r_{n-1}\partial) + r_{n-1}(id - r_{n-2}\partial)\partial\\
&= id - r_{n-1}\partial + r_{n-1}\partial - r_{n-1}r_{n-2}\partial^2\\
&= id
\end{aligned}$$

■

Definition 11.6.3 Suppose $f : (C_\bullet, \partial) \longrightarrow (D_\bullet, \partial')$. The **mapping cone** of f is the chain complex $Cone_\bullet(f), \partial^f$, where $Cone_n(f) = C_{n-1} \otimes D_n$ and $\partial^f : Cone_n(f) \longrightarrow Cone_{n-1}(f)$

$$\partial^f(c,d) = (-\partial c, fc + \partial' d)$$

$$\partial^f = \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix}$$

Exercise?? 11.6.C If $f : C_\bullet \longrightarrow D_\bullet$ is a chain map between two free chain complexes and $Cone_\bullet(f)$ is acyclic then prove f is a chain equivalence. ▪

Proof. Note that the definition of mapping cone implies $Cone_\bullet(f)$ to be a free chain complex. Then we can apply Exercise 11.6.B and there is a contracting chain homotopy Q such that

$$Q\partial^f + \partial^f Q = id$$

$$Q = \begin{pmatrix} p & g \\ r & -p' \end{pmatrix}$$

$$\begin{pmatrix} \partial & 0 \\ f & -\partial' \end{pmatrix} \begin{pmatrix} p & g \\ r & -p' \end{pmatrix} + \begin{pmatrix} p & g \\ r & -p' \end{pmatrix} \begin{pmatrix} \partial & 0 \\ f & -\partial' \end{pmatrix} = \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}$$

$$\begin{pmatrix} -\partial p - p\partial + gf & -\partial g + g\partial' \\ * & fg - \partial' p' - p'\partial' \end{pmatrix} = \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}$$

Then we know $g : D_\bullet \longrightarrow D_\bullet$ is a chain map

$p\partial + \partial p = gf - id$

$p'\partial' + \partial' p' = fg - id$. Thus f is a chain equivalence with inverse g. ■

Lemma 11.6.1 Let $f : C_\bullet \longrightarrow D_\bullet$. Then there is a LES

$$\cdots \longrightarrow H_{n+1}(Cone_\bullet(f)) \longrightarrow H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(D_\bullet) \longrightarrow H_n(Cone_\bullet(f)) \longrightarrow \cdots$$

Proof. Denote by $C_\bullet^+$ the chain complex $C_n^+ = C_{n-1}$. There is a SES

$$0 \longrightarrow D_\bullet \xrightarrow{i} Cone_\bullet(f) \xrightarrow{p} C_\bullet^+ \longrightarrow 0$$

with $i(d) = (0,d)$ and $p(c,d) = c$

Pass to the LES in homology

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(Cone_\bullet(f)) \arrow[r] & H_{n+1}(C_\bullet^+) \arrow[r, "\delta"] \arrow[d, equal] & H_n(D_\bullet) \arrow[r] & H_n(Cone_\bullet(f)) \arrow[r] & \cdots \\ & & H_n(C_\bullet) & & & \end{tikzcd}$$

It remains to check $\delta = H_n(f)$.

Note if c is a cycle in C_n. Then

$$\partial^f \circ p^{-1}(c) = (-\partial c, fc) = (0, fc) = i(fc)$$

$$\delta : \langle c \rangle \longmapsto \langle i^{-1}\partial^f p^{-1} c \rangle = \langle fc \rangle = H_n(f)\langle c \rangle$$

■

Exercise?? 11.6.D Suppose $f : C_\bullet \longrightarrow D_\bullet$ is a chain map between the two free chain complex . Then f is a chain equivalence iff

$$H_n(f) : H_n(C_\bullet) \longrightarrow H_n(D_\bullet)$$

is an isomorphism for all n,

Proof. If f is a chain equivalence then $H_n(f)$ is always a isomorphism. This does not require any freeness assumptions and we proved in last semester.

For the converse, if $H_n(f)$ is always an isomorphism, then the LES

$$\cdots \longrightarrow H_{n+1}(Cone_\bullet(f)) \longrightarrow H_n(C_\bullet) \xrightarrow{\cong} H_n(D_\bullet) \longrightarrow H_n(Cone_\bullet(f)) \longrightarrow \cdots$$

This implies $H_n(Cone_\bullet(f)) = 0, \forall n$. Then $Cone_\bullet(f)$ is acyclic, and we can conclude by Exercise 11.6.C. ∎

11.7 Tor and Ext

Definition 11.7.1 Suppose A is an abelian group, A **Free resolution** is an exact sequence of the form

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \longrightarrow 0,$$

where each F_i is a free abelian group. If moreover $F_i = 0, \forall i \geq 2$, we call it **Short free resolution**

$$0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$$

(We can easily generalize this definition to R-modules)

Proposition 11.7.1 Let A be an abelian group. Then there exists a short free resolution of A.

Proof. Let F be the free abelian group generated by all elements in A. There is a surjection from F to A by linearly extending the map sending basis element to itself. Let K denote the kernel of this map. K is an abelian subgroup of a free abelian group ($\mathbb{Z}$-module). A subgroup of a free abelian group is torsion free as a module. $\mathbb{Z}$ is a *PID*. If R is a *PID*, then an R-module is free iff it is torsion free (See Bosch section 4.2). Then we know in particular, K is a free abelian group. ∎

With this construction, we can define the Tor functor now:

Definition 11.7.2 Let A be an abelian group. Let $0 \to K \xrightarrow{f} F \to A \to 0$ be a short free resolution of A. Given any other abelian group B and apply the functor $\otimes B$ to it. We get an exact sequence

$$K \otimes B \xrightarrow{f \otimes id_B} F \otimes B \longrightarrow A \otimes B \longrightarrow 0$$

We define

$$\operatorname{Tor}(A,B) := \ker(f \otimes id_B).$$

It measure the failure of $\otimes B$ to be left exact. Tor (A,B) can be more generally defined in the category of R-modules, where R is a principal ideal ring, where short free resolution does exist.

