Summary for Algebraic Topology II

Notes by Lin-Da Xiao

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Contents

1 Feb 21th: Tor functor

Definition 1.1. *Suppose A is an abelian group, A* ***Free resolution*** *is an exact sequence of the form*

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \longrightarrow 0,$$

where each F_i is a free abelian group. If moreover $F_i = 0, \forall i \geq 2$, we call it ***Short free resolution***

$$0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$$

(We can easily generalize this definition to R-modules)

Proposition 1.2. *Let A be an abelian group. Then there exists a short free resolution of A.*

Proof. Let F be the free abelian group generated by all elements in A. There is a surjection from F to A by linearly extending the map sending basis element to itself. Let K denote the kernel of this map. K is an abelian subgroup of a free abelian group ($\mathbb{Z}$-module). A subgroup of a free abelian group is torsion free as a module. $\mathbb{Z}$ is a PID. If R is a PID, then an R-module is free iff it is torsion free (See Bosch section 4.2). Then we know in particular, K is a free abelian group. □

With this construction, we can define the Tor functor now:

Definition 1.3. *Let A be an abelian group. Let $0 \to K \xrightarrow{f} F \to A \to 0$ be a short free resolution of A. Given any other abelian group B, we define*

$$Tor(A, B) := \ker(f \otimes id_B)$$

Tor(A,B)

This definition is independent on the choice of short free resolution.

2 Feb 28th:

<u>Question</u>: Given X, Y what is the cohomology of $X \times Y$?

<u>Answer</u>:

$$H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) + \bigoplus_{k+\ell=n-1} \mathrm{Tor}(H_k, (X), H_\ell(Y))$$

We will discuss Elenberg-Zilber theorem along this line the next lecture.

Today, we will prove the Algebraic Kueneth Theorem

Definition 2.1. *Suppose* $(C_\bullet, \partial)$ *and* $(C'_\bullet, \partial')$ *are two non-negative chain complexes. We define the* ***tensor complex*** $(C_\bullet \otimes C'_\bullet, \Delta)$, *where*

$$(C_\bullet \otimes C'_\bullet)_n = \oplus_{i+j=n} C_i \otimes C'_j$$

and the differential Δ *is defined by*

$$\Delta(c_i \otimes c'_j) = \partial c_i \otimes c'_j + (-1)^i c_i \otimes \partial' c_j$$

First, note that $\Delta(c_i \otimes c'_j)$ does indeed belong to $(C_\bullet \otimes C'_\bullet)_{n-1}$. The reason for $(-1)^i$ is to make $\Delta^2 = 0$. $C_\bullet \otimes C'_\bullet$ is another non-negative chain complex.

Definition 2.2. *Suppose* $f_\bullet : C_\bullet \longrightarrow D_\bullet$ *and* $g_\bullet : C'_\bullet \longrightarrow D'_\bullet$ *are two morphism of chain complexes. Then we can define a chain map*

$$f \otimes g : C \otimes C' \longrightarrow D \otimes D'$$

by

$$(f \otimes g)_n = \sum_{i+j=n} f_i \otimes g_j$$

It is easy to check this is indeed a chain map.

Lemma 2.3. *If* $f' : C \longrightarrow C'$ *and* $g' : D \longrightarrow D'$ *are two more chain maps with* f *homotopic to* f' *and* g *homotopic to* g'. *Then* $f' \otimes g'$ *is homotopic to* $f \otimes g$.

Theorem 2.4. *(Algebraic Kuenneth Theorem) Let* (C, ∂) *and* (D, ∂') *be two non-negative free complex. Then for every* $n \geq 0$, *there is a split exact sequence*

$$0 \longrightarrow \oplus_{i+j=n} H_i(C) \otimes H_j(D) \longrightarrow H_N(C \otimes D) \longrightarrow \oplus_{k+\ell=n-1}\ Tor(H_k(C), H_\ell(D)) \longrightarrow 0$$

where ω *is the map* $\langle c_i \rangle \otimes \langle d_j \rangle \mapsto \langle c_i \otimes d_j \rangle$. *Thus there also exists a (non-natural) isomorphism*

$$H_n(C \times D) \cong \oplus_{i+j=n} H_i(C) \otimes H_j(D) + \oplus_{k+\ell=n-1}\ Tor(H_k, (C), H_\ell(D)$$

The proof requires two auxiliary results.

Proposition 2.5. *Let* $(E_\bullet, 0)$ *be a non-negative chain complex with all differential zero and* $(D_\bullet, \partial)$ *be any non-negative chain complex. Given* $i \geq 0$, *let* $D^i_\bullet$ *denote the chain complex where* $D^i_n = D_{n-i}$ *and the boundary map*

$$D^i_n \longrightarrow D^i_{n-1}$$

is just the map: $D_{n-i} \longrightarrow D_{n-i-1}$.

Then

$$H_n(E_\bullet \otimes D_\bullet) \cong \bigoplus_{i \geq 0} H_n(E_i \otimes D^i_\bullet)$$

Proof. (of the Proposition) Since $E_\bullet$ has no differentials

$$\begin{aligned}\Delta(e_i \otimes d_{n-i}) &= (-1)^i e_i \otimes \partial d_{n-i} \\ &= (-1)^i (id_E \otimes \partial)[e_i \otimes d_{n-i}]\end{aligned}$$

$$\begin{aligned}H_n(E_\bullet \otimes D_\bullet) &= \frac{ker\Delta}{im\Delta} \\ &= \bigoplus_{i\geq 0} \frac{ker(id_E \otimes \partial|_{D_{n-i}})}{im(id_E \otimes \partial|_{D_{n-i+1}})} \\ &= \bigoplus_{i\geq 0} H_n(E_i \otimes D_\bullet^i)\end{aligned}$$

