Autour de la Géométrie Algébrique Dérivée

groupe de travail—spring 2013
Institut de Mathématiques de Jussieu
Université Paris Diderot

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Introduction

The purpose of these notes is to provide a written background for the seminars taught by the working group organized by Gabriele Vezzosi in the Spring of 2013. The main goal is to give a wide introduction to Derived Algebraic Geometry, starting from the very foundations. In the present notes, we added more details with respect to the expositions, to make easier for the not (yet) specialised reader to go over them.

The work is organized as follows:

1. Chapter 1 contains a survey of the theory of model categories: the homotopy category, function complexes, left and right (Bousfield) localizations, standard simplicial localization and hammock localization. The relative seminar has been taught the 3/15/2013 by Mauro Porta and Brice Legrignou;

2. Chapter 2 contains a survey of all the known models for $(\infty, 1)$-categories, and the details of three of those constructions (quasicategories, Segal categories and Segal spaces). The relative seminar will be taught the 3/22/2013 by Yan Zhao and Valerio Melani;

3. Chapter 3 contains a survey of of the theory of simplicial presheaves and their application to the theory of (higher) stacks. The relative seminar has been taught the 3/29/2013 by Mauro Porta and Pietro Vertechi;

4.

Finally, the Appendices contain material which is somehow more foundational: we collected some basic results from the theory of simplicial sets, enriched category theory... Essentially because the authors wanted to learn all these theories in a better way.

This is a work in progress

EXPOSÉ **1**

Model categories

In this chapter we will introduce one of the main tools of this cycle of seminars, namely model categories. After an informal section concerning motivations coming from others areas of Mathematics, we will introduce the basic definitions and we will state the main theorems, sketching some proof. We will try, whenever possible and within our capacities, to explain the intuitions motivating the definitions, why we should expect a certain theorem to be true and so on; moreover, we selected four examples where we can test the result we will obtain:

1. simplicial sets **sSet**: this is the main example in homotopy theory; in a sense, we could say that simplicial sets stands to homotopy theory as sets stand to the whole mathematics (which can be seen as constructions on the topos **Set**, in a very broad sense);

2. topological spaces: this is the "continuous" version of simplicial sets; for many purpose there is no distinction from the homotopy theory for topological spaces and the one for simplicial sets;

3. chain complexes: this example relates homotopy theory to homological algebra. It gives the "correct motivation" for a bunch of facts; for example, that a homotopy of topological spaces gives rise to a chain homotopy between singular complexes;

4. groupoids: this will be needed in future chapters; it also provides an interesting example where the model structure is uniquely determined by the weak equivalences.

The main topics include the homotopy category, Quillen functors, Reedy categories and (co)simplicial resolutions, function complexes. The last part is devoted to the theory of localizations: Bousfield localization (and simplicial localization).

There is finally a complement section containing some selected topics which couldn't be exposed during the seminar. Some of them is just a collection of definitions; in those case, a proper reference is indicated.

Mauro Porta

1.1 Motivations and main ideas

From a historical point of view, the theory of Model Categories was first introduced by D. Quillen in his foundational work [Qui67] with the purpose of unifying several constructions which are in common to several areas of Mathematics. To my best knowledge, these areas are

1. Algebraic Topology;

2. Homological Algebra;

I will spend a few words for each of them, in order to explain the main ideas that led to the work of Quillen.

Algebraic Topology

The main goal of Algebraic Topology is the study (up-to homeomorphism) of topological spaces, with the aid of certain algebraic invariants. For our purpose, we can assume as definition of "algebraic invariant" the following:

Definition 1.1.1. An algebraic invariant for topological spaces is just a functor $\mathcal{H}\colon \mathbf{Top} \to \mathbf{A}$, where $\mathbf{A}$ is some algebraic category.

In fact, we will be more likely interested in a set of invariants (maybe with some mutual relations between); a perfect result would be to find such a set of (calculable) invariants describing completely topological spaces.

For example, we can consider singular (co)homology, or homotopy groups. For example: to obtain the singular homology, one first consider the cosimplicial object in **Top**

$$\{|\Delta^n|\}_{n\in\mathbb{N}}$$

and then defines for every $X \in \mathrm{Ob}(\mathbf{Top})$:

$$C_n^{\mathrm{sing}}(X) := \mathbb{Z}\mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X) \in \mathbf{Ab}$$

Each C_n^{sing} is, accordingly to our Definition 1.1.1, an algebraic invariant. The cosimplicial structure on $\{|\Delta^n|\}_{n\in\mathbb{N}}$ produces interesting properties of the set of functors $\{C_n^{\mathrm{sing}}\}$; for example, they can be arranged in a complex $\{C_n^{\mathrm{sing}}, d_n\}_{n\in\mathbb{N}}$. See [CSAM29, Ch. 8.2] (in particular Example 8.2.3) for the details of this construction; one then can use this complex to define the singular homology groups:

$$H_n^{\mathrm{sing}}(X) := H_n(C_\bullet^{\mathrm{sing}}, d_n)$$

Singular (co)homology groups are quite easy to compute, because we can use several tools (Excision, Mayer Vietoris). In the case where the space is a CW-complex, we can also use the powerful tool which is Cellular Homology (see [Hat01, Ch. 2.2] for more details on computation tools).

The homotopy groups construction shares the same philosophy, but it is more involved from a technical point of view. In this case, the group operation is not simply formal, but it reflects the structure of the topological space we are considering. This implies of course that homotopy groups carry more informations than singular homology groups, but they are also more difficult to compute.

Example 1.1.2. Let S be a compact orientable surface of genus $g \geq 2$. Then S cannot be a topological group. The proof relies on the simple fact that if $g \geq 2$ then

$$\pi_1(S) = \left\langle a_i, b_i \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

which is not abelian; on the other side a simple Eckmann-Hilton argument shows that the first fundamental group of every topological group is abelian.

Homotopy groups can identify almost completely certain kind of "good" spaces, namely CW-complexes. In fact, a classical result of Algebraic Topology, known as Whitehead's theorem says that:

Theorem 1.1.3. If a map $f : X \to Y$ between connected CW-complexes induces isomorphisms $f_*: \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence.

Proof. See [Hat01, Thm 4.5]. □

Since one usually encounters only CW complexes in applications to other areas of Mathematics, Theorem 1.1.3 can be considered as a really satisfying result.

Keeping this result in mind, we can pass to the problem of computation of homotopy groups. In this case, we can exploit the "weakness" of the set of functors $\{\pi_n\}_{n\in\mathbb{N}}$; first of all they are defined only up to homotopy equivalence, hence we can replace every space with an homotopically equivalent one. But we can do a subtler thing: if a map $f : X \to Y$ is such that the induced morphisms $f_*: \pi_n(X) \to \pi_n(Y)$ are isomorphisms for every n, then we can replace X with Y and compute the homotopy groups of Y. This is not a different technique if we work only with CW-complexes because of Theorem 1.1.3; however, if X is a general topological space, we can try to reduce to the CW-complex case, where more standard techniques are available. This is in fact always possible:

Theorem 1.1.4. For every topological space X, there exists a CW-complex Z and a map $f : X \to Y$ such that $f_*: \pi_n(X) \to \pi_n(Y)$ is an isomorphism for every n.

Proof. See [Hat01, Prop. 4.13]. □

The previous reasoning shows then that it may be worth of to introduce the following definition:

Definition 1.1.5. A *weak equivalence* in **Top** is a morphism $f : X \to Y$ such that the induced morphisms $f_*: \pi_n(X) \to \pi_n(Y)$ are isomorphisms for every n.

To formalize the technique sketched before, we would like to have a category where weak equivalences are invertible. The construction, however, is not completely trivial; to avoid the need to change universe, we can observe that Theorem 1.1.4 allows us to restrict ourselves to the full subcategory of CW-complexes; there Theorem 1.1.3 shows that inverting weak equivalences produces the same result as quotienting by homotopy, except that in this case it is perfectly clear that we are not enlarging our universe.

This construction motivates the construction of the homotopy category of a model category, as we will see later.

Homological Algebra

One could say in a very fancy way that Homological Algebra is the study of the lack of exactness of functors between abelian categories. Motivations, to my best knowledge, come from Algebraic Topology and Algebraic Geometry. In the first case, the relationship is self-evident: we attach to every topological space the complex of (co)chains, and we reason then in term of this complex. It becomes therefore useful to be able to manipulate chain complexes with general techniques. On the other side, in Algebraic Geometry, Homological Algebra shows up in a totally unexpected way.

Recall that a scheme $(X, \mathscr{O}_X)$ is said to be regular at a point x if the local ring $\mathscr{O}_{X,x}$ is regular, i.e.

$$\dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \text{dim.Krull}\, \mathscr{O}_{X,x}$$

The scheme is said to be regular if it is regular at each point. The hope is this notion of regularity is a local condition; however, it is not clear at all why we should be able to check the condition only over closed points. In fact, the proof of this fact relies on a theorem of Serre:

Theorem 1.1.6. If a noetherian local ring has finite global homological dimension, then it is a regular local ring.