Exercise?? 11.7.A Given $h : A \longrightarrow A'$ a homomorphism of abelian group define the induced morphism

$$\operatorname{Tor}(h,B) : \operatorname{Tor}(A,B) \longrightarrow \operatorname{Tor}(A',B).$$

Use this argument to show that $\operatorname{Tor}(A,B)$ is well-defined (independent on the choice of short free resolutions)

Proof. Let $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ and $0 \longrightarrow K' \longrightarrow F' \longrightarrow A' \longrightarrow 0$ be two short free resolutions of A and A' respectively, and denote by $C_\bullet$ and $C'_\bullet$ the corresponding chain complexes.

$$C_n = \begin{cases} F, n = 0 \\ K, n = 1 \\ 0, n \neq 0, 1 \end{cases}$$

with the only nontrivial boundary map $\partial : C_1 \longrightarrow C_0$ to be $f : K \longrightarrow F$ and $C'_\bullet$ is defined similarly. We have $H_0(C_\bullet) = A$ and $H_0(C'_\bullet) = A'$. We can therefore think of $h : A \longrightarrow A'$ as a homomorphism $H_0(C_\bullet) \longrightarrow H_0(C'_\bullet)$. Now we can invoke Corollary A.1.3, which tells us there exists a chain map $g_\bullet : C_\bullet \longrightarrow C'_\bullet$ with $H_0(g_\bullet) = h$.

Tensoring with B, we get a chain map $g_\bullet \otimes id_B : C_\bullet \otimes B \longrightarrow C'_\bullet \otimes B$ because $\otimes B$ is a functor. Now pass to the first homology group to get a map

$$H_1(g_\bullet \otimes id_B) : H_1(C_\bullet \otimes B) \longrightarrow H_1(C'_\bullet \otimes B).$$

However, by definition $H_1(C_\bullet \otimes B) = \operatorname{Tor}(A,B)$, we therefore have defined the morphism induced by $\operatorname{Tor}(\square, B)$

$$\operatorname{Tor}(h,B) : H_1(g_\bullet \otimes id_B)$$

In order to prove $\operatorname{Tor}(A,B)$ is well-defined, we consider the case where $A = A'$, again still by A.1.3, we obtain a chain map $g_\bullet : C_\bullet \longrightarrow C'_\bullet$ with $H_0(g_\bullet) = id_A$ and this chain map $g_\bullet : C_\bullet \longrightarrow C_\bullet$ is a chain equivalence thus induces chain equivalence $g_\bullet \otimes id_B$ and finally we have

$$H_1(C_\bullet \otimes B) = H_1(C'_\bullet \otimes B).$$

Therefore two different short free resolution determines identical $\operatorname{Tor}(A,B)$. ■

R The above exercise means basically for each fixed abelian group B, $\operatorname{Tor}(\square, B) : Ab \longrightarrow Ab$ is a covariant functor. In a similar vein, we can also fix the first variable of Tor and define $\operatorname{Tor}(A, \square) : Ab \longrightarrow Ab$ as a covariant functor.

Exercise?? 11.7.B Define $\operatorname{Tor}(A,h)$ for a given homomorphism $h : B \longrightarrow B'$.

Proof. This case is easier, the homomorphism $h : B \longrightarrow B'$ induces a morphism of chain complexes $id_{C_\bullet} \otimes h$ and thus induces the morphism $\operatorname{Tor}(A,h) := H_1(id_{C_\bullet} \otimes h) : \operatorname{Tor}(A,B) \longrightarrow \operatorname{Tor}(A,B')$ ■

Theorem 11.7.2 (Properties of Tor)
(1) If either A or B are torsion-free abelian groups then $\operatorname{Tor}(A,B) = 0$
(2) If $T(A)$ denotes the torsion subgroup of A then for any abelian group B, one has

$$\operatorname{Tor}(A,B) \cong \operatorname{Tor}(T(A),B).$$

(3) If $0 \longrightarrow B \longrightarrow B' \longrightarrow B'' \longrightarrow 0$ is exact, then for any A, there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}\,(A,B) \arrow[r] & \mathrm{Tor}\,(A,B') \arrow[r] & \mathrm{Tor}\,(A,B'') \arrow[dll] & \\ & A\otimes B \arrow[r] & A\otimes B' \arrow[r] & A\otimes B'' \arrow[r] & 0 \end{tikzcd}$$

If $0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$ is a short exact sequence, then for any abelian group B, there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}\,(A,B) \arrow[r] & \mathrm{Tor}\,(A',B) \arrow[r] & \mathrm{Tor}\,(A'',B) \arrow[dll] & \\ & A\otimes B \arrow[r] & A'\otimes B \arrow[r] & A''\otimes B \arrow[r] & 0 \end{tikzcd}$$

(4) For any two abelian groups A,B, $\mathrm{Tor}\,(A,B) \cong \mathrm{Tor}\,(B,A)$.

(5) If B is an abelian group and $\{A_\lambda | \lambda \in \Lambda\}$ is a (possibly uncountable) family of abelian groups then there is an isomorphism

$$\mathrm{Tor}\left(\bigoplus_{\lambda\in\Lambda} A_\lambda, B\right) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{Tor}\,(A_\lambda,B)$$

and

$$\mathrm{Tor}\left(B, \bigoplus_{\lambda\in\Lambda} A_\lambda\right) \cong \bigoplus \mathrm{Tor}\,(B,A_\lambda).$$

(6) For any $m \in \mathbb{N}$ and any abelian group B,

$$\mathrm{Tor}\,(\mathbb{Z}/m\mathbb{Z}, B) \cong \{b \in B | mb = 0\}$$

Proof. (1) A torsion free abelian group B is a free $\mathbb{Z}$-module, $\otimes B$ is there fore exact functor, in this case we have $\mathrm{Tor}\,(A,B) = 0$ for all A.

On the other hand, if A is free, we can choose a silly short free resolution: $0 \longrightarrow 0 \longrightarrow A \longrightarrow A \longrightarrow 0$. Then clearly, $\mathrm{Tor}\,(A,B) = 0$ for any B. We have not prove the case where A is merely torsion free, it will follows directly from (4) and the case B is torsion free.

Then we skip part (2) and prove the first statement of (3). Given an exact sequence $0 \longrightarrow B \longrightarrow B' \longrightarrow B'' \longrightarrow 0$. let $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ be short free resolution. Then we know $K \otimes \Box$ and $F \otimes \Box$ are exact functors. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K\otimes B \arrow[r] \arrow[d] & K\otimes B' \arrow[r] \arrow[d] & K\otimes B'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F\otimes B \arrow[r] & F\otimes B' \arrow[r] & F\otimes B'' \arrow[r] & 0 \end{tikzcd}$$

This means we have a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet \otimes B \longrightarrow C_\bullet \otimes B' \longrightarrow C_\bullet \otimes B'' \longrightarrow 0.$$

It induces a long exact sequence in homology which is just the desired six-term exact sequence in (3).

The second statement would follow directly from (4) and the first statement of (3).

