Notes of Readings in Topology and Geometry

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Contents

About the Notes:

This notes is a summary of personal reading of Topology and geometry.

1 Complex manifold

1.1 Complex structure, almost complex structure

Definition 1.1. *A complex valued function $f : \mathbb{C}^m \longrightarrow \mathbb{C}$ is **holomorphic** if $f = f_1 + if_2$ it satisfies the **Cauchy-Riemann relations** for $z^\mu = x^\mu + iy^\mu$,*

$$\frac{\partial f_1}{\partial x^\mu} = \frac{\partial f_2}{\partial y^\mu}$$

$$\frac{\partial f_2}{\partial x^\mu} = -\frac{\partial f_1}{\partial y^\mu}$$

2 Spectral sequences

Definition 2.1. *An **exact couple** is an exact sequence of Abelian groups of the form*

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

where i, j, k are group homomorphisms. Define $d : B \longrightarrow B$ by $d = j \circ k$. Then $d^2 = j(jk)k = 0$, so the homology group $H(B) = ker(d)/im(d)$ is wel-defined Abelian group.

Out of the exact couple, we can construct a **derived couple**

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd}$$

by setting

(i) $A' := i(A)$; $B' := H(B)$.

(ii) i' induced from i, i.e., $i'(i(a)) = i(i(a))$

(iii) If $a' = i(a) \in A'$ with $a \in A$, then $j'(a') := [j(a)] \in H(B)$.

(iv) k' is induced from k. Consider a comology calss $[b]$, $db = 0 \iff jkb = 0$, then $kb \in i(A)$. Define $k'[b] := kb \in i(A)$.

Proof. We can check that j' is wel-defined : $ia = ia_1 \implies [j(a)] = [j(a_1)]$. Indeed $i(a - a_1) = 0 \implies (a - a_1) \in im(k)$. $\exists z \in B$ s.t. $k(z) = a - a_1 \implies [j(a - a_1)] = [jk(z)] = [dz]$. Also k' is well defined: $[b] = [b_1] \implies b - b_1 = dz \implies k(b - b_1) = kjkz = 0$.

The derived sequence is indeed exact:

Exactness at $\xrightarrow{k'} A' \xrightarrow{i'}$: $i' \circ k'[b] = i'kb = ikb = 0$, we know $ker(i') \supseteq im(k')$. For the reverse inclusion, $i(a) \in ker(i') \implies iia = 0$ then $ia \in im(k)$ because the original exact couple is exact. $ia = kb \implies k'[b]$, hence $im(k') \supseteq ker(i')$.

Exactness at $\xrightarrow{i'} A' \xrightarrow{j'}$: $j' \circ i'(ia) = j'(iia) = [jia] = 0$. For the reverse inclusion, consider $ia \in ker(j'), j'(ia) = [ja] = 0 \implies ja = db \in B \implies j(a - kb) = 0 \implies a - kb = i(a_1) \implies i(a - kb) = ia = ii(a_1) = i'(ia_1)$.

Exactness at B': $k'j'(ia) = k'[ja] = kja = 0 \implies ker(k') \supseteq im(j')$. For the reverse inclusion, we can pick $[b] \in ker(k') \implies k'[b] = kb = 0 \implies b = ja$ for some $a \in A$. $[b] = j'(ia) \in im(j')$.

□

3 Fundamental groups

Definition 3.1. *Let X and Y be topological spaces and $f, g : X \longrightarrow Y$ continuous maps. A **homotopy** from f to g is a continuous map*

$$H : X \times [0,1] \longrightarrow Y, (x,t) \longmapsto H(x,t) = H_t(x)$$

such that $f(x) = H(x,0)$ and $g(x) = H(x,1)$ $\forall x \in X$. $f = H_0$ and $g = H_1$, $f \simeq g$

The homotopy relation is an equivalence relation on the set o f continuous maps $X \longrightarrow Y$. Given two homotopy $K : f \simeq g$ and $L : g \simeq h$, the product homotopy $K * L$

$$(K * L)(x,t) = \begin{cases} K(x,2t), & 0 \le t \le 1/2, \\ L(x,2t-1), & 1/2 \le t \le 1, \end{cases}$$

and shows $f \simeq h$.

The inverse homotopy is defined to be $H^-(x,t) := H(x, 1-t)$. Notice that product of homotopy and inverse homotopy is not constant homotopy.

The equivalence class of f is denoted $[f]$ and called the homotopy class of f . We denote by $[X, Y]$ the set of homotopy classes $[f]$ of maps $f : X \longrightarrow Y$. A homotopy $H_t : X \longrightarrow Y$ is said to be relative to $A \subset X$ if the restriction $H_t|_A$ does not dependont(is constant on A).We use the notation $H : f \longrightarrow g(relA)$in this case.

Quotient category means we identify some of the morphism. For each M$or\ (X, Y)$, we quotient a relation $R_{X,Y}$.