Proof. See [AK70], in particular Theorem 5.15 and Corollary 5.18. □

Other motivations come from Algebraic Geometry (like the considerations that lead to Verdier duality), but this is not the right place where to do a full review of them. Instead, we will shortly describe the main construction of elementary Homological Algebra, since it will be of some relevance in our next discussion about model categories.

Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories, and let's assume that F is left exact. Following Grothendieck ([Gro57], but cfr. also [CSAM29, Ch. II]) we say that a derived functor for F is a cohomological δ-functor which is universal in an appropriate sense. The classical existence theorem says that if $\mathcal{A}$ has enough injectives, then the derived functor always exists and it is obtained via the following procedure:

1. start with an object $A \in \mathrm{Ob}(\mathcal{A})$;
2. choose an injective resolution $A \to I^\bullet$ in $\mathbf{Ch}(\mathcal{A})$;
3. define $R^i F := H^i(F(I^\bullet))$.

One technical difficulty in the proof that this gives back a universal cohomological δ-functor is that our definition might depend on the choice of point 2. In fact,

1.2 Model categories

Now that the motivational part is more or less settled, we can start getting serious with the theory of model categories. For sake of completeness, we recall some basic definitions from Category Theory (for more details the reader is referred to [Mac71]):

Definition 1.2.1. Let $\mathcal{C}$ be any category. The arrow category of $\mathcal{C}$, denoted $\mathbf{Arr}(\mathcal{C})$ is by definition the comma category $(\mathcal{C} \downarrow \mathcal{C})$.

Remark 1.2.2. Explicitly, objects of $\mathbf{Arr}(\mathcal{C})$ are the arrows of $\mathcal{C}$ and morphisms are commutative squares.

Remark 1.2.3. We will denote the two natural projection functors $(\mathcal{C} \downarrow \mathcal{C}) \to \mathcal{C}$ by

$$\mathbf{d}_0, \mathbf{d}_1 \colon \mathbf{Arr}(\mathcal{C}) \to \mathcal{C}$$

Observe that the composition of arrows induces a functor

$$\circ \colon \mathbf{Arr}(\mathcal{C}) \times_{\mathbf{d}_1, \mathbf{d}_0} \mathbf{Arr}(\mathcal{C}) \to \mathbf{Arr}(\mathcal{C})$$

where $\mathbf{Arr}(\mathcal{C}) \times_{\mathbf{d}_1, \mathbf{d}_0} \mathbf{Arr}(\mathcal{C})$ is the pullback

$$\begin{tikzcd} \mathbf{Arr}(\mathcal{C}) \times_{\mathbf{d}_1, \mathbf{d}_0} \mathbf{Arr}(\mathcal{C}) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & \mathbf{Arr}(\mathcal{C}) \arrow[d, "\mathbf{d}_1"] \\ \mathbf{Arr}(\mathcal{C}) \arrow[r, "\mathbf{d}_0"] & \mathcal{C} \end{tikzcd}$$

Definition 1.2.4. Let $\mathcal{C}$ be any category. An arrow $f \in \mathrm{Arr}(\mathcal{C})$ is a retract of an arrow $g \in \mathrm{Arr}(\mathcal{C})$ if it is a retract of an object in $\mathbf{Arr}(\mathcal{C})$.

Explicitly, f is a retract of g if we are given a commutative diagram as the following:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}_A"] & C \arrow[r, "\beta"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "\gamma"] \arrow[rr, bend right, "\mathrm{id}_B"'] & D \arrow[r, "\delta"] & B \end{tikzcd}$$

Definition 1.2.5. Let $\mathcal{C}$ be any category and let I, J be subcategories of $\mathbf{Arr}(\mathcal{C})$. A functorial (I, J)-factorization for $\mathcal{C}$ is a *strict* section of the restriction of the composition functor

$$\circ\colon I \times_{\mathbf{d}_1, \mathbf{d}_0} J \to \mathbf{Arr}(\mathcal{C})$$

Definition 1.2.6. Let $\mathcal{C}$ be any category and let $i\colon A \to B$, $p\colon X \to Y$ be any two arrows in $\mathcal{C}$. We say that i has the left lifting property (LLP) with respect to p or, equivalently, that f has the right lifting property (RLP) with respect to i if and only if for each commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

the dotted lifting exists.

Definition 1.2.7. Let $\mathcal{C}$ be any category. A model structure on $\mathcal{C}$ is the given of three full subcategories W, Fib, Cofib of $\mathbf{Arr}(\mathcal{C})$ satisfying the following axioms:

MC1. $\mathcal{M}$ is (small) complete and (small) cocomplete.

MC2. if f, g, h are arrows satisfying $fg = h$ and two of them are in W, then so is the third;

MC3. W, Fib, Cofib are closed under retracts;

MC4. every arrow in $W \cap$ Fib has the RLP with respect to every arrow in Cofib and every arrow in Fib has the RLP with respect to every arrow in $W \cap$ Cofib;

MC5. there are functorial ($W \cap$ Cofib, Fib) and (Cofib, $W \cap$ Fib) factorizations in $\mathcal{C}$.

We will denote by ($\mathcal{C}$, W, Fib, Cofib) a category with a model structure; we will also say that the arrows in W are the *weak equivalences*, that those in Fib (resp. in Fib $\cap W$) are the fibrations (resp. trivial fibrations or acyclic fibrations) and that those in Cofib (resp. in Cofib$\cap W$) are the cofibrations (resp. trivial cofibrations or acyclic cofibrations) with respect to the given model structure.

Remark 1.2.8. In the original work of Quillen [Qui67] and in work of Dwyer and Spalinski [DS95] is required only the existence of *finite* limits and a (not necessarily functorial) factorization. In these notes we will follow the more modern habit (cfr. [MSM63], [Hir03], [DHK97]).

Example 1.2.9. Let $\mathcal{M}$ be a model category. Then $\mathcal{M}^{\mathrm{op}}$ carries a model category structure in a natural way: weak equivalences in $\mathcal{M}^{\mathrm{op}}$ are the same as in $\mathcal{M}$; fibrations in $\mathcal{M}^{\mathrm{op}}$ are the cofibrations of $\mathcal{M}$, and cofibrations of $\mathcal{M}^{\mathrm{op}}$ are the fibrations of $\mathcal{M}$. The check that this defines a model structure is straightforward and it is left to the reader.

Example 1.2.10. c

Remark 1.2.11. The previous example shows that the theory of model categories is self-dual. It follows that we can apply a duality argument to shorten proofs.

Remark 1.2.12. The axioms for a model category are overdetermined. As next Lemma will show, the knowledge of weak equivalences and fibrations completely determine the cofibrations.

Lemma 1.2.13. Let $\mathcal{M}$ be a model category. Then:

1. fibrations are exactly those arrows with the RLP with respect to all trivial cofibrations;

2. trivial fibrations are exactly those arrows with the RLP with respect to all cofibrations;

3. cofibrations are exactly those arrows with the LLP with respect to all trivial fibrations;

4. trivial cofibrations are exactly those arrows with the LLP with respect to all fibrations.

Sketch of the proof. The proof of 3. and 4. is dual to that of 1. and 2.; we will sketch 1., and 2. will be analogous. One inclusion is by definition; assume that f has the RLP with respect to all trivial cofibrations; factorize f as pi where i is a trivial cofibration and p is a fibration. Choose a lifting h in the diagram

$$\begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "p"'] \arrow[ur, "h"] & \bullet \end{tikzcd}$$

and observe now that the diagram

$$\begin{tikzcd} \bullet \arrow[r, "i"] \arrow[d, "f"] & \bullet \arrow[r, "h"] \arrow[d, "p"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "\mathrm{id}"] & \bullet \arrow[r, "\mathrm{id}"] & \bullet \end{tikzcd}$$

express f as retract of ~~i.~~This implies that f is a fibration. □

Corollary 1.2.14. Let $\mathcal{M}$ be a model category. Then

1. Fib is closed under pullback;

2. Cofib is closed under pushout.

Remark 1.2.15. As general philosophy, in a model category we care the most about weak equivalences. However, fibrations and cofibrations are useful at a technical level, since they allow particular constructions (see the homotopy category construction, for example). Also, it is absolutely not true that weak equivalences determine in general the whole model structure: we can endow **CGHaus** with at least two different model structures having the same weak equivalences (see the section about examples for the details). However, there are remarkable exceptions: **Cat** and **Grpd** have a uniquely determined model structure where weak equivalences are equivalences of categories. See the complements to this chapter for a detailed proof.

Before ending this section, we introduce a few more concepts that will turn useful later on.