Then we prove part (4). Given a short free resolution $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$. Because K, F are free, we know $\text{Tor}(B,K) = 0$ and $\text{Tor}(B,F) = 0$ from what we have proved in (1). (there is no circular deduction). Then, from the first statement of (3), we obtain an exact sequence

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \text{Tor}(B,A) \longrightarrow B \otimes K \longrightarrow B \otimes F \longrightarrow B \otimes A \longrightarrow 0.$$

Also from the definition of $\text{Tor}(A,B)$ the bottom row of the next diagram is exact.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d, "\cong"] & \text{Tor}(B,A) \arrow[r] \arrow[d] & B \otimes K \arrow[r] \arrow[d, "\cong"] & B \otimes F \arrow[r] \arrow[d, "\cong"] & B \otimes A \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \text{Tor}(A,B) \arrow[r] & K \otimes B \arrow[r] & F \otimes B \arrow[r] & A \otimes B \arrow[r] & 0
\end{tikzcd}$$

Then we know from five lemma that $\text{Tor}(B,A) \cong \text{Tor}(A,B)$

Up to now, we have totally proved (1) and (3) and we come to (2). For any abelian group, if $T(A)$ denote the torsion subgroup, $A/T(A)$ is torsion free. We therefore have $\text{Tor}(A/T(A),B) = 0$ for any abelian group from (1). Then we apply the second statement of (3) to the short exact sequence $0 \longrightarrow T(A) \longrightarrow A \longrightarrow A/T(A) \longrightarrow 0$ and first three terms of the six-term exact sequence is

$$0 \longrightarrow \text{Tor}(T(A),B) \longrightarrow \text{Tor}(A,B) \longrightarrow 0.$$

For each A_λ, we can find a short free resolution $0 \longrightarrow K_\lambda \longrightarrow F_\lambda \longrightarrow A_\lambda \longrightarrow 0$ and

$$0 \longrightarrow \oplus_\lambda K_\lambda \longrightarrow \oplus_\lambda F_\lambda \longrightarrow \oplus_\lambda A_\lambda \longrightarrow 0$$

is a free resolution of $\oplus_\lambda A_\lambda$. Then

$$\text{Tor}(\oplus_{\lambda \in \Lambda} A_\lambda, B) = \ker(\oplus_\lambda K_\lambda \otimes B \longrightarrow \oplus_\lambda F_\lambda \otimes B) = \bigoplus_\lambda \ker(K \otimes B \longrightarrow F \otimes B).$$

This prove the first statement of (5) and we can prove the second combined with (4).

(6), we can consider the short free resolution $0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$

$$\text{Tor}(\mathbb{Z}/m\mathbb{Z}, B) = \ker(B \xrightarrow{m} B)$$

■

Definition 11.7.3 Suppose A is an abelian group and let $0 \longrightarrow K \xrightarrow{f} F \longrightarrow A \longrightarrow 0$ be a short free resolution. Take another abelian group B and apply $\text{Hom}(\square, B)$, we can find an exact sequence

$$0 \longrightarrow \text{Hom}(A,B) \longrightarrow \text{Hom}(F,B) \xrightarrow{\text{Hom}(f,B)} \text{Hom}(K,B)$$

and we define $\text{Ext}(A,B) := \text{coker Hom}(f,B) = \text{Hom}(K,B)/im\ \text{Hom}(f,B)$. Thus $\text{Ext}(A,B)$ measures the failure for $\text{Hom}(\square, B)$ to be right exact.

Here is a more sophisticated way of viewing $\text{Ext}(A,B)$ consider a chain complex $C_1 = K, C_0 = F$, $\partial_1 : C_1 \longrightarrow C_0 = f : K \longrightarrow F$ and all other group zero. $H_0(C_\bullet) = A$. Now apply $\text{Hom}(\square, B)$ to a cochain complex $\text{Hom}(C_\bullet, B)$, the definition of $\text{Ext}(A,B)$ gives us immediately that

$$H^1(\text{Hom}(C_\bullet, B)) = \text{Ext}(A,B).$$

From this it follows that $\text{Ext}(\square, B)$ is a contravariant functor and it is well defined (independent of the choice of short free resolution.)

Definition 11.7.4 An abelian group D is said to be **divisible** if for every $b \in D$ and every $n \in \mathbb{N}$ there exists an $a \in D$ s.t., $na = b$

Theorem 11.7.3 (Properties of Ext)

For a fixed abelian group A, $\operatorname{Ext}(\square, A)$ is a contravariant functor and $\operatorname{Ext}(A, \square)$ is a covariant functor. Moreover,

(1) If A is a free group, then $\operatorname{Ext}(A,B) = 0, \forall B$. If D is a divisible abelian group, then $\operatorname{Ext}(A,D) = 0 \forall A$.

(2) If A is a finitely generated group with torsion subgroup $T(A)$ then $\operatorname{Ext}(A,\mathbb{Z}) = T(A)$

(3) $0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$ is exact, then for any B, there is an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}(A'',B) & \longrightarrow & \operatorname{Hom}(A',B) & \longrightarrow & \operatorname{Hom}(A,B) \\ & & & & & \swarrow & \\ \operatorname{Ext}(A'',B) & \longrightarrow & \operatorname{Ext}(A',B) & \longrightarrow & \operatorname{Ext}(A,B) & \longrightarrow & 0 \end{array}$$

If $0 \longrightarrow B \longrightarrow B' \longrightarrow B'' \longrightarrow 0$ is exact, then for any A, there is an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}(A,B) & \longrightarrow & \operatorname{Hom}(A,B') & \longrightarrow & \operatorname{Hom}(A,B'') \\ & & & & & \swarrow & \\ \operatorname{Ext}(A,B) & \longrightarrow & \operatorname{Ext}(A,B') & \longrightarrow & \operatorname{Ext}(A,B'') & \longrightarrow & 0 \end{array}$$

(4) If B is an abelian group and $\{A_\lambda | \lambda \in \Lambda\}$ is a collection of abelian group then

$$\operatorname{Ext}\left(\bigoplus_{\lambda\in\Lambda} A_\lambda, B\right) \cong \prod_{\lambda\in\Lambda} \operatorname{Ext}(A_\lambda, B)$$

$$\operatorname{Ext}\left(B, \bigoplus_{\lambda\in\Lambda} A_\lambda\right) \cong \prod_{\lambda\in\Lambda} \operatorname{Ext}(B, A_\lambda)$$

(5) For any $m \in \mathbb{N}$ and any B

$$\operatorname{Ext}(\mathbb{Z}/m\mathbb{Z}, B) \cong B/mB$$

Proof. $\operatorname{Ext}(A,\square)$ and $\operatorname{Ext}(\square,A)$ are covariant functor and contravariant functor respectively. The proof is identical to those in 11.7.A.

