LECTURE 28

Cochain complexes and cohomology

The last four lectures have all centred on how the homology functor interacts with the tensor product functor $\square \otimes A$. We now repeat this theme, only instead of the functor $\square \otimes A$ we use the functor $\mathrm{Hom}(\square, A)$.

To begin with though, consider the functor $\mathrm{Hom}(A, \square)\colon \mathsf{Ab} \to \mathsf{Ab}$ (i.e. with the $\square$ in the second position instead.) This is defined as one might expect: to an abelian group B it associates the abelian group $\mathrm{Hom}(A, B)$, and if $f\colon B \to B'$ is a homomorphism between two abelian groups then

$$\mathrm{Hom}(A, f)\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$$

is defined by sending $g\colon A \to B$ to $f \circ g\colon A \to B'$. It is routine to see that this is an additive well-defined functor.

However, when we try this with the functor $\mathrm{Hom}(\square, A)$, we come across a problem. In this case if $f\colon B \to B'$ is a homomorphism then there is a natural induced map that sends a homomorphism $g\colon B' \to A$ to $g \circ f\colon B \to A$. Denoting this homomorphism by $\mathrm{Hom}(f, A)$, we have

$$\mathrm{Hom}(f, A)\colon \mathrm{Hom}(B', A) \to \mathrm{Hom}(B, A).$$

But this goes the "wrong" way round! This means that $\mathrm{Hom}(\square, A)$ is *not* a functor (at least as we have defined functors so far). Luckily, this can be easily rectified, by taking a slightly more liberal-minded approach to the definition of a functor.

Definition 28.1. Let C be a category. The **opposite category** is the category C^{op} with

$$\mathrm{obj}(\mathsf{C}^{\mathrm{op}}) := \mathrm{obj}(\mathsf{C}),$$

and morphism sets given by,

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathsf{C}}(B, A), \qquad A, B \in \mathrm{obj}(\mathsf{C}).$$

The composition $\circ^{\mathrm{op}}$ in C^{op} is defined by

$$f \circ^{\mathrm{op}} g := g \circ f,$$

where $\circ$ is the composition in C. This makes sense, i.e. it defines a map

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) \times \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(B, C) \to \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, C).$$

One easily checks that C^{op} is a well-defined category; the identity element in $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, A)$ is just the identity element in $\mathrm{Hom}_{\mathsf{C}}(A, A)$, and associativity of $\circ^{\mathrm{op}}$ follows from associativity of $\circ$.

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We can use the notion of the opposite category to extend the definition of a functor.

Definition 28.2. Let C and D be categories. A **contravariant functor** $T\colon \mathsf{C} \to \mathsf{D}$ is simply a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$. Let us spell out exactly what this means: A contravariant functor associates to each $A \in \mathrm{obj}(\mathsf{C})$ an object $T(A) \in \mathrm{obj}(\mathsf{D})$, and to each morphism $A \xrightarrow{f} B$ in C a morphism $T(B) \xrightarrow{T(f)} T(A)$ in D which satisfies the following two axioms:

1. If $A \xrightarrow{f} B \xrightarrow{g} C$ in C then $T(C) \xrightarrow{T(g)} T(B) \xrightarrow{T(f)} T(A)$ in D and
$$T(g \circ f) = T(f) \circ T(g).$$
2. $T(\mathrm{id}_A) = \mathrm{id}_{T(A)}$ for every $A \in \mathrm{obj}(\mathsf{C})$.

In other words, a contravariant functor is defined in exactly the same way as a normal functor, apart from the fact that it *reverses* the directions of the arrows.

Remark 28.3. Contravariant functors are not really anything new, since they are just (normal) functors from the opposite category. In particular, up to remembering to reverse directions of arrows, all the abstract results we have proved about functors between categories continue to hold for contravariant functors too. As an easy test of the definitions, I invite you to explore what a natural transformation between two contravariant functors looks like.

Example 28.4. With this new terminology, if A is an abelian group then $\mathrm{Hom}(\square, A)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a contravariant functor.

The type of functor we have studied up to now (i.e. with the arrows pointing the right way round) is sometimes called a **covariant functor**. When no confusion is possible, we will normally refer to both covariant and contravariant functors simply as "functors".

Remark 28.5. If $T\colon \mathsf{C} \to \mathsf{C}$ is a contravariant functor from a given category to itself, then $T \circ T$ is a covariant functor (as reversing the arrows twice means they go in the right direction again). Thus $\mathrm{Hom}(\mathrm{Hom}(\square, A), A)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a functor. Taking $A = \mathbb{R}$ and restricting to real vector spaces, this is the functor that assigns to a vector space its double dual, cf. Theorem 21.5.