Definition 3.2. *Topological spaces and homotopy classes of maps form a quotient category of Top, which is called* ***homotopy category****, denoted h-Top. The composition of homotopy class is induced by composition of representing maps. The isomorphism in this category is homotopy equivalence.*

In the category of *h-Top*. Consider the Hom-functors. Given $f : X \longrightarrow Y$.

$$Hom(Z, \_)(f) = f_* : [Z, X] \longrightarrow [Z, Y] : g \longmapsto fg$$

$$Hom(\_, Z)(f) = f^* : [Z, X] \longrightarrow [Z, Y] : h \longmapsto hf$$

f is h-equivalence (isomorphism in the category *h-Top*) iff $Hom(\_, Z)(f)$ is always bijective. (Yoneda Lemma), similarly for $Hom(Z, \_)(f)$. Because we know for f_*, g_*, g_*f_*, 2 of the three maps are bijective implies that the third is bijective. This can be translated into homotopy category, where f, g, fg two of the three homotopy class being homotopy equivalence implies the third is also a homotopy equivalence.

Let P be a point. A map $P \longrightarrow Y$ can be identified as its image and a homotopy can be identified with path. Then the Hom-functor $[P, \_]$ can be identified as π_0

Proposition 3.3. *The product of paths has the following properties:*

(i) *Let* $\alpha : I \longrightarrow I$ *be continuous and* $\alpha(0) = 0$, $\alpha(1) = 1$. *Then* $u \simeq u\alpha$.

(ii) $u_1 * (u_2 * u_3) = (u_1 * u_2) * u_3$

(iii) $u_1 \simeq u_1'$ *and* $u_2 \simeq u_2'$ *implies* $u_1 * u_2 \simeq u_1' * u_2'$.

(iv) $u * u^-$ *is always defined and homotopic to the constant path.*

(v) $k_{u(0)} * u \simeq u \simeq u * k_{u(1)}$.

Proof. (i) Let $H : (s, t) \mapsto u(s(1 - t) + t\alpha(s))$ is homotopy from u to $u\alpha$.

(ii) choose

$$\alpha(t) = \begin{cases} 2t, & t \leq \frac{1}{4} \\ t + \frac{1}{4} & \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t+1}{2}, & \frac{1}{2} \leq t \leq 1 \end{cases}$$

we have $u_1 * (u_2 * u_3)\alpha = (u_1 * u_2) * u_3$, then we can apply (i)

(iii) Given $F_i : u_i \simeq U_i'$, then we can define the homotopy $G : u_1 * u_2 \simeq u_1' u_2'$

$$G(s,t) = \begin{cases} F_1(2s,t), & 0 \leq t \leq \frac{1}{2} \\ F_2(2s-1,t) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

(iv) The map $F : I \times I \longrightarrow X$ defined as

$$F(s,t) = \begin{cases} u(2s(1-t)), & 0 \leq t \leq \frac{1}{2} \\ u(2(1-s)(1-t)) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is the homotopy from $u * u^-$ to the constant path.

(v) use (i) again.

$\square$

This basically says that the homotopy class of path with a fixed point is a group.

From homotopy classes of paths in X, we obtain again a category denote $\Pi(X)$. The objects are the points of X. A morphism from x to y is a homotopy class of paths. It is called **Fundamental groupoid** of X. The automorphism group of the object x in this category is the fundamental group of X with base point x.

Proposition 3.4. *Let $H : X \times I \longrightarrow Y$ be a homotopy from f to g. Each $x \in X$ yields the path $H^x : t \mapsto H(x,t)$ and the morphism $[H^x]$ in $\Pi(Y)$ from $f(x)$ to $g(x)$. The H^x constitute a natural transformation $\Pi(H)$*

Proposition 3.5. *Let $f : X \longrightarrow Y$ be a homotopy equivalence. Then the functor $\Pi(f) : \Pi(X) \longrightarrow \Pi(Y)$ is an equivalence of categories. The induced maps between morphism sets $f_* : \Pi X(x,y) \longrightarrow \Pi Y(fx, fy)$ are bijections. IN particular,*

$$\pi_1(f) : \pi_1(X,x) \longrightarrow \pi_1(Y, f(x)), [\omega] \mapsto [f\omega]$$

is an isomorphism for each $x \in X$

Theorem 3.6. *(R. Brown). Let X_0 and X_1 be a subspace of X such that the interiors cover X. Let $i_\nu : X_{01} \hookrightarrow X_\nu$ and $j_\nu : X_\nu \hookrightarrow X$ be the inclusions. Then*

$$\begin{tikzcd} \Pi(X_{01}) \arrow[r, "\Pi(i_0)"] \arrow[d, "\Pi(i_1)"'] & \Pi(X_0) \arrow[d, "\Pi(j_0)"] \\ \Pi(X_1) \arrow[r, "\Pi(j_1)"] & \Pi(X) \end{tikzcd}$$

is a pushout (fibered coproduct) in the category of groupoids. Or $\Pi(X)$ is the colimit of diagrams indexed by $X_0 \supset X_{01} \subset X_1$

3.1 Spectral sequences of filtered complexes

Now, we want to construct the spectral sequence of a filtered complex.

