Lecture Notes for Algebraic Geomtry I

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About the notes

In this notes we mainly focus on the algebraically closed fields.

1 Classical varieties

1.1 Feb 27th: Algebraic sets and morphisms

https://imaginary.org/programs

Recall: $V(I) \subset \mathbb{A}^n = \{x | \forall f \in I, f(x) = 0\}$.

Definition 1.1. *Closed subspaces of $\mathbb{A}^n$ are called* ***affine algebraic sets*** *and irreducible algebraic sets are called* ***affine algebraic varieties***

Definition 1.2. *Given Y an affine algebraic set in $\mathbb{A}^n$, we define the* ***coordinate ring*** *$\mathcal{O}(Y)$ as $K[X_1, ..., X_n]/I(Y)$*

Definition 1.3. *Let $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ be affine algebraic sets. A* ***morphism*** *$X \longrightarrow Y$ of affine algebraic sets is a map $f : X \longrightarrow Y$ of the underlying sets such that there exist polynomials $f_1, ..., f_n \in k[T_1, ..., T_m]$ with $f(x) = (f_1(x), ..., f_n(x))$ for all $x \in X$.*

We denote the category of affine algebraic sets over K as Alg_K

Theorem 1.4. *Let $Y_1 \subset \mathbb{A}^n, X_1, ..., X_n$, $Y_2 \subset \mathbb{A}^m, T_1, , ..., T_m$ affine algebraic sets. There are bijections*

$$\begin{aligned}
&Hom_{K\text{-}alg}(\mathcal{O}(Y_2), \mathcal{O}(Y_1)) \\
&\overset{(*)}{\longleftrightarrow} \{(f_1, ..., f_m) \in K[X]^m | \forall x \in Y_1, (f_1(x), ..., f_m(x)) \in Y_2\} \\
&\overset{(**)}{\longleftrightarrow} \{f : Y_1 \longrightarrow Y_2 | \forall \varphi \in \mathcal{O}(Y_2), \varphi \circ f \text{ is in } \mathcal{O}(Y_1)\} \\
&= \ Hom_{Alg_K}(Y_1, Y_2)
\end{aligned}$$

Proof. <u>Key observation</u>

To give $(f_1, ..., f_m) \in K[X]^m$ is "the same " as giving a ring morphism $g_0 : K[T] \longrightarrow K[X] : T_i \mapsto f_i$, which gives by composition $g_1 = \pi_1 \circ g_0$, where $\pi_1 : K[X] \longrightarrow \mathcal{O}(Y_1)$ is the canonical projection.

$$g_1 : K[T] \longrightarrow \mathcal{O}(Y_1)$$

which has a factorization

$$\begin{tikzcd}
K[T] \arrow[r, "g_1"] \arrow[d, "\pi_2"] & \mathcal{O}(Y_1) \\
\mathcal{O}(Y_2) \arrow[ur, "g"] &
\end{tikzcd}$$

iff $g_1(I(Y_2)) = 0$, which means iff

$$g_1(\varphi) = \text{ “replace } T_i \text{ by } f_i \text{ in } \varphi\text{”}$$

belongs to $I(Y_1)$ if $\varphi \in I(Y_2)$, which means if $x \in Y_1$, then $g_1(\varphi)(x) = 0$. That means $\varphi(f_1(x), ..., f_m(x)) = 0$ for $\varphi \in I(Y_2)$, i.e., $(f_1(x), ..., f_m(x)) \in Y_2$. If $x \in Y_1$. In the statement, this gives the $(*)$ bijection. Any k-algebra morphism $\mathcal{O}(Y_1) \longrightarrow \mathcal{O}(Y_2)$ comes from $K[T] \longrightarrow \mathcal{O}(Y_1)$ s.t. it vanishes on $I(Y_2)$.

For the bijection $(**)$, suppose

$$g : Y_1 \xrightarrow{g} Y_2 \xrightarrow{\varphi} K$$

sends $\varphi(Y_2)$ to $\varphi \circ g \in \mathcal{O}(Y_1)$. Then we get

$$\begin{aligned} \mathcal{O}(Y_2) &\longrightarrow \mathcal{O}(Y_1) \\ \varphi &\longmapsto \varphi \circ g, \end{aligned}$$

which is a K-algebra morphism.

As for the reverse direction, given g. From $\mathcal{O}(Y_2) \longrightarrow \mathcal{O}(Y_1)$ to get a $g : Y_1 \longrightarrow Y_2$. We get a $\tilde{g} : Y_1 \longrightarrow Y_2$ in the second set

$$\tilde{g}(x) = (f_1(x), ..., f_m(x))$$

then we have $\varphi \circ g \in \mathcal{O}(Y_1)$ for $\varphi \in \mathcal{O}(Y_2)$. One checks that this shows that the first and third sets are the same. $\square$

Define morphism $Y_1 \longrightarrow Y_2$ by the second(and third) set. Composition in the obvious way and identity is a morphism. $\Longrightarrow$ get a category (Alg_K) of affine algebraic sets over K.

Corollary 1.5. *$Y \mapsto \mathcal{O}(Y)$, $g \mapsto [\varphi \mapsto \varphi \circ g]$ is a functor: $(Alg_K) \longrightarrow (K\text{-}Alg)^{opp}$.*

Facts: The “image” of this functor is the category of finitely generated K-algebras which are reduced.

Proof. A finitely generated reduced K-algebra. ($\exists n \geq 1$, so that $K[X_1, ..., X_n]/I \cong A$). Then “$A$ is reduced”$\Longleftrightarrow$ I is radical ideal. $\Longrightarrow A = \mathcal{O}(V(I))$, where $V(I) \subset \mathbb{A}^n$. $\square$

Corollary 1.6. *There is a equivalence of categories between*

$$(\textit{Algebraic sets over } K) \longleftrightarrow (\textit{finitely generated reduced } K\textit{-Algebras.})$$

Example 1.7.

(1) $\mathbb{A}^1 \longrightarrow V(Y^2 - X^3 - X^2) \subset \mathbb{A}^2$, $t \mapsto (t^2-1, t(t^2-1))$

(2) $\mathbb{A}^1 \longrightarrow V(Y^2 - X^3) \subset \mathbb{A}^2$: $t \longmapsto (t^2, t^3)$ *is a bijection but* <u>*Not*</u> *an isomorphism.*

(3) Assume K *with characteristic* $p > 0$, $K \supset \mathbb{F}_p$. $Y = V(f_1, ..., f_m)$ *where* $f_i \in \mathbb{F}_p[X] \subset K[X]$. *Consider the morphism:*

$$\begin{aligned} &Y \longrightarrow Y \\ &(x_1, .., x_n) \longmapsto (x_1^p, ..., x_n^p). \end{aligned}$$

It is bijective and homeomorphism but not an isomorphism if $dim(Y) \geq 1$.

Proposition 1.8. $Y = V(I) \subset \mathbb{A}^n$

(1) The points of Y *are in bijection with maximal ideals* $I \subset \mathcal{O}(Y)$ *by*

$$Y \ni x \longmapsto \{f \in \mathcal{O}(Y) | f(x) = 0\}$$

(2) We have a bijection

$$\mathcal{O}(Y) \longleftrightarrow \; Hom_{Alg_K}(Y, \mathbb{A}^1)$$

Proof. (1) $I_x := Ker(\mathcal{O}(Y) \longrightarrow K)$, $f \mapsto f(x)$, since the evaluation map is surjective $[1 \mapsto 1]$, we get an isomorphism

$$\mathcal{O}(Y)/I_x \xrightarrow{\sim} K,$$

so I_x is maximal in $\mathcal{O}(Y)$.

Conversely, if $I \subset \mathcal{O}(Y)$ is maximal, we get $I = I'/I(Y)$ for $I' \subset K[X]$ maximal.