□

Proof. (of Theorem) We will prove it in three steps:

Let's use the same notation as we did in the proof of the universal coefficient theorem. $B_n \subset Z_n \subset C_n$. $(Z_\bullet, 0)$ and $(B_\bullet^+, 0)$ are chain complexes with no differentials, where $B_n^+ = B_{n-1}$. $(H_\bullet, 0)$ be the chain complex. $i : Z_n \hookrightarrow C_n$, $j : B_n \hookrightarrow Z_n$, $d : C_n \longrightarrow B_{n-1}$, where d is the just the differential ∂ of $C_\bullet$ and we use p to denote the projection $Z_n \twoheadrightarrow H_n$. Then we have two short exact sequence of chain complexes

$$0 \longrightarrow Z_\bullet \xrightarrow{i_\bullet} C_\bullet \xrightarrow{D_\bullet} B_\bullet^+ \longrightarrow 0$$

$$0 \longrightarrow B_\bullet \xrightarrow{j_\bullet} Z_\bullet \xrightarrow{p_\bullet} H_\bullet \longrightarrow 0.$$

We tensor it with $D_\bullet$.

$$0 \longrightarrow Z_\bullet \otimes D_\bullet \xrightarrow{i_\bullet} C_\bullet \otimes D_\bullet \xrightarrow{D_\bullet} B_\bullet^+ \otimes D_\bullet \longrightarrow 0$$

$$0 \longrightarrow B_\bullet \otimes D_\bullet \xrightarrow{j_\bullet} Z_\bullet \otimes D_\bullet \xrightarrow{p_\bullet} H_\bullet \otimes D_\bullet \longrightarrow 0.$$

They are again short exact sequence of chain complexes because D is free Abelian group thus flat module.

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, bend left, "i"] & C_n \arrow[l, bend left, "r"] \arrow[r, "d"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

This sequence splits as B_{n-1} is free abelian. Thus $\exists$ a map $r : C_n \longrightarrow Z_n$ such that $r|_{Z_n}$ is the identity $r_\bullet : C_\bullet \longrightarrow Z_\bullet$.

Denote by μ the composition $p \circ r : C_\bullet \longrightarrow H$.

Claim: μ is a chain map from $(C_\bullet, \partial) \longrightarrow (H_\bullet, 0)$. Take $c \in C_{n+1}$ and check it commutes

$$\mu \circ \partial c = \mu \partial c = p \circ r \partial c = \langle \partial c \rangle = 0$$

and $0 \circ \mu c = 0$

Step 2: Define $\varphi = H_n(\mu \otimes id)$. $H_n(C_\bullet \otimes D_\bullet) \longrightarrow H_n(H_\bullet \otimes D_\bullet)$.

Claim: φ is an isomorphism.

It suffices to prove the diagram commutes and conclude by five lemma.

$$\begin{tikzcd}
H_{n+1}(B_\bullet^+ \otimes D_\bullet) \arrow[r, "\delta"] \arrow[d, "id"] & H_n(Z_\bullet \otimes D_\bullet) \arrow[r] \arrow[d, "id"] & H_n(C_\bullet \otimes D_\bullet) \arrow[r] \arrow[d, "\varphi"] & H_n(B_\bullet^+ \otimes D_\bullet) \arrow[r, "\delta"] \arrow[d, "id"] & H_{n-1}(Z_\bullet \otimes D_\bullet) \arrow[d, "id"] \\
H_n(B_\bullet \otimes D_\bullet) \arrow[r] & H_n(Z_\bullet \otimes D_\bullet) \arrow[r] & H_n(H_\bullet \otimes D_\bullet) \arrow[r, "\delta'"] & H_{n-1}(B_\bullet \otimes D_\bullet) \arrow[r] & H_{n-1}(Z_\bullet \otimes D_\bullet)
\end{tikzcd}$$

Step 3: We complete the proof

$$\begin{aligned} H_n(C_\bullet \otimes \otimes D_\bullet) &\cong H_n(H_\bullet \otimes D_\bullet) \\ &\cong \bigoplus_{i \geq 0} H_n(H_i(C_\bullet) \otimes D_\bullet^i) \end{aligned}$$

By the universal coefficient theorem, there is a split exact sequence

$$0 \longrightarrow H_i(C_\bullet) \otimes H_n(D_\bullet^i) \longrightarrow H_n(H_i(C_\bullet) \otimes D_\bullet^i) \longrightarrow \operatorname{Tor}(H_i(C_\bullet), H_{n-1}(D_\bullet^i)) \longrightarrow 0$$

If we get rid of the notation $D_\bullet^i$.

$$0 \longrightarrow H_i(C_\bullet) \otimes H_n(D_\bullet^i) \longrightarrow H_n(H_i(C_\bullet) \otimes D_\bullet^i) \longrightarrow \operatorname{Tor}(H_i(C_\bullet), H_{n-1-i}(D_\bullet)) \longrightarrow 0$$

Take the direct sum over i and use the fact that

□

3 Mar 2nd: Eilenberg-Zilber

Theorem 3.1. *(Eilenberg-Zilber) if X and Y are two topological spaces. There is a nontrivial chain equivalence*

$$\Omega_\bullet : C_\bullet(X \times Y) \longrightarrow C_\bullet(X) \otimes C_\bullet(Y)$$

which is unique up to chain homotopy

Digression on chain equivalences

Lemma 3.2. *Let $(C_\bullet, \partial)$ be a free chain complex. Then $C_\bullet$ is acyclic iff it has contracting chain homotopy*

Proof. A contracting homotopy means $Q : C_n \longrightarrow C_{n+1}$ s.t. $Q\partial + \partial Q = id$.

