

# **Algebraic Topology**

**A solution manual by and for stupid student**

**Vector\_Cat**

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## 1. Topological spaces



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## 2. Fundamental groups





### **3. Covering spaces**



## 4. Elementary homotopy theory

### 4.1 The mapping cylinder

**Definition 4.1.1** Given a continuous map  $f : X \rightarrow Y$  of topological spaces, one can define its **mapping cylinder** as a pushout (fibered coproduct)

$$\begin{array}{ccccc}
 & & B & & \\
 & \swarrow \exists! & & \searrow & \\
 & Z(f) & \xleftarrow{f \times id} & X \times I & \\
 & \uparrow r & & \uparrow i_0 & \\
 & Y \times I & & X & \\
 & \uparrow i_0 & \xrightarrow{f} & & \\
 & Y & & & 
 \end{array}$$

Set-theoretically, the mapping cylinder is usually represented as the quotient space  $(X \times I \amalg Y) / \sim$ , where  $f(x) \sim (x, 0)$ . We use  $Mf$  to denote it. (other notations are used including  $Mf$ ,  $M_f$  and  $\text{Cyl}(f)$ .)

Notice that it is  $Mf$  rather than  $Y \times I$  that plays the role of pushout because the map  $r$  is not unique. Our only restriction on  $r$  is  $r \circ j = id$ , where  $j : Mf \rightarrow Y \times I$  is the map that restricts to  $f \times id$  on  $X \times I$  and restricts to  $i_0$  on  $Y$ .

**R** Another equivalent definition is used in tom Dieck.

In the following, we consider  $X \amalg Y$  as subspace of  $Z(f)$  via the map  $J : J(x) = [(x, 0)]$  and  $J(y) = [y]$ . Then we consider a homotopy commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \alpha \downarrow & & \downarrow \beta \\
 X' & \xrightarrow{f'} & Y',
 \end{array}$$

where the diagram commutes up to a homotopy  $\Psi : f' \circ \alpha \simeq \beta \circ f$ .





