Summary for Algebraic Topology II

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2018 ETH

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1 21th Feb: Tor functor

Definition 1.1. *Suppose A is an abelian group, A* ***Free resolution*** *is an exact sequence of the form*

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \longrightarrow 0,$$

where each F_i is a free abelian group. If moreover $F_i = 0, \forall i \geq 2$, we call it ***Short free resolution***

$$0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$$

(We can easily generalize this definition to R-modules)

Proposition 1.2. *Let A be an abelian group. Then there exists a short free resolution of A.*

Proof. Let F be the free abelian group generated by all elements in A. There is a surjection from F to A by linearly extending the map sending basis element to itself. Let K denote the kernel of this map. K is an abelian subgroup of a free abelian group ($\mathbb{Z}$-module). A subgroup of a free abelian group is torsion free as a module. $\mathbb{Z}$ is a PID. If R is a PID, then an R-module is free iff it is torsion free (See Bosch section 4.2). Then we know in particular, K is a free abelian group. $\square$

With this construction, we can define the Tor functor now:

Definition 1.3. *Let A be an abelian group. Let $0 \to K \xrightarrow{f} F \to A \to 0$ be a short free resolution of A. Given any other abelian group B, we define*

$$Tor(A, B) := \ker(f \otimes id_B)$$

Tor(A,B)

This definition is independent on the choice of short free resolution.

2 28th Feb:

<u>Question</u>: Given X, Y what is the cohomology of $X \times Y$?

<u>Answer</u>:

$$H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) + \bigoplus_{k+\ell=n-1} \mathrm{Tor}(H_k, (X), H_\ell(Y))$$

We will discuss Elenberg-Zilber theorem along this line the next lecture.

Today, we will prove the Algebraic Kueneth Theorem

Definition 2.1. *Suppose $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ are two non-negative chain complexes. We define the* ***tensor complex*** *$(C_\bullet \otimes C'_\bullet, \Delta)$, where*

$$(C_\bullet \otimes C'_\bullet)_n = \oplus_{i+j=n} C_i \otimes C'_j$$

and the differential Δ is defined by

$$\Delta(c_i \otimes c'_j) = \partial c_i \otimes c'_j + (-1)^i c_i \otimes \partial' c_j$$

First, note that $\Delta(c_i \otimes c'_j)$ does indeed belong to $(C_\bullet \otimes C'_\bullet)_{n-1}$. The reason for $(-1)^i$ is to make $\Delta^2 = 0$. $C_\bullet \otimes C'_\bullet$ is another non-negative chain complex.

Definition 2.2. *Suppose $f_\bullet : C_\bullet \longrightarrow D_\bullet$ and $g_\bullet : C'_\bullet \longrightarrow D'_\bullet$ are two morphism of chain complexes. Then we can define a chain map*

$$f \otimes g : C \otimes C' \longrightarrow D \otimes D'$$

by

$$(f \otimes g)_n = \sum_{i+j=n} f_i \otimes g_j$$

It is easy to check this is indeed a chain map.

Lemma 2.3. *If $f' : C \longrightarrow C'$ and $g' : D \longrightarrow D'$ are two more chain maps with f homotopic to f' and g homotopic to g'. Then $f' \otimes g'$ is homotopic to $f \otimes g$.*

Theorem 2.4. *(Algebraic Kuenneth Theorem) Let (C, ∂) and (D, ∂') be two non-negative free complex. Then for every $n \geq 0$, there is a split exact sequence*

$$0 \longrightarrow \oplus_{i+j=n} H_i(C) \otimes H_j(D) \longrightarrow H_N(C \otimes D) \longrightarrow \oplus_{k+\ell=n-1}\ Tor(H_k(C), H_\ell(D)) \longrightarrow 0$$

where ω is the map $\langle c_i \rangle \otimes \langle d_j \rangle \mapsto \langle c_i \otimes d_j \rangle$. Thus there also exists a (non-natural) isomorphism

$$H_n(C \times D) \cong \oplus_{i+j=n} H_i(C) \otimes H_j(D) + \oplus_{k+\ell=n-1}\ Tor(H_k, (C), H_\ell(D)$$

The proof requires two auxiliary results.

