LECTURE 25

The Universal Coefficients Theorem

This lecture we'll begin by showing how to make Tor into a functor. This requires a few preliminaries.

Definition 25.1. Suppose $(C_\bullet, \partial)$ is a chain complex and A is an abelian group. Let us denote by $C_\bullet \otimes A$ the chain complex whose nth group is $C_n \otimes A$, and whose boundary operator is $\partial \otimes \mathrm{id}_A$. The fact that this is a chain complex (i.e. that $(\partial \otimes \mathrm{id}_A) \circ (\partial \otimes \mathrm{id}_A) = 0$) follows from additivity of $\square \otimes A$ (part (2) of Proposition 24.8). In this way we can view $\square \otimes A$ also as a functor $\mathsf{Comp} \to \mathsf{Comp}$.

Lemma 25.2. *Let $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ be two chain complexes, and let A be an abelian group.*

1. *Suppose $f_\bullet \colon C_\bullet \to C'_\bullet$ is a chain map. Let A be an abelian group. Then $f_\bullet \otimes \mathrm{id}_A \colon C_\bullet \otimes A \to C'_\bullet \otimes A$ is also a chain map.*
2. *Suppose $f_\bullet \colon C_\bullet \to C'_\bullet$ and $g_\bullet \colon C_\bullet \to C'_\bullet$ are two chain maps which are chain homotopic. Then the chain maps $f_\bullet \otimes \mathrm{id}_A$ and $g_\bullet \otimes \mathrm{id}_A$ are also chain homotopic.*

Proof. For the first statement, note that

$$(f_\bullet \otimes \mathrm{id}_A) \circ (\partial \otimes \mathrm{id}_A) = (f_\bullet \circ \partial) \otimes \mathrm{id}_A = (\partial' \circ f_\bullet) \otimes \mathrm{id}_A = (\partial' \otimes \mathrm{id}_A) \circ (f_\bullet \otimes \mathrm{id}_A).$$

For the second, if $P_\bullet$ is a chain homotopy from $f_\bullet$ to $g_\bullet$, that is, $\partial' P_\bullet + P_\bullet \partial = f_\bullet - g_\bullet$, then

$$\begin{aligned}(\partial' \otimes \mathrm{id}_A) \circ (P_\bullet \otimes \mathrm{id}_A) + (P_\bullet \otimes \mathrm{id}_A) \circ (\partial \otimes \mathrm{id}_A) &= \partial' P_\bullet \otimes \mathrm{id}_A + P_\bullet \partial \otimes \mathrm{id}_A \\ &= (\partial' P_\bullet + P_\bullet \partial) \otimes \mathrm{id}_A \\ &= (f_\bullet - g_\bullet) \otimes \mathrm{id}_A \\ &= f_\bullet \otimes \mathrm{id}_A - g_\bullet \otimes \mathrm{id}_A,\end{aligned}$$

so that $P_\bullet \otimes \mathrm{id}_A$ is a chain homotopy between $f_\bullet \otimes \mathrm{id}_A$ and $g_\bullet \otimes \mathrm{id}_A$. ∎

Now let us rephrase the definition of Tor in rather fancier language. Suppose $0 \to K \xrightarrow{f} F \to A \to 0$ is a short free resolution. Let us define a (rather stupid) chain complex $(C_\bullet, \partial)$ by setting:

$$C_n := \begin{cases} F, & n = 0, \\ K, & n = 1, \\ 0, & n \neq 0, 1. \end{cases} \tag{25.1}$$

Will J. Merry, Algebraic Topology II, Spring 2018, ETH Zürich. Last modified: February 27, 2018.

and defining the boundary map $\partial\colon C_1 \to C_0$ to be $f\colon K \to F$. Then this chain complex has the property that

$$H_0(C_\bullet) = F/\operatorname{im} f \cong A.$$

Note this chain complex is both free and acyclic in positive degrees. Now let us tensor this chain complex with B, forming a new complex $(C_\bullet \otimes B, \partial \otimes \mathrm{id}_B)$ (this complex is not free if B is not free.) This new complex has the property that

$$H_1(C_\bullet \otimes B) = \ker(\partial \otimes \mathrm{id}_B\colon C_1 \otimes B \to C_0 \otimes B) = \ker(f \otimes \mathrm{id}_B) = \mathrm{Tor}(A, B).$$