Let us see what happens when we apply the contravariant functor $\mathrm{Hom}(\square, A)$ to a chain complex $(C_\bullet, \partial)$. Applying $\mathrm{Hom}(\square, A)$ to $\partial\colon C_n \to C_{n-1}$ we get maps

$$d := \mathrm{Hom}(\partial, A)\colon \mathrm{Hom}(C_{n-1}, A) \to \mathrm{Hom}(C_n, A),$$

defined by

$$d\gamma(c) = \gamma(\partial c), \qquad \gamma \in \mathrm{Hom}(C_{n-1}, A),\ c \in C_n,$$

and hence we get a sequence

$$\cdots \longleftarrow \mathrm{Hom}(C_{n+1}, A) \xleftarrow{d} \mathrm{Hom}(C_n, A) \xleftarrow{d} \mathrm{Hom}(C_{n-1}, A) \longleftarrow \cdots$$

This doesn't immediately give us a chain complex, since the arrows are going the wrong way. But this is easily fixed. Let us define

$$\tilde{C}_{-n} := \mathrm{Hom}(C_n, A).$$

LEMMA 28.6. $(\tilde{C}_\bullet, d)$ *is a chain complex.*

Proof. We need only check that $d \circ d = 0$. But this is obvious: if $\gamma \in \tilde{C}_{-n} = \mathrm{Hom}(C_n, A)$ and $d\gamma = \delta$ then for any $c \in C_{n+2}$ we have

$$d\delta(c) = \delta(\partial c) = \gamma(\partial^2 c) = 0.$$

Thus $d\delta$ is zero[1] in $\mathrm{Hom}(C_{n+2}, A) = \tilde{C}_{-n-2}$. ∎

Nevertheless, negative indices are annoying.

So we introduce a notational "trick". Set:

$$C^n := \tilde{C}_{-n}.$$

Then d is a map $C^n \to C^{n+1}$. This gives us the notion of a **cochain complex**.

DEFINITION 28.7. A **cochain complex** is a sequence of abelian groups and homomorphisms

$$\dots \longrightarrow C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} C^{n+1} \longrightarrow \dots$$

for $n \in \mathbb{Z}$ which satisfies

$$d^2 = 0, \qquad \forall\, n \in \mathbb{Z}.$$

We refer to the entire complex as $(C^\bullet, d)$ or sometimes just $C^\bullet$. The maps d are called the **differentials**[2] of the cochain complex.

DEFINITION 28.8. The fact that $d^2 = 0$ means that if we define

$$Z^n = Z^n(C^\bullet) = \ker d\colon C^n \to C^{n+1}$$

and

$$B^n = B^n(C^\bullet) = \operatorname{im} d\colon C^{n-1} \to C^n$$

then

$$B^n \subseteq Z^n.$$

We call elements of Z_n n-**cocycles** and elements of B^n n-**coboundaries**. We define the n**th cohomology group** of the cochain complex $C^\bullet$ to be the quotient group

$$H^n = H^n(C^\bullet) := Z^n(C^\bullet) \Big/ B^n(C^\bullet).$$

If[3] $\gamma \in Z^n$ then we will continue to the use the notation $\langle\gamma\rangle$ to denote the class in H^n.

[1] This argument is implicitly using that $\mathrm{Hom}(\square, A)$ is an additive functor: $d^2 = \mathrm{Hom}(\partial, A)^2 = \mathrm{Hom}(\partial^2, A) = \mathrm{Hom}(0, A) = 0$; the last equality is only true due to additivity.

[2] The name "differential" is used (instead of "coboundary operator") because of *differential forms* in differential geometry, which gives rise to the *de Rham cohomology* of a manifold.

[3] I will usually use Greek letters for elements of cochain complexes, to help differentiate them from elements of chain complexes.

It is important to realise that (like with contravariant functors) we are not really doing anything new here: if $(C^\bullet, d)$ is a cochain complex then setting $\tilde{C}_{-n} := C^n$ gives us a chain complex, and the homology in degree $-n$ is the same as the cohomology in degree n:

$$H_{-n}(\tilde{C}_\bullet) = H^n(C^\bullet).$$

For this reason, we will *not* introduce the category of cochain complexes, since it is just the same as the category of chain complexes, modulo our trick of replacing n by $-n$. More importantly, this means that all the basic homological algebra results we already proved for chain complexes continue to hold for cochain complexes, without needing to reprove them. For instance, Theorem 11.5 still holds, which we will need at the end of the lecture[4].

As far as this course is concerned, the most important example of a cochain complex is the following:

DEFINITION 28.9. Let X be a topological space and let A be an abelian group. The **singular cochain complex of X with coefficients in A** is the cochain complex $C^\bullet(X; A)$ where $C^n(X; A) := \mathrm{Hom}(C_n(X), A)$. We denote by $Z^n(X)$ and $B^n(X)$ the cocycles and coboundaries of this complex.

