LECTURE 27

The Eilenberg-Zilber Theorem

In this lecture we state and prove the *Eilenberg-Zilber Theorem* which allows us to apply the Algebraic Künneth Theorem 26.5 from last lecture to the singular homology of a product of two spaces.

We begin however with a digression about chain equivalences[1] that we will need in the course of the proof of the Eilenberg-Zilber Theorem. Our first result is a partial converse to Corollary 10.26.

PROPOSITION 27.1. *Let $(C_\bullet, \partial)$ be a free chain complex. Then $(C_\bullet, \partial)$ is acyclic if and only if it has a contracting chain homotopy.*

Proof. Sufficiency was proved in Corollary 10.26. For the converse, assume that $H_n(C_\bullet) = 0$ for all n. Since $B_n = Z_n$, we have the following short exact sequence for every n:

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial} Z_{n-1} \to 0$$

Since Z_{n-1} is free abelian, this sequence splits, so let $r\colon Z_{n-1} \to C_n$ be such that $\partial \circ r_n = \mathrm{id}$. Observe that $\mathrm{id}_{C_n} - r_{n-1}\partial$ has image in Z_n. Indeed, if $c \in C_n$ then $\partial c = \partial c - \partial r_n \partial c = \partial c - \partial c = 0$. Now define

$$Q_n\colon C_n \to C_{n+1}, \qquad Q_n = r_n(\mathrm{id} - r_{n-1}\partial).$$

Then

$$\begin{aligned}
\partial Q_n + Q_{n-1}\partial &= \partial r_n(\mathrm{id} - r_{n-1}\partial) + r_{n-1}(\mathrm{id} - r_{n-2}\partial)\partial \\
&= \mathrm{id} - r_{n-1}\partial + r_{n-1}\partial - 0 \\
&= \mathrm{id}.
\end{aligned}$$

■

DEFINITION 27.2. Let $f\colon (C_\bullet, \partial) \to (D_\bullet, \partial')$ be a chain map. Given $n \in \mathbb{Z}$, define an abelian group

$$\mathrm{Cone}_n(f) := C_{n-1} \oplus D_n.$$

Define a map $\partial^f\colon \mathrm{Cone}_n(f) \to \mathrm{Cone}_{n-1}(f)$ by

$$\partial^f(c, d) = (-\partial c, fc + \partial' d).$$

Will J. Merry, Algebraic Topology II, Spring 2018, ETH Zürich. Last modified: March 02, 2018.

[1]This material originally appeared as Question 6 in the Algebraic Topology I Exam in January 2018! I am including this again here for those of you that either (a) did not take the exam, (b) took the exam but got it wrong, or (c) took the exam, got it right, but then forgot everything thirty seconds after the exam and have absolutely no recollection of it anymore...

In matrix form,

$$\partial^f = \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix}.$$

From this it is clear that $\partial^f \circ \partial^f = 0$, since

$$\partial^f \circ \partial^f = \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix} \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix} = \begin{pmatrix} -\partial^2 & 0 \\ -f\partial + \partial' f & (\partial')^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We call this the **mapping cone** of f.

Observe that if both $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ are free chain complexes then so is $\big(\mathrm{Cone}_\bullet(f), \partial^f\big)$.

Here is an easy result about mapping cones.

PROPOSITION 27.3. *Let $f\colon (C_\bullet, \partial) \to (D_\bullet, \partial')$ be a chain map between two free chain complexes. Assume that $\big(\mathrm{Cone}_\bullet(f), \partial^f\big)$ is acyclic. Then f is a chain equivalence.*

Proof. Since $\big(\mathrm{Cone}_\bullet(f), \partial^f\big)$ is free and acyclic, by Proposition 27.1 it has a contracting homotopy Q. Let us suggestively write Q in the form:

$$Q = \begin{pmatrix} p & g \\ r & -p' \end{pmatrix},$$

so that

$$\begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix} \begin{pmatrix} p & g \\ r & -p' \end{pmatrix} + \begin{pmatrix} p & g \\ r & -p' \end{pmatrix} \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

This gives us four equations:

$$\begin{pmatrix} -\partial p - p\partial + gf & -\partial g + g\partial' \\ \text{mess} & fg - \partial' p' - p'\partial' \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

Then $g\colon D_\bullet \to C_\bullet$ satisfies $-\partial g + g\partial' = 0$ and hence is a chain map. Moreover we obtain $p\partial + \partial p = gf - \mathrm{id}$ and $p'\partial' + \partial' p' = fg - \mathrm{id}$, which shows that f is a chain equivalence. ■

Our next result fits the mapping cone into a long exact sequence.

