

Notes of Readings in Topology and Geometry

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About the Notes:

This notes is a summary of personal reading of Topology and geometry.

1 Complex manifold

1.1 Complex structure, almost complex structure

Definition 1.1. A complex valued function $f : \mathbb{C}^m \rightarrow \mathbb{C}$ is **holomorphic** if $f = f_1 + if_2$ it satisfies the **Cauchy-Riemann relations** for $z^\mu = x^\mu + iy^\mu$,

$$\frac{\partial f_1}{\partial x^\mu} = \frac{\partial f_2}{\partial y^\mu}$$

$$\frac{\partial f_2}{\partial x^\mu} = -\frac{\partial f_1}{\partial y^\mu}$$

2 Spectral sequences

Definition 2.1. An **exact couple** is an exact sequence of Abelian groups of the form

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & \swarrow k & \searrow j \\ & B & \end{array}$$

where i, j, k are group homomorphisms. Define $d : B \rightarrow B$ by $d = j \circ k$. Then $d^2 = j(jk)k = 0$, so the homology group $H(B) = \ker(d)/\text{im}(d)$ is well-defined Abelian group.

Out of the exact couple, we can construct a **derived couple**

$$\begin{array}{ccc} A' & \xrightarrow{i'} & A' \\ & \swarrow k' & \searrow j' \\ & B' & \end{array}$$

by setting

- (i) $A' := i(A)$; $B' := H(B)$.
- (ii) i' induced from i , i.e., $i'(i(a)) = i(i(a))$

- (iii) If $a' = i(a) \in A'$ with $a \in A$, then $j'(a') := [j(a)] \in H(B)$.
- (iv) k' is induced from k . Consider a comology class $[b]$, $db = 0 \iff jkb = 0$, then $kb \in i(A)$. Define $k'[b] := kb \in i(A)$.

Proof. We can check that j' is well-defined: $ia = ia_1 \implies [j(a)] = [j(a_1)]$. Indeed $i(a - a_1) = 0 \implies (a - a_1) \in \text{im}(k)$. $\exists z \in B$ s.t. $k(z) = a - a_1 \implies [j(a - a_1)] = [jk(z)] = [dz]$. Also k' is well defined: $[b] = [b_1] \implies b - b_1 = dz \implies k(b - b_1) = kjkz = 0$.

The derived sequence is indeed exact:

Exactness at $\xrightarrow{k'} A' \xrightarrow{i'} i' \circ k'[b] = i'kb = ikb = 0$, we know $\ker(i') \supseteq \text{im}(k')$. For the reverse inclusion, $i(a) \in \ker(i') \implies iia = 0$ then $ia \in \text{im}(k)$ because the original exact couple is exact. $ia = kb \implies k'[b]$, hence $\text{im}(k') \supseteq \ker(i')$.

Exactness at $\xrightarrow{i'} A' \xrightarrow{j'} j' \circ i'(ia) = j'(iia) = [jia] = 0$. For the reverse inclusion, consider $ia \in \ker(j')$, $j'(ia) = [ja] = 0 \implies ja = db \in B \implies j(a - kb) = 0 \implies a - kb = i(a_1) \implies i(a - kb) = ia = ii(a_1) = i'(ia_1)$.

Exactness at B' : $k'j'(ia) = k'[ja] = kja = 0 \implies \ker(k') \supseteq \text{im}(j')$. For the reverse inclusion, we can pick $[b] \in \ker(k') \implies k'[b] = kb = 0 \implies b = ja$ for some $a \in A$. $[b] = j'(ia) \in \text{im}(j')$.

□

3 Fundamental groups

Definition 3.1. Let X and Y be topological spaces and $f, g : X \longrightarrow Y$ continuous maps. A **homotopy** from f to g is a continuous map

$$H : X \times [0, 1] \longrightarrow Y, (x, t) \longmapsto H(x, t) = H_t(x)$$

such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1) \forall x \in X$. $f = H_0$ and $g = H_1$, $f \simeq g$

The homotopy relation is an equivalence relation on the set of continuous maps $X \longrightarrow Y$. Given two homotopy $K : f \simeq g$ and $L : g \simeq h$, the product homotopy $K * L$

$$(K * L)(x, t) = \begin{cases} K(x, 2t), & 0 \leq t \leq 1/2, \\ L(x, 2t - 1), & 1/2 \leq t \leq 1, \end{cases}$$

and shows $f \simeq h$.

The inverse homotopy is defined to be $H^-(x, t) := H(x, 1 - t)$. Notice that product of homotopy and inverse homotopy is not constant homotopy.

The equivalence class of f is denoted $[f]$ and called the homotopy class of f . We denote by $[X, Y]$ the set of homotopy classes $[f]$ of maps $f : X \rightarrow Y$. A homotopy $H_t : X \rightarrow Y$ is said to be relative to $A \subset X$ if the restriction $H_t|_A$ does not depend (is constant on A). We use the notation $H : f \rightarrow g$ (rel A) in this case.