## **5. Cofibrations and fibrations**



## 6. Homotopy groups





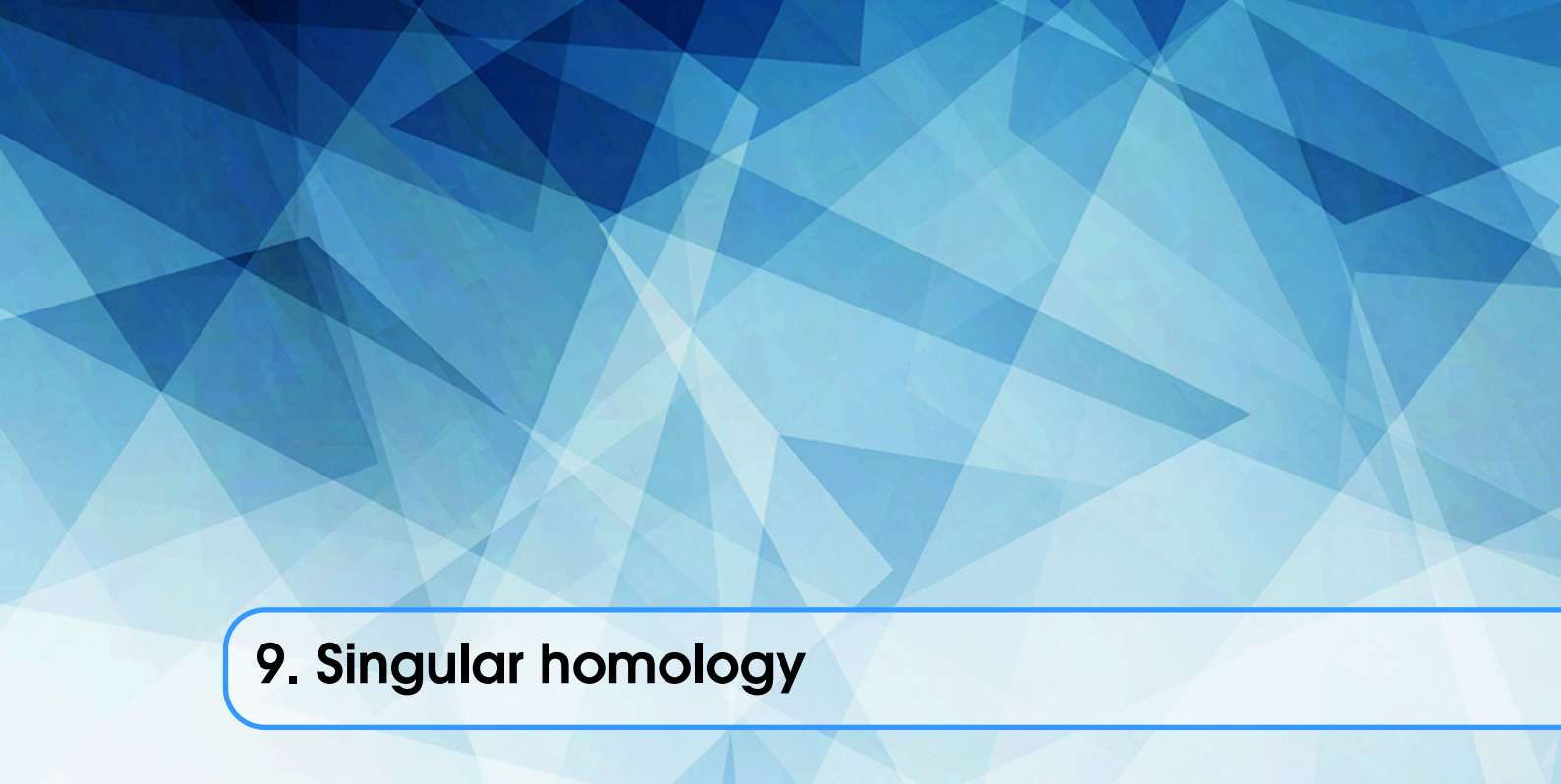
## **7. Stable homotopy. Daulity**



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## **8. Cell complexes**



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## 9. Singular homology



## 10. Homology

### 10.1 The Axioms of Eilenberg and Steenrod





# 11. Homological algebra

## 11.1 Diagrams

## 11.2 Exact sequences

## 11.3 Chain complex

## 11.4 Cochain complex

## 11.5 Natural chain maps and homotopies

## 11.6 Linear algebra of chain complexes

**Exercise?? 11.6.A** Tensor product is compatible with chain homotopy. Let  $s : f \simeq g : C \rightarrow C'$  be a chain homotopy. Then  $s \otimes id : f \otimes id \simeq g \otimes id : C \otimes D \rightarrow C' \otimes D$  is a chain homotopy. ■

*Proof.* Know:  $s\partial_C + \partial_{C'}s = f - g$

Want:  $(s \otimes id_D)\partial_{C \otimes D} + \partial_{C' \otimes D}(s \otimes id_D) = f \otimes id_D - g \otimes id_D$ .

$C \otimes D$  is generated by pure tensors like  $c'_n \otimes d_m$ , therefore we can check the formula on element  $c_n \otimes d_m \in C_n \otimes D_m$

$$\begin{aligned} & (s \otimes id_D)\partial_{C \otimes D}(c_n \otimes d_m) \\ &= (s \otimes id_D)(\partial_C c_n \otimes d_m + (-1)^n c_n \otimes \partial_D d_m) \\ &= s \circ \partial_C c_n \otimes d_m + (-1)^n s c_n \otimes \partial_D d_m \end{aligned}$$

and

$$\begin{aligned} & \partial_{C' \otimes D}(s \otimes id_D)(c_n \otimes d_m) \\ &= \partial_{C' \otimes D}(s c_n \otimes d_m) \\ &= \partial_{C'} s c_n \otimes d_m + (-1)^{\deg(sc_n)} s c_n \otimes \partial_D d_m, \end{aligned}$$

where  $\deg(sc_n) = n - 1$ . Then we have

$$\begin{aligned} & (\partial_{C' \otimes D}(s \otimes id_D) + (s \otimes id_D)\partial_{C \otimes D})(c_n \otimes d_m) \\ &= (s\partial_C + \partial_{C'}s)c_n \otimes d_m + 0 \\ &= (f \otimes id_D - g \otimes id_D)(c_n \otimes d_m) \end{aligned}$$

We are done. Also we can generalize this statemnt to

Let  $s : f \simeq g : C \longrightarrow C'$  and  $t : p \simeq q : D \longrightarrow D'$  be chain homotopies. Then  $s \otimes t : f \otimes p \simeq g \otimes q : C \otimes D \longrightarrow C' \otimes D'$  is a chain homotopy. We easily conclude by  $s \otimes id$  and  $id \otimes s$  are chain homotopy and composition of chain homotopies is a chain homotopy. ■

### 11.7 Tor and Ext

### 11.8 Universal coefficients

### 11.9 The Künneth Formula

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## **12. Cellular homology**



## **13. Partition of unity in homotopy**





More







## **Bibliography**

**Articles**

**Books**