Fix an abelian group B. We will show that $\mathrm{Tor}(\square, B)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a functor. We have already defined $\mathrm{Tor}(\square, B)$ on objects (i.e. abelian groups), so it remains to explain what it does to morphisms. Thus suppose $h\colon A \to A'$ is a homomorphism. We wish to define a homomorphism

$$\mathrm{Tor}(h, B)\colon \mathrm{Tor}(A, B) \to \mathrm{Tor}(A', B).$$

Let $0 \to K \to F \to A \to 0$ and $0 \to K' \to F' \to A' \to 0$ be two short free resolutions of A and A' respectively, and denote by $C_\bullet$ and $C'_\bullet$ the corresponding chain complexes, as described in (25.1), so that

$$H_0(C_\bullet) = A, \qquad H_0(C'_\bullet) = A'.$$

We can therefore think of $h\colon A \to A'$ as a map $H_0(C_\bullet) \to H_0(C'_\bullet)$. Now we invoke Theorem 22.7, which tells us there exists a chain map $g_\bullet\colon C_\bullet \to C'_\bullet$ with $H_0(g_\bullet) = h$.

Tensoring with B, we get a chain map $g_\bullet \otimes \mathrm{id}_B\colon : C_\bullet \otimes B \to C'_\bullet \otimes B$ by the first part of Lemma 25.2. Now pass to the first homology group to get a map

$$H_1(g_\bullet \otimes \mathrm{id}_B)\colon H_1(C_\bullet \otimes B) \to H_1(C'_\bullet \otimes B).$$

Since $H_1(C_\bullet \otimes B) = \mathrm{Tor}(A, B)$ and $H_1(C'_\bullet \otimes B) = \mathrm{Tor}(A', B)$, we can think of this a map

$$\mathrm{Tor}(h, B) := H_1(g_\bullet \otimes \mathrm{id}_B)\colon \mathrm{Tor}(A, B) \to \mathrm{Tor}(A', B).$$

We still haven't addressed the question as to why Tor is well defined (i.e. that it doesn't depend on the choice of short free resolution.) This is proved in the same way as the argument above: we start with two short free resolutions $0 \to K \to F \to A \to 0$ and $0 \to K' \to F' \to A \to 0$ of the same group A. Denote as before the two chain complexes by $C_\bullet$ and $C'_\bullet$. Now take h to be the identity.

Invoking Theorem 22.7 again, we obtain a chain map $g_\bullet\colon C_\bullet \to C'_\bullet$ with $H_0(g_\bullet) = \mathrm{id}_A$. Moreover, by Corollary 22.9, this chain map $g_\bullet$ is a chain equivalence. By the second part of Lemma 25.2, the tensored map $g_\bullet \otimes \mathrm{id}_B$ is also a chain equivalence. Thus in particular $H_1(g_\bullet \otimes \mathrm{id}_A)$ is an isomorphism, which implies that

$$H_1(C_\bullet \otimes B) \cong H_1(C'_\bullet \otimes B).$$

Thus $\mathrm{Tor}(A, B)$ is indeed independent of the choice of short free resolution.

We have therefore proved:

Proposition 25.3. *For each fixed abelian group B, $\operatorname{Tor}(\square, B)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a functor.*

Remark 25.4. In a similar vein, we can also fix the first variable of Tor: if A is any abelian group then $\operatorname{Tor}(A, \square)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a functor, and the value of $\operatorname{Tor}(A, \square)$ on B is isomorphic to the value of $\operatorname{Tor}(\square, B)$ on A. I will leave if to you as an exercise to guess how to define the induced map $\operatorname{Tor}(A, h)\colon \operatorname{Tor}(A, B) \to \operatorname{Tor}(A, B')$ for a given homomorphism $h\colon B \to B'$.

The next result is on Problem Sheet L.

Lemma 25.5. *Suppose B is a torsion-free abelian group. Then $\square \otimes B$ and $B \otimes \square$ are exact functors.*

The following theorem summarises the main properties of Tor.

Theorem 25.6 (Properties of Tor).