$(3)-(5)$ can be proved similar to those proof for Tor although we have $\operatorname{Ext}(A,B) \neq \operatorname{Ext} B, A$ in general.

For the first six-term exact sequence, we can need an auxiliary fact

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A \arrow[u] \arrow[r, "f"] & A' \arrow[u] \arrow[r, "g"] & A'' \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & F \arrow[u, "u"] \arrow[r, "i_1"] & F \oplus F'' \arrow[u, dashed, "u'"] \arrow[r, "p_1"] & F'' \arrow[u, "u''"] \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[u, "v"] \arrow[r, "i_2"] & K \oplus K'' \arrow[u, dashed, "v'"] \arrow[r, "p_2"] & K'' \arrow[u, "v''"] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Given the first and third column to be short free resolutions and the first row exact, we can construct the middle column to be a short free resolution of A' and each row is exact.

This is guaranteed by the Horseshoe lemma

For the second six-term exact sequence we consider $0 \longrightarrow K \xrightarrow{f} F \longrightarrow 0$ is a chain complex of free abelian group. Free module is a special case of projective module, hence $\mathrm{Hom}(K,\square)$ and $\mathrm{Hom}(F,\square)$ are exact functors.

Then we have the short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet \longrightarrow C'_\bullet \longrightarrow C''_\bullet \longrightarrow 0.$$

Because each row splits, we also have the SES of cochain complexes

$$0 \longrightarrow \mathrm{Hom}(C''_\bullet, B) \longrightarrow \mathrm{Hom}(C'_\bullet, B) \longrightarrow \mathrm{Hom}(C_\bullet, B) \longrightarrow 0$$

and we derive the six-term LES from it.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(K,B) & \longrightarrow & \mathrm{Hom}(K,B') & \longrightarrow & \mathrm{Hom}(K,B'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(F,B) & \longrightarrow & \mathrm{Hom}(F,B') & \longrightarrow & \mathrm{Hom}(F,B'') & \longrightarrow & 0 \end{array}$$

which is a short exact sequence of cochain complexes

$$0 \longrightarrow \mathrm{Hom}(C_\bullet, B) \longrightarrow \mathrm{Hom}(C_\bullet, B') \longrightarrow \mathrm{Hom}(C_\bullet, B'') \longrightarrow 0$$

(1) If A is a free abelian group, then the free resolution $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ splits then when applied with functor $\mathrm{Hom}(\square, B)$, we still get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A \oplus K, B) \xrightarrow{\mathrm{Hom}(f,B)} \mathrm{Hom}(K,B) \longrightarrow 0$$

whatever B is. $\mathrm{Hom}(f,B)(\varphi) = \varphi \circ f$, it is now surjective because for any $v \in \mathrm{Hom}(K,B)$, we can choose $0 \oplus v \in \mathrm{Hom}(A \oplus K, B)$ so that $(0 \oplus v) \circ f = v$.

For the second statement, we Claim: when D is divisible, $\mathrm{Hom}(\square, D)$ is exact.

It suffices to prove $\mathrm{Hom}(f,D)$ is surjective

$$0 \longrightarrow \mathrm{Hom}(A,D) \longrightarrow \mathrm{Hom}(F,D) \xrightarrow{\mathrm{Hom}(f,D)} \mathrm{Hom}(K,D) \longrightarrow 0$$

■

11.8 Universal coefficients

11.9 Algebraic Künneth formula

In this section we would prove an algebraic version of Künneth formula for free chain complexes. In the next section we would prove Eilenber-Zilber theorem and then derive the general Künneth formula as a corollary of the algebraic one.

Theorem 11.9.1 (Algebraic Künneth Theorem) Let (C,∂) and (D,∂') be two non-negative free complex. Then for every $n \geq 0$, there is a split exact sequence

$$0 \longrightarrow \oplus_{i+j=n} H_i(C_\bullet) \otimes H_j(D_\bullet) \xrightarrow{\omega} H_n(C_\bullet \otimes D_\bullet) \longrightarrow \oplus_{k+\ell=n-1} \mathrm{Tor}\,(H_k(C_\bullet), H_\ell(D_\bullet)) \longrightarrow 0$$

where ω is the map $\langle c_i\rangle \otimes \langle d_j\rangle \mapsto \langle c_i \otimes d_j\rangle$. Thus there also exists a (non-natural) isomorphism

$$H_n(C_\bullet \otimes D_\bullet) \cong \left(\bigoplus_{i+j=n} H_i(C_\bullet)\otimes H_j(D_\bullet)\right) \oplus \left(\bigoplus_{k+\ell=n-1} \operatorname{Tor}\left(H_k(C_\bullet), H_\ell(D_\bullet)\right)\right)$$

11.10 Eilenberg-Zilber theorem and Künneth formula

Theorem 11.10.1 (Eilenberg-Zilber) if X and Y are two topological spaces. There is a nontrivial chain equivalence

$$\Omega_\bullet : C_\bullet(X \times Y) \longrightarrow C_\bullet(X) \otimes C_\bullet(Y)$$

which is unique up to chain homotopy.

Proof. $Top \times Top$ is the category of pairs (X,Y) of topological spaces.

We will define two functor from $Top \times Top \longrightarrow Comp$

$$S_\bullet(X,Y) = C_\bullet(X,Y), \;\; T_\bullet(X,Y) = C_\bullet(X) \otimes C_\bullet(Y)$$

For models

$$\mathcal{M} = \{(\Delta^i, \Delta^j), i, j \geq 0\}$$

Claim: $S_\bullet$ and $T_\bullet$ are both acyclic in positive degree on $\mathcal{M}$ and free with basis contained in $\mathcal{M}$

$S_\bullet$, $H_n(S_\bullet(\Delta^i, \Delta^j)) = H_n(\Delta^i \times \Delta^j) = 0$, $\forall n > 0, \forall i, j$ (Acyclic in positive degrees)

$S_i : Top \times Top \longrightarrow Ab$

$S_i(X,Y) = C_i(X \times Y)$

subclaim: $\{(\Delta^i, \Delta^i)\}$ is a S_i-model set and the diagonal map $d_i : \Delta^i \longrightarrow \Delta^i \otimes \Delta^i \; x \mapsto (x,x)$ gives a model basis.

Indeed, if (X,Y) is any object in $Top \times Top$ and if $\sigma : \Delta^i \longrightarrow X \times Y$ is any singular simplex in $S_i(X \times Y) = C_i(X \times Y)$, then we can write $\sigma = (\sigma_x, \sigma_y) \circ d_i$, where $\sigma_x = p_X \circ \sigma$ be the composition of σ with $p_X : X \times Y \longrightarrow X$. $S_i(\tau)(d_i), \tau \in \operatorname{Hom}(\Delta^i \times \Delta^i, X \times Y)$ forms a basis of the free abelian group $S_i(X \times Y) = C_i(X \times Y)$.