Proposition 2.5. *Let $(E_\bullet, 0)$ be a non-negative chain complex with all differential zero and $(D_\bullet, \partial)$ be any non-negative chain complex. Given $i \geq 0$, let $D^i_\bullet$ denote the chain complex where $D^i_n = D_{n-i}$ and the boundary map*

$$D^i_n \longrightarrow D^i_{n-1}$$

is just the map: $D_{n-i} \longrightarrow D_{n-i-1}$.

Then

$$H_n(E_\bullet \otimes D_\bullet) \cong \bigoplus_{i \geq 0} H_n(E_i \otimes D^i_\bullet)$$

Proof. (of the Proposition) Since $E_\bullet$ has no differentials

$$\begin{aligned}\Delta(e_i \otimes d_{n-i}) &= (-1)^i e_i \otimes \partial d_{n-i} \\ &= (-1)^i (id_E \otimes \partial)[e_i \otimes d_{n-i}]\end{aligned}$$

$$\begin{aligned}H_n(E_\bullet \otimes D_\bullet) &= \frac{ker\Delta}{im\Delta} \\ &= \bigoplus_{i\geq 0} \frac{ker(id_E \otimes \partial|_{D_{n-i}})}{im(id_E \otimes \partial|_{D_{n-i+1}})} \\ &= \bigoplus_{i\geq 0} H_n(E_i \otimes D^i_\bullet)\end{aligned}$$

□

Proof. (of Theorem) We will prove it in three steps:

Let's use the same notation as we did in the proof of the universal coefficient theorem. $B_n \subset Z_n \subset C_n$. $(Z_\bullet, 0)$ and $(B^+_\bullet, 0)$ are chain complexes with no differentials, where $B^+_n = B_{n-1}$. $(H_\bullet, 0)$ be the chain complex. $i : Z_n \hookrightarrow C_n$, $j : B_n \hookrightarrow Z_n$, $d : C_n \longrightarrow B_{n-1}$, where d is the just the differential ∂ of $C_\bullet$ and we use p to denote the projection $Z_n \twoheadrightarrow H_n$. Then we have two short exact sequence of chain complexes

$$0 \longrightarrow Z_\bullet \xrightarrow{i_\bullet} C_\bullet \xrightarrow{D_\bullet} B^+_\bullet \longrightarrow 0$$

$$0 \longrightarrow B_\bullet \xrightarrow{j_\bullet} Z_\bullet \xrightarrow{p_\bullet} H_\bullet \longrightarrow 0.$$

We tensor it with $D_\bullet$.

$$0 \longrightarrow Z_\bullet \otimes D_\bullet \xrightarrow{i_\bullet} C_\bullet \otimes D_\bullet \xrightarrow{D_\bullet} B^+_\bullet \otimes D_\bullet \longrightarrow 0$$

$$0 \longrightarrow B_\bullet \otimes D_\bullet \xrightarrow{j_\bullet} Z_\bullet \otimes D_\bullet \xrightarrow{p_\bullet} H_\bullet \otimes D_\bullet \longrightarrow 0.$$

They are again short exact sequence of chain complexes because D is free Abelian group thus flat module.

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, bend left, "i"] & C_n \arrow[l, bend left, "r"] \arrow[r, "d"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

This sequence splits as B_{n-1} is free abelian. Thus $\exists$ a map $r : C_n \longrightarrow Z_n$ such that $r|_{Z_n}$ is the identity $r_\bullet : C_\bullet \longrightarrow Z_\bullet$.

Denote by μ the composition $p \circ r : C_\bullet \longrightarrow H$.

Claim: μ is a chain map from $(C_\bullet, \partial) \longrightarrow (H_\bullet, 0)$. Take $c \in C_{n+1}$ and check it commutes

$$\mu \circ \partial c = \mu\partial c = p \circ r\partial c = \langle \partial c \rangle = 0$$

and $0 \circ \mu c = 0$

Step 2: Define $\varphi = H_n(\mu \otimes id)$. $H_n(C_\bullet \otimes D_\bullet) \longrightarrow H_n(H_\bullet \otimes D_\bullet)$.