Nullstellensatz says $\exists (x_1, ..., x_n) \in \mathbb{A}^n$ s.t., $I' = (X_1 - x_1, ..., X_n - x_n)$.

Since $I' \supset I(Y)$, we get $(x_1, .., x_n) \in Y$. Then we check that $\mathcal{O}(Y) \longrightarrow \mathcal{O}(Y)/I \cong K$ is just given by $f \mapsto f(x_1, ..., x_n)$. That means $I = I_x$.

(2) We saw in 1.4, that there is a bijection between sets

$$\mathrm{Hom}_{Alg_k}(Y, \mathbb{A}^1) \longleftrightarrow \; \mathrm{Hom}_{K-alg}(\mathcal{O}(\mathbb{A}^1), \mathcal{O}(Y)).$$

But $\mathrm{Hom}_{K-alg}(\mathcal{O}(\mathbb{A}^1), \mathcal{O}(Y)) = \; \mathrm{Hom}_{K-alg}(K[X], \mathcal{O}(Y)) \cong \mathcal{O}(Y)$ (by $g : \mathcal{O}(\mathbb{A}^1) \longrightarrow \mathcal{O}(Y)$, $g \mapsto g(X)$) ☐

Projective Algebraic sets

Projective sets can have a good notion of "compactness".

N.B. Any $Y \in (Alg_K)$ is **quasi-compact**(open cover have a finite subcover).

Definition 1.9. $\mathbb{P}^n_K = \mathbb{P}^n$ *can be either defined as*

"the set of lines in $\mathbb{A}^{n+1}$ *that pass through the origin"*

or

"the equivalence classes of points in $K^{n+1}\backslash\{0\}$ *with the equivalence relation* $x \sim y$ *iff* $x = \lambda y$ *for some* $\lambda \in K$*" and we use the notion* $[x_0 : ... : x_n]$ *for the equivalence class of* $(x_0, .., x_n)$

These two definitions are equivalent:

Given a line $l \in \mathbb{A}^1 \longleftrightarrow$ hyperplane in K^{n+1}, corresponds to a equation

$$a_0X_0 + ... + a_nX_n = 0$$

with at least one of a_i non-zero.

Conversely, from $[x_0 : .. : x_n]$, we we get the corresponding hyperplane/line trivially.

Notes the following fact:

$$\mathbb{P}^n = \cup_{0\leq i\leq n} H_i,$$

where $H_i = \{[x_0, ..., x_n] | x_i \neq 0\}$ and there is a bijection

$$\begin{aligned}
&H_i \longrightarrow K^n \\
&[x_0 : ... : x_n] \longmapsto \left(\frac{x_0}{x_i}, ..., \frac{\widehat{x_i}}{x_i}, .., \frac{x_n}{x_i}\right) \\
&[y_1 : ... : y_{i-1} : 1 : y_i : ... : y_n] \hookleftarrow (y_1, ..., y_n)
\end{aligned}$$

We define from linear algebra some notions in $\mathbb{P}^n$ a line in $\mathbb{P}^n$ is the image by the projection $K^{n+1}\backslash\{0\} \longrightarrow \mathbb{P}^n$ of the two dimensional affine subspace.

Example 1.10. $l_1, l_2 \subset \mathbb{P}^2$ *lines* $l_1 \cap l_2$ *is a line if* l_1 *and* l_2 *are identical and would be a single point otherwise.*

Observation: If $f \in K[X_0, ..., X_{n+1}]$ is homogeneous, then for $x \in \mathbb{P}^n$, it makes no sense to speak of "$f(x) \in K$", but the zero-loci or the set where $f(x) \neq 0$ does make sense.

Definition 1.11. *A **projective algebraic set** $S \subset \mathbb{P}^n$ is*

$$S = \{x \in \mathbb{P}^n | f_1(x) = ... = f_m(x) = 0\},$$

where $f_1, ..., f_m$ are homogeneous of some degrees.

*An irreducible projective algebraic set is called a **projective variety***

Notation: $V(f_1, .., f_n)$

Example 1.12. $V(Y^2Z - X^3 - XZ^2) \subset \mathbb{P}^2$.

Let $0 \leq i \leq n$, then $S \cap H_i = \{[x_0 : ... : x_n] \in S | x_i \neq 0\}$ is , via the bijection $H_i \longrightarrow K^n$, in bijection with an affine algebraic set $S_1 \subset \mathbb{A}^n$ given by $\tilde{f}_1(y) = ... = \tilde{f}_m(y) = 0$, where $\tilde{f}_i(y_1, .., y_n) = f_i(y_1, ..., y_{i-1}, 1, y_i, ..., y_n)$

1.2 Mar 2nd: Projective algebraic sets and regular functions

Recall: $\mathbb{P}^n_K = K^{n-1} - \{0\}/\sim$, and $H_i := \{[x_0 : ... : x_n] | x_i \neq 0\}$ is in bijection with $\mathbb{A}^n$. $V(f_1, ..., f_m) = \{x \in \mathbb{P}^n | \forall i, f_i(x) = 0\}$, where $f_1, ..f_m$ are homogeneous.

More generally, we can define

$$V(I) = V(\text{homogeneous element of } I=) = V(\cup_{d\geq 0} I_d)$$

where I is an homogeneous ideal of $K[X_0, ..., X_n]$ that is $I = \oplus_{d\geq 0} I_d$, I_d the the degree d piece of $K[X_0, ..., X_n]$.

Conversely, given $S \subset \mathbb{P}^n$, we can define

$$I(S) := \text{ideal generated by homogeneous polynomials that vanishes on } S$$

Lemma 1.13. *This is a homogeneous ideal*

Proof. $f \in I(s) \implies f = \sum_{i\in I} g_i f_i$, where f_i is homogeneous and vanishes on S. We can expand each g_i as $\sum_j g_{ij}$, where each g_{ij} is homogeneous in $I(S)$. Then we know $f \in \otimes I(S)_d$ and the converse is clear. $\square$

Lemma 1.14. *The projective sets $V(I)$ where I is homogeneous form the closed sets of a topology. It is called the Zariski topology (same name for the induced topology on projective sets).*

Example 1.15. *$H_0 \subset \mathbb{P}^n$ and $\sigma : H_0 \cong \mathbb{A}^n$. Under this bijection, the Zariski topologies correspond σ is a homeomorphism*

$$f \in K[X_0, ..., X_n] \text{ homogeneous} \rightsquigarrow V(f) \subset \mathbb{P}^n$$

$$\tilde{f} = f(1, X_1, ..., X_n) \in K[X_1, .., X_n] \rightsquigarrow V(\tilde{f}) \subset \mathbb{A}^n$$

and $\sigma(V(f)) = V(\tilde{f})$.

Definition 1.16. *$Y \subset \mathbb{P}^n$ projective $S(Y) = K[X_0, ..., X_n]/I(Y)$, **homogeneous coordinate ring***

Note elements in $S(Y)$ are not functions on Y. The geometric meaning of $S(Y)$ will be explained latter with the language of schemes.

We now want to define morphisms of projective algebraic sets. We have to look at it more carefully because we can not simply copy the affine definition.

