Notes of B-V formalism in derived settings

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2018 ETH

Contents

1 Recap on BV formalism

$$\int_X e^{iS_0(X)/\hbar} f(x)dx$$

If S_0 is a Morse function on a finite dimensional manifold.

But usually we would have to work on infinite dimensional space.

<u>idea</u>:

1. embed X into a larger graded manifold V and extend $S_0(X)$ to a function on $S(x)$ on V and express the initial integral as

$$\int_{V \subset T^*[-1]V} e^{iS(y)/\hbar} f(y)dy$$

then deform V as a Lagrangian inside the odd cotangent bundle. In order to make the integral invariant, the new S has to satisfies the quantum master equation QME. At the oder $\hbar = 0$, QME reduces to the classical master equation

$$[S_0, S_0] = 0$$

We first given a heuristic version of BV-formalism of quantum field theory on "points", where the moduli space is finite dimensional.

Let M be a finite dimensional smooth manifold or affine variety. Let $S : M \longrightarrow \mathbb{A}^1$ be a smooth function. The critical locus of S

$$Crit(S) = graph(dS) \times_{T^*M} M$$

the fibered product

$$\begin{tikzcd} Crit(S) \arrow[r] \arrow[d] & M \arrow[d, "dS_0"] \\ M \arrow[r, "zero\ sections"] & T^*M \end{tikzcd}$$

is the intersection of graph of dS and the zero section inside T^*M.

Sometimes this intersection could be non-transitive, we want to define a derived version.

Traditionally, the BV-BRST complex of Lagrangian field theory is obtained in three steps

1. Find a Koszul-Tate complex to resolve the critical locus;

2. find a BRST complex to encode the gauge invariance

3. apply the homological perturbation theory to find a unified BV-differential.

$$s_{BV} = s_{KT} + s_{BRST} + ...$$

Choosing the derived critical locus is equivalent to inverting the Koszul-Tate resolution.

2 Derived functor, homotopy pushout

We skip here the introduction of model category but only remember that given a model category $\mathcal{C}$, the homotopy category $\gamma : \mathcal{C} \longrightarrow Ho(\mathcal{C})$ exists, which is the localization of $\mathcal{C}$ w.r.t the weak equivalence. Any functor $G : \mathcal{C} \longrightarrow \mathcal{B}$ which sends weak equivalences to isomorphisms would factor through γ.

Given three categories, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and two functors X, F.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "X"'] & \mathcal{C} \\ \mathcal{B} \arrow[ur, dashed, "R"'] & \end{tikzcd}$$

The **right Kan extension of** X **along** F consists of a functor $R : \mathcal{B} \longrightarrow \mathcal{C}$ and a natural transformation $\eta : RF \longrightarrow X$ which is **couniversal** with respect to the specification, in the sense that for any functor $M : \mathcal{B} \longrightarrow \mathcal{C}$ and a natural transformation $\mu : MF \longrightarrow X$, a unique natural transformation $\delta : M \longrightarrow R$ is defined and the diagram of functors commutes

$$\begin{tikzcd} RF \arrow[d, "\eta"'] & \\ X & MF \arrow[l, "\mu"] \arrow[ul, dashed, "\delta_F"'] \end{tikzcd}$$

where $\delta_F(a) = \delta(F(a)) \longrightarrow RF(a)$ for any object a of $\mathcal{A}$.

Similarly, we have a dual notion of **left Kan extension**.

Definition 2.1. *Let $\mathcal{C}$ be a model category and let $F : \mathcal{C} \longrightarrow \mathcal{B}$ by any functor. We call the right Kan extension of F along $\gamma : \mathcal{C} \longrightarrow Ho(\mathcal{C})$ the left derived functor of F. We will denote it by $(\mathbf{L}F, \eta)$, where η is a defining natural transformation in Kan extension.*

Dually, the left Kan extension of F along $\gamma : \mathcal{C} \longrightarrow Ho(\mathcal{C})$ the right derived functor of F.

In the case $F = (co)lim : \mathcal{C}^{\mathcal{D}} \longrightarrow \mathcal{C}$, where $\mathcal{D}$ is a diagram. We can define the homotopy limits and homotopy colimits: $\mathbf{R}lim$ and $\mathbf{L}colim$.

It would be long story to introduce the model structure on $\mathcal{C}^{\mathcal{D}}$, which we suppress here.

If $\mathcal{D}$ is chosen to be

$$\bullet \longrightarrow \bullet \longleftarrow \bullet,$$

the $\mathbf{R}lim$ is the **homotopy pullback**

The derived critical locus is now defined to be a homotopy pullback in the category $caga^{op}_{\leq 0}$

$$\begin{tikzcd}
dCrit(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & M \arrow[d, "dS"] \\
M \arrow[r, "0"] & T^*M
\end{tikzcd}$$

The detailed construction is discussed in the next section.

3 Derived everything

3.1 Recap on spaces, functor of point

First, we recall the definition of a topological manifold.