<u>Claim</u>: φ is an isomorphism.

It suffices to prove the diagram commutes and conclude by five lemma.

$$\begin{tikzcd}
H_{n+1}(B^+_\bullet \otimes D_\bullet) \arrow[r, "\delta"] \arrow[d, "id"] & H_n(Z_\bullet \otimes D_\bullet) \arrow[r] \arrow[d, "id"] & H_n(C_\bullet \otimes D_\bullet) \arrow[r] \arrow[d, "\varphi"] & H_n(B^+_\bullet \otimes D_\bullet) \arrow[r, "\delta"] \arrow[d, "id"] & H_{n-1}(Z_\bullet \otimes D_\bullet) \arrow[d, "id"] \\
H_n(B_\bullet \otimes D_\bullet) \arrow[r] & H_n(Z_\bullet \otimes D_\bullet) \arrow[r] & H_n(H_\bullet \otimes D_\bullet) \arrow[r, "\delta'"] & H_{n-1}(B_\bullet \otimes D_\bullet) \arrow[r] & H_{n-1}(Z_\bullet \otimes D_\bullet)
\end{tikzcd}$$

Step 3: We complete the proof

$$\begin{aligned} H_n(C_\bullet \otimes \otimes D_\bullet) &\cong H_n(H_\bullet \otimes D_\bullet) \\ &\cong \bigoplus_{i \geq 0} H_n(H_i(C_\bullet) \otimes D^i_\bullet) \end{aligned}$$

By the universal coefficient theorem, there is a split exact sequence

$$0 \longrightarrow H_i(C_\bullet) \otimes H_n(D^i_\bullet) \longrightarrow H_n(H_i(C_\bullet) \otimes D^i_\bullet) \longrightarrow \operatorname{Tor}(H_i(C_\bullet), H_{n-1}(D^i_\bullet)) \longrightarrow 0$$

If we get rid of the notation $D^i_\bullet$.

$$0 \longrightarrow H_i(C_\bullet) \otimes H_n(D^i_\bullet) \longrightarrow H_n(H_i(C_\bullet) \otimes D^i_\bullet) \longrightarrow \operatorname{Tor}(H_i(C_\bullet), H_{n-1-i}(D_\bullet)) \longrightarrow 0$$

Take the direct sum over i and use the fact that

$\square$

3 2nd Mar: Eilenberg-Zilber

Theorem 3.1. *(Eilenberg-Zilber) if X and Y are two topological spaces. There is a nontrivial chain equivalence*

$$\Omega_\bullet : C_\bullet(X \times Y) \longrightarrow C_\bullet(X) \otimes C_\bullet(Y)$$

which is unique up to chain homotopy

Digression on chain equivalences

Lemma 3.2. *Let $(C_\bullet, \partial)$ be a free chain complex. Then $C_\bullet$ is acyclic iff it has contracting chain homotopy*

Proof. $Q : C_n \longrightarrow C_{n+1}$ s.t. $Q\partial + \partial Q = id$ if such Q exists then $H_n(C_\bullet) = 0 \forall n$. That direction doesn't require $C_\bullet$ to be free

$$B_n \subseteq Z_n \subseteq C_n$$

If we assume $C_\bullet$ is acyclic then

$$B_n = Z_n, \forall n$$

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} Z_{n_1} \longrightarrow 0$$

Since Z_{n-1} is free abelian the sequence splits $\exists r_n : Z_{n-1} \longrightarrow C_n$ s.t. $\partial \circ r_n = id$. Note that $id - r_{n-1} \circ \partial$ jas image in Z_{n-1}, $c \in C_n$. $\partial(c - r_n \partial c) = \partial c - \partial c = 0$

Now define $Q_n : C_n \longrightarrow C_{n+1}$ by $Q_n = r_n(id - r_{n-1} \circ \partial)$. This works.

