Evanescent Tide

A solution manual by and for stupid student

Vector_Cat

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1.1 Towards schemes

Exercise?? 1.1.A Suppose that $\pi : X \longrightarrow Y$ is a continuous map of differentiable manifolds (as topological spaces — not a priori differentiable). Show that π is differentiable if differentiable functions pull back to differentiable functions, i.e., if pullback by π gives a map $\mathscr{O}_Y \longrightarrow \pi_* \mathscr{O}_X$. (Hint: check this on small patches. Once you figure out what you are trying to show, you will realize that the result is immediate.) ■

Proof. Recall the definition of differentiable maps: At every point $p \in X$, local charts $(U, \varphi) \ni p$ and $(V, \phi) \ni \pi(p)$, where $\varphi : U \longrightarrow \mathbb{R}^n$ and $\phi : V \longrightarrow \mathbb{R}^m$ are homeomorphisms. We say π is differentiable iff $\phi \circ \pi \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable. And check that if $\forall g$, $g \circ \pi$ is differentiable at p. $g \circ \phi^{-1} \circ (\phi \circ \pi \circ \varphi^{-1})$ is differentiable for all g. We can choose $g = \phi_i$ such that $\phi_i \circ \phi^{-1} = pr_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ is the projection map to the i-th coordinate. Then the map $\phi \circ \pi \varphi^{-1} = (pr_1 \circ \phi \circ \pi \varphi^{-1}, \ldots, pr_m \circ \phi \circ \pi \varphi^{-1})$ is differentiable. ■

Exercise?? 1.1.B Show that a morphism of differentiable manifolds $\pi : X \longrightarrow Y$ with $\pi(p) = q$ induces a morphism of stalks $\pi^\# : \mathscr{O}_{Y,q} \longrightarrow \mathscr{O}_{X,p}$. Show that $\pi^\#(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,\mathfrak{p}}$. In other words, if you pull back a function that vanishes at q, you get a function that vanishes at p. ■

Proof. $\mathscr{O}_{X,p}$ is the germ of smooth functions at p. $\pi^\# : \mathscr{O}_{Y,q} \longrightarrow \mathscr{O}_{X,p}; [(f, U)] \longmapsto [(f \circ \pi, \pi^{-1} U)]$. If $[(f, U)] \in \mathfrak{m}_{Y,q}$, $(f, U) \sim (0, V)$. In particulr, there is an open set $W \subseteq U \cap V$, so that $f|_W = 0$. $f \circ \pi|_{\pi^{-1} W} = 0 \Longrightarrow [f \circ \pi, \pi^{-1} W] \in \mathfrak{m}_{X,p}$. ■

1.2 The underlying sets of affine schemes

Exercise?? 1.2.A A SMALL EXERCISE ABOUT SMALL SCHEMES

(a) Describe the set $Spec(k[\varepsilon]/(\varepsilon^2))$. The ring $k[\varepsilon]/(\varepsilon^2)$ is called the ring of **dual numbers**, and will turn out to be quite useful. You should think of ε as a very small number, so small that its square is 0 (although it itself is not 0). It is a nonzero function whose value at all points is zero, thus giving our first example of functions not being determined by their values at points.

(b) Describe the set $k[x]_{(x)}$

Proof.

(a) Recall the isomorphism theorem, there is a one to one correspondence between $\{\mathfrak{p} \in Spec(A), \mathfrak{p} \supseteq \mathfrak{a}\}$ and $\{\mathfrak{q} \in Spec(A/\mathfrak{a})\}$. The only prime in $k[\varepsilon]$ that contains (ε^2) is (ε). Hence, $Spec(k[\varepsilon]/(\varepsilon^2)) = \{[(\varepsilon)]\}$

(b) There is a one to one correspondence,

$$\{\mathfrak{p} \in Spec(A) : \mathfrak{p} \cap S = \emptyset\} \longleftrightarrow \{\mathfrak{q} \in Spec(S^{-1}A)\}.$$

In this specific case, the multiplicative set is $S := k[x] - (x)$. Then the primes in $k[x]_{(x)}$ corresponds to those primes ideals contained in (x), which is just $(x) \cdot k[x]_{(x)}, (0)$

∎

Exercise?? 1.2.B Show that for $\mathbb{R}[X]/(p(X)) \cong \mathbb{C}$, where $p(x)$ is an irreducible quadratic polynomial.

Proof. For example, consider $p(X) = X^2 + aX + b$, where $a^2 - 4b < 0$. Then

$$A = \mathbb{R}[X]/(X^2 + aX + b) \cong \{f \in \mathbb{R}[x] : x^2 + ax + b = 0\},$$

hence every element in A can be written as $cx + d$, with $c, d \in \mathbb{R}$. However we know $\mathbb{C} \cong \mathbb{R} \oplus \mathbb{R}i$ as $\mathbb{R}$-algebra. Consider an $\mathbb{R}$-linear morphism of vector space

$$\varphi : x \longmapsto -\frac{a}{2} + \frac{\sqrt{4b - a^2}}{2} i, 1 \longmapsto 1$$

is surjective and injective. Check that φ is in fact a well-defined ring morphism. It suffices to check it on basis, we only need to check

$$-a\varphi(x) + b = \varphi(x^2) = \varphi(x)\varphi(x).$$

∎

Exercise?? 1.2.C Describe the set $\mathbb{A}^1_{\mathbb{Q}} =$ Spec $(\mathbb{Q}[X])$.

Proof. Each irreducible polynomial $p(X) \in \mathbb{Q}[X]$ corresponds to a splitting field. And $\mathbb{Q}[X]/(p(X)) \cong \mathbb{Q}[\alpha]$ where $\alpha \in \overline{Q}$ s.t. $p(\alpha) = 0$. Spec $(\mathbb{Q}[X])$ contains a generic point (0), closed point corresponding to $(x - q)$, where $q \in \mathbb{Q}$, and $p(X)$ irreducible, where $(p(X))$ corresponds an equivalent class of algebraic number and their Galois conjugates. ∎

Exercise?? 1.2.D If k is a field, show that $\operatorname{Spec} k[X]$ has infinitely many points. (Hint: Euclid's proof of the infinitude of primes of $\mathbb{Z}$.) ■

Proof. Polynomial ring over a field is Euclidean domain, which means every polynomial has a unique factorization as product of irreducible polynomials. Assume there are only finitely many prime ideals in $k[X]$, which means there are only finitely many irreducible polynomials $p_1, ..., p_n \in k[X]$. We can set $p = p_1 \cdots p_n + 1$. Assume p is reducible, then it must contain some p_i, but then we have a contradiction that $p_i | 1$. Then p is irreducible and we have infinite prime ideals (points). ■

Exercise?? 1.2.E Show that we have identified all the prime ideals of $\mathbb{C}[x,y]$. (Show that all prime ideals in $\mathbb{C}[x,y]$ are either of the form (0), $(x-a, y-b)$,$(f(x,y))$ where $f(x,y)$ is irreducible) ■

Proof. $\mathbb{C}[x,y]$ is a UFD. Then, a nonzero principal ideal is prime if and only if it is generated by a irreducible element.

Then it suffices to check the non-principal case. Suppose a prime $\mathfrak{p} \subseteq \mathbb{C}[x,y]$ is not principal ideal. We can find at least two elements $f(x,y), g(x,y)$ in $\mathfrak{p}$ with no common factor. We can regard $f(x,y)$ and $g(x,y)$ as polynomials of y with coefficients in $c(x)$.

$$f(x,y) = f_n(x)y^n + f_{n-1}(x)y^{n-1} + ... + f_0$$

$$g(x,y) = g_m(x)y^m + g_{m-1}(x)y^{m-1} + ... + g_0$$

Consider the Euclidean algorithm in $\mathbb{C}(x)[y]$, we can calculate the greatest common divisor of f, g when regard then as elements in $\mathbb{C}(x)[y]$.

Claim: Let R be a Euclidean domain, and $f, g \in \mathscr{R}[y]$ such that $gcd(f,g) = 1$ (in $R[y]$). They $gcd(f,g) = 1$ in $K[y]$, where K is the field of fractions of R. c Let $h \in K[y]$ such that $h|f$ and $h|g$ in $K[y]$. We wan to show that $deg(h) = 0$.

Let d be the product of all denominators of the coefficients of h, and $k = dh \in R[y]$. Then $k|df$ and $k|dg$ in $K[y]$, so there are $a, b \in R \backslash \{0\}$ such that $k|(ad)f$ and $k|bdg$ in $R[y]$. Write $adf = kp$ and $bdg = kq$ with $p, q \in R[y]$.

In the following, one denotes by $c(r)$ the greatest common divisor of the coefficients of $r \in R[y]$, and write $r = c(r)r_1$ where r_1 is primitive, that is , the greatest common divisor of its coefficients is 1.

From $adf = kp$ and $bdg = kq$ we get $adc(f) = c(k)c(p)$ and $bdc(g) = c(k)c(q)$. But $(ad)c(f)f_1 = c(k)c(p)k_1p_1$ and $bdc(g)g_1 = c(k)c(q)k_1q_1$, so $f_1 = k_1p_1$ and $g_1 = k_1q_1$. Thus we get $k_1|f_1|f$ and $k_1|g_1|g$, so $k_1 = 1$, and we are done.

$\mathbb{C}(x)$ is such a Euclidean domain. f, g has no common factor in $\mathbb{C}[x][y] \Longrightarrow gcd(f,g) = 1 \in \mathbb{C}(x)[y]$. $\exists u, v \in \mathbb{C}(x)[y]$ s.t.

$$uf + vg = 1.$$

Multiplying the product of denominators of coefficients in u, v, we can get an equality in $\mathbb{C}[x,y]$

$$u'f + v'g = h(x).$$

$h(x) \in (f(x,y), g(x,y)) \subseteq \mathfrak{p}$. $h(x)$ can split into product of linear polynomials in $\mathbb{C}[x]$, because $\mathfrak{p}$ is prime, at least one the linear factor $x - a$ is in $\mathfrak{p}$. Similarly we can prove that at least one linear factor $y-b \in \mathfrak{p}$. Thus, $(x-a, y-b) \in \mathfrak{p}$, but because $(x-a, y-b)$ is already maximal, we have every non-principal prime ideal is of the form $(x-a, y-b)$. ■

Exercise?? 1.2.F Show that the Nullstellensatz(Zariski's Lemma) implies the Weak Nullstellensatz.

Proof. Set $A := k[X_1, ..., X_d]$, with k being a algebraically closed field. Let $\mathfrak{m} \subseteq A$ be any maximal ideal, then $L = A/\mathfrak{m}$ is a field.

$$\begin{tikzcd} k \arrow[r] \arrow[rr, bend right, "j"'] & k[X_1, ..., X_d] = A \arrow[r, two heads, "q"] & L = A/\mathfrak{m} \end{tikzcd}$$

Note: L is a finitely generated k-algebra, generated by $q(X_1), ..., q(X_d)$

$$\begin{aligned} \text{Zariski's Lemma} &\Longrightarrow L/j(k) \text{ is finite field extension} \\ &\Longrightarrow L \cong k(k \text{ algebraically closed}) \end{aligned}$$

Set $x := (j^{-1}(q(X_1)), ..., j^{-1}(q(X_d))) \in \mathbb{C}^d$. Check $\mathfrak{m} = \mathfrak{m}_x := (X_1 - x_1, ..., X_d - x_d)$. We know j is surjective because q is, and j is always injective because k is a field (ring morpihsm of fields are injective). Suppose $P \in \mathfrak{m} \Longrightarrow q(P) = 0 \Longrightarrow j^{-1}(P(q(X))) = 0 \Longrightarrow P(j^{-1}(q(X))) = P(x) = 0$, hence $\mathfrak{m}_x \in \mathfrak{m}$, but we already know $\mathfrak{m}_x$ is maximal, then $\mathfrak{m} = \mathfrak{m}_x$. ■

Exercise?? 1.2.G Any integral domain A which is a finite k-algebra (i.e., a k-algebra that is a finite-dimensional vector space over k) must be a field.

Proof. For a nonzero element $x \in A$, we xA is a finite dimensional subspace of A. If $\dim xA \subsetneq \dim A$, then $x^2 A \subsetneq xA$ otherwise $\forall z \in A, \exists y \in A xz = x^2 y, \Longrightarrow x(z - xy) = 0 \Longrightarrow z = xy$ because A is an integral domain contradicting to $xA \neq A$. Then we have a descending chain of vector spaces $\cdots x^3 A \subsetneq x^2 A \subsetneq xA \subsetneq A$. Because A is finite dimensional, there is an $n \in \mathbb{Z}$ so that $x^n A = 0 \Longrightarrow x^n = 0$ contradiction. Therefore, multiplying x must be an isomorphism.

In particular, the above argument is not true for general finitely generated k-algebras. $k[X]$ is not Artinian, there exists an infinitely descending chain of ideals $(X) \supsetneq (X^2) \supsetneq (X^3) \supsetneq ...$ Multiplying X is not an isomorphism on $k[X]$.

Now, assume $A = k[X_1, ..., X_d]/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ so that the residue ring A is finite k-algebra. Such $\mathfrak{p}$ exists by Nullstellensatz. We already proved that $0 \neq x \in A$ multiplies as an isomorphism, hence x is not contained in any proper ideal in A. $\mathfrak{p}$ must be maximal ideal. Hence A is a field. ■

Exercise?? 1.2.H Describe the maximal ideal of $\mathbb{Q}[x,y]$ corresponding to points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Describe the maximal ideal of $\mathbb{Q}[x,y]$ corresponding to points $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. What are the residue fields in each case?

Proof. $(x^2 - 2, y^2 - 2)$ is not a maximal ideal, geometrically it consists of two points.

$(\sqrt{2}, \sqrt{2}) \sim (-\sqrt{2}, -\sqrt{2})$ correspond to $(x - y, x^2 - 2) = (x - y, y^2 - 2)$.

$(-\sqrt{2}, \sqrt{2}) \sim (\sqrt{2}, -\sqrt{2})$ correspond to $(x + y, x^2 - 2) = (x + y, y^2 - 2)$.

The residue field in both case is $\mathbb{Q}[x,y]/(x-y,y^2-2)=\mathbb{Q}[y]/[y^2-2]\cong\mathbb{Q}[\sqrt{2}]\cong\mathbb{Q}[x,y]/(x+y,y^2-2)$. And in particular, the quotient ring of (x^2-2,y^2-2) is

$$\mathbb{Q}[x,y]/(x^2-2,y^2-2)\cong\mathbb{Q}[\sqrt{2}]\times\mathbb{Q}[\sqrt{2}]$$

■

Exercise?? 1.2.I Consider the map of sets $\phi:\mathbb{C}^2\longrightarrow\mathbb{A}^2_{\mathbb{Q}}$ defined as follows. (z_1,z_2) is sent to the prime ideal of $\mathbb{Q}[x,y]$ consisting of polynomials vanishing at (z_1,z_2).

(a) What is the image of (π,π^2)

(b) Show that ϕ is surjective.

Proof. (a) There is an injective morphism from $\varphi:\mathbb{Q}[x,y]\hookrightarrow\mathbb{C}[x,y]$, it induces a morphism on $\operatorname{Spec}\mathbb{C}[x,y]\longrightarrow\operatorname{Spec}\mathbb{Q}[x,y]$. In particular, the point (π,π^2) maps as

$$(x-\pi,y-\pi^2)\longrightarrow(x-\pi,y-\pi^2)\cap\mathbb{Q}[x,y]$$

The problem reduces to find two polynomials $p(x,y),q(x,y)\in\mathbb{C}[x,y]$ what is the polynomial

$$p(x,y)(x-\pi)+q(x,y)(y-\pi^2)\in\mathbb{Q}[x,y].$$

We can find

$$(x+\pi)(x-\pi)-(y-\pi^2)=x^2-y\in\mathbb{Q}[x,y]$$

$(x^2-y)\subseteq(x-\pi,y-\pi^2)\cap\mathbb{Q}[x,y]$. The tricky part is the reverse inclusion.
Suppose there is a polynomial

$$P(x,y)=\sum_{m,n}a_{m,n}x^my^n,a_{m,n}\in\mathbb{Q}$$

so that $P(\pi,\pi^2)=0$. $P(\pi,\pi^2)$ is equal to rational polynomial F of π alone.

$$F(\pi)=\sum_k(\sum_{m+2n=k}a_{m,n})\pi^k$$

Because π is transcendental, each coefficients of F should be zero.

$$\sum_n a_{k-2n,n}=0$$

Then we reorder the summation of P

$$\begin{aligned}P(x,y)&=\sum_k\left(\sum_n a_{k-2n,n}x^{k-2n}y^n\right)\\&=\sum_k\left(\sum_n a_{k-2n,n}x^{k-2n}(y-x^2+x^2)^n\right)\\&=\sum_k\left(\sum_n a_{k-2n,n}x^k\right)+\sum_k\left(\sum_n a_{k-2n,n}\sum_{j=1}\binom{n}{j}x^{k-2j}(y-x^2)^j\right)\\&=0+\sum_k\left(\sum_n a_{k-2n,n}\sum_{j=1}\binom{n}{j}x^{k-2j}(y-x^2)^j\right)\end{aligned}$$

All such $P(x,y)\in(x^2-y)$.

(b) ϕ is basically φ^* for $\varphi : \mathbb{Q}[x,y] \hookrightarrow \mathbb{C}[x,y]$.
Want: φ^* is surjective.
Consider the special case where $\mathfrak{p}$ is maximal. Then $\mathbb{Q}[x,y]/\mathfrak{p}$ is a finitely generated field extension, hence is finite algebraic field extension by Nullstellensatz. It embeds into $\mathbb{C}$, with a,b being the images of x,y.

$$(x-a, y-b) \cap \mathbb{Q}[x,y] = \mathfrak{p}$$

$\mathfrak{p} \subseteq (x-a, y-b) \cap \mathbb{Q}[x,y]$ is trivially true since $\mathfrak{p} \subseteq \varphi_* \varphi^*(\mathfrak{p})$. And the reverse inclusion holds because $\mathfrak{p}$ is maximal.
For the prime ideal $(0) \subseteq \mathbb{Q}[x,y]$, we can find two algebraically independent transcendental number ξ, δ s.t., $p(\xi,\delta) = 0 \longrightarrow p = 0$ given that $p \in \mathbb{Q}[x,y]$.
For non-zero non-maximal ideal $\mathfrak{p}$, we consider the residue field

$$K := Frac\,(\mathbb{Q}[x,y]/\mathfrak{p}) \cong \mathbb{Q}[x,y]_{\mathfrak{p}}/\mathfrak{p}\mathbb{Q}[x,y]_{\mathfrak{p}},$$

where $\mathfrak{p}\mathbb{Q}[x,y]_{\mathfrak{p}}$ is the maximal ideal in the local ring $\mathbb{Q}[x,y]_{\mathfrak{p}}$ and we have

$$\mathfrak{p} = \mathfrak{p}\mathbb{Q}[x,y]_{\mathfrak{p}} \cap \mathbb{Q}[x,y]$$

We have a chain of injections

$$\mathbb{Q}[x,y] \hookrightarrow \mathbb{Q}[x,y]_{\mathfrak{p}} \hookrightarrow K \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{C}[x,y].$$

By the similar argument, let $\xi, \delta \in \mathbb{C}$ be the images of $\frac{x}{1}, \frac{y}{1}$, we have

$$(x-\xi, y-\delta) \cap \mathfrak{p}\mathbb{Q}[x,y]_{\mathfrak{p}} = \mathfrak{p}\mathbb{Q}[x,y]_{\mathfrak{p}} \cap \mathbb{Q}[x,y]$$

because $\mathfrak{p}\mathbb{Q}[x,y]_{\mathfrak{p}} \cap \mathbb{Q}[x,y]$ is maximal. Hence,

$$(x-\xi, y-\delta) \cap \mathbb{Q}[x,y] = \mathfrak{p}\mathbb{Q}[x,y]_{\mathfrak{p}} \cap \mathbb{Q}[x,y] \cap \mathbb{Q}[x,y] = \mathfrak{p}.$$

■

Exercise?? 1.2.J Suppose A is a ring, and I an ideal of A. Let $\varphi : A \longrightarrow A/I$. Show that φ^{-1} gives an inclusion-preserving bijection between prime ideals of A/I and prime ideals of A containing I. Thus we can picture $\operatorname{Spec} A/I$ as a subset of $\operatorname{Spec} A$. ■

Proof. Claim1: $\mathfrak{p} \subseteq A/I$ is prime iff $\varphi^{-1}(\mathfrak{p})$ is prime.
$\varphi(x)\varphi(y) = \varphi(xy) \in \mathfrak{p}$, $xy \in \varphi^{-1}\mathfrak{p}$. If $\varphi^{-1}\mathfrak{p}$ is prime, at least one of x,y is contained in $\varphi^{-1}\mathfrak{p} \Longrightarrow$ at least one of $\varphi(x), \varphi(y)$ is contained in $\mathfrak{p}$.
$xy \in \varphi^{-1}\mathfrak{p} \Longrightarrow \varphi(xy) \in \mathfrak{p}$. If $\mathfrak{p}$ is prime at least one of $\varphi(x)$ and $\varphi(y)$ is contained in $\mathfrak{p} \Longrightarrow x$ or y is contained in $\mathfrak{p}$.
Claim2: φ induces a inclusion-preserving bijection between ideals of A/I and ideals of A containing I.
Consider an ideal $\mathfrak{a} \in A/I$. $\varphi^{-1}\mathfrak{a}$ is an ideal in A. In particular, because $\varphi(I) = [0] \in \mathfrak{a} \Longrightarrow I \subseteq \mathfrak{a}$.

$$\mathfrak{a} \longmapsto \varphi^{-1}\mathfrak{a}$$

is an bijection because we have the inverse, for $I \subseteq \mathfrak{q} \subseteq A$ $\varphi : \mathfrak{q} \longmapsto \mathfrak{q}/I$, where $\mathfrak{q}/I$ is a well-defined ideal in A/I.

And φ^{-1} preserves the inclusion of ideals (it preserves the proper inclusion in fact).

Consider $\mathfrak{a} \subsetneq \mathfrak{b} \subseteq A/I \Longrightarrow \exists [x] \in \mathfrak{b}$ s.t. $[x] \notin \mathfrak{a} \Longrightarrow$ the representative element $x \in \varphi^{-1}\mathfrak{b}, x \notin \varphi^{-1}\mathfrak{a}$. ■

Exercise?? 1.2.K Suppose S is a multiplicative subset of A. Describe an order-preserving bijection of the prime ideals of $S^{-1}A$ with the prime ideals of A that don't meet the multiplicative set S.

The claim should contain the following points

(a) $\mathfrak{p}$ prime $\Longleftrightarrow \iota_*\mathfrak{p}$ prime,

(b) $\iota^*\iota_*\mathfrak{p} = \mathfrak{p}$, (True for only prime ideal $\mathfrak{p}$ which satisfies $\mathfrak{p} \cap S = \emptyset$. If $\mathfrak{p} \cap S \neq \emptyset$, $\iota^*\iota_*\mathfrak{p} = A$)

(c) $\iota_*(\mathfrak{a}) = S^{-1}A \Longleftrightarrow \mathfrak{a} \cap S \neq \emptyset$, (True for any ideals)

(d) $\iota_*\iota^*\mathfrak{q} = \mathfrak{q}$ (True for any ideal of $S^{-1}A$, not necessarily prime)

■

Proof. (a) As for point (a), $\mathfrak{p}$ prime $\overset{?}{\Longrightarrow} \iota_*\mathfrak{p}$ prime. Consider $\frac{a}{s} \cdot \frac{b}{t} \in \iota_*\mathfrak{p}$, then $\frac{ab}{st} = \frac{c}{u}, c \in \mathfrak{p}, u \in S$, then $\exists v \in S : abuv = cstv$, where $uv \in S$ $cstv \in \mathfrak{p}$, $uv \notin \mathfrak{p} \Longrightarrow ab \in \mathfrak{p} \Longrightarrow$ at least one of $a, b \in \mathfrak{p} \Longrightarrow$ at least one of $\frac{a}{s}, \frac{b}{t} \in \iota_*\mathfrak{p}$.

(b) $\iota^*\iota_*\mathfrak{p} \supseteq \mathfrak{p}$ is a general fact. For the converse inclusion, $\iota^*\iota_*\mathfrak{p} = \iota^{-1}(\iota_*\mathfrak{p}) \overset{?}{\subseteq} \mathfrak{p}$, choose an $a \in \iota^{-1}(\iota_*\mathfrak{p})$. $\iota(a) = \frac{a}{1} \in \iota_*\mathfrak{p} \Longrightarrow \exists b \in \mathfrak{p}, s \in S$ s.t. $\frac{a}{1} = \frac{b}{s} \Longrightarrow ast = bt \in \mathfrak{p}$ and $s, t \in S \subseteq A - \mathfrak{p} \Longrightarrow a \in \mathfrak{p}$ because $\mathfrak{p}$ is a prime ideal.

(c) $\iota_*(\mathfrak{a}) = S^{-1}A \Longleftrightarrow \exists a \in \mathfrak{a}, s \in S$ s.t. $a/s = 1/1 \Longleftrightarrow \exists t \in S$ s.t. $\mathfrak{a} \ni ta = ts \in S$, then $\mathfrak{a} \cap S \neq \emptyset$. Conversely, $\mathfrak{a} \cap S \neq \emptyset$, exists an $a \in \mathfrak{a}, a = s \in S$, then $a/s = 1/1$.

(d) $\iota_*(\iota^*(\mathfrak{b})) \subset \mathfrak{b}$ in general. For the converse inclusion, if $a/s \in \mathfrak{b}$, then $a/s \cdot s/1 = a/1 \in \mathfrak{b}$, which means $a \in \iota^*(\mathfrak{b}) \Longrightarrow a/s \in \iota_*(\iota^*(\mathfrak{b}))$. This claims means every ideal in $S^{-1}A$ is extension of an ideal in A.

The fact that $\iota^*\iota_*\mathfrak{p} = \mathfrak{p}$ and $\iota_*\iota^*\mathfrak{q}$ means that the correspondence preserves the inclusion and proper inclusion. ■

Exercise?? 1.2.L Show that these two rings are isomorphic.

$$(\mathbb{C}[x,y]/(xy))_{[x]} \cong \mathbb{C}[x]_x$$

■

Proof. $A := \mathbb{C}[x,y]/(xy)$, $S := \{[x]^n | n \geq 0, n \in \mathbb{Z}\}$. For the morphism $\iota : A \longrightarrow S^{-1}A$,

$$a \longmapsto \frac{a}{1}$$

$$\frac{a}{1} \sim \frac{0}{1} \Longleftrightarrow \exists s \in S \text{ s.t. } sa = 0$$

the kernel is $\{a \in A | \exists n \geq 0 \text{ s.t. } [x]^n a = 0\}$. $[y] \in \ker \iota$. A general element $\frac{a}{s} \in S^{-1}A$ should be

$$\frac{a}{s} = \frac{p([x],[y])}{[x]^n} \sim \frac{1}{[x]^n} p\left(\frac{[x]}{1}, \frac{[y]}{1}\right) = \frac{1}{[x]^n} p\left(\frac{[x]}{1}, 0\right) = \frac{q([x])}{[x]^n},$$

where polynomial $q(X) := p(X,0)$. And then we can define the isomorphism

$$\begin{aligned} \phi : (\mathbb{C}[x,y]/(xy))_{[x]} &\longrightarrow \mathbb{C}[x]_x \\ \frac{q([x])}{[x]^n} &\longmapsto \frac{q(x)}{x^n} \end{aligned}$$

■

Exercise?? 1.2.M If $\phi : B \longrightarrow A$ is a map of rings, and $\mathfrak{p}$ is a prime ideal of A, show that $\phi^{-1}\mathfrak{p}$ is a prime ideal of B. ■

Proof. Consider two elements $x, y \in B$ s.t. $xy \in \phi^{-1}\mathfrak{p}$. Then $\phi(xy) \in B \implies \phi(x)\phi(y) \in \mathfrak{p}$. Because $\mathfrak{p}$ is prime ideal, at least one of $\phi(x), \phi(y) \in \mathfrak{p} \implies$ at least one of x, y is contained in $\phi^{-1}\mathfrak{p}$. ■

Exercise?? 1.2.N Let B be a ring.

(a) Suppose $I \subseteq B$ is an ideal. Show that the map $\operatorname{Spec} B/I \longrightarrow \operatorname{Spec} B$ is the inclusion of prime ideals that containing I.

(b) Suppose $S \subseteq B$ is a multiplicative set. Show that the map $\operatorname{Spec} S^{-1}B \longrightarrow \operatorname{Spec} B$ is the inclusion of prime ideals that does not intersect with S.

■

Proof. The detailed proof of the correspondences are already contained in 1.2.J and 1.2.K. ■

Exercise?? 1.2.O Consider the map of complex manifolds sending $\mathbb{C} \longrightarrow \mathbb{C}$ via $x \mapsto y = x^2$. We interpret the "source" $\mathbb{C}$ as the "x-line", and the "target" $\mathbb{C}$ the "y-line". You can picture it as the projection of the parabola $y = x^2$ in the xy-plane to the y-axis. Interpret the corresponding map of rings as given by $\mathbb{C}[y] \longrightarrow \mathbb{C}[x]$ by $y \mapsto x^2$. Verify that the preimage (the fiber) above the point $a \in \mathbb{C}$ is the point(s) $\pm\sqrt{a} \in \mathbb{C}$, using the definition given above. ■

Proof. We have defined the morphism of rings

$$\begin{aligned} \phi : \mathbb{C}[y] &\longrightarrow \mathbb{C}[x] \\ p(y) &\longmapsto p(x^2) \end{aligned}.$$

The only prime ideals in $\mathbb{C}[x]$ are (0) and $(x-b)$.

$$\phi^*(0) = \{p \in \mathbb{C}[y] : p(x^2) = 0\} = (0)$$

$$\phi^*(x-b) = \{p \in \mathbb{C}[y] : p(x^2) \in (x-b)\} = \{p \in \mathbb{C}[y] : p(b^2) = 0\} = (y-b^2).$$

Then $\phi^* :\ \operatorname{Spec} \mathbb{C}[x] \longrightarrow \operatorname{Spec} \mathbb{C}[y]$ is totally described. On each traditional point $y-a$, there are two points $(x - \pm\sqrt{a})$ in the preimage. ■

Exercise?? 1.2.P Suppose k is a field,and $f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$are given. Let $\phi : k[y_1, \ldots, y_n] \longrightarrow k[x_1, \ldots, x_m]$ be the ring morphism defined by $y_i \mapsto f_i$.

(a) Show that ϕ induces a map of sets $\operatorname{Spec} k[x_1, \ldots, x_m]/I \longrightarrow \operatorname{Spec} k[y_1, \ldots, y_n]/J$ for any ideals $I \subseteq k[x_1, \ldots, x_m]$ and $J \subseteq k[y_1, \ldots, y_n]$ such that $\phi(J) \subseteq I$.

(b) Show that the map of part (a) sends the point $(a_1, \ldots, a_m) \in k^m$ (or more precisely, $[(x_1 -$

$a_1,\dots,x_m - a_m)] \in$ Spec $k[x_1,\dots,x_m])$ to

$$(f_1(a_1,\dots,a_m),\dots,f_n(a_1,\dots,a_m)) \in k^n$$

Proof. (a) In fact, part (a) has nothing to do with k-algebras. We can choose any two rings $\phi : B \longrightarrow A$ and two ideals $J \subseteq B, I \subseteq A$ and $\phi(J) \subseteq I$ and prove that ϕ induces a morphism Spec $(A/I) \longrightarrow$ Spec (B/J).

We direct define a ring morphism induced by ϕ

$$\begin{aligned} \tilde{\phi} : B/J &\longrightarrow A/I \\ b+J &\longmapsto \phi(b)+I. \end{aligned}$$

$\tilde{\phi}$ is well-defined because if we choose another representative b' s.t. $b-b' \in J$, we have $\phi(b)-\phi(b') = \phi(b-b') \in I \Longrightarrow \phi(b)+I = \phi(b')+I$. Then problem reduces to the basic case, where $\tilde{\phi}$ induces morphism

$$\tilde{\phi}^* : \text{ Spec } (A/I) \longrightarrow \text{ Spec } (B/J).$$

(b) The point (maximal ideal) $(x_1-a_1,\dots,x_m-a_m)$ is mapped to $\phi^{-1}(x_1-a_1,\dots,x_m-a_m)$

$$\begin{aligned} \phi^{-1}(x_1-a_1,\dots,x_m-a_m) &= \{p \in k[y_1,\dots,y_n] : \phi(p) = p[f_1,\dots,f_n] \in (x_1-a_1,\dots,x_m-a_m)\} \\ &= \{p \in k[y_1,\dots,y_n] : p(f_1(a_1,\dots,a_m),\dots,f_n(a_1,\dots a_m)) = 0\} \\ &= (y_1 - f_1(a_1,..,a_m),\dots,y_n - f_n(a_1,\dots,a_m)). \end{aligned}$$

■

Exercise?? 1.2.Q Consider the map of sets $\pi : \mathbb{A}^n_{\mathbb{Z}} \longrightarrow$ Spec $\mathbb{Z}$, given by the ring map $\mathbb{Z} \longrightarrow \mathbb{Z}[x_1,\dots,x_n]$. If p is prime, describe a bijection between the fiber $\pi^{-1}([(p)])$ and $\mathbb{A}^n_{\mathbb{F}_p}$. (Can you interpret the fiber over $[(0)]$ as $\mathbb{A}^n_k$ for some field k?)

Proof. π is induced by the canonical inclusion $\iota : \mathbb{Z} \hookrightarrow \mathbb{Z}[x_1,..,x_n]$. In particular, for each prime ideal $\mathfrak{q} \in \mathbb{A}^n_{\mathbb{Z}}$

$$\pi(\mathfrak{q}) = \mathfrak{q} \cap \mathbb{Z}.$$

Hence $\mathbb{A}^n_{\mathbb{Z}}$ can be partitioned as

$$\mathbb{A}^n_{\mathbb{Z}} = \pi^{-1}([(0)]) \cup \left(\bigcup_{p \text{ prime}} \pi^{-1}([(p)]) \right).$$

For each $\mathfrak{p} \in \mathbb{A}^n_{\mathbb{Z}}$, $\mathfrak{p} \in \pi^{-1}([(p)]) \Longleftrightarrow \mathfrak{p} \cap \mathbb{Z} = (p)$.
$\overset{(*)}{\Longleftrightarrow} \mathfrak{p} \supseteq (p)\mathbb{Z}[x_1,\dots,x_n] =: \iota_*((p))$.

$$\mathbb{Z}[x_1,\dots,x_n]/(p)\mathbb{Z}[x_1\dots,x_n] \cong \mathbb{F}_p[x_1,\dots,x_n].$$

There is a one to one correspondence of prime ideals in $\mathbb{F}_p[x_1,\dots,x_n]$ and prime ideals in $\mathbb{Z}[x_1,\dots,x_n]$ that containing $(p)\mathbb{Z}[x_1,\dots,x_n]$. Hence $\pi^{-1}([(p)]) =$ Spec $\mathbb{F}_p[x_1,\dots,x_n] = \mathbb{A}^n_{\mathbb{F}_p}$.

The $\Longleftarrow$ part of $(*)$ is questionable. We have to prove each prime ideal $\mathfrak{q}$ that contains $(p)\mathbb{Z}[x_1,..,x_n]$ must have the property that $\mathfrak{q}\cap\mathbb{Z}=(p)$. $\mathfrak{q}$ is such a prime ideal, $\iota_*\mathfrak{q}\supseteq\iota_*((p)\mathbb{Z}[x_1,..,x_n])$ but (p) is already maximal in $\mathbb{Z}$, hence $\iota_*\mathfrak{q}=\mathfrak{q}\cap\mathbb{Z}=(p)$.

As for the fiber over $[(0)]$, $\mathfrak{p}\in\pi^{-1}([(0)])\Longleftrightarrow\mathfrak{p}\cap\mathbb{Z}=0$. Considering the multiplicative set $S:=\mathbb{Z}^{\times}$, it is equivalent to $S\cap\mathfrak{p}=\emptyset$. There is a one to one correspondence between the primes $S^{-1}\mathbb{Z}[x_1,..,x_n]$ and the primes in $\mathbb{Z}[x_1,..,x_n]$ which does not intersect S and $S^{-1}\mathbb{Z}[x_1,...,x_n]\cong\mathbb{Q}[x_1,...,x_n]$, thus

$$\pi^{-1}([(0)])\cong\mathbb{A}^n_{\mathbb{Q}}.$$

■

Exercise?? 1.2.R Ring elements that have a power that is 0 are called **nilpotents**.

(a) Show that if I is an ideal of nilpotents, then the inclusion $\operatorname{Spec} B/I\longrightarrow\operatorname{Spec} B$ of Exercise 1.2.J is a bijection. Thus nilpotents don't affect the underlying set.

(b) Show that the nilpotents of a ring B form an ideal. This ideal is called the **nilradical**, and is denoted $\mathfrak{N}=\mathfrak{N}(B)$.

Proof. (a) Each element of I is nilpotent $x\in I$, $x^n=0\in\mathfrak{p}$. $\Longrightarrow x\in\mathfrak{p}$ because $\mathfrak{p}$ is prime. $I\subseteq\mathfrak{p}$. Each prime ideals contain I, hence $\operatorname{Spec} B/I\longrightarrow\operatorname{Spec} B$ is a bijection.

(b) Two nilpotents $x,y\in B$. $x^n=0$ and $y^m=0$. $\forall a\in A,(ax)^n=0$ and $(ax+by)^{n+m}=0$. Hence, all nilpotents form an ideal. ■

Exercise?? 1.2.S Prove :

$$\mathfrak{N}(A)=\bigcap_{\text{primes in } A}\mathfrak{p}$$

Proof. Denote by $\mathfrak{N}'$ the intersection of all prime ideals of A. For any nilpotent element $f\in A$ with $n>0$ s.t. $f^n=0$, We have $f^n\in\mathfrak{p}$ for every prime ideal $\mathfrak{p}$. Hence $f\in\mathfrak{p}\Longrightarrow$ conclude $f\in\mathfrak{N}'$.

Conversely, suppose $f\in A$ is not nilpotent Define $\Sigma:=\{\mathfrak{a}\subset A \text{ ideals}|\forall n>0: f^n\notin\mathfrak{a}\}$. We will apply Zorn's lemma. We have

1. $(0)\in\Sigma$, so Σ is nonempty,
2. Σ is partially ordered by inclusion.
3. For any chain $(\mathfrak{a}_i)_{i\in I}\subset\Sigma$, the set $\mathfrak{a}:=\cup_{i\in I}\mathfrak{a}_i$ is an ideal and

for all $n>0$, we have $f^n\notin\mathfrak{a}$, hence $\mathfrak{a}\in\Sigma$. By Zorn's lemma we conclude that there is a maximal element $\mathfrak{p}\in\Sigma$.

