Summary for Algebraic Topology II

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Contents

1 21th Feb: Tor functor

Definition 1.1. *Suppose A is an abelian group, A* ***Free resolution*** *is an exact sequence of the form*

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \longrightarrow 0,$$

where each F_i is a free abelian group. If moreover $F_i = 0, \forall i \geq 2$, we call it ***Short free resolution***

$$0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$$

(We can easily generalize this definition to R-modules)

Proposition 1.2. *Let A be an abelian group. Then there exists a short free resolution of A.*

Proof. Let F be the free abelian group generated by all elements in A. There is a surjection from F to A by linearly extending the map sending basis element to itself. Let K denote the kernel of this map. K is an abelian subgroup of a free abelian group ($\mathbb{Z}$-module). A subgroup of a free abelian group is torsion free as a module. $\mathbb{Z}$ is a PID. If R is a PID, then an R-module is free iff it is torsion free (See Bosch section 4.2). Then we know in particular, K is a free abelian group. □

With this construction, we can define the Tor functor now:

Definition 1.3. *Let A be an abelian group. Let $0 \to K \xrightarrow{f} F \to A \to 0$ be a short free resolution of A. Given any other abelian group B, we define*

$$Tor(A, B) := \ker(f \otimes id_B)$$

Tor(A,B)

This definition is independent on the choice of short free resolution.

2 28th Feb:

Question: Given X, Y what is the cohomology of $X \times Y$?

Answer:

$$H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) + \bigoplus_{k+\ell=n-1} \mathrm{Tor}(H_k,(X), H_\ell(Y))$$

We will discuss Elenberg-Zilber theorem along this line the next lecture.

Today, we will prove the Algebraic Kueneth Theorem

Definition 2.1. *Suppose* $(C_\bullet, \partial)$ *and* $(C'_\bullet, \partial')$ *are two non-negative chain complexes. We define the* ***tensor complex*** $(C_\bullet \otimes C'_\bullet, \Delta)$, *where*

$$(C_\bullet \otimes C'_\bullet)_n = \oplus_{i+j=n} C_i \otimes C'_j$$

and the differential Δ *is defined by*

$$\Delta(c_i \otimes c'_j) = \partial c_i \otimes c'_j + (-1)^i c_i \otimes \partial' c_j$$

First, note that $\Delta(c_i \otimes c'_j)$ does indeed belong to $(C_\bullet \otimes C'_\bullet)_{n-1}$. The reason for $(-1)^i$ is to make $\Delta^2 = 0$. $C_\bullet \otimes C'_\bullet$ is another non-negative chain complex.

Definition 2.2. *Suppose* $f_\bullet : C_\bullet \longrightarrow D_\bullet$ *and* $g_\bullet : C'_\bullet \longrightarrow D'_\bullet$ *are two morphism of chain complexes. Then we can define a chain map*

$$f \otimes g : C \otimes C' \longrightarrow D \otimes D'$$

by

$$(f \otimes g)_n = \sum_{i+j=n} f_i \otimes g_j$$

It is easy to check this is indeed a chain map.

Lemma 2.3. *If* $f' : C \longrightarrow C'$ *and* $g' : D \longrightarrow D'$ *are two more chain maps with* f *homotopic to* f' *and* g *homotopic to* g'. *Then* $f' \otimes g'$ *is homotopic to* $f \otimes g$.

Theorem 2.4. *(Algebraic Kuenneth Theorem) Let* (C, ∂) *and* (D, ∂') *be two non-negative free complex. Then for every* $n \geq 0$, *there is a split exact sequence*

$$0 \longrightarrow \oplus_{i+j=n} H_i(C) \otimes H_j(D) \longrightarrow H_N(C \otimes D) \longrightarrow \oplus_{k+\ell=n-1}\ Tor(H_k(C), H_\ell(D)) \longrightarrow 0$$

where ω *is the map* $\langle c_i \rangle \otimes \langle d_j \rangle \mapsto \langle c_i \otimes d_j \rangle$. *Thus there also exists a (non-natural) isomorphism*

$$H_n(C \times D) \cong \oplus_{i+j=n} H_i(C) \otimes H_j(D) + \oplus_{k+\ell=n-1}\ Tor(H_k, (C), H_\ell(D)$$

The proof requires two auxiliary results.

Proposition 2.5. *Let* $(E_\bullet, 0)$ *be a non-negative chain complex with all differential zero and* $(D_\bullet, \partial)$ *be any non-negative chain complex. Given* $i \geq 0$, *let* $D^i_\bullet$ *denote the chain complex where* $D^i_n = D_{n-i}$ *and the boundary map*

$$D^i_n \longrightarrow D^i_{n-1}$$

is just the map: $D_{n-i} \longrightarrow D_{n-i-1}$.