$$\begin{aligned}
\partial Q_n + Q_{n-1}\partial &= \partial r_n(id - r_{n-1}\partial) + r_{n-1}(id - r_{n-2}\partial)\partial \\
&= id - r_{n-1}\partial + r_{n-1}\partial - r_{n-1}r_{n-2}\partial^2 \\
&= 0
\end{aligned}$$

□

Definition 3.3. *Suppose* $f : (C_\bullet, \partial) \longrightarrow (D_\bullet, \partial')$. *The* ***mapping cone*** *of* f *is the chain complex* $Cone_\bullet(f), \partial^f$, *where* $Cone_n(f) = C_{n-1} \otimes D_n$ *and* $\partial^f : Cone_n(f) \longrightarrow Cone_{n-1}(f)$

$$\partial^f(c, d) = (-\partial c, fc + \partial' d)$$

$$\partial^f = \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix}$$

Note if $C_\bullet$ and $D_\bullet$ are free chain complex, so is the cone.

Lemma 3.4. *If* $f : C_\bullet \longrightarrow D_\bullet$ *is a chain map between two free chain complexes and* $Cone_\bullet(f)$ *is acyclic then* f *is a chain equivalence.*

Proof. If $Cone_\bullet(f)$ is acyclic, there exists Q s.t.

$$Q\partial^f + \partial^f Q = id$$

$$Q = \begin{pmatrix} p & g \\ r & -p' \end{pmatrix}$$

$$\begin{pmatrix} \partial & 0 \\ f & -\partial' \end{pmatrix} \begin{pmatrix} p & g \\ r & -p' \end{pmatrix} + \begin{pmatrix} p & g \\ r & -p' \end{pmatrix} \begin{pmatrix} \partial & 0 \\ f & -\partial' \end{pmatrix} = \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}$$

$$\begin{pmatrix} -\partial p - p\partial + gf & -\partial g + g\partial' \\ * & fg - \partial' p' - p'\partial' \end{pmatrix} \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}$$

Then we know $g : D_\bullet \longrightarrow D_\bullet$ is a chain map

$p\partial + \partial p = gf - id$

$p'\partial' + \partial' p = fg - id$. Thus f is a chain equivalence with inverse g. □

Lemma 3.5. *Let $f : C_\bullet \longrightarrow D_\bullet$. Then there is a LES*

$$\cdots \longrightarrow H_{n+1}(Cone_\bullet(f)) \longrightarrow H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(D_\bullet) \longrightarrow H_n(Cone_\bullet(f)) \longrightarrow \cdots$$

Proof. Denote by $C^+_\bullet$ the chain complex $C^+_n = C_{n-1}$. There is a SES

$$0 \longrightarrow D_\bullet \xrightarrow{i} Cone_\bullet(f) \xrightarrow{p} C^+_\bullet \longrightarrow 0$$

with $i(d) = (0, d)$ and $p(c, d) = c$

Pass to the LES in homology

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_{n+1}(Cone_\bullet(f)) & \longrightarrow & H_{n+1}(C^+_\bullet) & \longrightarrow & H_n(D_\bullet) & \longrightarrow & H_n(Cone_\bullet(f)) & \longrightarrow \cdots \\ & & & \| & & & & & \\ & & & H_n(C_\bullet) & & & & & \end{array}$$

It remains to check $\delta = H_n(f)$.

Note if c is a cycle in C_n. Then

$$\partial^f \circ p^{-1}(c) = (-\partial c, fc) = (0, fc) = i(fc)$$

$$\delta : \langle c \rangle \longmapsto \langle i^{-1}\partial^f p^{-1} c \rangle = \langle fc \rangle = H_n(f)\langle c \rangle$$

□

Proposition 3.6. *Suppose $F : C_\bullet \longrightarrow D_\bullet$ is a chain map between the two free chain complex . Then F is a chain equivalence iff*

$$H_n(f) : H_n(C_\bullet) \longrightarrow H_n(D_\bullet)$$

is an isomorphism for all n,

Proof. If f is a chain equivalence then $H_n(f)$ is always a isomorphism. This does not require any freeness assumptions and we proved in last semester.