We show that $\mathfrak{p}$ is a prime ideal. For any $x,y\notin\mathfrak{p}$, consider the ideals $\mathfrak{p}+(x),\mathfrak{p}+(y)$. They strictly contain $\mathfrak{p}$ and are thus not in Σ. Let $n,m>0$ s.t. $f^n\in\mathfrak{p}+(x),f^m\in\mathfrak{p}+(y)$. We conclude that $f^{n+m}\in\mathfrak{p}+(xy)$, so $\mathfrak{p}+(xy)\notin\Sigma$. Hence $xy\notin\mathfrak{p}$, which means, $\mathfrak{p}$ is a prime ideal so $f\notin\mathfrak{N}'$. ■

Exercise?? 1.2.T Suppose we have a polynomial $f(x)\in k[x]$. Instead, we work in $k[x,\varepsilon]/(\varepsilon^2)$. What then is $f(x+\varepsilon)$?

Proof. First we check what happens to simple examples like x^2 and x^3

$$(x+\varepsilon)^2=x^2+2x\varepsilon+\varepsilon^2=x^2+2x\varepsilon$$

and

$$(x+\varepsilon)^3 = x^3 + 3x^2\varepsilon.$$

For general term x^n, we have $(x+\varepsilon)^n = x^n + nx^{n-1}\varepsilon$, which can be linearly extended to general polynomial f.

$$f(x+\varepsilon) = f(x) + f'(x)\varepsilon$$

■

1.3 Visualizing schemes I: generic points

1.4 The underlying topological space of an affine scheme

Exercise?? 1.4.A Check that the x-axis is contained in $V(xy,yz) \subsetneq \operatorname{Spec} \mathbb{C}[x,y,z]$.

Proof. x-axis is defined to be $(y-0,z-0)$. $\{xy,yz\} \subset (xy,yz) \subseteq (y,z)$ by definition $(y,z) \in V(xy,yz)$. ■

Exercise?? 1.4.B Show that if (S) is the ideal generated by S, then $V(S) = V((S))$.

Proof. $V(S) := \{[\mathfrak{p}] \in \operatorname{Spec} A : S \subset \mathfrak{p}\}$ and $V((S)) := \{[\mathfrak{p}] \in \operatorname{Spec} A : (S) \subset \mathfrak{p}\}$.
If $\mathfrak{q} \in V(S) \implies S \subset \mathfrak{q} \implies (S) \subset \mathfrak{q} \implies \mathfrak{q} \in V((S))$, we have $V(S) \subseteq V((S))$.
If $\mathfrak{q} \in V((S)) \implies (S) \subset \mathfrak{q} \implies S \subseteq \mathfrak{q} \implies \mathfrak{q} \in V(S)$, we have $V((S)) \subseteq V(S)$. ■

Exercise?? 1.4.C

(a) Show that $\emptyset$ and $\operatorname{Spec} A$ are both open subsets of $\operatorname{Spec} A$.
(b) If I_i is a collection of ideals (as i runs over some index set), show that $\cap_i V(I_i) = V(\sum_i I_i)$. Hence the union of any collection of open sets is open.
(c) Show that $V(I_1) \cup V(I_2) = V(I_1 \cdot I_2)$. Hence the intersection of any finite number of open sets is open.

Proof. (a) $\emptyset = \operatorname{Spec} A - V((0))$ and $\operatorname{Spec} A = \operatorname{Spec} A - V((1))$, hence they are both open sets.

(b) $\mathfrak{p} \in \cap_i V(I_i) \implies \mathfrak{p} \supseteq I_i \forall i$. Recall that $\sum_i I_i$ is defined to be $\sum_i a_i$, where $a_i \in I_i$ and only finitely many of they are non-zero. (this is basically the smallest ideal that contains $\cup_i I_i$). Hence, $\mathfrak{p} \supseteq \sum_i I_i$. Then, we have $\cap_i V(I_i) \subseteq V(\sum_i I_i)$.
The reverse inclusion is easier. $\mathfrak{p} \supseteq \sum_i I_i \supseteq \cup_i I_i$. We can conclude that $\cap_i V(I_i) = V(\sum_i I_i)$ and consider the de morgen law, union of arbitrary collection of open set is still open.

(c) $\mathfrak{p} \in V(I_1) \cup V(I_2) \implies \mathfrak{p} \in V(I_1)$ or $\mathfrak{p} \in V(I_2)$. $\implies \mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$. In either case $\mathfrak{p} \supseteq I_1 I_2$ because $I_1 I_2$ is a subset in $I_1 \cap I_2$. $V(I_1) \cup V(I_2) \subseteq V(I_1 \cdot I_2)$.
For the reverse inclusion, $\mathfrak{p} \in V(I_1 \cdot I_2)$, we have $\mathfrak{p} \supseteq I_1 \cdot I_2$.
Want: $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$.
Suppose it is not the case, $\mathfrak{p} \not\supseteq I_1$ and $\mathfrak{p} \not\supseteq I_2$. Then there are $x_1 \in I_1$ and $x_2 \in I_2$ such that both are not in $\mathfrak{p}$. $x_1 x_2 \in I_1 \cdot I_2 \subseteq \mathfrak{p}$ but $\mathfrak{p}$ is prime ideal, contradiction.

■

Exercise?? 1.4.D Show that $\sqrt{I}$ is an ideal and

$$\sqrt{\sqrt{I}} = \sqrt{I}.$$

Show that prime ideals are radical

$$\sqrt{\mathfrak{p}} = \mathfrak{p}$$

■

Proof. Two element $x, y \in \sqrt{I}$. $x^n \in I$ and $y^m \in I$. $\forall a \in A, (ax)^n \in I$ and $(ax+by)^{n+m} \in I$. Hence, radical is an ideal.

$\sqrt{\mathfrak{a}} \supseteq \mathfrak{a}$ is trivially true. For the reverse inclusion, consider an element $x \in \sqrt{\sqrt{I}}$. $x^n \in \sqrt{I}$, then $(x^n)^m \in I \Longrightarrow x \in \sqrt{I}$.

$x^n \in \mathfrak{p} \Longrightarrow x \in \mathfrak{p}$ because $\mathfrak{p}$ is prime ideal. In fact we have $\sqrt{\mathfrak{q}^n} = \mathfrak{q}$ because

$$\mathfrak{p} \subseteq \sqrt{\mathfrak{q}^n} \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}.$$

■

Exercise?? 1.4.E Prove that taking radical commutes with finite intersection of ideals. ■

Exercise?? 1.4.F Prove that

$$\sqrt{\mathfrak{a}} = \bigcap_{\text{primes} \supseteq \mathfrak{a}} \mathfrak{p}$$

■

Exercise?? 1.4.G Describe the topological space $\mathbb{A}^1_k$ ■

Proof. The points of the topological space is$[(0)]$ and $[(x-a)], \forall a \in k$. It is almost the cofinite topology on k^1 but there is a new point $[(0)]$.

Claim: Every point $[\mathfrak{p}] \in \mathbb{A}^1_n$, every open neighborhood of $[\mathfrak{p}]$ contains $[(0)]$.

The claim holds for the case $\mathfrak{p} = (0)$. For $\mathfrak{p} = (x-a)$. An open neighborhood of $[\mathfrak{p}] = [(x-a)]$ is of the form $\mathbb{A}^1_k - V(S)$. which means $[\mathfrak{p}] \notin V(S) \Longrightarrow S \not\subset \mathfrak{p}$.

Want: $\mathbb{A}^1_k - V(S) \ni [(0)] \Longleftrightarrow S \not\subset (0)$, which is a direct fact from $S \not\subset \mathfrak{p}$ and $(0) \subset \mathfrak{p}$

Then we know every open neighborhood of every point in $\operatorname{Spec} k[x]$ contains (0). In other words, the closure of (0) is the whole affine line. ■

Exercise?? 1.4.H A ring morphism $\phi : B \longrightarrow A$ induces a map π on the spectrum. By showing that closed sets pull back to closed sets, show that $\pi = \phi^* : \operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ is a continuous map. Interpret Spec as a contravariant functor *Rings*$\longrightarrow$*Top*. ■

Proof. Assume S is a subset in B. Consider a closed set $V(S)$ in $\operatorname{Spec} B$, we will verify that $\pi^{-1}V(S)$ is also closed.

$$\pi^{-1}V(S) = \{[\mathfrak{p}] \in \operatorname{Spec} A : \pi(\mathfrak{p}) \supset S\} = \{[\mathfrak{p}] \in \operatorname{Spec} A : \phi^*(\mathfrak{p}) \supset S\}$$

Claim: $\pi^{-1}V(S) = V(\phi(S))$

In fact $\mathfrak{p} \supset \phi(S) \iff \phi^{-1}(\mathfrak{p}) \supset S$, therefore,

$$\pi^{-1}V(S) = \{[\mathfrak{p}] \in \operatorname{Spec} A : \phi^*(\mathfrak{p}) \supset S\} = \{[\mathfrak{p}] \in \operatorname{Spec} A : \mathfrak{p} \supset \phi(S)\} = V(\phi(S)).$$

The preimage of a closed set is always closed. π is continuous.

Now, we can interpret Spec as a contravariant functor form *Rings* to *Top*

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\operatorname{Spec}"] & A \arrow[d, "\operatorname{Spec}"] \\ \operatorname{Spec} B & \operatorname{Spec} A \arrow[l, "\operatorname{Spec}(\phi)=\pi"'] \end{tikzcd}$$

■

Exercise?? 1.4.I Suppose that $I, S \subset B$ are an ideal and multiplicative subset respectively.

(a) Show that $\operatorname{Spec} B/I$ is naturally a closed subset of $\operatorname{Spec} B$. If $S = 1, f, f^2, \ldots (f \in B)$, show that $\operatorname{Spec} S^{-1}B$ is naturally an open subset of $\operatorname{Spec} B$. Show that for arbitrary S, $\operatorname{Spec} S^{-1}B$ need not be open or closed. (Hint: $\operatorname{Spec}\mathbb{Q} \subset \operatorname{Spec}\mathbb{Z}$, or possibly Figure 3.5.)

(b) Show that the Zariski topology on $\operatorname{Spec} B/I$ (resp. $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion in $\operatorname{Spec} B$. (Hint: compare closed subsets.)

■

Proof. (a) Prime ideals in B/I are the prime ideals in B that contains I by 1.2.J.

$$\operatorname{Spec} B/I = \{[\mathfrak{p}] \in B : \mathfrak{p} \supset I\} = V(I)$$

which a closed set in $\operatorname{Spec} B$. For multiplicative set generated by one element $S := \{1, f, f^2, \ldots\}$. Recall 1.2.K. The prime ideals in $S^{-1}B$ are just the prime ideals in B that do not intersect S. Because $\mathfrak{p}$ is prime, and $1 \notin \mathfrak{p}$, $\mathfrak{p} \cap S \neq \emptyset \implies f^n \in \mathfrak{p} \implies f \in \mathfrak{p}$, thus $\mathfrak{p} \supset S - \{1\}$. Then we know

$$\operatorname{Spec} S^{-1}B = \operatorname{Spec} B - \{[\mathfrak{p}] \in \operatorname{Spec} B : \mathfrak{p} \ni f\} = D(f),$$

which is an open set.

In general, $\operatorname{Spec} S^{-1}B$ is neither open nor closed in $\operatorname{Spec} B$. For example consider $\operatorname{Spec}\mathbb{Q} \subset \operatorname{Spec}\mathbb{Z}$. $\operatorname{Spec}\mathbb{Q}$ corresponds to the generic point $[(0)]$ in $\mathfrak{p}$, which is neither open nor closed. Or we can consider the morphism $\operatorname{Spec}\mathbb{C}[x,y]_{(x,y)}$, which corresponds to prime ideals in $\mathbb{C}[x,y]$ that are contained in (x,y), which is neither open nor closed.

(b) Denote the projection $\phi : B \longrightarrow B/I$. Consider a closed set in $\operatorname{Spec} B/I$, call it $V_{B/I}(D)$, where D is a subset in B.

$$\begin{aligned} V_{B/I}(D) &= \{[\mathfrak{p}] \in B/I : \mathfrak{p} \supset D\} \\ &\cong \{[\mathfrak{q}] \in B : \mathfrak{q} \supset I \text{ and } \mathfrak{q} \supset \phi^{-1}D\} \\ &= \{[\mathfrak{q}] \in B : \mathfrak{q} \supset I + (\phi^{-1}D)\} \\ &= \operatorname{Spec}(B/I) \cap V_B(\phi^{-1}D). \end{aligned}$$

For the localization map $\iota: B \longrightarrow S^{-1}B$. Consider a closed set in $\operatorname{Spec} B$, call it $V_{S^{-1}B}(P)$, where P is a subset in $S^{-1}B$.

$$\begin{aligned} V_{S^{-1}B}(P) &= \{[\mathfrak{p}] \in \operatorname{Spec}(S^{-1}B) : \mathfrak{p} \supset P\} \\ &\cong \{[\mathfrak{q}] \in B : \mathfrak{q} \cap S = \emptyset, \text{ and } \mathfrak{q} \supset \iota^{-1}P\} \\ &= \{[\mathfrak{q}] \in B : \mathfrak{q} \cap S = \emptyset\} \cap \{[\mathfrak{q}] \in B : \mathfrak{q} \supset \iota^{-1}P\} \\ &= \operatorname{Spec} S^{-1}B \cap V_B(\iota^{-1}P). \end{aligned}$$

We have verified that Zariski topologies on $\operatorname{Spec} B/I$ and $\operatorname{Spec} S^{-1}B$ are the induced topology on by Zariski topology on $\operatorname{Spec} B$.

■

R In particular, if I is an ideal in the nilradical, the $\operatorname{Spec} B$ and $\operatorname{Spec} B/I$ are homeomorphic. $\iota: \operatorname{Spec} B/I \longrightarrow \operatorname{Spec} B$. It is continuous 1.4.H and bijective 1.2.R. A continuous bijection is not necessarily homeomorphism. But we have a theorem that **bijective continuous map is homeomorphism iff it is closed or open**. In our case, ι is closed.

Exercise?? 1.4.J Suppose $I \subset B$ is an ideal. Show that f vanishes on $V(I)$ if and only if $f \in \sqrt{I}$ ■

Proof. Assume f vanishes on $V(I)$, then

$$f \mod \mathfrak{p} = 0, \ \forall \mathfrak{p} \supset I,$$

which is equivalent to

$$f \in \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} = \sqrt{I}$$

■

Exercise?? 1.4.K Describe the topological space $\operatorname{Spec} k[x]_{(x)}$. ■

Proof. The only two prime ideals in $k[x]_{(x)}$ are (0) and $\mathfrak{m} := (x)k[x]_{(x)}$, of which $[(0)]$ is generic point and $[\mathfrak{m}]$ is closed point. ■

1.5 A base of the Zariski topology on schemes: Distinguished open sets

Exercise?? 1.5.A Show that the distinguished open sets form a base for the (Zariski) topology. (Hint: Given a subset $S \subset A$, show that the complement of $V(S)$ is $\cup_{f \in S} D(f)$.) ■

Proof. Recall one of the equivalent definitions of base of topology:

Base of a topology is a subset B of topology τ such that every open set in τ can be obtained by union of some elements of B.

Each open set in the Zariski topology is the complement of some closed set $V(S)$.

Claim: $\operatorname{Spec} A - V(S) = \cup_{f \in S} D(f)$

$[\mathfrak{p}] \in \cup_{f \in S} D(f) \Longleftrightarrow [\mathfrak{p}] \in D(f)$ for some $f \in S$.

$\Longleftrightarrow \exists f \in S, f \notin \mathfrak{p} \Longrightarrow \mathfrak{p} \not\supset S \Longleftrightarrow [\mathfrak{p}] \in \operatorname{Spec} A - V(S)$. ■

Exercise?? 1.5.B Suppose $f_i \in A$ as i runs over some index set J. Show that $\cup_{i\in J} D(f_i) = \operatorname{Spec} A$ if and only if $(\{f_i\}_{i\in J}) = A$, or equivalently and very usefully, if there are $\{a_i\}_{i\in J}$, all but finitely many 0, such that $\sum_{i\in J} a_i f_i = 1$. ■

Proof. "$\Longrightarrow$": We know by definition $D(f_i) = \operatorname{Spec} A - V((f_i))$.

$\cup_{i\in J} D(f_i) = A \Longrightarrow \cap V((f_i)) = \emptyset. \Longrightarrow V(\sum_i (f_i)) = \emptyset$.

In particular, for principal ideals $\sum_i (f_i) = (f_1, f_2, \ldots f_i, ..)$.

Any proper ideal is contained in some maximal ideal $\mathfrak{m}$, hence $\sum_i (f_i)$ can be a proper ideal, other wise there is at least one element $[\mathfrak{m}] \in V(\sum_i (f_i))$. We know $\sum_i (f_i) = (f_1, f_2, \ldots f_i, ..) = A = (1)$, which means there exists a summation such that $\sum_{i\in J} a_i f_i = 1$ with finitely many a_i nonzero.

"$\Longleftarrow$": We know $\sum_{i\in J} a_i f_i = 1$ and hence $\sum_{i\in J} (f_i) = A$. Then we know

$$\operatorname{Spec} A - \cup_i D(f_i) = \cap_i (\operatorname{Spec} A - D(f_i)) = \cap_i V((f_i)) = V(\sum_{i\in J} (f_i)) = \emptyset.$$

■

Exercise?? 1.5.C Show that if $\operatorname{Spec} A$ is an infinite union of distinguished open sets $\cup_{i\in J} D(f_j)$, then in fact it is a union of a finite number of these, i.e., there is a finite subset J' so that $\operatorname{Spec} A = \cup_{j\in J'} D(f_j)$. ■

Proof. Recall 1.5.B,

$$\cup_{i\in J} D(f_i) = \operatorname{Spec} A \Longleftrightarrow \sum_{j\in J} a_j f_i = 1.$$

There are only finitely many of a_j nonzero. We can choose a finite subset $J' \subset J$ such that $\sum_{j\in J'} a_j f_i = 1$. Then we have $\cup_{i\in J'} D(f_i) = \operatorname{Spec} A$. ■

Exercise?? 1.5.D Show that $D(f) \cap D(g) = D(fg)$. ■

Proof. Pick $[\mathfrak{p}] \in D(f) \cap D(g)$, then $[\mathfrak{p}] \in D(f)$ and $D(g)$ $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, then by $\mathfrak{p}$ being prime, we have $fg \notin \mathfrak{p} \Longrightarrow [\mathfrak{p}] \in D(fg)$. $D(f) \cap D(g) \subset D(fg)$.

For the reverse inclusion, pick $\mathfrak{q} \in D(fg)$, then $fg \notin \mathfrak{p}$, again by primality, we have $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, hence $[\mathfrak{p}] \in D(f) \cap D(g)$. ■

Exercise?? 1.5.E Show that $D(f) \subset D(g)$ if and only if $f^n \in (g)$ for some $n \geq 1$, or equivalently, if and only if the image of g is an invertible element of A_f. ■

Proof. $D(f) \subset D(g) \Longleftrightarrow V((f)) \supset V((g))$. $V((g))$ is the closed set where g vanishes. The derived inclusion means is equivalent to saying that f vanishes on $V((g))$. By Exercise 1.4.J, we know it is equivalent to

$$f \in \sqrt{(g)}$$

$$\Longleftrightarrow (g) \cap S_f \neq \emptyset,$$

where $S_f := \{1, f, f^2, \ldots, \}$.

$$\Longleftrightarrow S_f^{-1}(g) = A_f$$

which is equivalent to "the image of g is invertible in A_f". ■

Exercise?? 1.5.F Show that $D(f)=\emptyset$ if and only if $f\in\mathfrak{N}(A)$

Proof. $D(f)=\emptyset \Longrightarrow \forall\mathfrak{p}\in \operatorname{Spec} A$, $f \mod \mathfrak{p}=0 \Longrightarrow f\in\cap_{\text{prime}}\mathfrak{p}=\mathfrak{N}(A)$.

For $\Longleftarrow$ direction, consider an element $g\in\mathfrak{N}(A)$, $(0)\subset(g)\subset\mathfrak{N}(A)$.

$$\operatorname{Spec} A=V((0))\supset V((g))\supset V\left(\sqrt{(0)}\right)=\operatorname{Spec} A$$

$D(g)=\operatorname{Spec} A-V((g))=\emptyset$. ■

1.6 Topological (and Noetherian) properties

Exercise?? 1.6.A $A=A_1\times A_2\times\cdots\times A_n$, describe $\operatorname{Spec} A_1\coprod\operatorname{Spec} A_2\coprod\cdots\coprod\operatorname{Spec} A_n\longrightarrow\operatorname{Spec} A$ as a homeomorphism for which each $\operatorname{Spec} A_i$ is mapped onto a distinguished open subset $D(f_i)$ of $\operatorname{Spec} A$. Thus $\operatorname{Spec}\prod_{i=1}^n A_i=\coprod_{i=1}^n\operatorname{Spec} A_i$ as topological spaces.

Proof. We can induct on n and reduce to the spacial case $n=2$. Consider $A=A_1\times A_2$.

<u>Claim</u>: $\mathfrak{a}\in A$ is an ideal iff $\mathfrak{a}=\mathfrak{a}_1\times\mathfrak{a}_2$, for some ideals $\mathfrak{a}_1\in A_1$ and $\mathfrak{a}_2\in A_2$.

"$\Longleftarrow$" direction is trivial. Let's consider the "$\Longrightarrow$" direction. $\mathfrak{a}$ is an ideal in A, therefore $\mathfrak{a}=S\times T$, where S,T are general subset in A_1,A_2. $\mathfrak{a}$ should be closed under multiplication and linear summation of $A_1\times 0$, which implies that S is an ideal in A_1, similarly we can prove that T is an ideal in A_2.

<u>Claim</u>: $\mathfrak{p}\in A$ is a prime ideal iff $\mathfrak{p}=\mathfrak{p}_1\times A_2$ or $\mathfrak{p}=A_1\times\mathfrak{p}_2$.

We already know that the ideal $\mathfrak{p}=\mathfrak{q}_1\times\mathfrak{q}_2$ and the quotient ring $A/\mathfrak{p}=A_1/\mathfrak{q}_1\times A_2/\mathfrak{q}_2$ is integral domain. Recall that $(1,0)\times(0,1)=(0,0)$, We must have $\mathfrak{q}_1=A_2$ or $\mathfrak{q}_2=A_2$. The reverse direction is trivial.

Now, set $f_1=(1,0)$ and $f_2=(0,1)$ and denote the two canonical projections $\phi_i:A_1\times A_2\longrightarrow A_i$. We have

$$\begin{aligned} D(f_1)&=\{[\mathfrak{p}]\in\operatorname{Spec} A: f_1=(1,0)\notin\mathfrak{p}\}\\ &=\{\mathfrak{p}_1\times A_2:\mathfrak{p}_1\in\operatorname{Spec} A_1\}\end{aligned}$$

and

$$\begin{aligned}\pi_1:=\operatorname{Spec}(\phi_1):\operatorname{Spec} A_i&\longrightarrow\operatorname{Spec} A\\ [\mathfrak{p}_1]&\longmapsto\phi_1^{-1}(\mathfrak{p}_1)=\mathfrak{p}_1\times A_2.\\ V(S)&\longrightarrow V(S)\times A_2\end{aligned}$$

Hence,

$$D(f_i)=\pi_i(\operatorname{Spec} A_i).$$

Each π_i is continuous bijective. Furthermore, π_i is closed map as shown above. Then π_i is homeomorphism. Recall the universal property of coproduct (disjoint union in *Top*), there is a unique morphism from $\operatorname{Spec} A_1\coprod\operatorname{Spec} A_2$ to $\operatorname{Spec} A$, which is denoted by $\pi_1\coprod\pi_2$. This morphism is also continuous bijective and closed.

These proof can be generalized to finite product and finite coproduct without much difficulty. ■

Theorem 1.6.1 $\operatorname{Spec} A$ is not connected if and only if A is isomorphic to the product of nonzero rings A_1 and A_2.

Proof. One direction has been proved in 1.6.A. Now we just focus on the other direction. Follow the hint, we call a pair of idempotents a_1, a_2 such that $a_1^2 = a_1$, $a_2^2 = a_2$, $a_1 + a_2 = 1$ and $a_1 a_2 = 0$ **complimentary idempotents**.

We can see there are a pair of complimentary idempotents in $A_1 \times A_2$, $(1,0)$ and $(0,1)$.

Claim: If $\operatorname{Spec} A$ not connected, there is a pair of complimentary idempotents in A.

A is not connected, it can decompose into disjoint union of two open sets or equivalently two closed sets. Suppose they are $V(I_1)$ and $V(I_2)$, we have

$$V(I_1) \cup V(I_2) = V(I_1 I_2) = \operatorname{Spec} A$$

and

$$V(I_1) \cap V(I_2) = V(I_1 + I_2) = \emptyset.$$

At most can find a pair of element $f_1 \in I_1$ and $f_2 \in I_2$, such that $f_1 + f_2 = 1$ and $f_1 f_2$ is nilpotent.

Lemma: Every nontrivial idempotent in $A/\mathfrak{N}$ lifts to a unique nontrivial idempotents in A.

With this lemma, assume e_1, e_2 are the nontrivial idempotents in A. We have $(e_1) \cap (e_2) = (e_1) \cdot (e_2)$ and $(e_1) + (e_2) = (1)$, we can use the Chinese remainder theorem

$$A = A/((e_1) \cdot (e_2)) \cong \frac{A}{(e_1)} \times \frac{A}{(e_2)} \cong (e_2) \times (e_1)$$

Now, we prove the lemma. $f \neq 0,1$ $f(1-f)$ is nilpotent. $\exists n \geq 0, f^n(1-f)^n = 0$. Because f^n and $(1-f)^n$ are coprime, we can still use the Chinese remainder theorem

$$A \cong \frac{A}{(f^n)} \times \frac{A}{(1-f)^n}.$$

The preimage of $(1,0)$ is unique and idempotent. (This step in fact gives us the desired proof that A should be a product of rings but we want to reduce it to the simplest case)

Or simpler, start with $x + y = 1$ and xy nilpotent. $(xy)^m = 0$.

$$1 = (x+y)^{2m} = \underbrace{x^{2m} + \cdots + \binom{2m}{m+1} x^{m+1} y^{m-1}}_{e_1} + \underbrace{\binom{2m}{m} x^m y^m + \cdots + y^{2m}}_{e_2}$$

and these e_1, e_2 satisfies

$$e_1 e_2 = 0, \quad e_1^2 = e_1, \quad e_2^2 = e_2.$$

■

Exercise?? 1.6.B

(a) Show that in an irreducible topological space, any nonempty open set is dense. (For this reason, you will see that unlike in the classical topology, in the Zariski topology, nonempty open sets are all "huge".)

(b) If X is a topological space, and Z (with the subspace topology) is an irreducible subset, then the closure $\overline{Z}$ in X is irreducible as well.

Proof. (a) Recall that in topology $\overline{U \cup V} = \overline{U} \cup \overline{V}$. (The analogous statement for intersection of closure is not true.) In an irreducible topological space X, consider an open set U, if $\overline{U}$ is not the whole space X, we have $V := X - \overline{U} \neq \emptyset$, then we have $\overline{U} \cup \overline{V} = X$, contradiction.

Also an open subset U in a irreducible space X is irreducible with the subset topology. If $U = A \cup B$ with A, B closed in U, then taking the closure in X yields $X = \overline{U} = \overline{A} \cup \overline{B}$ because U is dense in X. This forces $\overline{A} = X$. By assumption A is closed in U and this means $A = \overline{A} \cap U = X \cap U = U$. It follows that U is irreducible in subspace topology on it.
In the special case $X = \operatorname{Spec} A$. We only need to verify that every nonempty distinguished open set is dense in Zariski's topology.
Consider a point $[\mathfrak{p}] \in V((f)) = \operatorname{Spec} A - D(f)$, then without loss of generality, there is an open neighborhood of the form $D(g) \ni [\mathfrak{p}]$.
Claim: $D(f) \cap D(g) \neq \emptyset$.
Assume $D(f) \cap D(g) = \emptyset$, then $\operatorname{Spec} A - D(f) \cap D(g) = (\operatorname{Spec} A - D(f)) \cup (\operatorname{Spec} A - D(g)) = V((f)) \cup V((g)) = \operatorname{Spec} A$ contradicting to our assumption that $\operatorname{Spec} A$ is irreducible.
A by-product of this fact is that every product of non-nilpotent is non-nilpotent.

(b) If the closure of Z in X is reducible, $\overline{Z} = U \cup V$, where U, V are closed set with induced topology in $\overline{Z}$ (They are not necessarily closed in X. Denote the closure of U, V in X by $\overline{U}$, $\overline{V}$. We can write $Z = (\overline{U} \cap Z) \cup (\overline{V} \cap Z)$, where $\overline{U} \cap Z$ and $\overline{V} \cap Z$ are closed set because Z is endowed with subset topology.

∎

Exercise?? 1.6.C If A is an integral domain, show that $\operatorname{Spec} A$ is irreducible. (Hint: pay attention to the generic point $[(0)]$.)

Proof. Assume $\operatorname{Spec} A$ is reducible and can be written as $\operatorname{Spec} A = V(I_1) \cup V(I_2)$. $\operatorname{Spec} A = V(I_1) \cup V(I_2) = V(I_1 I_2)$, which means all $I_1 I_2$ vanishes on each point $[\mathfrak{p}]$, i.e. $I_1 I_2 \subseteq \cap \mathfrak{p} = \mathfrak{N}$. $V(I_1)$ and $V(I_2)$ are proper closed subsets, none of I_1, I_2 is contained in $\mathfrak{N}$. In particular they are non-zero ideals. There exist non-nilpotents $x, y \in I_1, I_2$ such that xy is nilpotent. In particular $x^n y^n = 0$, where x^n, y^n are non-zero zero-divisors. Which contradicts the hypothesis that A is integral domain. ∎

Exercise?? 1.6.D Show that an irreducible topological space is connected.

Proof. Assume a topological space X is not connected. Then $X = U \coprod V$ where U, V are both closed and open. Hence we also have $X = U \cup V$, which means X is reducible. ∎

Exercise?? 1.6.E Give (with proof!) an example of a ring A where $\operatorname{Spec} A$ is connected but reducible.

Proof. Follow the hint, consider the ring

$$A := \mathbb{C}[x, y]/(xy)$$

$\operatorname{Spec} A$ is connected because A is not of the form $A_1 \times A_2$. The only idempotents in A are 0 and 1.
It is reducible. Indeed,

$$\operatorname{Spec} A = V((x)) \cup V((y)),$$

each is proper closed subset. ∎

Exercise?? 1.6.F

(a) Suppose $I = (wz - xy, wy - x^2, xz - y^2) \subset k[w,x,y,z]$. Show that $\operatorname{Spec} k[w,x,y,z]/I$ is irreducible, by showing that $k[w,x,y,z]/I$ is an integral domain.

(b) Note that the generators of the ideal of part (a) may be rewritten as the equations ensuring that

$$\operatorname{rank}\begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} \leq 1$$

i.e., as the determinants of submatrices. Generalize this to $2 \times n$ variables.

Proof. (a) Follow the hint, consider an morphism

$$\begin{aligned} k[w,x,y,z] &\longrightarrow k[a,b] \\ w \longmapsto a^3, y &\longmapsto ab^2, \\ x \longmapsto a^2b, z &\longmapsto b^3. \end{aligned}$$

The kernel is just I, and the image is $k[a^3, a^2b, ab^2, b^3]$. Thus, we have established the isomorphism

$$k[w,x,y,z]/I \cong k[a^3, a^2b, ab^2, b^3] \subset k[a,b]$$

a subring of integral domain is always an integral domain.

(b) Generalize this to $2 \times n$ matrices

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \leq 1$$

and the ideal I is generated by the determinants of submatrices.

$$\begin{aligned} k[x_0, \ldots, x_n]/I &\cong k[a^n, a^{n-1}b, \ldots, b^n] \\ x_i &\longmapsto a^{n-i}b^i \end{aligned}$$

$A := k[x_0, \ldots, x_n]/I$ is again integral domain and hence $\operatorname{Spec} A$ is irreducible ■

Exercise?? 1.6.G

(a) Show that $\operatorname{Spec} A$ is quasicompact.

(b) Show that in general $\operatorname{Spec} A$ can have nonquasicompact open sets.

Proof. (a) Consider an open cover of $\operatorname{Spec} A = \cup_{i \in J} U_i$. Recall the definition of base of topology, each open set U_i is a union of $\cup_{j \in J^{(i)}} D(f_j^{(i)})$.

$$\operatorname{Spec} A = \cup_{i \in J} \cup_{j \in J^{(i)}} D(f_j^{(i)})$$

by Exercise 1.5.C, there is a finite subcover

$$\operatorname{Spec} A = \cup_{(i,j) \in J'} D(f_j^{(i)}),$$

where J' is a finite index subset in $\cup_{i \in J} J^{(i)}$. Then there are finitely many $i \in J$ s.t., $(i,j) \in J'$, denote it by I'. Then because $U_i \supset D(f_j^{(i)})$, we have a finite subcover of the initial cover

$$\operatorname{Spec} A = \cup_{i \in I'} U_i.$$

(b) Consider the ring $A := k[x_1, x_2, \ldots]$ and the maximal ideal $\mathfrak{m} = (x_1, x_2, \ldots)$. There is an open subset

$$U = \operatorname{Spec} A - V(\mathfrak{m})$$

with open cover

$$U = \cup_{i \in \mathbb{Z}} D(x_i) = \{[\mathfrak{p}] \in \operatorname{Spec} A \text{ s.t. at least one of } x_i \notin \mathfrak{p}\}$$

There is no finite subcover because $[\mathfrak{p}_i] := [(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots)] \in U$ but $[\mathfrak{p}_i] \notin U - D(x_i)$. ■

Exercise?? 1.6.H

(a) If X is a topological space that is a finite union of quasicompact spaces, show that X is quasicompact.

(b) Show that every closed subset of a quasicompact topological space is quasicompact. ■

Proof. (a) Assume $X = \cup_{0 \le i \le n} X_i$, every open cover of X is in particular an open cover of X_i. Then we can select finite subcovers such that they cover each X_i. It is also a finite sub cover that covers X.

(b) $Y \subset X$ is closed subspace of X. Given any cover of $Y = \cup_i U_i$. For each U_i we can find open subset $V_i \subset X$ such that $V_i \cap Y = U_i$, which is a consequence of subset topology. Then we adjoint the complement of Y. $\cup_i V_i \cup (X - Y)$ is an open cover of X, it has a finite subcover because X is quasicompact. This finite subcover of $\cup_i V_i \cup (X - Y)$ induces a finite subcover of $\cup_i U_i$.

■

Exercise?? 1.6.I Show that the closed points of $\operatorname{Spec} A$ correspond to the maximal ideals. ■

Proof. $\{[\mathfrak{p}]\}$ is a closed subset of $\operatorname{Spec} A$, which means $[\mathfrak{p}] = V(S)$, where S is a subset of A.

$V(S)$ consists of only one point. By definition it is equivalent to "$\mathfrak{p}$ is the only prime such that $\mathfrak{p} \supset S$". (S) the the ideal generated by S, we have $V(S) = V((S))$. Each ideal is contained in some maximal ideals. The only possibility is that (S) is maximal and $\mathfrak{p} = (S)$. ■

Exercise?? 1.6.J

(a) Suppose that k is a field, and A is a finitely generated k-algebra. Show that closed points of $\operatorname{Spec} A$ are dense, by showing that if $f \in A$, and $D(f)$ is a nonempty (distinguished) open subset of $\operatorname{Spec} A$, then $D(f)$ contains a closed point of $\operatorname{Spec} A$.

(b) Show that if A is a k-algebra that is not finitely generated the closed points need not be dense. ■

Proof. (a) We already showed that $\operatorname{Spec} A_f \cong D(f) \subset \operatorname{Spec} A$. A every nonzero ring has a maximal ideal, There is a closed point in $\operatorname{Spec} A_f$, we have to prove that the pull back of this closed point is also a closed point in $\operatorname{Spec} A$.

Assume $\mathfrak{p} \subset A$ such that $\mathfrak{p}A_f$ is maximal ideal in A_f. $\mathfrak{p}A_f$ contains no units $f/1 \notin \mathfrak{p}A_f \implies f \notin \mathfrak{p}$. We have the isomorphism

$$\frac{A_f}{\mathfrak{p}A_f} \cong (A/\mathfrak{p})_f.$$

Notice that A_f is also finitely generated k-algebra, by Nullstellensatz, $(A_f/\mathfrak{p}_f)$ is the finite field extension of k, hence so is $(A/\mathfrak{p})_f$.

The k-integral domain $A/\mathfrak{p}$ is a subring of $(A/\mathfrak{p})_f$. (It is a k-vector space of a finite dimensional k-vector space). Then we know k-integral domain $A/\mathfrak{p}$ must be a finite dimensional k-vector space. Then it must be a field by exercise 1.2.G. $\mathfrak{p}$ is also a maximal ideal and $[\mathfrak{p}] \in D(f)$. Hence, the closed points is dense in $\operatorname{Spec} A$.

(b) Consider the ring $A := k[x]_{(x)}$, it has only one maximal ideal $\mathfrak{m} = (x)k[x]_{(x)}$ and one generic point $[(0)]$. The closed point $[\mathfrak{m}]$ is not dense because $[0]$ is open. This ring is not finitely generated. Consider the set $1/(1+x), 1/(1+x+x^2), 1/(1+x+x^2+x^3), \ldots$, each can not be expressed as a polynomial. (In fact $k[x]_{(x)}$ is not even a finitely generated $k[x]$-algebra.)

■

Exercise?? 1.6.K Suppose k is an algebraically closed field, and $A = k[x_1, \ldots, x_n]/I$ is a finitely generated k-algebra with $\mathfrak{N}(A) = \{0\}$. Consider the set $X = \operatorname{Spec} A$ as a subset of $\mathbb{A}_k^n$. The space $\mathbb{A}_k^n$ contains the "classical" points k^n. Show that functions on X are determined by their values on the closed points. ■

Proof. Suppose f, g are two distinct functions on X.

$f - g \neq 0$, i.e., $f - g \notin \mathfrak{N}(A) = \{0\}$. functions are determined by their value at points. Want: Functions are determined by their value at closed points.

Claim: If f and g are distinct functions, then $f-g$ is nowhere zero on an open set $D(f-g)$.

By definition, $D(f-g)$ is an the set of points on which $f-g$ does not vanish. It is not empty because $f-g \neq 0$ implies $f-g$ is not nilpotents in our setting.

In particular, A is finitely generated k-algebra, the closed point is dense. $D(f-g)$ contains at least one closed point because closed points is dense in $\operatorname{Spec} A$. Hence we know distinct function f, g must have different values on at least one closed point.

Without the hypothesis $\mathfrak{N}(A) = \{0\}$, we can not argue like in topology " functions agreeing on a dense subset must agree on the whole set". ■

Exercise?? 1.6.L If $X = \operatorname{Spec} A$, show that $[\mathfrak{q}]$ is a specialization of $[\mathfrak{p}]$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. Hence show that $V(\mathfrak{p}) = \overline{\{[\mathfrak{p}]\}}$. ■

Proof. By definition, "$[\mathfrak{q}]$ is a specialization of $[\mathfrak{p}]$" means

$[\mathfrak{q}] \in \overline{\{[\mathfrak{p}]\}}$.

$\iff$ every open neighborhood of $[\mathfrak{q}]$ contains $[\mathfrak{p}]$.