As for T_i, we quote the exercise, for any $(X,Y) \in Top \times Top$, $C_i(X) \otimes C_j(Y)$ is free abelian with ba

$T_i(X \times Y) = (C_\bullet(X) \otimes C_\bullet(Y))$. $T_i(X,Y)$ is the tensor product of the free groups and thus is free. $\{(\ell_i, \ell_j) | i + j = n\}$ is a T_n-model basis.

The last thing to check is that $T_\bullet(\Delta^i, \Delta^j)$ is acyclic in positive degrees

$$H_n(C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j)) = 0, \forall n > 0.$$

We can not compute this! However we can cheat

$$H_n(C_\bullet(\Delta^i)) = H_n(\Delta^i) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Consider the chain complex

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \cdots$$

$C_\bullet(\Delta^i)$ has the same homology as this complex. Thus $C_\bullet(\Delta^i)$ is equivalent to the complex and $C_\bullet(\Delta^j)$ is also chain equivalent to it (By Exercise 11.6.D). $C_\bullet(\Delta^i)\otimes C_\bullet(\Delta^j)$ is chain equivalent to

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{Z}\otimes\mathbb{Z} \longrightarrow 0\cdots$$

Thus $H_n(C_\bullet(\Delta^i)\otimes C_\bullet(\Delta^j)) = H_n(\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}\otimes\mathbb{Z} \longrightarrow 0\cdots)$. We then know $T_\bullet(\Delta^i,\Delta^j)$ is indeed acyclic in positive degrees.

We have now verified that the hypotheses of the Acyclic Models Theorem and its corollary are satisfied. Define $\Theta : H_0\circ S_\bullet \longrightarrow H_0\circ T_\bullet$ is a natural equivalence.

$$\Theta(X\times Y) : H_0(C_\bullet(X\times Y)) \longrightarrow H_0(C_\bullet(X)\otimes C_\bullet(Y))$$

$$\langle c_{(x,y)}\rangle \mapsto \langle c_x\rangle\otimes\langle c_y\rangle$$

where $c_{(x,y)} : \Delta^0 \longrightarrow (x,y)$ is the constant map to point (x,y). It is indeed a natural transformation

$$\begin{array}{ccc} H_0(S_\bullet(X\times Y)) & \xrightarrow{\Theta(X\times Y)} & H_0(T_\bullet(X\times Y)) \\ {\scriptstyle H_0(S_\bullet(f,g))}\downarrow & & \downarrow{\scriptstyle H_0(T_\bullet(f,g))} \\ H_0(S_\bullet(W\times Z)) & \xrightarrow{\Theta(W\times Z)} & H_0(T_\bullet(W\times Z)) \end{array}$$

By algebraic Künneth formula 11.9.1, we know $H_0(C_\bullet(X\times Y)) \cong H_0(C_\bullet(X)\otimes C_\bullet(Y))$ and $\Theta(X,Y)$ is an isomorphism of abelian groups. The map $\langle c_x\rangle\otimes\langle c_y\rangle \mapsto \langle c_{(x,y)}\rangle$ gives the inverse of $\Theta(X,Y)$, therefore we know Θ is a natural equivalence.

By the acyclic models theorem A.1.1

$$\Omega_\bullet : S_\bullet \longrightarrow T_\bullet$$

is a natural chain equivalence such that $H_0(\Omega_\bullet) = \Theta$

We therefore find a chain equivalence when apply it to $X\times Y$

$$\Omega_\bullet(X,Y) : C_\bullet(X\times Y) \longrightarrow C_\bullet(X)\otimes C_\bullet(Y)$$

These two chain complex have isomorphic homologies. ■

Corollary 11.10.2 (Künneth formula) As a result, we can apply the algebraic Künneth formula here and derive the Künneth formula for product of topological spaces.

Then for every $n\geq 0$, there is a split exact sequence

$$0\longrightarrow\oplus_{i+j=n}H_i(C_\bullet(X))\otimes H_j(C_\bullet(Y))\xrightarrow{\omega}H_n(C_\bullet(X)\otimes C_\bullet(Y))\longrightarrow\oplus_{k+\ell=n-1}\operatorname{Tor}(H_k(C_\bullet(X)),H_\ell(C_\bullet(Y)))\longrightarrow 0$$

where ω is the map $\langle c_x\rangle\otimes\langle c_y\rangle \mapsto \langle c_x\otimes c_y\rangle$. Thus there also exists a (non-natural) isomorphism

$$\begin{aligned} H_n(X\times Y) &= H_n(C_\bullet(X\times Y)) \\ &\cong H_n(C_\bullet(X)\otimes C_\bullet(Y)) \\ &\cong \left(\bigoplus_{i+j=n} H_i(C_\bullet(X))\otimes H_j(C_\bullet(Y))\right)\oplus\left(\bigoplus_{k+\ell=n-1}\operatorname{Tor}(H_k(C_\bullet(X)),H_\ell(C_\bullet(Y)))\right) \end{aligned}$$

12. Cellular homology

13. Partition of unity in homotopy

14. Bundles

15. Manifolds

16. Homology of manifolds

17. Cohomology

17.1 Axiomatic approach

First we state the Eilenberg-Steenord axioms for cohomology. A **cohomology theory** is a family of contravariant functor $h^n | n \in \mathbb{Z}$ from the category of pair of topological spaces Top^2 to the category of R-modules $R-Mod$ together with a family of natural transformations $\delta^n | n \in \mathbb{Z}$ s.t. they satisfies the following axioms.

(1) **Homotopy invariance**. Homotopy maps induces same homomorphism.

(2) **Exact sequence**. For each pair (X,A), the sequence

$$\cdots \longrightarrow h^{n-1}(A,\emptyset) \xrightarrow{\delta} h^n(X,A) \longrightarrow h^n(X,\emptyset) \longrightarrow h^n(A,\emptyset) \xrightarrow{\delta} \cdots$$

is exact and the unspecified arrows are induced by the inclusions.

(3) **Excision.** Let (X,A) be a pair and $U \subset A$ such that $\overline{U} \subset A^\circ$. Then the inclusion $(X\backslash U, A\backslash U) \longrightarrow (X,A)$ induces an isomorphism between $h^n(X,A) \cong h^n(X\backslash U, A\backslash U)$

We say the cohomology theory is **ordinary** if it in addition satisfies the 4th axiom

(4) **Dimension**. $h^n(pt) = 0$ for $n \neq 0$.