A topological variety is an topological space together with an open cover $\{U_i\}_{i\in I}$ such that for each $i \in I$ there exists a homeomorphism from U_i to an open set in $\mathbb{R}^{n_i}$, each integer $n_i \geq 0$ depends on i.

We can give a fancy definition of topological manifolds

Consider the coequilizer

$$Colim \left(\coprod_{(i,j)\in I^2} U_{i,j} \rightrightarrows \coprod_{i\in I} U_i \right),$$

where the upper morphism is induced by $U_{i,j} \longrightarrow U_i$ while the second morphism is induced by the morphism $U_{i,j} \longrightarrow U_j$. The morphism from $\coprod_i U_i \longrightarrow X$ would factorize through

$$Colim \left(\coprod_{(i,j)\in I^2} U_{i,j} \rightrightarrows \coprod_{i\in I} U_i \right) \longrightarrow X$$

Lemma 3.1. *The morphism*

$$Colim\left(\coprod_{(i,j)\in I^2} U_{i,j} \rightrightarrows \coprod_{i\in I} U_i\right) \longrightarrow X$$

is an isomorphism.

Proof. Consider Y another topological space with a morphism

$$Colim\left(\coprod_{(i,j)\in I^2} U_{i,j} \rightrightarrows \coprod_{i\in I} U_i\right) \xrightarrow{f} Y$$

$f_i := f|_{U_i}$ and $f_i|_{U_{i,j}} = f_j|_{U_{i,j}}$. We can define an map g from X to Y such that they agree pointwisely. $g|_{U_i} = f_i$. The restriction to each open set in the open cover is continuous, we therefore know g is itself continuous.

Hence, there is a morphism from X to Y and continuous map g is unique because it has to agree with f pointwisely.

X satisfies the universal property of coequalizer hence is isomorphic to the coequalizer. □

We can consider it in an even fancier way. Use $\mathcal{C}$ to denote the full subcategory of topological manifold. One consider the Yoneda embedding from the category $\mathcal{C}$ to the category of presheaves over $\mathcal{C}$, where $PSh(\mathcal{C})$ denote the functor category $[\mathcal{C}^{op}, Sets]$

$$h_{\_} : TopMfd \longrightarrow PSh(\mathcal{C})$$

$$X \longmapsto h_X$$

where $h_X(Y) := \mathrm{H}om_{TopMfd}(Y, X)$ for al $Y \in \mathcal{C} \subset TopMfd$

Lemma 3.2. *The functor $h_{\_}$ defined above is fully faithful.*

Proof. refer to any proof of Yoneda lemma. □

(This functor is not necessarily essentially surjective) These lemma means $TopMfd$ is equivalent to a subcategory of presheaves over $\mathcal{C}$. We all now trying to characterize this subcategory.

We start by making $\mathcal{C}$ a Grothendieck site.

Definition 3.3. *We can specify that certain collections of maps with a common codomain should cover their codomain. A family of morphisms $\{U_i \longrightarrow U\}_{i\in I}$ is called a covering family is each morphism $U_i \longrightarrow U$ is an open immersion and the induced morphism on the coproduct $\coprod_{i\in I} U_i \longrightarrow U$ is surjective. This definition gives the neighborhood system for a pretopology (**Grothendieck pretopology**), we denote the associated topology τ*

Lemma 3.4. *For every $X \in TopMfd$, the presheaf $h_X \in PSh(\mathcal{C})$ is a sheaf with the specified topology τ*

Definition 3.5. *We say a functor $F : \mathcal{C} \longrightarrow Sets$ is **representable** if it is naturally isomorphic to h_X for some object $X \in \mathcal{C}$.*

A morphism $f : F \longrightarrow G$ of $Sh(\mathcal{C}, \tau)$ is a local homeomorphism if for each $X \in \mathcal{C}$ and all morphism $h_X \longrightarrow G$, the sheaf $F \times_G \times h_X$ is representable by $Y \in TopMfd$, and the induced morphism $Y \longrightarrow X$ by projection $F \times_G h_X \cong h_Y \longrightarrow X$ is a local homeomorphism of topological spaces.

*A morphism of sheaves $Sh(\mathcal{C}, \tau)$ is an **open immersion** if it is a monomorphism and a local homeomorphism.*

Proposition 3.6. *A sheaf $F \in Sh(\mathcal{C}, \tau)$ is representable by a topological manifold if there exists a family of objects $\{U_i\}_{i\in I}$ in $\mathcal{C}$ and a morphism of sheaves*

$$p : \coprod_{i\in I} h_{U_i} \longrightarrow F$$

such that the following two conditions holds

1. *p is an epimorphism.*

2. *For all $i \in I$, the morphism $U_i \longrightarrow F$ is an open immersion.*

Generally, we can identifies the category of $TopMfd$ with its image in $Sh(\mathcal{C}, \tau)$

In context of algebraic geometry, things are more famous.

Schemes can be characterized as representable sheaves $Sh(Aff, \tau)$, where τ is the canonical Grothendieck topology.