If such Q exists then $H_n(C_\bullet) = 0 \forall n$. That direction doesn't require $C_\bullet$ to be free

$$B_n \subseteq Z_n \subseteq C_n$$

If we assume $C_\bullet$ is acyclic then

$$B_n = Z_n, \forall n$$

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} Z_{n_1} \longrightarrow 0$$

Since Z_{n-1} is free abelian the sequence splits $\exists r_n : Z_{n-1} \longrightarrow C_n$ s.t. $\partial \circ r_n = id$. Note that $id - r_{n-1} \circ \partial$ jas image in Z_{n-1}, $c \in C_n$. $\partial(c - r_n \partial c) = \partial c - \partial c = 0$

Now define $Q_n : C_n \longrightarrow C_{n+1}$ by $Q_n = r_n(id - r_{n-1} \circ \partial)$. This works.

$$\begin{aligned} \partial Q_n + Q_{n-1}\partial &= \partial r_n(id - r_{n-1}\partial) + r_{n-1}(id - r_{n-2}\partial)\partial \\ &= id - r_{n-1}\partial + r_{n-1}\partial - r_{n-1}r_{n-2}\partial^2 \\ &= 0 \end{aligned}$$

□

Definition 3.3. *Suppose $f : (C_\bullet, \partial) \longrightarrow (D_\bullet, \partial')$. The* ***mapping cone*** *of f is the chain complex $Cone_\bullet(f), \partial^f$, where $Cone_n(f) = C_{n-1} \otimes D_n$ and $\partial^f : Cone_n(f) \longrightarrow Cone_{n-1}(f)$*

$$\partial^f(c, d) = (-\partial c, fc + \partial' d)$$

$$\partial^f = \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix}$$

Note if $C_\bullet$ and $D_\bullet$ are free chain complex, so is the cone.

Lemma 3.4. *If $f : C_\bullet \longrightarrow D_\bullet$ is a chain map between two free chain complexes and $Cone_\bullet(f)$ is acyclic then f is a chain equivalence.*

Proof. If $Cone_\bullet(f)$ is acyclic, there exists Q s.t.

$$Q\partial^f + \partial^f Q = id$$

$$Q = \begin{pmatrix} p & g \\ r & -p' \end{pmatrix}$$

$$\begin{pmatrix} \partial & 0 \\ f & -\partial' \end{pmatrix} \begin{pmatrix} p & g \\ r & -p' \end{pmatrix} + \begin{pmatrix} p & g \\ r & -p' \end{pmatrix} \begin{pmatrix} \partial & 0 \\ f & -\partial' \end{pmatrix} = \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}$$

$$\begin{pmatrix} -\partial p - p\partial + gf & -\partial g + g\partial' \\ * & fg - \partial' p' - p'\partial' \end{pmatrix} \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}$$

Then we know $g : D_\bullet \longrightarrow D_\bullet$ is a chain map

$p\partial + \partial p = gf - id$

$p'\partial' + \partial' p = fg - id$. Thus f is a chain equivalence with inverse g. □

Lemma 3.5. *Let $f : C_\bullet \longrightarrow D_\bullet$. Then there is a LES*

$$\cdots \longrightarrow H_{n+1}(Cone_\bullet(f)) \longrightarrow H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(D_\bullet) \longrightarrow H_n(Cone_\bullet(f)) \longrightarrow \cdots$$

Proof. Denote by $C_\bullet^+$ the chain complex $C_n^+ = C_{n-1}$. There is a SES

$$0 \longrightarrow D_\bullet \xrightarrow{i} Cone_\bullet(f) \xrightarrow{p} C_\bullet^+ \longrightarrow 0$$

with $i(d) = (0, d)$ and $p(c, d) = c$

Pass to the LES in homology

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(Cone_\bullet(f)) \arrow[r] & H_{n+1}(C_\bullet^+) \arrow[r] \arrow[d, equal] & H_n(D_\bullet) \arrow[r] & H_n(Cone_\bullet(f)) \arrow[r] & \cdots \\ & & H_n(C_\bullet) & & & \end{tikzcd}$$

It remains to check $\delta = H_n(f)$.

Note if c is a cycle in C_n. Then

$$\partial^f \circ p^{-1}(c) = (-\partial c, fc) = (0, fc) = i(fc)$$

$$\delta : \langle c \rangle \longmapsto \langle i^{-1}\partial^f p^{-1} c \rangle = \langle fc \rangle = H_n(f)\langle c \rangle$$

□

Proposition 3.6. *Suppose $F : C_\bullet \longrightarrow D_\bullet$ is a chain map between the two free chain complex . Then F is a chain equivalence iff*

$$H_n(f) : H_n(C_\bullet) \longrightarrow H_n(D_\bullet)$$

is an isomorphism for all n,

Proof. If f is a chain equivalence then $H_n(f)$ is always a isomorphism. This does not require any freeness assumptions and we proved in last semester.

For the converse, if $H_n(f)$ is always an isomorphism, then the LES

$$\cdots \longrightarrow H_{n+1}(Cone_\bullet(f)) \longrightarrow H_n(C_\bullet) \xrightarrow{\cong} H_n(D_\bullet) \longrightarrow H_n(Cone_\bullet(f)) \longrightarrow \cdots$$

This implies $H_n(Cone_\bullet(f)) = 0, \forall n$. Then $Cone_\bullet(f)$ is acyclic, and we can conclude by the previous lemma. □

Recap on Acyclic models.

Definition 3.7. *Suppose $\mathcal{C}$ is a category and $T_\bullet : \mathcal{C} \longrightarrow Comp$ is a functor. A family of **models** in $\mathcal{C}$ is simply a subset of $obj(\mathcal{C})$*

Fix $n \in \mathbb{Z}$ and consider $T_n : \mathcal{C} \longrightarrow Ab$

$$T_n(\mathcal{C}) = (T_\bullet(\mathcal{C}))_{nth\ group}$$

A T_n model set χ is simply a choice of element $x_\lambda \in T_n(M_\lambda)$ for each λ $\mathcal{M} = \{M_\lambda | \lambda \in \Lambda\}$

We say that the model is free if the following condition holds.