Definition 1.17. *$Y \subset \mathbb{P}^n$ projective, let $V \subset Y$ be an open subsets of Y.*

*(1) $f : V \longrightarrow K$ continuous is called **regular** on Y if $\forall x \in Y$, $\exists U$ open $x \in U$, $\exists f_1, f_2 \in K[X_0, ..., X_n]$ homogeneous of same degree such that $f_2(x) \neq 0$ for all $x \in U$ and $f(x) = \frac{f_1(x)}{f_2(x)}$ for $x \in U \cap Y$*

*(2) Y_1, Y_2 are projective sets in $\mathbb{P}^n, \mathbb{P}^m$, $f : Y_1 \longrightarrow Y_2$ is a **morphism** if f is continuous and for any $U \subset Y_1$ open and any $\varphi : U \longrightarrow K$ regular, the composite $\varphi \circ f : f^{-1}(U) \longrightarrow K$ is regular.*

Note: IN (2), one can not restrict to φ regular on Y_2 because often the space of such function is reduced to K

Proposition 1.18. *For $\mathbb{P}^n$, the space of functions regular on $\mathbb{P}^n$ is K.*

Proof. The case $n = 1$ implies the general case: if $f : \mathbb{P}^n \longrightarrow K$ regular, and $x \neq y$ in $\mathbb{P}^n$, the line joining x to y in $\mathbb{P}^n$ is "isomorphic" to $\mathbb{P}^1$ and $f|_L$ is regular so constant, hence $f(x) = f(y)$.

For $n = 1$, suppose x, y are arbitrary points and let $U \ni x$, $V \ni y$ be open neighbourhoods such that $f|_U = f_1(x)/f_2(x)$ and $f|_V = g_1(x)/g_2(x)$ where f_1, f_2, g_1, g_2 are homogeneous polynomials and f_1, f_2 have the same degree as well as g_1, g_2. We may assume that f_1 and f_2 are coprime and also g_1, g_2 are coprime. Hence on $U \cap V$,

$$f_1 g_2 = g_1 f_2.$$

We know that $U \cap V$ is infinite so this implies $f_1 = g_1$ and $f_2 = g_2$. Since x and y were arbitrary points we conclude that $f = f_1(x)/f_2(x)$ on all of $\mathbb{P}^1$ hence f is a constant. □

Concretely: To say that $f : Y_1 \subset \mathbb{P}^n \longrightarrow Y_2 \subset \mathbb{P}^m$ is a morphism of projective algebraic sets. It reduces to $\forall x \in Y_1, \exists U$ open containing x s.t. there exists $f_0, ..., f_m \in K[X_0, ..., X_{n+1}]$ homogeneous of same degree, with no common zero in U, such that $\forall y \in U \cap Y_1$, $f(y) = [f_0(y) : ... : f_m(y)]$. It is easy to see that if f is of this form, then it is a morphism.

The converse is left as an exercise.

Example 1.19. *(1) Let $g \in Gl_n(K), n \geq 1$. Define*

$$f_g : \mathbb{P}^n \longrightarrow \mathbb{P}^n$$

$$[x_0 : ... : x_n] \longmapsto [g(x_0, ..., x_n)]$$

is a morphism. In fact, it is an isomorphism. $f_g^{-1} = f_{g^{-1}}$. It also has some other properties: $f_g = f_{\lambda g}, \lambda \neq 0$ and we get an induced group morphism

$$\begin{array}{ccc} PGL_{n+1}(K) & = & GL_{n+1}(K)/K^{\times} \\ \downarrow & & \\ Aut_{proj}(\mathbb{P}^n) & & \end{array}$$

which is an isomorphism. A special case is $Aut_{hol}(\mathbb{CP}^1) = PGL_2(\mathbb{C})$

$$g \longmapsto \left[z \mapsto \frac{az+b}{cz+d} \right]$$

(2) $K = \mathbb{C}$. One can do holomorphic geometry (using holomorphic functions instead of polynomials). IN $\mathbb{C}^n$,we get a much more complicated picture [e.g. $V(\sin z)$] is a an infinite sets in $\mathbb{P}^n_{\mathbb{C}}$, however Chow proved that the holomorphic sets and the projective algebraic sets are the same (Serre "GAGA" principle compares many different invariant of both categories.)

(3) Consider the map $S := V(Y^2Z - X^3 - XZ^2) \xrightarrow{f} \mathbb{P}^1$, $[x : y : z] \mapsto [y : z]$.
<u>Claim</u>, this is a morphism of projective sets.
This means that there is no solution to $Y^2Z - X^3 - XZ^2 = 0$ with $Y = Z = 0$. (But $[x : y : z] \mapsto [x : z]$ is not a morphism because $[0 : 1 : 0] \in S$). f is surjective but not injective $[x : y : z]$ and $[x : -y : z]$ have same image. This works in field k with char$k \neq 2$.

(4) $\mathbb{P}^1 \xrightarrow{v} \mathbb{P}^2$, $[x : y] \mapsto [x^2 : xy : y^2]$ (special case of Veronese embedding). This is a morphism. The image of v is equal to $[y_0 : y_1 : y_2], \mathbb{P}^2$. $S = V(Y_1^2 - Y_0Y_2)$. In fact, σ gives an isomorphism $\sigma : \mathbb{P}^1 \longrightarrow S$ with inverse given by

$$\tau : S \longrightarrow \mathbb{P}^1$$

$$[y_0 : y_1 : y_2] \mapsto \begin{cases} [Y_1 : Y_2] & \text{if } Y_2 \neq 0 \\ [Y_0 : Y_1] & \text{if } Y_0 \neq 0 \end{cases}$$

τ is a morphism defined on all of S, because if $[y_0 : y_1 : y_2] \in S$ satisfies $y_0 = y_2 = 0$, it would implie $y_1^2 = y_0 y_2 = 0 \Longrightarrow y_1 = 0$

$$\tau \circ \sigma([x : y]) = \tau([x^2 : xy : y^2]) \quad = \begin{cases} [xy : y^2] = [x : y], y \neq 0 \\ [x^2 : xy] = [x : y], x \neq 0 \end{cases}$$

therefore $\tau \circ \sigma = id_{\mathbb{P}^1}$ and $\sigma \circ \tau = id_S$ can proved similarly

One can not find $f_0 : f_1$ in $K[Y_0, Y_1, Y_2]$ s.t. $\tau([y_0 : y_1 : y_2] = [f_0(Y) : f_1(Y)]$ for all $Y \in S$

1.3 Mar 5th: Exercise class

The content covered can be found in Hartshorne, p50ff Proposition 7.4 and Theorem 7.5.

1.4 Mar 6th: Rational/birational maps

$Y \subset \mathbb{A}^n$ algebraic if Y is irreducible, then $\mathcal{O}(Y)$ is an integral domain. Let $K(Y)$ be its quotient field. What is the geometric meaning of $K(Y)$? It is called the **function field** of Y.

We will see

Theorem 1.20. *For Y_1, Y_2 affine varieties (irreducible) $K(Y_1) \cong K(Y_2)$ as fields $\Longleftrightarrow \exists U_1 \subset Y_1$ open dense subset and $\exists U_2 \subset Y_2$ open dense subset such that U_1 and U_2 are isomorphic.*

Definition 1.21. *(Quasi-affine and quasi-projective) varieties)*

1. ***quasi-affine variety*** *V is an open subset $V \subset Y$, where $Y \subset \mathbb{A}^n$ is an affine variety. [$V \neq \emptyset \Longrightarrow V$ dense in $Y \Longrightarrow V$ irreducible]. It is given by the Zariski's topology from Y.*
   (1') $V \subset Y \subset \mathbb{P}^n$ where V is an open subset of Y is ***quasi-projective****, where Y is projective variety.*

2. *A regular function $f : V \longrightarrow K = \mathbb{A}^1$, where V is quasi-affine is an f such that for all $x \in V$, $\exists U \subset V$ open containing x s.t., $\forall x \in V$, $f(x) = \frac{f_1(x)}{f_2(x)}$ where $f_1, f_2 \in \mathcal{O}(\mathbb{A}^n)$ and $f_2(x) \neq 0$ on U.*
   (2') V is quasi-projective variety a regular function f is $\frac{f_1(x)}{f_2(x)}$ f_i homogeneous of same degrees.