Definition 1.2.16. Let $\mathcal{M}$. Then

1. an object $X \in \mathrm{Ob}(\mathcal{M})$ is said to be cofibrant if the map $\emptyset \to X$ is a cofibration;

2. an object $X \in \mathrm{Ob}(\mathcal{M})$ is said to be fibrant if the map $X \to *$ is a fibration;

3. a cofibrant approximation to an object $X \in \mathrm{Ob}(\mathcal{M})$ is a pair $(\widetilde{X}, i)$ where $\widetilde{X}$ is a cofibrant object and $i : \widetilde{X} \to X$ is a weak equivalence;

4. a fibrant approximation to an object $X \in \mathrm{Ob}(\mathcal{M})$ is a pair $(\widehat{X}, j)$ where $\widehat{X}$ is a fibrant object and $j : X \to \widehat{X}$ is a weak equivalence;

5. a cofibrant approximation to an arrow $f : X \to Y$ is the given of cofibrant approximations $(\widetilde{X}, i_X)$, $(\widetilde{Y}, i_Y)$ to X and Y and an arrow $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ such that the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "i_X"] \arrow[d, "\widetilde{f}"] & X \arrow[d, "f"] \\ \widetilde{Y} \arrow[r, "i_Y"] & Y \end{tikzcd}$$

commutes;

6. a fibrant approximation to an arrow $f : X \to Y$ is the given of two fibrant approximations $(\widehat{X}, j_X)$ and $(\widehat{Y}, j_Y)$ and an arrow $\widehat{f} : \cancel{RX \to RY}$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & \widehat{X} \arrow[d, "\widehat{f}"] \\ Y \arrow[r] & \widehat{Y} \end{tikzcd}$$

commutes.

Proposition 1.2.17. Let $\mathcal{M}$ be a model category. Then:

1. every object $X \in \mathrm{Ob}(\mathcal{M})$ has a *functorial* cofibrant approximation $(\widetilde{X}, i_X)$ where i_X is a trivial fibration;

2. if $(\widetilde{X}, i_X)$, $\widetilde{X}', i'_X)$ are cofibrant approximations to an object $X \in \mathrm{Ob}(\mathcal{M})$, and moreover i'_X is a fibration, then there is a weak equivalence $f : \widetilde{X} \to \widetilde{X}'$;

3. every morphism in $\mathcal{M}$ has a fibrant approximation.

Sketch of the proof. 1. is a consequence of the factorization axiom. 2. and 3. follows from the lifting properties of cofibrations with respect to trivial fibrations. □

Remark 1.2.18. We leave to the reader to state the dual of Proposition 1.2.17.

Remark 1.2.19. Proposition 1.2.17 is the analogue, in our abstract setting of Theorem 1.1.4 (cellular approximation for topological spaces). We will return on this analogy in the examples.

Lemma 1.2.20. Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a category with a subcategory $S \subset \mathbf{Arr}(\mathcal{M})$ containing all the identities and satisfying the 2-out-of-3 axiom. If $F : \mathcal{M} \to \mathcal{C}$ is a functor that takes acyclic cofibrations between cofibrant objects to elements of S, then F takes every weak equivalence between cofibrant objects to elements of S. Dually, if F takes acyclic fibrations between fibrant objects to elements of S, then F takes every weak equivalence between fibrant objects to elements of S.

Proof. Let $f : A \to B$ be a generic weak equivalence between cofibrant objects. Factor the map

$$\langle f, 1_B \rangle : A \sqcup B \to B$$

as

$$\cancel{A\sqcup} \xrightarrow{q} C \xrightarrow{p} B$$

with p an acyclic fibration. Since A and B are cofibrant, stability of cofibrations for cobase change (Corollary 1.2.14) implies that the two maps

$$A \xrightarrow{i_0} A \sqcup B \xleftarrow{i_1} B$$

are cofibrations. By the 2-out-of-3 axiom, both $q \circ i_0$ and $q \circ i_1$ are weak equivalences; the hypothesis now imply that $F(q \circ i_0)$ and $F(q \circ i_1)$ are elements of S. Since $F(p \circ q \circ i_1) = F(1_B)$ is in S, it follows that $F(p)$ is in S. Therefore $F(f) = F(p) \circ F(q \circ \cancel{i_2})$ is in S. □

1.3 The homotopy category

Exposition of the problem

As we will see later on in this cycle of seminars, model categories gives a powerful framework to deal with higher homotopies. In this sense, their theory is absolutely necessary for the theory of $(\infty, 1)$-categories as exposed, for example, in [HTT]. The homotopy category is a way to extract (first order) homotopical informations from the model category $\mathcal{M}$. In any case, the passage from the model category to its homotopy category produces a loss of informations. We will see later on how to associate other invariants to any model category.

Besides this general philosophy about the relationship between a model category and its homotopy category, we can give a more concrete idea of what we are going to do: roughly speaking, the main goal is to describe a general procedure to invert weak equivalences in an arbitrary model category $\mathcal{M}$, without enlarging the Grothendieck universe we fixed at the beginning; the way to do that, will be to imitate the general procedure described in 1.1.

Let's get serious now:

Definition 1.3.1. Let $\mathbb{U} \subset \mathbb{V}$ be Grothendieck universes and let $\mathcal{C}$ be a $\mathbb{U}$-small category. Let $S \subset \mathrm{Arr}(\mathcal{C})$ be a set of arrows. A $\mathbb{V}$-*localization of* $\mathcal{C}$ *with respect to* S is a $\mathbb{V}$-small category $\mathcal{C}[S^{-1}]$ together with a functor $F_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that

1. for all $s \in S$, $F_S(s)$ is an isomorphism;
2. for any other $\mathbb{V}$-small category $\mathcal{A}$ and any functor $G : \mathcal{C} \to \mathcal{A}$ such that $G(s)$ is an isomorphism for each $s \in S$, there is a functor $G_S : \mathcal{C}[S^{-1}] \to \mathcal{A}$ and a natural isomorphism
$$\eta_G : G_S \circ F_S \simeq G$$
3. for any $\mathbb{V}$-small category $\mathcal{A}$, the induced functor
$$F_S^* : \mathbf{Funct}(\mathcal{C}[S^{-1}], \mathcal{A}) \to \mathbf{Funct}(\mathcal{C}, \mathcal{A})$$
is fully faithful.

Remark 1.3.2. This definition differs a little from those given in [GZ67] and in [CSAM29] because of the natural isomorphism η_G. However, this matches better the philosophy of category theory; a similar definition can be found in [KS06, Ch. 7.1].

Remark 1.3.3. The purpose of point 3. is to ensure the uniqueness of the factorization G_S, as well as that of the natural isomorphism η_G.

Remark 1.3.4. It can be shown that given $\mathbb{U}$, $\mathcal{C}$ and S as in the previous definition there is always $\mathbb{V}$ such that a $\mathbb{V}$-localization exists. This is proved for example in [GZ67, p. I.1].

Remark 1.3.5. Let $\mathbb{U} \subset \mathbb{V}$ be Grothendieck universes. If $\mathcal{C}$ and S are as in Definition 1.3.1 and $\mathcal{C}[S^{-1}]$ is a $\mathbb{V}$-localization of $\mathcal{C}$ at S, then, for every Grothendieck universe $\mathbb{V} \subset \mathbb{W}$, $\mathcal{C}[S^{-1}]$ is also a $\mathbb{W}$-localization of $\mathcal{C}$ at S.

With this terminology, the main goal of this section becomes to provide a proof of the following theorem:

Theorem 1.3.6. Let $\mathbb{U}$ be a Grothendieck universe and let $\mathcal{M}$ be a model category. Then there exists a $\mathbb{U}$-localization of $\mathcal{M}$ with respect to the set of weak equivalences W.

Localizing subcategories

To prove Theorem 1.3.6, we will follow the exposition given in [DHK97], using the refined version that can be found in [Rie12]. First of all, let us introduce a definition:

Definition 1.3.7. Let $\mathcal{C}$ be a category and let $\mathcal{C}_0, \mathbf{W} \subset \mathcal{C}$ be subcategories. We say that $\mathcal{C}_0$ is a left (resp. right) deformation retract of $\mathcal{C}$ with respect to $\mathbf{W}$ if there exists a functor $R\colon \mathcal{C} \to \mathcal{C}_0$ and a natural transformation $s\colon R \to \mathrm{Id}_{\mathcal{C}}$ (resp. $s\colon \mathrm{Id}_{\mathcal{C}} \to R$) such that:

1. R sends $\mathbf{W}$ into $\mathbf{W} \cap \mathcal{C}_0$;
2. for every object $C \in \mathrm{Ob}(\mathcal{C})$, the map s_C is in $\mathbf{W}$;
3. for every object $C_0 \in \mathrm{Ob}(\mathcal{C}_0)$, the map s_{C_0} is in $\mathbf{W} \cap \mathcal{C}_0$.

The pair (R, s) is called a left (resp. right) deformation retraction from $\mathcal{C}$ to $\mathcal{C}_0$ with respect to $\mathbf{W}$. If $\mathbf{W} = \mathcal{C}$, we will say that (R, s) is an absolute deformation retraction of $\mathcal{C}$ to $\mathcal{C}_0$.

Lemma 1.3.8. Let $\mathcal{C}$ be a category and let $\mathcal{C}_0, \mathbf{W} \subset \mathcal{C}$ be subcategories. Let $R\colon \mathcal{C} \to \mathcal{C}_0$ be an absolute left deformation retraction. Assume that for every object $C \in \mathrm{Ob}(\mathcal{C})$ the map s_C is in $\mathbf{W}$; if $\mathbf{W}$ satisfies the 2-out-of-3 then R sends $\mathbf{W}$ into $\mathbf{W} \cap \mathcal{C}_0$. If $\mathcal{C}_0$ is a full subcategory, then for every $C_0 \in \mathrm{Ob}(\mathcal{C}_0)$, the map s_{C_0} is in $\mathbf{W} \cap \mathcal{C}_0$.