Definition 3.7. *Let K be differential complex with differential operator D. A subcomplex K' is a subgroup in K such that $DK' \subseteq K'$. A sequence of subcomplex*

$$K = K_0 \supseteq K_1 \supseteq K_2 ...$$

*is called a filtration of K. A differential complex with specified filtration is called a filtered complex with **associated graded complex***

$$Gr_\bullet K = \oplus_{p=0}^{\infty} K_p / K_{p+1}$$

For filtered complex K, let A be the group

$$A = \oplus_{p \in \mathbb{Z}} K_p.$$

A is again a differential complex with differential operator $\oplus D$. Define ι to be the inclusion $A \hookrightarrow A$ induced by $K_{p+1} \hookrightarrow K_p$. Then we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} A \xrightarrow{j} B := Gr_\bullet K \longrightarrow 0.$$

If A, K are themselves graded chain complex (A different grading from the associated grading w.r.t. to filtration, we use upper index to distinguish it from the filtration index), we have a long exact sequence of cohomology groups

$$\longrightarrow H^k(A^\bullet) \xrightarrow{i} H^k(A) \xrightarrow{j_1} H^k(B) \xrightarrow{k_1} H^{k+1}(A) \longrightarrow ..$$

Consider that $H(A) = \oplus_k H^k(A)$, we have the exact couple

$$\begin{tikzcd} H(A) \arrow[rr, "i"] & & H(A) \arrow[dl, "j_1"] \\ & H(B) \arrow[ul, "k_1"] & \end{tikzcd} := \begin{tikzcd} A_1 \arrow[rr, "i"] & & A_1 \arrow[dl, "j_1"] \\ & B_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

From now on we will suppress the subscript of i_n because by definition, $i_n(i_{n-1}...(i(a))) = i^n(a)$. Even if they are not graded, we can still artificially construct the short exact sequence of chain complex

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "D"] & A \arrow[r, "j"] \arrow[d, "D"] & B \arrow[r] \arrow[d, "D"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "D"] & A \arrow[r, "j"] \arrow[d, "D"] & B \arrow[r] \arrow[d, "D"] & 0 \\ 0 \arrow[r] & \vdots \arrow[r, "i"] & \vdots \arrow[r, "j"] & \vdots \arrow[r] & 0 \end{tikzcd}$$

And it still gives the above exact couple.

Then we have all the derived exact couples and label it with r meaning it is the r-th derived exact couple of the first one:

$$\begin{tikzcd} A_r \arrow[rr, "i"] & & A_r \arrow[dl, "j_r"] \\ & B_r \arrow[ul, "k_r"] & \end{tikzcd} .$$

Consider the special case where the filtration terminates after K_3.

$$K_{-1} = K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq 0$$

$$\begin{tikzcd} A_1 := & H(K_0) \arrow[d, equal] & \oplus & H(K_1) \arrow[ll, bend right, "i"'] & \oplus & H(K_2) \arrow[ll, bend right, "i"'] & \oplus & H(K_3) \arrow[ll, bend right, "i"'] \\ A_2 := i(A_1) = & iH(K_0) \arrow[d, equal] & \oplus & iH(K_1) \arrow[ll, bend right, "\supseteq"'] \arrow[d, equal] & \oplus & iH(K_2) \arrow[ll, bend right, "i"'] & \oplus & iH(K_3) \arrow[ll, bend right, "i"'] \\ A_3 := i(A_2) = & i^2H(K_0) \arrow[d, equal] & \oplus & i^2H(K_1) \arrow[ll, bend right, "\supseteq"'] \arrow[d, equal] & \oplus & i^2H(K_2) \arrow[ll, bend right, "\supseteq"'] \arrow[d, equal] & \oplus & i^2H(K_3) \arrow[ll, bend right, "i"'] \\ A_4 := i(A_3) = & i^3H(K_0) & \oplus & i^3H(K_1) \arrow[ll, bend right, "\supseteq"'] & \oplus & i^3H(K_2) \arrow[ll, bend right, "\supseteq"'] & \oplus & i^3H(K_3). \arrow[ll, bend right, "\supseteq"'] \end{tikzcd}$$

Because $iH(K_1) \subseteq H(K_0)$ and i act as identity on $H(K_0)$, we know i act as inclusion on $iH(K_1)$, hence $iH(K_1) = i^2H(K_1)$. Similarly, we can say $i^n(K_i)$ stabilizes when $n \geq 3$, hence $A_4 = A_5 = ...A_\infty$.

$$\begin{tikzcd} A_4 \arrow[rr, "i"] & & A_4 \arrow[dl, "j_4"] \\ & B_4 \arrow[ul, "k_4"] & \end{tikzcd}.$$