$\iff$ every distinguished open $D(f)$ containing $[\mathfrak{q}]$ contains $[\mathfrak{p}]$.

$\iff$ every f not vanishing on $[\mathfrak{q}]$ neither vanishes on $[\mathfrak{p}]$.

$\iff A - \mathfrak{q} \subset A - \mathfrak{p}$.

$\iff \mathfrak{q} \supset \mathfrak{p}$.

$\iff [\mathfrak{q}] \in V(\mathfrak{p})$. ■

Exercise?? 1.6.M Verify that $[(y-x^2)] \in \mathbb{A}^2$ is a generic point for $V(y-x^2)$ ▪

Proof. $\overline{\{[(y-x^2)]\}} = V(y-x^2)$ as $(y-x^2)$ is prime ideal. ∎

Exercise?? 1.6.N Suppose $[\mathfrak{p}]$ is a generic point for the closed subset K. Show that it is "near every point $[\mathfrak{q}]$ of K "(every neighborhood of $[\mathfrak{q}]$ contains $[\mathfrak{p}]$), and "not near any point $[\mathfrak{n}]$ not in K" (there is a neighborhood of $[\mathfrak{n}]$ not containing $[\mathfrak{p}]$). ▪

Proof. $\overline{\{[\mathfrak{p}]\}} = K$ topologically means every open neighborhood of $[\mathfrak{q}] \in K$ contains $[\mathfrak{p}]$. On the other hand assume, $[\mathfrak{n}] \notin K$ hence $[\mathfrak{n}] \notin \overline{\{[\mathfrak{p}]\}}$ topologically means there is an open neighborhood of $[\mathfrak{n}]$ that does not contain $[\mathfrak{p}]$. ∎

Exercise?? 1.6.O (EVERY TOPOLOGICAL SPACE IS THE UNION OF IRREDUCIBLE COMPONENTS). Show that every point x of a topological space X is contained in an irreducible component of X. ▪

Proof. The point $\{x\}$ as a subset is irreducible.

Claim: Every irreducible subset $Z \subset X$ is contained in an irreducible component.

Let $Z \subset X$ be irreducible. Consider the set Σ of irreducible subsets $Z \subset Z_\alpha \subset X$. Note that Σ is nonempty since $Z \in \Sigma$. There is a partial ordering on Σ coming from inclusion: $\alpha \leq \alpha' \iff Z_\alpha \subset Z_{\alpha'}$. Choose a maximal totally ordered subset $\Sigma' \subset \Sigma$, and let $Z' = \cup_{\alpha \in \Sigma'} Z_\alpha$. We claim that Z' is irreducible. Namely, suppose that $Z' = T_1 \cup T_2$ is a union of two closed subsets of Z'. For each $\alpha \in \Sigma'$ we have either $Z_\alpha \subset T_1$ or $Z_\alpha \subset T_2$, by irreducibility of Z_α. Suppose that for some $\alpha_0 \in \Sigma'$ we have $Z_{\alpha_0} \subset T_2$ (say, if not we're done anyway). Then, since Σ' is totally ordered we see immediately that $Z_\alpha \subset T_2$ for all $\alpha \in \Sigma'$. Hence $Z = T_2$.

Unlike connected component, a point x does not determine a unique connected component. ∎

Exercise?? 1.6.P Show that $\mathbb{A}^2_{\mathbb{C}}$ is a Noetherian topological space: any decreasing sequence of closed subsets of $\mathbb{A}^2_{\mathbb{C}} =$ Spec $\mathbb{C}[x,y]$ must eventually stabilize. Note that it can take arbitrarily long to stabilize. Show that $\mathbb{C}^2$ with the classical topology is not a Noetherian topological space. ▪

Proof. Consider a chain of closed subsets $V(S_0) \supset V(S_1) \supset V(S_2) \supset \cdots$. If it does not stabilize, w.l.o.g, we can assume each inclusion is strict inclusion. Assume there is a strictly descending chain $V(I_0) \supsetneq V(I_1) \supsetneq V(I_2) \supsetneq \cdots$. W.l.o.g. we assume each I_i is radical ideal.

$\forall \mathfrak{q} \supset I_1, \mathfrak{q} \supset I_0$, and $\exists \mathfrak{p} \supset I_0, \mathfrak{p} \not\supset I_1$

$I_1 = \sqrt{I_1} \supset I_0$ and $I_1 \not\subset \sqrt{I_0} = I_0$. $I_1 \supsetneq I_0$. We have a strictly ascending chain of radical ideals.

But this is impossible for $\mathbb{C}[x,y]$. We intensionally prove this without using Noetherian property of $\mathbb{C}[x,y]$. Our proof is basically the special case of Hilbert Basis theorem.

Claim: each ideal $\mathfrak{a}$ in $A := \mathbb{C}[x,y]$ is finitely generated.

Pick an element $\mathfrak{a} \ni f = \sum a_{n,m} x^n y^m$. Denote the maximal power of x by N and maximal power of y by M.

Construct an A-module

$$M := \oplus_{i \leq N, j \leq M} A x^i y^j$$

it is a finite dimensional $\mathbb{C}$-vector space.

$$M \cap \mathfrak{a} + (f) = \mathfrak{a}$$

$\supset$ is trivial. Consider an element $\mathfrak{a} \ni g = bx^{k_1}y^{l_1} + cx^{k_2}y^{l_2}..., b, c \neq 0$ with $k_i > N$ or $l_i > M$ and k_1 is maximal among k_i, l_2 is maximal among l_i

$$g - \left(\frac{b}{a}x^{k_1-n}y^{l_1-m} + \frac{c}{a}x^{k_2-n}y^{l_2-m}\right) f = 0 + \text{terms with lower powers in } x,\, y.$$

We can induct on k_1, l_2 until they finally goes down to N, M. $g \in (f) + M \cap \mathfrak{a}$.

Then we claim: if the each ideal is finitely generated, then every ascending chain of ideal stabilize

Let $I_0 \subseteq I_1 \subseteq I_2 \subseteq ...$ Want:show that $\exists n_0, I_n = I_{n_0} \forall n \geq n_0$. Define $I' := \cup_n I_n$. We know that every ideal of A is finitely generated. Then Assume I' to be finitely generated by r elements $\{x_1, ..., x_r\}$, with $x_j \in I_{n_j}$. Choose $n_0 = \max\{n_1, ..., n_r\}$, then we have $x_1, ..., x_r \in I_{n_0} \implies I = I_{n_0}$. $\implies I_n = I_{n_0}, \forall n \geq n_0$.

$\mathbb{C}^2$ under traditional topology is not Noetherian, we can consider the a chain of balls

$$B_n := \left\{|x^2| + |y^2| \leq \frac{1}{n^2}\right\},$$

which gives a strictly descending chain of closed sets. ■

Exercise?? 1.6.Q Show that every connected component of a topological space X is the union of irreducible components of X. Show that any subset of X that is simultaneously open and closed must be the union of some of the connected components of X. If X is a Noetherian topological space, show that the union of any subset of the connected components of X is always open and closed in X. ■

Proof. "Every connected component of a topological space X is the union of irreducible components of X:" By 1.6.D, a irreducible component is connected, hence it is contained in some connected component. But every point in X is contained in some irreducible component. A point x in a connected component $C \subset X$. Every x is contained in some irreducible component Z_x 1.6.O. Every connected set containing x is a subset of the connected component C. Hence, $Z_x \subset C$ and $C = \cup_{x \in C} Z_x$.

"A clopen subset in X must be some union of connected component": If C is a clopen subset in X, then $C \subset \cup_{x \in C} C_x$, where C_x is the connected component in X that contains x. (This inclusion is true for arbitrary subset C). For the reverse inclusion, consider an element $y \in C_x$. If $y \notin C$, because C is closed, there is an open neighborhood U_y of y such that $U_y \cap C = \emptyset$. $U_y \cap C_x$ is open, and $A := \cup_y U_y \cap C_x$ is open, $B := C_x \cap C$ is also open, and $A \cap B = \emptyset, A \cup B = C_x$, which contradicts the fact that C_x is connected. Then we know every point y of C_x must be contained in C. A clopen set is the union of the connected components that intersect C.

The reverse statement is not true for general topological space, but when the topological space is in addition Noetherian, we have "union of arbitrary connected components in X is clopen": Any union of open set is always open, we only need to prove that $\cup_i C_i$ is closed. Consider $X - C_0 \supset X - C_0 \cup C_1 \supset ...$, is a descending chain of closed subsets, it will stabilize after n_0, In other words, $\cup_i C_i = \cup_{0 \leq i \leq n_0} C_i$. A finite union of closed sets in closed. Done.

■

Exercise?? 1.6.R Show that a ring A is Noetherian if and only if every ideal of A is finitely generated. ■

Proof. In fact we have,

(Lemma) The following characterizations are equivalent:

(a) A satisfies the **ascending chain condition on ideals (ACC)** (All the sequence $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$ stabilizes, i.e. $\exists n_0$ s.t. $\mathfrak{a}_n = \mathfrak{a}_{n_0} \forall n \geq 0$)

(b) Every ideal of A is finitely generated.

(c) $\{$ideals in $A\}$ satisfies the **maximal property**: i.e. Every subset contains a maximal element. That is : For any nonempty collection S of ideals in A, $\exists \mathfrak{a} \in S$ s.t. $\forall \mathfrak{b} \in S \Longrightarrow \mathfrak{b} \not\supset \mathfrak{a}$

(a)$\Longrightarrow$(b). Let $\mathfrak{a}$ be an ideal. we may assume that $\mathfrak{a}$ is **NOT** finitely generated. Inductively construct $x_1, x_2, x_3 \ldots \in \mathfrak{a}$ such that $(x_1) \neq 0$ and $\mathfrak{a} \supsetneqq (x_1, x_2) \supsetneqq (x_1)$ an also $\mathfrak{a} \supsetneqq (x_1, x_2, x_3) \supsetneqq (x_1, x_2)$, but then this sequence contradict the **ACC**.

(a)$\Longrightarrow$(c). Let $\emptyset \neq S \subseteq \{$ideals in $A\}$. If S violates the maximal property, then start from arbitrary ideal $\mathfrak{a}_1$, we can find $\mathfrak{a}_1 \subsetneqq \mathfrak{a}_2 \in S$. Similarly, we can find $\mathfrak{a}_{j+1} \supsetneqq \mathfrak{a}_j, \forall j \in \mathbb{Z}_{\geq 0}$ by the countable choice axiom. Then the ACC fails.

(c)$\Longrightarrow$(a). If ACC fails, $\exists \mathfrak{a}_1 \subsetneqq \mathfrak{a}_2 \subsetneqq \ldots$ Take $S := \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \ldots\}$. Then S violates maximal property.

(b)$\Longrightarrow$(a). Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$ Want:show that $\exists n_0, \mathfrak{a}_n = \mathfrak{a}_{n_0} \forall n \geq n_0$. Define $\mathfrak{a} := \cup_n \mathfrak{a}_n$. We know that every ideal of A is finitely generated. Then $\mathfrak{a}$ is also finitely generated by assumption (b). Then Assume it to be finitely generated by r elements $\{x_1, \ldots, x_r\}$, with $x_j \in \mathfrak{a}_{n_j}$. Choose $n_0 = \max\{n_1, \ldots, n_r\}$, then we have $x_1, \ldots, x_r \in \mathfrak{a}_{n_0} \Longrightarrow \mathfrak{a} = \mathfrak{a}_{n_0}. \Longrightarrow \mathfrak{a}_n = \mathfrak{a}_{n_0}, \forall n \geq n_0$. ■

Exercise?? 1.6.S If A is Noetherian, show that $\operatorname{Spec} A$ is a Noetherian topological space. Describe a ring A such that $\operatorname{Spec} A$ is not a Noetherian topological space. ■

Proof. Let $(V_i)_{i \in \mathbb{N}}$ be a descending chain of closed subsets of $\operatorname{Spec} A$. For every $i \in \mathbb{N}$, let $I_i \in A$ be an ideal such that $V(I_i) = V_i$. We conclude that $V(I_i) \supset V(I_{i+1})$ for all $i \in \mathbb{N}$. W.l.o.g. we assume each I_i is radical ideal.

$\forall \mathfrak{q} \supset I_{i+1}, \mathfrak{q} \supset I_i$

$I_{i+1} = \sqrt{I_{i+1}} \supset I_i$. We have an ascending chain of radical ideals.

By the Noetherian property of A, this chain of ideals must stabilize. Hence the descending chain of closed subsets also stabilize, therefore $\operatorname{Spec} A$ is a Noetherian topological space.

The converse is not true. Consider the ring $A := k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$ for a field k. Let $\mathfrak{p} \subset A$ be prime idal. Then the ideal $(x_1, x_2, \ldots)$ is contained in $\mathfrak{p}$. But the ideal $(x_1, x_2, \ldots)$ is already maximal. Hence $\operatorname{Spec} A$ is only one point and trivially Noetherian topological space. But A is not Noetherian because $(x_1, x_2, ..)$ is not finitely generated.

As for an example of non-Noetherian topological space, we can consider the ring $A := k[x_1, x_2, \ldots]$. A is not a Noetherian ring and there is a strictly ascending chain of ideals $(x_1) \subsetneqq (x_1, x_2) \subsetneqq \ldots$. It give rise to a strictly descending chain of closed subsets

$$V(x_1) \supsetneqq V(x_1, x_2) \supsetneqq \cdots$$

■

Exercise?? 1.6.T Show that every open subset of a Noetherian topological space is quasicompact. Hence if A is Noetherian, every open subset of $\operatorname{Spec} A$ is quasicompact. ■

Proof. Let $Y \subset X$ be an open subset of X with the induced topology on Y, Then Y is also Noetherian. Consider an open covering $\cup_{i \in I} V_i = Y$ of Y. is it true that every cover has countable subcover?. Without loss of generality, we assume $I = \mathbb{N}$ and define

$$U_n = \bigcup_{0 \leq i \leq n} V_i.$$

U_n form an ascending chain of open subsets. Since Y is Noetherian, we conclude the chain stabilizes and we find the finite subcover. ■

R A closed subset of Noetherian space is always quasicompact, because a Noetherian space is itself quasicompact.

Exercise?? 1.6.U Show that if M is a Noetherian A-module, then any submodule of M is a finitely generated A-module. ■

Proof. The proof is just identical to the different characterization of Noetherian rings.

If the submodule M' of M is not finitely generated, then we can inductively construct $M' \supsetneq Ax_1$, $M' \supsetneq Ax_1 \oplus Ax_2$.... But this violate the ascending chain condition on M. ■

Exercise?? 1.6.V If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is exact, show that M' and M'' are Noetherian if and only if M is Noetherian. ■

Proof. "$\Longrightarrow$" Use ACC. Let $N_1 \subseteq N_2 \subseteq \ldots$ be submodules of M. Want: show that $\exists n_0 : (n \geq n_0) \Longrightarrow N_n = N_{n_0}$. Consider $N_j'' :=$ Image of N_j in M''. $N_1'' \subseteq N_2'' \subseteq \ldots$ By ACC of M'', $N_{n_0}'' = N_n'' \forall n \geq n_0$. Do the same for $N_j' := M' \cap N_j$ $(M' \hookrightarrow M)$
Need: if $N_i \subseteq N_j \subseteq M$ and $N_i'' = N_j'', N_i' = N_j'$, then $N_i = N_j$. (Five Lemma)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_i' & \longrightarrow & N_i & \longrightarrow & N_i'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & N_j' & \longrightarrow & N_j & \longrightarrow & N_j'' & \longrightarrow & 0
\end{array}$$

For the $\Longleftarrow$ direction, M is Noetherian. Because every ascending chain in M' can be identified as ascending chain in M, M' is Noetherian. Also for every ascending chain of submodules in M'', it's preimage is an ascending chain of submodule in M, hence the preimage chain will stabilize, hence the chain in M'' would also stabilize. ■

Exercise?? 1.6.W Show that if A is a Noetherian ring, then $A^{\oplus n}$ is a Noetherian A-module. ■

Proof. When we regard A as an A-module, the submodule of A is just the ideals in A. Hence A is a Noetherian ring iff A is a Noetherian A-module. Then $A \oplus A$ is Noetherian A-module by 1.6.V because

$$0 \longrightarrow A \longrightarrow A \oplus A \longrightarrow A \longrightarrow 0$$

and then we can induct on n, by

$$0 \longrightarrow A^{\oplus n-1} \longrightarrow A^{\oplus n} \longrightarrow A \longrightarrow 0$$

■

Exercise?? 1.6.X Show that if A is a Noetherian ring and M is a finitely generated A-module, then M is a Noetherian module. Hence by Exercise 1.6.U, any submodule of a finitely generated module over a Noetherian ring is finitely generated. ■

Proof. Suppose M is generated by $\{x_1,\ldots,x_n\}$ We always have a SES

$$0 \longrightarrow Ker(\varphi) \longrightarrow A^n \xrightarrow{\varphi} M \longrightarrow 0,$$

then apply 1.6.V. ■

1.7 The function $I(\cdot)$, taking subsets of $\operatorname{Spec} A$ to ideals of A

Exercise?? 1.7.A Let $A = k[x,y]$. If $S = \{[(y)],[(x,y-1)]\}$, then $I(S)$ consists of those polynomials vanishing on the y-axis, and at the point $(0,1)$. Give generators for this ideal. ■

Proof. By definition, $I(S) = \cap_{[\mathfrak{p}]\in S}\mathfrak{p} = (y)\cap(x,y-1)$. Hence $I(S)$ can be identified as polynomials in $k[x,y]$ that vanishes on the both y-axis and $(0,1)$. The generators of this ideal are $\{xy, y(y-1)\}$ ■

Exercise?? 1.7.B Suppose $S \subset \mathbb{A}^3_{\mathbb{C}}$ is the union of the three axes. Give generators for the ideal $I(S)$. We will see in Chapter 12 that this ideal is not generated by less than three elements. ■

Proof. S geometrically is the union of three axes. $I(S)$ is the polynomial that vanishes on all the three axis. The points in S are of the form $[(0)],[(x,y)],[(y,z)],[(x,z)],[(x-a,y,z)],[(x,y-b,z)],[(x,y,z-c)]$. The union of corresponding ideals would be $(x,y)\cap(y,z)\cap(x,z)$. The ideal is finitely generated by (xy,yz,zx). The inclusion $(x,y)\cap(y,z)\cap(x,z) \supset (xy,yz,zx)$ is trivial. For the reverse inclusion. Consider an element $g \in (x,y)\cap(y,z)\cap(x,z)$, g can't contain terms like x,y,z,x^n,y^n,z^n because they are not element in the intersection. Then the lowest degree terms of g should be xy,yz,xz, and all other terms can be generated by xy,yz,zx. ■

Exercise?? 1.7.C Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S. ■

Proof. "$V(I(S)) \supset \overline{S}$": By definition, $V(I(S))$ is closed set. And $S \subset V(I(S))$, because $[\mathfrak{p}] \in S, I(S) = \cap_{[\mathfrak{q}]\in S}\mathfrak{q}\ I(S) \mod \mathfrak{p} = 0, \Longrightarrow I(S) \subset \mathfrak{p} \Longrightarrow [\mathfrak{p}] \in V(I(S))$. Closure $\overline{S}$ is the smallest closed set that contains S, therefore $\overline{S} \subset V(I(S))$.

"$V(I(S)) \subset \overline{S}$": We need to verify that open neighborhood of each point in $V(I(S))$ intersects with S. Consider a point $[\mathfrak{q}] \in V(I(S))$, we have $\mathfrak{q} \supset I(S)$. Assume a non-empty distinguished open $D(f)$ contains $[\mathfrak{q}] \Longleftrightarrow f \notin \mathfrak{q} \Longrightarrow f \notin I(S) \Longrightarrow f$ does not vanish on every point in S, $\exists[\mathfrak{n}] \in S$, s.t. $f \notin [\mathfrak{n}] \Longleftrightarrow D(f) \ni [\mathfrak{n}]$, therefore $D(f)\cap V(I(S)) \neq \emptyset$. Hence $V(I(S)) \subset \overline{S}$. ■

Exercise?? 1.7.D Prove that if $J \subset A$ is an ideal, then $I(V(J)) = \sqrt{J}$. ▪

Proof. Suppose $J \subset A$ is an ideal. By definition, $I(V(J))$ is the set of functions that vanish on $V(J)$, $I(V(J)) = \cap_{[\mathfrak{p}] \in V(J)} \mathfrak{p}$. Also recall that $[\mathfrak{p}] \in V(J) \iff \mathfrak{p} \supset J$, we have

$$I(V(J)) = \cap_{[\mathfrak{p}] \in V(J)} \mathfrak{p} = \cap_{\mathfrak{p} \supset J} \mathfrak{p} = \sqrt{J}.$$

■

Exercise?? 1.7.E Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between irreducible closed subsets of $\operatorname{Spec} A$ and prime ideals of A. From this conclude that in $\operatorname{Spec} A$ there is a bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$ (where a point determines an irreducible closed subset by taking the closure). Hence each irreducible closed subset of $\operatorname{Spec} A$ has precisely one generic point any irreducible closed subset Z can be written uniquely as $\overline{\{z\}}$. ▪

Proof. "S irreducible closed $\implies I(S)$ is prime":

Assume $I(S)$ is not prime. $\iff \exists x, y \notin I(S)$ but $xy \in I(S)$, which means x, y each does not vanish on every point in S but xy vanishes on each point in S.

$$V((x)) \not\supset S$$

$$V((y)) \not\supset S$$

but

$$V((xy)) = V((x)) \cup V((y)) \supset S$$

Then

$$S = (V((x)) \cap S) \cup (V((y)) \cap S),$$

where $(V((x)) \cap S)$ and $(V((y)) \cap S)$ are non-empty and closed. Contradiction.

"$\mathfrak{p}$ is prime $\implies V(\mathfrak{p})$ is irreducible closed."

Assume $V(\mathfrak{p})$ is reducible, it can be written as union of two non-empty closed subsets, $V(\mathfrak{p}) = W \cup Z$. Assume $\mathfrak{a}, \mathfrak{b}$ radical ideals such that $W = V(\mathfrak{a})$, $Z = V(\mathfrak{b})$, we have $\mathfrak{a} \supsetneq \mathfrak{p}$ and $\mathfrak{b} \supsetneq \mathfrak{p}$

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{p})$$

By the theorem about bijection between closed sets and radical ideals in A.

$$\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} = \sqrt{\mathfrak{a}\mathfrak{b}} = \mathfrak{p}.$$

But $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{p}$ and $\mathfrak{a} \subset \mathfrak{p}$. Then we find elements $x \in \mathfrak{a} \backslash \mathfrak{p}$, $y \in \mathfrak{b} \backslash \mathfrak{p}$, but $xy \in \mathfrak{p}$. Contradiction.

This bijection is also inclusion reversing (just consider them as ordinary closed subsets and radical ideals).

■

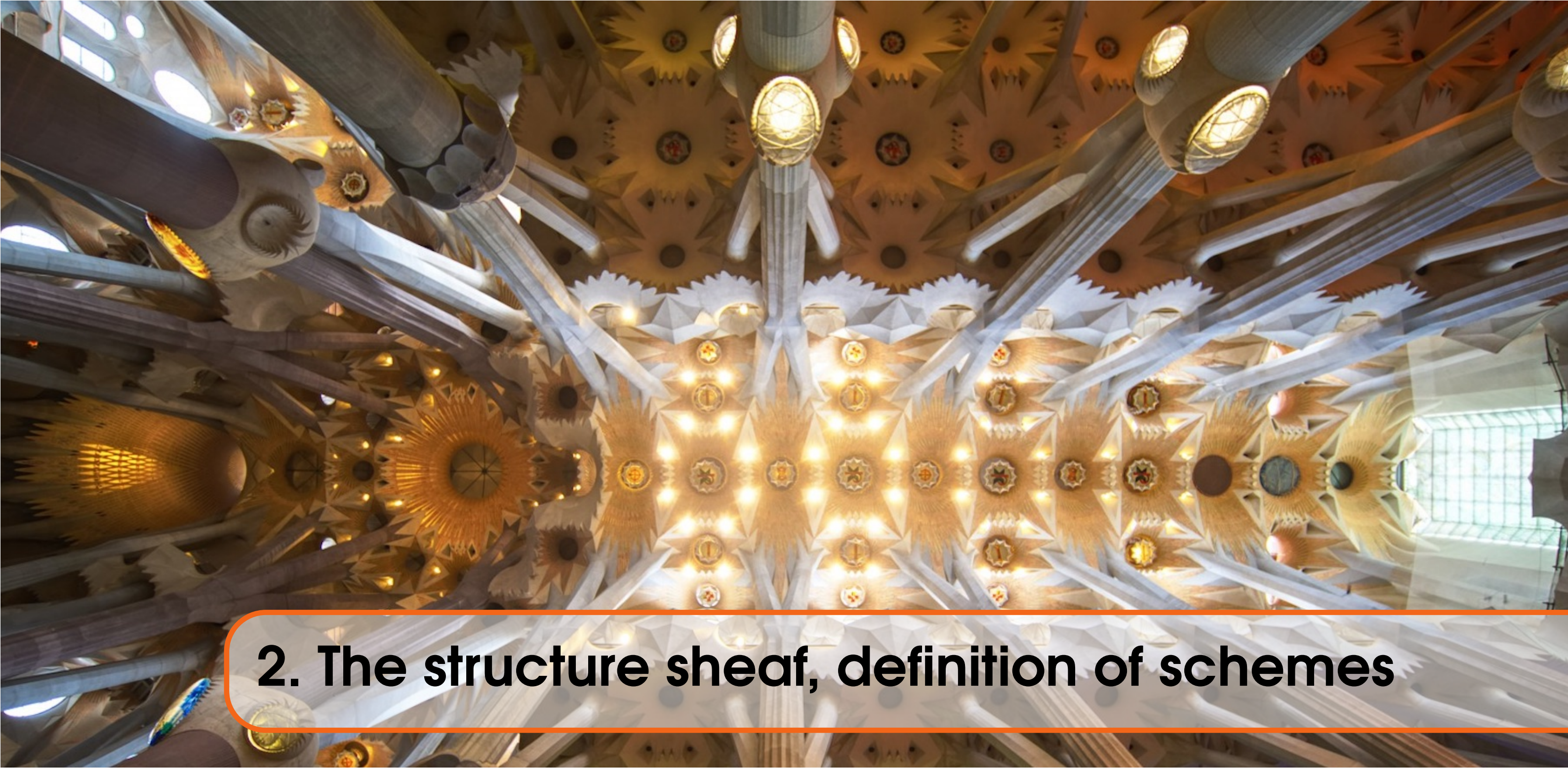
Exercise?? 1.7.F A prime ideal of a ring A is a **minimal prime ideal** (or more simply, **minimal prime**) if it is minimal with respect to inclusion. (For example, the only minimal prime of $k[x,y]$ is (0).) If A is any ring, show that the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal prime ideals of A. In particular, $\operatorname{Spec} A$ is irreducible if and only if A has only one minimal prime ideal; this generalizes Exercise 1.6.C. ▪

Proof. $\mathfrak{n}$ is minimal prime
$\iff$ there is no strictly smaller prime $\mathfrak{p} \subset \mathfrak{n}$.
$\iff$ There is no strictly bigger irreducible closed subset that contains $V(\mathfrak{n})$. By the bijection in 1.7.E.
$\iff$ $V(\mathfrak{n})$ is maximal irreducible closed subset (irreducible component)

In particular $\operatorname{Spec} A$ is irreducible iff $\operatorname{Spec} A$ is itself the unique irreducible component. Then $I(\operatorname{Spec} A)$ is prime and contained in any other irreducible sets. ■

Exercise?? 1.7.G What are the minimal prime ideals of $A := k[x,y]/(xy)$ (where k is a field)? ▪

Proof. Geometrically, three are two irreducible components of $\operatorname{Spec} A$, the x-axis and y-axis, corresponding to (y) and (x) respectively. ■



2. The structure sheaf, definition of schemes

2.1 The structure sheaf of an affine scheme

Definition 2.1.1 Define $\mathscr{O}_{\operatorname{Spec} A}(D(f))$ to be localization of A at the multiplicative set S, where

$$S := \{\text{All functions that do not vanish outside } V(f) \text{ (Do not vanish on } D(f))\}.$$

(i.e., those $g \in A$ such that $V(g) \subset V(f)$ or equivalently $D(f) \subset D(g)$)

In particular, $\mathscr{O}_{\operatorname{Spec} A}(\emptyset) = \{0\}$, where localize at the multiplicative set of functions g such that $V(g) \subset \operatorname{Spec} A$. This multiplicative set includes 0, hence the localization is $\{0\}$ ring.

Exercise?? 2.1.A Show that the natural map $A_f \longrightarrow \mathscr{O}_{\operatorname{Spec} A}(D(f))$ is an isomorphism.

Proof. In particular, $S_f := \{1, f, f^2, \ldots\}$ is a multiplicative subset of the multiplicative set T in the definition of $\mathscr{O}_{\operatorname{Spec} A}(D(f))$, where

$$T := \{\text{All functions that do not vanish outside } V(f) \text{ (Do not vanish on } D(f))\}.$$

$S_f \subset T$. There is a natural homomorphism

$$A \xrightarrow{S_f^{-1}} A_f \xrightarrow{\tilde{T}^{-1}} \mathscr{O}_{\operatorname{Spec} A}(D(f)),$$

where we have denoted the image of T in A_f by $\tilde{T}$. $g \in S \iff D(f) \subset D(g)$
$\iff T^{-1}g$ is invertible in A_f by Exercise 1.5.E.
$\iff \tilde{T} \subseteq A_f^{\times}$
$\iff \tilde{T}^{-1}$ is an isomorphism. $A_f \cong \mathscr{O}_{\operatorname{Spec} A}(D(f))$. ■

Exercise?? 2.1.B Prove the base identity axiom for any distinguished open $D(f)$.

Proof. Consider the $D(f) = \cup_{i\in I} D(f_i)$. We already showed that $\operatorname{Spec} A_f \cong D(f)$ as topological spaces 1.4.I. If $D(f) = \cup_{i\in I} D(f_i) = \cup_{i\in I} D(f_i) \cap D(f) = \cup_{i\in I} D(f_i f)$.

$D(f_i f) \cong \operatorname{Spec} A_{ff_i}$ A_{ff_i} is the localization of A_f at the image of f_i. $D(f_i f)$ corresponds to the point $[\mathfrak{q}] \in \operatorname{Spec} A_f$ such that $\mathfrak{q} \notin \frac{f_i}{1}$.

Then $D(f) = \cup_{i\in I} D(f_i) \subset \operatorname{Spec} A \iff \operatorname{Spec} A_f = \cup_{i\in I} D(f_i/1)$

$\mathscr{O}_{\operatorname{Spec} A}(D(f)) \cong A_f = \mathscr{O}_{\operatorname{Spec} A_f}(\operatorname{Spec} A_f)$. The function restricts to 0 on each $D(f_i)$ iff its restriction to $D(f_i/1)$ vanishes.

Then the problem reduces to the proved case $D(f) = \operatorname{Spec} A$. ■

Exercise?? 2.1.C Alter this argument appropriately to show base gluability for any distinguished open $D(f)$. ■

Proof. Again, we regard $D(f) \cong \operatorname{Spec} A_f$.

Then $D(f) = \cup_{i\in I} D(f_i) \subset \operatorname{Spec} A \iff \operatorname{Spec} A_f = \cup_{i\in I} D(f_i/1)$.

$\mathscr{O}_{\operatorname{Spec} A}(D(f)) \cong A_f = \mathscr{O}_{\operatorname{Spec} A_f}(\operatorname{Spec} A_f)$.

The base gluability follows from the special case we have proved for $\operatorname{Spec} A = D(f)$. ■

Exercise?? 2.1.D Suppose M is an A-module. Show that the following construction describes a sheaf M on the distinguished base. Define $\tilde{M}(D(f))$ to be the localization of M at the multiplicative set of all functions that do not vanish outside of $V(f)$. Define restriction maps $\operatorname{res}_{D(f),D(g)}$ in the analogous way to $\mathscr{O}_{\operatorname{Spec} A}$. Show that this defines a sheaf on the distinguished base, and hence a sheaf on $\operatorname{Spec} A$. Then show that this is an $\mathscr{O}_{\operatorname{Spec} A}$-module. ■

Proof. Define $\tilde{M}_{\operatorname{Spec} A}(D(f))$ to be localization of M at the multiplicative set S, where

$$S := \{\text{All functions that do not vanish outside } V(f) \text{ (Do not vanish on } D(f)\}.$$

<u>Claim</u>: $\tilde{M}(D(f)) \cong M_f$.

In particular, $S_f := \{1, f, f^2, \ldots\}$ is a multiplicative subset of the multiplicative set S.

There is a natural homomorphism

$$M \xrightarrow{S_f^{-1}} M_f \xrightarrow{\tilde{S}^{-1}} \tilde{M}(D(f)),$$

where we have denoted the image of S in A_f by $\tilde{S}$. $g \in S \iff D(f) \subset D(g)$

$\iff T^{-1}g$ is invertible in A_f by Exercise 1.5.E.

$\iff \tilde{S} \subseteq A_f^{\times}$

$\iff \tilde{S}^{-1}$ is an isomorphism. $M_f \cong \tilde{M}(D(f))$.

Define the restriction map:

$$\operatorname{res}_{D(f),D(g)} : \tilde{M}(D(f)) \longrightarrow \tilde{M}(D(g))$$

it is the further localization of modules. The restriction does generate a presheaf on distinguished base because of the functorial property of localization.

Base identity axiom: Consider the special case where $D(f) = \operatorname{Spec} A$. Then $\operatorname{Spec} A = \cup_{i\in I} D(f_i)$, there is a finite subcover, say $\cup_{i=1,\ldots,n} D(f_i) = \operatorname{Spec} A$, i.e., $(f_1, \ldots, f_n) = A$.

Suppose $m \in M = \tilde{M}(\operatorname{Spec} A)$ such that $\operatorname{res}_{\operatorname{Spec} A, D(f_i)} m = 0$. Then $f_i^{l_i} m = 0, \forall i$. We can choose $N \geq \max\{l_i\}$ such that $f_i^N m = 0$. Also $(f_1^N, ..., f_n^N) = A$ ($\cup_i D(f_i^N) = A$), then we know exist $r_i \in A$ s.t. $\sum_i r_i f_i^N = 1$

$$m = \sum_i (r_i f_i^N) m = 0.$$

The restriction map is injective, hence the base identity axiom holds for $D(f) = \operatorname{Spec} A$.

As for general $D(f)$, we can replace A by A_f so that $D(f) = \operatorname{Spec} A_f$. The problem reduces to what we have proved.

Base gluability axiom: Suppose again $\cup_{i \in I} D(f_i) = \operatorname{Spec} A$. We assume in addition I is a finite index set. We have elements

$$\frac{m_i}{f_i^{l_i}} \in M_{f_i} = \tilde{M}(D(f_i))$$

Letting $g_i = f_i^{l_i}$ to simplify the notation. m_i/g_i and m_j/g_j agree when restricted to intersection $D(f_i) \cap D(f_j) = D(g_i) \cap D(g_j) = D(g_i g_j)$, which means

$$(g_i g_j)^{n_{ij}} (g_j m_i - g_i m_j) = 0 \in A.$$

By setting $N = \max\{n_{ij}\}$, we have

$$(g_i g_j)^N (g_j m_i - g_i m_j) = 0 \in A.$$

Set $b_i = m_i g_i^N$ and $h_i = g_j^{N+1}$, the overlap condition simplifies to

$$b_i h_j - b_j h_i = 0.$$

Note that $\cup_i D(f_i) = \cup_i D(h_i) = \operatorname{Spec} A$ implies that $\exists r_i \in A$ s.t., $\sum_i r_i h_i = 1$. Define

$$r = \sum_i r_i b_i.$$

Then

$$r h_j = \sum_i r_i b_i h_j = \sum_i r_i h_i b_j = b_j$$

$$g_j^N (g_j r - m_j) = g_j^N (f_j^{l_j} r - m_j) = 0$$

$$\operatorname{res}_{\operatorname{Spec} A, D(f_j)} r = \frac{r}{1} = \frac{m_j}{f_j^{l_j}} \in \tilde{M}(D(f_j)), \forall j$$

For infinite index set I, choose a finite subset $\{1, .., n\} \subset I$ with $\operatorname{Spec} A = \cup_i D(f_i)$, we can construct r as above. We will show that for $z \in I - \{1, ..., n\}$, r restricts to desired element $m_z \in M_{f_z}$. Repeat the entire process for $\{1, ..., n, z\}$ in place of $\{1, .., n\}$ and we get a $r' \in M$ which restricts to $m_i \in M_{f_i}$ for $i = 1, .., n$. Then by identity axiom $r = r'$. Hence r restricts to m_z as desired.

We have checked the base gluability in the special case $D(f) = \operatorname{Spec} A$. For general $D(f)$, we can replace A with A_f so that $D(f) = \operatorname{Spec} A_f$ and the problem reduces to the special case we have solved above.

We have up to now proved that $\tilde{M}$ is a sheaf on the distinguished base. We can recover form it a sheaf on $\operatorname{Spec} A$.