1. *If either A or B are torsion-free abelian groups then $\operatorname{Tor}(A, B) = 0$.*
2. *If[1] $T(A)$ denotes the torsion subgroup of A then for any abelian group B one has $\operatorname{Tor}(A, B) = \operatorname{Tor}(T(A), B)$.*
3. *If $0 \to B \to B' \to B'' \to 0$ is a short exact sequence, then there is an exact sequence*

$$0 \to \operatorname{Tor}(A, B) \to \operatorname{Tor}(A, B') \to \operatorname{Tor}(A, B'') \to A \otimes B \to A \otimes B' \to A \otimes B'' \to 0. \tag{25.2}$$

Similarly if $0 \to A \to A' \to A'' \to 0$ is a short exact sequence, then there is an exact sequence

$$0 \to \operatorname{Tor}(A, B) \to \operatorname{Tor}(A', B) \to \operatorname{Tor}(A'', B) \to A \otimes B \to A' \otimes B \to A'' \otimes B \to 0. \tag{25.3}$$

4. *For any two abelian groups A, B, $\operatorname{Tor}(A, B) \cong \operatorname{Tor}(B, A)$.*
5. *If B is an abelian group and $\{A_\lambda \mid \lambda \in \Lambda\}$ is a (possibly uncountable) family of abelian groups then there is an isomorphism*

$$\operatorname{Tor}\Big(\bigoplus_{\lambda \in \Lambda} A_\lambda, B\Big) \cong \bigoplus_{\lambda \in \Lambda} \operatorname{Tor}(A_\lambda, B).$$

and similarly

$$\operatorname{Tor}\Big(B, \bigoplus_{\lambda \in \Lambda} A_\lambda\Big) \cong \bigoplus_{\lambda \in \Lambda} \operatorname{Tor}(B, A_\lambda).$$

6. *For any $m \in \mathbb{N}$ and any abelian group B,*

$$\operatorname{Tor}(\mathbb{Z}_m, B) \cong \{b \in B \mid mb = 0\}.$$

[1] This is the reason for the name "Tor".

Proof. If A is free then we can choose a silly short free resolution: $0 \to 0 \to A \to A \to 0$. Then clearly $\mathrm{Tor}(A, B) = 0$ for any abelian group B. If B is torsion-free then by Lemma 25.5, for any short free resolution $0 \to K \to F \to A$ of A, the sequence $0 \to K \otimes B \to F \otimes B \to A \otimes B \to 0$, so that $\mathrm{Tor}(A, B) = 0$ in this case too. This proves part (1) in the case where A is free and the case where B is torsion-free. We have not yet done the case where A is merely torsion-free; we will do this after proving part (4).

Let us skip part (2) for now and prove the first statement of part (3). Suppose $0 \to B' \to B'' \to B'' \to 0$ is exact. Let $0 \to K \to F \to A \to 0$ be a short free resolution of A. Let $C_\bullet$ denote the chain complex as defined in (25.1), so that $\mathrm{Tor}(A, B) = H_1(C_\bullet \otimes B)$ and similarly for the other two. Then since K and F are free, by Lemma 25.5 again, the following diagram has exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K \otimes B \arrow[r] \arrow[d] & K \otimes B' \arrow[r] \arrow[d] & K \otimes B'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \otimes B \arrow[r] & F \otimes B' \arrow[r] & F \otimes B'' \arrow[r] & 0
\end{tikzcd}$$

This means that

$$0 \to C_\bullet \otimes B \to C_\bullet \otimes B' \to C_\bullet \otimes B'' \to 0$$

is a short exact sequence of chain complexes. The desired sequence (25.2) is then simply the last six terms of the long exact sequence in homology (Theorem 11.5) associated to this short exact sequence. This proves the first statement of part (3). We will prove the second statement of part (3) after we have proved part (4).

Suppose $0 \to K \to F \to A \to 0$ is a short free resolution of A. Then for any abelian group B, since K and F are free, from part (1) we know that $\mathrm{Tor}(B, K) = \mathrm{Tor}(B, F) = 0$. We then apply (25.2) to the exact sequence $0 \to K \to F \to 0$ to obtain an exact sequence

$$0 \to 0 \to 0 \to \mathrm{Tor}(B, A) \to B \otimes K \to B \otimes F \otimes B \otimes A \to 0.$$

By definition of $\mathrm{Tor}(A, B)$, the bottom row of the next diagram is exact:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Tor}(B, A) \arrow[r] & B \otimes K \arrow[r] \arrow[d, "\cong"] & B \otimes F \arrow[r] \arrow[d, "\cong"] & B \otimes A \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathrm{Tor}(A, B) \arrow[r] & K \otimes B \arrow[r] & F \otimes B \arrow[r] & A \otimes B \arrow[r] & 0
\end{tikzcd}$$

The vertical arrows are isomorphisms by natural commutativity of the tensor product (part (1) of Proposition 24.8). Thus by the Five Lemma (Proposition 11.3) we also have an isomorphism $\mathrm{Tor}(B, A) \cong \mathrm{Tor}(A, B)$. This proves part (4).