Sometimes, we also refer to the cohomology theories satisfying only the first three axioms as **generalized cohomologies**. Notationally , we write $h^n(X,\emptyset) = h^n(X)$.

Exercise?? 17.1.A (exact sequence of a triple), prove that given a triple of topological spaces, (X,A,B), $B \subset A \subset X$. Prove there is a exact sequence of triple

$$\cdots \longrightarrow h^{n-1}(A,B) \xrightarrow{\delta} h^n(X,A) \longrightarrow h^n(X,B) \longrightarrow h^n(A,B) \xrightarrow{\delta} \cdots$$

all the unspecified arrows are induced by inclusions. ▪

Proof. For diagram chasing proof, confer [1, Chapter 4, section 8, theorem 5]. We will give here a spectral sequence proof here following the StackExchange question ■

17.2 Cohomological universal coefficients theorems

18. Duality

19. Characteristic classes

20. Homology and homotopy

21. Bordism

A. Acyclic models and model categories

A.1 Acyclic models theorem

In algebraic topology, the acyclic models theorem can be used to show that two homology theories are isomorphic and usually applying it would great simplify the proof. It can be thought of as a "universal pattern" of homology theories.

Definition A.1.1 Let $\mathscr{C}$ be a category. A family of **models** in $\mathscr{C}$ is simply an indexed subset $\mathcal{M} = \{M_\lambda | \lambda \in \Lambda\}$ of $obj(\mathscr{C})$.

Definition A.1.2 Let $\mathscr{C}$ be a category with family of models $\mathcal{M} = \{M_\lambda | \lambda \in \Lambda\}$. Suppose $T : \mathscr{C} \longrightarrow Ab$ is a functor. A T-**model set** χ is a choice of elements $x_\lambda \in T(M_\lambda)$ for each λ:

$$\chi = \{x_\lambda \in T(M_\lambda) | \lambda \in \Lambda\}$$

Definition A.1.3 Let $\mathscr{C}$ be a category with family of models $\mathcal{M} = \{M_\lambda | \lambda \in \Lambda\}$. Suppose $T : \mathscr{C} \longrightarrow Ab$ is a functor. We say that T is **free with basis in** $\mathcal{M}$ if the following condition holds:

1. $T(C)$ is a free abelian group $\forall C \in \mathscr{C}$
2. There is a T-model set $\chi = \{x_\lambda \in T(M_\lambda) | \lambda \in \Lambda\}$ s.t.

$$\{T(f)(x_\lambda) | f \in \operatorname{Hom}(M_\lambda, C), \lambda \in \Lambda\}$$

   is a basis for the free abelian group $T(C)$.

We call χ a **model basis** for T.

We say $T_\bullet : \mathscr{C} \longrightarrow Comp$ if free with basis in $\mathcal{M}$ if each T_n is free with basis in $\mathcal{M}$.

Definition A.1.4 $T_\bullet : \mathscr{C} \longrightarrow Comp$, we say $T_\bullet$ is **non-negative** if $T_n(C) = 0$ for all $n < 0$ and $\forall C$. $T_\bullet$ is **acyclic in the positive degrees on** C or C **is $T_\bullet$-acyclic** if $H_n(T_\bullet(C)) = 0, \forall n > 0$.

■ **Example A.1.1** Take $\mathscr{C} = Top$, $\mathscr{M} = \{\Delta^n | n \geq 0\}$. $T_\bullet$ is the singular chain functor.

$$\mathscr{C}_\bullet : Top \longrightarrow Comp$$

$$X \mapsto C_\bullet(X)$$

By definition, $T_\bullet$ is free with basis in $\mathscr{M}$. Then $T_\bullet$ is non-negative because $C_\bullet$ is non-negative, ✓. Also, Δ^n is $T_\bullet$-acyclic $H_n(C_\bullet(\Delta^i)) = H_n(\Delta^i) = 0, \forall n > 0$✓. (We say $\mathscr{M}$ is $T_\bullet$-acyclic) ■

Theorem A.1.1 Suppose $\mathscr{C}$ is a category with models $\mathscr{M}$. Suppose $T_\bullet, S_\bullet : \mathscr{C} \longrightarrow Comp$ are two functors such that both $T_\bullet$ and $S_\bullet$ are non-negative. Assume further $T_\bullet$ is free with basis in $\mathscr{M}$ and $S_\bullet$ is acyclic in the positive degree on each element $M \in \mathscr{M}$.

Suppose

$$\Theta : H_0 \circ T_\bullet \longrightarrow H_0 \circ S_\bullet$$

is a natural transformation. $\exists$ a natural chain morphism $\Psi_\bullet : T_\bullet \longrightarrow S_\bullet$ which is unique up to natural chain homotopy and has $H_0(\Psi_\bullet) = \Theta$.

Corollary A.1.2 We will be mostly interested in the case where both $S_\bullet$ and $T_\bullet$ are free with basis $\mathscr{M}$ and that each model $M \in \mathscr{M}$ is both $S_\bullet$-acyclic and $T_\bullet$-acyclic. In this case if $\Theta : H_0 \circ T_\bullet \longrightarrow H_0 \circ S_\bullet$ is a natural equivalence then every natural chain map $\Phi_\bullet$ inducing Θ is natural chain equivalence.

Corollary A.1.3 Suppose $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ are two non-negative chain complexes. Assume $C_\bullet$ is free and that $D_\bullet$ is acyclic in positive degrees. Then given any homomorphism $h : H_0(C_\bullet) \longrightarrow H_0(D_\bullet)$, there exists a chain map $f_\bullet : C_\bullet \longrightarrow D_\bullet$ over h and this chain map is unique up to chain homotopy.

Moreover, if both non-negative chain complexes $C_\bullet$ and $D_\bullet$ are free and acyclic, and $h : H_0(C_\bullet) \longrightarrow H_0(D_\bullet)$ is an isomorphism, then the chain map $g_\bullet$ is a chain equivalence.

Both corollaries can be directly derived, the first from the definition of chain equivalence and in the second, we choose the category with only one object.