Observe that $\tilde{M}(D(f)) = M_f$ which is automatically an A_f-module ($\mathscr{O}_{\operatorname{Spec} A}$-module). $\tilde{M}$ is $\mathscr{O}_{\operatorname{Spec} A}(D(f))$-module on the distinguished base. And this would also extend to a $\mathscr{O}_{\operatorname{Spec} A}$-module. ■

2.2 Visualizing schemes II: nilpotents

2.3 Definition of schemes

Exercise?? 2.3.A Describe a bijection between the isomorphisms $\operatorname{Spec} A \longrightarrow \operatorname{Spec} A'$ and the ring isomorphisms $A' \longrightarrow A$. Hint: the hardest part is to show that if an isomorphism $\pi : \operatorname{Spec} A \longrightarrow \operatorname{Spec} A'$ induces an isomorphism $\pi^{\#} : A' \longrightarrow A$, which in turn induces an isomorphism $\rho : \operatorname{Spec} A \longrightarrow \operatorname{Spec} A'$, then $\pi = \rho$. First show this on the level of points; this is (surprisingly) the trickiest part. Then show $\pi = \rho$ as maps of topological spaces. Finally, to show $\pi = \rho$ on the level of structure sheaves, use the distinguished base. Feel free to use insights from later in this section, but be careful to avoid circular arguments. Even struggling with this exercise and failing (until reading later sections) will be helpful. ▪

Proof. For simplicity, we replace the notion of topological space $\operatorname{Spec} A$, $\operatorname{Spec} A'$ by X, X' respectively. Recall what it means by an isomorphism of schemes. An isomorphism of ringed space

$$\pi : X \cong X' (\text{homeomorphism})$$

and

$$\pi_* \mathscr{O}_X \cong \mathscr{O}_{X'} (\text{sheaf isomorphism}).$$

$A = \mathscr{O}_X(\pi^{-1}(X')) = \pi_* \mathscr{O}_X(X') = \mathscr{O}_{X'}(X') = A'$. It induces an morphism

$$\pi^{\#} : A' \longrightarrow A$$

$$\mathscr{O}_{X'}(X') \ni a' \mapsto a' \in \pi_* \mathscr{O}_X(X').$$

This $\pi^{\#}$ in turn would induces a isomorphism $\rho : \operatorname{Spec} A \longrightarrow \operatorname{Spec} A'$.

$$\rho : [\mathfrak{p}] \mapsto [(\pi^{\#})^{-1} \mathfrak{p}]$$

We need to show that $\pi = \rho$.

$$\pi : [\mathfrak{p}] \mapsto ?$$

TO BE ADDED ∎

Exercise?? 2.3.B Suppose $f \in A$. Show that under the identification of $D(f)$ in $\operatorname{Spec} A$ with $\operatorname{Spec} A_f$, there is a natural isomorphism of ringed spaces

$$(D(f), \mathscr{O}_{\operatorname{Spec} A}|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathscr{O}_{\operatorname{Spec} A_f}).$$

Hint: notice that distinguished open sets of $\operatorname{Spec} A_f$ are already distinguished open sets in $\operatorname{Spec} A$. ▪

Proof. By exercise 1.4.I, we know $\operatorname{Spec} A_f \subset \operatorname{Spec} A$ and the Zariski topology on $\operatorname{Spec} A_f$ is the subspace topology induced by Zariski topology of $\operatorname{Spec} A$. Hence $D(f) \cong \operatorname{Spec} A_f$ as topological spaces. In particular, the distinguished base of $\operatorname{Spec} A_f$ is induced by distinguished base of $\operatorname{Spec} A$ intersects with $D(f)$. Specifically,

$$D(f) \cap D(f_i) \cong D(f_i/1) \subset \operatorname{Spec} A_f$$

On the other hand, by exercise 2.1.A, We know $A_f \cong \mathscr{O}_{\operatorname{Spec} A}(D(f))$. We need to verify that $\mathscr{O}_{\operatorname{Spec} A_f} \cong \mathscr{O}_{\operatorname{Spec} A}|_{D(f)}$

$$\begin{aligned}\mathscr{O}_{\operatorname{Spec} A}|_{D(f)}(D(f)\cap D(g)) = \mathscr{O}_{\operatorname{Spec} A}(D(f)\cap D(g)) = \mathscr{O}_{\operatorname{Spec} A}(D(fg)) \cong A_{fg} \\ = (A_f)_{g/1} = \mathscr{O}_{\operatorname{Spec} A_f}(D(g/1)).\end{aligned}$$

Because f is invertible in A_f. $D(g/f^n) = D(g/1)\cap D(1/f^n) = D(g/1)\cap D(1/1) = D(g/1)$. (All distinguished open in $\operatorname{Spec} A_f$ are of the form.) We know the two sheaves coincide on the distinguished base, which means they are isomorphic. ■

Exercise?? 2.3.C If X is a scheme, and U is any open subset, prove that $(U, \mathscr{O}_X|_U)$ is also a scheme.

Proof. Recall the definition of scheme, at any point $x \in X$, there is an open neighborhood V such that $(V, \mathscr{O}_X|_V)$ is an affine scheme. Assume $(V, \mathscr{O}_X|_V) \cong (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$. This isomorphism says the subset topology on V coincides with the Zariski topology. $U\cap V$ is also an open set in $\operatorname{Spec} A$.

<u>Claim</u>: There exists $x \in D(f) \subset U\cap V$ such that $(D(f), \mathscr{O}_X|_{D(f)})$ is an affine scheme.

Such $D(f)$ always exists because distinguished open form a base of Zariski topology and $D(f)$ can be regarded as open set in X. On the other hand, $\mathscr{O}_X|_{D(f)} = (\mathscr{O}_X|_U)|_{D(f)} = (\mathscr{O}_X|_V)|_{D(f)} \cong (\mathscr{O}_{\operatorname{Spec} A})|_{D(f)} \cong \mathscr{O}_{\operatorname{Spec} A_f}$. The last isomorphism by 2.3.B

In $(U, \mathscr{O}_X|_U)$, there is an open $U \supset D(f) \ni x$ such that

$$(D(f), (\mathscr{O}_X|_U)|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathscr{O}_{\operatorname{Spec} A_f}).$$

which means $(U, \mathscr{O}_X|_U)$ is also a scheme. ■

Exercise?? 2.3.D Show that if X is a scheme, then the affine open sets form a base for the Zariski topology.

Proof. $x \in X$, and U is an open neighborhood of x. We need to verify that there is an affine open set $x \in W \subset U$.

In Exercise 2.3.C, we proved that $(U, \mathscr{O}_X(U))$ is open subscheme of $(X, \mathscr{O}_X)$. Then for each $x \in U$, there is an affine open set W **in** U. But the Zariski topology of U is just the subset topology induced by the Zariski topology of X. We know W is also a open set in X. Finally, $\mathscr{O}_X|_W = (\mathscr{O}_X|_U)|_W$. W is an affine open in X iff W is an affine open in X. ■

Exercise?? 2.3.E The disjoint union of schemes is defined as you would expect: it is the disjoint union of sets, with the expected topology, with the expected sheaf.

(a) Show that the disjoint union of a finite number of affine schemes is also an affine scheme.

(b) (a first example of a non-affine scheme) Show that an infinite disjoint union of (nonempty) affine schemes is not an affine scheme.

Proof. (a) In Exercise 1.6.A, we see that for finite index set I:

$$\coprod_{i\in I} \operatorname{Spec} A_i \cong \operatorname{Spec} \prod_i A_i$$

and we need to describe the structure sheaf and verify that

$$\mathscr{O}_{\coprod_i \operatorname{Spec} A_i} \cong \mathscr{O}_{\operatorname{Spec} \prod_i A_i}.$$

Consider the inclusion map $\iota_i : \operatorname{Spec} A_i \hookrightarrow \coprod_i \operatorname{Spec} A_i$

$$\mathscr{O}_{\coprod_i \operatorname{Spec} A_i} := \prod_i (\iota_i)_* \mathscr{O}_{\operatorname{Spec} A_i}$$

For $U = \coprod_i U_i \subset \coprod \operatorname{Spec} A_i$,

$$\left(\prod_i (\iota_i)_* \mathscr{O}_{\operatorname{Spec} A_i}\right)(U) = \prod_i \mathscr{O}_{\operatorname{Spec} A_i}(\iota_i^{-1} U) = \prod_i \mathscr{O}_{\operatorname{Spec} A_i}(U_i).$$

On the other hand, the structure sheaf $\mathscr{O}_{\operatorname{Spec} \prod_i A_i}$ has the down to earth definition. We need to verify that they agree on $D(f)$ for $f = (f_1, .. f_i, ..) \in \prod_i A_i$. It is not hard to check

$$D(f) = \coprod_i D(f_i).$$

$$\mathscr{O}_{\operatorname{Spec} \prod_i A_i}(D(f)) = (\prod_i A_i)_f \stackrel{?}{=} \prod_i (A_i)_{f_i} = \prod_i \mathscr{O}_{\operatorname{Spec} A_i}(D(f_i)) = \mathscr{O}_{\coprod_i \operatorname{Spec} A_i}(D(f))$$

The equality under question mark need to be checked. We just verify the universal property directly.

$$\begin{tikzcd}
R & & \\
& (A \times B)_{(f,g)} \arrow[ul, dashed, "\exists u"'] & B_g \arrow[l, "\beta"] \arrow[ull, "\phi"'] \\
& A_f \arrow[u, "\alpha"'] \arrow[uul, "\psi"] &
\end{tikzcd}$$

where

$$\alpha : \frac{a}{f^n} \mapsto \frac{(a, g^n)}{(f,g)^n}$$

$$\beta : \frac{b}{g^m} \mapsto \frac{(f^m, b)}{(f,g)^m}$$

Given ϕ and ψ, there is a unique morphism u to make diagram commute.

$$u : \frac{(a,b)}{(f,g)^k} \mapsto \psi\left(\frac{a}{f^k}\right)\phi\left(\frac{b}{g^k}\right)$$

(b) Exercise 1.6.G shows that the topological space of affine scheme is quasicompact but the infinite disjoint union of affine scheme has infinite many connected components hence can't be quasicompact. For example the cover $\cup_i \operatorname{Spec} A_i$ does not have finite subcover.

■

Exercise?? 2.3.F Show that the stalk of $\mathscr{O}_{\operatorname{Spec} A}$ at the point $[\mathfrak{p}]$ is the local ring $A_{\mathfrak{p}}$. ■

Proof.

$$\mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]} = \varinjlim_{[\mathfrak{p}]\in U} \mathscr{O}_{\operatorname{Spec} A}(U)$$

without loss of generality, we can assume each U is a distinguished open $D(f)$ for some f. $D(f) \ni [\mathfrak{p}]$ iff $f \notin \mathfrak{p}$ or equivalently $f \in S_{\mathfrak{p}} = A - \mathfrak{p}$.

Claim:

$$\mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]} = \varinjlim_{[\mathfrak{p}]\in D(f)} \mathscr{O}_{\operatorname{Spec} A}(D(f)) = A_{\mathfrak{p}},$$

The index category is now a subset of distinguished base which contain $[\mathfrak{p}]$ and the morphism are inclusions.

$$\begin{tikzcd}
A_{\mathfrak{p}} \arrow[rrr, dashed] & & & R \\
& \mathscr{O}_{\operatorname{Spec} A}(D(f)) = A_f \arrow[ul, "l_f"] \arrow[urr, "\phi_f"'] \arrow[r] & \mathscr{O}_{\operatorname{Spec} A}(D(g)) = A_g \arrow[ull, "l_g"] \arrow[ur, "\phi_g"'] &
\end{tikzcd}$$

If there is a ring R with all morphisms from A_f to it. The image of all $f \in S_{\mathfrak{p}}$ is invertible in R, equivalently, $S_{\mathfrak{p}} \subset R^{\times}$. Then there is a morphism from A to R by universal property of localization.

Equivalently we have general formula

$$S^{-1}M = \varinjlim_{f\in S} M_f.$$

The index category is now the partially ordered set S, with $f \geq g$ iff $f \in \sqrt{(g)}$, see 1.5.E ■

Exercise?? 2.3.G

(a) If f is a function on a locally ringed space X, show that the subset of X where f vanishes is closed. (Hint: show that if f is a function on a ringed space X, show that the subset of X where the germ of f is invertible is open.)

(b) Show that if f is a function on a locally ringed space that vanishes nowhere, then f is invertible.

■

Proof. (a) f is a function on a ringed space X ($f \in \Gamma(X, \mathscr{O}_X)$). A germ of f at p is denoted f_p, assume f_p is invertible. $\exists g_p \in \mathscr{O}_{X,p}$ such that $f_p g_p = 1 \in \mathscr{O}_{X,p}$, which means $\exists U \ni p$ open such that $(f|_U)(g|_U) = 1 \in \mathscr{O}_X(U)$. Then all f_q is invertible for $q \in U$. This means the set in X where germ of f is invertible is open. In the case of locally ringed space, the complement of subset where f vanishes is just the set where germ of f is invertible. (We don't have good notion of function vanishing on general ringed space.)

(b) f vanishes nowhere, then the germ of f is invertible everywhere. Notice that $\{f_p\}_{p\in X}$ consists of compatible germs, then their inverse $\{g_p\}_{p\in X}$ also consists of compatible germs. Check it by hand: for each $g_p, p \in X$, there is an open $V_p \subset X$, and $\tilde{g}_p \in \mathscr{O}_X(V_p)$ such that $g_q = \tilde{g}_p|q$. We can just choose $\tilde{g}_p$ to be the representative in the equality $f_p g_p = 1$. And by Exercise **??**. There is a unique global section to glue to and we name it $g \in \mathscr{O}_X(X)$. Finally, $fg = 1$, because $f_p g_p = 1_p = 1, \forall p \in X$ by **??**.

■

2.4 Three examples

Exercise?? 2.4.A Show that you can glue an arbitrary collection of schemes together. Suppose we are given:

- schemes X_i (as i runs over some index set I, not necessarily finite),
- open subschemes $X_{ij} \subset X_i$ with $X_{ii} = X_i$,
- isomorphisms $f_{ij} : X_{ij} \longrightarrow X_{ji}$ with f_{ii} the identity such that
- (the cocycle condition) the isomorphisms "agree on triple intersections", i.e., $f_{ik}|_{X_{ij}\cap X_{ik}} = f_{jk}|_{X_{ji}\cap X_{jk}} \circ f_{ij}|_{X_{ij}\cap X_{ik}}$ (so implicitly, to make sense of the right side, $f_{ij}(X_{ik}\cap X_{ij}) \subset X_{jk}$).

(The cocycle condition ensures that f_{ij} and f_{ji} are inverses. In fact, the hypothesis that f_{ii} is the identity also follows from the cocycle condition.) Show that there is a unique scheme X (up to unique isomorphism) along with open subsets isomorphic to the X_i respecting this gluing data in the obvious sense. (Hint: what is X as a set? What is the topology on this set? In terms of your description of the open sets of X, what are the sections of this sheaf over each open set?) ■

Proof. Confer Exercise **??** so see that we can glue together sheaves on an open cover. The part of structure sheaf on X is constructed similarly.

The isomorphism of schemes $f_{ij} : X_{ij} \longrightarrow X_{ji}$ can be considered as a homeomorphism of topological spaces $f_{ij} : X_{ij} \longrightarrow X_{ji}$ (abuse notion here) together with an isomorphism of sheaves: $h_{ij} : \mathscr{O}_{X_j}|_{X_{ji}} \longrightarrow (f_{ij})_*\mathscr{O}_{X_i}|_{X_{ij}}$.

The cocycle condition on topological space still means

$$f_{ik}|_{X_{ij}\cap X_{ik}} = f_{jk}|_{X_{ji}\cap X_{jk}} \circ f_{ij}|_{X_{ij}\cap X_{ik}}$$

and for structure sheaves it means

$$(f_{ik}^{-1})_* h_{ik}|_{X_{ij}\cap X_{ik}} = (f_{jk}^{-1})_* h_{jk}|_{X_{ji}\cap X_{jk}} \circ (f_{ij}^{-1})_* h_{ij}|_{X_{ij}\cap X_{ik}}$$

For the topological space

$$X = \coprod_i X_i / \sim,$$

where $x \sim y$ iff $x \in X_{ij}, y \in X_{ji}$ and $f_{ij}x = y$ for some i, j. X is endowed with quotient topology. Denote the quotient map $\pi : \coprod_i X_i \longrightarrow X$

$$\tau_X = \{U \subset X : \pi^{-1}(U) \in \tau_{\coprod_i X_i}\}.$$

Then $X_i \overset{\iota_i}{\hookrightarrow} \coprod_j X_j \overset{\pi}{\longrightarrow} X$ and we denote the composition $g_i := \pi \circ \iota_i$. $U_i := g_i(X_i)$ form an open cover of the topological space X. g_i is a homeomorphism. The structure sheaf $\mathscr{O}_{X_i}$ push forward to $\mathscr{O}_{U_i} := g_{i*}\mathscr{O}_{X_i}$ on U_i.

Given $x \in X_{ij}$, $g_i(x) = g_j(f_{ij}x)$ because $x \sim f_{ij}x$. Then we can identify $g_i(X_{ij})$ and $g_j(X_{ji})$ as $U_{ij} = U_i \cap U_j$. The cocycle condition of topological homeomorphisms means simply that we can identify $g_i(X_{ij}\cap X_{ik}) = g_j(X_{ji}\cap X_{jk}) = g_k(X_{kj}\cap X_{ki})$ as $U_{ijk} = U_i \cap U_j \cap U_k$. The sheave isomorphism h_{ij} induces isomorphism of sheaves on U_{ij}:

$$\begin{tikzcd}
\mathscr{O}_{X_j}|_{X_{ji}} \arrow[r, "h_{ij}"] \arrow[d, "(g_j)_*"'] & (f_{ij})_*\mathscr{O}_{X_i}|_{X_{ij}} \arrow[d, "(g_j)_*"] \\
(g_j)_*\mathscr{O}_{X_j}|_{X_{ji}} = \mathscr{O}_{U_j}|_{U_{ij}} \arrow[r, "\phi_{ji}"] & \mathscr{O}_{U_i}|_{U_{ij}} = (g_j)_*(f_{ij})_*\mathscr{O}_{X_i}|_{X_{ij}},
\end{tikzcd}$$

where we have used $g_j \circ f_{ij} = g_i$. Also notice that g_i are isomorphism on X_i.

$$\begin{aligned}\phi_{ji} &= (g_j)_* h_{ij} (g_j^{-1})_* \\ &= (g_j)_* (f_{ij})_* (f_{ij}^{-1})_* h_{ij} (g_j^{-1})_* \\ &= (g_i)_* (f_{ij}^{-1})_* h_{ij} (g_j^{-1})_*\end{aligned}$$

Then the cocycle relation translate to the standard cocycle relation in **??**, $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on $U_i \cap U_j \cap U_k$. Then we can glue up and get a sheaf $\mathscr{O}_X$ such that $\mathscr{O}_X|_{U_i} = \mathscr{O}_{U_i} = g_{i*}\mathscr{O}_{U_i}$ ■

Exercise?? 2.4.B Show that affine line with doubled origin is not affine scheme. ■

Proof. Denote the affine line with doubled origin by X. X is constructed by gluing $X_1 := \mathbb{A}_k^1 = \operatorname{Spec} k[u]$ and $X_2 := \mathbb{A}_k^1 = \operatorname{Spec} k[t]$ along $X_{12} = \operatorname{Spec} [u, 1/u]$ and $X_{21} = \operatorname{Spec} [t, 1/t]$. They glue up by isomorphism $X_{12} \cong X_{21}$, $u \longleftrightarrow t$. Denote the quotient map $\pi : X_1 \coprod X_2 \longrightarrow X$.

For an open set $W \subset X$

$$\mathscr{O}_X(W) = \mathscr{O}_{X_1}(\pi^{-1}(W) \cap X_1) \times_{\mathscr{O}_{X_1}(\pi^{-1}(W)\cap X_1 \cap X_2) = \mathscr{O}_{X_2}(\pi^{-1}(W)\cap X_1\cap X_2)} \mathscr{O}_{X_2}(\pi^{-1}(W) \cap X_2),$$

where the notion means fibered product.

We have

$$\begin{aligned}\Gamma(X, \mathscr{O}_X) &= \mathscr{O}_{X_1}(\pi^{-1}(X) \cap X_1) \times_{\mathscr{O}_{X_1}(\pi^{-1}(X)\cap X_{12}) = \mathscr{O}_{X_2}(\pi^{-1}(X)\cap X_{21})} \mathscr{O}_{X_2}(X \cap X_2) \\ &= \mathscr{O}_{X_1}(X_1) \times_{\mathscr{O}_{X_1}(X_{12}) = \mathscr{O}_{X_2}(X_{21})} \mathscr{O}_{X_2}(X_2) \\ &= k[u] \times_{k[u,1/u] = k[t,1/t]} k[t] \\ &= k[t]\end{aligned}$$

The structure sheaf of X is identical as a single affine line. Then X can not be affine because, $X \ncong \operatorname{Spec} \mathscr{O}_X(X)$. ■

Exercise?? 2.4.C Do the same construction with $\mathbb{A}^1$ replaced by $\mathbb{A}^2$. You will have defined the affine plane with doubled origin. Describe two affine open subsets of this scheme whose intersection is not an affine open subset. ■

Proof. X is constructed by gluing $X_1 = \mathbb{A}^2 = \operatorname{Spec} k[s,t]$ and $X_2 = \mathbb{A}^2 = \operatorname{Spec} k[u,v]$ along

$$X_{12} = D(s) \cup D(t) = \operatorname{Spec} k[s,t,1/s] \cup \operatorname{Spec} k[s,t,1/t]$$

and

$$X_{21} = D(u) \cup D(v) = \operatorname{Spec} k[u,v,1/u] \cup \operatorname{Spec} k[u,v,1/v]$$

via $s \longleftrightarrow u, t \longleftrightarrow v$. (or we can exchange the role of s,t, which will not cause any change)

$$\mathscr{O}_X(W) = \mathscr{O}_{X_1}(\pi^{-1}(W) \cap X_1) \times_{\mathscr{O}_{X_1}(\pi^{-1}(W)\cap X_1 \cap X_2) = \mathscr{O}_{X_2}(\pi^{-1}(W)\cap X_1\cap X_2)} \mathscr{O}_{X_2}(\pi^{-1}(W) \cap X_2),$$

In particular,

$$\begin{aligned}\Gamma(X, \mathscr{O}_X) &= \mathscr{O}_{X_1}(\pi^{-1}(X) \cap X_1) \times_{\mathscr{O}_{X_1}(\pi^{-1}(X)\cap X_{12}) = \mathscr{O}_{X_2}(\pi^{-1}(X)\cap X_{21})} \mathscr{O}_{X_2}(X \cap X_2) \\ &= \mathscr{O}_{X_1}(X_1) \times_{\mathscr{O}_{X_1}(X_{12}) = \mathscr{O}_{X_2}(X_{21})} \mathscr{O}_{X_2}(X_2) \\ &= k[s,t] \times_{k[s,t] = k[u,v]} k[u,v] \\ &= k[s,t],\end{aligned}$$

where we know $\mathscr{O}_{X_1)} = k[s,t]$ from example 4.4.1

$U_1 = \pi(X_1)$ and $U_2 = \pi(X_2)$ are affine open but their intersection $U_{12} = \pi(X_{12}) = \pi(X_{21})$ is the affine plain without origin, which is not affine. ■

Exercise?? 2.4.D Check that the gluing of projective space $\mathbb{P}^n$ satisfies the cocycle relation, as painlessly as possible. ■

Proof. The "Charts" of projective space

$$X_i := \operatorname{Spec} k[x_{0/i}, x_{1/i}, ..., x_{n/i}]/(x_{i/i} - 1),$$

$$X_{ij} := D(x_{j/i}) \subset X_i$$

$$X_{ij} \cap X_{ik} = D(x_{j/i}) \cap D(X_{k/i}) = D(x_{j/i} x_{k/i}) \subset X_i$$

and we have the isomorphism

$$\begin{aligned} f_{ij} : X_{ij} &\longrightarrow X_{ji} \\ x_{k/i} &\longmapsto x_{k/j}/x_{k/i}. \end{aligned}$$

The cocycle relation on triple intersection should be

$$f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{X_{ji} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}.$$

Notice that

$$X_{ij} = \operatorname{Spec} k[x_{0/i}, x_{1/i}, ..., x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1),$$

and

$$X_{ij} \cap X_{ik} = \operatorname{Spec} k[x_{0/i}, x_{1/i}, ..., x_{n/i}, x_{j/i}^{-1}, x_{k/i}^{-1}]/(x_{i/i} - 1)$$

$f_{jk} \circ f_{ij}$:

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, ..., x_{n/i}, x_{j/i}^{-1}, x_{k/i}^{-1}]/(x_{i/i} - 1) \longrightarrow \operatorname{Spec} k[x_{0/k}, x_{1/k}, ..., x_{n/k}, x_{j/k}^{-1}, x_{i/k}^{-1}]/(x_{k/k} - 1)$$

$$f_{jk} \circ f_{ij} : x_{l/i} \mapsto x_{l/j}/_{i/j} \mapsto (x_{l/k}/x_{j/k})/(x_{i/k}/x_{j/k}) = x_{l/k}/x_{i/k} = f_{ik}(x_{l/i})$$

■

Exercise?? 2.4.E Show that the only functions on $\mathbb{P}^n_k$ are constants ($\Gamma(\mathbb{P}^n_k, \mathscr{O}) \cong k$), and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. ■

Proof. Consider two open sets X_i and X_j defined in last exercise. Also, we denote $\mathbb{P}^n_k$ by X.

$$\Gamma(X_i \cup X_j, \mathscr{O}_X) = \mathscr{O}_{X_i}(X_i) \times_{\mathscr{O}_X(X_{ij}) = \mathscr{O}_X(X_{ji})} \mathscr{O}_{X_j}(X_j)$$

polynomials in $\mathscr{O}_{X_i}(X_i)$ and $\mathscr{O}_{X_j}(X_j)$ that agree on the intersection X_{ij}.

$$f \in k[x_{0/i}, x_{1/i}, ..., x_{n/i}]/(x_{i/i} - 1)$$

and

$$g \in k[x_{0/j}, x_{1/j}, ..., x_{n/j}]/(x_{j/j} - 1)$$

agree when restricted to X_{ij}.

$$f(x_{0/i}, x_{1/i}, ..., x_{n/i}) = g\left(\frac{x_{0/i}}{x_{j/i}}, \frac{x_{1/j}}{x_{j/i}}, ..., \frac{x_{n/j}}{x_{j/i}}\right) \in \operatorname{Spec} k[x_{0/i}, x_{1/i}, ..., x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1)$$

Then $x_{j/i}$ can not appear in f. $h \in \mathscr{O}_X(X)$ can restrict to any pair of open sets and agree on the intersection, then must be constant. This solution is intuitive, but a more rigorous statement is: Given ω be the base of topology by choosing the open sets contained in a single X_i.

$$\mathscr{O}_X(U) = \varprojlim_{B\in\omega, B\subset U} \mathscr{O}_X(B)$$

and then check the universal property of k.

■

Exercise?? 2.4.F Show that if k is algebraically closed, the closed points of $\mathbb{P}^n_k$ may be interpreted in the traditional way: the points are of the form $[a_0, \ldots, a_n]$, where the a_i are not all zero, and $[a_0, \ldots, a_n]$ is identified with $[\lambda a_0, \ldots, \lambda a_n]$ where $\lambda \in k^\times$. ■

Proof. The closed point in $X := \mathbb{P}^n_k$ is a point $p \in X$ which is also closed subset of X. X is a scheme, for each x, we can find an open set $X_i \ni x$ and $X_i, \mathscr{O}_X|_{X_i}$ is an affine scheme. We can choose X_i as in the former exercise. The closed point in X is also a closed point in X_i. By Nullstellensatz, the closed point (maximal ideal) in $k[t_1, \ldots, t_n]$ is exactly of the form $(t_1 - a_1, \ldots, t_n - a_n)$, where k is algebraically closed. In this case

$$\mathfrak{m} = (x_{0/i} - a_0, .., x_{i-1/i} - a_{i-1}, x_{i/i} - a_i, x_{i+1/i} - a_{i+1}, \ldots, x_{n/i} - a_n). \ \ x_{i/i} = a_i = 1$$

The same closed point can be identified in another open set X_j as

$$(x_{0/j} - b_0, .., x_{j-1/j} - b_{j-1}, x_{j/j} - b_j, x_{j+1/j} - a_{j+1}, \ldots, x_{n/j} - a_n), \ \ x_{j/j} = b_i = 1,$$

$(b_0, .., b_j = 1, .., b_n) = \frac{1}{a_j}(a_0, \ldots, a_i = 1, \ldots, a_n)$. This justifies our identification of closed point as $[a_0, \ldots, a_n]$ ■

2.5 Projective schemes, and the *Proj* construction

Exercise?? 2.5.A Consider $\mathbb{P}^2_k$, with projective coordinates x_0, x_1, and x_2. (The terminology "projective coordinate" will not be formally defined until §4.5.8, but you should be able to solve this problem anyway.) Think through how to define a scheme that should be interpreted as $x_0^2 + x_1^2 - x_2^2 = 0$ "in $\mathbb{P}^2_k$". ■

Proof. In the open set X_2, there is an affine scheme cut out by $x_{0/2}^2 + x_{1/2}^2 - 1 = 0$. It is a closed subset in $\operatorname{Spec} k[x_{0/2}, x_{1/2}]$ and can be identified as $\operatorname{Spec} k[x_{0/2}, x_{1/2}]/(x_{0/2}^2 + x_{1/2}^2 - 1)$ as shown in 1.4.I.

In all three open charts, we can find

$$V_0 := \operatorname{Spec} k[x_{1/0}, x_{2/0}]/(1 + x_{1/0}^2 - x_{2/0}^2),$$

$$V_1 := \operatorname{Spec} k[x_{0/1}, x_{2/1}]/(x_{0/1}^2 + 1 - x_{2/1}^2),$$

$$V_2 := \operatorname{Spec} k[x_{0/2}, x_{1/2}]/(x_{0/2}^2 + x_{1/2}^2 - 1)$$

$$V_{01} = \operatorname{Spec} k[x_{1/0}, x_{2/0}, x_{1/0}^{-1}]/(1 + x_{1/0}^2 - x_{2/0}^2)$$

$$V_{10} = \operatorname{Spec} k[x_{0/1}, x_{2/1}, x_{0/1}^{-1}]/(x_{0/1}^2 + 1 - x_{2/1}^2)$$

and so on. We find that the gluing isomorphisms of X_i induces well behaved isomorphisms on V_i

$$g_{01} : V_{01} \longrightarrow V_{10}$$

induced by

$$f_{10} : x_{1/0} \mapsto x_{1/1}/x_{0/1}, x_{2/0} \mapsto x_{2/1}/x_{0/1},$$

where $x_{i/i}$ are dummy variables that equals 1. f_{01} induces well-defined morphism of the quotient rings because

$$\begin{aligned} f_{01} : (1 + x_{1/0}^2 - x_{2/0}^2) \longrightarrow & (1 + (x_{1/1}/x_{0/1})^2 + (x_{2/1}/x_{0/1})^2) \\ &= (1/x_{0/1})^2 (x_{0/1}^2 + 1 - x_{2/1}^2) \\ &= (x_{0/1}^2 + 1 - x_{2/1}^2) \end{aligned}$$

g_{ij} also satisfies the cocycle relation because f_{ij} does.

Hence V_i glue up to a closed subscheme in $\mathbb{P}_k^2$

■

Exercise?? 2.5.B More generally, consider $\mathbb{P}_A^n$, with projective coordinates $x_0, \ldots, x_n$. Given a collection of homogeneous polynomials $f_i \in A[x_0, \ldots, x_n]$, make sense of the scheme "cut out in $\mathbb{P}_A^n$ by the f_i."

Proof. Given a homogeneous polynomial $f_i \in A[x_0, \ldots, x_n]$, we can consider map $x_j \to x_{j/j} = 1$ and $x_k \to x_{k/j}$ so that we get a polynomial in $F_{i,(j)} \in A[x_{i/j}, \ldots]$ which no longer homogeneous. Then we can construct the closed subset in affine chart X_i

$$U_{i,(j)} := \operatorname{Spec} A[x_{1/j}, \ldots, x_{n/j}]/(x_{j/j} - 1, F_{i,(j)})$$

and

$$\begin{aligned} U_{i,(j);k,(l)} &= \operatorname{Spec} A[x_{1/j}, \ldots, x_{n/j}, x_{l/j}^{-1}]/(x_{j/j} - 1, F_{i,(j)}, F_{k,(j)}) \\ U_{k,(l);i,(j)} &= \operatorname{Spec} A[x_{1/l}, \ldots, x_{n/l}, x_{j/l}^{-1}]/(x_{l/l} - 1, F_{i,(l)}, F_{k,(l)}). \end{aligned}$$

Again the gluing map of X_i

$$h_{jl} : x_{m/j} \mapsto x_{m/l}/x_{j/l}$$

would induce well-defined morphism on the quotient ring. Because

$$h_{jl} : F_{k,(j)} \mapsto \frac{1}{x_{j/l}^{n_{jl}}} F_{k,(l)},$$

where n_{jl} is the degree of $x_{l;j}$ in $F_{k,(j)}$. Denote the induced morphism $g_{i,(j);k,(l)}$. We can similarly verify that the induced morphism satisfies the cocycle relation because h_{jl} does.

$$g_{i,(j);k,(l)} = g_{p,(j);k,(l)} \circ g_{i,(j);p,(l)}$$

Hence we can glue the affine schemes to get a closed subscheme in $\mathbb{P}_A^n$. ■

Exercise?? 2.5.C

(a) Show that an ideal I is homogeneous if and only if it contains the degree n piece of each of its elements for each n.

(b) Show that the set of homogeneous ideals of a given $\mathbb{Z}$-graded ring $S_\bullet$ is closed under sum, product, intersection, and radical.

(c) Show that a homogeneous ideal $I \subset S_\bullet$ is prime if $I \neq S_\bullet$, and if for any homogeneous $a, b \in S_\bullet$, if $ab \in I$, then $a \in I$ or $b \in I$.

Proof. (a) Want:"I is homogeneous ideal" $\iff$ "I contains each degree n piece of each element for each degree"
$\Longrightarrow$: A homogeneous ideal I is generated by homogeneous elements $\{a_i \in S_{n(i)}\}_{i \in I}$, where $n(i)$ is the degree of a_i. A general element looks like $b = \sum_i s_i a_i$. s_i are not necessarily homogeneous, but we can decompose s into homogeneous pieces and verify that homogeneous pieces of b are still generated by a_i which are contained in I.
$\Longleftarrow$: $b = \sum_n b_n \in S$, where b_n is homogeneous of degree n. We can simply choose the generating set to be the homogeneous pieces of each of the element in I. Then I is generated by homogeneous elements. Then we can decompose I as $\oplus_{n \in \mathbb{Z}} I_n$. And S/I has a natural $\mathbb{Z}$-grading.

(b) I, J are homogeneous ideals in $S_\bullet$. It is easy to verify that the $I + J$, $I \cdot J$ and $I \cap J$ are generated by homogeneous elements. As for radicals,

$$L = \sqrt{I}$$

$x \in L \iff x^n \in L$. $x = \sum_{i \in \mathbb{Z}} x_i$ only finitely many of x_i are not zero. Assume d the highest degree of x, $deg\ x_d$ is larger than any other homogeneous piece. Specifically, $x_d^n \in I$ which means $x_i \in L$. Then $x - x_d \in L$, we can induct on this procedure until we find each homogeneous piece of x is contained in L. Which means I is homogeneous by (a).

(c) The only if direction is trivial. We only need to check the if direction.
Consider general elements $x = \sum_i^d x_i$ and $y = \sum_i^k y_j$, they are finite sums by definition. We can check

$$\sum_i^d x_i \cdot \sum_j^k y_j = \sum_n \sum_{i+j=n} x_i y_j.$$

Assume neither x nor y is contained in I. Then at least one x_d and y_k are not in I. Assume d, k to be maximal with this property.
We can subtract from x the part $\sum_{n>d} x_n$ such that $\tilde{x} = x - \sum_{n>d} x_n \notin I$. Similarly, set $\tilde{y} = y - \sum_{m>k} y_m \notin I$. Then we have $\tilde{x}\tilde{y} \in I$ and the leading homogeneous piece being $x_d y_k$. Then $x_d y_k \in I$, $\Longrightarrow x_d$ or $y_k \notin I$. Contradiction.

■

Exercise?? 2.5.D

(a) Show that a graded ring $S_\bullet$ over A is a finitely generated graded ring (over A) if and only if $S_\bullet$ is a finitely generated graded A-algebra, i.e., generated over $A = S_0$ by a finite number of homogeneous elements of positive degree. (Hint for the forward implication: show that the generators of S_+ as an ideal are also generators of $S_\bullet$ as an algebra.)

(b) Show that a graded ring $S_\bullet$ over A is Noetherian if and only if $A = S_0$ is Noetherian and $S_\bullet$ is a finitely generated graded ring.

Proof. (a) $\Longrightarrow$: $S_\bullet$ is a finitely generated graded ring over A, which means $S_0 = A$ and the irrelevant ideal S_+ is finitely generated ideal. Such a graded ring is naturally an A-algebra.

Assume S_+ is finitely generated by as $(s_{1,1},\ldots,s_{n,m_j})$ each of degree higher than $s_{i,k}, k \le m_j$ is of degree i. Then a general element in S_1 can be expressed as

$$\sum_j^{m_1} a_j s_{1,j}.$$

Each element in S_2 can be expressed as

$$\sum_j^{m_2} a_j s_{2,j} + \sum_{j,k}^{m_1} b_{jk} s_{1,j} s_{1,k}.$$

And it will also work for any degree, which means S is generated as polynomial $A[s_{i,j}]$ (quotient some relations).

$\Longleftarrow$ If a graded ring $S_\bullet$ is finitely generated as an A-algebra ($S_0 = A$), $S_+ = S_\bullet - A$ is finitely generated.

(b) $\Longleftarrow$: By (a), $S_\bullet$ being a finitely generated graded ring with $S_0 = A$ means $S_\bullet$ is finitely generated A-algebra. Grading is only an extra structure on S, we can proceed by Hilbert basis theorem. If A is Noetherian, the finitely generated A-algebra is Noetherian.

$\Longrightarrow$: Graded ring $S_\bullet$ is Noetherian, then every ideal of $S_\bullet$ is finitely generated, specifically, S_+ is finitely generated. On the other hand $S_\bullet/S_+ \cong A$, is a homomorphic image of a Noetherian ring. Hence A must also be Noetherian.

■

Exercise?? 2.5.E Suppose $f \in S_+$ is homogeneous.

(a) Give a bijection between the prime ideals of $((S_\bullet)_f)_0$ and the homogeneous prime ideals of $(S_\bullet)_f$. Hint: Avoid notational confusion by proving instead that if A is a $\mathbb{Z}$-graded ring with a homogeneous invertible element f in positive degree, then there is a bijection between prime ideals of A_0 and homogeneous prime ideals of A. Using the ring map $A_0 \longrightarrow A$, from each homogeneous prime ideal of A we find a prime ideal of A_0. The reverse direction is the harder one. Given a prime ideal $P_0 \subset A_0$, define $P \subset A$ (a priori only a subset) as $\oplus Q_i$, where $Q_i \subset A_i$, and $a \in Q_i$ if and only if $a^{\deg f}/f^i \in P_0$. Note that $Q_0 = P_0$. Show that $a \in Q_i$ if and only if $a^2 \in Q_{2i}$; show that if $a_1, a_2 \in Q_i$ then $a_1^2 + 2a_1a_2 + a_2^2 \in Q_{2i}$ and hence $a_1 + a_2 \in Q_i$; then show that P is a homogeneous ideal of A; then show that P is prime.

(b) Interpret the set of prime ideals of $((S_\bullet)_f)_0$ as a subset of $\operatorname{Proj} S_\bullet$.

Proof. (a) Follow the hint, we avoid the notation trouble by working in a $\mathbb{Z}$-graded ring A with invertible homogeneous element f with $\deg f > 0$.

Claim: There is a bijection between prime ideals of A_0 and homogeneous prime ideal of A. There is a natural inclusion $\iota : A_0 \hookrightarrow A$. For a given homogeneous prime ideal P in A we can find a prime ideal $\iota^* P \in A_0$ which is just $P \cap A_0$ which is a prime ideal in A_0.

The reverse direction is harder. Consider a prime ideal $P_0 \in A_0$, define $P \subset A$ as $P = \oplus Q_i$, where $Q_i \subset A_i$ and a homogeneous element $a \in Q_i$ iff $a^{\deg f}/f^i \in P_0$. Note that $P_0 = Q_0$.