1. *$T_n(C)$ is a free abelian group $\forall C \in \mathcal{C}$*

2. *There is a T_n-model set $\{x_\lambda | \lambda \in \Lambda\}$ s..t*

$$\{T_n(f)()x_\lambda | f \in Hom(M_\lambda, C), \lambda \in \Lambda\}$$

   is a basis for the free abelian group $T_n(C)$.

$f : M_\lambda \longrightarrow C$ is a morphism in $\mathcal{C}$ $T_n(f) : T(M_\lambda) \longrightarrow T_n(C)$ is a homomorphism between two abelian groups. $T(M_\lambda) \in T_n(f)(x_\lambda)$ does indeed belong to $T_n(C)$. A baissi for $T_n(C)$ is obtained by letting f run over all of $Hom(M_\lambda, C)$ and letting λ run over Λ.

We say $T_\bullet : \mathcal{C} \longrightarrow Comp$ if free with basis in $\mathcal{M}$ if each T_n is free with basis in $\mathcal{M}$

Definition 3.8. *$T_\bullet \mathcal{C} \longrightarrow Comp$, we say $T_\bullet$ is**non-negative** if $T_n(C) = 0$ forall $n < 0$ and $\forall C$. $T_\bullet$ is **acyclic in the positive degrees on** C if $H_n(T_\bullet(C)) = 0, \forall n > 0$.*

Suppose $T_\bullet C \longrightarrow Comp$. $H_0 \circ T_\bullet \mathcal{C} \longrightarrow Ab$.

Theorem 3.9. *Suppose $\mathcal{C}$ is a categroy with omdels $\mathcal{M}$. Supose $S_\bullet, T_\bullet : \mathcal{C} \longrightarrow Comp$ are 2 functors such that S and T are non-negative and acyclic in positive degree on every model, and both S and T are free with basis in $\mathcal{M}$.*

Suppose

$$\Theta : H_0 \circ S_\bullet \longrightarrow H_0 \circ T_\bullet$$

is a natural equivalence. $\exists$ a natural cahin equivalence $\Psi_\bullet : S_\bullet \longrightarrow T_\bullet$ which isn unique up to chain homotopy and has $H_0(\Psi_\bullet) = \Theta$

Example 3.10. *Take $\mathcal{C} = Top$, $\mathcal{M} = \{\Delta^n | n \geq 0\}$. T is the singular chain functor.*

$$\mathcal{C}_\bullet : Top \longrightarrow Comp$$

$$X \mapsto C_\bullet(X)$$

$C_\bullet$ *is non-negative,* ✓. $H_n(C_\bullet(\Delta^i)) = H_n(\Delta^i) =$.

<u>Claim</u>: C_n is free with basis in Δ^n

Choose an element $x \in C_n(\Delta^n)$. Take x to be the identity map $\Delta^n \longrightarrow \Delta^n$, write this as $\ell_n : \Delta^n \longrightarrow \Delta^n$. Think of the identity map as an element of $C_n(\Delta^n)$ if σ is any $n-$simplex in any topological space $C_n(\sigma)(\ell_n) = \sigma \circ \ell_n = \sigma$

$\{C_n(\sigma)(\ell_n) | \sigma : \Delta^n \longrightarrow X\}$ is basis for the free abelian group $C_n(X)$.

Eilenberg-Zilber $Top \times Top$ is the category of pairs (X, Y) of topological spaces.

We will define two functor from $Top \times Top \longrightarrow Comp$ $S_\bullet(X, Y) = C_\bullet(X, Y)$. $T_\bullet(X, Y) = C_\bullet(X) \otimes C_\bullet(Y)$

For models

$$\mathcal{M} = \{(\Delta^i, \Delta^j), i, j \geq 0\}$$

<u>Claim</u>: S and T are both acyclic in positive degree on $\mathcal{M}$ and free with basis in $\mathcal{M}$

$S_\bullet$, $H_n(S_\bullet(\Delta^i, \Delta^j)) = H_n(\Delta^i \times \Delta^j) = 0$, $\forall n > 0, \forall i, j$

$S_i : Top \times Top \longrightarrow Ab$

$S_i(X, Y) = C_i(X \times Y)$

<u>Claim</u>: $\{(\Delta^i, \Delta^i)\}$ is a S_i-model set and a basis is $d_i : \Delta^i \otimes \Delta^i$ the diagonal map $x \mapsto (x, x)$ gives a basis

$$\sigma : \Delta^i \longrightarrow X \times Y$$

we can write $\sigma = (\sigma_x, \sigma_y) \circ d_i$, where $\sigma_x = p_X \circ \sigma$ be the composition of σ with $p_X : X \times Y \longrightarrow X$.

$\sigma = S_i(\sigma)(d_i)$ so that $\{s_i(\sigma)(d_i \| \sigma : \Delta^i \longrightarrow X \times Y\}$ is a basis of the free abelian group $C_i(X \times Y)$. $T_i(X \times Y) = (C_\bullet(X) \otimes C_\bullet(Y))$. $T_i(X, Y)$ is the tensor product of the free groups and so is free. $\{(\ell_i, \ell_j) | i + j = n\}$ is a T_n-model basis.