Furthermore, since $i : A_4 \longrightarrow A_4$ is the inclusion, the map $k_4 : B_4 \longrightarrow A_4$ must be a zero map, hence the differential $d_4 := j_4k_4 = 0$ and $B_5 = H_{d_4}(B_4) = B_4$. $B - r$ also stabilize after B_4. $B_4 = B_5 = ... = B_\infty$.

$$\begin{tikzcd} A_\infty \arrow[rr, "i_\infty:\subseteq"] & & A_\infty \arrow[dl, "j_\infty"] \\ & B_\infty \arrow[ul, "k_\infty=0"] & \end{tikzcd}.$$

$k_\infty = 0 \implies B_\infty$ is the quotient of i_∞. In other words, B_∞ is the associated graded complex of the filtration

$$H(K) = H(K_0) \supseteq iH(K_1) \supseteq iiH(K_2) \supseteq iiiH(K_3).$$

In general consider a filtration of complex K with differential D.

$$K = K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq ...$$

It induces a sequence in cohomology

$$H(K) = H(K_0) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \xleftarrow{i} ...$$

Set $F_p := i^pH(K_p)$ be the image of $H(K_p)$ in $H(K)$. It gives a filtration of $H(K)$

$$H(K) = F_0 \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq ...$$

A filtration $K_\bullet$ is of **length** l if the descending chain terminates after K_l. If $K_\bullet$ is of finite length, then A_r and B_r eventually stabilize and B_∞ is the associated graded complex $\oplus F_p/F_{p+1}$ of $F_\bullet H(K)$.

Definition 3.8. *It is customary to write E_r for B_r. A sequence of differential complex $\{E_r, d_r\}$ in which $E_{r+1} = H_{d_r}(E_r)$ is called a **spectral sequence**. If E_r eventually stabilize, we denote the stationary value E_∞. If $E_\infty \cong Gr_\bullet H$ of some filtered complex H.*

Now assume K is a graded differential complex $K = \oplus_n K^n$, with filtration $K_\bullet$. Then each graded piece K^n is filtered complex with filtration $K^n_p = K^n \cap K_p$.

Theorem 3.9. *If $K = \oplus_n K^n$ is a graded filtered complex with filtration $\{K_p\}$ and let $H_D(K)$ denote the cohomology of K with a filtration $\{F_p\}$ induced by $\{K_p\}$. Suppose that for each fixed grading index n, the filtration $\{K^n_p\}$ is of finite length. Then the short exact sequence*

$$0 \longrightarrow \oplus_{p\in\mathbb{Z}} K_{p+1} \longrightarrow \oplus_{p\in\mathbb{Z}} K_p \longrightarrow \oplus_{p\in\mathbb{Z}} K_p/K_{p+1} \longrightarrow 0$$

induces a spectral sequence converging to $H_D(K)$.

Proof. We have the exact couple

$$\begin{tikzcd} A_r \arrow[rr, "i"] & & A_r \arrow[dl, "j_r"] \\ & B_r \arrow[ul, "k_r"] & \end{tikzcd},$$

where $A_r = i^{r-1}H(K_p)$, if $r \geq p$, $i^r H(K_p) = F_p$. (When $r \geq p+1$, the map $i : i^r H(K_p) \longrightarrow i^r H(K_p)$ is an inclusion).

Recall that k_1 is the connecting map $k_1 : H^*(B) \longrightarrow H^{*+1}(A)$. k_r would send $B^d_r \longrightarrow A^{n+1}_r$, while i, j_r would fix n.

For a fixed grading index n, assume the length of the filtration $\{K^n_p\}$ is $l(n)$. When $r \geq l(n+1)+1$, for every p we have

$$i^r H^{n+1}(K_p) = F^{n+1}_p$$

$$A^{n+1}_r = \oplus_p F^{n+1}_p$$

and the map

$$i : i^r H^{n+1}(K_p) \longrightarrow i^r H^{n+1}(K_p)$$

is inclusion. Hence

$$i : A^{n+1}_r \longrightarrow A^{n+1}_r \quad i : F^{n+1}_{p+1} \longrightarrow F^{n+1}_p$$

is injective and

$$k_r : B^n_r \longrightarrow A^{n+1}_r$$

is zero map. We have

$$0 \longrightarrow \oplus_p F^n_{p+1} \xrightarrow{i} \oplus_p F^n_p \longrightarrow B^n_\infty \longrightarrow 0$$

Then we know

$$B_\infty^n = \oplus_{p\le l(n)} F_p^n/F_{p+1}^n$$

and

$$B_\infty = \oplus_n B_\infty^n = \oplus_p F_p/F_{p+1} = Gr_\bullet H_D(K)$$

□

3.2 Spectral sequences of double complex

4 Model categories

Model categories were an abstraction of homotopy theory. They especially useful when we only care about topological spaces up to some weak form of equivalence. For example, homotopy equivalence is a prototype of this "weak equivalence".