Want: P is an ideal. Indeed, for $a \in P$, $b \in A$. $a = \sum_i a_i$, $b = \sum_j b_j$, where each $a_i \in Q_i$ and $b_j \in A_j$. $a_i b_i \in A_{2i}$.

$$\frac{(a_i b_j)^{\deg f}}{f^{i+j}} = \frac{a_i^{\deg f}}{f^i} \frac{b_i^{\deg f}}{f^j} \in P_0.$$

Then product is contained in P_0 because $a_i^{\deg f} f^i \in P_0$ and $b_j^{\deg f} f^j \in A_0$ and P_0 is a prime ideal in A_0. Then we have

$$\frac{(a_i b_j)^{\deg f}}{f^{i+j}} \in Q_{i+j}.$$

Similarly, we can prove $a \cdot b = \sum_i a_i \cdot \sum_i b_i \in P$.
subclaim: $a \in Q_i$ iff $a^2 \in Q_i$. one direction is clear and we only need to prove the only if part.

$$\frac{(a^2)^{\deg f}}{f^{2i}} = \frac{(a)^{\deg f}}{f^i} \frac{(a)^{\deg f}}{f^i} \in P_0 \Longrightarrow \frac{(a)^{\deg f}}{f^i} \in P_0$$

because P_0 is prime ideal.
On the other hand, given $a_1, a_2 \in Q_i$ We have P is an ideal in A. We know $a_1^2 + 2a_1a_2 + a_2^2 \in Q_{2i}$. Hence by the subclaim, $a_1 + a_2 \in Q_i$.
Then we know P is closed under summation and multiplication by A. (P is an ideal.)
Want: P is homogeneous ideal. Recall 2.5.C, part (a), we only need to prove that P contains the homogeneous piece of each of its element of each degree. Assume $a \in P$ then $a = \sum_i a_i$ and $a_i \in Q_i$ by definition of P. We know P is homogeneous ideal.
Want: P is a prime ideal in A. See 2.5.C, part (c). A homogeneous ideal is prime iff it is prime for homogeneous elements. Suppose $a \in A_i, b \in A_j$ and we have $ab \in Q_{ij}$, then by definition

$$\frac{a^{\deg f}}{f^i} \frac{b^{\deg f}}{f^j} \in P_0$$

then either $a \in Q_i$ or $b \in Q_j$ because P_0 is prime ideal.
We still need to prove that this indeed gives a bijection. The map $\alpha : I \longrightarrow I \cap A_0$ as described above and the map $\beta : P_0 \mapsto P$. $\alpha \circ \beta : P_0 \mapsto P_0$. $\beta \circ \alpha : I \mapsto I \cap A_0 \mapsto ?I$.

$$\beta(I \cap A_0) \supset I$$

because $x \in I_j$ then $x^{\deg f}/f^j \in I \cap A_0$.
For the reverse inclusion, consider an element $y \in \beta(I \cap A_0)_j$ then by definition $y^{\deg f}/f^j \in I \cap A_0$, then $y^{\deg f} = y^{\deg f}/f^j \cdot f^j \in I_{j \deg f} \Longrightarrow y \in I_j$ because I is prime ideal.

(b) We have proved there is bijection between "prime ideals in $((S_\bullet)_f)_0$" and homogeneous prime ideals in $(S_\bullet)_f$. Recall the property of localization, there is a one to one homogeneous prime ideals in $(S_\bullet)_f$ and homogeneous prime ideals in $S_\bullet$ that does not intersect $\{f, f^2, \ldots\} \subset S_+$. (Localization preserves the homogeneity of prime ideals.) In particular, these homogeneous prime ideals do not contain S_+, therefore they can be interpreted as subset in $\operatorname{Proj} S_\bullet$.

■

Exercise?? 2.5.F Show that $D(f)$ "is" (or more precisely, "corresponds to") the subset $\operatorname{Spec}((S_\bullet)_f)_0$ you described in Exercise 2.5.E. For example, the $D(x_i)$ are the standard open sets covering projective space.

Proof. $D(f)$ is the projective distinguished open set and

$$\begin{aligned} D(f) &= \operatorname{Proj} S_\bullet - V(f) \\ &= \{\text{Homogeneous prime ideals that do not contain } f \text{ or } S_+\} \\ &= \{\text{Homogeneous prime ideal that do not contain } f\} (f \in S_+) \end{aligned}$$

And we have the correspondences:

$$\begin{aligned}
&\{\text{Homogeneous prime ideals in } S_\bullet \text{ that do not contain } f\}\\
&\longleftrightarrow (\text{by localization property})\\
&\{\text{Homogeneous prime ideals in } (S_\bullet)_f\}\\
&\longleftrightarrow (\text{by Exercise 2.5.E})\\
&\{\text{Prime ideals in } ((S_\bullet)_f)_0\}\\
&= \operatorname{Spec}((S_\bullet)_f)_0
\end{aligned}$$

■

Exercise?? 2.5.G Verify that the projective distinguished open sets $D(f)$ (as f runs through the homogeneous elements of S_+) form a base of the Zariski topology. ■

Proof. As in the affine case, the closed set of Zariski's topology is of the form

$$V(T) = V(\overline{T}).$$

where $I := \overline{T}$ is a homogeneous ideal generated by T. $\overline{T} \subset S_+$.

It forms a topology

$$\cap_i V(J_i) = V(\sum_i J_i)$$

and

$$V(I) \cup V(J) = V(I \cdot J)$$

and

$$IV(J) = \sqrt{J}$$

both are well-defined because {homogeneous ideals} is closed under intersection, product, addition and taking radicals.

Claim: $\{D(f)\}$ where f is homogeneous element in S_+ form a base of the topology.

Given a point $[P]$ in an open set $\operatorname{Proj} S_\bullet - V(I)$, P is a homogeneous prime ideal in $S_\bullet$ that does not contains I. We can find $f \in I \backslash P$. Then by definition

$$V(f) \supset V(I)$$

and

$$D(f) \subset \operatorname{Proj} S_\bullet - V(I).$$

Also P does not contain f, hence $D(f) \ni [P]$.

Which means $D(f)$ as f ranges over S_+ form a base of the Zariski's topology. ■

Exercise?? 2.5.H Fix a graded ring $S_\bullet$.

(a) Suppose I is any homogeneous ideal of $S_\bullet$ contained in S_+, and f is a homogeneous element of positive degree. Show that f vanishes on $V(I)$ (i.e.,$V(I) \subset V(f)$) if and only if $f^n ?I$ for some n. (Hint: Mimic the affine case 1.4.J,) In particular, as in the affine case (Exercise 1.5.E), if $D(f) \subset D(g)$, then $f^n \in (g)$ for some n, and vice versa. (Here g is also homogeneous of positive degree.)

(b) If $Z \subset \operatorname{Proj} S_\bullet$, define $I(Z) \subset S_+$. Show that it is a homogeneous ideal of $S_\bullet$. For any two subsets, show that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.

(c) For any subset $Z \subset \operatorname{Proj} S_\bullet$, show that $V(I(Z)) = \overline{Z}$.

Proof. (a) If $f^n \in I$ for some n, then $f \in P$ for all P homogeneous prime containing I because P is prime. Then f vanishes on $V(I)$.

For the reverse direction, f vanishes on $V(I)$, $V(I) \subset V(f)$. It means all homogeneous prime ideals that containing I but not S_+ contains f:

$$f \in \cap_{P \supset I, P \not\supset S_+} P.$$

Notice when f is homogeneous of positive degree, $f \in S_+$, f is automatically contained in any homogeneous prime that contains S_+. the condition f vanishes on $V(I)$ in fact is equivalent to

$$f \in \cap_{P \supset I} P.$$

Note that I itself is homogeneous.

Claim: The radical of a homogeneous ideal I is the intersection of all homogeneous prime ideal that contains I.

In general, we use P to denote homogeneous prime

$$\cap_{P \supset I} P \supset \cap_{\mathfrak{p} \supset I} \mathfrak{p}.$$

For the reverse inclusion, consider $\mathfrak{p}^h$ to be the the homogeneous ideal generated by the homogeneous element of $\mathfrak{p}$. $\mathfrak{p}^h \subset \mathfrak{p}$ is prime on homogeneous elements then $\mathfrak{p}^h$ is homogeneous prime ideal by 2.5.C. If $\mathfrak{p}$ is any prime containing I, then $\mathfrak{p}^h$ also contains I, because $\mathfrak{p}^h$ contains the homogeneous elements of I, and I as a homogeneous ideal is generated by these homogeneous elements. We have

$$\cap_{P \supset I} P \subset \cap_{\mathfrak{p} \supset I} \mathfrak{p}^h \subset \cap_{\mathfrak{p} \supset I} \mathfrak{p}.$$

Then we know the intersection of homogeneous primes that containing a homogeneous ideal I is just the radical of I.

$$f \in \cap_{P \supset I} P = \cap_{\mathfrak{p} \supset I} \mathfrak{p} = \sqrt{I}.$$

Hence exists $n \geq 0$ such that $f^n \in I$.

(b) $I(Z)$ is defined to be the element of $S_\bullet$ that vanishes on all of $[P] \in Z$. $Z \subset \operatorname{Proj} S_\bullet$, means each $[P] \in Z$. P is homogeneous prime that does not contain S_+.

$f \in I(Z) \Longleftrightarrow f \in \cap_{[P] \in Z} P$.

Denote the degree i piece of f by f_i. $f \in \cap_{[P] \supset Z} P \Longrightarrow f_i \in P \ \forall [P] \in Z$, because each P is homogeneous, this in turn means $f_i \in I(Z)$. $I(Z)$ contains each homogeneous piece of each element in it, which implies that $I(Z)$ is a homogeneous ideal.

For two subsets Z_1 and Z_2 in $\operatorname{Proj} S_\bullet$,

$$f \in I(Z_1 \cup Z_2)$$

$$\Longleftrightarrow f \in P, \forall [P] \in Z_1 \cup Z_2$$

$$\Longleftrightarrow f \in P, \forall [P] \in Z_1 \text{ AND } f \in Q, \forall [Q] \in Z_2$$

$$\Longleftrightarrow f \in I(Z_1) \cap I(Z_2)$$

(c) "$V(I(Z)) \supset \overline{Z}$": By definition, $V(I(Z))$ is closed set. And $Z \subset V(I(Z))$, because $[P] \in Z, I(Z) = \cap_{[Q]\in Z} Q$ $I(Z) \mod P = 0, \Longrightarrow I(Z) \subset P \Longrightarrow [P] \in V(I(Z))$. Closure $\overline{Z}$ is the smallest closed set that contains Z, therefore $\overline{Z} \subset V(I(Z))$.
"$V(I(Z)) \subset \overline{Z}$": We need to verify that open neighborhood of each point in $V(I(Z))$ intersects with Z. Consider a point $[Q] \in V(I(Z))$, we have $Q \supset I(Z)$. Assume a non-empty distinguished open $D(f)$ contains $[Q] \Longleftrightarrow f \notin Q \Longrightarrow f \notin I(Z) \Longrightarrow f$ does not vanish on every point in Z, $\exists [P] \in Z$, s.t. $f \notin P \Longleftrightarrow D(f) \ni [P]$, therefore $D(f) \cap V(I(Z)) \neq \emptyset$. Hence $V(I(Z)) \subset \overline{Z}$.

■

Exercise?? 2.5.I Fix a graded ring $S_\bullet$, and a homogeneous ideal I. Show that the following are equivalent.

(a) $V(I) = \emptyset$.

(b) For any f_i (as i runs through some index set) generating I, $\cup_i D(f_i) = \operatorname{Proj} S_\bullet$.

(c) $\sqrt{I} \supset S_+$.

Proof. $(a) \Longrightarrow (b)$: Assume I is finitely generated by $\{f_i\}_{i\in J}$. $V(I) = \emptyset$, means $\forall [P] \in \operatorname{Proj} S_\bullet$, $P \not\supset I$. For each $[P] \in \operatorname{Proj} S_\bullet$, $P \not\ni f_i$ for some $f_i \in I \Longrightarrow [P] \in D(f_i)$ for some f_i, which means $\operatorname{Proj} S_\bullet \subset \cup_{i\in J} D(f_i) \subset \operatorname{Proj} S_\bullet$.

$(b) \Longrightarrow (c)$: $\operatorname{Proj} S_\bullet = \cup_{i\in J} D(f_i)$ means each homogeneous prime ideal that does not contain S_+ in $S_\bullet$ does not contain I. The contrapositive says each homogeneous prime ideal contain I would contain S_+:

$$\cap_{P\supset I} P \supset S_+.$$

By Exercise 2.5.H, we know $\sqrt{I} = \cap_{P\supset I} P$ for homogeneous ideal I.

$(c) \Longrightarrow (a)$,

$$\sqrt{I} = \cap_{P\supset I} P \supset S_+$$

means for each point $[P] \in \operatorname{Proj} S_\bullet$, P does not contain I. $\Longrightarrow [P] \notin V(I)$.

$\Longrightarrow V(I) = \emptyset$. ■

Exercise?? 2.5.J Suppose some homogeneous $f \in S_+$ is given. Via the inclusion

$$D(f) = \operatorname{Spec}((S_\bullet)_f)_0 \hookrightarrow \operatorname{Proj} S_\bullet$$

of Exercise 2.5.F, show that the Zariski topology on $\operatorname{Proj} S_\bullet$ restricts to the Zariski topology on $\operatorname{Spec}((S_\bullet)_f)_0$

Proof. We can compare the closed subsets of these two topological space. Consider a subset $I \subset ((S_\bullet)_f)_0$

$$\begin{aligned}((S_\bullet)_f)_0 \supset V'(I) &= \{[\mathfrak{p}] \in \operatorname{Spec}((S_\bullet)_f)_0 : I \subset \mathfrak{p}\}\\ &\longleftrightarrow \{[P] \in \operatorname{Proj}(S_\bullet)_f : I^h \subset P\}\\ &(I^h \text{ is the homogeneous ideal generated by } I.)\\ &\longleftrightarrow \{[Q] \in \operatorname{Proj} S_\bullet : \iota^*(I^h) \subset Q, Q \not\ni f\}\\ &(\iota^*(I^h) \text{ denote the preimage of } I^h \text{ under localization})\end{aligned}$$

where we use $\mathfrak{p}$ to denote ordinary prime ideal and P to mean homogeneous prime ideal. Also $\iota : S_\bullet \hookrightarrow (S_\bullet)_f$ is the standard inclusion of localization.

$$\begin{aligned}((S_\bullet)_f)_0 \supset V'(D) &\longleftrightarrow \{[Q] \in \operatorname{Proj} S_\bullet : \iota^*(I^h) \subset Q, Q \not\ni f\} \\ &\longleftrightarrow \{[Q] \in \operatorname{Proj} S_\bullet : Q \supset \iota^*(I^h)\} \cap \{[Q] \in \operatorname{Proj} S_\bullet : Q \not\ni f\} \\ &= V(\iota^*(I^h)) \cap D(f)\end{aligned}$$

For illustration, consider previous example $S := k[x_0, x_1, x_2]$, $S_{x_0} := k[x_0, x_1, x_2, x_0^{-1}]$, $(S_{x_0})_0 := k[x_1/x_0, x_2/x_0]$. $I = (1 + x_{1/0}^2 - x_{2/0}^2)_0$ and $I^h = (1 + x_{1/0}^2 - x_{2/0}^2)$.

$$V'(I) = V_0 := \operatorname{Spec} k[x_{1/0}, x_{2/0}]/(1 + x_{1/0}^2 - x_{2/0}^2),$$

$\iota^*(I^h) = (x_0^2 + x_1^2 - x_2^2)$.

$$V'(I) \longleftrightarrow D(x_0) \cap V((x_0^2 + x_1^2 - x_2^2))$$

■

Exercise?? 2.5.K If $f, g \in S_+$ are homogeneous and nonzero, describe an isomorphism between $\operatorname{Spec}((S_\bullet)_{fg})_0$ and the distinguished open subset $D(g^{\deg f}/f^{\deg g})$ of $\operatorname{Spec}((S_\bullet)_f)_0$. ■

Proof. Recall that

$$\operatorname{Spec}((S_\bullet)_{fg})_0 \cong D(fg) \subset \operatorname{Proj} S_\bullet$$

Then problem reduce to describe the intersection of $D(fg)$ with $\operatorname{Spec}((S_\bullet)_f)_0$.

$$\begin{aligned}\operatorname{Spec}((S_\bullet)_{fg})_0 &= \{[\mathfrak{q}] \in \operatorname{Spec}((S_\bullet)_{fg})_0 :\} \\ &\longleftrightarrow \{[Q] \in \operatorname{Proj}(S_\bullet)_{fg}\} \\ &\longleftrightarrow \{[Q] \in \operatorname{Proj} S_\bullet : Q \not\ni fg\} \\ &\longleftrightarrow \{[Q] \in \operatorname{Proj} S_\bullet : Q \not\ni f\} \cap \{[Q] \in \operatorname{Proj} S_\bullet : Q \not\ni g\} \\ &\longleftrightarrow \{[P] \in \operatorname{Proj}(S_\bullet)_f : P \not\ni \frac{g}{1}\} \\ &\longleftrightarrow \{[\mathfrak{p}] \in \operatorname{Spec}((S_\bullet)_f)_0 : \mathfrak{p} \not\ni \frac{g^{\deg f}}{f^{\deg g}}\}\end{aligned}$$

We only have to explain the last bijection. $\mathfrak{p} = P \cap ((S_\bullet)_f)_0$ and if $P \not\ni g/1$, we have $\mathfrak{p} \not\ni g^{\deg f}/f^{\deg g}$. For the reverse direction, we have to recall the construction in 2.5.E, $g/1 \notin P_{\deg g}$ iff $g^{\deg f}/f^{\deg g} \notin \mathfrak{p} = P_0$. ■

Before finishing the definition, we have to collect some results about graded ring here in order to construct the isomorphism of sheaves on intersections.

Theorem 2.5.1 Let $S_\bullet$ be a graded ring. Let $f \in S_\bullet$ homogeneous of positive degree.

1. If $g \in S$ homogeneous of positive degree and $D(g) \subset D(f)$, then
   (a) f is invertible in $(S_\bullet)_g$, and $f^{\deg(g)}/g^{\deg(f)}$ is invertible in $S_{(g)}$,
   (b) $g^e = af$ for some $e \geq 1$ and $a \in S$ homogeneous,
   (c) there is a canonical $S_\bullet$-algebra map $(S_\bullet)_f \to (S_\bullet)_g$,
   (d) there is a canonical $(S_\bullet)_0$-algebra map $((S_\bullet)_f)_0 \to ((S_\bullet)_g)_0$ compatible with the map $(S_\bullet)_f \to (S_\bullet)_g$,

(e) the map $((S_\bullet)_f)_0 \to ((S_\bullet)_g)_0$ induces an isomorphism

$$(((S_\bullet)_f)_0)_{g^{\deg(f)}/f^{\deg(g)}} \cong ((S_\bullet)_g)_0,$$

(f) there are compatible canonical $S_\bullet)_f$ and $((S_\bullet)_f)_0$-module maps $(M_\bullet)_f \to (M_\bullet)_g$ and $((M_\bullet)_f)_0 \to ((M_\bullet)_g)_0$ for any graded $S_\bullet$-module $M_\bullet$, and

(g) the map $((M_\bullet)_f)_0 \to ((M_\bullet)_g)_0$ induces an isomorphism

$$(((M_\bullet)_f)_0)_{g^{\deg(f)}/f^{\deg(g)}} \cong ((M_\bullet)_g)_0.$$

This is a horrible notation, where the first and third subscripts means grading while second and forth means localizations.

2. Any open covering of $D_+(f)$ can be refined to a finite open covering of the form $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$.
3. Let $g_1, \dots, g_n \in S$ be homogeneous of positive degree. Then $D_+(f) \subset \bigcup D_+(g_i)$ if and only if $g_1^{\deg(f)}/f^{\deg(g_1)}, \dots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$.

Proof. We only prove $1.(a) - 1.(e)$ and the remaining part can be found at Stack Project 26.8.

(a) f is is invertible in $(S_\bullet)_g$ by 1.5.E. And $f^{\deg g}/g^{\deg f}$ is an invertible element an element in $((S_\bullet)_g)_0$ because it is not contained in any prime ideal in $((S_\bullet)_g)_0$. ($f/1$ is not contained in any prime ideal in $(S_\bullet)_g$ because $f/1$ is invertible.)

(b) Then the inverse of f in $(S_\bullet)_g$ can be written as a'/g^d and $fa'/g^d = 1/1$. We can replace a' by its homogeneous part a'' and $(fa'' - g^d)/1 = 0/1$ $g^k(fa - g^d) = 0$. Then after defining $a := a''g^k$, we get $fa = g^e$, for $e = k+d$.

(c) The morphism exists by universal property of localization. It maps b/f^n to $a^n b/g^{ne}$.

(d) This clearly induce a map of the degree 0 ring, $((S_\bullet)_f)_0 \to ((S_\bullet)_g)_0$.

(e) We need to look the ring morphism $((S_\bullet)_f)_0 \to ((S_\bullet)_g)_0$ in detail. It is induced by

$$\phi : \left(\frac{b}{f^n}\right)_0 \mapsto \left(\frac{a^n b}{g^{ne}}\right)_0$$

In particular

$$\frac{g^{\deg f}}{f^{\deg g}} \mapsto \frac{a^{\deg g}}{g^{e \deg g - \deg f}}$$

Because $g^{\deg f}/f^{\deg g}$ is invertible in $(S_\bullet)_0$, it induces a morphism

$$\tilde{\phi} : (((S_\bullet)_f)_0)_{g^{\deg f}/f^{\deg g}} \longrightarrow ((S_\bullet)_g)_0$$

$$\begin{aligned}
\left(\frac{b}{f^n}\right)_0 \Big/ \left(\frac{g^{\deg f}}{f^{\deg g}}\right)^m &\mapsto \phi\left(\frac{b}{f^n}\right)_0 \Big/ \phi\left(\frac{g^{\deg f}}{f^{\deg g}}\right)^m \\
&= \left(\frac{(a^n b/g^{ne})}{(a^{\deg g}/g^{e \deg g - \deg f})^m}\right)_0 \\
&= \left(\frac{a^{n - m\deg g} b}{g^{m \deg f - me \deg g + ne}}\right)_0 \\
&= \left(\frac{b}{g^{m \deg f} f^{n - m \deg g}}\right)_0
\end{aligned}$$

This morphism is surjective because f is invertible in $(S_\bullet)_g$. Then we can look at the kernel of this ring morphism. If

$$\left(\frac{b}{g^{m\deg f}f^{n-m\deg g}}\right)_0 = 0,$$

It means exactly the degree 0 part of b/f^n is zero. Hence we have the isomorphism. ■

Exercise?? 2.5.L By checking that these gluings behave well on triple overlaps (see Exercise **??**), finish the definition of the scheme $\operatorname{Proj} S_\bullet$. ■

Proof. $D(f)$ give an open cover of the topological space of $X := \operatorname{Proj} S_\bullet$. We identify the $\mathscr{O}_{X,f}$ to be the structure sheaf of $\operatorname{Spec}((S_\bullet)_f)_0$: $\mathscr{O}_{\operatorname{Spec}((S_\bullet)_f)_0}$.

$D(f)\cap D(g) = D(fg)$, we have the isomorphism of sheaves:

$$\phi_{f,g} : \mathscr{O}_{X,f}|_{D(fg)} \longrightarrow \mathscr{O}_{X,g}|_{D(fg)}.$$

$$\mathscr{O}_{\operatorname{Spec}((S_\bullet)_f)_0}|_{D(g^{\deg f}/f^{\deg g})} \longrightarrow \mathscr{O}_{\operatorname{Spec}((S_\bullet)_g)_0}|_{D(f^{\deg g}/g^{\deg f})}$$

We want to verify the cocycle relation on $D(fgh)$. We verify the cocycle relation on a base of topology. The distinguished base of $\operatorname{Spec}(S_\bullet)_{fgh})_0$ is also induced by distinguished open in $\operatorname{Proj} S_\bullet$.

Want:

$$\phi_{f,h}(D(fghk)) = \phi_{g,h}(D(fghk))\circ\phi_{f,g}(D(fghk)), \forall k \text{ homogeneous in } S_+.$$

Notice that sheaf morphism is commutative with restriction maps. It suffices to check

$$\phi_{f,h}(D(fgh)) = \phi_{g,h}(D(fgh))\circ\phi_{f,g}(D(fgh)).$$

$$\begin{aligned}\mathscr{O}_{X,f}(D(fg)) &= \mathscr{O}_{\operatorname{Spec}((S_\bullet)_f)_0}(D(g^{\deg f}/f^{\deg g}))\\ &= ((S_\bullet)_f)_0)_{g^{\deg f}/f^{\deg g}},\end{aligned}$$

Claim:

$$((S_\bullet)_f)_0)_{g^{\deg f}/f^{\deg g}} \cong ((S_\bullet)_{fg})_0$$

This isomorphism is already discussed in Theorem 2.5. We have the isomorphisms

$$((S_\bullet)_f)_0)_{(gh)^{\deg f}/f^{\deg g+\deg h}} \cong ((S_\bullet)_{fgh})_0.$$

Similarly, we have other two isomorphisms and the $\phi_{f,h}(D(fgh))$ is the composition of these ring isomorphism

$$\begin{tikzcd}
 & & \mathscr{O}_{X,h}(D(fgh)) = ((S_\bullet)_h)_0)_{(fg)^{\deg h}/h^{\deg f+\deg g}} \arrow[dl] \\
\mathscr{O}_{X,f}(D(fgh)) = ((S_\bullet)_f)_0)_{(gh)^{\deg f}/f^{\deg g+\deg h}} \arrow[r] \arrow[urr, "\phi_{f,h}(D(fgh))"] \arrow[drr, "\phi_{fg}(D(fgh))"'] & ((S_\bullet)_{fgh})_0 & \\
 & & \mathscr{O}_{X,g}(D(fgh)) = ((S_\bullet)_g)_0)_{(fh)^{\deg g}/g^{\deg f+\deg h}} \arrow[ul] \arrow[uu, "\phi_{g,h}(D(fgh))"']
\end{tikzcd}$$

This means we can legally glue the structure sheaf of distinguished opens and there is a unique way to define the structure sheaf of $\operatorname{Proj} S_\bullet$. ■

Exercise?? 2.5.M (Some will find this essential, others will prefer to ignore it.) (Re)interpret the structure sheaf of $\operatorname{Proj} S_\bullet$ in terms of compatible germs. ▪

Proof. We have to first discuss what the stalk of the structure sheaf $\mathscr{F} := \mathscr{O}_{\operatorname{Proj} S_\bullet}$ is.

By definition

$$\mathscr{F}_{[P]} = \varinjlim_{[P]\in U} \mathscr{F}(U),$$

where $[P] \in \operatorname{Proj} S_\bullet$ and P is a homogeneous prime in $S_\bullet$. Assume $[P] \in D(f)$ for a homogeneous element $f \in S_\bullet$, we define the structure sheaf on $D(f)$ to be the structure sheaf of $\operatorname{Spec}((S_\bullet)_f)_0$. P corresponds to an ordinary prime $\mathfrak{p} \in ((S_\bullet)_f)_0$.

Recall the form of stalks of a affine scheme, we have the isomorphism

$$\mathscr{F}_{[P]} \cong (((S_\bullet)_f)_0)_\mathfrak{p} =: (S_{(f)})_\mathfrak{p}.$$

This notation is terrifying, the second subscript means localize at an element and the forth subscript means localize at a prime $\mathfrak{p}$. And we will abbreviate it a little by writing $((S_\bullet)_f)_0$ as $S_{(f)}$, do not confuse it with localize at a prime ideal (f).

For an open set $U \subset \operatorname{Proj} S_\bullet$, U is covered by $\{D(g)\}_{g\in G}$. Each stalk at $[P]$ should be interpreted as the equivalence class

$$\mathscr{F}_{[P]} := \coprod_{(g,\mathfrak{p})\in P} (S_{(g)})_{\mathfrak{p}_g} / \sim_{[P]}$$

where $(g,\mathfrak{p}) \in P$ means $D(g) \ni [P]$ and $P \cap S_{(g)} = \mathfrak{p}$.

$$(S_{(g)})_\mathfrak{p} \ni s_{g,\mathfrak{p}} \sim_{[P]} s_{f,\mathfrak{q}} (S_{(f)})_{\mathfrak{q}_f} \text{ iff } (g,\mathfrak{p}) \in P, (f,\mathfrak{q}) \in P \text{ and } s_{q,\mathfrak{p}}/1 = s_{f,\mathfrak{q}}/1 \in S_{(fg)}.$$

Then we can interpret the sections in $\mathscr{F}(U)$ via compatible germs

$$\left\{ \prod_{[P]} s_{[P]} \in \prod_{[P]\in U} \mathscr{F}_{[P]} : \begin{array}{c} \forall [Q] \in U, \exists D(f) \ni [Q] \text{ and } \tilde{s} \in S_{(f)} \text{ s.t.} \\ \tilde{s}|_\mathfrak{q} / \sim_{[K]} = s_{[K]}, \forall (f,\mathfrak{q}) \in K, \forall [K] \in D(g) \cap U \end{array} \right\}$$

where $\tilde{s}|_\mathfrak{q}$ means taking $\tilde{s}/1 \in (S_{(f)})_\mathfrak{q}$. ∎

Exercise?? 2.5.N Check that defining $\mathbb{P}^n_A$ as $\operatorname{Proj} A[x_1,\ldots,x_n]$ agrees with our earlier construction by patching up specific affine schemes. (How do you know that the $D(x_i)$ cover $\operatorname{Proj} A[x_0,\ldots,x_n]$?) ▪

Proof. We have already checked the new definition $\operatorname{Proj} A[x_1,..,x_n]$ is compatible with patching up all $D(f)$. $D(x_i)$ is only a special case. We only need to check $D(x_i)$ indeed covers Proj $V(x_i) = \operatorname{Proj} A[x_1,\ldots,x_n] - D(x_i)$ and

$$\cap_i^n V(x_i) = V\left(\sum_i^n (x_i)\right) = V((x_1,\ldots,x_n))$$

But $(x_1,\ldots,x_n)$ is S_+ in the case $S_\bullet = A[x_1,..,x_n]$. We have

$$\operatorname{Proj} A[x_1,..,x_n] - \cup_i D(x_i) = \cap_i V(x_i) = V(S_+) = \emptyset.$$

∎

Exercise?? 2.5.O Suppose that k is an algebraically closed field. We know from Exercise 4.4.F that the closed points of $\mathbb{P}^n_k$, are in bijection with the points of classical projective space. With our new definition of projective space, a point of classical projective space corresponds to a homogeneous prime ideal of $k[x_0,\ldots,x_n]$. Which homogeneous prime ideal is it? ▪

Proof. Assume $a_0 \neq 0$, $[a_0,\ldots,a_n]$ corresponds to $(x_{1/0}-a_1/a_0,\ldots,x_{n/0}-a_n/a_0)$ in the affine chart $D(x_0)$.

$(x_{1/0}-a_1/a_0,\ldots,x_{n/0}-a_n/a_0)\subset(k[x_0,\ldots,x_n]_{x_0})_0$ would corresponds to a homogeneous prime ideal in $k[x_0,\ldots,x_n]$, as shown in 2.5.C. We can construct the homogeneous prime by constructing it degree by degree and we omit the process and only list the final result (because we know it)

$$\left(x_1-\frac{a_1}{a_0}x_0,\ldots,x_n-\frac{a_n}{a_0}x_0\right)$$

■

Exercise?? 2.5.P If $S_\bullet$ is generated in degree 1, and $f\in S_+$ is homogeneous, explain how to define $V(f)$ “in” $\operatorname{Proj} S_\bullet$ the vanishing scheme of f. (Warning: f in general isn?t a function on $\operatorname{Proj} S_\bullet$). Hence define $V(I)$ for any homogeneous ideal I of S_+. ▪

Proof. It is customary to define the topological space of $V(f)$ as the the homogeneous prime ideals containing f. The trouble is how define the structure sheaf on $V(f)$.

In the affine case $V(I)=\operatorname{Spec} A/I$. In the projective space assume I or f homogeneous, we can still define the vanishing scheme to be

$$\operatorname{Proj} S_\bullet/I,\quad \operatorname{Proj} S_\bullet/(f),$$

which are topologically homeomorphic to $V(I)$ and $V(f)$ in $\mathbb{P}S_\bullet$.

These two are well-defined schemes, the subtlety lies in how to interpret them as subschemes of $\operatorname{Proj} S_\bullet$. These would be a long story and the key word is closed subscheme. ■

Exercise?? 2.5.Q Suppose k is algebraically closed. Describe a natural bijection between one-dimensional subspaces of V and the closed points of $\mathbb{P}V$. Thus this construction canonically (in a basis-free manner) describes the one-dimensional subspaces of the vector space V. ▪

Proof. We define $\mathbb{P}V$ to be $\operatorname{Proj}(\operatorname{Sym}^\bullet V^\vee)$ and we already know $\operatorname{Sym}^\bullet V^\vee\cong k[x_0,\ldots,x_n]$, where $x_0,..,x_n$ form a basis of $V^\vee$.

$\operatorname{Sym}^\bullet V^\vee$ can be naturally regarded as functions on V by sending x_i to $x_i(v)$.

A closed point in $\operatorname{Proj} k[x_0,\ldots,x_n]$ is of the form $[a_0,\ldots,a_n]$ or $\left(x_1-\frac{a_1}{a_0}x_0,\ldots,x_n-\frac{a_n}{a_0}x_0\right)$. We can define the corresponding one dimensional vector space by

$$\left\{v\in V:\left(x_1-\frac{a_1}{a_0}x_0,\ldots,x_n-\frac{a_n}{a_0}x_0\right)v=0\right\}.$$

In return, we can define the closed point to be the ideal of functions in $\operatorname{Sym}^\bullet V^\vee$ that vanishes on the one dimensional linear space. This gives a natural bijection. ■

3. Some properties of schemes

3.1 Topological properties

Exercise?? 3.1.A Show that $\mathbb{P}^n_k$ is irreducible. ▪

Proof. In this case $S_\bullet = k[x_1,..,x_n]$ and $S_+ = (x_1,...,x_n)$. Assume $\mathbb{P}^n_k$ can be written as union of two proper closed subsets: $V(I_1)\cup V(I_2)$, where I_1, I_2 are two homogeneous ideal in $k[x_1,..,x_n]$, also we require I_1, I_2 not contain S_+.

(0) is still a homogeneous prime ideal in $k[x_1,..,x_n]$ because it is integral domain and it does not contain the irrelevant ideal. W.l.o.g, assume $V(I_1) \ni [(0)]$, which means $I_1 \subset (0) \implies I_1 = (0)$. But $V(I_1) = V((0)) = \operatorname{Proj} S_\bullet = \mathbb{P}^n_k$, which contradicts the assumption that $V(I_1)$ is a proper closed subset. ■

Exercise?? 3.1.B Exercise 1.7.E showed that there is a bijection between irreducible closed subsets and points for affine schemes (the map sending a point p to the closed subset $\overline{\{p\}}$ is a bijection). Show that this is true of schemes in general. ▪

Proof. Given a scheme X, we want a bijection

$$\begin{aligned} X &\longrightarrow \{Z \subset X : Z \text{ closed irreducible}\} \\ z &\longmapsto \overline{\{z\}}. \end{aligned}$$

We already showed this bijection exists for affine schemes. This map can be naturally extended to general scheme, with the Zariski topology.

We only need to show

- $\overline{\{z\}}$ is irreducible in X, and
- $\overline{\{z\}} \neq \overline{\{x\}}$ if $z \neq x$, and
- Each irreducible closed subset in X is of the form.

Each $\{z\}$ is irreducible in X, the closure of a irreducible subset is irreducible by 1.6.B.

If $\overline{\{z\}} = \overline{\{x\}}$, they are covered by some affine open subsets U_i. By definition of closure, each U_i contain z, x. $\overline{\{z\}} \cap U_i = \overline{\{z\}} \cap U_i$ are the closure of z, x in the subset topology of U_i, they are irreducible in U_i because $\{z\}, \{x\}$ are irreducible in U_i with subset topology. Then because the bijection exists in affine schemes, there is a unique generic point corresponding to $\overline{\{z\}} \cap U_i$, $z = x$ in U_i, which means $z = x$ in X.

Given an irreducible subset $Z \subset X$. Let U be an affine open subset such that $Z \cap U \neq \emptyset$. $Z \cap U$ is an open set in Z with subspace topology on Z. By 1.6.B, $Z \cap U$ is dense and irreducible in Z, with subspace topology. Then it is also irreducible in U with subspace topology on U. Hence it corresponds to a unique generic point in U. $\exists z \in U \cap Z$ s.t. $Z \cap U = \overline{\{z\}}'$ is the closure of z in $Z \cap U$. However, we already know $Z \cap U$ is dense in Z, therefore $\overline{\{z\}} = Z$.

We have established to bijection between the irreducible closed subsets and generic points of the irreducible closed subsets. ■

Exercise?? 3.1.C Prove that if X is a scheme that has a finite cover $X = \cup_{i=1}^n \operatorname{Spec} A_i$ where A_i is Noetherian, then $X$is a Noetherian topological space (We will soon call a scheme with such a cover a **Noetherian scheme**) Hint: show that a topological space that is a finite union of Noetherian subspaces is itself Noetherian. ■

Proof. We have shown in 1.6.S that affine scheme corresponding to a Noetherian ring is Noetherian topological space.

It suffices to check finite union of Noetherian subspaces $X = \cup_i^n X_i$ is itself Noetherian.

A topological space is called Noetherian if it satisfies the descending chain condition for closed subsets. Given an descending chain of closed subsets in X.

$$Z_1 \supset Z_2 \supset \cdots \supset Z_k \supset \cdots$$

$Z_k = \cup_i^n (X_i \cap Z_k)$, $X_i \cap Z_k$ would give a descending chain of closed subsets in the subspace X_i with subspace topology on X_i. Each would stabilize after m_i. Choose $m = \max\{m_i\}_{1 \le i \le n}$, the descending chain $\{Z_k\} = \{\cup_i^n (X_i \cap Z_k)\}$ would also stabilize after m. ■

Exercise?? 3.1.D Show that a scheme X is quasicompact if and only if it can be written as a finite union of affine open subschemes. (Hence $\mathbb{P}_A^n$ is quasicompact for any ring A.) ■

Proof. One direction is easy, if a scheme X can be written as union of affine open subschemes X_i. Given a open cover of X, its intersection with affine open subschemes also given an open cover of X_i. Affine open subscheme is quasicompact, we can select a finite subcover of these intersection covers, and the union gives a finite subcover of the initial cover.

For the reverse direction, consider a quasicompact scheme X. By definition, for each point x, there is an affine open subscheme U_x contain x. These affine open subscheme give an open cover of X. $\cup_{x \in X} U_x$, we can select a finite subcover $\cup_{x \in I} U_x$ of X. Then the scheme X is the union $\cup_{x \in I} U_x$. Then X can be written as a finite union of affine open subschemes.

In particular, $\mathbb{P}_A^n$ is quasicompact. ■

Exercise?? 3.1.E QUASICOMPACT SCHEMES HAVE CLOSED POINTS. Show that if X is a quasicompact scheme, then every point has a closed point in its closure. Show that every nonempty closed subset of X contains a closed point of X. In particular, every nonempty quasicompact scheme has a closed point. ■

Proof. We already showed that $\overline{\{z\}}$ is a irreducible subset in X. Hence it suffices to show only every nonempty closed subset of a quasicompact scheme X contains a closed point of X.

Given a closed subset $Z \subset X$. Observe that a point $p \in Z$ is closed in Z iff p is closed in X.