The last thing to check is that $T_\bullet(\Delta^i, \Delta^j)$ is acyclic in positive degrees

$$H_n(C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j)) = 0, \forall n > 0.$$

We can not compute this! However we can cheat

$$H_n(C_\bullet(\Delta^i)) = H_n(\Delta^i) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Consider the chain complex

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \cdots$$

$C_\bullet(\Delta^i)$ has the same homology as this complex. Thus $C_\bullet(\Delta^i)$ is equivalenct to the complex and $C_\bullet(\Delta^j)$ is also chain equivalent to it. $C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j)$ is chain equivalent to

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z} \longrightarrow 0 \cdots$$

Thus $H_n(C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j)) = H_n(0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z} \longrightarrow 0 \cdots)$

<u>Want</u>: $\Theta : H_0 \circ S_\bullet \longrightarrow H_0 \circ T_\bullet$ is a natural equivalence.

$$(x, y) \mapsto x \otimes y$$

$$H_0(C_\bullet(X \times Y)) \longrightarrow H_0(C_\bullet(X) \otimes C_\bullet(Y))$$

By the Acylic model theorem

$$\Omega_\bullet : S_\bullet \longrightarrow T_\bullet$$

is a natural chain equivalence

$$\Omega_\bullet : C_\bullet(X \times Y) \longrightarrow C_\bullet(X) \otimes C_\bullet(Y)$$

Corollary 3.11. *Kueneth formula. Let X and Y be otpological spaces then for $n \geq 0$*

There is a split exact sequence

4 Mar 7th: Cochain complexes and cohomology

5 Mar 9th: Universal coefficient theorem for cohomology

Definition 5.1. *Suppose A is an abelian group and let*

$$0 \longrightarrow K \xrightarrow{f} F \longrightarrow A \longrightarrow 0$$

is a short free resolution. Take another abelian group and apply $Hom(\Box, B)$, we can find an exact sequence

$$0 \longrightarrow Hom(A, B) \longrightarrow Hom(F, B) \xrightarrow{Hom(f,B)} Hom(K, B)$$

and we define $Ext(A, B) := coker\ Hom(f, B) = Hom(K, B)/\ im\ Hom(f, B)$. Thus $Ext(A, B)$measures the failure for $Hom(\Box, B)$ to be right exact.

Here is a more sophisticated way of viewing $\operatorname{Ext}(A, B)$ consider a chain complex $C_1 = K$, $C_0 = F$, $\partial_1 : C_1 \longrightarrow C_0 = f : K \longrightarrow F$ and all other group zero. $H_0(C_\bullet) = A$. Now apply $\operatorname{Hom}(\Box, B)$ to a cochain complex $\operatorname{Hom}(C_\bullet, B)$, the definition of $\operatorname{Ext}(A, B)$ gives us immediately that

$$H^1(\operatorname{Hom}(C_\bullet, A)) = \operatorname{Ext}(A, B).$$

From this it follows that $\operatorname{Ext}(\Box, B)$ is a contravariant functor and it is well defined (independent of the choice of short free resolution.)

Definition 5.2. *An abelian group D is said to be* ***divisible*** *if for every $b \in D$ and every $n \in \mathbb{N}$ there exists an $a \in D$ s.t., $na = b$*

Theorem 5.3. *(Properties of Ext)*

For a fixed abelian group A, $Ext(\Box, A)$ is a contravariant functor and $Ext(A, \Box)$ is a covariant funcotr. Moreover,

(1) If F is a free group, then $Ext(F, B) = 0, \forall B$. If D is a divisible abelian group, then $Ext(A, D) = 0 \forall A$.

(2) If A is a finitely generated group with torsion subgroup $T(A)$ then $Ext(A, \mathbb{Z}) = T(A)$

(3) $0 \longrightarrow A \longrightarrow A' \longrightarrow A'' \longrightarrow 0$ is exact, then for any B, there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Hom(A'', B) \arrow[r] & Hom(A', B) \arrow[r] & Hom(A, B) \arrow[dll] & \\ & Ext(A'', B) \arrow[r] & Ext(A', B) \arrow[r] & Ext(A, B) \arrow[r] & 0 \end{tikzcd}$$

If $0 \longrightarrow B \longrightarrow B' \longrightarrow B'' \longrightarrow 0$ is exact, then for any A, there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Hom(A, B) \arrow[r] & Hom(A, B') \arrow[r] & Hom(A, B'') \arrow[dll] & \\ & Ext(A, B) \arrow[r] & Ext(A, B') \arrow[r] & Ext(A, B'') \arrow[r] & 0 \end{tikzcd}$$

(4) If B is an abelian group and $\{A_\lambda | \lambda \in \Lambda\}$ is a collection of abelian group then

$$Ext\left(\bigoplus_{\lambda \in \Lambda} A_\lambda, B\right) \cong \prod_{\lambda \in \Lambda} Ext(A_\lambda, B)$$

$$Ext\left(B, \prod_{\lambda} A_\lambda\right) \cong \prod_{\lambda\in\Lambda} Ext(B, A_\lambda)$$

(5) For any $m \in \mathbb{N}$ and any B

$$Ext(\mathbb{Z}_m, B) \cong B/mB$$

Proof. Every thin g apart from (1) and (2) all follow identically to the corresponding statements about Tor. These two new statements are left as exercise. □

Three universal coefficient theorems

Let $C_\bullet$ be a chain complex and A be an abelian group, form $\mathrm{Hom}(C_\bullet, A)$ as a cochain complex.

we define a natural chain map ζ as follows

$$\zeta : H^n(\mathrm{Hom}(C_\bullet, A)) \longrightarrow \mathrm{Hom}(H_n(C_\bullet), A)$$

$$\zeta\langle\gamma\rangle\langle c\rangle = \gamma(c) \in A$$

We need to check this is well defined.

Suppose γ, γ' to be cocycles s.t. $\langle\gamma\rangle = \langle\gamma'\rangle$, c, c' to be cycles s.t. $\langle c\rangle = \langle c'\rangle$.