3. *If V_1, V_2 are <u>Varieties</u> (of any of the four types), then $f : V_1 \longrightarrow V_2$ is a* ***morphism*** *if for al open $U \subset V_2$ all $\varphi : U \longrightarrow K$ regular, the composition $\varphi \circ f : f^{-1}U \longrightarrow K$ is also regular.*

N.B.

1. This makes sense because if $U \subset V_2$, where V_2 is quasi affine U open, $\Longrightarrow$ $U \subset V_2 \subset Y$ so U is also quasi-affine in $\mathbb{A}^n$
2. <u>Exercise</u> If f is regular on V, then f is continuous $V \longrightarrow \mathbb{A}^1$. (check that $f^{-1}(\{a\})$ is closed, use that closedness is a local condition.)
3. In the (quasi)-affine case, it is enough to check that $\varphi \circ f$ is regular on V_1 for φ regular on V_2.
4. <u>Notation</u>:
$$\mathcal{O}(V) = \{f : V \longrightarrow K | \text{ regular}\}$$
This is a ring of with unity, and because of the condition that for open $V \subset Y$ in a variety Y, either $\mathcal{O}(V) = 0, V \neq \emptyset$ or V is dense in Y, $\Longrightarrow \mathcal{O}(V)$ integral domain.

Example 1.22.

1. *$GL_n(K) = \{x \in M_{n\times n}(K) | \det(x) \neq 0\} \subset \mathbb{A}^{n^2}$ is quasi-affine since* $\det$ *: $M_{n\times n}(K) \longrightarrow K$ is continuous and not emptyset.*
2. *In fact, for any $0 \neq f \in \mathcal{O}(\mathbb{A}^n)$*
$$U_f = \{x \in \mathbb{A}^n | f(x) \neq 0\}$$
is a quasi-affine variety.

<u>Fact</u>: There is an isomorphism
$$\sigma = \begin{cases} U_f \longrightarrow Y = \{(x,y) \in \mathbb{A}^{n+1} | yf(x) = 1\} \\ \quad x \longmapsto \left(x, \dfrac{1}{f(x)}\right) \end{cases}$$
with inverse $(x,y) \stackrel{\pi}{\longmapsto} x$. (Indeed, $\pi \circ \sigma = Id_{U_f}$, $\sigma \circ \pi = Id_Y$) and π is a morphism: Consider $\varphi \in \mathcal{O}(U_f)$
$$Y \xrightarrow{u} U_f \xrightarrow{\varphi} K$$

then $\varphi \circ \pi(x,y) = \varphi(x)$.

Indeed, for any $x \in U_f$, $\exists f_1, f_2 \in \mathcal{O}(\mathbb{A}^2)\varphi(x) = \frac{f_1(x)}{f_2(x)}, f_2(x) \neq 0$, *one can show: assume* $f_2(x) = f(x)^d$ *then*

$$\varphi(x) = \frac{f_1(x)}{f(x)^d} = f_1(x)y^d$$

for $(x,y) \in Y$, *so this is regular.*

(2) σ *is a morphism*

$$U_f \xrightarrow{\sigma} Y \xrightarrow{\varphi} K$$

$\varphi \in \mathcal{O}(Y) = K[X_1, ..., X_n, Y]/(Yf(x) = 1)$

$$\begin{aligned} \varphi \circ \sigma(x) = \varphi(x, 1/f(x)) &= \left(\sum_j a_j Y^j\right)|_{Y=1/f(X)} \\ &= \sum_j a_j(x)/f(x)^j \in \mathcal{O}(U_f) \end{aligned}$$

3. $\mathbb{P}^n = \cup_{0 \leq i \leq n} H_i$, *with* $H_i = \{[x_0 : ... : x_n] | x_i \neq 0\}$, $H_i \subset \mathbb{P}^n\}$ *open , so quasi-projective. The map*

$$\begin{cases} H_i \xrightarrow{f_i} \mathbb{A}^n \\ [x_0 : ... : x_n] \longmapsto (\frac{x_0}{x_1}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}x_i}{,}..., \frac{x_n}{x_i}) \end{cases}$$

is an isomorphism.

Definition 1.23. *Y variety,* $K(Y) = \{(U,f) | \emptyset \neq U \subset Y$ *open* $, f \in \mathcal{O}(U)\}/\sim$, *where* $(U_1, f_1) \sim (U_2, f_2)$ *iff* $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$

Fact: $\sim$ is an equivalence relation. We define

$$(U_1, f_1) + (U_2, f_2) = (U_1 \cap U_2, f_1 + f_2)$$

$$0 := (Y, 0), \quad 1 := (Y, 1)$$

Proposition 1.24. *Y is quasi-affine, $U \subset Y$ open nonempty.*

1. $\mathcal{O}(Y) \hookrightarrow \mathcal{O}(U) \hookrightarrow K(Y)$
   $f \longmapsto f|_U \mapsto (U, f)$

2. *$K(Y)$ is a field, and identifies with the fraction field of $\mathcal{O}(Y)$ and of $\mathcal{O}(U)$.*

3. *if Y is an affine variety, then $\mathcal{O}(Y)$ as defined above coincides with $\mathcal{O}(Y) = K[X_1, ..., X_n]/I(Y)$ as defined in previous sections.*

   (3') If $Y = U_f$ for $0 \neq f$ in $\mathcal{O}(\mathbb{A}^n)$, then $\mathcal{O}(Y) = \{f_1/f^d | f_1 \in \mathcal{O}(\mathbb{A}^n, d \geq 0\} = \mathcal{O}(\mathbb{A}^n)_f$ the localization at f.

Proof. (1), The morphism $\mathcal{O}(Y) \longrightarrow \mathcal{O}(U) \longrightarrow K(Y)$ are injective because any $U \subset Y, \neq \emptyset$ is dense.

(2) Let $(U, f) \neq 0$ in $K(Y)$, then $\exists x_0 \in U$, $f(x_0) \neq 0$ in a $V \subset U$, $x_0 \in V$

$$f(x) = \frac{f_1(x)}{f_2(x)}, f_1, f_2 \in \mathcal{O}(\mathbb{A}^n), f_2 \neq 0 \text{ in } V$$

in particular, $f_1(x_0) \neq 0$ and $(U \cap \{f_1(x) \neq 0\}, \frac{f_2(x)}{f_1(x)}) \in K(Y)$, where $U \cap \{f_1(x) \neq 0\} \neq \emptyset$ is the inverse of (U, f) in $K(Y)$.

By (1), $K(Y) \supset \mathcal{O}(Y)$.

Let $(U, f) \in K(Y)$, pick $x \in Y$ so that around x, $f(x) = \frac{f_1}{f_2}$, $f_i \in \mathcal{O}(\mathbb{A}^n)$, then $(U, f) = \frac{(Y, f_1)}{(Y, f_2)}$, so $K(Y)$ is the fraction field of $\mathcal{O}(Y)$.

(3) Write $\mathcal{O}'(Y) = K[X]/I(Y)$. Note $K[X.., Y]/I(Y)$ identifies to a ring of functions on Y, the claim is that this ring is $\mathcal{O}(Y)$.

Observation: For $x \in Y$, to say that $f : Y \longrightarrow K$ is "regular at x" means precisely that $f \in \mathcal{O}'(Y)_{I_x}$, where $I_x = \{f \in \mathcal{O}'(Y) | f(x) = 0\}$. (Localization at a maximal ideal)

So

$$\begin{aligned}\mathcal{O}(Y) &= \bigcap_{x \in Y} \mathcal{O}'(Y)_{I_x} \\ &= \bigcap_{\mathfrak{m} \subset \mathcal{O}'(Y)} \mathcal{O}'(Y)_{\mathfrak{m}} \\ &= \mathcal{O}'(Y)\end{aligned}$$

the second equality from Nullstellensatz and the third from commutative algebra.