To prove Theorem A.1.1, we need to first quote some two lemmas

Lemma A.1.4 Let $\mathscr{C}$ be a category with family of models $\mathscr{M} = \{M_\lambda | \lambda \in \Lambda\}$. Assume $S, T : \mathscr{C} \longrightarrow Ab$ are functors and assume T is free with basis in $\mathscr{M}$. Let $\chi := \{x_\lambda \in T(M_\lambda) | \lambda \in \Lambda\}$ denote the model basis for T. Choose element $y_\lambda \in S(M_\lambda)$ for each $\lambda \in \Lambda$, and set $\Upsilon := \{y_\lambda \in S(M_\lambda) | \lambda \in \Lambda\}$. Then there exists a unique natural transformation $\Phi : T \longrightarrow S$ such that

$$\Phi(M_\lambda)(x_\lambda) = y_\lambda, \forall \lambda \in \Lambda$$

Proof. Because T is free with basis $\mathscr{M}$, we know for each $C \in \mathscr{C}$, $T(C)$ is free abelian group and

$$\{T(f)(x_\lambda) | f \in \mathrm{Hom}(M_\lambda, C), \lambda \in \Lambda\}$$

is a basis for the free abelian group $T(C)$. For fixed $\lambda \in \Lambda$ and fixed object $C \in obj(\mathscr{C})$, we have a commutative diagram for every morphism $f : M_\lambda \longrightarrow C$

$$\begin{tikzcd} T(M_\lambda) \arrow[r, "T(f)"] \arrow[d, "\Phi(M_\lambda)"'] & T(C) \arrow[d, "\Phi(C)"] \\ S(M_\lambda) \arrow[r, "S(f)"'] & S(C) \end{tikzcd}$$

We have $\Phi(C) \circ T(f)(x_\lambda) = S(f)(y_\lambda)$. Since $T(f)(x_\lambda)$ forms a basis of $T(C)$, we know $\Phi(C)$ is uniquely determined, therefore Φ is unique if it exists.

It indeed exists. Fix any object $C \in \mathscr{C}$, then by assumption $\{T(f)(x_\lambda)\}$ form a basis of $T(C)$ and $\Phi(C)(T(f)(x_\lambda)) = S(f)(x_\lambda)$ by the universal property of free abelian group, there exists a unique homomorphism $\Phi(C) : T(C) \longrightarrow S(C)$ that restricts to it on basis.

We have proved each individual $\Phi(C)$ exists and is unique. It only lefts to check that such specified Φ is indeed a natural transformation

$$\begin{tikzcd} T(A) \arrow[r, "T(g)"] \arrow[d, "\Phi(A)"'] & T(B) \arrow[d, "\Phi(B)"] \\ S(A) \arrow[r, "S(g)"'] & S(B) \end{tikzcd}$$

Given a typical basis element $T(f)(x_\lambda)$ for some $\lambda \in \Lambda$ and $f \in \operatorname{Hom}(M_\lambda, A)$. Then

$$S(g) \circ \Phi(A)(T(f)(x_\lambda)) = S(g)S(f)y_\lambda = S(g \circ f)y_\lambda.$$

But also going the other way round:

$$\Phi(B) \circ T(g)(T(f)(x_\lambda)) = \Phi(B)(Tg \circ f)(x_\lambda) = S(g \circ f)(x_\lambda).$$

Thus Φ is indeed a natural transformation. ■

Lemma A.1.5 Let $\mathscr{C}$ be category with family of models $\mathcal{M}$. Suppose given six functors $T_i, S_i : \mathscr{C} \longrightarrow Ab$, $i = 0, 1, 2.$ together with six natural transformations as pictured below

$$\begin{tikzcd} T_2 \arrow[r, "\Phi_2"] & T_1 \arrow[r, "\Phi_1"] \arrow[d, "\Theta_1"] & T_0 \arrow[d, "\Theta_0"] \\ S_2 \arrow[r, "\Psi_2"'] & S_1 \arrow[r, "\Psi_1"'] & S_0 \end{tikzcd}$$

Assume that

1. For every object $C \in obj(\mathscr{C})$, the composition $\Phi_1(C) \circ \Phi_2(C) : T_2(C) \longrightarrow T_0(C)$ is the zero homomorphism.
2. The bottom row is exact on $\mathcal{M}$, in the sense that for every model $M \in \mathcal{M}$, one has $im\,\Psi_2(M) = \ker \Psi_1(M)$.
3. The diagram commutes for every object $C \in obj(\mathscr{C})$.
4. T_2 is free with basis in M.

Then there exists a natural transformation $\Gamma : T_2 \longrightarrow S_2$ such the the first square commutes for every object of $\mathscr{C}$.

$$\begin{tikzcd} T_2 \arrow[r, "\Phi_2"] \arrow[d, dashed, "\Gamma"] & T_1 \arrow[r, "\Phi_1"] \arrow[d, "\Theta_1"] & T_0 \arrow[d, "\Theta_0"] \\ S_2 \arrow[r, "\Psi_2"'] & S_1 \arrow[r, "\Psi_1"'] & S_0 \end{tikzcd}$$

Proof. Let $\chi = \{x_\lambda \in T_2(M_\lambda) | \lambda \in \Lambda\}$ denote a model basis for T_2. Then for each $\lambda \in \Lambda$ we have a commutative diagram in *Ab* such that both top row and bottom row are chain complex and the bottom row is exact. Also $T_2(M_\lambda)$ is free

$$\begin{tikzcd} T_2(M_\lambda) \arrow[r, "f_2"] \arrow[d, dashed, "\gamma"] & T_1(M_\lambda) \arrow[r, "f_1"] \arrow[d, "t_1"] & T_0(M_\lambda) \arrow[d, "t_0"] \\ S_2(M_\lambda) \arrow[r, "g_2"'] & S_1(M_\lambda) \arrow[r, "g_1"'] & S_0(M_\lambda) \end{tikzcd}$$

$im\, t_1 \circ f_2 \subset im\, g_2$ because $g_1 \circ t_1 \circ f_2 = h \circ f_1 \circ f_2 = 0$, hence $im\, t_1 \circ f_2 \subset \ker g_1 = im\, g_2$. For each x_λ, there is a $y_\lambda \in S_2(M_\lambda)$ such that $g_2(y_\lambda) = t_1 \circ f_2(x_\lambda)$ because g_2 is surjective. By universal property of free module, we get a unique morphism $\gamma : T_2(M_\lambda) \longrightarrow S_2(M_\lambda)$ such that $\gamma(x_\lambda) = y_\lambda$. (But because y_λ are not unique, γ is not the unique morphism that makes the triangle commute)

$$\begin{tikzcd} & T_2(M_\lambda) \arrow[dl, dashed, "\exists\gamma"'] \arrow[d, "t_1 \circ f_2"] & \\ S_2(M_\lambda) \arrow[r, "g_2"] & im\, g_2 \arrow[r] & 0. \end{tikzcd}$$

(It also makes the first square of the previous diagram commutes). We then know by A.1.4 there exists a unique natural transformation $\Gamma : T_2 \longrightarrow S_2$ such that

$$\Gamma(M_\lambda)(x_\lambda) = y_\lambda, \forall \lambda \in \Lambda.$$

It remains to check thus constructed Γ makes the functor diagram commutes. If we define $z_\lambda := \Psi_2(M_\lambda)(y_\lambda) = \Theta_1(M_\lambda)(\Phi(M_\lambda)(x_\lambda))$, $\Theta_1 \circ \Phi_2$ and $\Psi \circ \Gamma$ are two natural transformations that sends x_λ to z_λ. Then we know the first square of functor diagram commutes by the uniqueness in A.1.4. ∎

Finally we come back to the proof of Theorem A.1.1.