Claim: $\gamma(c) = \gamma'(c')$.

$$\begin{aligned} \gamma' &= \gamma + d\delta \\ c' &= c + \partial a \\ \gamma'(c') &= \gamma(c) + d\delta(c) + \gamma(\partial a) + d\delta(\partial a) \\ &= \gamma(c) + \delta(\partial c) + d\gamma(a) + d\delta(\partial a) \\ &= \gamma(c) \end{aligned}$$

Theorem 5.4. *(The dual universal coefficients theorem)*

Let $(C_\bullet, \partial)$ be a free chain complex and let A be an abelian group. Then for every n there is a split exact sequence

$$0 \longrightarrow Ext(H_{n-1}(C_\bullet, A)) \longrightarrow H^n(Hom(C_\bullet, A)) \xrightarrow{\zeta} Hom(H_n(C_\bullet)mA) \longrightarrow 0,$$

where ζ is the map defined above.

$$H^n(Hom(C_\bullet, A)) \cong Hom(H_n(C_\bullet), A) \oplus Ext(H_n C_\bullet, A).$$

This specialize to $C_\bullet = C_\bullet(X)$ for X a topological space

Proof. Go through the proof of universal coefficient theorem and

(a) erase $\square \otimes A$ and write $\operatorname{Hom}(\square, A)$

(b) erase Tor and write Ext

(c) reverse arrow when needed.

$\square$

Definition 5.5. *A topological space X is of* ***finite type*** *is $H_n(X)$ is finitely generated for all n. This covers all spaces we have looked at so far.*

Corollary 5.6. *Let X be of finite type Denote by $T_n(X)$ the torsion subgroup of $H_n(X)$. Then $\forall n \geq 0$*

$$H^n(X) \cong H_n(X)/T_n(X) \oplus T_{n-1}(X)$$

Proof. For any finitely generated group A, $\operatorname{Hom}(A, \mathbb{Z}) \cong A/T(A)$[in problem sheet] $\operatorname{Ext}(H_{n-1}(X), \mathbb{Z}) \cong T_{n-1}(X)$ by property (2) of Ext theorem. $\square$

If $C_\bullet$ and $D_\bullet$ be two chain complex there is a natural chain map to male $Hom(C_\bullet, D_\bullet)$ to a cochain complex, and one could mimic what we did in Lecture 26 to obtain an algebraic Kuenneth lemma. But it is useless, because there is no analogous Eilenberg-Zilber theorem for cochain complex.

Proposition 5.7. *Let X be a topological space of finite type. Then there exists a free non-negative chain complex $(E_\bullet, \epsilon)$ such that $(C_\bullet(X), \partial) \cong (E_\bullet, \epsilon)$ and such that each E_n is finitely generated.*

Proof. Let $p : Z_n(X) \longrightarrow H_n(X)$, $c \mapsto \langle c \rangle$. Since $H_n(X)$ is finitely generated, $\exists$ finitely generated subgroup $F_n \subset Z_n(X)$ s.t., $p|_{F_n}$ is surjective. Note F_n is free (As $Z_n(X)$ is free). $F_n' = \ker p|_{F_n} : F_n \longrightarrow H_n(X)$,

$$E_n = F_n \oplus F_n'$$

E_n is indeed free and finitely generated. $\epsilon : E_n \longrightarrow E_{n-1}$ $(c, c') \mapsto (c', 0)$, then obviously, $\epsilon^2 = 0$, $(E_\bullet, \epsilon)$ is a chain complex.

$$H_n(E_\bullet) = \frac{ker\epsilon : E_n \longrightarrow E_{n-1}}{\mathrm{im}\epsilon E_{n+1} \longrightarrow E_n} = \frac{F_n}{F_{n-1}'} = H_n(X). \tag{*}$$

Now let us build a chain map

$$f : (E_\bullet, \epsilon) \longrightarrow (C_\bullet(X), \partial)$$

Since F_n' is free abelian, there exists a homomorphism

$$g : F_n' \longrightarrow C_{n+1}(X)$$

s.t. $\partial g(c') = c', \forall c' \in F_n'$. (Lemma 22.3) in ATI notes.

Define

$$f : E_n \longrightarrow C_n(X)$$
$$(c, c') \longmapsto c + g(c')$$

<u>Claim</u>: $f \circ \epsilon = \partial \circ f$

$$f\epsilon(c, c') = f(c', 0) = c'$$

$\partial f(c, c') = \partial c + \partial g(c)$. But $F_n \subset Z_n(X)$ by assumption so $\partial f(c, c') = c' = f\epsilon(c, c')$. Thus f is a chain map and we get an induced map.

$$H_n(f) : H_n(E_\bullet) \longrightarrow H_n(X)$$

which is isomorphism by $(*)$, then f is a chain equivalence because E_n is free.

□

Lemma 5.8. *Let $E_\bullet$ be a chian complex s.t. each E_n is finitely generated, and let A be an abelian group. Then there is an isomorphism of cochain complexes*

$$Hom(E_\bullet, \mathbb{Z}) \otimes A \cong \ Hom(E_\bullet, A)$$

Proof.

$$h : \ \mathrm{Hom}(E_n, \mathbb{Z}) \longrightarrow \ \mathrm{Hom}(E_n, A)$$
$$\gamma \otimes a \longmapsto [c \mapsto \gamma(c)a]$$

$E_n \ni h(\gamma \otimes a)(c) = \gamma(c) \cdot a$, where $\gamma(c) \in \mathbb{Z}$ and this is multiplication in A. This is clearly a chain map but hwy is it an isomorphism?

induct on the rank of E_n if rank $E_n = 1$ then $E_n \cong \mathbb{Z}$ and $\mathbb{Z} \otimes A \cong A$.