Now we can go back and finish the proof of part (1): if A is torsion-free then for any abelian group B, using part (4) we have $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A) = 0$, since we have already shown that $\mathrm{Tor}(\square, A)$ vanishes on torsion-free groups.

We can also now prove part (2). Note that for any abelian group A, if $T(A)$ denotes the torsion subgroup then $A/T(A)$ is torsion-free. Thus by part (1) (which we have now completely proved) we have $\mathrm{Tor}(A/T(A), B) = 0$ for any abelian group B.

We now apply part (3) to the short exact sequence $0 \to T(A) \to A \to A/T(A) \to 0$. The first three terms of (25.2) become

$$0 \to \operatorname{Tor}(T(A), B) \to \operatorname{Tor}(A, B) \to 0.$$

This proves part (2). The second sequence (25.3) in part (3) also easily follows from (25.2), given part (4).

The proof of part (5) is easy: if $0 \to K_\lambda \to F_\lambda \to A_\lambda \to 0$ is a short free resolution of A_λ then

$$0 \to \bigoplus_{\lambda \in \Lambda} K_\lambda \to \bigoplus_{\lambda \in \Lambda} F_\lambda \to \bigoplus_{\lambda \in \Lambda} A_\lambda \to 0$$

is a short free resolution of $\bigoplus_{\lambda \in \Lambda} A_\lambda$. This proves the first statement of part (5), and the second statement then follows by applying part (4).

Finally, the proof of part (6) is also easy. For this we use the short free resolution $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}_m \to 0$ of $\mathbb{Z}_m$. The desired result then follows by applying part (3) of Proposition 24.8. This finally finishes the proof of the theorem. ■

Remark 25.7. Theorem 25.6 allows one to compute the value of Tor in some easy situations. For instance, $\operatorname{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$, where d is the greatest common divisor of m and n.

Finally some topology:

Definition 25.8. Let X be a topological spaces and let A be an abelian group. Let $C_\bullet(X)$ denote the singular chain complex, and set $C_\bullet(X; A) := C_\bullet(X) \otimes A$. Thus a typical n-chain in $C_n(X; A)$ has the form $\sum a_i \otimes \sigma_i$, where $a_i \in A$ and $\sigma_i \colon \Delta^n \to X$ is a singular n-simplex in X. We define the **nth singular homology of X with coefficients in A** to be the group

$$H_n(X; A) := H_n(C_\bullet(X; A)).$$

In the same way, if $X' \subseteq X$ is a subspace we define the chain complex[2] $C_\bullet(X, X'; A)$ and the relative homology with coefficients in A.

It follows from part (4) of Proposition 24.8 that taking $A = \mathbb{Z}$ recovers the normal singular homology groups:

$$H_n(X, X'; \mathbb{Z}) = H_n(X, X').$$

Remark 25.9. Let A be an abelian group. We define a **homology theory** $(\mathcal{H}_\bullet, \delta)$ **with coefficients in** A in exactly the same way as we defined a (normal) homology theory in Definition 21.9, apart from the dimension axiom is replaced by:

- If $\{*\}$ is a one-point space then $\mathcal{H}_n(*) = 0$ for all $n > 0$ and $\mathcal{H}_0(*) = A$.

Singular homology with coefficients in A is then an example of a homology theory with coefficients in A.

[2] Pay attention to the difference between the comma and the semi-colon in $H_n(X, X'; A)$!

We will see shortly why homology with coefficients is useful. For instance, taking $A = \mathbb{Z}_2$ is often particularly pleasant, as this allows one to ignore all $\pm$ signs that crop up in formulae. On Problem Sheet L you will see that taking $\mathbb{Z}_2$ coefficients gives one an easier way to prove Theorem 15.12, that an odd map has an odd degree.