Quotient category means we identify some of the morphism. For each $\text{Mor}(X, Y)$, we quotient a relation $R_{X,Y}$.

Definition 3.2. *Topological spaces and homotopy classes of maps form a quotient category of Top , which is called **homotopy category**, denoted $h\text{-Top}$. The composition of homotopy class is induced by composition of representing maps. The isomorphism in this category is homotopy equivalence.*

In the category of $h\text{-Top}$. Consider the Hom -functors. Given $f : X \rightarrow Y$.

$$\text{Hom}(Z, _)(f) = f_* : [Z, X] \rightarrow [Z, Y] : g \mapsto fg$$

$$\text{Hom}(_, Z)(f) = f^* : [Z, X] \rightarrow [Z, Y] : h \mapsto hf$$

f is h -equivalence (isomorphism in the category $h\text{-Top}$) iff $\text{Hom}(_, Z)(f)$ is always bijective. (Yoneda Lemma), similarly for $\text{Hom}(Z, _)(f)$. Because we know for f_*, g_*, g_*f_* , 2 of the three maps are bijective implies that the third is bijective. This can be translated into homotopy category, where f, g, fg two of the three homotopy class being homotopy equivalence implies the third is also a homotopy equivalence.

Let P be a point. A map $P \rightarrow Y$ can be identified as its image and a homotopy can be identified with path. Then the Hom -functor $[P, _]$ can be identified as π_0

Proposition 3.3. *The product of paths has the following properties:*

- (i) Let $\alpha : I \rightarrow I$ be continuous and $\alpha(0) = 0, \alpha(1) = 1$. Then $u \simeq u\alpha$.
- (ii) $u_1 * (u_2 * u_3) = (u_1 * u_2) * u_3$
- (iii) $u_1 \simeq u'_1$ and $u_2 \simeq u'_2$ implies $u_1 * u_2 \simeq u'_1 * u'_2$.
- (iv) $u * u^-$ is always defined and homotopic to the constant path.
- (v) $k_{u(0)} * u \simeq u \simeq u * k_{u(1)}$.

Proof. (i) Let $H : (s, t) \mapsto u(s(1-t) + t\alpha(s))$ is homotopy from u to $u\alpha$.

(ii) choose

$$\alpha(t) = \begin{cases} 2t, & t \leq \frac{1}{4} \\ t + \frac{1}{4}, & \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t+1}{2}, & \frac{1}{2} \leq t \leq 1 \end{cases}$$

we have $u_1 * (u_2 * u_3)\alpha = (u_1 * u_2) * u_3$, then we can apply (i)

(iii) Given $F_i : u_i \simeq U'_i$, then we can define the homotopy $G : u_1 * u_2 \simeq u'_1 u'_2$

$$G(s, t) = \begin{cases} F_1(2s, t), & 0 \leq t \leq \frac{1}{2} \\ F_2(2s - 1, t), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

(iv) The map $F : I \times I \longrightarrow X$ defined as

$$F(s, t) = \begin{cases} u(2s(1 - t)), & 0 \leq t \leq \frac{1}{2} \\ u(2(1 - s)(1 - t)), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is the homotopy from $u * u^-$ to the constant path.

(v) use (i) again.

□

This basically says that the homotopy class of path with a fixed point is a group.

From homotopy classes of paths in X , we obtain again a category denote $\Pi(X)$. The objects are the points of X . A morphism from x to y is a homotopy class of paths. It is called **Fundamental groupoid** of X . The automorphism group of the object x in this category is the fundamental group of X with base point x .

Proposition 3.4.

Proposition 3.5. *Let $f : X \longrightarrow Y$ be a homotopy equivalence. Then the functor $\Pi(f) : \Pi(X) \longrightarrow \Pi(Y)$ is an equivalence of categories. The induced maps between morphism sets $f_* : \Pi X(x, y) \longrightarrow \Pi Y(fx, fy)$ are bijections. IN particular,*

$$\pi_1(f) : \pi_1(X, x) \longrightarrow \pi_1(Y, f(x)), [\omega] \mapsto [f\omega]$$

is an isomorphism for each $x \in X$

Theorem 3.6. (*R. Brown*). Let X_0 and X_1 be a subspace of X such that the interiors cover X . Let $i_\nu : X_{01} \hookrightarrow X_\nu$ and $j_\nu : X_\nu \hookrightarrow X$ be the inclusions. Then

$$\begin{array}{ccc} \Pi(X_{01}) & \xrightarrow{\Pi(i_0)} & \Pi(X_0) \\ \Pi(i_1) \downarrow & & \downarrow \Pi(j_0) \\ \Pi(X_1) & \xrightarrow{\Pi(j_1)} & \Pi(X) \end{array}$$

is a pushout (fibered coproduct) in the category of groupoids. Or $\Pi(X)$ is the colimit of diagrams indexed by $X_0 \supset X_{01} \subset X_1$

3.1 Spectral sequences of filtered complexes

Now, we want to construct the spectral sequence of a filtered complex.