Then

$$H_n(E_\bullet \otimes D_\bullet) \cong \bigoplus_{i \geq 0} H_n(E_i \otimes D^i_\bullet)$$

Proof. (of the Proposition) Since $E_\bullet$ has no differentials

$$\begin{aligned}\Delta(e_i \otimes d_{n-i}) &= (-1)^i e_i \otimes \partial d_{n-i} \\ &= (-1)^i (id_E \otimes \partial)[e_i \otimes d_{n-i}]\end{aligned}$$

$$\begin{aligned}H_n(E_\bullet \otimes D_\bullet) &= \frac{ker\Delta}{im\Delta} \\ &= \bigoplus_{i \geq 0} \frac{ker(id_E \otimes \partial|_{D_{n-i}})}{im(id_E \otimes \partial|_{D_{n-i+1}})} \\ &= \bigoplus_{i \geq 0} H_n(E_i \otimes D_\bullet^i)\end{aligned}$$

□

Proof. (of Theorem) We will prove it in three steps:

Let's use the same notation as we did in the proof of the universal coefficient theorem. $B_n \subset Z_n \subset C_n$. $(Z_\bullet, 0)$ and $(B_\bullet^+, 0)$ are chain complexes with no differentials, where $B_n^+ = B_{n-1}$. $(H_\bullet, 0)$ be the chain complex. $i : Z_n \hookrightarrow C_n$, $j : B_n \hookrightarrow Z_n$, $d : C_n \longrightarrow B_{n-1}$, where d is the just the differential ∂ of $C_\bullet$ and we use p to denote the projection $Z_n \twoheadrightarrow H_n$. Then we have two short exact sequence of chain complexes

$$0 \longrightarrow Z_\bullet \xrightarrow{i_\bullet} C_\bullet \xrightarrow{D_\bullet} B_\bullet^+ \longrightarrow 0$$

$$0 \longrightarrow B_\bullet \xrightarrow{j_\bullet} Z_\bullet \xrightarrow{p_\bullet} H_\bullet \longrightarrow 0.$$

We tensor it with $D_\bullet$.

$$0 \longrightarrow Z_\bullet \otimes D_\bullet \xrightarrow{i_\bullet} C_\bullet \otimes D_\bullet \xrightarrow{D_\bullet} B_\bullet^+ \otimes D_\bullet \longrightarrow 0$$

$$0 \longrightarrow B_\bullet \otimes D_\bullet \xrightarrow{j_\bullet} Z_\bullet \otimes D_\bullet \xrightarrow{p_\bullet} H_\bullet \otimes D_\bullet \longrightarrow 0.$$

They are again short exact sequence of chain complexes because D is free Abelian group thus flat module.

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, bend left, "i"] & C_n \arrow[l, bend left, "r"] \arrow[r, "d"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

This sequence splits as B_{n-1} is free abelian. Thus $\exists$ a map $r : C_n \longrightarrow Z_n$ such that $r|_{Z_n}$ is the identity $r_\bullet : C_\bullet \longrightarrow Z_\bullet$.

Denote by μ the composition $p \circ r : C_\bullet \longrightarrow H$.

Claim: μ is a chain map from $(C_\bullet, \partial) \longrightarrow (H_\bullet, 0)$. Take $c \in C_{n+1}$ and check it commutes

$$\mu \circ \partial c = \mu \partial c = p \circ r \partial c = \langle \partial c \rangle = 0$$

and $0 \circ \mu c = 0$

Step 2: Define $\varphi = H_n(\mu \otimes id)$. $H_n(C_\bullet \otimes D_\bullet) \longrightarrow H_n(H_\bullet \otimes D_\bullet)$.

<u>Claim</u>: φ is an isomorphism.

It suffices to prove the diagram commutes and conclude by five lemma.

$$\begin{tikzcd}
H_{n+1}(B^+_\bullet \otimes D_\bullet) \arrow[r, "\delta"] \arrow[d, "id"] & H_n(Z_\bullet \otimes D_\bullet) \arrow[r] \arrow[d, "id"] & H_n(C_\bullet \otimes D_\bullet) \arrow[r] \arrow[d, "\varphi"] & H_n(B^+_\bullet \otimes D_\bullet) \arrow[r, "\delta"] \arrow[d, "id"] & H_{n-1}(Z_\bullet \otimes D_\bullet) \arrow[d, "id"] \\
H_n(B_\bullet \otimes D_\bullet) \arrow[r] & H_n(Z_\bullet \otimes D_\bullet) \arrow[r] & H_n(H_\bullet \otimes D_\bullet) \arrow[r, "\delta'"] & H_{n-1}(B_\bullet \otimes D_\bullet) \arrow[r] & H_{n-1}(Z_\bullet \otimes D_\bullet)
\end{tikzcd}$$

Step 3: We complete the proof

$$\begin{aligned} H_n(C_\bullet \otimes \otimes D_\bullet) &\cong H_n(H_\bullet \otimes D_\bullet) \\ &\cong \bigoplus_{i \geq 0} H_n(H_i(C_\bullet) \otimes D^i_\bullet) \end{aligned}$$

By the universal coefficient theorem, there is a split exact sequence

$$0 \longrightarrow H_i(C_\bullet) \otimes H_n(D^i_\bullet) \longrightarrow H_n(H_i(C_\bullet) \otimes D^i_\bullet) \longrightarrow \operatorname{Tor}(H_i(C_\bullet), H_{n-1}(D^i_\bullet)) \longrightarrow 0$$

If we get rid of the notation $D^i_\bullet$.

$$0 \longrightarrow H_i(C_\bullet) \otimes H_n(D^i_\bullet) \longrightarrow H_n(H_i(C_\bullet) \otimes D^i_\bullet) \longrightarrow \operatorname{Tor}(H_i(C_\bullet), H_{n-1-i}(D_\bullet)) \longrightarrow 0$$

Take the direct sum over i and use the fact that

$\square$