Definition 4.1. ***weak homotopy equivalences*** *are map from X to Y that induces isomorphism on each homotopy groups, $\pi_n(X,x) \cong \pi_n(Y, f(x))$. We often denote weak equivalences by $\xrightarrow{\sim}$.*

Definition 4.2. *Let $\mathcal{C}$ be any category. The arrow category of $\mathcal{C}$, denoted $\mathbf{Arr}(\mathcal{C})$ is defined to be a category where objects are the arrows of $\mathcal{C}$ and morphism are commutative squares.*

Definition 4.3. *Let $\mathcal{C}$ be any category. An arrow $f \in Arr(\mathcal{C})$ is a retract of an arrow $g \in Arr(\mathcal{C})$ if it is a retract of an object in $\mathbf{Arr}(\mathcal{C})$.*

Explicitly, f is a retract of g if we are given a commutative diagram as the following:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "id_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "id_B"'] & D \arrow[r] & B
\end{tikzcd}$$

Definition 4.4. *Let $\mathcal{C}$ be any category. A* ***model structure*** *on $\mathcal{C}$ is the given three full subcategories $W, FIB, COFIB$ of $\mathbf{Arr}(\mathcal{C})$ satisfying the following axioms:*

MC1 *$\mathcal{C}$ is (small) complete and (small) cocomplete;*

MC2 *if f, g, h are arrows s.t. $fg = h$ and two of them are in W, then so is the third;*

MC3 *$W, FIB, COFIB$ are closed under retracts;*

MC4 *every arrow in $W \cap FIB$ has the RLP with respect to every arrow in $COFIB$ and every arrow in FIB has the RLP with respect to every arrow in $W \cap COFIB$;*

MC5 *there are functorial $(W \cap COFIB, FIB)$ and $(COFIB, W \cap FIB)$ factorization in $\mathcal{C}$.*

Definition 4.5. *By Axiom **MC1**, we know a model category have initial object $\emptyset$ and final object $*$. We see an object $A \in \mathcal{C}$ is cofibrant if $\emptyset \longrightarrow A$ is a cofibration and we say $B \in \mathcal{C}$ is fibrant if $B \longrightarrow *$ is a fibration.*

Example 4.6. *The category Ch_R can be given the structure of a model category by defining a map $f : M \longrightarrow N$ to be*

(i) a weak equivalence if f induces isomorphisms on homology groups,

(ii) a cofibration if for each $k \geq 0$ the map $f_k : M_k \Longrightarrow N_k$ is a monomorphism with projective R-module as its cokernel, and

(iii) a fibration if for each $k \geq 1$ the map $f_k : M_k \longrightarrow N_k$ is an epimorphism.

The initial object in Ch_R is the zero complex. The cofibrant object in Ch_R are the projective chain complexes. The homotopy category $Ho(Ch_r)$ is equivalent to the category whose objects are these cofibrant chain complexes and whose morphisms are ordinary chain homotopy classes of maps.

Lemma 4.7. *Let $\mathcal{C}$ be a model categroy. Then :*

(i) fibrations are exactly those arrows with RLP with respect to all acyclic cofibrations;

(ii) acyclic fibrations are exactly those arrows with RLP with respect to all cofibrations;

(iii) cofibrations are exactly those arrows with LLP with respect to all acyclic fibrations;

(iv) acyclic cofibrations are exactly those arrows with LLP with respect to all fibrations;

Proof. We sketch the proof of (i): One inclusion is by definition: every fibration has RLP with every acyclic cofibration. For the reverse inclusion, consider an arrow f, by axiom $MC5$, f can be factorized in to $p \circ i$ where i is an acyclic cofibration and p is a fibration.

$$\begin{tikzcd} \bullet \arrow[r, "id"] \arrow[d, "i"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "p"] \arrow[ur, "h"] & \bullet \end{tikzcd}$$

and observe that the diagram

$$\begin{tikzcd} \bullet \arrow[r, "i"] \arrow[d, "f"] & \bullet \arrow[r, "h"] \arrow[d, "p"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "id"'] & \bullet \arrow[r, "id"'] & \bullet \end{tikzcd}$$

expresses f as a retract of p. This implies that f is a fibration. $\square$

Corollary 4.8. *Let $\mathcal{C}$ be a model category. Then*

(i) FIB is closed under pull back;

(ii) COFIB is closed under pushout.