PROPOSITION 27.4. *Let $f\colon (C_\bullet, \partial) \to (D_\bullet, \partial')$ be a chain map. Then there is an exact sequence*

$$\cdots \to H_{n+1}\big(\mathrm{Cone}_\bullet(f)\big) \to H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(D_\bullet) \to H_n\big(\mathrm{Cone}_\bullet(f)\big) \to \dots$$

Proof. Let $C_\bullet^+$ denote the same chain complex as $C_\bullet$ but the groups (and differentials) shifted by one: $C_n^+ := C_{n-1}$.Then there is a short exact sequence of chain complexes

$$0 \to D_\bullet \xrightarrow{i} \mathrm{Cone}_\bullet(f) \xrightarrow{p} C_\bullet^+ \to 0,$$

where $i\colon d \mapsto (0, d)$ and $p\colon (c, d) \mapsto c$. This gives us a long exact sequence in homology:

$$\dots H_{n+1}\big(\mathrm{Cone}_\bullet(f)\big) \to H_{n+1}(C_\bullet^+) \xrightarrow{\delta} H_n(D_\bullet) \to H_n\big(\mathrm{Cone}_\bullet(f)\big) \to \dots$$

It is clear that $H_{n+1}(C^+_\bullet) = H_n(C_\bullet)$, and it remains to see that under this identification the connecting homomorphism δ is just $H_n(f)$. For this we recall that for a cycle $c \in C_n$, one has $\partial^f(c, 0) = (-\partial c f c) = (0, fc) = i(fc)$, and hence from the definition of δ (cf. (11.3))

$$\delta\colon \langle c\rangle \mapsto \langle i^{-1}\partial^f p^{-1}(c)\rangle = \langle fc\rangle = H_n(f)\langle c\rangle.$$

This completes the proof. ∎

We can now use the mapping cone construction to obtain a partial converse to Proposition 10.24.

Proposition 27.5. *Let $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ be two free chain complexes. Let $f_\bullet\colon C_\bullet \to D_\bullet$ denote a chain map. Then f is a chain equivalence if and only if $H_n(f_\bullet)\colon H_n(C_\bullet, \partial) \to H_n(D_\bullet, \partial')$ is an isomorphism for every $n \in \mathbb{Z}$.*

Proof. Necessity was proved in Proposition 10.24. For sufficiency, we use the exact sequence from Proposition 27.4. Since $H_n(f)$ is an isomorphisms, we must have $H_n\big(\mathrm{Cone}_\bullet(f)\big) = 0$ for all n. Thus $\mathrm{Cone}_\bullet(f)$ is acyclic, and hence by Proposition 27.3 we see that f is a chain equivalence as desired. ∎

The next result allows us to use the Künneth Theorem to compute the homology of the product of two spaces. This is our first example of a theorem which can be proved directly using a rather lengthy and horrible argument, but has a nice short proof using the Acyclic Models Theorem 23.8.

Theorem 27.6 (Eilenberg-Zilber). *Let X and Y be topological spaces. Then there is a natural chain equivalence*

$$\Omega_\bullet\colon C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$$

which is unique up to chain homotopy. Thus for all $n \geq 0$, we have

$$H_n(X \times Y) \cong H_n(C_\bullet(X) \otimes C_\bullet(Y)).$$

Proof. We will apply the Acyclic Models Theorem from Lecture 23. Let $\mathsf{Top} \times \mathsf{Top}$ denote the cateogry with objects all ordered pairs (X, Y) of topological spaces and morphisms all ordered pairs of continuous maps. (Note: we do *not* require Y to be a subspace of X; this is *not* the same as the category Top^2.)

Now we define a family of models $\mathcal{M}$ for $\mathsf{Top} \times \mathsf{Top}$. Let

$$\mathcal{M} := \left\{(\Delta^i, \Delta^j) \mid i, j \geq 0\right\}.$$

We define two functors

$$S_\bullet, T_\bullet\colon \mathsf{Top} \times \mathsf{Top} \to \mathsf{Comp}$$

by

$$S_\bullet(X, Y) := C_\bullet(X \times Y), \qquad T_\bullet(X, Y) := C_\bullet(X) \otimes C_\bullet(Y).$$

We claim that for all $n \geq 0$, both S_n and T_n are free with basis contained in $\mathcal{M}$, and moreover that every model (Δ^i, Δ^j) is both $S_\bullet$-acyclic in positive degrees and $T_\bullet$-acyclic in positive degrees.

Let's start with $S_\bullet$. Let $d_i \colon \Delta^i \to \Delta^i \times \Delta^i$ denote the diagonal map $x \mapsto (x, x)$. Thus $d_i \in C_i(\Delta^i \times \Delta^i) = S_i(\Delta^i, \Delta^i)$. We claim that $\mathcal{X}_i := \{d_i\}$ is an S_i-model basis. Indeed, if (X, Y) is any object in $\mathsf{Top} \times \mathsf{Top}$ and if $\sigma \colon \Delta^i \to X \times Y$ is any singular n-simplex in $S_i(X, Y) = C_i(X \times Y)$ then we can write $\sigma = (\sigma_X \times \sigma_Y) \circ d_i$, where $\sigma_X = p_X \circ \sigma$ and $\sigma_Y = p_Y \circ \sigma$, and $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ are projections. Conversely, given any pair of singular n-simplices $\tau \colon \Delta^i \to X$ and $\tau' \colon \Delta^i \to Y$, the composition $(\tau \times \tau') \circ d_i$ is a singular n-simplex in $X \times Y$. Thus $\{d_i\}$ is indeed a model basis for S_i.