Another useful choice is $A = \mathbb{R}$; this gives homology with **real coefficients**. This theory is particularly useful in differential geometry (when X is a manifold). Since $\mathbb{R}$ is torsion-free, we will see shortly that one always has

$$H_n(X; \mathbb{R}) = H_n(X) \otimes \mathbb{R}.$$

Here is the main theorem of today's lecture:

THEOREM 25.10 (The Universal Coefficients Theorem). *Let X be a topological space and let A be an abelian group. Then for every $n \geq 0$ there is an exact sequence*

$$0 \to H_n(X) \otimes A \xrightarrow{\omega} H_n(X; A) \to \operatorname{Tor}(H_{n-1}(X), A) \to 0, \tag{25.4}$$

where ω is the map $\langle c \rangle \otimes a \mapsto \langle c \otimes a \rangle$. Moreover this sequence splits, and hence

$$H_n(X; A) \cong H_n(X) \otimes A \oplus \operatorname{Tor}(H_{n-1}(X), A). \tag{25.5}$$

REMARK 25.11. The splitting of the sequence (25.4) is not natural, and hence the isomorphism (25.5) is also not natural (cf. Remark 12.17.)

In fact we will prove a more general statement:

THEOREM 25.12 (The Universal Coefficients Theorem II). *Let $(C_\bullet, \partial)$ denote a free chain complex and let A denote an abelian group. Then for every $n \geq 0$, there is an exact sequence*

$$0 \to H_n(C_\bullet) \otimes A \xrightarrow{\omega} H_n(C_\bullet \otimes A) \to \operatorname{Tor}(H_{n-1}(C_\bullet), A) \to 0,$$

where ω is the map $\langle c \rangle \otimes a \mapsto \langle c \otimes a \rangle$. Moreover this sequence splits, and hence

$$H_n(C_\bullet \otimes A) \cong H_n(C_\bullet) \otimes A \oplus \operatorname{Tor}(H_{n-1}(C_\bullet), A).$$

We will need the following lemma, whose proof is again on Problem Sheet L. It tells us that split exact sequences are *always* preserved by additive functors.

LEMMA 25.13. *If $T \colon \mathsf{Ab} \to \mathsf{Ab}$ is an additive functor and $0 \to A \to B \to C \to 0$ is any split exact sequence, then $0 \to T(A) \to T(B) \to T(C) \to 0$ is also a split exact sequence.*

Proof of Theorem 25.12. The proof is even more tedious and long-winded than the proof of Theorem 25.6, and we will break the proof up into four steps.

1. Let $B_n \subseteq Z_n \subseteq C_n$ denote the boundaries and cycles respectively. For notational clarity, we denote by d the map $\delta \colon C_n \to B_{n-1}$ (i.e. when we think of the target as B_{n-1} rather than C_n). Let us also denote by $i \colon Z_n \hookrightarrow C_n$ the inclusion. Then we have an exact sequence

$$0 \to Z_n \xrightarrow{i} C_n \xrightarrow{d} B_{n-1} \to 0 \tag{25.6}$$

Since B_{n-1} is a subgroup of the free abelian group C_n, B_{n-1} is itself free, and hence by part (3) of Problem F.6, the exact sequence (25.6) splits. To keep the notation simple, for the rest of the proof we write id for id_A. Now tensor with A to obtain another split exact sequence

$$0 \to Z_n \otimes A \xrightarrow{i \otimes \mathrm{id}} C_n \otimes A \xrightarrow{d \otimes \mathrm{id}} B_{n-1} \otimes A \to 0, \tag{25.7}$$

where we are using Lemma 25.13. Next, we can view $Z_\bullet$ as a subcomplex of $C_\bullet$ where all the differentials are zero. Define a chain complex $B_\bullet^+$ to be the chain complex whose nth group $B_n^+ := B_{n-1}$ (!) and again with all the differentials zero. Then we can assemble the short exact sequences (25.7) into a short exact sequence of chain complexes:

$$0 \to Z_\bullet \otimes A \xrightarrow{i_\bullet \otimes \mathrm{id}} C_\bullet \otimes A \xrightarrow{d_\bullet \otimes \mathrm{id}} B_\bullet^+ \otimes A \to 0.$$