Definition 3.7. *A **derived scheme** is a pair $(X, \mathcal{O})$ consisting of topological space and a sheaf $\mathcal{O}$ of commutative ring spectra on X such that the*

1. *pair $(X, \pi_0\mathcal{O})$ is a scheme and*

2. *each $\pi_k\mathcal{O}$ is a quasi-coherent $\pi_0\mathcal{O}$-module.*

From the homotopical point of view, we note that a derived scheme X defines a functor

$$h_X : dAff^{op} = cdga_{\leq 0} \longrightarrow Sets$$

Furthermore, we have the following lemma

Lemma 3.8. *h_X sends each quasi-isomorphism of $cdga_{\leq 0}$ to an isomorphism in $Sets$.*

Recall the model structure on $cdga_{\leq}$, the weak equivalence are just the quasi-isomorphisms, i.e., the functor h_X factors through the homotopy category $Ho(cdga_{\leq 0})$. Following the spirit of functor of points, we can regard X as a locally representable sheaf in $Sh(Ho(cdga_{\leq 0}))$.

3.2 Stacks, derived stacks

Roughly speaking, a Stack is a sheaf that takes values in categories rather than sets.

Definition 3.9. *A category $\mathcal{B}$ with a functor F to a category $\mathcal{C}$ is called a **fibered category over** $\mathcal{C}$ if for any morphism $G : X \longrightarrow Y$ in $\mathcal{C}$ and any object $y \in \mathcal{B}$ s.t. $F(y) = Y$, there is a poullback $g : x \longrightarrow y$ of y by F, i.e. $F(g) = G$.*

Definition 3.10. *The category $\mathcal{B}$ is called a **prestack** over a category $\mathcal{C}$ with a Grothendieck topology if it is fibered over $\mathcal{C}$ and*

for any object $U \in \mathcal{C}$ and object $x, y \in \mathcal{B}$ with image U, the functor from objects over U to sets taking $[F : V \longrightarrow U]$ to $Hom(F^*x, F^*y)$ is a sheaf.

*The category $\mathcal{B}$ is called a **stack** over the category $\mathcal{C}$ with a Grothendieck topology if it is a prestack over $\mathcal{C}$ and every descent datum is effective.*

*A **descent datum** consists roughly of a covering of an object V of C by family V_i, elements x_i in the fiber over V_i and morphism f_{ji} between the restrictions of x_i and x_j to $V_{ij} = V_i \times_V V_j$ satisfying the compatibility condition $f_{ki} = f_{kj} f_{ji}$. The descent datum is called effective if the elements x_i are essentially the pullbacks of an element x with image V.*

The descent condition here is just a derived version of the usual sheaf axioms. The fiber functor $F : \mathcal{B} \longrightarrow \mathcal{C}$ can be regarded a sheaf on $\mathcal{B}$ with value in $\mathcal{C}$.

For example if we tak $\mathcal{C} = Grpds$ the stack is called 1-stack.

We will jump through the story of n-stacks and go directly to $\infty - stacks$.

3.3 Derived critical locus

$$\begin{tikzcd}
dCrit(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\!\!\!\swarrow"] & M \arrow[d, "dS"] \\
M \arrow[r, "\underline{0}"] & T^*M
\end{tikzcd}$$

Mostly, it would be easier to analyze it in terms of functions on it. We will mostly work in the affine case only to convince ourselves.

Let R be a commutative k-algebra, and P a projective R-module of finite type. Let $S := Sym_R(P^\vee)$ the symmetric algebra on on the R-dual $P^\vee$. S is a commutative R-algebra. Consider $\wedge^\bullet P^\vee$ be the exterior algebra of $P^\vee$ as an R-module and we can construct a non-positively graded S-module $S \otimes_R \wedge^\bullet P^\vee$ graded by

$$(S \otimes_R \wedge^\bullet P^\vee)_m := S \otimes_R \wedge^{-m} P^\vee$$

. This $S \otimes_R P^\vee$ is naturally a graded commutative S-algebra and can be endowed with a degree 1 differential d. The differential d is induced by a homomorphism

$$\begin{aligned} h : & R \longrightarrow Hom_R(P, P) \cong P^\vee \otimes P \\ & 1_R \longmapsto \sum_i \alpha_i \otimes x_i \end{aligned}$$

$$d(a \otimes (\beta \wedge \cdots \wedge \beta_{n+1})) = \sum_j a \cdot \alpha_j \otimes \sum_k (-1)^k \beta_k(x_j)(\beta_1 \wedge \cdots \wedge \hat{\beta}_k \wedge \cdots \wedge \beta_{n+1})$$

$(S \otimes_R, d)$ is commutative differential non-positively graded algebra over S. We call it Koszul cdga and denote it by $K(R; P)$.

Proposition 3.11. *The cohomology of Koszul cdga $K(R; P)$ is zero in degrees ≤ 0, and $H^0(K(R; P)) \cong R$.*

Proof. See for example this notes □

4 The BRST on $dCrit(S)$ = the BV-BRST on $Crit(S)$