For the inductive steps, we just use that both $\otimes$ and Hom respects $(B \oplus B')$ □

Theorem 5.9. *(Cohomological universal coefficients theorem) Let X be a topological space of finite type and let A be an abelian group. Then $\forall n \geq 0$, there are split exact sequences.*

$$0 \longrightarrow H^n(X) \otimes A \longrightarrow H^n(X, A) \longrightarrow \ Tor(H^{n+1}, A) \longrightarrow 0$$

Proof. By the proposition, $\exists$ a free and finitely generated chain complex $E_\bullet$ which is chain equivalent to $C_\bullet(X)$. Set $E^\bullet = \operatorname{Hom}(E_\bullet.\mathbb{Z})$. Then $E^\bullet$ is free, finitely generated cochina complex, thus by the UCT, there is a split short exact sequence.

$$0 \longrightarrow H^n(E^\bullet) \otimes A \longrightarrow H^n(E^\bullet \otimes A) \longrightarrow \operatorname{Tor}(H^{n+1}(E^\bullet), A) \longrightarrow 0$$

$$H^n(E^\bullet) = H^n(\operatorname{Hom}(E_\bullet, \mathbb{Z})) \cong H^n(\operatorname{Hom}(C_n(X), \mathbb{Z})) = H^n(X)$$
$$E^\bullet \otimes A = \operatorname{Hom}(E, \mathbb{Z}) \otimes A = \operatorname{Hom}(E_\bullet, A) \cong \operatorname{Hom}(C_\bullet(X), A)$$ □

Theorem 5.10. *(Kuenneth Formula for cohomology)*

Let X and Y be topological spaces of finite type. Then $\forall n \geq 0$, $\exists$a split short exact sequence.

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \longrightarrow H^n(X \times Y) \longrightarrow \bigoplus_{k+\ell=n+1} Tor(H^k(X), H^\ell(Y)) \longrightarrow 0$$

Proof. Let $E_\bullet, F_\bullet$ be two finitely generated free chain complexes equivalent to $C_\bullet(X), C_\bullet(Y)$ respectively. □

6 Mat 16th: Ring structure on cohomology.

Definition 6.1. *A ring R is an abelian group with a additional operation called "multiplication" which is associative and distributive over the abelian group structure. There is a multiplicative identity.*

Remark 6.2. *If $1 = 0$, we have $R = \{0\}$ is the zero ring.*

we omit a lot of stuffs on ring theory here and only state a lemma that we will use later

Lemma 6.3. *Suppose R is a graded ring and I is a homogeneous graded ideal. then quotient ring is again a graded ring.*

$$R/I = \bigoplus_n (R^n + I)/I$$

Definition 6.4. *Let R be a ring. Let X be a topological space. $H^*(X, R)$ is the total cohomology*

$$\bigoplus_{n \geq 0} H^n(X; R)$$

Goal: If R is commutative, we will show $H^*(X;R)$ is a graded ring (NOT necessarily commutative.)

Suppose A is an abelian group. How to make A a ring?

If R is a ring, then $\mathrm{Hom}(A,R)$ is naturally a ring. $f,g \in \mathrm{Hom}(A,R)$, fg is defined as $(fg)(a) = f(a)g(a) \forall a \in A$. We will use this to endow $H^*(X;R)$ with a ring structure.

Let's recall the face maps from last semester. The face maps

$$\begin{aligned} \epsilon_i^n : & \Delta^{n-1} \longrightarrow \Delta^n \\ & (s_0, s_1.., s_{n-2}) \mapsto (s_0, ...s_{i-1}, 0, s_i, ..., s_{n-2}) \end{aligned}$$

maps the standard $n-1$ simplex onto the i-th face of the standard n-simplex.

Definition 6.5. *Let $0 \le i \le n$, the ith **front face***

$$\begin{aligned} F_i^n : \Delta^i \longrightarrow \Delta^n \\ (s_0, ..., s_{i-1}) \mapsto (s_0, ..., s_{i-1}, 0, 0, .., 0) \end{aligned}$$

*the ith **back face***

$$\begin{aligned} B_i^n : \Delta^i \longrightarrow \Delta^n \\ (s_0, ..., s_{i-1}) \mapsto (0, 0, .., 0, s_0, ..., s_{i-1}) \end{aligned}$$

Lemma 6.6.

1. $\epsilon_0^{n+1} = B^{n+1})n$ $\epsilon_{n+1}^{n+1} = F_n^{n+1}$
2. $B_{m+k}^n \circ B_k^{m+k} = B_k^n$, $F_{m+k}^n \circ F_k^{m+k} = F_k^n$
3. $$\epsilon_i^{n+1} \circ F_m^n = \begin{cases} F_{m+1}^{n+1} \circ \epsilon_i^{m+1} & i \le m \\ F_m^{n+1} & i \ge m+1 \end{cases}$$

Let $\alpha \in C^n(H,R) = \mathrm{Hom}(C_n(X),R)$, $\beta \in C^m(X,R) = \mathrm{Hom}(C_m(X),R)$. Define $\alpha \smile \beta$ the cup product of α and β to the elemetnof $C^{n+m}(X,R)$ defined by

$$\alpha \smile \beta(\sigma) = \alpha(\sigma \circ F^n)\beta(\sigma \circ B_m)$$

$F_n : \Delta^n \longrightarrow \Delta^{n+m}, \sigma \circ F_n : \Delta^n \longrightarrow \Delta^{n+m} \longrightarrow X$ is a singular n-simplex. $\alpha(\sigma \circ F_n)$ is a well-defined element of R. Similarly, $\beta(\sigma \circ B_m)$ is an element in R . Then $\alpha(\sigma \circ F_m)\beta(\sigma \circ B_m)$ is an element of R. it is then clear that $\alpha \smile \beta$ extends by linearity to define an element of $\mathrm{Hom}(C_{n+m}(X),R) = C^{n+m}(X,R)$

Proposition 6.7. *Let R be commutative. Then $C^*(X;R) = \bigoplus_{n\geq 0} C^n(X,R)$ is a graded ring under the cup product*

Proof. Let $\alpha \in C^n$, $\beta, \gamma \in C^m$.