X is a finite union of affine opens $\cup_i^n U_i$ by 3.1.D. Consider a finite sub-collection of $\{U_i\}_{1\leq i\leq n}$ that intersects with Z, w.l.o.g, we assume it is $\{U_i\}_{1\leq i\leq m}$. Then a point $p \in Z$ is closed iff p is closed in each $\{U_i\}_{1\leq i\leq m}$ that contains z.

A closed point in U_1 corresponds to maximal ideals in $\operatorname{Spec} A_1$. $Z \cap U_1$ is not empty, hence contains at least one $[\mathfrak{p}]$ and $\overline{\{[\mathfrak{p}]\}} \subset Z \cap U_1$. $\mathfrak{p}$ is contained in a maximal ideal $\mathfrak{m}$. $[\mathfrak{m}] \in \overline{\{[\mathfrak{p}]\}}$, which means there is at least one closed point in $Z \cap U_1$.

We can find a closed point (in U_1) $z_1 \in Z \cap U_1$. If z_1 is also closed in other U_i, we are done.

If not, we can find in $\overline{\{z_1\}} \cap U_2$ a closed point z_2 such that z_2 is closed in U_2. $z_2 \notin U_1$ because if $z_2 \in U_1$, $\overline{\{z_1\}} \cap U_1 \neq \emptyset$, which means z_1 is not closed point in U_1 in the first place.

If z_2 is closed in all U_i (besides U_1) that contains it, we are done. If not, assume z_2 is not closed point in U_3. $\overline{\{z_2\}} \subset \overline{\{z_1\}}$. We can find $z_3 \in \overline{\{z_2\}} \cap U_3$, z_3 can't lie in U_1 or U_2, because it would imply z_1 not close point in U_1, z_2 not close point in U_2.

We can induct on these process until we find z_m is closed in all U_i that contains it. The process would terminate because we started with a finite cover with affine opens.

Any closed subset in quasicompact scheme contains a closed point. In particular, a quasicompact scheme contains a closed point and the closure of each point in a quasicompact scheme contains a closed point. ■

Exercise?? 3.1.F Show that a scheme is quasiseparated if and only if the intersection of any two affine open subsets is a finite union of affine open subsets. ■

Proof. The $\Longrightarrow$ direction is trivial, affine open subsets are quasicompact, if the scheme is quasiseparated, then the intersection of any two affine open subsets A, B would be quasicompact and by 3.1.D this intersection should be finite union of affine open subsets in $A \cap B$. Because $A \cap B$ is endowed with subspace topology of X, $A \cap B$ is also finite union of affine open subsets in X.

For the reverse direction. In a scheme any quasicompact subset is a finite union of affine open subsets, $Y = \cup_i^n U_i$ and $Z = \cup_j^m V_j$. Then the intersection of Y and Z should be

$$Y \cap Z = \cup_{1\leq i\leq n, 1\leq j\leq m} U_i \cap V_j.$$

If each $U_i \cap V_j$ can be written as finite union of affine open subsets, $Y \cap Z$ can be written as finite union of affine open subsets, which is equivalent to $Y \cap Z$ is quasicompact by 3.1.D ■

Exercise?? 3.1.G Show that affine schemes are quasiseparated. ■

Proof. We have to firstly characterize a quasicompact open subset in $\operatorname{Spec} A$. (Do not try to characterize all affine opens in an affine scheme in general, it is difficult. The hint seems to be a trap)

Any open subset U, in $\operatorname{Spec} A$ can written as a union of distinguished base. If in addition this set is quasicompact, we can write it as a finite union of $D(f_i)$.

$$Y \cap Z = \cup_i^n D(f_i) \cap \cup_j^m D(g_j) = \cup_{1 \le i \le n, 1 \le j \le m} D(f_i) \cap D(g_j) = \cup_{1 \le i \le n, 1 \le j \le m} D(f_i g_j)$$

which can be written as a finite union of affine opens. Then $Y \cap Z$ is quasicompact by 3.1.D.

Then intersection of any two quasicompact opens is also quasicompact. Affine scheme is quasiseparated. ■

Exercise?? 3.1.H Show that a scheme X is quasicompact and quasiseparated if and only if X can be covered by a finite number of affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets. ■

Proof. $\Longrightarrow$ direction: X can be written as a finite union of affine open subsets because X is quasicompact by 3.1.D. If in addition X is quasiseparated, each intersection of these affine opens can be written as a finite union of affine open subsets by 3.1.F.

$\Longleftarrow$ direction: Assume X can be written as a finite union of affine open subsets such that each pairwise intersection can be written as finite union of affine open subsets. Then X is quasicompact by 3.1.D.

The quasiseparatedness part of the reverse direction is tricky. It can be cleanly proved using the diagonal morphism arguments, as shown in the first answer to this StackExchange question. We will give a more topological proof here.

Notice that we have proved affine schemes are quasicompact and quasiseparated. The specified affine opens are quasicompact and quasiseparated in the subspace topology. Given Z, Y two quasicompact subsets in X. Each is covered by finitely many of specified affine opens.

$$Z \cap Y = \cup_i (Z \cap Y \cap U_i) = \cup_i (Z \cap U_i) \cap (Y \cap U_i)$$

<u>Claim</u>: $Z \cap U_i$, $Y \cap U_i$ are quasicompact in U_i for all i.

If we have the claim, we know $(Z \cap U_i) \cap (Y \cap U_i)$ is quasicompact because U_i is quasiseparated. And a finite union of quasicompact space is always quasicompact. Hence we are done.

Then the problem reduces to prove the claim:

<u>proof of the claim</u>: Z is quasicompact in X, therefore can be written as finite union of affine opens $Z = \cup_k Z_k$.

$$Z = \cup_{k,i} (Z_k \cap U_i)$$

Z_k is affine open we can choose a distinguished base of $W_{k,i,\alpha} := D(f_{k,i,\alpha})$, each $W_{k,i,\alpha}$ is quasicompact.

$Z_k \cap U_i$ is an open set in Z_k, hence can be written as union of $W_{k,\alpha}$ (not necessarily finite). We have

$$Z = \cup_{k,i} (Z_k \cap U_i) = \cup_{k,i} \cup_{\alpha \in L_{ki}} W_{k,i,\alpha},$$

where L_{ki} is not finite in general. But because Z is quasicompact, there is a finite subcover, we can choose a finite subset F_{ki} for each L_{ki} so that

$$Z = \cup_{k,i} \cup_{\alpha \in F_{ki}} W_{k,i,\alpha}.$$

(Now $\cup_{\alpha \in F_{ki}} W_{k,\alpha}$ no longer cover $Z_k \cap U_i$ in general) Intersect it with U_i, we get

$$Z \cap U_1 = \cup_{k,i} \cup_{\alpha \in F_{ki}} W_{k,i,\alpha} \cap U_1.$$

The condition that "$U_i \cap U_j$ can be written as a finite union of affine opens $\cup_{l \in M_{ij}} V_{ij;l}$ " would be used in the next step, where M_{ij} is a finite index set depending on i, j.

$$Z \cap U_1 = \cup_{k,i} \cup_{\alpha \in F_{ki}} W_{k,i,\alpha} \cap U_1 \overset{(*)}{=} \cup_{k,i} \cup_{\alpha \in F_{ki}} W_{k,i,\alpha} \cap U_1 \cap U_i = \cup_{k,i} \cup_{\alpha \in F_{ki}} \cup_{l \in M_{1i}} W_{k,i,\alpha} \cap V_{1i;l}.$$

where $(*)$ holds because each $W_{k.i.\alpha} \subset U_i$. $W_{k,i,\alpha}$ and $V_{1i;l}$ are quasicompact subset in U_i and their intersection is quasicompact because U_i is quasiseparated. $Z \cap U_1$ is finite union of quasicompact subsets, therefore is compact. We proved the claim there fore the quasiseparatedness part of the reverse direction. ■

Exercise?? 3.1.I Show that all projective A-schemes are quasicompact and quasiseparated. ■

Proof. By definition a projective A-scheme X is defined to be $\operatorname{Proj} S_\bullet$, where $S_\bullet$ is a finitely generated graded ring over A. A graded ring is finitely generated iff S_+ is finitely generated. Assume $S_+ = (x_1, ..., x_n)$, where each x_i is homogeneous, we can consider an open cover of X:

$$D(x_i).$$

We already showed that $D(x_i)$ is affine open subset that is isomorphic to $\operatorname{Spec}((S_\bullet)_{x_i})_0$.

$$D(x_i) \cap D(x_j) = D(x_i x_j).$$

In summary, $X = \operatorname{Proj} S_\bullet$ can be covered by a finite union of affine open subsets and each intersection of these affine open subsets can be written again as finite union of affine open subsets. Then we can conclude that $X = \operatorname{Proj} S_\bullet$ is quasicompact and quasiseparated by 3.1.H. ■

Exercise?? 3.1.J Let $X = \operatorname{Spec} k[x_1, x_2, ...]$, and let U be $X - [\mathfrak{m}]$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, ...)$. Take two copies of X, glued along U ("affine ∞-space with a doubled origin", see Example 4.4.5 and Exercise 2.4.C for "finite-dimensional" versions). Show that the result is not quasiseparated. ■

Proof. We glue two copies of $\operatorname{Spec} k[x_1, x_2, ...]$ X, X'along the open

$$U := \operatorname{Spec} k[x_1, x_2, ...] - [\mathfrak{m}],$$

and denote the resulting scheme as $\mathbb{A}^{\infty}_{(2)}$. We showed in Exercise 1.6.G(b) that U is not quasicompact in $\operatorname{Spec} k[x_1, x_2, ...]$.

Denote the the quotient map $\pi : X \coprod X' \longrightarrow \mathbb{A}^{\infty}_{(2)}$. $\mathbb{A}^{\infty}_{(2)}$ is equipped with quotient topology, which make the quotient map continuous. The images $\pi(X)$ and $\pi(X')$ in $\mathbb{A}^{\infty}_{(2)}$ are quasicompact $\operatorname{Spec} B$ is quasicompact for any ring B. And $\pi(U)$ is not quasicompact because $\cup_{i \in \mathbb{Z}} \pi(D(x_i))$ is an open cover of $\pi(U)$ with no finite subcover.

$\pi(X) \cap \pi(X') = \pi(U)$. Intersection of two quasicompact open subsets is not quasicompact, therefore $\mathbb{A}^{\infty}_{(2)}$ is not quasiseparated. ■

3.2 Reducedness and integrality

Exercise?? 3.2.A (REDUCEDNESS IS A stalk-local PROPERTY, I.E., CAN BE CHECKED AT STALKS). Show that a scheme is reduced if and only if none of the stalks have nonzero nilpotents. Hence show that if f and g are two functions (global sections of $\mathscr{O}_X$) on a reduced scheme that agree at all points, then $f = g$. ■

Proof. "$\Longrightarrow$": A scheme X is reduced then $\mathscr{O}_X(U)$ is reduced for all open subsets $U \subset X$. Assume $[(f \in \mathscr{O}_X(U); U)] \in \mathscr{O}_{X,p}$ is nilpotent, which means $f|_V^n = 0 \in \mathscr{O}_X(V)$ for some open neighborhood V of p. But $\mathscr{O}_X(V)$ is reduced ring, $\Longrightarrow f|_V = 0$ and $[(f \in \mathscr{O}_X(U); U)] = 0$, which implies there is no non-zero nilpotents in $\mathscr{O}_{X,p}$.

"$\Longleftarrow$": Given $f \in \mathscr{O}_X(U)$, assume $f^n = 0$. Then its image to stalks are also nilpotent. The condition that stalks are reduced implies the image of f to each stalk is zero. By **??**, we know the morphism

$$\mathscr{O}_X(U) \longrightarrow \prod_{p \in U} \mathscr{O}_{X,p}$$

is injective, therefore we know $f = 0 \in \mathscr{O}_X(U)$.

We have to clarify what it means by "agree on all points". It means given any affine open subset U, $(U, \mathscr{O}_X(U)) \cong (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$, $f|_U \equiv g|_U \mod \mathfrak{p}, \forall [\mathfrak{p}] \in \operatorname{Spec} A$.

$$f|_U \equiv g|_U \mod \mathfrak{p}$$

means $f|_U \equiv g|_U$ divers only by a nilpotent. Because $\mathscr{O}_X(U)$ is reduced, we have $f|_U = g|_U$. f, g agree when restricted to a base of topology, then f, g have same germs on each stalks which means $f = g$. ■

R We should have mentioned this before. When $f = g + n$, where $n \in \sqrt{(0)}$, they are generally not equal when taking germs

$$\frac{n}{1} \neq \frac{0}{1} \in A_{\mathfrak{p}}.$$

For example, $A = k[x, \varepsilon]/(\varepsilon^2)$, $f = x^2, g = x^2 + \varepsilon$. $\varepsilon/1 \neq 0/1 \in A_{(\varepsilon)}$.

Exercise?? 3.2.B If A is a reduced ring, show that $\operatorname{Spec} A$ is reduced. Show that $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are reduced. ■

Proof. We state a stronger version:

Spec A is reduced scheme iff A is a reduced ring.

One direction is already contained in the definition of reduced scheme.

For the other direction, we prove an algebraic fact. Any localization of a reduced ring is reduced.

Given any multiplicative subset $S \subset A$. Assume $a/s \in S^{-1}A$ is nilpotent $(a/s)^n = 0$, there exists $t \in S$ s.t. $ta^n = 0 \in A$, but then we have $(ta)^n = 0 \in A$ and A being reduced, hence $ta = 0$. $\Longrightarrow a/s = 0 \in S^{-1}A$.

Exercise 3.2.A also translate to: A is reduced iff $A_{\mathfrak{p}}$ is reduced for all $[\mathfrak{p}] \in \operatorname{Spec} A$. We will give a algebraic proof here. Remember that taking radical is commutative with localization, $\mathfrak{N}(A)_{\mathfrak{p}} = \mathfrak{N}(A_{\mathfrak{p}})$. Also notice that being zero is a local property for A-modules, each $A_{\mathfrak{p}}$ begin reduced implies $\mathfrak{N}(A)$ is zero.

We even have a stronger version: A is reduced iff $A_{\mathfrak{m}}$ is reduced for all maximal ideals in A. (Because the "being zero" local property of A-module says an A module M is zero iff $M_{\mathfrak{m}}$ is zero module for all maximal ideals in A.)

But note that being reduced can not be checked only on closed points for general schemes, we need quasicompactness, see Exercise 3.2.D below.

Then $\mathbb{A}_k^n$ is reduced because $k[x_1,\dots,x_n]$ is reduced. On the other hand, $\mathbb{P}_k^n$ is reduced because it can be covered by $\mathbb{A}_k^n$ and hence each stalk is reduced. ■

Exercise?? 3.2.C Show that $\left(k[x,y]/(y^2,xy)\right)_x$ has no nonzero nilpotent elements. (Possible hint: show that it is isomorphic to another ring, by considering the geometric picture. Exercise 1.2.L may give another hint.) Show that the only point of $\operatorname{Spec} k[x,y]/(y^2,xy)$ with a nonreduced stalk is the origin.

Proof. Geometrically $\operatorname{Spec}\left(k[x,y]/(y^2,xy)\right)_{\bar{x}}=$ the distinguished open $D([x])$ in $\operatorname{Spec} k[x,y]/(y^2,xy)$. It is equivalent to the non-vanishing loci of x in $V(y^2,xy)$ in the affine plain, which is the affine line with origin removed. This geometric picture gives us some intuition

Claim:

$$\left(k[x,y]/(y^2,xy)\right)_{\bar{x}}=k[x]_x,$$

where $\bar{x}$ is the image of x in the quotient ring. A general element in $k[x,y]/(y^2,xy)$ is of the form

$$p(\bar{x})+a\bar{y},$$

where p is a polynomial in $k[x]$, a is a constant in k and $\bar{x}\bar{y}=0$ and $\bar{y}^2=0$. A general element in $\operatorname{Spec}\left(k[x,y]/(y^2,xy)\right)_{\bar{x}}$ is of the form

$$\frac{p(\bar{x})}{\bar{x}^n}.$$

$\bar{y}/1=0$ because $\bar{x}\bar{y}=0$. There is a well-define ring morphism

$$\frac{p(\bar{x})}{\bar{x}^n}\mapsto\frac{p(x)}{x^n},$$

Which is obviously an isomorphism.

$k[x]_x$ is reduced because $k[x]$ is reduced. We then know there is no nonzero nilpotents in $\left(k[x,y]/(y^2,xy)\right)_{\bar{x}}$.

Consider the ring of $\operatorname{Spec}\left(k[x,y]/(y^2,xy)\right)$. The prime ideals of this ring corresponds to prime ideals in $k[x,y]$ that contains (y^2,xy). Hence a general prime ideal in this ring is of the form $(\bar{x}-a,\bar{y})$ or $(\bar{y})$. When $a\neq 0$, $\bar{x}\notin(\bar{x}-a,\bar{y})$. $\bar{x}$ is invertible in $\left(k[x,y]/(y^2,xy)\right)_{(\bar{x}-a,\bar{y})}$, hence it is further localization of $\left(k[x,y]/(y^2,xy)\right)_{\bar{x}}$ and should be reduced.

When $a=0$, we focus on the ring

$$\left(k[x,y]/(y^2,xy)\right)_{(\bar{x},\bar{y})}.$$

The multiplicative set consists of elements like $b(\bar{x})+c\bar{y}$, where $b(\bar{x})$ is a polynomial of $\bar{x}$ and the constant term $b_0\neq 0$. $\bar{y}(b\bar{x}+c\bar{y})^n=b_0^n\bar{y}\neq 0$, therefore $\bar{y}/1\neq 0/1$. And $\bar{y}/1$ is nilpotent.

$\left(k[x,y]/(y^2,xy)\right)_{(\bar{x},\bar{y})}$ contains at least one nonzero nilpotent $\bar{y}/1$, hence is nonreduced. ■

Exercise?? 3.2.D If X is a quasicompact scheme, show that it suffices to check reducedness at closed points. Hint: Do not try to show that reducedness is an open condition (see Remark 5.2.2). Instead show that any nonreduced point has a nonreduced closed point in its closure, using Exercise 3.2.E. (This result is interesting, but we won't use it.) ▪

Proof. Follow the hint, we prove the contrapositive: If a quasicompact scheme is not reduced then at least one of its closed points has non-reduced stalk.

By definition, we know if X is not reduced then at least one of its stalk is not reduced by 3.2.A. Denote this non-reduced point by p, then there is a closed point $z \in \overline{\{p\}}$ by 3.1.E. (I suppose the hint in the body of this problem is wrong). The stalk $\mathscr{O}_{X,p}$ is a localization of the stalk $\mathscr{O}_{X,z}$ at closed point. $\mathscr{O}_{X,p}$ is not reduced, therefore $\mathscr{O}_{X,z}$ is also non-reduced. ■

Exercise?? 3.2.E Suppose X is quasicompact, and f is a function that vanishes at all points of X. Show that there is some n such that $f^n = 0$. Show that this may fail if X is not quasicompact. (This exercise is less important, but shows why we like quasicompactness, and gives a standard pathology when quasicompactness doesn't hold.) ▪

Proof. f is a global section on a quasicompact scheme X that vanishes at each point.

X is quasicompact and can be written as a finite union of affine open subschemes $\cup_i U_i$, where each $U_i \cong \operatorname{Spec} A_i$. We then have $f|_{U_i} \mathscr{O}_X(U_i) = A_i$. $f|_{U_i}$ vanishes on each point $[\mathfrak{p}] \in \operatorname{Spec} A_i$ and hence is an element in the nilpotent $\mathfrak{A}_i$, $\Longrightarrow f_{U_i}^{n_i} = 0$ for some $n_i \in \mathbb{Z}$. There are only finitely many such affine opens, we can choose $n := \max_i\{n_i\}$ and then $f^n|_{U_i} = 0 \in \mathscr{O}_X(U_i)$ and there is only one element 0 to glue to in the global section.

For the counterexample when X is not quasicompact, we follow the hint and consider the union of infinite disjoint union of $\operatorname{Spec} A_n$, where $A_n = k[\varepsilon]/(\varepsilon^n)$. Each $\operatorname{Spec} A_n$ consists of only one point $[(\varepsilon)]$. Consider the function $f = \coprod_n \varepsilon$. We know f vanishes at each each point but is not nilpotent in the global section. ■

Exercise?? 3.2.F Show that a scheme X is integral if and only if it is irreducible and reduced. (Thus we picture integral schemes as: "one piece, no fuzz") ▪

Proof. In a scheme X is integral, $\mathscr{O}_X(U)$ is integral for all open subsets, hence $\mathscr{O}_X(U)$ is also reduced because integral domain has no nonzero zero divisors.

An integral scheme should be irreducible. Assume contrarily X is reducible, and can be written as union of two closed subsets $X = Y \cup Z$. Define the complements $U := X - Y$ and $V = X - Z$, we know U, V are nonempty opens and their have empty intersection. The structure sheaf $\mathscr{O}_X(U \cup V) = \mathscr{O}_X(U) \times \mathscr{O}_X(V)$ which is not integral in general.

For the reverse direction, we would use the hint 2.3.G.

<u>Claim</u>: Any open subset U in a irreducible space X is irreducible with the subset topology.

For any other nonempty open subset $U \subset X$. By 1.6.B, U is dense in X. Assume U is reducible and can be written as $(Y \cap U) \cup (Z \cap U) = (Y \cup Z) \cap U$, where Y, Z are nonempty proper closed subsets in X. We also know $Y \cup Z \supset U$. Hence $\overline{Y \cup Z} = \overline{Y} \cup \overline{Z} = Y \cup Z \supset \overline{U} = X$, which means X is reducible, contradiction.

Given $f, g \in \mathscr{O}_X(U)$ and $fg = 0 \in \mathscr{O}_X(U)$. Recall 2.3.G, the set where f, g vanishes is closed. (Notice here vanishes means $f_p g_p \in \mathfrak{m}_p$). U could be covered by the closed set V f vanishes and the closed set W where g vanishes. Then because X is irreducible, we know W or V must be the whole space X. Assume f vanishes on every point in X, we know $f = 0 \in \mathscr{O}_X(U)$ because U is a reduced scheme 3.2.A. ■

Exercise?? 3.2.G Show that an affine scheme $\operatorname{Spec} A$ is integral if and only if A integral domain. ■

Proof. The "only if" direction is obvious, because $\Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) = A$.

Consider the "if" direction. We simply have $\operatorname{Spec} A$ is irreducible if A is an integral domain by 1.6.C. Also an integral domain is always reduced ring, hence $\operatorname{Spec} A$ is reduced by 3.2.B. We know $\operatorname{Spec} A$ is irreducible and reduced if A is integral domain, hence $\operatorname{Spec} A$ is integral by 3.2.F. ■

Exercise?? 3.2.H Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} A$ is any nonempty affine open subset of X. Show that the stalk at η, $\mathscr{O}_{X,\eta}$ is naturally identified with $K(A)$, the fraction field of A. This is called the **function field** $K(X)$ of X. It can be computed on any nonempty open set of X, as any such open set contains the generic point. The reason for the name: we will soon think of this as the field of *rational functions* on X ■

Proof. Suppose X is an integral scheme. X is itself irreducible closed and hence corresponds to a unique generic point by 3.1.B. This generic point η is contained in any open subset of X. Specifically, it is contained in affine open $U := \operatorname{Spec} A$ and corresponds to $[(0)] \in \operatorname{Spec} A$, where A is an integral domain. $\mathscr{O}_{X,\eta} \cong A_{[(0)]} = K(A)$, where localization at the prime ideal (0) is isomorphic to the fraction field $K(A)$. ■

Exercise?? 3.2.I Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V}$: $\mathscr{O}_X(U) \longrightarrow \mathscr{O}_X(V)$ are inclusions so long as $V \neq \emptyset$. Suppose $\operatorname{Spec} A$ is any nonempty affine open subset of X (so A is an integral domain). Show that the natural map $\mathscr{O}_X(U) \longrightarrow \mathscr{O}_{X,\eta} = K(A)$ (where U is any nonempty open subset) is an inclusion. ■

Proof. In fact, the first question relies on the second. Because we have the composition of maps

$$\mathscr{O}_X(U) \xrightarrow{\operatorname{res}_{U,V}} \mathscr{O}_X(V) \longrightarrow \mathscr{O}_{X,\eta},$$

it suffices to prove $\mathscr{O}_X(U) \longrightarrow \mathscr{O}_{X,\eta}$ is injective. ($\operatorname{res}_{U,V}$ has to be injective because its composition with an injection is again an injection.)

Claim: $\mathscr{O}_X(U) \longrightarrow K(X)$ is injective.

Assume a section $f \in \mathscr{O}_X(U)$ and $f_\eta = 0$. Want: $f = 0 \in \mathscr{O}_X(U)$. It suffices to prove that $f_\eta \implies f|_W = 0$ for all affine open subsets contained in U.

For an affine open $W = \operatorname{Spec} A$, the natural morphism

$$\mathscr{O}_X(W) \longrightarrow \mathscr{O}_{X,\eta} = K(A)$$

as shown in 3.2.H. This map must be the canonical inclusion because the stalk at $[(0)]$ is a further localization of any $A_f = \mathscr{O}_{\operatorname{Spec} A}(D(f))$.

We have prove both questions. ■

3.3 Properties of schemes that can be checked "affine-locally"

Exercise?? 3.3.A Show that locally Noetherian schemes are quasiseparated. ▪

Proof. In a locally Noetherian scheme, each affine open is isomorphic to $\operatorname{Spec} A$ for some Noetherian ring A. By 3.1.F, we can check whether any intersection of two affine open can be written as finite union of affine opens.

Given $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ for A, B Noetherian rings, consider then intersection $U \cap V$. By proposition 5.3.1, we know $U \cap V$ can be written as union of open sets such that are simultaneously distinguished open subschemes in $\operatorname{Spec} A$ and $\operatorname{Spec} B$. But notice by 1.6.T that open subset of Noetherian space is quasicompact, we can cover $U \cap V$ with finitely many simultaneous distinguished opens, which are affine opens.

We don't need Prop 5.3.1 to prove this statement but it is more clear to see which of the affine opens are selected. ∎

Exercise?? 3.3.B Show that a Noetherian scheme has a finite number of irreducible components. (Hint: Proposition 3.6.15.) Show that a Noetherian scheme has a finite number of connected components, each a finite union of irreducible components. ▪

Proof. Claim: The underline topological space of a Noetherian scheme is Noetherian.

Note that a Noetherian scheme is locally Noetherian and quasicompact, we know a Noetherian scheme X can be covered by finitely many Noetherian affine subschemes.

subclaim: A finite union of Noetherian subspace is also Noetherian.

We can prove the subclaim directly by checking the descending chain condition on closed subsets. Given a chain $V_0 \supset V_1 \supset \cdots$, each V_i can be covered by finitely many U_j, where U_j is Noetherian subsapce. It renders a descending chain of closed subset in U_j with subset topology, hence will stabilize after n_j. Choose $N = \max\{n_j\}$, we have $V_n = \cup_n (V_N \cap U_j)$ would stabilize when $n \geq N$.

Then in particular, we know a Noetherian scheme is Noetherian topological space and by Prop 3.6.15, we know it has only finitely many irreducible components.

Exercise 1.6.Q already showed that connected components in topological space X are unions of irreducible components of X. But in the case of Noetherian scheme, there are only finitely many irreducible components, hence X is the union of finitely many connected components and each connected component is union of finitely many irreducible components. ∎

Exercise?? 3.3.C Show that a Noetherian scheme X is integral if and only if X is nonempty and connected and all stalks $\mathscr{O}_{X,p}$ are integral domains. Thus in "good situations", integrality is the union of local (stalks are integral domains) and global (connected) conditions. ▪

Proof. We follow the hint and recall Exercise 3.2.F says integral = irreducible + reduced. Note that "being reduced" is stalk-local 3.2.A.

$\Longrightarrow$ direction is easy:

"Irreducible" alone guarantees that X is connected. And also note that localization of integral domain is also integral domain because $\frac{a}{s}\frac{b}{t} = \frac{0}{1} \Longrightarrow \exists r \in S, rab = 0 \Longrightarrow ab = 0$.

We want to prove the reverse direction under Noetherian hypothesis.

Each stalk being integral domain alone implies each stalk is reduced, hence the scheme X is reduced.

Want: under Noetherian hypothesis, connected + stalks being integral domain $\implies X$ is irreducible.

X is Noetherian, we know each connected component is union of finitely many irreducible components. We know X is connected and hence X is the union of finitely many irreducible components, and some of the pair-wise intersections of these irreducible components are nonempty to make sure their union is connected.

Assume Y, Z are two of these irreducible components and their intersection is non-empty. $p \in Y \cap Z$. Claim: $\mathscr{O}_{X,p}$ is not integral domain.

Choose an affine open $U \ni p$, $U = \operatorname{Spec} A$ is the union of $U \cap Y$ and $U \cap Z$, U is not irreducible and A is not integral domain 1.6.C. Assume $U \cap Y = V(\mathfrak{q}_1)$ and $U \cap Z = V(\mathfrak{q}_2)$, where $\mathfrak{q}_1, \mathfrak{q}_2$ are two prime ideals in A, they should be minimal prime because there is a bijection 1.7.F between the minimal primes and irreducible components in $\operatorname{Spec} A$. $V(\mathfrak{q}_1) \cap V(\mathfrak{q}_2) = V(\mathfrak{q}_1 + \mathfrak{q}_2)$. $p = [\mathfrak{p}] \in V(\mathfrak{q}_1 + \mathfrak{q}_2) \implies \mathfrak{q}_1 + \mathfrak{q}_2 \subset \mathfrak{p}$. The stalk at p, is $\mathscr{O}_{X,p} = A_{\mathfrak{p}}$. Then there are at least two minimal prime ideals in $A_{\mathfrak{p}}$, therefore $A_{\mathfrak{p}}$ can not be integral domain.

In short, we have **the minimal primes in $\mathscr{O}_{X,p}$ is in one to one correspondence with the irreducible components that pass through** p.

The only possibility is that X is irreducible. Combined with our proof that X is reduced, we know X is integral scheme. ■

Exercise?? 3.3.D

(a) (quasiprojective implies finite type) If X is a quasiprojective A-scheme (Definition 4.5.9), show that X is of finite type over A. If A is furthermore assumed to be Noetherian, show that X is a Noetherian scheme, and hence has a finite number of irreducible components.

(b) Suppose U is an open subscheme of a projective A-scheme. Show that U is locally of finite type over A. If A is Noetherian, show that U is quasicompact, and hence quasiprojective over A, and hence by (a) of finite type over A. Show this need not be true if A is not Noetherian. Better: give an example of an open subscheme of a projective A-scheme that is not quasicompact, necessarily for some non-Noetherian A.

Proof.

(a) a quasiprojective A-scheme X is a quasicompact open subscheme of projective A-scheme Y. A projective A-scheme is of the form $Y := \operatorname{Proj} S_\bullet$, where $S_\bullet$ is finitely generated graded ring over A. It is locally of finite type over A. Because it is covered by affine open sets $\operatorname{Spec}((S_\bullet)_f)_0$, where $(S_\bullet)_f$ is finitely generated over A because it is generated by $\{S_\bullet, 1/f\}$ hence its degree zero piece is finitely generated over A.

Recall the Affine Communication Lemma 5.3.2 and also check that "being finite over A" is indeed an affine-local property. Then $Y = \cup_{i \in I} \operatorname{Spec} B_i$ and $\operatorname{Spec} B_i$ is of finite generated algebra over A, $\implies$ any affine open $U \subset X$ is of finite type over A. In particular X can be cover covered by affine open $\operatorname{Spec}_{j \in J} B_j$, where B_j is finitely generated A-algebra, therefore X is of locally finite type over A. In addition X is quasicompact, we know X is of finite type over A.

If A is furthermore assumed to be Noetherian, then each affine open of X is isomorphic to $\operatorname{Spec} B_i$, where B_i is Noetherian because B_i is finitely generated over a Noetherian ring A, hence is also Noetherian by Hilbert basis theorem. Then it has finite number of irreducible components by 3.3.B.

(b) For same reasons, an open subscheme in a scheme of locally of finite type over A is locally of finite type over A, therefore any open subscheme U of a projective A-scheme Y is locally of finite type over A.
If A is Noetherian, we know U is locally Noetherian. Also notice that a projective A-scheme is quasicompact and quasiseparated by 3.1.I.
A projective A-scheme over a Noetherian ring A is a Noetherian scheme.
By 1.6.T, an open subset of a Noetherian topological space is quasicompact. We know U is quasicompact and locally Noetherian, hence is Noetherian. U is also quasiprojective and of finite type over A.
As for the counterexample. Consider the silly example 4.5.11, $\operatorname{Proj} A[T] \cong \operatorname{Spec} A$, where $x_0 = T$ is of degree 1. Also recall 1.6.G, $\operatorname{Spec} A$ can have non-quasicompact open subscheme.

■

R The part of (locally)Noetherian and quasicompactness does not depend on (Projective), hence we have in general:
Any open or closed subscheme of a (locally) Noetherian scheme is (locally) Noetherian.

Exercise?? 3.3.E

(a) Showthat $\operatorname{Spec} k[x_1,\dots,x_n]/I$ is an affine k-variety if and only if $I \subset k[x_1,\dots,x_n]$ is a radical ideal.

(b) Suppose $I \subset k[x_0,\dots,x_n]$ is a radical graded ideal. Show that $\operatorname{Proj} k[x_0,\dots,x_n]/I$ is a projective k-variety. (Caution: The example of $I = (x_0^2, x_0x_1, \dots, x_0x_n)$ shows that $\operatorname{Proj} k[x_0,\dots,x_n]/I$ can be a projective k-variety without I being radical.)

■

Proof. (a) $X :=\operatorname{Spec} k[x_1,\dots,x_n]/I$ is of finite type over k.
k is Noetherian, $\operatorname{Spec} k[x_1,\dots,x_n]/I$ is a projective k-scheme. We know open subscheme of a projective A-scheme over a Noetherian ring is of finite type over A by Exercise 3.3.D.
Want: $\operatorname{Spec} k[x_1,\dots,x_n]/I$ is reduced iff I is radical ideal.
Recall 3.2.B, an affine scheme $\operatorname{Spec} A$ is reduced iff A is reduced. The nilradical of $k[x_1,\dots,x_n]/I$ is just $\sqrt{I}$. We know $k[x_1,\dots,x_n]/I$ is reduced iff I is radical.

(b) We define projective k-variety to be reduced k-scheme. $\operatorname{Proj} k[x_1,\dots,x_n]/I$ is automatically of finite type over k by 3.3.D.
We know I is radical homogeneous ideal, then $S_\bullet := k[x_1,\dots,x_n]/I$ is a reduced graded ring, hence $((S_\bullet)_f)_0$ is reduced and so is any further localizations. We have checked any stalks of $\operatorname{Proj} k[x_1,\dots,x_n]/I$ is reduced ring, hence it is a reduced projective k-scheme, i.e., a projective k-variety.

■

Exercise?? 3.3.F Show that a point of a locally finite type k-scheme is a closed point if and only if the residue field of the stalk of the structure sheaf at that point is a finite extension of k. Show that the closed points are dense on such a scheme (even though it needn't be quasicompact, cf. Exercise 3.1.E). ■

Proof. We have a stronger version, for a locally finite type k-scheme X, the followings are equivalent:

(a) the point $p \in X$ is closed point
(b) the field extension $k \hookrightarrow K(\mathscr{O}_{X,p})$ is finite
(c) the field extension $k \hookrightarrow K(\mathscr{O}_{X,p})$ is algebraic

p is contained in an affine open subset $U = \operatorname{Spec} A$ of X, where A is a k-algebra and it corresponds to a maximal ideal $\mathfrak{m}$ in A. By Hilbert Nullstellensatz, $K(\mathscr{O}_{X,p}) \cong A/\mathfrak{m}$ is a finite field extension of k.

A finite field extension is automatically an algebraic field extension. These proves $(a) \Longrightarrow (b), (c)$.

For $(c) \Longrightarrow (a)$, p is a point in $\operatorname{Spec} A$, and $\mathfrak{p}$ is the corresponding prime ideal. $K(\mathscr{O}_{X,p}) \cong Frac(A/\mathfrak{p})$ is algebraic over k, (It is integral over k, hence the integral domain $A/\mathfrak{p}$ is integral over k.) Recall a lemma in commutative algebra, Lemma 1.9 in [1]. Or one can use directly [Stacks, Lemma 00GS]

B is integral over A, and both of them are integral domains. Then A is a field $\Longleftrightarrow$ B is a field.

We know then $A/\mathfrak{p}$ is a field, therefore $\mathfrak{p}$ is a maximal ideal ideal, which means p is a closed point.

Claim:Let k be a field. Let X be a k-scheme locally of finite type. Let U be a non-empty affine open subset of X. Then any closed point of U is a closed point of X.

In an affine open subset $U \subset X$, the residue field at a closed point p is finite field extension of k, which means it is also a closed point in X.

Hence, there is a closed point in each affine open, therefore the closed point is dense in X. ■

Exercise?? 3.3.G Suppose X is a reduced, finite type $\mathbb{C}$-scheme. Define the corresponding complex analytic prevariety X_{an}. (The definition of an analytic prevariety is the same as the definition of a variety without the Hausdorff condition.) ■

Proof. It would be a long story, **To Be Added**. The major reference should be Chapter$4-5$ of [2] ■

Exercise?? 3.3.H Finish the proof of Proposition 5.3.3(a). ■

Proof. $A = (f_1, \ldots, f_n)$. It remains to show that, given a strictly increasing chain of ideals $I_1 \subsetneq I_2 \subsetneq \cdots$ in A, we can construct a strictly increasing chain of ideals in A_{f_i} by

$$I_{i,1} \subset I_{i,2} \subset \cdots,$$

where $I_{i,j} = I_j \otimes_A A_{f_i}$. It suffices to show that for each j,

$$I_{i,j} \subsetneq I_{i,j+1},$$

for some i. (Notice that this does mean there is a strictly increasing chain is some A_{f_i}) But it means there is at least a chain $I_{i,1} \subset I_{i,2} \subset \cdots$, not stabilize in the Noetherian ring A_{f_i}, contradiction.

Each $I_{i,j}$ could be interpreted as localization of the A-module I_j at f_i by **??**

Claim: $A = (f_1, \dots, f_n)$. Consider an A-module M, it is zero iff $M_{f_i} = 0$ for all f_i.

We only need the part: $\forall f_i, M_{f_i} = 0 \Longrightarrow M = 0$. Assume $M_{f_i} = 0$ for all i, choose $m \in M$, exists $n_i \geq 0$ s.t., $f_i^n m = 0$. On the other hand $1 = \sum_{i=1}^n r_i f_i$, choose $N \geq \sum_{i=1}^n n_i + 1$. $1 \cdot m = 1^N m = (\sum_i r_i f_i)^N = \sum_i a_i f_i^{n_i} m = 0$. Hence $M = 0$.

Or equivalently, we can prove it by observing that each $M_{\mathfrak{p}}$ is further localization of some M_{f_i}. All $M_{f_i} = 0$ implies all $M_{\mathfrak{p}} = 0$, which in return implies $M = 0$ [Stacks, Lemma 00HN].