Definition 3.7. Let K be differential complex with differential operator D . A subcomplex K' is a subgroup in K such that $DK' \subseteq K'$. A sequence of subcomplex

$$K = K_0 \supseteq K_1 \supseteq K_2 \dots$$

is called a filtration of K . A differential complex with specified filtration is called a filtered complex with **associated graded complex**

$$Gr_\bullet K = \bigoplus_{p=0}^{\infty} K_p / K_{p+1}$$

For filtered complex K , let A be the group

$$A = \bigoplus_{p \in \mathbb{Z}} K_p.$$

A is again a differential complex with differential operator $\oplus D$. Define ι to be the inclusion $A \hookrightarrow A$ induced by $K_{p+1} \hookrightarrow K_p$. Then we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} A \xrightarrow{j} B := Gr_\bullet K \longrightarrow 0.$$

If A, K are themselves graded chain complex (A different grading from the associated grading w.r.t. to filtration, we use upper index to distinguish it from the filtration index), we have a long exact sequence of cohomology groups

$$\longrightarrow H^k(A^\bullet) \xrightarrow{i} H^k(A) \xrightarrow{j_1} H^k(B) \xrightarrow{k_1} H^{k+1}(A) \longrightarrow \dots$$

Consider that $H(A) = \bigoplus_k H^k(A)$, we have the exact couple

$$\begin{array}{ccc}
H(A) & \xrightarrow{i} & H(A) \\
\swarrow k_1 & & \swarrow j_1 \\
& H(B) &
\end{array}
:=
\begin{array}{ccc}
A_1 & \xrightarrow{i} & A_1 \\
\swarrow k_1 & & \swarrow j_1 \\
& B_1 &
\end{array}$$

From now on we will suppress the subscript of i_n because by definition, $i_n(i_{n-1} \dots (i(a))) = i^n(a)$. Even if they are not graded, we can still artificially construct the short exact sequence of chain complex

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & A & \xrightarrow{j} & B \longrightarrow 0 \\
& & \downarrow D & & \downarrow D & & \downarrow D \\
0 & \longrightarrow & A & \xrightarrow{i} & A & \xrightarrow{j} & B \longrightarrow 0 \\
& & \downarrow D & & \downarrow D & & \downarrow D \\
0 & \longrightarrow & \vdots & \xrightarrow{i} & \vdots & \xrightarrow{j} & \vdots \longrightarrow 0
\end{array}$$

And it still gives the above exact couple.

Then we have all the derived exact couples and label it with r meaning it is the r -th derived exact couple of the first one:

$$\begin{array}{ccc}
A_r & \xrightarrow{i} & A_r \\
\swarrow k_r & & \swarrow j_r \\
& B_r &
\end{array}$$

Consider the special case where the filtration terminates after K_3 .

$$K_{-1} = K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq 0$$

$$\begin{array}{lcl}
A_1 := & & \begin{array}{ccccccc}
& & \xleftarrow{i} & & \xleftarrow{i} & & \xleftarrow{i} \\
H(K_0) & \oplus & H(K_1) & \oplus & H(K_2) & \oplus & H(K_3) \\
\parallel & & \xleftarrow{\supseteq} & & \xleftarrow{i} & & \xleftarrow{i} \\
A_2 := i(A_1) = & & iH(K_0) & \oplus & iH(K_1) & \oplus & iH(K_2) & \oplus & iH(K_3) \\
\parallel & & \xleftarrow{\supseteq} & & \parallel & \xleftarrow{\supseteq} & \parallel & \xleftarrow{i} & \\
A_3 := i(A_2) = & & i^2H(K_0) & \oplus & i^2H(K_1) & \oplus & i^2H(K_2) & \oplus & i^2H(K_3) \\
\parallel & & \xleftarrow{\supseteq} & & \parallel & \xleftarrow{\supseteq} & \parallel & \xleftarrow{\supseteq} & \\
A_4 := i(A_3) = & & i^3H(K_0) & \oplus & i^3H(K_1) & \oplus & i^3H(K_2) & \oplus & i^3H(K_3).
\end{array}
\end{array}$$

Because $iH(K_1) \subseteq H(K_0)$ and i act as identity on $H(K_0)$, we know i act as inclusion on $iH(K_1)$, hence $iH(K_1) = i^2H(K_1)$. Similarly, we can say $i^n(K_i)$ stabilizes when $n \geq 3$, hence $A_4 = A_5 = \dots A_\infty$.