Distributive: $\alpha \smile (\beta+\gamma) = \alpha \smile \beta + \alpha \smile \gamma$. Take $\sigma : \Delta^{n+m} \longrightarrow X$ and

$$\begin{aligned}(\alpha \smile (\beta+\gamma))(\sigma) &= \alpha(\sigma \circ F_n)[(\beta+\gamma)(\sigma \circ B_m)] \\ &= \alpha(\sigma \circ F_n)[\beta(\sigma \circ B_m) + \gamma(\sigma \circ B_m)] \\ &= \alpha(\sigma \circ F)\beta(\sigma \circ B_m) + \alpha(\sigma \circ F)\gamma(\sigma \circ B_m) \\ &= \alpha \smile \beta(\sigma) + \alpha \smile \gamma(\sigma)\end{aligned}$$

The same argument shows $(\beta+\gamma) \smile \alpha = \beta \smile \alpha + \gamma \smile \alpha$

Associativity take $\alpha \in C^n, \beta \in C^m, \gamma \in C^p$ and $\sigma : \Delta^{n+m+p} \longrightarrow X$.

$$(\alpha \smile \beta) \smile \gamma(\sigma) = \alpha(\sigma \circ F_{n+m} \circ F_n)\beta(\sigma \circ F_{n+m} \circ F_n) \cdot \gamma(\sigma \circ B_p)$$

$$\alpha \smile (\beta \smile \gamma)(\sigma) = \alpha(\sigma \circ F_n)\beta(\sigma \circ B_{m+p} \circ F_m) \cdot \gamma(\sigma \circ B_{m+p} \circ B_p)$$

By the face relation lemma the above two equal.

identity: Define $\nu(x) = 1_R, \forall x \in X$ ∎

how does this ring structure behave with respect to continuous map? Take

$$f : X \longrightarrow Y$$

$$f_\# : C_\bullet(X) \longrightarrow C_\bullet(Y)$$
$$f^\# : C^\bullet(X,R) \longrightarrow C^\bullet(X,R)$$

Claim: $f^\#(\alpha \smile \beta) = f^\#(\alpha) \smile f^\#(\beta)$

$$\begin{aligned}f^\#(\alpha \smile \beta)(\sigma) &= \alpha \smile \beta(f_\#\sigma) \\ &= (\alpha \smile \beta)(f \circ \sigma) \\ &= \alpha(f_\#(\sigma \circ F_n))\beta(f_\#(\sigma \circ B_m)) \\ &= f^\#(\alpha) \smile f^\#(\beta)(\sigma)\end{aligned}$$

Corollary 6.8. *There is a contravariant functor*

$$C^*(\Box, R) : TOP \longrightarrow Gr - Rings.$$

The ring structure is not very helpful, because it does not descend to the homotopy category.

We will see now that $\smile$ induces an operation on cohomology

$$\langle\alpha\rangle \smile \langle\beta\rangle = \langle\alpha \smile \beta\rangle$$

and induces a ring structue on $H^*(X;R)$ that does indeed respect homotopy.

Theorem 6.9. *(Ring structure on cohomology) If R is a commutative ring, then $H^*(\Box, R) : h - Top \longrightarrow GrRings$ is a well-defined functor.*

Proposition 6.10. $d(\alpha \smile \beta) = d\alpha \smile \beta + (-1)^n \alpha \smile d\beta$

Proof.

$$\begin{aligned} d(\alpha \smile \beta)(\sigma) &= (\alpha \smile \beta)(\partial\sigma) \\ &= \sum_{i=0}^{n+m+1} (-1)^i (\alpha \smile \beta)(\sigma \circ \epsilon_i) \\ &= \sum_{i=0}^{n} (-1)^i \alpha(\sigma \circ \epsilon_i \circ F_n)\beta(\sigma \circ \epsilon \circ B_m) + \sum_{i=n+1}^{m+n+1} (-1)^i \alpha(\sigma \circ \epsilon_i \circ F_n)\beta(\sigma \circ \epsilon_i \circ B_m) \end{aligned}$$

the first sum becomes $(d\alpha \smile \beta)(\sigma)$ and the second sum is $(-1)^n(\alpha \smile d\beta)(\sigma)$ $\Box$

Assuming the proposition , lets prove the theorem

Proof.

$$Z^* = \bigoplus Z^n(X, R)$$

$$B^* = \bigoplus B^n(X, R)$$

If $\alpha, \beta \in Z^*$ then

$$d(\alpha \smile \beta) = d\alpha \smile \beta + (-1)^? \alpha \smile d\beta = 0$$

Thus Z^* is a graded subring of C^* if $\alpha \in Z^n$ and $\beta \in B^m$ say $\beta = d\gamma$ then

$$\alpha \smile \beta = \alpha \smile d\gamma = \pm\alpha \smile \gamma + \alpha \smile d\gamma = \pm d(\alpha \smile \gamma)$$

thus $\alpha \smile \beta \in B^{n+m}$, similarly $\beta \smile \alpha \in B^{n+m}$. B^* is a homogeneous two sided ideal of C^*.

Thus by ht elemma at the start of the lecture,

$$H^*(X;R) = Z^*/B^*$$

has a graded ring structure, where

$$\langle\alpha\rangle \smile \langle\beta\rangle = \langle\alpha \smile \beta\rangle$$

$\Box$