We apply the above claim to $I_{j+1}/I_j \neq 0$, therefore we have for each j,

$$I_{i,j+1}/I_{i,j} \neq 0$$

for some i. ■

Exercise?? 3.3.I In text, we want to prove the proposition

"Given $A = (f_1, \dots, f_n)$, then A is a finitely generated B-algebra iff each A_{f_i} is a finitely generated B-algebra."

with some "partition of unity argument". Make this argument precise.

Proof. One direction is clear, $\operatorname{Spec} A_{f_i}$ is generated by $A, 1/f_i$. If A is a finitely generated B-algebra, so is A_{f_i}.

A is generated by f_i. $1 = \sum_i c_i f_i$ for $c_i \in A$. A_{f_i} is finitely generated as B-algebra and we assume its generators are $\{r_{ij}/f_i^{k_j}\}$.

Claim: $\{f_i\} \cup \{c_i\} \cup \{r_{ij}\}$ generate A as a B-algebra.

r is an element in A and restricts to $r/1 \in A_{f_i}$, which could be written as

$$\sum_j b_{ij} \frac{r_{ij}}{f_i^{k_{ij}}} = \frac{\sum_j b_{ij} r_{ij} f_i^{K_{ij}}}{f_i^{Q_i}} =: \frac{P_i(r_{ij}, f_i)}{g_i},$$

where we define $P_i := \sum_j b_{ij} r_{ij} f_i^{K_{ij}} \in B[r_{ij}, f_i]$ and $g_i := f_i^{Q_i}$ for simplicity.

P_i/g_i and P_j/g_j would agree on $A_{g_i g_j}$, which means

$$(g_i g_j)^{m_{ij}} (P_i g_j - P_j g_i) = 0.$$

By taking $m = \max\{m_{ij}\}$, (the indexing set is finite), we can simplify the notation

$$(g_i g_j)^m (P_i g_j - P_j g_i) = 0,$$

for all i, j.

Let $T_i := g_i^m P_i$ and $h_j := g_j^{m+1}$, we have

$$T_i h_j - T_j h_i = 0.$$

Now, because $D(f_i^n) = D(f_i)$, we have $\operatorname{Spec} A = \cup_i D(h_i)$ and this implies $1 = \sum_i s_i h_i$, where each s_i can be written as a polynomial of c_i, f_i. Define

$$r' := \sum_i s_i T_i.$$

r' restricts to each T_i/h_i, because

$$r'h_j = \sum_i s_i T_i h_j = \sum_i s_i h_i T_j = T_j.$$

In other words, it restricts to each

$$\sum_j b_{ij} \frac{r_{ij}}{f_i^{k_{ij}}}$$

and by the identity axiom $r = r'$, which can be expressed as a polynomial in $B[\{f_i\} \cup \{c_i\} \cup \{r_{ij}\}]$ ■

3.4 Normality and factoriality

Exercise?? 3.4.A Show that integrally closed domains behave well under localization: if A is an integrally closed domain, and S is a multiplicative subset not containing 0, show that $S^{-1}A$ is an integrally closed domain. ■

Proof. Assume $x \in K(A)$ is integral over $S^{-1}A$. It satisfies a monic polynomial

$$x^n + \frac{a_{n-1}}{s_{n-1}} x^{n-1} + \cdots + \frac{a_0}{s_0} = 0.$$

Define $s := \prod_i s_i$ and if multiply the above equation by s^n, we get a polynomial of sx with coefficients in A. Because A is integral closed, we know $sx \in A$, hence $x \in S^{-1}A$, which means $S^{-1}A$ is integral closed. ■

Exercise?? 3.4.B Show that a Noetherian scheme is normal if and only if it is the finite disjoint union of integral Noetherian normal schemes. ■

Proof. $\Longrightarrow$ direction: A Noetherian scheme X is normal at least implies each stalks of this scheme is integral. Recall each connected component is open subscheme and open subscheme of Noetherian scheme is Noetherian (Remark 3.3).

By 3.3.C, each connected component of X_i is a integral scheme. Also by 3.3.B, a Noetherian scheme has only finitely many connected components.

Then we know X is a disjoint finite union of integral Noetherian normal schemes.

$\Longleftarrow$ direction is clear. Because normality is stalk-local, we know a finite union of integral Noetherian normal schemes is normal and it is Noetherian because it can be covered by finite Noetherian affine opens. ■

Exercise?? 3.4.C If A is an integral domain, show that $A = \cap A_{\mathfrak{m}}$, where the intersection runs over all maximal ideals of A. ■

Proof. If we interpret them as subsets in $K(A)$, we have clearly

$$A \subset \cap A_{\mathfrak{m}}.$$

As for the reverse direction, we claim:

$$K(A) \backslash A \subset K(A) \backslash \cap A_{\mathfrak{m}}.$$

If nonempty, pick $s \in K(A)\backslash A$, we can construct the ideal of denominators of s

$$I_s := \{r \in A : rs \in A\}$$

$I_s \neq A$ because $1 \notin I_s$. Then $I_s \subset \mathfrak{m}$ for some maximal ideal. $s \notin A_{\mathfrak{m}}$. ■

Exercise?? 3.4.D One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend $A = k[w,x,y,z]/(wz - xy)$ (which we first met in Example 4.4.12, and which we will later recognize as the cone over the quadric surface), and $w/y = x/z \in K(A)$. Show that the ideal of denominators of this element of $K(A)$ is (y,z). ■

Proof. Denote the element $s := w/y = x/z$. we easily check that $(y,z) \subset I_s$.

Assume $t \in I_s$, we have

$$tw/y = tx/z \in A,$$

hence $tw = ya$. Then t can not contain monomials in w,x. Furthermore, t can not contain terms with only w,x because $w^n x^m$ can not be a multiple of y. As a result, $t \in (y,z)$. ■

Exercise?? 3.4.E Show that any nonzero localization of a unique factorization domain is a unique factorization domain. ■

Proof. Recall the definition of a UFD. A UFD is an integral domain in which every non-zero element x of R can be written as a product of irreducible elements and a unit and this product is unique up to order of irreducible elements and units.

An equivalent characterization is "an integral domain R in which every non-zero element can be written as a product of unit and prime elements of R".

We prove the statement based on the second characterization. $S \subset R$ is a multiplicative set and let T be the set of all prime elements that divides an element of S, and let M be the set of all prime element not in T.

Claim: $p \in T$ iff image of p in $S^{-1}R$ is a unit.

Indeed, if $p \in T$, then there exists $s \in S$, such that $p|s$; Let $x \in R$ with $px = s$. Then

$$\frac{p}{1} \cdot \frac{x}{s} = \frac{ps}{s} \cdot \frac{x}{s} = 1_{S^{-1}R}.$$

For the "if" part, consider $p/1$ a unit, there is an inverse v/s, $\exists t \in S$ s.t., $pvt = st$, $pv = s$ because $0 \notin S$ and R is integral domain.

Claim: if $p \in M$, then $p/1 \in S^{-1}R$ is prime.

Assume $p/1|(a/s)(b/t)$, $(pc)/(r) = (ab)/(st)$, because R is iintegral domain, $pcst = abr$. Then we know $p|ab$ or $p|r$ because p is prime, but we assumed $p \nmid s \forall s \in S$. Then we have $p|a$ or $p|b$ because p is prime, which means $p/1|a/s$ or $p/1|b/t$ in return.

Now given $a = up_1^{b_1} \cdots p_r^{b_r} q_1^{c_1} \cdots q_t^{c_t}$ be a prime factorization of a, where $p_i \in T$ and $q_j \in M$. We have

$$\frac{a}{s} = \frac{u}{s}\left(\frac{p_1 s}{s}\right)^{b_1} \cdots \left(\frac{p_r s}{s}\right)^{b_r} \left(\frac{q_1 s}{s}\right)^{c_1} \cdots \left(\frac{q_t s}{s}\right)^{c_t}.$$

Then we know each $a/s \in S^{-1}R$ can be factorized into product of a unit and prime elements.

This proof is a improved version of This Answer, in that we don't need to verify the uniqueness of factorization. ■

Exercise?? 3.4.F Show that unique factorization domains are integrally closed. Hence factorial schemes are normal, and if A is a unique factorization domain, then $\operatorname{Spec} A$ is normal. ▪

Proof. Assme R is a UFD, denote the fraction field by $K(R)$. Let $x \in K(R)^{\times}$, say $x = r/s$, $gcd(x,s) = 1$, $r,s \in R, s \neq 0$. Suppose $\exists a_1,..,a_n \in R$ s.t.,

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0.$$

Then after multiplying it by s^n, set

$$r^n = -(a_1 r^{n-1} s + a_2 r^{n-2} s^2 + \ldots + a_n s^n)$$

$\Longrightarrow s | r^n$, $gcd(r^n, s) = 1, \Longrightarrow s \in R^{\times} \Longrightarrow x \in R.$

Then we have "all stalks of a scheme X are UFD" $\Longrightarrow$ "all stalks of a scheme X are normal", which means factorial schemes are normal.

If A is a UFD, each stalk is a localization of A is therefore UFD by 3.4.E . $\operatorname{Spec} A$ is factorial $\Longrightarrow$ $\operatorname{Spec} A$ is normal. ■

Exercise?? 3.4.G Show that the following schemes are normal: $\mathbb{A}^n_k$, $\mathbb{P}^n_k$, $\operatorname{Spec} \mathbb{Z}$. (As usual, k is a field. Although it is true that if A is integrally closed then A[x] is as well, this is not an easy fact, so do not use it here.) ▪

Proof. $\operatorname{Spec} \mathbb{Z}$ is normal because $\mathbb{Z}$ is UFD.

$\mathbb{A}^n_k =$ $\operatorname{Spec} k[x_1,..,x_n]$, where the polynomial ring over a field k is UFD.

$\mathbb{P}^n_k =$ $\operatorname{Proj} k[x_0,\ldots,x_n]$ can covered by $k[x_{0/i},\ldots,x_{n/i}](x_{i/i}-1)$, which are UFDs, there fore we know each stalk is UFD. Then we know $\mathbb{P}^n_k$ is factorial and hence normal. ■

Exercise?? 3.4.H Suppose A is a unique factorization domain with 2 invertible, and $z^2 - f$ is irreducible in $A[z]$.

(a) Show that if $f \in A$ has no repeated prime factors, then $\operatorname{Spec} A[z]/(z^2 - f)$ is normal. Hint: $B := A[z]/(z^2 - f)$ is an integral domain, as $(z^2 - f)$ is prime in $A[z]$. Suppose we have monic $F(T) \in B[T]$ so that $F(T) = 0$ has a root α in $K(B) \setminus K(A)$. Then by replacing $F(T)$ by $\bar{F}(T)F(T)$, we can assume $F(T) \in A[T]$. Also, $\alpha = g + hz$ where $g,h \in K(A)$. Now α is the root of $Q(T) = 0$ for monic $Q(T) = T^2 - 2gT + (g^2 - h^2 f) \in K(A)[T]$, so we can factor $F(T) = P(T)Q(T)$ in $K(A)[T]$. By Gauss's lemma, $2g, g^2 - h^2 f \in A$. Say $g = r/2$, $h = s/t$ (s and t have no common factors, $r,s,t \in A$). Then $g^2 - h^2 f = (r^2 t^2 - 4s^2 f)/4t^2$. Then t is invertible.

(b) Show that if $f \in A$ has repeated prime factors, then $\operatorname{Spec} A[z]/(z^2 - f)$ is not normal. ▪

Proof. (a) In fact, the hint already gives the detailed solution. We only write here some further explanations.

$A[z]/(z^2 - f)$ is integral domain because $z^2 - f$ is irreducible and hence is prime element in UFD.

B is equivalent to formally adjoining a pair of roots $\pm\sqrt{f}$. We can take the "conjugate" of a monic $F(T)$ be mapping the coefficients $a + b\sqrt{f}$ to $a - b\sqrt{f}$ and we denote the conjugate polynomial $\bar{F}(T)$. Then $\bar{F}(T)F(T) \in A[T]$.

Assume $\alpha \in K(B)\backslash K(A)$ is a root of $F(T)=0$ hence a root of $\bar{F}(T)F(T)=0$, We replace $F(T)$ by $\bar{F}(T)F(T)$ from now on. Want: α is contained in B.
The root α can be written as $g+hz$ or equivalently $(g+h\sqrt{f})$, where $g,h \in K(A)$.
Then α is the root of $(T-g-h\sqrt{f})(T-g+h\sqrt{f}) = T^2-2gT+g^2-h^2f \in K(A)[T]$, so we can factor $F(T)$ as $P(T)Q(T) \in K(A)[T]$. Recall Gauss' lemma for UFD:
Given A is a UFD, and $K(A)$ its fraction field. $F(T) \in A[T]$, if it factors in $K(A)[T]$, then it factors in $A[T]$.
Then we know $2g, g^2-h^2f \in A$. Because 2 is invertible in A, we know $g \in A$ hence $h^2f \in A$. Assume $h = r/s$ and $gcd(r,s)=1$, we have $r^2f/s^2 \in A$, because f is square free, the only possibility is $s \in A^{\times}$. Then we know $h \in A$, therefore $\alpha = g+hz = g+h\sqrt{f} \in B$.
We have proved that $B = A[z]/(z^2-f)$ is integral closed, therefore $\operatorname{Spec} A[z]/(z^2-f)$ is normal.

(b) Assume $f \in A$ has repeated prime factors.
Want: $B := A[z]/(z^2-f)$ and $B_{\mathfrak{p}}$ is not integral closed for some $\mathfrak{p} \in \operatorname{Spec} B$. By Proposition 5.4.2, it is equivalent to prove B is not integral closed.
Want:$F(T) \in B[T]$, find an root of $F(T)=0$, $\beta \in K(B)\backslash K(A)$ s.t., $\beta \notin B$.
Like in (a), we can restrict to the case $F(T) \in A[T]$. Assume p is the repeated prime element in f, i.e, $p^2|f$.
We can choose the polynomial $F(T) := T^2 - 1/p^2 f$, $\sqrt{f}/p$ is a root of $F(T)$ in $K(B)$ and is not an element in B.

■

Exercise?? 3.4.I Show that the following schemes are normal:

(a) $\operatorname{Spec} \mathbb{Z}[x]/(x^2-n)$ where n is a square-free integer congruent to 3 modulo 4. Caution: the hypotheses of Exercise 3.4.H do not apply, so you will have to do this directly. (Your argument may also show the result when 3 is replaced by 2. A similar argument shows that $\mathbb{Z}[(1+\sqrt{n})/2]$ is integrally closed if $n \equiv 1 (\mod 4)$ is square-free.)
(b) $\operatorname{Spec} k[x_1,\ldots,x_n]/(x_1^2+x_2^2+\cdots+x_m^2)$ where $\operatorname{char} k \neq 2, n \geq m \geq 3$.
(c) $\operatorname{Spec} k[w,x,y,z]/(wz-xy)$ where char $k \neq 2$. This is our cone over a quadric surface example from Example 4.4.12 and Exercise 3.4.D. Hint: Exercise 3.4.J may help. (The result also holds for char $k=2$, but don't worry about this.)

Proof. (a) $\mathbb{Z}[x]/(x^2-n) \cong B := \mathbb{Z}[\sqrt{n}]$. Assume a monic polynomial $P(T) \in B[T]$, we can take the conjugate $\bar{P}(T)$ and $\bar{P}(T)P(T) \in \mathbb{Z}[T]$. W.l.o.g, we can set $P(T) \in \mathbb{Z}[T]$. Assume $\alpha = g+h\sqrt{n} \in K(B)\backslash\mathbb{Q}$ is a root of $P(T)=0$, where $g,h \in \mathbb{Q}$. α is also a root of $G(T) = T^2-2gT+(g^2-h^2n) \in \mathbb{Q}[T]$. $P(T) = R(T)G(T)$.
Again, by Gauss's lemma, we have $G(T) \in \mathbb{Z}[T]$.

$$2g \in \mathbb{Z}$$

and

$$g^2 - h^2 n \in \mathbb{Z}$$

Assume $g = k/2$ and $h = r/s$, $k,r,s \in \mathbb{Z}$, we have

$$\frac{k^2}{4} - \frac{r^2 n}{s^2} \in \mathbb{Z}$$

$n = 4e+3$ and $k = 2q$ or $k = 2q+1$. The later case is ruled out. If $k = 2q+1$, the equation reduces to

$$\frac{1}{4} - \frac{r^2 n}{s^2} = m \in \mathbb{Z}$$

Then we know $r^2 n = s^2/4 + ms^2$ and $gcd(r,s) = 1$ and n is square-free, we know $4|s^2$, assume $s^2 = 4t^2$, therefore

$$3r^2 \equiv t^2 \pmod 4,$$

where $gcd(r,t) = 1$. We know r,t are coprime and have same parity. r and t can't be both even number and have to be both odd. Assume $r = 2u+1$ and $t = 2v+1$, then the equation reduces to

$$3 \equiv 1 \pmod 4.$$

We get the contradiction. As a result, $k = 2q$ and $r^2 n/s^2 \in \mathbb{Z}$, which means $s = 1$. $g,h \in \mathbb{Z}$ $\beta = g + h\sqrt{f} \in B$.

The identical argument works when n congruent to 2 modulo 4.

(b) $\operatorname{char} k \neq 2$, then 2 is invertible in k. For example, in the case $k[x_1,x_2,x_3]/(x_1^2+x_2^2+x_3^2)$, regard $k[x_1,x_2,x_3]$ as $A[x_1]$, where $A = k[x_2,x_3]$, A is a UFD, 2 is invertible in A, $x_2^2+x_3^2$ is square-free in A, and $x_1^2+(x_2^2+x_3^2)$ is irreducible in $A[x_1]$. We can now use 3.4.H to prove that $k[x_1,x_2,x_3]/(x_1^2+x_2^2+x_3^2)$ is integrally closed. The identical argument can be used to prove for arbitrary $n \geq m \geq 3$.

(c) We can diagonalize the quadratic form $wz - xy$.

$$wz - xy = \left(\frac{w+z}{2}\right)^2 - \left(\frac{w-z}{2}\right)^2 - \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2.$$

Then it reduces to a special case of (b) above.

■

Exercise?? 3.4.J Suppose k is an algebraically closed field of characteristic not 2. (The hypothesis that k is algebraically closed is not necessary, so feel free to deal with this more general case.)

(a) Show that any quadratic form in n variables can be "diagonalized" by changing coordinates to be a sum of at most n squares.

(b) Show that the number of squares appearing depends only on the quadratic. For example, $x^2+y^2+z^2$ cannot be written as a sum of two squares.

Proof. In linear algebra courses, we have discussed the diagonalization of real symmetric matrices and Hermitian matrices. This exercise is a simple generalization to fields with non-2 characteristics.

(a) Consider a quadratic form

$$Q_n(\mathbf{x}) := \sum_{1\leq i,j\leq n} a_{ij} x_i x_j,$$

with coefficients in k. Follow the hints, we can induct on n. We firstly prove for $n = 2$,

$$Q_2(\mathbf{x}) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2.$$

Assume $a_{11}, a_{22} = 0$ and $2a_{12} \neq 0$ (which means $a_{12} \neq 0$ because $\operatorname{char} k \neq 2$), we have $Q_2(\mathbf{x}) = a_{12}(x_1+x_2)^2 - a_{12}x_1^2 - a_{12}x_2^2$. Else, one of $a_{11}, a_{22} \neq 0$, w.l.o.g, $a_{11} \neq 0$, we get $Q_2(\mathbf{x}) = a_{11}(x_1 - a_{12}a_{11}^{-1}x_2)^2 + (a_{22} - a_{12}^2/a_{11})x_2^2$.

Assume now all Q_n can be diagonalized, consider a quadratic form Q_{n+1}, it can be written as

$$Q_{n+1}(\mathbf{x},x_{n+1}) = P_n(\mathbf{x}) + 2c_{1,n+1}x_1x_{n+1} + \ldots + 2c_{n,n+1}x_nx_{n+1} + c_{n+1,n+1}x_{n+1}^2.$$

Up to reordering, we require $c_{n+1,n+1} \neq 0$. By induction hypothesis, we can write $P_n(\mathbf{x})$ as $\sum_{i=1}^n b_{ii}y_i^2$, with with possibility all $b_{ii} = 0$ and $\mathbf{y} = A^{-1}\mathbf{x}$, where A is a linear transforamtion.

$$\begin{aligned} Q_{n+1}(\mathbf{x},x_{n+1}) &= \mathbf{y}^T B\mathbf{y} + 2\sum_{i=1}^n\sum_{j=1}^n A_{ij}c_{i,n+1}y_jx_{n+1} + c_{n+1,n+1}x_{n+1}^2 \\ &= \mathbf{y}^T B\mathbf{y} + 2\sum_{j=1}^n \tilde{c}_{j,n+1}y_jx_{n+1} + c_{n+1,n+1}x_{n+1}^2, \end{aligned}$$

where $\tilde{c}_{j,n+1} := \sum_{i=1}^n A_{ij}c_{i,n+1}$.

$$Q_{n+1}(\mathbf{x},x_{n+1}) = c_{n+1,n+1}\left(x_{n+1}^2 + \sum_{j=1}^n \frac{\tilde{c}_{j,n+1}}{c_{n+1,n+1}}y_j\right)^2 + \sum_{i=1}^n\left(b_{ii} - \frac{c_{j,n+1}^2}{c_{n+1,n+1}}\right)y_i^2.$$

There are at most $n+1$ terms.

(b) Given a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T M\mathbf{x}$, under a transformation of basis, $\tilde{\mathbf{x}} = A\mathbf{x}$, we can write $Q(\tilde{\mathbf{x}}) = \tilde{\mathbf{x}}^T \tilde{M}\tilde{\mathbf{x}}$, where

$$\tilde{M} = A^T MA.$$

Assume $\mathbf{v}$ is in the null space of M, then $A^{-1}v$ is in the null space of $\tilde{M}$, because A is a non-degenerate matrix, we know the dimension of null space is invariant under the change of basis. Hence, the rank of matrix M is preserved under the base change. For quadratic forms, the rank is just the number of independent square terms.

■

Exercise?? 3.4.K (RINGS CAN BE INTEGRALLY CLOSED BUT NOT UNIQUE FACTORIZATION DOMAINS, ARITHMETIC VERSION). Show that $\mathbb{Z}[\sqrt{-5}]$ is integrally closed but not a unique factorization domain. ■

Proof. This is a classical counter-example.

$A := \mathbb{Z}[\sqrt{-5}]$ is integrally closed by 3.4.I(a) because

$$-5 \equiv 3 \mod 4.$$

It is not unique factorization domain because for example $6 = 2\times 3$ and $6 = (1-\sqrt{-5})(1+\sqrt{-5})$. ■

Exercise?? 3.4.L (RINGS CAN BE INTEGRALLY CLOSED BUT NOT UNIQUE FACTORIZATION DOMAINS, GEOMETRIC VERSION). Suppose char $k \neq 2$. Let $A = k[w,x,y,z]/(wz - xy)$, so Spec A is the cone over the smooth quadric surface (cf. Exercises 4.4.12 and 3.4.D).

(a) Show that A is integrally closed. (Hint: Exercises 3.4.I(c) and 3.4.J)

(b) Show that A is not a unique factorization domain. (Clearly $wz = xy$. But why are w, x, y, and z irreducible? Hint: A is a graded integral domain. Show that if a homogeneous element factors, the factors must be homogeneous.)

■

Proof. For (a), we can diagonalize the quadratic form $wz - xy$.

$$wz - xy = \left(\frac{w+z}{2}\right)^2 - \left(\frac{w-z}{2}\right)^2 - \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2.$$

We know $A \cong k[x_1,x_2,x_3,x_4]/(x_1^2 - x_2^2 - x_3^2 + x_4^2)$, then we know it is integrally closed by 3.4.I(b).

As for (b), we clearly have $wz = xy$. Then it suffices to prove w,x,y,z are irreducible.

Indeed, we can regard A as a graded integral domain because it is a quotient of the polynomial ring by a homogeneous prime ideal.

Claim: If a homogeneous element factors, the factors must be homogeneous.

An element f is of homogeneous degree n iff each monomial of f is of degree n.

Now assume $f = gh$, where g,h are not homogeneous. Assume $g = g_0 + g_1$, where g_0 is the highest degree term in g and g_1 is the rest. The degree of monomials in g_0 is strictly higher than those in g_1. We have $f = g_0h + g_1h$, but it is impossible for the two summands to have same highest degrees.

Then we know x is irreducible, because the only degree 1 homogeneous terms are x,y,z,w and we won't have the relation $w = \lambda x$. ■

Exercise?? 3.4.M Suppose A is a k-algebra, and l/k is a finite extension of fields. (Most likely your proof will not use finiteness; this hypothesis is included to avoid distraction by infinite-dimensional vector spaces.) Show that if $A \otimes_k l$ is a normal integral domain, then A is a normal integral domain as well.

Proof. Here the notion "normal integral domain" is the same as "integrally closed". A can be identified as a subring of $A \otimes_k l$, therefore $A \otimes_k l$ is an integral domain implies that A being an integral domain.

The proof is sketched in the hint, we just follow it step by step.

Fix a k-basis for l, $b_1 = 1, \ldots, b_d$. There is a dual basis $v_1 = 1, \ldots, v_d$ such that $trace_{l/k}(v_i b_j) = \delta_{ij}$. We can extend the trace to $Tr : A \otimes_k l \longrightarrow A : a \otimes x \mapsto a \cdot trace_{l/k}(x)$. The map Tr is nondegenerate $Tr(a \otimes x) = 0 \Longrightarrow a = 0$ or $x = 0$.

Claim: $1 \otimes b_1, \ldots, 1 \otimes b_d$ forms a free A-basis for $A \otimes_k l$.

Assume an element $a \otimes x \in A \otimes_k l$, where $x = \sum_i x_i b_i$. We have $a \otimes x = a \otimes \sum_i x_i b_i$ and it equals to $\sum_i (x_i a)(1 \otimes b_i)$. It indeed forms an generating set of A-module $A \otimes_k l$. And we can show that $A \otimes_k l$ is a free A-module with this basis. Assume $\sum_i r_i(1 \otimes b_i) = 0$, $\sum_i r_i \otimes b_i = 0$. Consider $Tr((1 \otimes v_j) \cdot (\sum_i r_i \otimes b_i)) = r_j = 0$. We have proved that $A \otimes_k l$ is a free A-module with the specified basis.

Claim: The diagram commutes and the composition map $A \longrightarrow K(A) \otimes_k l$ is injective.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & K(A) \arrow[d] \\
A \otimes_k l \arrow[r] & K(A) \otimes_k l
\end{tikzcd}$$

$A \longrightarrow K(A)$ is injective and l is a flat k-module. Hence the composition map is injective.

Claim: $K(A) \otimes_k l \cong K(A \otimes_k l)$.

We know $A \otimes_k l \subset K(A) \otimes_k l \subset K(A \otimes_k l)$, then it suffices to prove only $K(A) \otimes_k l$ is a field.

A general term in $K(A)\otimes_k l$ is of the form $\frac{a}{c}\otimes x$, $1\otimes b_i$ gives a $K(A)$-basis of $K(A)\otimes_k l$. (it comes from our first claim). By 1.2.G, a finite k-algebra which is an integral domain must be a field. We therefore have $K(A)\otimes_k l$ is a field.

Claim: $K(A)\cap A\otimes_k l = A$, where we regard both $K(A)$ and $A\otimes_k l$ as subrings of $K(A\otimes_k l)$.

$A\subset K(A)\cap A\otimes_k l$ is automatically true. $K(A)\otimes_k l$ is a $K(A)$-vector space with basis $1\otimes b_i$. The subspace $K(A)\cong K(A)\otimes_k 1$ is generated by $1\otimes 1$. $K(A)\cap A\otimes_k l$ is an A-submodule of $A\otimes_k l$ generated by $1\otimes 1$, therefore is contained in A.

Now we have finished our preparation. Given $P(T)$ a monic polynomial in $A[T]$. We can regard $P(T)$ as an element in $K(A\otimes_k l)[T]$. By assumption, $A\otimes_k l$ is normal integral closed. The root of $P(T)=0$ in $K(A\otimes_k l)$, $\alpha\in A\otimes_k l$. Assume further more α is a root in $K(A)$, we then have $\alpha\in K(A)\cap A\otimes_k l = A$. Whence, we can conclude that A is normal integral domain.

■

Exercise?? 3.4.N (UFD-NESS IS NOT AFFINE-LOCAL). Let $A=(\mathbb{Q}[x,y]_{x^2+y^2})_0$ denote the homogeneous degree 0 part of the ring $\mathbb{Q}[x,y]_{x^2+y^2}$. In other words, it consists of quotients $f(x,y)/(x^2+y^2)^n$, where f has pure degree $2n$. Show that the distinguished open sets $D\left(\frac{x^2}{x^2+y^2}\right)$ and $D\left(\frac{y^2}{x^2+y^2}\right)$ cover $\operatorname{Spec} A$. Show that $A_{\frac{x^2}{x^2+y^2}}$ and $A_{\frac{y^2}{x^2+y^2}}$ are unique factorization domains. Finally, show that A is not a unique factorization domain. ■

Proof. By 1.5.B, we know $\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}=1\Longrightarrow D\left(\frac{x^2}{x^2+y^2}\right)\cup D\left(\frac{y^2}{x^2+y^2}\right) = \operatorname{Spec} A$.

A general element of $A_{\frac{x^2}{x^2+y^2}}$ is of the form

$$\frac{\frac{f(x,y)}{(x^2+y^2)^n}}{\frac{x^{2m}}{(x^2+y^2)^m}}=\frac{f(x,y)/x^{2n}}{(x^2+y^2)^{n-m}(x^2)^{m-n}}=\frac{g(t)}{(1+t^2)^{n-m}}$$

where $t=\frac{y}{x}$ and $g(t)=f(x,y)/x^{2n}$. Hence we know both $A_{x^2/(x^2+y^2)}$ and $A_{y^2/(x^2+y^2)}$ are isomorphic to $\mathbb{Q}[t]_{t^2+1}$, which is a localization of UFD, by 3.4.E, they are also UFDs.

Finally, A is not a UFD, for example

$$\left(\frac{xy}{x^2+y^2}\right)^2=\frac{x^2}{x^2+y^2}\cdot\frac{y^2}{x^2+y^2},$$

where both

$$\frac{xy}{x^2+y^2}$$

and

$$\frac{x^2}{x^2+y^2}$$

are irreducible because $x/(x^2+y^2), y/(x^2+y^2)\notin A$ because they are not of degree 0. ■

3.5 The crucial points of a scheme: Associated points and primes

Exercise?? 3.5.A Suppose f is a function on $\operatorname{Spec} k[x,y]/(y^2,xy)$(i.e.,$f\in k[x,y]/(y^2,xy)$). Show that $\operatorname{Supp} f$ is either the empty set, or the origin, or the entire space. ■

Proof. Denote the ring $k[x,y]/(xy,y^2) =: A$ f is a global section on $\operatorname{Spec} k[x,y]/(xy,y^2)$. By definition

$$Supp(f) := \{p \in X = \operatorname{Spec} A : f_p \neq 0 \text{ in } \mathscr{O}_{X,p}\}$$

$Supp\, f$ is the point $[\mathfrak{p}]$ where $f/1 \neq 0$ in $A_{\mathfrak{p}}$. $f/1 \neq 0$ means $fs \neq 0 \forall s \in A - \mathfrak{p}$.

A general element f in $k[x,y]/(y^2,xy)$ is of the form

$$q(\bar{x}) + a\bar{y},$$

where q is a polynomial in $k[X]$, a is a constant in k and $\bar{x}\bar{y} = 0$ and $\bar{y}^2 = 0$.

1. Assume $q = 0$ and $a = 0$, then $f = 0$. In this trivial case the support of f is empty set because $fs = 0$ for whatever $s \in A$.

2. In the case $f = q(\bar{x}) + a\bar{y}$, $q \neq 0, a \neq 0$

$$\bar{y}^2 f = 0$$

which implies $\bar{y}^2 \in \mathfrak{p}$ for $[\mathfrak{p}] \in Supp\, f$. But in fact, all the prime ideals in A contains $\bar{y}$.

Claim: if $f = q(\bar{x}) + a\bar{y}$ with $q \neq 0, a \neq 0$, then $Supp\, f = \operatorname{Spec} A$.

Indeed the point of $\operatorname{Spec} A$ is of the form $(\bar{x} - b, \bar{y})$ or $(\bar{y})$, for any $\mathfrak{p}_b = (\bar{x} - b, \bar{y})$, the elements in multiplicative subset $S_b = A - \mathfrak{p}_b$ are of the form $g(\bar{x}) + c\bar{y}$ where $g(b) \neq 0$. $(g(\bar{x}) + c\bar{y})(q(\bar{x}) + a\bar{y}) \neq 0$. For $\mathfrak{p} = (\bar{y})$, the multiplicative set are of the form $g(\bar{x})$ and $g(\bar{x})(q(\bar{x}) + c\bar{y}) \neq 0$.

3. Assume f is $a\bar{y}$, f is not supported on $(\bar{y})$ because $\bar{x} \in A - (\bar{y})$. f is neither supported over $(\bar{x} - b, \bar{y})$ since $\bar{x} \in A - (\bar{x} - b, \bar{y})$.

f is only supported on the origin $(\bar{x}, \bar{y})$, where the multiplicative set are just the constants.

We have discussed all the possible cases thus finished the proof. ∎

Exercise?? 3.5.B (ASSUMING **(A)**) Suppose A is an integral domain. Show that the generic point $[0]$ is the only associated point of $\operatorname{Spec} A$ ▪

Proof. **Confusion: when we talk about the associated point of $\operatorname{Spec} A$, does it mean we choose $M = A$?**

We use the definition **(A)** in page 167 in the case $M = A$ and always assume A to be Noetherian when discussing associated points.

Given A a Noetherian integral domain, $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ is a Noetherian integral scheme by 3.2.G, which means it is irreducible and reduced by 3.2.F.

The global sections of $\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}$ are just elements in A. Now each section a is supported on the whole scheme $\operatorname{Spec} A$. The only irreducible component is $\operatorname{Spec} A$ itself. Hence the generic point is the only associated point of $\operatorname{Spec} A$. ∎

Exercise?? 3.5.C (ASSUMING **(A)**) Show that if A is reduced, $\operatorname{Spec} A$ has no embedded points. ▪

Proof. What we want to show is

"the associated points of a scheme $\operatorname{Spec} A$ for a reduced ring A are generic points of the irreducible components of $\operatorname{Spec} A$"

Follow the hint.

(i): In the case where A is integral domain, we know the only associated point is the generic point of $\operatorname{Spec} A$ by Exercise 3.5.B.

(ii) For a general reduced ring A, we Claim:

If f is a nonzero function on a reduced affine scheme, then $Supp\, f = \overline{D(f)}$

Indeed, $D(f) = \{[\mathfrak{q}] \in \operatorname{Spec} A | f \notin \mathfrak{q}\}$, $[\mathfrak{p}] \in D(f) \iff f \in A - \mathfrak{p}$, $f/1 \neq 0 \in A_{\mathfrak{p}}$ because $A - \mathfrak{p}$ is multiplicatively closed and $A - \mathfrak{p}$ does not contain 0. Then we know $D(f) \subset Supp\, f$, therefore $\overline{D(f)} \subset Supp\, f$ because $Supp\, f$ is closed. For the reverse inclusion, consider $[\mathfrak{p}] \in Supp\, f$ $\implies fs \neq 0, \forall s \in A - \mathfrak{p}$. Suppose $[\mathfrak{p}] \in D(g)$ for some $g \in A$, $g \notin \mathfrak{p} (g \in A - \mathfrak{p})$, $D(f) \cap D(g) = D(fg)$. Then we know $fg \neq 0$. fq is not nilpotent because A is a reduced ring, therefore $D(fg) \neq \emptyset$. Useful exercises are 1.5.D and 1.5.F

Claim: $\overline{D(f)}$ is the union of irreducible component that intersects $D(f)$.

By Proposition 3.6.15, we know in Noetherian topological space, a non-empty closed set can be expressed uniquely as a finite union of irreducible closed subsets.

Apply it to $\overline{D(f)}$, it can be expressed as union of $\cup_i D_i$. What we need to prove is that the irreducible closed subset D_i are in fact a irreducible components X_i in $\operatorname{Spec} A$. Assume by contradiction, $D_i \subsetneq Z_i \subset X_i$ is not a maximal irreducible subset, where Z_i is a minimal irreducible closed subset that contains D_i. (i.e. there is no intermediate irreducible closed subset between D_i and Z_i). By 1.7.E, there is a reverse inclusion of generic point $\mathfrak{p}_d \supsetneq \mathfrak{p}_z$, where $\overline{\mathfrak{p}_d} = D_i$ and $\overline{\mathfrak{p}_z} = Z_i$. (This also means $\mathfrak{p}_d$ is one of associated primes of A corresponding to f)

Consider the ring $A/\mathfrak{p}_z$, $\mathfrak{p}_d/\mathfrak{p}_z$ is a non-zero prime ideal in the quotient ring.

The canonical projection $A \longrightarrow A/\mathfrak{p}_z$ induces a continuous map $\operatorname{Spec} A/\mathfrak{p}_z \longrightarrow \operatorname{Spec} A$, which can be identified as the inclusion of a closed subset.

We know $\overline{\mathfrak{p}_d/\mathfrak{p}_z} = \overline{\mathfrak{p}_d} \cap \operatorname{Spec} A/\mathfrak{p}_z$. On the other hand $\overline{\mathfrak{p}_d} \subset \overline{D(f)}$, $\overline{\mathfrak{p}_d/\mathfrak{p}_z} = \overline{\mathfrak{p}_d} \cap \operatorname{Spec} A/\mathfrak{p}_z \cap \overline{D(f)} = \overline{\mathfrak{p}_d} \cap \overline{D([f])}$, where $[f]$ is the equivalence class of f in $A/\mathfrak{p}_z$. Every subspace of Noetherian space is also Noetherian by 1.6.T. We know $\overline{D([f])}$ is a subspace of Noetherian space $\overline{D(f)}$. Still by Proposition 3.6.15, both $\overline{D([f])}$ and $\overline{D(f)}$ can expressed uniquely as a union of their irreducible components. Also because each irreducible closed subset is contained in an irreducible component.

We then know Lemma: **the irreducible component Y_i of a subspace Y in Noetherian space X is the intersection of $Y \cap X_i$ with the irreducible component of X.**

As a result, $\overline{\mathfrak{p}_d/\mathfrak{p}_z}$ is the irreducible component of $\overline{D([f])} = Supp\, [f]$.

$\mathfrak{p}_d/\mathfrak{p}_z$ is an associated point of $A/\mathfrak{p}_z$, where $A/\mathfrak{p}_z$ is an integral domain, we then apply (i). $\mathfrak{p}_d/\mathfrak{p}_z$ can not be an embedded point $(\mathfrak{p}_d/\mathfrak{p}_z = 0)$ contradicting to our setting $\mathfrak{p}_d \subsetneq \mathfrak{p}_z$.

Now we are done.

■

Exercise?? 3.5.D (ASSUMING **(A)**). Suppose $m \in M$. Show that $Supp\, m$ is the closure of those associated points of M where m has non-zero germ.(Hint: $Supp\, m$ is a closed set containing the points described, and thus their closure. Why does it contain no other points?)