Proof. Given a fibration f and consider the following diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "j"'] & * \arrow[r] \arrow[d, "q"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[ur, "h"] \arrow[urr, "h'"'] & \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

where the right square is the pullback diagram funder g. Consider an arbitrary acyclic cofibration j, then there exists a h' because f is RLP w.r.t j. h' must factor through h because $*$ is a pull back. As a result q has RLP w.r.t every acyclic cofibration. Thus we know q is a fibration by the lemma above. $\square$

Definition 4.9. *Let $\mathcal{C}$ be a model category. Then A cofibrant approximation to an object $X \in Obj(\mathcal{C})$ is a pair $(\tilde{X}, i)$ where $\tilde{X}$ is cofibrant and $i : \tilde{X} \longrightarrow X$ is a weak equivalence;*

a fibrant approximation to an object $X \in Obj(\mathcal{C})$ is a pair $(\hat{X}, j)$ where $\hat{X}$ is a fibrant object and $j : X \longrightarrow \hat{X}$ is a weak equivalence;

Proposition 4.10. *(i) every object $X \in Obj(\mathcal{C})$ has a cofibrant approximation $(\tilde{X}, i_X)$ where i_X is a trivial fibration;*

(ii) if $(\tilde{X}, i_X)$ $(\tilde{X}', i'_X)$ are cofibrant approximations there is a weak equivalence $f : \tilde{X} \longrightarrow \tilde{X}'$;

(iii) every morphism in $\mathcal{C}$ has fibrant approximation.

Proof. (i) Every morphism to X factorizes as $p \circ q$ where q is a cofibration and p is a acyclic fibration. In particular, the initial object has a morphism ending in X, $\emptyset \longrightarrow X$ factorizes as $i_X \circ q$ where i_X is a acyclic fibration and $q : \emptyset \longrightarrow \tilde{X}$ is a cofibration. By definition $\tilde{X}$ is cofibrant.

(ii) Consider the diagram

$$\begin{tikzcd}
\emptyset \arrow[r, "q"] \arrow[d, "p"'] & \tilde{X}' \arrow[d, "i'_X"] \\
\tilde{X} \arrow[r, "i_X"] \arrow[ur, dashed, "f"] & X
\end{tikzcd}$$

where i_X, i'_X are acyclic fibrations and p, q are cofibrations. Because each cofibration has the LLP with acyclic fibrations, there exists a map $f : \tilde{X} \longrightarrow \tilde{X}'$. And this map f is a weak equivalence because. Because i_X, i'_X are weak equivalence and $i'_X f = i_X$, by the 2 of 3 axiom for weak equivalences, f is also a weak equivalence.

(iii): Here we use the notion $(\hat{X}, j_X)$ to denote the functorial fibration approximation where j_X is an acyclic cofibration. And we consider $(\hat{Y}, j)$ an ordinary fibration approximation of Y. Consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"'] \arrow[d, "j_X"'] \arrow[dr] & Y \arrow[d, "j"'] \\
\hat{X} \arrow[r, dashed] \arrow[d, "p_X"'] & \hat{Y} \arrow[dl, "p"] \\
*
\end{tikzcd}$$

where the composition $j \circ \alpha$ gives a map from X to an fibrant object $\hat{Y}$, therefore $X, \hat{X}, *, \hat{Y}$ form a commutative square. Note that j_X is an acyclic cofibration and p is a fibration. There is a lift $h : \hat{X} \longrightarrow \hat{Y}$ because fibration has RLP w.r.p acyclic cofibrations. This lift h gives the fibrant approximation of α. □

Lemma 4.11 (Ken Brown's Lemma)**.** *Let $\mathcal{C}$ be a model category and let $\mathcal{B}$ be a category with subcategory $S \subset \mathbf{Arr}(\mathcal{C})$ containing all the identities and satisfies the 2-out-of-3 axiom. If $F : \mathcal{C} \longrightarrow \mathcal{B}$ is a functor hat takes acyclic cofibrations* *between cofibrant objects to elements of S, then F takes fibrations between fibrant objects to elements of S, then F takes every weak equivalence between fibrant objects to elements of S.*

Remark 4.12. *Basically, this lemmas says a functor sending acyclic (co)fibrations between (co)fibrant objects to S ("weak equivalences in $\mathcal{B}$"), would send all weak equivalence in $\mathcal{C}$ to "weak equivalence in $\mathcal{B}$"*

Proof. Let $f : A \longrightarrow B$ be a generic weak equivalence between cofibrant objects between cofibrant objects. By **MC1** all colimits exists in $\mathcal{C}$, there the coproduct $A \sqcup B \in Obj(\mathcal{C})$. By universal property of coproduct there exists a morphism $\langle f, id_B \rangle : A \sqcup B \longrightarrow B$. We can factorize this map as

$$A \sqcup B \xrightarrow{q} C \xrightarrow{p} B$$

with p and acyclic fibration and q a cofibration. Since A and B are cofibrant, stability of cofibrations under pushout implies i_0, i_1 are both cofibrations

$$\begin{tikzcd}
 & C \arrow[dr, "p"] & \\
A \sqcup B \arrow[ur, "q"] \arrow[rr, bend left, "{\langle f, id_B \rangle}"] & & B \arrow[ll, "i_1"] \\
A \arrow[u, "i_0"] \arrow[urr, "f"] & & \emptyset \arrow[ll] \arrow[u]
\end{tikzcd} .$$

Then qi_0 and qi_1 are both weak equivalences because of 2-out-of-3 axiom by considering the the following two triangles.