2. Now consider the long exact sequence in homology associated to this short exact sequence. Since $Z_\bullet$ and $B_\bullet^+$ have no differentials, we have

$$H_n(Z_\bullet \otimes A) = Z_n \otimes A, \qquad H_n(B_\bullet^+ \otimes A) = B_{n-1} \otimes A,$$

and hence if δ is the connecting homomorphism of the long exact sequence, we can write it as:

$$\dots B_n \otimes A \xrightarrow{\delta} Z_n \otimes A \to H_n(C_\bullet \otimes A) \to B_{n-1} \otimes A \xrightarrow{\delta} Z_{n-1} \otimes A \to \dots$$

Thus for every n there is an exact sequence

$$0 \to Z_n \otimes A / \operatorname{im} \delta \xrightarrow{\omega} H_n(C_\bullet \otimes A) \to \ker \delta \to 0, \tag{25.8}$$

where ω is the map induced by $H_n(i_\bullet \otimes \mathrm{id})$:

$$\omega\colon z \otimes a + \operatorname{im} \delta \mapsto H_n(i_\bullet \otimes \mathrm{id})(z \otimes a) = \langle z \otimes a \rangle$$

(recall i is just an inclusion.) Let us identify the connecting homomorphism δ. From Theorem 11.5, we have for a generator $b \otimes a \in B_{n-1} \otimes A$ that

$$\delta(b \otimes a) = (i \otimes \mathrm{id})^{-1}(\partial \otimes \mathrm{id})(d \otimes \mathrm{id})^{-1}(b \otimes a),$$

which is just $b \otimes a$ again, only now regarded as an element of $Z_{n-1} \otimes A$. Thus if $j\colon B_\bullet \hookrightarrow Z_\bullet$ is the inclusion, then $\delta = j \otimes \mathrm{id}$. This means we can rewrite (25.8) as

$$0 \to (Z_n \otimes A)/\operatorname{im}(j \otimes \mathrm{id}) \xrightarrow{\omega} H_n(C_\bullet \otimes A) \to \ker(j \otimes \mathrm{id}) \to 0. \tag{25.9}$$

3. The definition of homology gives exact sequences

$$0 \to B_{n-1} \xrightarrow{j} Z_{n-1} \to H_{n-1}(C_\bullet) \to 0.$$

In fact, this is a short free resolution, since (as we have already observed), both B_{n-1} and Z_{n-1} are free. Thus $\operatorname{Tor}(H_{n-1}(C_\bullet), A) = \ker(j \otimes \mathrm{id})$. Next, apply (25.3) to this short exact sequence, and use the fact that $\mathrm{Tor}(Z_{n-1}, A) = 0$ as Z_{n-1} is free to obtain exact sequences

$$0 \to \mathrm{Tor}(H_{n-1}(C_\bullet), A) \to B_{n-1} \otimes A \xrightarrow{j \otimes \mathrm{id}} Z_{n-1} \otimes A \to H_{n-1}(C_\bullet) \otimes A \to 0.$$

This tells us that (replacing $n-1$ with n) that

$$(Z_n \otimes A)/\operatorname{im}(j \otimes \mathrm{id}) = \operatorname{coker}(j \otimes \mathrm{id}) = H_n(C_\bullet) \otimes A.$$

Thus we can rewrite (25.9) as

$$0 \to H_n(C_\bullet) \otimes A \xrightarrow{\omega} H_n(C_\bullet \otimes A) \to \mathrm{Tor}(H_{n-1}(C_\bullet), A) \to 0, \tag{25.10}$$

which is what we were trying to prove.

4. It remains to show that (25.10) splits. For this, let $r_n \colon C_n \to Z_n$ denote a splitting of (25.6) (such an r_n exists by part (1) of Problem F.6). Then the composition

$$\ker(\partial \otimes \mathrm{id}) \subseteq C_\bullet \otimes A \xrightarrow{r_\bullet \otimes \mathrm{id}} Z_n \otimes A \to H_n(C_\bullet) \otimes A$$

maps $\operatorname{im}(\partial \otimes \mathrm{id})$ to zero and hence induces a map $\rho \colon H_n(C_\bullet \otimes A) \to H_n(C_\bullet) \otimes A$ with $\rho \circ \omega$ the identity on $H_n(C_\bullet) \otimes A$. This finally completes the proof. ■