(3') Similarly, using characterization of maximal ideals in $A_f, f \neq 0$ □

Definition 1.25. *K(Y) is called the fraction or function field of Y*

Example 1.26. $K(\mathbb{A}^n) = K(\mathbb{P}^n) = K(X_1, .., X_n)$

Definition 1.27. *(Rational maps) Y_1, Y_2 varieties. A* ***rational map*** *$f : Y_1 \dashrightarrow Y_2$ is a pair $(U, \tilde{f})$ where $U \neq \emptyset$ in Y_1 and $\tilde{f} : U \longrightarrow Y_2$ is a morphism with $(U, \tilde{f}) = (U', \tilde{f}')$ iff*

$$\tilde{f}|_{U \cap U'} = \tilde{f}'|_{U \cap U'}$$

[Check: this is coherent, i.e., this is an equivalence relation]

Definition 1.28. *$f : Y_1 \dashrightarrow Y_2$ is a **dominant** if its image $\tilde{f}(U) \subset Y_2$ is dense.*

Example 1.29. *(1) there is a bijection $\{Y \dashrightarrow \mathbb{A}^1\} = K(Y)$*

$$\begin{array}{cc} (U, \tilde{f}) & (U, f) \\ \tilde{f} : U \longrightarrow \mathbb{A}^1 \textit{ morphism} & f : U \longrightarrow K \textit{ regular} \end{array}$$

So it is enough to check

$$Hom_{Var}(U, \mathbb{A}^1) = \mathcal{O}(U)$$

Left as exercise

(2) $Y, f_1, f_2, f_3 \in \mathcal{O}(Y)$

$$\begin{cases} Y \dashrightarrow \mathbb{P}^2 \\ x \longmapsto [f_1(x) : f_2(x) : f_2(x)] \end{cases}$$

defined on $\{x | f_i(x)$ are not all zero$\}$, which is open if any of the 3 sections is non-zero.

Theorem 1.30. *Y_1, Y_2 varieties*

$\exists\{Y_1 \overset{f}{\dashrightarrow} Y_2 | f$ dominant$\}$

$\overset{bij}{\longleftrightarrow}$

$K(Y_2) \longrightarrow K(Y_1)$

Corollary 1.31. *Y_1, Y_2 varieties. Y_1 and Y_2 are birational*

iff $K(Y_1)$ is isomorphic to $K(Y_2)$

iff $\exists U \subset Y_1$ open $\neq \emptyset$ $\exists V \subset Y_2$, open $\neq \emptyset$ so that U and V are isomorphic as varieties.

Corollary 1.32. *Any variety Y of dimension $d \geq 0$ is birational to a hypersurface $V \subset \mathbb{P}^{d+1}$*

Proof. (1) Given $Y_1 \overset{f}{\dashrightarrow} Y_2$ dominant, we want a morphism $K(Y_2) \longrightarrow K(Y_1)$.

Let $(U, \tilde{f}) = f$, $(V, \varphi), \varphi : V \longrightarrow K$ in $K(Y_2)$

$$\varphi \circ f : \tilde{f}^{-1}(V) \longrightarrow K$$

is in $K(Y_1)$, provided $\tilde{f}^{-1}(V)$ is dense, it is enough that $\tilde{f}^{-1}(V) \neq \emptyset$, $\tilde{f}(U) \cap V \neq \emptyset$, since V is open and $\tilde{f}(U)$ is dense.

(2) Given $i : K(Y_2) \longrightarrow K(Y_1)$. Let $\tilde{Y_2} \subset Y_2 \subset \mathbb{A}^n$ open quasi-affine so that $K(Y_2) = K(\tilde{Y_2}) = Frac(\mathcal{O}(\tilde{Y_2}))$

Let $X_1, .., X_n$ be the coordinates in $\mathbb{A}^n$ as elements of $\mathcal{O}(\tilde{Y_2})$, then let

$$f_j = i(X_j) \in K(Y_1)$$

$f_j \longleftrightarrow (U_j, \tilde{f}_j)$ with $U_j \subset Y_1$ dense and $\tilde{f}_j \in \mathcal{O}(U_j)$. Then $f_j \longleftrightarrow (U, \tilde{f}_j)$, $U := U_1 \cap ... \cap U_n$ still dense.

Define $U \longrightarrow \tilde{Y_2} \hookrightarrow Y_2$ by

$$x \longmapsto (\tilde{f}_1(x), ..., \tilde{f}_n(x)).$$

This is a rational map $Y_1 \dashrightarrow Y_2$

□

1.5 Mar 9th: Continue and Nonsingular varieties

Recall

Theorem 1.33. *Y_1, Y_2 varieties*

$$\{dominant\ Y_1 \dashrightarrow Y_2\} \longleftrightarrow \{K(Y_2 \hookrightarrow K(Y_1)\}$$

Corollary 1.34. *The followings are equivalent:*

- *Y_1 and Y_2 are birational*
- *the function field $K(Y_1)$ and $K(Y_2)$ are isomorphic*
- *$\exists \emptyset \neq U \subset Y_1$, $\emptyset \neq V \subset Y_2$ and isomorphism between U and V*

Proof. The last condition implies the second because $K(Y_1) = Frac(\mathcal{O}(U)) \cong Frac(\mathcal{O}(V)) = K(Y_2)$. Assume we have rational maps

$$Y_1 \overset{f_2}{\dashrightarrow} Y_2 \overset{f_2}{\dashrightarrow} Y_1$$

with $f_2 \circ f_1 = id_{Y_1}, f_2 \circ f_1 = id_{Y_2}$.

Let $f_1 = (U', \tilde{f}_1), f_2 = ()V', \tilde{f}_2$

$$\begin{tikzcd}
Y_1 \arrow[r, dashed, "f_1"] & Y_2 \arrow[r, dashed, "f_2"] & Y_1 \\
U' \arrow[u, hook] \arrow[ur, "\tilde{f}_1"'] & V' \arrow[u, hook] \arrow[ur, "\tilde{f}_2"'] &
\end{tikzcd}$$

$$f_2 \circ f_1 = (Y_1, Id_{Y_1})$$

so $\tilde{f}_2(\tilde{f}_1(x)) = x$ if $\tilde{f}_1(x) \in V'$. Similarly, $f_1 \circ f_1 = (\tilde{f}_2^{-1}(U'), \tilde{f}_1 \circ \tilde{f}_2)$. Define $U = \tilde{f}_1^{-1}(\tilde{f}_2^{-1}(U')) \subset U'$, which is a dense open subset. Also we have $V = \tilde{f}_2^{-1}(\tilde{f}_1^{-1}(V'))$.

Claim: $U \xrightarrow{\tilde{f}_1} V \xrightarrow{\tilde{f}_2} U$ and then $\tilde{f}_1|_U, \tilde{f}_2|_U$ are reciprocal isomorphism.

We check that if $x \in U$, then $\tilde{f}_1(x) \in V$. Let $y = \tilde{f}_1(x) \in V'$ so $\tilde{f}_2(y) = \tilde{f}_2(\tilde{f}_1(x)) = x$ so $\tilde{f}_1(\tilde{f}_2(y)) = \tilde{f}_1(x) \in V' \Longrightarrow y \in V$. Similarly for f_2. □

Definition 1.35. *A **rational variety** Y is a variety Y birational to $\mathbb{P}^n$ for some n (or to $\mathbb{A}^n$). BY the theorem above we know $\exists n, K(Y) \cong K(X_1, ..., X_n)$.*

*A **unirational variety** Y is a variety s.t. there is a dominant $\mathbb{P}^n \dashrightarrow Y$ for some n, by theorem above $\exists n, K(Y) \hookrightarrow K(X_1, ..., X_n)$ We obviously have*

$$\textit{Unirational} \Longleftarrow \textit{rational}$$

but

$$\textit{Unirational} \overset{?}{\Longrightarrow} \textit{rational}$$

For char $= 0$: $\dim Y = 1$ or 2, Luroth and some italian showed that unirational curves or surfaces are rational.