For the converse, if $H_n(f)$ is always an isomorphism, then the LES

$$\cdots \longrightarrow H_{n+1}(Cone_\bullet(f)) \longrightarrow H_n(C_\bullet) \xrightarrow{\cong} H_n(D_\bullet) \longrightarrow H_n(Cone_\bullet(f)) \longrightarrow \cdots$$

This implies $H_n(Cone_\bullet(f)) = 0, \forall n$. Then $Cone_\bullet(f)$ is acyclic, and we can conclude by the previous lemma. □

Recap on Acyclic models.

Definition 3.7. *Suppose $\mathcal{C}$ is a category and $T_\bullet : \mathcal{C} \longrightarrow Comp$ is a functor. A family of **models** in $\mathcal{C}$ is simply a subset of $obj(\mathcal{C})$*

Fix $n \in \mathbb{Z}$ and consider $T_n : \mathcal{C} \longrightarrow Ab$

$$T_n(\mathcal{C}) = (T_\bullet(\mathcal{C}))_{nth\ group}$$

A T_n model set χ is simply a choice of element $x_\lambda \in T_n(M_\lambda)$ for each λ $\mathcal{M} = \{M_\lambda | \lambda \in \Lambda\}$

We say that the model is free if the following condition holds.

1. *$T_n(C)$ is a free abelian group $\forall C \in \mathcal{C}$*
2. *There is a T_n-model set $\{x_\lambda | \lambda \in \Lambda\}$ s..t*

$$\{T_n(f)()x_\lambda | f \in Hom(M_\lambda, C), \lambda \in \Lambda\}$$

is a basis for the free abelian group $T_n(C)$.

$f : M_\lambda \longrightarrow C$ is a morphism in $\mathcal{C}$ $T_n(f) : T(M_\lambda) \longrightarrow T_n(C)$ is a homomorphism between two abelian groups. $T(M_\lambda) \in T_n(f)(x_\lambda)$ does indeed belong to $T_n(C)$. A baissi for $T_n(C)$ is obtained by letting f run over all of $Hom(M_\lambda, C)$ and letting λ run over Λ.

We say $T_\bullet : \mathcal{C} \longrightarrow Comp$ if free with basis in $\mathcal{M}$ if each T_n is free with basis in $\mathcal{M}$

Definition 3.8. *$T_\bullet \mathcal{C} \longrightarrow Comp$, we say $T_\bullet$ is**non-negative** if $T_n(C) = 0$ forall $n < 0$ and $\forall C$. $T_\bullet$ is **acyclic in the positive degrees on** C if $H_n(T_\bullet(C)) = 0, \forall n > 0$.*

Suppose $T_\bullet C \longrightarrow Comp$. $H_0 \circ T_\bullet \mathcal{C} \longrightarrow Ab$.

Theorem 3.9. *Suppose $\mathcal{C}$ is a categroy with omdels $\mathcal{M}$. Supose $S_\bullet, T_\bullet : \mathcal{C} \longrightarrow Comp$ are 2 functors such that S and T are non-negative and acyclic in positive degree on every model, and both S and T are free with basis in $\mathcal{M}$.*

Suppose

$$\Theta : H_0 \circ S_\bullet \longrightarrow H_0 \circ T_\bullet$$

is a natural equivalence. $\exists$ a natural cahin equivalence $\Psi_\bullet : S_\bullet \longrightarrow T_\bullet$ which isn unique up to chain homotopy and has $H_0(\Psi_\bullet) = \Theta$

Example 3.10. *Take $\mathcal{C} = Top$, $\mathcal{M} = \{\Delta^n | n \geq 0\}$. T is the singular chain functor.*

$$\mathcal{C}_\bullet : Top \longrightarrow Comp$$

$$X \mapsto C_\bullet(X)$$

$C_\bullet$ *is non-negative,* $\checkmark$. $H_n(C_\bullet(\Delta^i)) = H_n(\Delta^i) =$.