Next, since $\Delta^i \times \Delta^j$ is convex, it follows that any model $(\Delta^i, \Delta^j) \in \mathcal{M}$ is $S_\bullet$-acyclic in positive degrees. (This is Corollary 13.3.)

Now let us move onto $T_\bullet$. By Problem L.2 (and its solution), for any $(X, Y) \in \mathsf{Top} \times \mathsf{Top}$, $C_i(X) \otimes C_j(Y)$ is free abelian with basis $\{\sigma \otimes \tau \mid \sigma \in C_i(X),\ \tau \in C_j(Y)\}$. By Example 23.4, the functor C_i is free with model basis $\{\ell_i\}$, with $\ell_i \colon \Delta^i \to \Delta^i$ the identity map (thought of as a singular i-simplex in Δ^i). It follows that T_n is free with basis contained in $\mathcal{M}$: a T_n-model basis is

$$\{\ell_i \otimes \ell_j \mid i + j = n\}.$$

The proof that each model is $T_\bullet$-acyclic in positive degrees is much harder, and this is the reason we first carried out the digression above. By Corollary 13.3 again, combined with Proposition 27.5 above, we see that $C_\bullet(\Delta^i)$ is *chain equivalent* to the chain complex

$$\dots 0 \to 0 \to 0 \to \underset{\text{in degree } 0}{\mathbb{Z}} \to 0 \to 0 \to 0 \to \dots$$

Thus by Lemma 26.7, $T_\bullet(\Delta^i, \Delta^j) = C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j)$ is chain equivalent to the chain complex

$$\dots 0 \to 0 \to 0 \to \underset{\text{in degree } 0}{\mathbb{Z} \otimes \mathbb{Z}} \to 0 \to 0 \to 0 \to \dots$$

Thus in particular $H_n(T_\bullet(\Delta^i, \Delta^j)) = 0$ for all $n > 0$.

We have now verified that the hypotheses of the Acyclic Models Theorem and its corollary (Corollary 23.9) are satisfied. We now define a natural equivalence $\Theta \colon H_0 \circ S_\bullet \to H_0 \circ T_\bullet$. For this note that by the Algebraic Künneth Theorem 26.5,

$$H_0(T_\bullet(X, Y)) \cong H_0(X) \otimes H_0(Y),$$

and moreover in degree 0 this isomorphism *is* natural. Meanwhile $H_0(S_\bullet(X, Y)) = H_0(X \times Y)$. The path components of $X \times Y$ are of the form $X' \times Y'$ where X' is a path component of X and Y' is a path component of Y. Thus there is a natural equivalence $H_0(X \times Y) \cong H_0(X) \otimes H_0(Y)$ (cf. Proposition 8.3), that is, a natural equivalence

$$\Theta(X, Y) \colon H_0(S_\bullet(X, Y)) \to H_0(T_\bullet(X, Y)).$$

We can now apply Corollary 23.9 to obtain a natural chain equivalence $\Omega_\bullet \colon S_\bullet \to T_\bullet$, which is unique up to chain homotopy and which satisfies $H_0(\Omega_\bullet) = \Theta$. This completes the proof. ∎

Putting the pieces together, we obtain our desired result, which is usually known as the *Künneth Formula* (in contrast to the Algebraic Künneth Theorem proved last lecture.)

Corollary 27.7 (The Künneth Formula). *Let X and Y be topological spaces. Then for every $n \geq 0$ there is a split exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \to H_n(X \times Y) \to \bigoplus_{k+l=n-1} \mathrm{Tor}\big(H_k(X), H_l(Y)\big) \to 0.$$

Thus

$$H_n(X \times Y) \cong \left(\bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \right) \oplus \left(\bigoplus_{k+l=n-1} \mathrm{Tor}\big(H_k(X), H_l(Y)\big) \right).$$

This gives us (yet another) way to compute the homology of $S^n \times S^m$ (recall we already saw two ways to do this in Problem I.6 and the discussion just after Corollary 20.9). We obtain immediately:

Example 27.8. Let $m, n \geq 1$. If $m \neq n$ then

$$H_i(S^m \times S^n) = \begin{cases} \mathbb{Z}, & i = 0, m, n, m+n, \\ 0, & \text{otherwise.} \end{cases}$$

If $m = n$ then

$$H_i(S^m \times S^n) = \begin{cases} \mathbb{Z}, & i = 0, 2m, \\ \mathbb{Z} \oplus \mathbb{Z}, & i = m, \\ 0, & \text{otherwise.} \end{cases}$$

However now we can also compute more complicated spaces, such as $\mathbb{R}P^m \times \mathbb{R}P^n$. A collection of examples for you to try is on Problem Sheet M.