$$\begin{tikzcd}
C \arrow[dr, "p"] & \\
A \arrow[u, "q \circ i_0"] \arrow[r, "f"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
C & \\
A \arrow[u, "q \circ i_0"] \arrow[r, "f"] & B \arrow[ul, "q \circ i_1"]
\end{tikzcd} .$$

By hypothesis, $F(q \circ i_0)$ and $F(q \circ i_0)$ are elements of S. Since $F(p \circ q \circ i_1) = F(id_B)$ is in S. By 2-out-of-3 hypothesis of S, It follows that $F(p) \in S$. Hence $F(f) = F(p) \circ F(q \circ i_0)$ is in S. $\square$

4.1 The homotopy category

Definition 4.13. *Let $\mathbb{U} \subset \mathbb{V}$ be Grothendieck universe and let $\mathcal{C}$ be a $\mathbb{U}$-small category. Let $S \subset Arr(\mathcal{C})$ be a set of Arrows. A $\mathbb{V}$-**localization of $\mathcal{C}$ with***

respect to *S is a $\mathbb{V}$-small category $\mathcal{C}[S^{-1}]$ together with a functor $F_S : \mathcal{C} \longrightarrow \mathcal{C}[S^{-1}]$ such that*

(i) for all $s \in S$, $F_S(s)$ is an isomorphism;

(ii) for any other $\mathbb{V}$-small category $\mathcal{A}$ and any functor $\mathcal{C} \longrightarrow \mathcal{A}$ such that $G(s)$ is an isomorphism for each $s \in S$, there is a functor $G_S : \mathcal{C}[S^{-1}] \longrightarrow \mathcal{A}$ and a natural isomorphism

$$\eta_G : G_S \circ F_S \cong G$$

(iii) for any $\mathbb{V}$-small category $\mathcal{A}$, the induces functor

$$F_S^* : Func(\mathcal{C}[S^{-1}], \mathcal{A}), \longrightarrow Func(\mathcal{C}, \mathcal{A})$$

is fully faithful.

Theorem 4.14. *Let $\mathbb{U}$ be a Grothendieck universe and let $\mathcal{C}$ be a model category. Then there exists a $\mathbb{U}$-localization of $\mathcal{C}$ with respect to the set of Weak equivalences W.*

Definition 4.15. *Let $\mathcal{C}$ be a category and let $\mathcal{C}_0$, $\mathbf{W} \subset \mathcal{C}$ be subcategories. We say that $\mathcal{C}_0$ is a left (resp. right) deformation retract of $\mathcal{C}$ with respect to $\mathbf{W}$ if there exits a functor $R : \mathcal{C} \longrightarrow \mathcal{C}_0$ and a natural transformation $s : \iota R \longrightarrow Id_{\mathcal{C}}$ (resp. $s : Id_{\mathcal{C}} \longrightarrow \iota R$), where $\iota : \mathcal{C}_0 \longrightarrow \mathcal{C}$ is the inclusion functor. such that:*

1. *R sends $\mathbf{W}$ into $\mathbf{W} \cap \mathcal{C}_0$;*

2. *for every object $C \in Obj(\mathcal{C})$, the map s_C is in $\mathbf{W}$;*

3. *for every object $C_0 \in Obj(\mathcal{C}_0)$, the map s_{C_0} is in $\mathbf{W} \cap \mathcal{C}_0$.*

The pair (R, s) is called a left (resp. right) deformation retraction from $\mathcal{C}$ to $\mathcal{C}_0$ with respect to $\mathbf{W}$. When $\mathbf{W} = \mathcal{C}$, we say (R, s) is an absolute deformation retraction of $\mathcal{C}$ to $\mathcal{C}_0$.

Lemma 4.16. *Let $\mathcal{C}$ be a category and let $\mathcal{C}_0, \mathbf{W} \subset \mathcal{C}$ be subcategories. Let $R : \mathcal{C} \longrightarrow \mathcal{C}_0$ be an absolute left deformation retraction. Assume that for every object $C \in Obj(\mathcal{C})$, the map s_C is in $\mathbf{W}$; if $\mathbf{W}$ satisfies the 2-out-of-3 axiom, then R sends $\mathbf{W}$ into $\mathbf{W} \cap \mathcal{C}_0$. If $\mathcal{C}_0$ is a full subcategory, then for every $C_0 \in Obj(\mathcal{C}_0)$ the map s_{C_0} is in $\mathbf{W} \cap \mathcal{C}_0$.*

Proof. Let $f : A \longrightarrow B$ be in $\mathbf{W}$, consider the diagram of natural transformation of s defined in the absolute deformation retraction (R, s).

$$\begin{tikzcd} R(A) \arrow[r, "s_A"] \arrow[d, "R(f)"'] & A \arrow[d, "f"] \\ R(B) \arrow[r, "s_B"] & B \end{tikzcd}$$