Proof. Let $f : A \to B$ be an arrow in $\mathbf{W}$; consider the commutative square

$$\begin{tikzcd} R(A) \arrow[r, "s_A"] \arrow[d, "R(f)"] & A \arrow[d, "f"] \\ R(B) \arrow[r, "s_B"] & B \end{tikzcd}$$

Then $f \circ s_A$ and s_B are in $\mathbf{W}$; the 2-out-of-3 implies that $R(f)$ is in $\mathbf{W}$. The second statement is trivial, since $s_{C_0} : R(C_0) \to C_0$ is an arrow between objects of $\mathcal{C}_0$. □

Proposition 1.3.9. Let $\mathcal{C}$ be a category and let $\mathcal{C}_0, \mathbf{W} \subset \mathcal{C}$ be subcategories; write $\mathbf{W}_0 := \mathcal{C}_0 \cap \mathbf{W}$. Let (R, s) be a left (or right) deformation retraction of $\mathcal{C}$ to $\mathcal{C}_0$ with respect to $\mathbf{W}$. Let $\mathbb{V}$ be a Grothendieck universe where the localizations $\mathcal{C}[\mathbf{W}^{-1}]$ exist. Then

1. the induced inclusion $\mathcal{C}_0[\mathbf{W}_0^{-1}] \to \mathcal{C}[\mathbf{W}^{-1}]$ is an equivalence of categories;

2. $\mathcal{C}[\mathbf{W}^{-1}]$ exists if and only if $\mathcal{C}_0[\mathbf{W}_0^{-1}]$ does.

Proof. Let $j_0 \colon \mathcal{C}_0 \to \mathcal{C}$ be the inclusion functor. The universal property of localization show that both j_0 and R define functors $\widetilde{R} \colon \mathcal{C}[\mathbf{W}^{-1}] \to \mathcal{C}_0[\mathbf{W}_0^{-1}]$ and $\widetilde{\jmath}_0 \colon \mathcal{C}_0[\mathbf{W}_0^{-1}] \to \mathcal{C}[\mathbf{W}^{-1}]$. The natural transformation $s \colon j_0 R \to \mathrm{Id}_{\mathcal{C}}$ define a natural transformation

$$\widetilde{s} \colon \widetilde{\jmath}_0 \circ \widetilde{R} \to \mathrm{Id}_{\mathcal{C}[\mathbf{W}^{-1}]}$$

By construction, $\widetilde{s}_C = \cancel{s_C}$. Therefore, $\widetilde{s}$ is a natural isomorphism. Condition 3. in Definition 1.3.7 shows that s restricts to a natural transformation $Rj_0 \to \mathrm{Id}_{\mathcal{C}_0}$. For the same reasons of above, this natural transformation lifts to a natural isomorphism $\widetilde{R} \circ \widetilde{\jmath}_0 \to \mathrm{Id}_{\mathcal{C}_0[\mathbf{W}_0^{-1}]}$. This gives the desired equivalence of categories. The second statement is an obvious consequence of the first. □

Let now $\mathcal{M}$ be a model category. Introduce the following notations:

Notation. Let $\mathcal{M}$ be a model category. We will consider the following subcategories:

- $\mathcal{M}_c$, the full subcategory whose objects are the cofibrant objects of $\mathcal{M}$;
- $\mathcal{M}_f$, the full subcategory whose objects are the fibrant objects of $\mathcal{M}$;
- $\mathcal{M}_{cf}$, the full subcategory whose objects are both fibrant and cofibrant objects of $\mathcal{M}$.

Using the factorization axiom **MC5** and Lemma 1.3.8 it is immediate to prove the following:

Proposition 1.3.10. For every model category $\mathcal{M}$:

1. $\mathcal{M}_{cf}$ and $\mathcal{M}_c$ are left deformation retracts of $\mathcal{M}_f$ and $\mathcal{M}$ with respect to weak equivalences;
2. $\mathcal{M}_{cf}$ and $\mathcal{M}_f$ are right deformation retracts of $\mathcal{M}_c$ and $\mathcal{M}$.

Proof. We will show that $\mathcal{M}_c$ is a left deformation retraction of $\mathcal{M}$ with respect to weak equivalences. The other statements are similar. Let's fix an initial object $\emptyset$; there exists a functor

$$F \colon \mathcal{M} \to \mathbf{Arr}(\mathcal{M})$$

sending an object A to the (unique) arrow $\emptyset \to A$; this assignment extends easily to arrows, and it is functorial. Introduce next the $(\textsc{Cofib}, \textsc{Fib} \cap W)$-factorization functor

$$G \colon \mathbf{Arr}(\mathcal{M}) \to \mathbf{Arr}(\mathcal{M}) \times_{\mathbf{d}_1, \mathbf{d}_0} \mathbf{Arr}(\mathcal{M})$$

Finally, denote by π_0 and π_1 the projection functors

$$\pi_i \colon \mathbf{Arr}(\mathcal{M}) \times_{\mathbf{d}_1, \mathbf{d}_0} \mathbf{Arr}(\mathcal{M}) \to \mathbf{Arr}(\mathcal{M})$$

Consider the functor

$$Q := \mathbf{d}_0 \circ \pi_1 \circ G \circ F \colon \mathcal{M} \to \mathcal{M}_c$$

For each object $A \in \mathrm{Ob}(\mathcal{M})$ we have an arrow

$$p_A := \pi_1(G(F(A)) \colon Q(A) \to A$$

which is a trivial fibration. It's clear that the family $\{p_A\}_{A \in \mathrm{Ob}(\mathcal{M})}$ defines a natural transformation $j_c \circ Q \to \mathrm{Id}_{\mathcal{M}}$ (here $j_c \colon \mathcal{M}_c \to \mathcal{M}$ denotes the natural inclusion). This, together with Lemma 1.3.8, implies that $\mathcal{M}_c$ is a left deformation retract of $\mathcal{M}$. □

Combining Propositions 1.3.9 and 1.3.10 it follows that we only need to show the existence of $\mathrm{Ho}(\mathcal{M}_{cf})$. This will be accomplished in the following paragraph.

Remark 1.3.11. In absence of functorial factorization Proposition 1.3.10 doesn't need to be true. However, it can be shown that even in that case the localization of a model category exists. For a proof which doesn't make use of the functoriality of factorization, the reader is referred to [DS95, Section 4].

Homotopy relations

To construct the localization of $\mathcal{M}_{cf}$ with respect to weak equivalences, we will need some machinery.

Definition 1.3.12. Let $\mathcal{M}$ be a model category and let $X \in \mathrm{Ob}(\mathcal{M})$ be any object.

1. A cylinder object for X is a factorization of the fold map

$$\nabla : X \sqcup X \overset{i}{\hookrightarrow} X \times I \overset{\sim}{\to} X$$

in a cofibration followed by a weak equivalence.

2. A path object for X is a factorization of the diagonal map

$$\Delta : X \overset{\sim}{\to} X^I \twoheadrightarrow X \times X$$

in a weak equivalence followed by a fibration.

Remark 1.3.13. A functorial cylinder (path) object always exist, thanks to the factorization axiom. Moreover, we can require the map $X \times I \to X$ (resp. $X \to X^I$) to be a trivial fibration (resp. a trivial cofibration). However, it is important to remark that a cylinder (path) object is *any* factorization of the fold (diagonal) map.

Notation. If $\mathcal{M}$ is a model category and $X \times I$ is a cylinder object for X, we will denote by $\mathrm{in}_k : X \to X \times I$ the two arrows making the diagram

$$\begin{tikzcd}
X \arrow[d, "i_0"] \arrow[dr, bend left, "\mathrm{in}_0"] & \\
X \sqcup X \arrow[r, hook] & X \times I \\
X \arrow[u, "i_1"] \arrow[ur, bend right, "\mathrm{in}_1"'] &
\end{tikzcd}$$

commutative. Observe that the 2-out-of-3 axiom implies that in_k is always a weak equivalence. Similarly, we denote the dual maps for path objects as pr_k.

We can use cylinder and path objects to introduce the notion of homotopy between two maps:

Definition 1.3.14. Let $\mathcal{M}$ be a model category and let $f, g : A \to X$ be two arrows.