Proof. of theorem A.1.1

1. Such $\Phi_\bullet$ exists: We need to construct natural transformations $\Phi_n : S_n \longrightarrow T_n$ such that the following diagram commutes.

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & T_2 \arrow[r, "\partial"] \arrow[d, dashed, "\Phi_2"] & T_1 \arrow[r, "\partial"] \arrow[d, dashed, "\Phi_1"] & T_0 \arrow[r] \arrow[d, dashed, "\Phi_0"] & H_0(T_\bullet) \arrow[r] \arrow[d, "\Theta"] & 0 \\ \cdots \arrow[r, "\partial'"] & S_2 \arrow[r, "\partial'"] & S_1 \arrow[r, "\partial'"] & S_0 \arrow[r] & H_0(S_\bullet) \arrow[r] & 0 \end{tikzcd}$$

With Lemma A.1.5 in hand, we can induct on n to concatenate the "ladders".

For $n = 0$, we have

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, dashed, "\exists\Phi_0"'] & H_0(T_\bullet) \arrow[r] \arrow[d] & 0 \arrow[d] \\ S_0 \arrow[r] & H_0(S_\bullet) \arrow[r] & 0 \end{tikzcd}$$

■

If we find $\Phi_i, 0 \le i \le n$ such that makes the first n ladder commute, we can find a Φ_{n+1} that makes the diagram commute

$$\begin{tikzcd} T_{n+1} \arrow[r] \arrow[d, dashed, "\exists\Phi_{n+1}"'] & T_n \arrow[r] \arrow[d, "\Phi_n"] & T_{n-1} \arrow[d, "\Phi_{n-1}"] \\ S_{n+1} \arrow[r] & S_n \arrow[r] & S_{n-1} \end{tikzcd}$$

2. $H_0(\Phi_\bullet) = \Theta$: This is a direct result of the corresponding diagram of chain complexes. On each object C, $H_0(\Phi_\bullet)(C) : H_0(T_\bullet(C)) \longrightarrow H_0(S_\bullet(C)) : \langle c \rangle \mapsto \langle \Phi_0(c) \rangle$ Then we have $H_0(\Phi_\bullet)(C) = \Theta(C)$ because the right most square commutes when applied to object C.

3. $\Phi_\bullet$ is unique up to natural chain homotopy Suppose now we have two such maps $\Psi_\bullet, \Phi_\bullet : T_\bullet \longrightarrow S_\bullet$. We need to find natural transformation $\Upsilon_n : T_n \longrightarrow S_{n+1}$ for $n \ge -1$ such that

$$\partial' \Upsilon_n + \Upsilon_{n-1} \partial = \Phi_n - \Psi_n.$$

Denote the difference $\Phi_n - \Psi_n =: \Xi_n$. We define $\Upsilon_{-1} = 0$ and proceed inductively. We have a diagram

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, dashed, "\exists\Upsilon_0"'] & T_0 \arrow[r] \arrow[d, "\Xi_0"] & 0 \arrow[d, "0"] \\ S_1 \arrow[r] & S_0 \arrow[r] & H_0(S_\bullet), \end{tikzcd}$$

where we have used A.1.5 again. Inductively, If we have constructed Υ_{n-1}, we have the following diagram

$$\begin{tikzcd} T_n \arrow[r, "id"] \arrow[d, dashed, "\exists\Upsilon_n"'] & T_n \arrow[r] \arrow[d, "\Xi_n - \Upsilon_{n-1}\circ\partial"] & 0 \arrow[d, "0"] \\ S_{n+1} \arrow[r, "\partial'"] & S_n \arrow[r] & S_{n-1} \end{tikzcd}$$

By induction hypothesis, the above diagram commutes because

$$\begin{aligned} &\partial'(\Xi_n - \Upsilon_{n-1}\partial) \\ &\quad = \partial'\Xi_n - \partial'\Upsilon_{n-1}\partial \\ &\quad = \partial'\Xi_n - (\Xi_{n-1} - \Upsilon_{n-1}\partial)\partial \\ &\quad = \partial'\Xi_n - \Xi_{n-1}\partial \\ &\quad = 0. \end{aligned}$$

Hence, we can construct Υ_n by A.1.5.

A.2 Model categories

Model categories catch the essence of homotopy. In order to study homotopic invariant, we need machinery to construct (weak) homotopy invariant functors.

Definition A.2.1 A **model category** on a category $\mathscr{C}$ consists of three distinguished classes of morphisms:**weak equivalences, fibrations** and **cofibrations** and two functorial factorizations (α,β) and (γ,δ) subject to the following axioms. Each classes of morphisms contain all the identities and are closed under composition of morphisms.

MC1: Finite limits and colimits exists in $\mathscr{C}$

MC2: If f,g are maps in $\mathscr{C}$ such that $g\circ f$ is defined and if two of the three maps $f,g,g\circ f$ are weak equivalences, then so is the third.

MC3: If f is a retract of g and g is a fibration, cofibration or weak equivalence, then so if f.

MC4: Given a commutative diagram of the

Similar with the localization of rings, we can inverse morphisms for some collection of morphisms.

Definition A.2.2 Localization of a category: Given a category $\mathscr{C}$ and some class W of morphisms in $\mathscr{C}$. (similar to localization of rings, we don't need W to be multiplicatively closed, because the construction will naturally lead to multiplicative closure of W). We can define it by universal property: there is a natural localization functor $\mathscr{C}\longrightarrow W^{-1}\mathscr{C}$ and given any other category $\mathscr{D}$, a functor $F:\mathscr{C}\longrightarrow\mathscr{D}$ factors uniquely through $\mathscr{C}\longrightarrow W^{-1}\mathscr{C}$ iff F sends every morphism in W to an isomorphism in $\mathscr{D}$

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\iota_W"'] & \mathscr{D} \\ W^{-1}\mathscr{C} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

B. Derived functors and derived categories

B.1 More on Tor and Ext

Bibliography

Articles

Books

[1] Edwin H Spanier. *Algebraic topology*. Volume 55. 1. Springer Science & Business Media, 1989 (cited on page 49).