The hypothesis says $s_A, s_B \in \mathbf{W}$, therefore $f \circ s_A \in \mathbf{W}$. Then by the 2-out-of-3 property of $\mathbf{W}$, we know $R(f) \in \mathbf{W}$. Combined with the fact $R : \mathcal{C} \longrightarrow \mathcal{C}_0$, it follows $R(\mathbf{W}) \subset \mathbf{W} \cap \mathcal{C}_0$; The second statemnt is clear because $\mathrm{M}or_{\,\mathcal{C}_0}(R(C_0), C_0) = \mathrm{M}or_{\,\mathcal{C}}(R(C_0), C_0)$, therefore $s_{C_0} \in \mathcal{C}_0$. □

Remark 4.17. *The above lemma says, under suitable choice of subcategory* $\mathbf{W}$*, the deformation retraction along* $\mathbf{W}$ *is "equivalent" to an absolute deformation retraction.*

Lemma 4.18. *Let* $\mathcal{C}$ *be a category and* $\mathcal{C}, \mathbf{W} \subset \mathcal{C}$ *subcategories. Let* (R, s) *be a left(resp. right) deformation retraction w.r.t* $\mathbf{W}$*. Denote* $\mathbf{W} \cap \mathcal{C}_0$ *as* $\mathbf{W}_0$*. Let* $\mathbb{V}$ *be the universe where* $\mathcal{C}[\mathbf{W}^{-1}]$ *exists.*

1. *The induced inclusion* $\mathcal{C}_0[\mathbf{W}_0^{-1}] \subset \mathcal{C}[\mathbf{W}^{-1}]$ *is an equivalence of categories.*
2. $\mathcal{C}[\mathbf{W}^{-1}]$ *exists iff* $\mathcal{C}_0[\mathbf{W}_0^{-1}]$ *exists.*

Proof. The second statement obviously comes from the first.

Consider the inclusion functor: $j_0 : \mathcal{C}_0 \longrightarrow \mathcal{C}$ and the left deformation functor $R : \mathcal{C} \longrightarrow \mathcal{C}_0$. The universal property of localization of categories implies both R, j_0 descend to functors between the localizations

$$\tilde{j}_0 : \mathcal{C}_0[\mathbf{W}_0^{-1}] \longrightarrow \mathcal{C}[\mathbf{W}^{-1}]$$

and

$$\tilde{R} : \mathcal{C}[\mathbf{W}^{-1}] \longrightarrow \mathcal{C}_0[\mathbf{W}_0^{-1}].$$

By the definition of of deformation retraction $s : Rj_0 \longrightarrow id_{\mathcal{C}}$ is a natural transformation and for all object $C \in Obj(\mathcal{C})$, $s_C \in \mathbf{W}$. The natural transformation s descends to a natural transformation $\tilde{s} : \tilde{R}\tilde{j}_0 \longrightarrow id_{\mathcal{C}[\mathbf{W}^{-1}]}$ where

$$\tilde{s}_{F_{\mathbf{W}}(C)} = F_{\mathbf{W}}(s_C)$$

where $F_{\mathbf{W}}$ is the defining functor of localization. By definition of localization $F_{\mathbf{W}}(s_C)$ is isomorphism because $s_C \in \mathbf{W}$. Hence, by definition, $\tilde{s}$ is a natural isomorphism. The third condition in the definition of deformation retraction means s restricts to a natural transformation $s_0 : j \circ R \longrightarrow id_{\mathcal{C}_0}$. With identical argument we can prove that there is a natural isomorphism between $\tilde{s}_0 : \tilde{j}_0\tilde{R} \longrightarrow id_{\mathcal{C}_0[\mathbf{W}_0^{-1}]}$.

We have established the desired equivalence of categories. □

Definition 4.19. *Let $\mathcal{M}$ be a model category.*

1. *$\mathcal{M}_c$ denote the full subcategory of $\mathcal{M}$ consisting of cofibrant objects.*
2. *$\mathcal{M}_f$ denote the full subcategory of $\mathcal{M}$ consisting of fibrant objects.*
3. *$\mathcal{M}_{cf}$ denote the full subcategory of $\mathcal{M}$ consisting of objects that are both cofibrant and fibrant.*

Proposition 4.20.

1. *$\mathcal{M}_{cf}, \mathcal{M}_c$ are left deformation retracts of $\mathcal{M}_f$ and $\mathcal{M}$ with respect to the weak equivalences.*
2. *$\mathcal{M}_{cf}, \mathcal{M}_f$ are right deformation retracts w.r.t the weak equivalence.*

5 Fibered category

6 Stacks

7 Formalization renormalization

Definition 7.1. *A **parametrix** for the Laplacian D on a manifold is a symmetric distribution P on M^2 such that $(D \otimes 1)P - \delta_M$ is a smooth function on M^2, where δ_M stands for the δ-distribution on the diagonal of M.*

Locally it means

$$D_x P_{x,y} - \delta_{x,y}$$

is a smooth function on M^2