$$\begin{array}{ccc} A_4 & \xrightarrow{i} & A_4 \\ & \swarrow k_4 \quad \nwarrow j_4 & \\ & B_4 & \end{array} .$$

Furthermore, since $i : A_4 \rightarrow A_4$ is the inclusion, the map $k_4 : B_4 \rightarrow A_4$ must be a zero map, hence the differential $d_4 := j_4 k_4 = 0$ and $B_5 = H_{d_4}(B_4) = B_4$. $B - r$ also stabilize after B_4 . $B_4 = B_5 = \dots = B_\infty$.

$$\begin{array}{ccc} A_\infty & \xrightarrow{i_\infty : \subseteq} & A_\infty \\ & \swarrow k_\infty = 0 \quad \nwarrow j_\infty & \\ & B_\infty & \end{array} .$$

$k_\infty = 0 \implies B_\infty$ is the quotient of i_∞ . In other words, B_∞ is the associated graded complex of the filtration

$$H(K) = H(K_0) \supseteq iH(K_1) \supseteq iiH(K_2) \supseteq iiiH(K_3).$$

In general consider a filtration of complex K with differential D .

$$K = K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$$

It induces a sequence in cohomology

$$H(K) = H(K_0) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \xleftarrow{i} \dots$$

Set $F_p := i^p H(K_p)$ be the image of $H(K_p)$ in $H(K)$. It gives a filtration of $H(K)$

$$H(K) = F_0 \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

A filtration K_\bullet is of **length** l if the descending chain terminates after K_l . If K_\bullet is of finite length, then A_r and B_r eventually stabilize and B_∞ is the associated graded complex $\oplus F_p / F_{p+1}$ of $F_\bullet H(K)$.

Definition 3.8. It is customary to write E_r for B_r . A sequence of differential complex $\{E_r, d_r\}$ in which $E_{r+1} = H_{d_r}(E_r)$ is called a **spectral sequence**. If E_r eventually stabilize, we denote the stationary value E_∞ . If $E_\infty \cong Gr_\bullet H$ of some filtered complex H .

Now assume K is a graded differential complex $K = \oplus_n K^n$, with filtration K_\bullet . Then each graded piece K^n is filtered complex with filtration $K_p^n = K^n \cap K_p$.

Theorem 3.9. *If $K = \oplus_n K^n$ is a graded filtered complex with filtration $\{K_p\}$ and let $H_D(K)$ denote the cohomology of K with a filtration $\{F_p\}$ induced by $\{K_p\}$. Suppose that for each fixed grading index n , the filtration $\{K_p^n\}$ is of finite length. Then the short exact sequence*

$$0 \longrightarrow \oplus_{p \in \mathbb{Z}} K_{p+1} \longrightarrow \oplus_{p \in \mathbb{Z}} K_p \longrightarrow \oplus_{p \in \mathbb{Z}} K_p / K_{p+1} \longrightarrow 0$$

induces a spectral sequence converging to $H_D(K)$.

Proof. We have the exact couple

$$\begin{array}{ccc} A_r & \xrightarrow{i} & A_r \\ & \nwarrow k_r \quad \nearrow j_r & \\ & B_r & \end{array},$$

where $A_r = i^{r-1}H(K_p)$, if $r \geq p$, $i^r H(K_p) = F_p$. (When $r \geq p+1$, the map $i : i^r H(K_p) \longrightarrow i^r H(K_p)$ is an inclusion).

Recall that k_1 is the connecting map $k_1 : H^*(B) \longrightarrow H^{*+1}(A)$. k_r would send $B_r^d \longrightarrow A_r^{n+1}$, while i, j_r would fix n .

For a fixed grading index n , assume the length of the filtration $\{K_p^n\}$ is $l(n)$. When $r \geq l(n+1) + 1$, for every p we have

$$i^r H^{n+1}(K_p) = F_p^{n+1}$$

$$A_r^{n+1} = \oplus_p F_p^{n+1}$$

and the map

$$i : i^r H^{n+1}(K_p) \longrightarrow i^r H^{n+1}(K_p)$$

is inclusion. Hence

$$i : A_r^{n+1} \longrightarrow A_r^{n+1} \quad i : F_{p+1}^{n+1} \longrightarrow F_p^{n+1}$$

is injective and

$$k_r : B_r^n \longrightarrow A_r^{n+1}$$

is zero map. We have

$$0 \longrightarrow \oplus_p F_{p+1}^n \xrightarrow{i} \oplus_p F_p^n \longrightarrow B_\infty^n \longrightarrow 0$$

Then we know

$$B_{\infty}^n = \oplus_{p \leq l(n)} F_p^n / F_{p+1}^n$$

and

$$B_{\infty} = \oplus_n B_{\infty}^n = \oplus_p F_p / F_{p+1} = Gr_{\bullet} H_D(K)$$

3.2 Spectral sequences of double complex

□