1. A left homotopy from f to g is a pair $(A \times I, H)$ where $A \times I$ is a cylinder object for A and $H : A \times I \to X$ is a map making the diagram

$$\begin{tikzcd}
A \arrow[d, "\mathrm{in}_0"'] \arrow[dr, bend left, "f"] & \\
A \times I \arrow[r, "H"] & X \\
A \arrow[u, "\mathrm{in}_1"] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

commutative. We say that f is left homotopic to g if a left homotopy from f to g exists; in this case we write $f \overset{l}{\sim} g$;

2. A right homotopy from f to g is a pair (X^I, K) where X^I is a path object for X and $K : A \to X^I$ is a map making the diagram

$$\begin{tikzcd} & X \\ A \arrow[ur, bend left, "f"] \arrow[r, "K"] \arrow[dr, bend right, "g"'] & X^I \arrow[u, "\mathrm{pr}_0"'] \arrow[d, "\mathrm{pr}_1"] \\ & X \end{tikzcd}$$

commutative. We say that g is right homotopic to g if a right homotopy from f to g exists; in this case we write $\cancel{f \overset{g}{\sim} g.}$

Lemma 1.3.15. If $f : A \to B$ is left homotopic to a weak equivalence, then f is a weak equivalence.

Proof. Choose a cylinder object $A \times I$ for A and a homotopy $H : A \times I \to B$. Since $\mathrm{in}_0, \mathrm{in}_1$ are weak equivalences, and $H \circ \mathrm{in}_1$ is a weak equivalence by hypothesis, it follows from the 2-out-of-3 axiom that H is a weak equivalence. Therefore, $f = K \circ \mathrm{in}_0$ is a weak equivalence too. □

Lemma 1.3.16. Let $\mathcal{M}$ be a model category.

1. If A is a cofibrant object, then $\overset{l}{\sim}$ defines an equivalence relation on $\mathrm{Hom}_{\mathcal{M}}(A, X)$ for every object $X \in \mathrm{Ob}(\mathcal{M})$;

2. if X is a fibrant object, then $\overset{r}{\sim}$ defines an equivalence relation on $\mathrm{Hom}_{\mathcal{M}}(A, X)$ for every object $A \in \mathrm{Ob}(\mathcal{M})$.

Definition 1.3.17. Let $\mathcal{M}$ be a model category. Let $A, X \in \mathrm{Ob}(\mathcal{M})$; we denote by $\pi^l(A, X)$ the quotient of $\mathrm{Hom}_{\mathcal{M}}(A, X)$ under the equivalence relation *generated* by left homotopy. Similarly, we will denote by $\pi^r(A, X)$ the quotient of $\mathrm{Hom}_{\mathcal{M}}(A, X)$ by the equivalence relation *generated* by right homotopy.

Lemma 1.3.18. Let $\mathcal{M}$ be a model category.

1. If A is cofibrant and $p : X \to Y$ is an acyclic fibration or a weak equivalence between fibrant objects, then the map
$$p_* : \mathrm{Hom}_{\mathcal{M}}(A, X) \to \mathrm{Hom}_{\mathcal{M}}(A, Y)$$
induces a bijection
$$p_* : \pi^l(A, X) \to \pi^l(A, Y)$$

2. If X is fibrant and $i : A \to B$ is an acyclic cofibration of a weak equivalence between cofibrant objects, then the map
$$i^* : \mathrm{Hom}_{\mathcal{M}}(B, X) \to \mathrm{Hom}_{\mathcal{M}}(A, X)$$
induces a bijection
$$i^* : \pi^r(B, X) \to \pi^r(A, X)$$

Lemma 1.3.19. Let $\mathcal{M}$ be a model category.

1. If X is fibrant then composition in $\mathcal{M}$ induces a map
$$\pi^l(A', A) \times \pi^l(A, X) \to \pi^l(A', X)$$

2. If A is cofibrant the composition in $\mathcal{M}$ induces a map

$$\pi^r(A,X) \times \pi^r(\cancel{A},X') \to \pi^r(A,X')$$

Lemma 1.3.20. Let $\mathcal{M}$ be a model category and let $f,g : A \to X$ be maps.

1. If A is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$;
2. if X is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.

Notation. If A is cofibrant and X is fibrant, previous lemma shows that the two relations $\overset{l}{\sim}$ and $\overset{r}{\sim}$ on $\mathrm{Hom}_{\mathcal{M}}(A,X)$ coincide. In this case we will denote both by $\sim$ and we will refer to it as the homotopy equivalence relation.

With these notations we have the following:

Corollary 1.3.21. The homotopy relation on morphisms of $\mathcal{M}_{cf}$ is an equivalence relation and it is compatible with composition. Hence the category $\mathcal{M}_{cf}/\sim$ exists.

The following proposition is the key result in our proof of Theorem 1.3.6. It is the equivalent, in our abstract setting, of Whitehead's Theorem 1.1.3.

Theorem 1.3.22. Let $\mathcal{M}$ be a model category and let $f : A \to X$ be a map between objects which are both fibrant and cofibrant. Then f is a weak equivalence if and only if it is a homotopy equivalence.

Sketch of the proof. Suppose that $f : A \to B$ is a weak equivalence of objects in $\mathcal{M}_{cf}$. Then Lemma 1.3.18 shows that for any other fibrant and cofibrant object X we have an induced bijection

$$f_* : \pi(X,A) \to \pi(X,B)$$

For $X = B$ we find $g : B \to A$ such that $fg \sim \mathrm{id}_B$; then $fgf \sim f$ and taking $X = A$ we can cancel f obtaining $gf \sim \mathrm{id}_A$. Thus f is a homotopy equivalence.

Conversely, suppose that f is a homotopy equivalence. Factor f as

$$A \xrightarrow{g} C \xrightarrow{p} B$$

with g acyclic cofibration. Since C is fibrant and cofibrant, it follows that g is a homotopy equivalence. Let $f' : B \to A$ be a homotopy inverse for f and choose a left homotopy

$$H : B \times I \to B$$

from ff' to 1_B. Since B is cofibrant the map $\mathrm{in}_0 : B \to B \times I$ is an acyclic cofibration; therefore we can choose a lifting $H' : B \times I \to C$ in the following diagram:

$$\begin{tikzcd}
B \arrow[r, "gf'"] \arrow[d, "\mathrm{in}_0"'] & C \arrow[d, "p"] \\
B \times I \arrow[r, "H"'] \arrow[ur, dotted, "H'"] & B
\end{tikzcd}$$

Set

$$q := H' \circ \mathrm{in}_1$$

Then $pq = 1_B$ and H' is a left homotopy from gf' to q. If g' is a homotopy inverse for g we get $p \sim pgg' \sim fg'$, i.e.

$$qp \sim (gf')(fg') \sim 1_C$$

Lemma 1.3.15 implies now that qp is a weak equivalence. But the diagram

$$\begin{tikzcd} C \arrow[r, "1_C"] \arrow[d, "p"] & C \arrow[r, "1_C"] \arrow[d, "qp"] & C \arrow[d, "p"] \\ B \arrow[r, "q"] & C \arrow[r, "p"] & B \end{tikzcd}$$

expresses p as retract of qp. □

Corollary 1.3.23. Let $\mathcal{M}$ be a model category; the quotient map $\gamma\colon \mathcal{M}_{cf} \to \mathcal{M}_{cf}/\sim$ is the localization of $\mathcal{M}_{cf}$ with respect to weak equivalences.

Proof. Let $F\colon \mathcal{M}_{cf} \to \mathcal{C}$ be a functor sending every weak equivalence to an isomorphism. Let $f, g\colon A \to B$ be homotopic maps. Choose a cylinder object for A

$$A \sqcup A \xrightarrow{i} A \times I \xrightarrow{p} A$$

and a (left) homotopy $H\colon A \times I \to B$ from f to g. Since p is a weak equivalence $F(p)$ is an isomorphism; therefore:

$$F(p) \circ F(\mathrm{in}_0) = F(p \circ \mathrm{in}_0) = F(p \circ \mathrm{in}_1)F(p) \circ F(\mathrm{in}_1)$$

and thus $F(\mathrm{in}_0) = F(\mathrm{in}_1)$. It follows that

$$F(f) = F(H\mathrm{in}_0) = F(H)F(\mathrm{in}_0) = F(H)F(\mathrm{in}_1) = F(H \circ \mathrm{in}_1) = F(g)$$

The universal property of the quotient therefore produces a unique morphism

$$\overline{F}\colon \mathcal{M}_{cf}/\sim \to \mathcal{C}$$

and a unique natural isomorphism $t\colon \overline{F} \circ \gamma \to F$. Universality follows from universal property of the quotient, with considerations similar to the ones above. □

We are now ready to provide a proof of Theorem 1.3.6:

Proof of Theorem 1.3.6. With Proposition 1.3.10 we were reduced to prove that the localization of $\mathcal{M}_{cf}$ with respect to weak equivalences exists. This is done in Corollary 1.3.23. □

Complement: homotopy and liftings

Before developing some examples, we want to discuss some lifting criterion and uniqueness (up-to-homotopy) property. All the objects will be objects in a given model category $\mathcal{M}$.

Proposition 1.3.24. Let $i\colon A \to B$ be a cofibration, $p\colon X \to Y$ a fibration. Then for each commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, shift left, "h_1"] \arrow[ur, dotted, shift right, "h_2"'] & Y \end{tikzcd}$$

and each pair of liftings h_1, h_2, we have:

1. if p is a trivial fibration, h_1 is left homotopic to h_2;
2. if i is a trivial cofibration, h_1 is right homotopic to h_2.