Proof. Recall what we mean by the support of a section m

$$Supp\,(m) = \{\mathfrak{p} \in X : m_{\mathfrak{p}} \neq 0\}.$$

By definition in $(\mathbf{A})$, associated points are the generic points of irreducible components of $Supp\,(n)$ for some n. In particular the associated described in the problem is contained in $Supp\,(m)$ hence contains the closure of those associated point.

Denote the set of associated point where m has nonzero germ by S. We want to prove $Supp\,(m) - \overline{S} = \emptyset$. Choose a point $\mathfrak{q} \in Supp\,(m)$, want $U \ni \mathfrak{q} \implies U \cap S \neq \emptyset$. Consider $D(f) \ni \mathfrak{q} \implies D(f) \cap Supp\,(m) \neq \emptyset$. $D(f)$ intersects with as least one irreducible component W of $Supp\,(m)$ by definition (A), it corresponds to an associated point $\mathfrak{p}$ and $\overline{\mathfrak{p}} = W$. We know $D(f) \cap W \neq \emptyset, \implies D(f) \ni \mathfrak{p}. \implies D(f) \cap S \neq \emptyset$. ■

Exercise?? 3.5.E (Assuming $(\mathbf{A})$ and $(\mathbf{B})$) Show that the locus on $\operatorname{Spec} A$ of points $[\mathfrak{p}]$ where $\mathscr{O}_{\operatorname{Spec} A,[\mathfrak{p}]} = A_{\mathfrak{p}}$ is nonreduced is the closure of those associated points of $\operatorname{Spec} A$ whose stalks are nonreduced. (Hint: why do points in the closure of these associated points all have nonreduced stalks? Why can't any other point have a nonreduced stalk?)

Proof. Follow the hint, we first show that the points in the closure of those specified associated point all have nonreduced stalk.

Let $\{\mathfrak{n}_i\}$ be the points with nonreduced stalks and let $\{\mathfrak{p}_j\}$ be the associated points with nonreduced stalks. What we want to show is basically the equality

$$\overline{\cup_j [\mathfrak{p}_j]} = \{[\mathfrak{n}_i]\}$$

By $(\mathbf{B})$, the closure of finite union is the finite union of closures.

$$\cup_j \overline{[\mathfrak{p}_j]} \stackrel{?}{=} \{[\mathfrak{n}_i]\}$$

The containment $\supset$ follows from the following fact: if $f \in A_{\mathfrak{n}}$ is a nilpotent, consider $Supp\,(f)$. $0 \neq f \in Nil(A_{\mathfrak{n}})$ means f is contained in all prime ideals in $\mathfrak{n}$ and also $\mathfrak{n} \in Supp\, f$.

Choose $\mathfrak{q}$ to the associated point corresponding to the irreducible component containing $[\mathfrak{n}]$. Then $f/1$ is nilpotent in $A_{\mathfrak{q}}$. This is because $\mathfrak{q} \subset \mathfrak{n}$ and

$$f \in Nil(A_{\mathfrak{n}}) = \cap_{prime\ \mathfrak{a} \in \mathfrak{n}} \mathfrak{a} \subset \cap_{prime\ \mathfrak{b} \in \mathfrak{q}} \mathfrak{b} = Nil(A_{\mathfrak{q}}).$$

The reverse inclusion is easier because assume $\mathfrak{a} \in \cup \overline{[\mathfrak{p}_i]}$, this means some $A_{\mathfrak{p}}$ is a further localization of $A_{\mathfrak{a}}$, where $A_{\mathfrak{p}}$ is nonreduced. We then know $A_{\mathfrak{a}}$ is nonreduced because we have shown before that localization of a reduced ring is reduced in Exercise 3.2.B. ■

III Morphisms

4. Morphisms of schemes

4.1 Introductions

4.2 Morphism of ringed space

Definition 4.2.1 A **morphism of ringed spaces** $\pi : X \longrightarrow Y$ is a continuous map of topological spaces along with a map $\mathscr{O}_Y \longrightarrow \pi_* \mathscr{O}_X$, which we think of as a "pullback map".

Exercise?? 4.2.A (Glue morphisms of ringed spaces). Suppose $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $\pi_i : U_i \longrightarrow Y$ that "agree on overlaps", i.e. $\pi_i \mid_{U_i \cap U_j} = \pi_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed space $\pi : X \longrightarrow Y$ such that $\pi|_{U_i} = \pi_i$. ■

Sol. Denote the morphisms of ringed spaces as $(\pi_i, \tilde{\pi}_i)$ The topological part has been shown in Exercise **??**, which says continuous maps from a topological space X to Y forms sheaf.

Suppose the topological part glue to a unique continuous map π. We denote pushforward sheaves by $\pi_* \mathscr{O}_X$. Given an open set $V \subset Y$, on the open set level

$$\pi_{i,*} \mathscr{O}_{U_i}(V) = \mathscr{O}_{U_i}(\pi_i^{-1}(V)) = \mathscr{O}_X((\pi|_{U_i})^{-1}(V)) = \mathscr{O}_X(\pi^{-1}(V) \cap U_i)$$

For each element $f \in \mathscr{O}_Y(V)$ $\pi_i(V)(f) \in \mathscr{O}_X(\pi^{-1}(V) \cap U_i)$ are compatible, thus glue to a unique element in $\mathscr{O}_X(\pi^{-1}(V)) = \pi_* \mathscr{O}_X(V)$. As discussed before, we can value-wisely define the morphism $\tilde{\pi}(V)$ and it is easy to check thus defined $\tilde{\pi}(V)$ is well-defined ring morphism and that $\tilde{\pi}$ forms a sheaf morphism. $(\pi, \tilde{\pi})$ forms the desired morphism of ringed spaces. ■

Exercise?? 4.2.B $\mathscr{O}$-module push forward. Given a morphism of ringed spaces $\pi : X \longrightarrow Y$, show that sheaf push forward induces a functor $\mathsf{Mod}_{\mathscr{O}_X} \longrightarrow \mathsf{Mod}_{\mathscr{O}_Y}$. ■

Proof. Given a $\mathscr{O}_X$-module $\mathscr{F}$, and a morphism of ringed spaces $(\pi, \tilde{\pi}) : X \longrightarrow Y$. The $\mathscr{F}$ as a sheaf of abelian groups automatically pushforward to $\pi_* \mathscr{F}$. It suffices to check the $\mathscr{O}_Y$-module structure on $\pi_*\mathscr{F}$.

$$\begin{array}{ccccc}
\mathscr{O}_Y(U)\times\pi_*\mathscr{F}(U) & \xrightarrow{\tilde{\pi}(U)\times id} & \pi_*\mathscr{O}_X(U)\times\pi_*\mathscr{F}(U) & \longrightarrow & \pi_*\mathscr{F}(U) \\
\downarrow \mathrm{res}_{U,V}\times\mathrm{res}_{U,V} & & \downarrow \mathrm{res}_{U,V}\times\mathrm{res}_{U,V} & & \downarrow \mathrm{res}_{U,V} \\
\mathscr{O}_Y(V)\times\pi_*\mathscr{F}(V) & \xrightarrow{\tilde{\pi}(V)\times id} & \pi_*\mathscr{O}_X(U)\times\pi_*\mathscr{F}(U) & \longrightarrow & \pi_*\mathscr{F}(U)
\end{array}$$

The right square commutes because $\pi_*\mathscr{F}$ is a $\pi_*\mathscr{O}_X$-module. The right square commutes by the definition of morphism of ringed spaces, therefore the outer square commutes, which indicates the $\mathscr{O}_Y$-module structure.

We still denote the functor π_*. It remains to check the commutative diagram

$$\begin{array}{ccccc}
\mathscr{F} & \xrightarrow{\phi} & \mathscr{G} & \xrightarrow{\psi} & \mathscr{H} \\
\downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* \\
\pi_*\mathscr{F} & \xrightarrow{\pi_*(\phi)} & \pi_*\mathscr{G} & \xrightarrow{\pi_*(\psi)} & \pi_*\mathscr{H},
\end{array}$$

where $\pi_*(\phi)(V)$ is defined to be $\phi(\pi^{-1}(V))$ and $\pi_*(\psi\circ\phi)(V) = \psi\circ\phi(\pi^{-1}(V)) = \psi(\pi^{-1}(V))\circ\phi(\pi^{-1}(V)) = \pi_*(\psi)\circ\pi_*(\phi)(V)$ This works for all opens, we thus have $\pi_*(\psi\circ\phi) = \pi_*(\psi)\circ\pi_*(\phi)$.

It still remains to check π_* is a well defined morphism of $\mathscr{O}_Y$ modules, but the procedure is similar. We just omit it. ■

Exercise?? 4.2.C Given a morphism of ringed spaces $\pi : X \longrightarrow Y$ with $\pi(p) = q$, show that there is a map of stalks $(\mathscr{O}_Y)_q \longrightarrow (\mathscr{O}_X)_p$ ▪

Sol. A morphism of ringed spaces $\pi : X \longrightarrow Y$ consists of the data $(\pi, \tilde{\pi})$ where

$$\tilde{\pi} : \mathscr{O}_Y \longrightarrow \pi_*\mathscr{O}_X$$

Exercise **??** shows that it induces a morphism on the stalks $\mathscr{O}_{Y,q} \longrightarrow (\pi_*\mathscr{O}_X)_q$.

On the other hand Exercise **??** says the pushforward induces a morphism on the stalks $(\pi_*\mathscr{O}_X)_q \longrightarrow \mathscr{O}_{X,p}$ if $\pi(p) = q$

The composition of the above two morphisms gives the desired map. ■

Exercise?? 4.2.D Suppose $\pi^\# : B \longrightarrow A$ is a morphism of rings. Define a morphism of ringed space $\pi :\ \operatorname{Spec} A \longrightarrow\ \operatorname{Spec} B$ as follows. The map of topological spaces was given by Exercise 1.4.H. To describe a morphism of sheaves $\mathscr{O}_{\operatorname{Spec} B} \longrightarrow \pi_*\mathscr{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} B$, it suffices to describe a morphism of sheaves on the distinguished base of $\operatorname{Spec} B$. On $D(f) \subset\ \operatorname{Spec} B$, we define

$$\mathscr{O}_{\operatorname{Spec} B}(D(g)) \longrightarrow \mathscr{O}_{\operatorname{Spec} A}(\pi^{-1}D(g)) = \mathscr{O}_{\operatorname{Spec} A}(D(\pi^\# g))$$

by $B_g \longrightarrow A_{\pi^\# g}$. verify that this makes sense (e.g., is independent of g), and that this describes a morphism of sheaves on the distinguished base. (This is the third in a series of exercises. We saw that a morphism of rings induces a map of sets in 3.2.9, a map of topological spaces in Exercise 1.4.H, and now a map of ringed spaces here.) ▪

Sol. We have shown that $A_f \cong \mathscr{O}_{\operatorname{Spec} A}(D(f))$ in Exercise 4.1.A. Choose another representative of $D(g)$, say h. Then we know $\sqrt{(h)} = \sqrt{(g)}$, which means h is invertible in B_g and g is invertible in B_h. i.e.

$$B_g \cong B_h$$

It also means $\pi^\#(h)$ is invertible in $(\pi^\#(g))$ and $\pi^\#(g)$ is invertible in $(\pi^\#(h))$, i.e.

$$A_{\pi^\#(g)} = A_{\pi^\#(h)}.$$

Then this map is indeed independent on the choice of representative of $D(g)$.

$$\mathscr{O}_{\operatorname{Spec} B}(D(g)) = B_g \longrightarrow \pi_* \mathscr{O}_{\operatorname{Spec} A}(D(g)) = A_{\pi^\#(g)}$$

gives a well-defined morphism of sheaves. ■

Exercise?? 4.2.E (Not all map of affine ringed spaces are induced by ring morphisms). Recall (Exercise 3.4.K) that $\operatorname{Spec} k[y]_{(y)}$ has two points, $[(0)]$ and $[(y)]$, where the second point is closed, and he first is not. Describe a map of ringed spaces $\operatorname{Spec} k(x) \longrightarrow \operatorname{Spec} k[y]_{(y)}$ sending the unique point of $\operatorname{Spec} k(x)$ to the closed point $[(y)]$, where pullback map on global sections sends k to k by the identity, and sends y to x. Show that this map of ringed spaces is not of the form described in Exercise 4.2.D. ■

Sol. Denote $B := k[y]_{(y)}$ and $A := k(x)$. The exotic choice topologically sends the unique point $\eta = [(0)]$ in $\operatorname{Spec} k(x)$ to the closed point $[(y)]$. These topological space discrete, making it a continuous map. The only open sets on $\operatorname{Spec} A$ are $\{[(0)],[(y)]\},\{[(0)]\}$ and $\emptyset$.

$$\phi: \operatorname{Spec} A = \{\eta\} \longrightarrow \operatorname{Spec} B = \{[(0)],[(y)]\}$$

$$\phi_* \mathscr{O}_{\operatorname{Spec} A}(\{[(0)],[(y)]\}) = \mathscr{O}_{\operatorname{Spec} A}(\eta) = A$$

$$\phi_* \mathscr{O}_{\operatorname{Spec} A}(\{[(0)]\}) = \mathscr{O}_{\operatorname{Spec} A}(\emptyset) = \{0\}$$

where $\{0\}$ is the final object in Rings. The corresponding maps on

$$\tilde{\phi}(\{[(0)],[(y)]\}) : B \longrightarrow A(k \mapsto k, y \mapsto x),$$

$$\tilde{\phi}(\{[(0)]\}) : B \longrightarrow 0.$$

This does not corresponds to a ring morphism as in Exercise 4.2.D. Assume by contradiction it is, then $\{[(0)]\} = D(y)$ and $\phi^\#$ has been given by the morphism on global sections. However $\phi^\#(y) = x$ and $A_{\pi^\#(y)} = A_x = A \neq 0$. ■

4.3 From locally ringed spaces to morphisms of schemes

In this section we will use the notion $\pi^\#$ to denote the induced morphism on stalks.

Exercise?? 4.3.A Show that morphisms of locally ringed spaces glue (cf. Exercise 4.2.A). (Hint: your solution to Exercise 4.2.A may work without change.) ■

Sol. Without change of the proof, we only need to check the glued morphism of ringed spaces is in fact a morphism of locally ringed spaces.

This is easy because for each pair of points $p, q, \pi(p) = q$, we can check that

$$\tilde{\pi}_p = \tilde{\pi}_{i\,p}$$

simply because $\tilde{\pi}$ restricts to $\tilde{\pi}_i$. Each π_i induces local morphisms on stalks, therefore π also induces local morphisms on stalks. ■

Exercise?? 4.3.B

(a) Show that $\operatorname{Spec} A$ is a locally ringed space.

(b) Show that the morphism of ringed spaces $\pi : \operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ defined by a ring morphism $\pi^{\#} : B \longrightarrow A$ is a morphism of locally ringed spaces.

Sol. (a) $\operatorname{Spec} A$ is a locally ringed space because we have shown in Exercise 2.3.F that

$$\mathscr{O}_{\operatorname{Spec} A, \mathfrak{p}} = A_{\mathfrak{p}}$$

and $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

(b) In Exercise 4.2.D, we have given explicitly on the distinguished base. We only need to should that the map on stalks indeed sends maximal ideals to maximal ideals.
Given two prime ideals $\mathfrak{q} \subset B$ and $\mathfrak{p} \subset A$, $\pi([\mathfrak{p}]) = [\mathfrak{q}]$ means $(\pi^{\#})^{-1}(\mathfrak{p}) = \mathfrak{q}$, recall that pull back of ideals preserve prime ideals. This means the $\pi^{\#}(B - \mathfrak{q}) \subset A - \mathfrak{p}$ and hence composition map $B \longrightarrow A_{\mathfrak{p}}$ sends $B - \mathfrak{q}$ into invertible element in $A_{\mathfrak{p}}$.

$$\begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr] & A \arrow[d] \\ B_{\mathfrak{q}} \arrow[r, dashed, "\phi"] & A_{\mathfrak{p}} \end{tikzcd}$$

There is a ring morphism from $B_{\mathfrak{q}}$ to $A_{\mathfrak{p}}$ by universal property of localization.

$$\phi : \frac{b}{s} \mapsto \frac{\pi^{\#}(b)}{\pi^{\#}(s)}.$$

We use ϕ to avoid abuse of notations.
We want to show that

$$\phi(\mathfrak{q}B_{\mathfrak{q}}) \subset \mathfrak{p}A_{\mathfrak{p}}.$$

This is true because

$$\phi\left(\sum_i q_i \frac{b_i}{s_i}\right) = \sum_i \frac{\pi^{\#}(q_i b_i)}{\pi^{\#}(s_i)} \in \mathfrak{p}A_{\mathfrak{p}}.$$

■

Proposition 4.3.1 If $\pi : \operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ is a morphism of locally ringed spaces then it is the morphism induced by the map $\pi^{\#} : B = \Gamma(\operatorname{Spec} B, \mathscr{O}_{\operatorname{Spec} B}) \longrightarrow \Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) = A$.

Exercise?? 4.3.C Show that a morphism of schemes $\pi : X \longrightarrow Y$ is a morphism of ringed spaces that looks locally like morphisms of affine schemes. Precisely, if $\operatorname{Spec} A$ is an affine open subset of X and $\operatorname{Spec} B$ is an affine open subset of Y, and $\pi(\operatorname{Spec} A) \subset \operatorname{Spec} B$, then the induced morphism of ringed spaces is a morphism of affine schemes. (In case it helps, note: note: if $W \subset X$ and $Y \subset Z$ are both open embeddings of ringed spaces, then any morphism of ringed spaces $X \longrightarrow Y$ induces a morphism of ringed spaces $W \longrightarrow Z$, by composition $W \longrightarrow X \longrightarrow Y \longrightarrow Z$.) Show that it suffices to check on a set $(\operatorname{Spec} A_i, \operatorname{Spec} B_i)$ where the $\operatorname{Spec} A_i$ form an open cover of X and the $\operatorname{Spec} B_i$ from an open cover of Y. ■

Sol. By affine open subset, we mean a open subscheme which is affine. There is an open embedding from $\operatorname{Spec} A$ to X. In particular, the $\mathscr{O}_X|_{\operatorname{Spec} A} = \mathscr{O}_{\operatorname{Spec} A}$. By definition, "morphism of schemes" is a morphism of locally ringed spaces. Restricting to the affine opens, the morphism of schemes induces a morphism of locally ringed spaces from $\operatorname{Spec} A$ to $\operatorname{Spec} B$. By proposition above, we know it is induced by a ring morphism $B \longrightarrow A$.

On the other hand, given the set $(\operatorname{Spec} A_i, \operatorname{Spec} B_i)$, s.t. $\pi(\operatorname{Spec} A_i) \subset \operatorname{Spec} B_i$ and A_i, B_i covers X, Y respectively. The morphism of affine schemes $\operatorname{Spec} A_i \longrightarrow \operatorname{Spec} B_i$ induces morphism of locally ringed spaces $\operatorname{Spec} A_i \longrightarrow Y$ by composition. We have checked that morphism of locally ringed spaces glue, hence there is a morphism of locally ringed spaces from $X \longrightarrow Y$. This criterion is valid. ■

Exercise?? 4.3.D Show that the category of rings and the opposite category of affine schemes are equivalent. ■

Sol. We give a pair of functors which are reciprocal quasi-inverse.

$$\operatorname{Spec}, \Gamma,$$

where Γ is the global section functor, both are contravariant.

$$\begin{tikzcd} B \arrow[r, "\operatorname{Spec}"] \arrow[d, "\pi'"] & \operatorname{Spec} B \arrow[r, "\Gamma"] \arrow[d, "(\pi, \tilde{\pi})"] & B \arrow[d, "\pi^\#"] \\ A \arrow[r, "\operatorname{Spec}"] & \operatorname{Spec} A \arrow[r, "\Gamma"] & A \end{tikzcd}$$

The above diagram is easy to verify and wesee from it

$$\Gamma \circ \operatorname{Spec} = Id_{\mathsf{Rings}}$$

$$\begin{tikzcd} (U, \mathscr{O}_U) \cong \operatorname{Spec} B \arrow[r, "\Gamma"] \arrow[d] & B \arrow[r, "\operatorname{Spec}"] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ (V, \mathscr{O}_V) \cong \operatorname{Spec} A \arrow[r, "\Gamma"] & A \arrow[r, "\operatorname{Spec}"] & \operatorname{Spec} A \end{tikzcd}$$

This means

$$\operatorname{Spec} \circ \Gamma \cong Id_{\mathsf{Affn}}.$$

Because both of them are contravariant functors, this indicates a equivalence of categories between Rings and Affn^{op}.

■

Exercise?? 4.3.E (This exercise can give you some practice with understanding morphisms of schemes by cutting up into affine open sets.) Make sense of the following sentence: "$\mathbb{A}_k^{n+1} \backslash \{0\} \longrightarrow \mathbb{P}_k^n$ given by

$$(x_0, x_1, \ldots, x_n) \mapsto [x_0, x_1, \ldots, x_n]$$

is a morphism of schemes." (Can you generalize to the case where k is replaced by a general ring B?)

Sol. $\mathbb{A}_k^{n+1} \backslash \{0\}$ can be covered by affine opens like

$$\operatorname{Spec} A_i \simeq D(x_i), \quad A_i := k[x_0, \ldots, x_n]_{x_i}.$$

On the other hand $\mathbb{P}_k^n$ can be covered by affine opens

$$\operatorname{Spec} B_i \simeq \mathbb{A}_k^n, B_i := k[x_{0/i}, \ldots, x_{k/i}, \ldots, x_{n/i}]/(x_{i/i} - 1) \cong k[y_0, \ldots, y_{n-1}].$$

We denote the morphism

$$f : (x_0, \ldots, x_n) \mapsto [x_0, \ldots, x_n]$$

and observe that $f_i := f|_{D(x_i)}$ maps $D(x_i)$ into $\operatorname{Spec} B_i$. Locally, it is a morphism of affine scheme induced by

$$B_i \longrightarrow A_i,$$
$$x_{k/i} \mapsto \frac{x_k}{x_i}, x_{i/i} \mapsto 1$$

which is a well-defined ring morphism.

Then it suffice to check that behaves well when we glue up the affine charts. It is equivalent to check that

$$f_i|_{D(x_i) \cap D(x_j)} = f_j|_{D(x_i) \cap D(x_j)}.$$

Note that $D(x_i) \cap D(x_j) = D(x_i x_j) = \operatorname{Spec} A_{ij} = \operatorname{Spec} k[x..]_{x_i x_j}$ and both $f_i|_{D(x_i) \cap D(x_j)}$ and $f_j|_{D(x_i) \cap D(x_j)}$ map into

$$\operatorname{Spec} B_{ij} := \operatorname{Spec} k[x_{0/i}, \ldots, x_{n/i}, 1/x_{j/i}]/(x_{i/i} - 1)$$

and

$$\operatorname{Spec} B_{ji} = \operatorname{Spec} k[x_{0/j}, \ldots, x_{n/j}, 1/x_{i/j}]/(x_{j/j} - 1)$$

. The above two are isomorphic by the map $x_{k/j} = x_{k/i}/x_{j/i}$. $f_i|_{D(x_i) \cap D(x_j)}$ is induced by

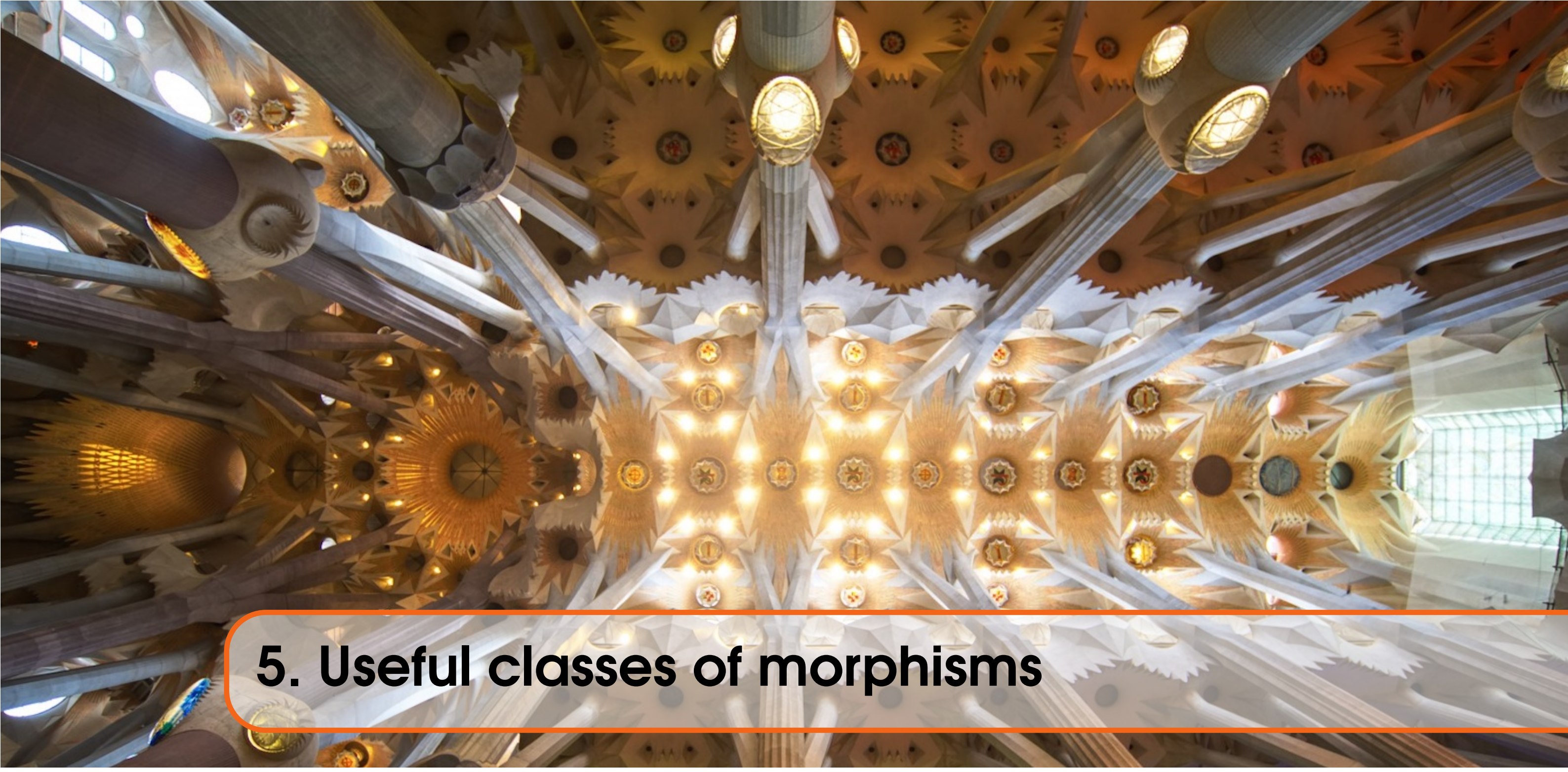
$$\operatorname{Spec} B_{ij} \mapsto \operatorname{Spec} A_{ij}$$

$$x_{k/i} \mapsto \frac{x_k x_j}{x_i x_j}, 1/x_{j/i} \mapsto \frac{x_i^2}{x_j x_i}$$

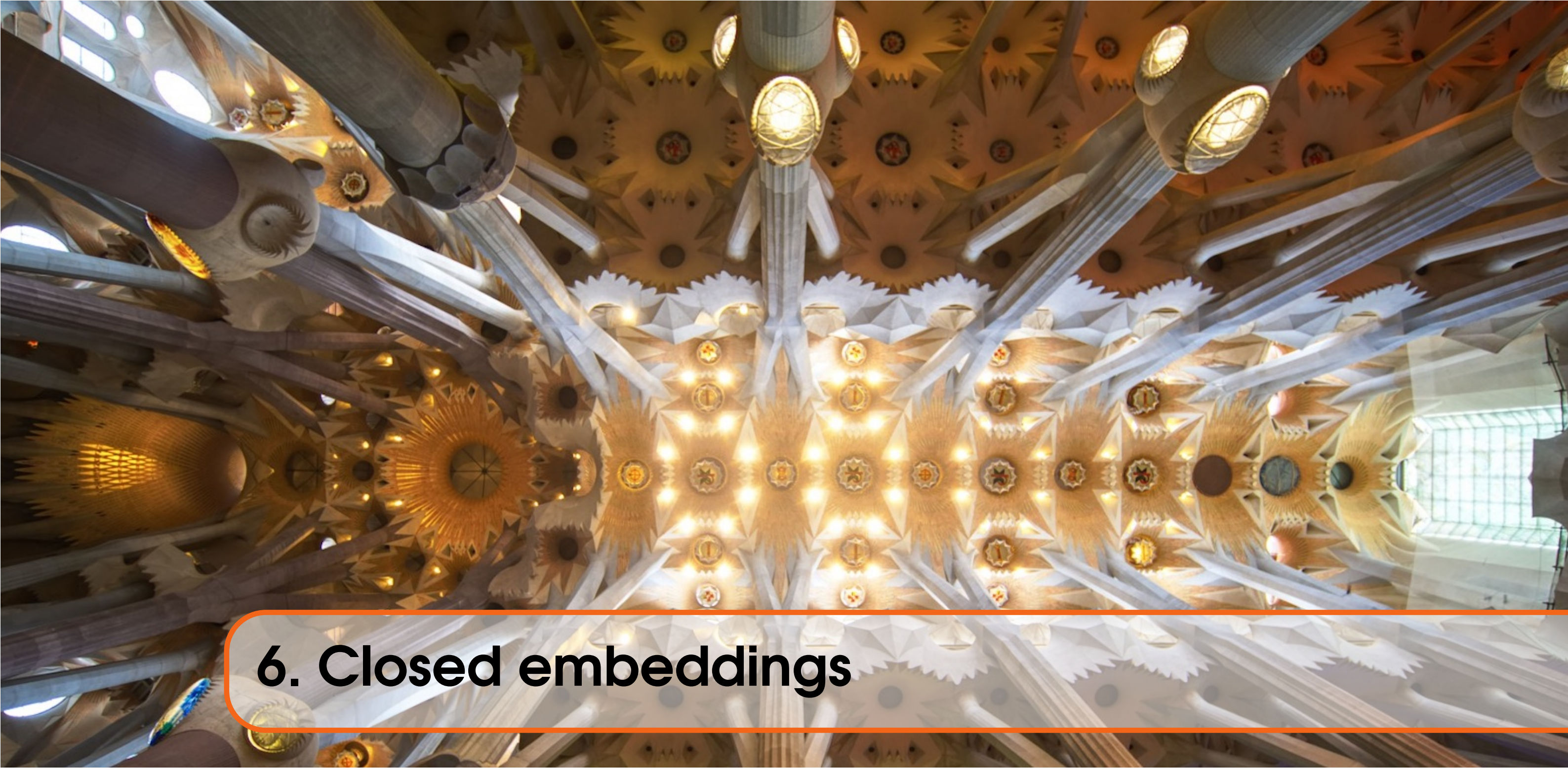
Together with the isomorphism above, $f_i|_{D(x_i) \cap D(x_j)}$ can be identified with $f_j|_{D(x_i) \cap D(x_j)}$. ■

Exercise?? 4.3.F Show that morphism $X \longrightarrow \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \longrightarrow \Gamma(X, \mathscr{O}_X)$.

Sol. ■

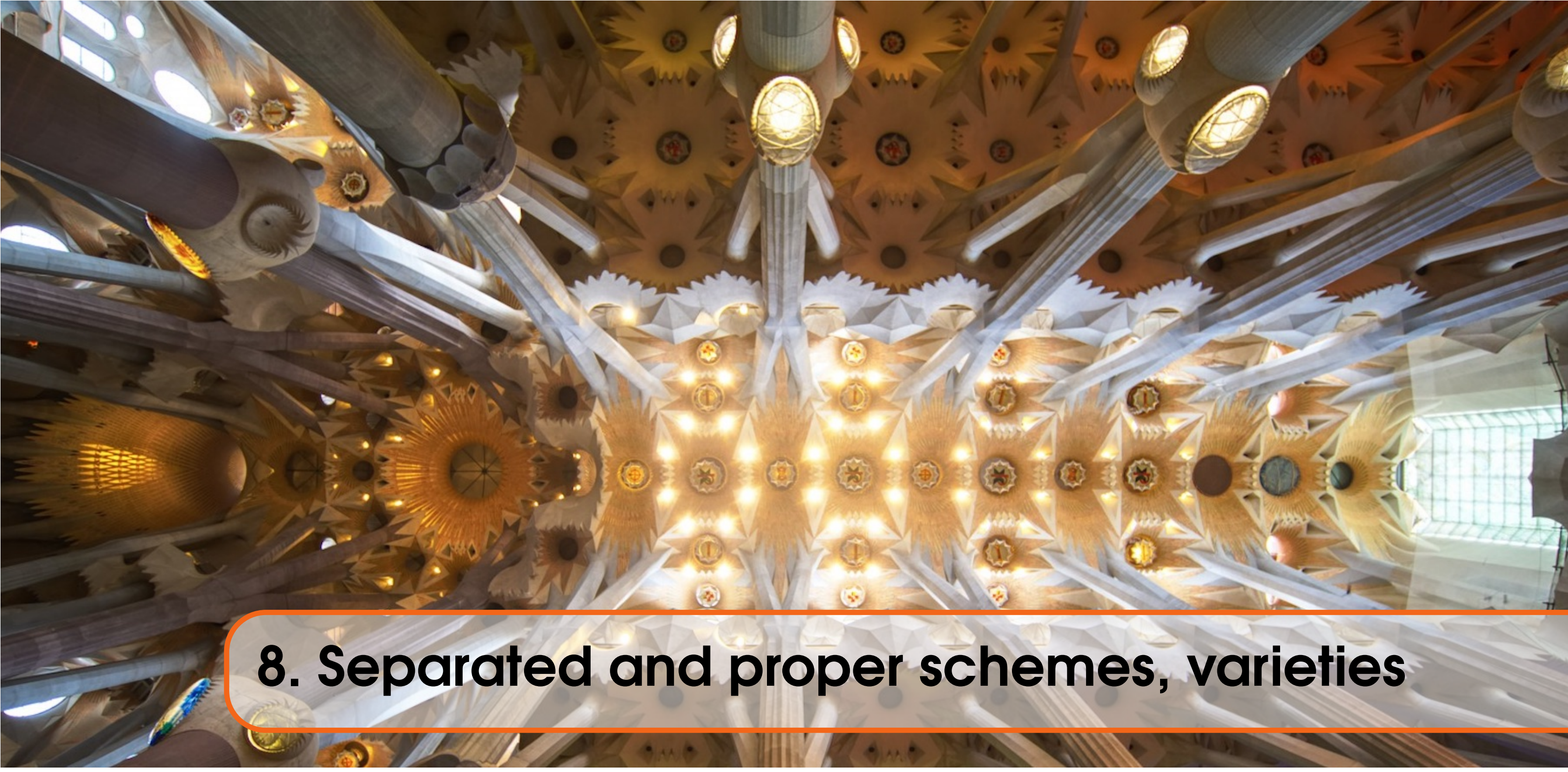


5. Useful classes of morphisms



6. Closed embeddings

7. Fibered product of schemes



8. Separated and proper schemes, varieties

IV

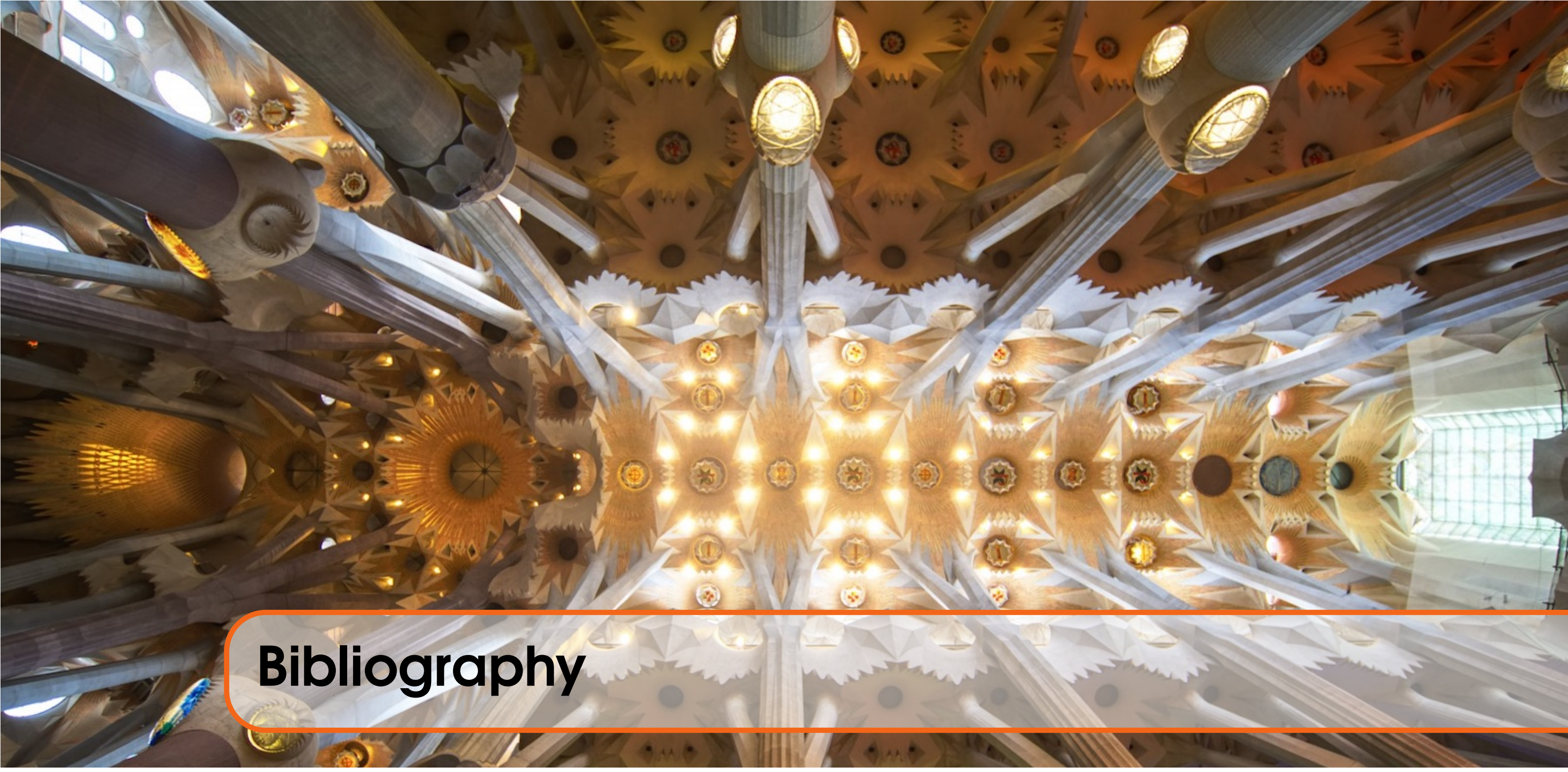
“Geometric” properties: Dimension and smoothness

V

Quasicoherent sheaves

VI

More



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