First example in char 0 of non-rational unirational varieties were provided by Clemens-Griffith: certain cubic hypersurfaces in $\dim 3$.

Iskovskih-Manin "general " quantic hypersurfaces of $\dim 3$.

Corollary 1.36. *Any variety Y is birational to a hypersurface in $\mathbb{P}^{\dim(Y)+1}$ or $\mathbb{A}^{\dim(Y)+1}$.*

Proof. Let $d = \dim(Y) = \dim(\mathcal{O}(Y))$. Then a fact in commutative algebra says $K(Y)$ is a finite separable extension of $K(X_1, ..., X_d) =: E$. By the primitive element theorem, there exists $\alpha :\in K(Y)$ such that $K(Y) = E(\alpha)$. Let $f \in E[T]$ be the minimal polynomial of α.

Write
$$f = \sum_{i=0}^{n} a_i T^i = \sum_{i=0}^{n} \frac{b_i}{c_i} T^i,$$
where $a_i \in E$ and $a_i, b_i \in A = K[X_1, ...X_d]$

$\Longrightarrow \tilde{f}(\alpha) = 0$ where $\tilde{f} = (\prod c_i) f \in A[T] = K[X_1, ..., X_d, T]$. Define $\tilde{Y} = V(\tilde{f}) \subset \mathbb{A}^{d+1}$. This is what we wanted.

(1) $\tilde{Y}$ is an irreducible hypersurface.

(2) $\tilde{Y}$ is birational to $Y \iff K(\tilde{Y}) = K(Y)$

Step 1: Need $\tilde{f}_1 \in K[X_1, ..., X_d, T]$ irreducible. Suppose $\tilde{f} = \tilde{f}_1\tilde{f}_2, \tilde{f}_i \in A[T] \implies E \ni f = \frac{\tilde{f}_1}{\prod c_i}\tilde{f}_2$ factors in $E[T]$, since f is irreducible in $E[T]$, one of $\deg(\tilde{f}_1)$ or $\deg(\tilde{f}_2)$ is zero

$\implies \tilde{f}$ is irreducible.

Step (2): $\mathcal{O}(\tilde{Y}) = K[X, T]/(\tilde{f})$. We have an injective morphism

$$\begin{cases} \mathcal{O}(\tilde{Y}) \longrightarrow K(Y) = E(\alpha) \\ \quad X_i \longmapsto X_i \\ \quad T \longmapsto \alpha \end{cases}$$

so the fraction field $K(\tilde{Y})$ injects into $K(Y)$. The image of $K(\tilde{Y})$ contains $X_1, ..., X_d$ and α hence it contains $E(\alpha)$, i.e., $K(\tilde{Y}) = K(Y)$ □

Nonsingular varieties

Concrete geometric definition:

Definition 1.37. *$Y \subset \mathbb{A}^n$ affine variety $\dim Y = d$, $x \in Y$. We say Y is* ***nonsingular*** *at x if, for any generating set $\underline{f} := (f_1, ..., f_m)$ of $I(Y)$, the Jacobian matrix at x*

$$J_{\underline{f}}(x) = \left(\frac{\partial f_i(x)}{\partial x_j}\right)_{1\le i\le m, 1\le j\le n} \in M_{m\times n}(K)$$

has rank $n - d$ if this holds for all x, then we say Y is ***nonsingular****.*

Key fact: It suffices to check the rank of $J_F(x)$ for some generating set.

Indeed suppose $\underline{h} = (h_1, .., h_k)$ also generate $I(Y)$ so

$$f_i = \sum_{\ell=1}^{k} g_{i\ell} h_\ell,$$

where $g_{i\ell} \in \mathcal{O}(\mathbb{A}^n)$, $\frac{\partial f_i}{\partial x_j} = \sum_{\ell=1}^{k} \frac{\partial g_{i\ell}}{\partial x_j} h_\ell + \sum_{\ell=1}^{k} g_{i\ell} \frac{h_\ell}{x_j}$

At x where $h_\ell(x) = 0$, we get

$$\frac{\partial f_i}{x_j}(x) = \sum_{\ell=1}^{k} \frac{\partial g_{i\ell}}{\partial x_j} h_\ell$$

$$\implies J_{\underline{f}}(x) = M J_{\underline{h}}(x)$$

so rank $J_{\underline{f}}(x) \le$ rank $J_{\underline{h}}(x)$. Exchanging $\underline{f}, \underline{h}$, we get the equality.

Example 1.38.

(1) If $K = \mathbb{C}$, the implicit function theorem says that around a point where $J_{\underline{f}}(x)$ has rank $n-d$, then $V(f_1, ..., f_m)$ is diffeomorphic to $\mathbb{C}^d$

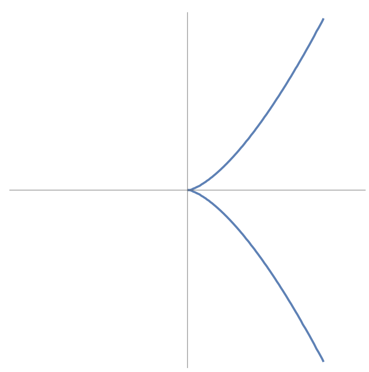
(2) Let $Y = V(f)$, f irreducible in $\mathbb{A}^n$. Then $x \in V(f)$ is nonsingular $\iff$ $(\partial f(x)/\partial x_1, ..., \partial f(x)/\partial x_n) \neq 0$

We have a singular point $\iff$ the system of $n+1$ equations

$$\begin{cases} f(x) = 0 \\ \dfrac{\partial f}{\partial x_1}(x) = 0 \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(x) = 0 \end{cases}$$

has a solution. For instance

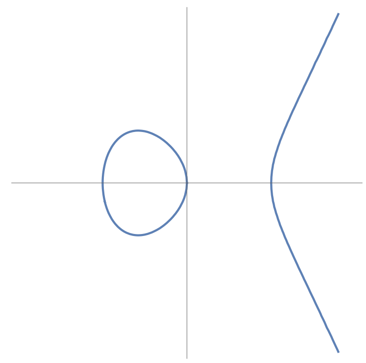
$$Y^2 = X^3$$



$$\begin{cases} f = Y^2 - X^3 \\ \dfrac{\partial f}{\partial X} = -3X^2 \\ \dfrac{\partial f}{\partial Y} = 2Y \end{cases}$$

so $X = Y = 0$ is the only singular point.

$$Y^2 = X^3 - X$$



$$\begin{cases} f = Y^2 - X^3 + X \\ \dfrac{\partial f}{\partial X} = -3X^2 + 1 = 0 \\ \dfrac{\partial f}{\partial Y} = 2Y = 0 \end{cases}$$

If char $k \neq 2$, $\Longrightarrow Y = 0$, $X^3 - X = 0$ $X = 0, -1, 1$ do not satisfy the system of solutions. In the case char $= 2$, $(1, 0) \in Y$ is singular.

The intrinsic characterization was found by Zariski.