Proof. We will prove the first statement. Choose a cylinder object for B:

$$B \sqcup B \xrightarrow{j} B \times I \xrightarrow{r} B$$

and consider the diagram

$$\begin{tikzcd} B \sqcup B \arrow[rr, "h_1 \sqcup h_2"] \arrow[d, "j"'] & & X \arrow[d, "p"] \\ B \times I \arrow[r, "r"'] \arrow[urr, dotted, "H"] & B \arrow[r, "g"'] & Y \end{tikzcd}$$

The lifting exists by hypothesis, since j is a cofibration and p is a trivial fibration. Clearly,

$$H \circ \mathrm{in}_0 = h_1, \quad H \circ \mathrm{in}_1 = h_2$$

□

Corollary 1.3.25. Let X, Y be given objects; let $(\widetilde{X}, i)$ and $(\widetilde{Y}, j)$ be cofibrant approximations such that the j is a fibration. Then any map $f : X \to Y$ has a lifting $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ and this lifting is unique up to right homotopy.

Proof. We obtain $\widetilde{f}$ choosing a lifting in the following diagram:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d, "j"] \\ \widetilde{X} \arrow[r, "f \circ i"'] \arrow[ur, dotted, "\widetilde{f}"] & Y \end{tikzcd}$$

Uniqueness up to right homotopy is then a consequence of Proposition 1.3.24. □

Corollary 1.3.26. Let X be a given object. If (W, i) and (W', j) are two cofibrant approximations such that j is a fibration, there is a weak equivalence $f : W \to W'$ such that $j \circ f = i$; moreover f is unique up to right homotopy.

Proof. Apply Corollary 1.3.25 to the identity $\mathrm{id}_X : X \to X$. □

Remark 1.3.27. The situation described in previous results is somehow standard: really often, in homotopy theory, we cannot ask for uniqueness, but only for uniqueness up-to-homotopy. The idea is that, from a homotopical point of view, it doesn't really matter, so we have a "virtual uniqueness". An important example is the following: if we are working up-to-homotopy, it doesn't really matter to know the composition of f and g; we just need to know a composition up-to-homotopy; obviously, the requirement is that every two admissible compositions are reciprocally homotopic "at any order". The correct formulation of these ideas is by no means trivial; it is somehow intuitive to try to reduce to a "contractibility property", but it took several decades to reach a satisfactory formulation.

To the best of my knowledge, after the work of Quillen it has been standard to interpret this ~~conctractibility~~ in term of the classifying space of a category. For example, if we are given maps as in Proposition 1.3.24, then the category whose objects are the diagonal fillings and whose morphisms are the right-homotopies will have conctractible classifying space. This shows that the theory we are developing is working well, since it produces the results that we intuitively expect. However, this theory has a great disadvantage: it requires a lot of constructions and a lot of comparison results. This defect has been overcome in the theory of $(\infty, 1)$-categories via quasicategories; the exposition given by Lurie in [HTT] is perfectly organic and doesn't need any "external" construction: for example, we have the notion of contractibility of a quasicategory, and if S is a quasicategory, we can consider the subcategory spanned by the diagonal fillings; the previous result can be restated by saying that this quasicategory is contractible, without need to invoke the classifying space construction. This is, in my opinion, one of the principal strengths of quasicategories.

1.4 Examples

In this section we collect some of the easiest examples of model categories. Each example will be organized in the following way:

1. description of the model structure with a sketch of the verification of the axioms;
2. explicit computation of cylinder objects and path objects (at least in some good case);
3. explicit computation of the Hom in the homotopy category $\mathrm{Ho}(\mathcal{M})$.

Simplicial sets

The theory of simplicial sets are at the core of homotopy theory. They are the first purely algebraic (combinatorial) model for the homotopy category of topological spaces we discovered. Appendix A contains a brief summary of this theory; more specific references are given there. Here we will simply describe the usual model structure given to **sSet**.

Theorem 1.4.1. The following classes of maps in **sSet** define a model structure:

- weak equivalences are exactly those morphisms inducing isomorphisms between all the homotopy group (of the geometric realization);
- fibrations are Kan fibrations;
- cofibrations are injections on objects.

Remark 1.4.2. It's possible to give a purely combinatorial description of the model structure on **sSet**. To describe weak equivalences avoiding geometric realization, one has to use the notion of minimal Kan complex and minimal fibration. For an exposition of these notions, the reader is referred to [May69] or to [GJ99, p. I.10].

From this moment on, when we refer to a model structure on **sSet** we will tacitly mean that of Theorem 1.4.1. The proof of this Theorem is hard and we will simply outline the main ideas involved there. For the details, the reader is referred to the first chapter of [GJ99] or to the third chapter

Sketch of the proof. Coming soon... □

At this point we can do some computation. Let's start with the computation of a cylinder object.

Proposition 1.4.3. If $K \in \mathbf{sSet}$ is a simplicial set, $K \times \Delta^1$ is a cylinder object for K.

Proof. For $e \in \{0,1\}$ define the maps $\mathrm{in}_e \colon K \to K \times \Delta^1$ by

$$\begin{tikzcd}
 & K \\
K \arrow[ur, bend left, "\mathrm{id}_K"] \arrow[r, "\mathrm{in}_e"] \arrow[dr, bend right, "f_e"'] & K \times \Delta^1 \arrow[u, "p"] \arrow[d, "q"] \\
 & \Delta^1
\end{tikzcd}$$

where $f_e \colon K \to \Delta^0 \xrightarrow{d^e} \Delta^1$. This gives an injective map

$$i \colon K \sqcup K \to K \times \Delta^1$$

Then

$$p \circ i \circ i_0 = p \circ \mathrm{in}_0 = \mathrm{id}_K$$
$$p \circ i \circ i_1 = p \circ \mathrm{in}_1 = \mathrm{id}_K$$

which shows that $p \circ i$ is a factorization of the fold map $\nabla \colon K \sqcup K \to K$. Since i is injective, it is also a cofibration. Since geometric realization commutes with products, we obtain that

$$|p| \colon |K| \times |\Delta^1| \to |K|$$

which is a homotopy equivalence ($|\Delta^1|$ is contractible), hence p is a weak equivalence. □

We will see later that this proposition holds in full generality: the key point is that $\{\Delta^n\}_{n \in \mathbb{N}}$ is a cosimplicial object in **sSet**, and **sSet** is enriched over itself.

Proposition 1.4.4. Let K be a simplicial set. Then $\mathbf{hom}(\Delta^1, K)$ is a path object for K.

Simplicial sets have another model structure, called the *Joyal model structure*. We won't use it in this chapter, but we will use it dealing with quasicategories (and the equivalence with simplicial categories).

Definition 1.4.5. Let $S \in$ **sSet**; we define a simplicial category $\mathfrak{C}[S]$ as ...

Theorem 1.4.6. **sSet** has a model structure where:

- cofibrations is a monomorphism;
- a map $f \colon S \to S'$ is a weak equivalence if and only if the induced functor $\mathfrak{C}[S] \to \mathfrak{C}[S']$ is an equivalence of simplicial categories;
- fibrations are maps with the RLP with respect to trivial cofibrations.

Topological spaces

Among the most important examples of model category there is **Top**.

Definition 1.4.7. A map of topological spaces $p \colon X \to Y$ is said to be a *Serre fibration* if, for each CW-complex A, the map p has the RLP with respect to the inclusion $A \times 0 \to A \times [0,1]$.

Theorem 1.4.8. **Top** has a model structure where

- weak equivalences are weak homotopy equivalences;
- fibrations are Serre fibrations;
- cofibrations are the maps with LLP with respect to acyclic fibrations.

Since $I := [0,1]$ is contractible, it follows that $A \times [0,1]$ retracts onto $A \times 0$; in particular, every map $X \to *$ is a Serre fibration, that is:

Corollary 1.4.9. With the model structure of Theorem 1.4.8, in **Top** every object is fibrant.

On the other side one can prove:

Lemma 1.4.10. Every cellular inclusion $A \to B$ is a cofibration.

At this point, we can do a computation. Assume that A is a CW-complex and let X be an arbitrary topological space. Since A is cofibrant by previous proposition, and X is fibrant because every object is fibrant, it follows that

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top})}(A, X) \sim \mathrm{Hom}_{\mathbf{Top}}(A, X)/\sim$$

Proposition 1.4.11. If A is a CW-complex, $A \times [0,1]$ is a cylinder object for A.

Proof. Consider the inclusions

$$A \sqcup A \to A \times [0,1]$$

at level 0 and 1. This is a cofibration because it is a cellular inclusion. Then we can introduce the projection:

$$p : A \times [0,1] \to A$$

This is a homotopy equivalence, hence also a weak equivalence. Moreover, it is a fibration: given a diagram

$$\begin{tikzcd} B \times 0 \arrow[r, "f"] \arrow[d, "g"] & A \times [0,1] \arrow[d, "p"] \\ B \times [0,1] \arrow[r, "h"] & A \end{tikzcd}$$

simply define a lifting $H : B \times [0,1] \to A \times [0,1]$ by

$$H(b,t) = (h(b,t), q(f(b,0)))$$

where $q : A \times [0,1] \to [0,1]$ is the second projection. □

Using this result we can also identify the homotopy relation for maps starting from a CW-complex:

Proposition 1.4.12. If A is a CW-complex and $f, g : A \to X$ are continuous map, then f is homotopic to g if and only if they are homotopic in the topological sense.