Claim: C_n is free with basis in Δ^n

Choose an element $x \in C_n(\Delta^n)$. Take x to be the identity map $\Delta^n \longrightarrow \Delta^n$, write this as $\ell_n : \Delta^n \longrightarrow \Delta^n$. Think of the identity map as an element of $C_n(\Delta^n)$ if σ is any $n-$simplex in any topological space $C_n(\sigma)(\ell_n) = \sigma \circ \ell_n = \sigma$

$\{C_n(\sigma)(\ell_n) | \sigma : \Delta^n \longrightarrow X\}$ is basis for the free abelian group $C_n(X)$.

Eilenberg-Zilber $Top \times Top$ is the category of pairs (X, Y) of topological spaces.

We will define two functor from $Top \times Top \longrightarrow Comp$ $S_\bullet(X, Y) = C_\bullet(X, Y)$. $T_\bullet(X, Y) = C_\bullet(X) \otimes C_\bullet(Y)$

For models

$$\mathcal{M} = \{(\Delta^i, \Delta^j), i, j \geq 0\}$$

Claim: S and T are both acyclic in positive degree on $\mathcal{M}$ and free with basis in $\mathcal{M}$

$S_\bullet$, $H_n(S_\bullet(\Delta^i, \Delta^j)) = H_n(\Delta^i \times \Delta^j) = 0$, $\forall n > 0, \forall i, j$

$S_i : Top \times Top \longrightarrow Ab$

$S_i(X, Y) = C_i(X \times Y)$

Claim: $\{(\Delta^i, \Delta^i)\}$ is a S_i-model set and a basis is $d_i : \Delta^i \otimes \Delta^i$ the diagonal map $x \mapsto (x, x)$ gives a basis

$$\sigma : \Delta^i \longrightarrow X \times Y$$

we can write $\sigma = (\sigma_x, \sigma_y) \circ d_i$, where $\sigma_x = p_X \circ \sigma$ be the composition of σ with $p_X : X \times Y \longrightarrow X$.

$\sigma = S_i(\sigma)(d_i)$ so that $\{s_i(\sigma)(d_i \| \sigma : \Delta^i \longrightarrow X \times Y\}$ is a basis of the free abelian group $C_i(X \times Y)$. $T_i(X \times Y) = (C_\bullet(X) \otimes C_\bullet(Y))$. $T_i(X, Y)$ is the tensor product of the free groups and so is free. $\{(\ell_i, \ell_j) | i + j = n\}$ is a T_n-model basis.

The last thing to check is that $T_\bullet(\Delta^i, \Delta^j)$ is acyclic in positive degrees

$$H_n(C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j)) = 0, \forall n > 0.$$

We can not compute this! However we can cheat

$$H_n(C_\bullet(\Delta^i)) = H_n(\Delta^i) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Consider the chain complex

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \cdots$$

$C_\bullet(\Delta^i)$ has the same homology as this complex. Thus $C_\bullet(\Delta^i)$ is equivalenct to the complex and $C_\bullet(\Delta^j)$ is also chain equivalent to it. $C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j)$ is chain equivalent to

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z} \longrightarrow 0 \cdots$$

Thus $H_n(C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j)) = H_n(0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z} \longrightarrow 0 \cdots)$

Want: $\Theta : H_0 \circ S_\bullet \longrightarrow H_0 \circ T_\bullet$ is a natural equivalence.

$$(x, y) \mapsto x \otimes y$$

$$H_0(C_\bullet(X \times Y)) \longrightarrow H_0(C_\bullet(X) \otimes C_\bullet(Y))$$

By the Acylic model theorem

$$\Omega_\bullet : S_\bullet \longrightarrow T_\bullet$$

is a natural chain equivalence

$$\Omega_\bullet : C_\bullet(X \times Y) \longrightarrow C_\bullet(X) \otimes C_\bullet(Y)$$

Corollary 3.11. *Kueneth formula. Let X and Y be otpological spaces then for $n \geq 0$*

There is a split exact sequence