The **singular cohomology of X with coefficients in A** is the cohomology of this complex. Taking $A = \mathbb{Z}$, we obtain the **singular cohomology** $H^\bullet(X)$ of X.

PROPOSITION 28.10. *Singular cohomology with coefficients in A defines a contravariant functor* $\mathsf{Top} \to \mathsf{Ab}$.

Proof. If $f\colon X \to Y$ is a continuous map, then we can define

$$f^n_\#\colon C^n(Y; A) \to C^n(X; A), \qquad f^n_\#(\gamma)(\sigma) := \gamma(f^\#_n(\sigma)) = \gamma(f \circ \sigma),$$

for $\gamma\colon C_n(Y) \to A$ and $\sigma\colon \Delta^n \to X$ a singular n-simplex in X. If $\gamma \in Z^n(Y)$ then we claim that $f^n_\#\gamma \in Z^n(X)$. Indeed,

$$d(f^n_\#\gamma)(\sigma) = (f^n_\#\gamma)(\partial\sigma) = \gamma(f^\#_n \partial\sigma) \overset{(*)}{=} \gamma(\partial f^\#_{n-1}\sigma) = (d\gamma)(f^\#_{n-1}\sigma) = 0,$$

where $(*)$ used the fact that we already know that $f^\#$ is a chain map $C_\bullet(X) \to C_\bullet(Y)$ (Proposition 7.20). Similarly if $\gamma \in B^n(Y)$ then $f^n_\#\gamma \in B^n(X)$. Thus $f_\#$ induces a map $H^n(f)\colon H^n(Y; A) \to H^n(X; A)$. One easily sees that $H^n(\mathrm{id}_X) = \mathrm{id}_{H^n(X)}$ and that if $f\colon X \to Y$ and $g\colon Y \to Z$ then

$$H^n(f \circ g) = H^n(g) \circ H^n(f)\colon H^n(Z; A) \to H^n(X; A).$$

■

Let us now investigate the analogue of Eilenberg-Steenrod axioms for cohomology.

[4]**Warning:** Not everything works in exactly the same way though! For instance, since $\square \otimes B$ is only right exact, tensoring a cochain complex is slightly different to tensoring a chain complex. We will investigate this more next lecture.

PROPOSITION 28.11 (The dimension axion for cohomology). *Let X be a one-point space. Then for any abelian group A,*

$$H^n(X) = \begin{cases} A, & n = 0, \\ 0, & n \geq 0. \end{cases}$$

Proof. Recall from the proof of the dimension axiom for homology (see the solution to Problem D.3) that if X is a one-point space then $C_n(X) \cong \mathbb{Z}$ for all $n \geq 0$, and that $\partial \colon C_n(X) \to C_{n-1}(X)$ is an isomorphism when n is even and positive, and zero if n is odd. Dualising, it follows immediately that $H^n(X; A) = 0$ for all $n \geq 1$.

So let us look at $H^0(X; A)$. We have

$$C_1(X) \xrightarrow{\partial = 0} C_0(X) \xrightarrow{0} 0.$$

Thus applying $\mathrm{Hom}(\square, A)$ we get

$$0 \to \mathrm{Hom}(C_0(X), A) \xrightarrow{d=0} \mathrm{Hom}(C_1(X), A),$$

and hence

$$H^0(X; A) = \ker(d \colon \mathrm{Hom}(C_0(X), A) \to \mathrm{Hom}(C_1(X), A) = \mathrm{Hom}(C_0(X), A) \cong \mathrm{Hom}(\mathbb{Z}, A).$$

But $\mathrm{Hom}(\mathbb{Z}, A) \cong A$, since a homomorphism $\varphi \colon \mathbb{Z} \to A$ is uniquely determined by $\varphi(1) \in A$. ∎

THEOREM 28.12. *If $f, g \colon X \to Y$ are homotopic then they induce the same homomorphism $H^n(Y; A) \to H^n(X; A)$ for all $n \geq 0$.*

Proof. Recall from Lecture 8 that the main step in the proof of the homotopy axiom was to show that the following claim: If X is a topological space and we define inclusions $\imath, \jmath \colon X \hookrightarrow X \times I$ by

$$\imath(x) := (x, 0), \qquad \jmath(x) := (x, 1).$$

Then there exists a chain homotopy $P \colon C_n(X) \to C_{n+1}(X \times I)$ such that

$$\partial P + P \partial = \jmath^{\#} - \imath^{\#}$$

(this was Proposition 8.5.) Applying the functor $\mathrm{Hom}(\square, A)$ to P, we get a map $Q := \mathrm{Hom}(P, A)$. One checks that Q satisfies.

$$dQ + Qd = \jmath_{\#} - \imath_{\#}.$$

This allows us to finish the proof in exactly the same way as we did in Theorem 8.9: Let $F \colon X \times I \to Y$ be a homotopy from f to g. Then

$$f = F \circ \imath, \qquad g = F \circ \jmath.$$

Thus as H^n is a contravariant functor, we have

$$H^n(f) = H^n(F \circ \imath) = H^n(\imath) \circ H^n(F) = H_n(\jmath) \circ H^n(F) = H_n(g).$$

∎

Now let us define relative cohomology and prove the analogue of the long exact sequence axiom. First, let us note the following result, whose proof is on Problem Sheet N.