Finally, we can anticipate that the adjoint pair composed by geometric realization and singular complex induce an equivalence between Ho(**Top**) and Ho(**sSet**). This will give an example of Quillen equivalence. We will return on this point in the next section.

Before passing on, we should at least remark that **Top** can be endowed with another model structure, where cofibrations are the inclusions $A \hookrightarrow X$ such that A is a closed subspace of X and the pair (X, A) has the homotopy lifting property. These are called Hurewicz cofibrations. The weak equivalences are unchanged. This gives an example of a model category whose structure is not completely determined by weak equivalences.

Chain complexes over $\mathbf{Mod}_R$

This is a purely algebraic example of model category, and establishes a strong relation with homological algebra. The intuition coming from this example will be useful as analogy in the construction of the total derived functor.

We will restrict ourselves to consider complexes *bounded below*.

Definition 1.4.13. A morphism of complexes $f_\bullet : M_\bullet \to N_\bullet$ is said to be a *quasi-isomorphism* if, for each $n \in \mathbb{N}$, $H_i(f_\bullet) : H_i(M_\bullet) \to H_i(N_\bullet)$ is an isomorphism.

Remark 1.4.14. Recall from [CSAM29, Ch. I] that every chain-equivalence is also a quasi-isomorphism. The converse is false: for example, a complex is split-exact if and only if its identity is chain-homotopic to the null map; however, there are exact complexes which are not split exact; for each such complex, the map to the zero complex is a quasi isomorphism, but it is not a chain-equivalence.

For each R-module A and each $n \in \mathbb{N}$, $n \geq 1$ define the complex $((D_n(A))_\bullet, d_\bullet)$ by

$$(D_n(A))_i = \begin{cases} 0 & \text{if } i \neq n-1, n \\ A & \text{if } i = n-1, n \end{cases}, \qquad d_i = \begin{cases} 0 & \text{if } i \neq n-1 \\ \mathrm{id}_A & \text{if } i = n-1 \end{cases}$$

Clearly this gives rise to a functor $D_n : \mathbf{Mod}_R \to \mathbf{Ch}(R)$. Let now $\Pi_n : \mathbf{Ch}(R) \to \mathbf{Mod}_R$ be the functor projecting everything to the n-th position. The following lemma is straightforward:

Lemma 1.4.15. For each $n \in \mathbb{N}$, $n \geq 1$ we have the following adjunction relations:

$$D_n \dashv \Pi_n \dashv D_{n+1}$$

In particular D_n takes projective objects into projective objects.

Sketch of the proof. The first statement is clear by inspection. The second statement follows easily from the adjunction $D_n \dashv \Pi_n$ and the fact that Π_n is an exact functor. □

Theorem 1.4.16. Let R be a ring. The category of (bounded below) chain complexes $\mathbf{Ch}(R)$ has a model structure where

- a weak equivalence is a quasi-isomorphism;
- a cofibration is a map $f_\bullet : M_\bullet \to N_\bullet$ such that f_k is a monomorphism with projective cokernel for every degree k;
- a fibration is a map $f_\bullet : M_\bullet \to N_\bullet$ which is an epimorphism in every strictly positive degree.

Proof. Sketch of the proof Axioms **MC1** – **MC3** are straightforward. Let's prove the RLP of cofibrations with respect to acyclic fibrations. Consider a diagram

$$\begin{tikzcd} A_\bullet \arrow[r, "g_\bullet"] \arrow[d, "i_\bullet"'] & X_\bullet \arrow[d, "p_\bullet"] \\ B_\bullet \arrow[r, "h_\bullet"] & Y_\bullet \end{tikzcd} \tag{1.1}$$

where $i_\bullet$ is a cofibration and p an acyclic fibration. Since $H_0(p_\bullet)$ is an isomorphism, it follows that $p_0 : X_0 \to Y_0$ is onto, and so $p_\bullet : X_\bullet \to Y_\bullet$ is onto. Now the idea is to split $B_k = A_k \oplus P_k$, where P_k is a projective module (possible because $i_\bullet$ is a cofibration) and use the projection $B_k \to A_k$ to build inductively a lifting $B_k \to X_k$.

Assume now that in the diagram (1.1) $i_\bullet$ is an acyclic cofibration and $p_\bullet$ a fibration. Let $P_\bullet$ be the cokernel of $i_\bullet$; then the long exact sequence in homology shows that $P_\bullet$ is acyclic; by assumption each P_k is a projective R-module. It can be shown with an induction argument that

$$P_\bullet \simeq \bigoplus_{k\geq 1} D_k(Z_{k-1}(P))$$

and that each $Z_k(P)$ is a projective R-module. It follows from Lemma 1.4.15 that $P_\bullet$ is a projective object in $\mathbf{Ch}(R)$. The short exact sequence

$$0 \to A_\bullet \xrightarrow{i_\bullet} B_\bullet \to P_\bullet \to 0$$

is split. This allows to build the desired lifting.

Axiom **MC5** follows by a standard small object argument. □

Example 1.4.17. For each R-module A, let $(P_\bullet, f_\bullet)$ be a cofibrant replacement for $J_0(A)$. The map $f_\bullet \colon P_\bullet \to K(A,0)$ is a quasi-isomorphism; in particular $P_\bullet$ is exact in strictly positive degrees. Moreover, it is cofibrant, hence each object is projective. It follows that $P_\bullet$ is a *projective resolution* of A.

Remark 1.4.18. The tensor product of chain complexes is defined as $A_\bullet \otimes_{\mathbf{Ch}(R)} B_\bullet := \mathrm{Tot}^{\oplus}(A_\bullet \otimes_R B_\bullet)$, where $A_\bullet \otimes_R B_\bullet$ is the natural double complex associated to $A_\bullet$ and $B_\bullet$. This tensor product endows $\mathbf{Ch}(R)$ with a monoidal structure. See the appendixes for the details.

Example 1.4.19. Consider the complex:

$$\Delta^1_R \colon \ \ldots \to 0 \to R \xrightarrow{d} R^2$$

where

$$d = \begin{pmatrix} -\mathrm{id}_R \\ \mathrm{id}_R \end{pmatrix}$$

Consider the map

$$h_\bullet \colon \Delta^1_R \to K(R,0)$$

defined by

$$h_0 = \begin{pmatrix} \mathrm{id}_R & \mathrm{id}_R \end{pmatrix}$$

and

$$h_n = 0 \quad \text{if } n > 0$$

We claim that $h_\bullet$ is a chain-equivalence. In fact, consider the map

$$g\bullet \colon K(R,0) \to \Delta^1_R$$

defined in degree 0 to be

$$g_0 = \begin{pmatrix} \mathrm{id}_R \\ 0 \end{pmatrix}$$

and $g_n = 0$ for $n > 0$. Then setting $f_\bullet := g_\bullet \circ h_\bullet$ we obtain the following map:

$$\begin{tikzcd}
\cdots \arrow[r] & R \arrow[r, "\begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}"] \arrow[d, "0"'] & R^2 \arrow[d, "\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ 0 & 0 \end{pmatrix}"] \arrow[dl, dotted, "s"'] \\
\cdots \arrow[r] & 0 \arrow[r, "\begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}"'] & R^2
\end{tikzcd}$$

Using as basis for R^2 the vectors

$$u_0 := \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad u_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

define $s \colon R^2 \to R$ by the conditions

$$s(u_0) = 1, \quad s(u_1) = 0$$

Then a routine check shows that s is the required chain homotopy between $f_\bullet$ and $\mathrm{id}_{\Delta^1_R}$. Since $h_\bullet \circ g_\bullet = \mathrm{id}_{K(R,0)}$, we completely proved our statement.

Lemma 1.4.20. Let $A_\bullet$ be a cofibrant object in $\mathbf{Ch}(R)$. Then $A_\bullet \otimes_{\mathbf{Ch}(R)} \Delta^1_R$ is a cylinder object for $A_\bullet$. More generally, let $A_\bullet$ be any object in $\mathbf{Ch}(R)$ and let $(\widetilde{A}_\bullet, f_\bullet)$ be a cofibrant approximation of $A_\bullet$; then $\widetilde{A}_\bullet \otimes \Delta^1_R$ is a cylinder object for $A_\bullet$.