Definition 1.39. *$x \in Y$ variety*

*(1) The **local ring** of Y at x*

$$\begin{aligned} \mathcal{O}_{Y,x} &= \{f \in K(Y) | f \text{ defined at } x\} \\ &= \{ \text{ regular functions on some } U \ni x\}/(f_1 \sim f_2 \text{ if they coincide on } U_{f_1} \cap U_{f_2}) \end{aligned}$$

if Y is affine, then $\mathcal{O}_{Y,x} = \{f_1/f_2 \in K(Y) \mid f_i \in \mathcal{O}(Y), f_2(x) \neq 0\} = \mathcal{O}(Y)_{\mathfrak{m}_x}$, where $\mathfrak{m}_x = \{f \in \mathcal{O}(Y) | f(x) = 0\}$ is the maximal ideal corresponding to x.

$$\mathcal{O}(Y) \subset \mathcal{O}_{Y,x} \subset K(Y)$$

Definition 1.40. *$Y \subset \mathbb{A}^n$ affine $x \in Y$. The (Zariski) cotangent spaces of Y at x is the K-vector space*

$$\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2,$$

where $\mathfrak{m}_{Y,x} \subset \mathcal{O}_{Y,x}$ is the maximal ideal

Remark 1.41. *$\mathcal{O}_{Y,x}$ is a local ring, it has a unique maximal ideal $\mathfrak{m}$ which is $\mathcal{O}_x\mathcal{O}_{Y,x}$ in the affine case. Moreover $\mathcal{O}_{Y,x}/\mathfrak{m} = K$ by $f \mapsto f(x)$.*

N.B. Intuitively, the Taylor expansion of $f \in \mathcal{O}_{Y,x}$ about $x \in \mathfrak{m}_{Y,x}$ is

$$f(X) = f(x) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x)(X - x_j) + \dots$$

if $f \in \mathfrak{m}_{Y,x}$ then $f(x) = 0$ and terms of order ≥ 2 belongs to $\mathfrak{m}_{Y,x}^2$, so f has image

$$\sum \frac{\partial f}{\partial x_j} dX_j \in \mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2$$

where $dX_j = X - x_j$.

Theorem 1.42. *(Zariski) $Y \subset \mathbb{A}^n, x \in Y$, the followings are equivalent*

(1) x is non-singular

(2) $\dim(Y) = \dim_K \mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2$

Remark 1.43. *One can show $\dim_K \mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2 \geq \dim(Y)$ so the question is whether it is larger or not.*

Definition 1.44. *A local ring $\mathcal{O}$ with maximal ideal $\mathfrak{m}$ is callled **regular** if*

$$\dim \mathcal{O} = \dim_k \mathfrak{m}/\mathfrak{m}^2$$

where $k = \mathcal{O}/\mathfrak{m}$ is the residue field.

1.6 Mar 13th-A: Continue and proofs

Recall: $x \in Y \subset \mathbb{A}^n$, $d = \dim(Y)$affine variety, and Y is non-singular at x iff

$$rank\ J_{\underline{f}}(x) = n - d.$$

Theorem 1.45. *(Zariski) The followings are equivalent*

(1) Y is non-singular at x

(2) $\dim(Y) = \dim_K(\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2)$, where $\mathfrak{m}_{Y,x}$ is the maximal ideal in the local ring $\mathcal{O}_{Y,x} := \mathcal{O}(Y)_{\tilde{\mathfrak{m}}_{Y,x}}$ with $\tilde{\mathfrak{m}}_{Y,x} = \{f \in \mathcal{O}(Y) \mid f(x) = 0\}$.

Proof. $I = I(Y), x = (x_1, ..., x_n) \in \mathbb{A}^n$.

Let $I_x := (X_1 - x_1, ..., X_n - x_n) \subset \mathcal{O}(\mathbb{A}^n)$ so that $\tilde{\mathfrak{m}}_{Y,x} = I_x/I$.

Fact: there is an isomorphism of K-vector spaces

$$\theta : \begin{cases} I_x/I_x^2 \longrightarrow K^n \\ f \longmapsto \left(\dfrac{\partial f(x)}{\partial x_j}\right)_{1\leq j\leq n} \end{cases}$$

$f \in I_x^2 \Longrightarrow f = \sum_{i,j} h_{ij}(X_i - x_i)(X_j - x_j)$, and in general

$$f = \sum_i^n (X_i - x_i)\frac{\partial f}{\partial x_i}(x) + I_x^2.$$

Let $(f_1, ..., f_m)$ be a generating set of I. Then $(\theta(f_1), ..., \theta(f_m))$ are the columns of $J_{\underline{f}}(x)$ and if $f \in I$,

$$f = \sum_j g_j f_j$$

$$\Longrightarrow \frac{\partial f}{\partial x_i}(x) = \sum_{j=1}^n g_j(x)\frac{\partial f_j}{x_i}(x)$$

$\Longrightarrow$ the span of $\theta(f_j)$ is $\theta(I_x/I_x^2)$, so

$$rank\ J_{\underline{f}}(x) = \dim_K \theta(I/I_x^2) = \dim_K(I/I_x^2)$$

Consider

$$0 \longrightarrow I_x^2 + I/I_x^2 \longrightarrow I_x/I_x^2 \longrightarrow I_x/(I + I_x^2) \longrightarrow 0$$

Then $= rank\ J_{\underline{f}}(x) + \dim_K(I_x/I + I_x^2)$. so
x is non-singular iff $d = \dim_K(I_x/I + I_x^2)$. $[]I_x$ depends on $Y \hookrightarrow \mathbb{A}^n]$
Consider

$$\begin{tikzcd} I_x \arrow[r] \arrow[rr, bend right, "\varphi"'] & \tilde{\mathfrak{m}}_{Y,x} \subset \mathfrak{m}_{Y,x} \arrow[r] & \mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2 \end{tikzcd}$$

Note $\varphi(I + I_x^2) = 0$ so we get a K-linear map

$$I_x/(I + I_x^2) \longrightarrow \mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2$$

Claim: This is an isomorphism [$\Longrightarrow$ the theorem]. (a) φ is surjective: $h \in \mathfrak{m}_{Y,x} \subset \mathcal{O}_{Y,x} \subset K(Y)$, $\Longrightarrow h = \frac{h_1}{h_2}$, with $h_1, h_2 \in \mathcal{O}(Y)$ and $h_2(x) \neq 0, h_1(x) = 0$. Then

$$h - \frac{h_1}{h_2(x)} = h_1\left(\frac{h_2(x) - h_2}{h_2(x)h_2}\right) \in \mathfrak{m}_{Y,x}^2$$

$$\Longrightarrow [h] = \varphi\left(\frac{h_1}{h_2(x)}\right),$$

where $\frac{h_1}{h_2(x)} \in I_x$, so φ is surjective.

(b) $\ker(\varphi) = I + I_x^2 \subset I_x$ (Intuitively, the restriction of f on Y vanishes to order 2 ar x).

Precisely:

$$\mathcal{O}_{Y,x} = (\mathcal{O}(\mathbb{A}^n)/I)_{I_x/I} = \mathcal{O}(\mathbb{A}^n)_{I_x}/I\mathcal{O}(\mathbb{A}^n)_{I_x}$$

the last equality from commutative algebra. $\varphi(f) = 0$ means that $f \mod I$ belongs to $(I_x^2)_{I_x}$ which is an ideal in $\mathcal{O}(\mathbb{A}^n)_{I_x}$ generated by I_x^2

$$f \mod I = \sum_{i,j}(X_i - x_i)(X_j - x_j)h_{ij}$$

$\theta(f \mod I) = 0 \Longrightarrow f \in I + I_x^2$. □

Theorem 1.46. *Let $Y \subset \mathbb{A}^n$ affine variety. Then $Y^\circ = \{x \in Y \mid Y$ non-singular at $x\}$ is dense open subset.*

Corollary 1.47. *Any variety Y is birational to a non-singular variety.*

Proof. (of theorem)
Let $S = Y - Y^\circ = \{$ singular points $\}$. Then we know

(1) S is closed in Y, indeed fixing $(f_1, ..., f_m)$ generating $I(Y)$

$$S = \{x \mid rank\ J_{\underline{f}}(x) \neq n - d\}$$

One can show that $rank\ J_{\underline{f}}(x \leq n - d)$. So

$$\begin{aligned} S &= \{x \mid rank\ J_{\underline{f}}(x) < n - d\} \\ &= \{x \in Y \mid \text{ for all minors } M \text{ of } J_{\underline{f}} \text{ of size } n - d \text{ are degenerate } \det(M) = 0.\} \end{aligned}$$

is a closed algebraic set in $\mathbb{A}^n$.