Proposition 28.13. *Let A be an abelian group. The contravariant functor* $\mathrm{Hom}(\square, A)$ *is left exact. That is, if $B \xrightarrow{f} B' \xrightarrow{g} B'' \to 0$ is exact then*

$$9 \to \mathrm{Hom}(B'', A) \xrightarrow{\mathrm{Hom}(g,A)} \mathrm{Hom}(B', A) \xrightarrow{\mathrm{Hom}(f,A)} \mathrm{Hom}(B, A)$$

is exact.

In fact, we will need the following result, whose proof is an immediate consequence of Lemma 25.13 (cf. the solution to Problem L.4.)

Lemma 28.14. *If $T\colon \mathsf{Ab} \to \mathsf{Ab}$ is an additive contravariant functor and $0 \to A \to B \to C \to 0$ is any split exact sequence, then $0 \to T(C) \to T(B) \to T(A) \to 0$ is also a split exact sequence.*

We then use:

Proposition 28.15. *Let $X' \subseteq X$ and let A be an abelian group. Then for every $n \geq 0$ there is a short exact sequence of abelian groups:*

$$0 \to \mathrm{Hom}\big(C_n(X)/C_n(X'), A\big) \to \mathrm{Hom}(C_n(X), A) \to \mathrm{Hom}(C_n(X'), A) \to 0.$$

Thus there is also a short exact sequence of complexes:

$$0 \to \mathrm{Hom}\big(C_\bullet(X)/C_\bullet(X'), A\big) \to C^\bullet(X; A) \to C^\bullet(X'; A) \to 0.$$

Proof. The group $C_n(X)/C_n(X')$ is a free abelian group: a basis is given by all cosets of the form $\sigma + C_n(X')$ where $\sigma\colon \Delta^n \to X$ has $\operatorname{im}\sigma \nsubseteq X'$. Thus by Problem F.6, the sequence $0 \to C_n(X') \to C_n(X) \to C_n(X)/C_n(X') \to 0$ is a split exact sequence. By Lemma 28.14, the sequence is still split exact after applying $\mathrm{Hom}(\square, A)$, and then by Problem E.5 the sequence of complexes is also exact. ■

Definition 28.16. Let $X' \subseteq X$ be a subspace, and let A be an abelian group. We define the **relative cohomology groups with coefficients in** A, written $H^n(X, X'; A)$ of the pair (X, X') to be the cohomology of the complex $C^\bullet(X, X'; A) = \mathrm{Hom}\big(C_\bullet(X)/C_\bullet(X'), A\big)$.

It now follows directly from Theorem 11.5 that there is a long exact sequence in cohomology.

Theorem 28.17 (The exact sequence axiom for cohomology). *Let $X' \subseteq X$ and let A be an abelian group. Then there is an exact sequence*

$$\cdots \to H^n(X, X'; A) \to H^n(X; A) \to H^n(X'; A) \xrightarrow{\delta} H^{n+1}(X, X'; A) \dots$$

Moreover the connecting homomorphisms $\delta\colon H^n(X'; A) \to H^{n+1}(X, X'; A)$ are natural.

The homotopy axiom extends to relative cohomology. The remaining axiom is excision. For this we will just state the result. The proof is an easy adaption of the proof of excision for homology (Theorem 14.8.)

THEOREM 28.18 (The excision axiom for cohomology). *Assume that X_1, X_2 are subspaces of X such that $X = X_1^\circ \cup X_2^\circ$. Let A be an abelian group. Then the inclusion $\imath\colon (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ induces an isomorphism in cohomology:*

$$H^n(\imath)\colon H^n(X, X_2; A) \to H^n(X_1, X_1 \cap X_2; A) \qquad \forall\, n \geq 0.$$

I will leave it up to you to formulate the precise axioms for a *cohomology theory with coefficients in A*, and check that we have now verified that singular cohomology with coefficients in A satisfies these axioms (modulo that, as with singular homology, we can't actually verify the weak equivalence axiom yet, as explained in item (2) of the remarks after Definition 21.9)

On Problem Sheet N, you are asked to state and prove the analogue of the Mayer-Vietoris sequence (Theorem 14.9) for cohomology.