(b), $S \neq Y (\Longrightarrow Y^\circ \neq \emptyset$ and open, so is dense).

If $S = Y$, then by the theorem of Zariski, the set of non-singular points in an open set of a hypersurface birational to Y would be empty. This means that we may assume $Y = V(f) \subset \mathbb{A}^{d+1}$ with f non-zero irreducible. Then

$$V(f) \supset S = \left\{x \in \mathbb{A}^{d+1} | 0 = f(x = \frac{\partial f}{x_1}(x) = ... = \frac{\partial f}{x_d}(x)\right\}$$

so if $S = V(f)$, $\frac{\partial f}{\partial x_1} \in I(V(f)) = f\mathcal{O}(\mathbb{A}^{d+1}) = fK[X_1, .., X_{d+1}]$

$$\implies \text{in char } = 0, \text{ comparing degrees, we have contradiction}$$

$\implies$ in chat $p \neq 0$, we get $\frac{\partial f}{\partial x_i} = 0$ for $1 \leq i \leq d$, $\implies f \in K[x_1^p, .., X_d^p] \implies f = g^p$, contradicting the irreducibility. □

2 Schemes

In this chapter we will mainly follow chap 2 of Hartshorne and chap 1 of Eisenbud-Harris.

2.1 Mar 13th-B: Affine schemes

Motivations

Serious probelms with classical approach occur in late 1950's

(1) Intrinsic definitions (Without embeddings in $\mathbb{A}^n$ or $\mathbb{P}^n$)

(2) Construction of various algebraic varieties especially Jacobian variety of a curve, especially w.r.t. base field (is the Jacobian of a curve given by equation with coefficients in the same field?)

(3) Reduction modulo p of a variety given by equation in $\mathbb{Z}[X_1, .., X_n]$

To attack (1), Serre started from

$$\{\text{alg. set } Y \subset \mathbb{A}^n\} \longleftrightarrow \{\text{fin.gen. reduced } K\text{-algebra}\}$$
$$Y \mapsto \mathcal{O}(Y)$$
$$\{\text{maximal ideals in } A\} \leftarrow\!\shortmid A.$$

Grothendieck tried to remove the restriction on the algebras and managed to interpret it geometrically.

$$\{\text{affine schemes }\} \longleftrightarrow \{\text{all commutative rings.}\}$$

To each ring A, we will associate a geometric object called its **spectrum** denoted $\operatorname{Spec}(A)$.

(1) $\operatorname{Spec} A$ is a set. $\operatorname{Spec} A \neq \{$ maximal ideals $\}$ because this choice is not functorial. If $A_1 \xrightarrow{f} A_2$, we want $\operatorname{Spec}(A_2) \overset{f^*}{\operatorname{Spec}}(A_1)$ which would have to be $f^*(\mathfrak{m}) = f^{-1}(\mathfrak{m}) \subset A_1$. But $f^{-1}(\mathfrak{m})$ is <u>NOT</u> necessarily maximal.

Example 2.1. *A is an integral domain*

$$\{0\} \subset A \hookrightarrow Frac(A) \supset \{0\} \textit{ maximal}$$

Definition 2.2. *Spec* $A := \{$ *prime ideals* $\mathfrak{p} \subset A$ $\}$

Fact: If $f : A_1 \longrightarrow A_2$ is a ring morphism then $\mathfrak{p} \mapsto f^{-1}\mathfrak{p}$ gives map of sets

$$\operatorname{Spec} A_2 \longrightarrow \operatorname{Spec} A_1$$

Proof.

$$A_1 \xrightarrow{f} A_2/\mathfrak{p}$$

$$f^{-1}\mathfrak{p} \mapsto 0$$

leads to an injective map

$$A_1/f^{-1}\mathfrak{p} \hookrightarrow A_2/\mathfrak{p}$$

, then $A/f^{-1}\mathfrak{p}$ is an integral domain and $f^*(\mathfrak{p})$ is therefore a prime ideal. □

Definition 2.3. *If* $\mathfrak{p} \in$ *Spec* A*. the fraction field of* $A/\mathfrak{p}$ *is called the residue field at* $\mathfrak{p}$*, denoted* $\kappa(\mathfrak{p})$*.*

If $a \in A$, then a defines a function $\tilde{a} : \operatorname{Spec} A \longrightarrow \coprod_{\mathfrak{p} \in \operatorname{Spec}(A)} \kappa(\mathfrak{p})$, $\mathfrak{p} \mapsto a \bmod \mathfrak{p}$

(2) Spec A as a topological space

Definition 2.4. *For any set* $S \subset A$*, let* $V(S) = \{\mathfrak{p} \in \textit{Spec}\,(A) | S \subset \mathfrak{p}\}$*:*

Note:

(1) $V(S) = V($ ideals generated by S $)$

(2) Not always true that $V(S) = V($ finitely many elements$)$

(3) $V(S) = \{\mathfrak{p} \in \operatorname{Spec} A | \forall x \in S, \tilde{x}(\mathfrak{p}) = 0 \in \kappa(\mathfrak{p})\}$

Lemma 2.5.

(1) The sets $V(I)$*,* I *ideal in* A*, from the closed set s of a topology on Spec* A *(called the Zariski topology).*

(2) $V(I) \subset V(J) \Longleftrightarrow \sqrt{J} \subset \sqrt{I}$

(3) If $f : A_1 \longrightarrow A_2$ is a ring morphism, then

$$f^* : \text{ Spec}\,(A_2) \longrightarrow \text{ Spec}\,(A_1)$$

is continuous.

Proof. (1) $\emptyset = V(A) = V(\{1\})$ $\operatorname{Spec} A = V(\{0\})$.

$$\begin{aligned}\cap_{i\in X} V(I_i) &= \{\mathfrak{p} \in \text{ Spec}\,(A) | I_i \subset \mathfrak{p} \text{ for every } i\} \\ &= \{\mathfrak{p} \in \text{ Spec}\,(A) | \sum I_i \subset \mathfrak{p}\} \\ &= V\left(\sum_{i\in X} I_i\right)\end{aligned}$$

$$\begin{aligned}V(I) \cup V(J) &= \{\mathfrak{p} \in \text{ Spec}\,(A) | I \subset \mathfrak{p} \text{ or } J \subset \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \text{ Spec}\, A | IJ \subset \mathfrak{p}\} (\text{because } \mathfrak{p} \text{ prime}) \\ &= V(IJ)\end{aligned}$$

(2) recall the definition of radicals of an ideal

$$\begin{aligned}\sqrt{I} := \{x \in A | \exists k \geq 0, x^k \in I\} \quad &= \cap_{I\subset\mathfrak{p}, \mathfrak{p}\in \text{ Spec}\, A} \mathfrak{p} \\ = \cap_{\mathfrak{p}\in V(I)} \mathfrak{p}\end{aligned}$$

then if $V(J) \subset V(I)$, we get $\sqrt{I} \subset \sqrt{J}$.

Conversely, if $\sqrt{I} \subset \sqrt{J}$ then for $\mathfrak{p} \in V(J)$, then $I \subset \sqrt{I} \subset \sqrt{J} \subset \mathfrak{p} \Longrightarrow \mathfrak{p} \in V(I)$.

(3) $\square$