Proof. Explicitly

$$(A_\bullet \otimes_{\mathbf{Ch}(R)} \Delta^1_R)_n = A_{n-1} \oplus A_n \oplus A_n$$

and the differential is given by

$$\begin{pmatrix} -d & 0 & 0 \\ \mathrm{id} & d & 0 \\ -\mathrm{id} & 0 & d \end{pmatrix}$$

Let's consider the map $g_\bullet \colon A_\bullet \otimes_{\mathbf{Ch}(R)} \Delta^1_R \to A_n$ defined by

$$g_n = \begin{pmatrix} \mathrm{id} & 0 & \mathrm{id} \end{pmatrix}$$

Introduce also $f_\bullet \colon B_\bullet \oplus B_\bullet \to A_\bullet \otimes_{\mathbf{Ch}(R)} \Delta^1_R$ setting

$$f_n = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

Inspection shows that $f_\bullet$ and $g_\bullet$ are chain maps. Moreover, they factorize the fold map

$$\nabla \colon B_\bullet \oplus B_\bullet \to B_\bullet$$

The map $g_\bullet$ is clearly surjective in every degree; in particular it is a fibration. Moreover, it is a quasi-isomorphism: using the map $h_\bullet$ defined in Example 1.4.19 we obtain a map

$$\mathrm{id} \otimes_{\mathbf{Ch}(R)} h_\bullet \colon B_\bullet \otimes_{\mathbf{Ch}(R)} \Delta^1_R \to B_\bullet \otimes_{\mathbf{Ch}(R)} K(R,0) \simeq B_\bullet$$

which is still a chain equivalence. Finally, $f_\bullet$ is a cofibration because $A_\bullet$ is assumed to be cofibrant. □

Remark 1.4.21. Observe that $A_\bullet \times \Delta^1_R$ is, in the notations of [CSAM29, Ch. 1.5], the mapping cylinder of $\mathrm{id}_{A_\bullet}$.

Corollary 1.4.22. Let $A_\bullet$ be a cofibrant object in $\mathbf{Ch}(R)$; let $f_\bullet, g_\bullet \colon A_\bullet \to B_\bullet$ be two chain-maps. Then a left homotopy from $f_\bullet$ to $g_\bullet$ is precisely a chain-homotopy.

Remark 1.4.23. We can now interpret Corollaries 1.3.25 and 1.3.26 in term of the usual results of homological algebra: any map between objects can be lifted to a map between projective resolutions, and the lifting is unique up-to-homotopy.

Proposition 1.4.24. $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Ch}(R))}(K(A,n),K(B,m)) = \mathrm{Ext}_R^{m-n}(A,B)$.

Proof. If $n = 0$ and $m > 0$, choose a cofibrant replacement for $K(A,0)$; as we saw, this is a projective resolution $P_\bullet \to A$. Then $P_\bullet$ is cofibrant, hence left homotopy on

$$\mathrm{Hom}_{\mathbf{Ch}(R)}(P_\bullet, K(B,m)) = \{\alpha\colon P_m \to B \mid \alpha \circ d = 0\}$$

coincides with chain homotopy. This is exactly the homology of

$$\mathrm{Hom}_R(P_{m+1},B) \to \mathrm{Hom}_R(P_m,B) \to \mathrm{Hom}_R(P_{m-1},B)$$

i.e. $\mathrm{Ext}_R^m(A,B)$. □

Groupoids

The category of groupoids will play a central role in these seminars, so we will explain in detail how to derive a model structure on **Grpd** starting from that on **sSet**. For technical details, we will refer to the Appendix.

Theorem 1.4.25. **Grpd** has a model structure where

- weak equivalences are equivalences of categories;
- fibrations are the functors with the RLP with respect to the map $\Delta^0_{\mathbf{Grpd}} \to \Delta^1_{\mathbf{Grpd}}$;
- cofibrations are functors which are injections on objects.

Proposition 1.4.26. Let $\mathcal{G}$ be a groupoid. A cylinder object for $\mathcal{G}$ is $\mathcal{G} \times \Delta^1_{\mathbf{Grpd}}$.

Proof. We obviously have maps for $k \in \{0,1\}$

$$\mathrm{in}_k\colon \mathcal{G} \to \mathcal{G} \times \{k\} \subset \mathcal{G} \times \Delta^1_{\mathbf{Grpd}}$$

inducing a map $i\colon \mathcal{G} \sqcup \mathcal{G} \to \mathcal{G} \times \Delta^1$, which is clearly injective on objects and hence a cofibration. The canonical projection map

$$\mathcal{G} \times \Delta^1 \to \mathcal{G}$$

is obviously a fibration and an equivalence of categories. Therefore we have a cylinder object. □

Corollary 1.4.27. Let $F_1, F_2\colon \mathcal{G} \to \mathcal{H}$ be functors between groupoids. They are left homotopic if and only if there is a natural transformation (hence a natural isomorphism) between them.

Proof. This is an easy consequence of Proposition 1.4.26 (using a standard reformulation of natural transformation). □

Categories

The category of all small categories **Cat** has a model structure which doesn't differ much from that of groupoids. However, we state it as a theorem for future references:

Theorem 1.4.28. **Cat** has a model structure where:

- weak equivalences are equivalences of categories;
- cofibrations are functors injective on objects;
- fibrations are functors with the RLP with respect to the map $\Delta^0_{\mathbf{Grpd}} \to \Delta^1_{\mathbf{Grpd}}$.

1.5 Quillen adjunctions and total derived functors

The goal of this section is to introduce a notion of morphism between model categories. It is a subtle question to decide how much of the model structure must be preserved by a functor; the naïf idea is probably to consider functors $F : \mathcal{M} \to \mathcal{N}$ sending fibrations, cofibrations and weak equivalences of $\mathcal{M}$ in their correspondents of $\mathcal{N}$. However, it turns out that this notion is too much restrictive, and there aren't many examples of such functors.

What is really done is to consider adjunction pairs $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$, where F is required to preserve cofibrations and G is required to preserve fibrations. It is easy to give several different formulations of this property; the result is what it's called a *Quillen adjunction*. First of all, we will describe these different reformulations; next, we will give the notion of *Quillen equivalence*. In the subsequent paragraph, dedicated to the notion of derived functor, we explain how a Quillen adjunction induces an adjunction between the homotopy categories and we prove that this is an equivalence if the starting adjunction was a Quillen equivalence.

We chose this order for the exposition because it seems us more logic: it is known that Quillen equivalences preserve many other constructions a part from the homotopy category; for example, they preserve also mapping spaces.

Quillen adjunctions and equivalences

Definition 1.5.1. Let $\mathcal{M}, \mathcal{N}$ be model categories. An adjoint pair

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

is said to be a Quillen adjunction if:

1. F preserves cofibrations;
2. G preserves fibrations.

Lemma 1.5.2. Let $(F, G, \varphi) : \mathcal{A} \to \mathcal{B}$ be an adjunction of categories. Let $f : A_1 \to A_2$ be an arrow in $\mathcal{A}$ and $g : B_1 \to B_2$ be an arrow in $\mathcal{B}$; then f has the LLP with respect to $G(g)$ if and only if g has the RLP with respect to $F(f)$.

Proof. Recall that in our notations φ is the natural isomorphism

$$\varphi_{A,B} : \mathrm{Hom}_{\mathcal{B}}(F(A), B) \to \mathrm{Hom}_{\mathcal{A}}(A, G(B))$$

Consider the two commutative diagrams

$$\begin{tikzcd} A_1 \arrow[r, "\alpha"] \arrow[d, "f"'] & G(B_1) \arrow[d, "G(g)"] \\ A_2 \arrow[r, "\beta"'] \arrow[ur, dotted, "h"] & G(B_2) \end{tikzcd} \qquad \begin{tikzcd} F(A_1) \arrow[r, "\gamma"] \arrow[d, "F(f)"'] & B_1 \arrow[d, "g"] \\ F(A_2) \arrow[r, "\delta"'] \arrow[ur, dotted, "k"] & B_2 \end{tikzcd}$$

Assume that f has the RLP with respect to $G(g)$. Starting with the diagram on the right, set $\alpha = \varphi(\gamma)$, $\beta = \varphi(\delta)$; the diagram on the left commutes thanks to adjunction properties. Therefore there is a diagonal lifting $h : A_2 \to G(B_1)$; write ψ for φ^{-1}. Then

$$\psi(h) : F(A_2) \to B_1$$

and

$$\alpha = \psi(\varphi(\alpha)) = \psi(h \circ f) = \psi(h) \circ F(f)$$
$$\beta = \psi(\varphi(\beta) = \psi(G(g) \circ h) = g \circ \psi(h)$$

The other statement is dual. □

Example 1.5.3. The adjoint pair $\pi_f : \mathbf{sSet} \rightleftarrows \mathbf{Grpd} : N$ of Theorem **??** is a Quillen pair. In fact ... (this is essentially the content of [Hol07, Lemma 3.3])

Corollary 1.5.4. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be an adjunction between model categories. The following statements are equivalent:

1. (F, G) is a Quillen pair;
2. F preserves cofibrations and acyclic cofibrations;
3. G preserves fibrations and acyclic fibrations;
4. F preserves acyclic cofibrations and G preserves acyclic fibrations.

Corollary 1.5.5. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen pair. Then F takes weak equivalences between cofibrant objects in weak equivalences. Dually, G takes weak equivalences between fibrant objects in weak equivalences.

Proof. This follows from Lemma 1.2.20 and Corollary 1.5.4. □

Definition 1.5.6. Let $(F, G, \varphi) : \mathcal{M} \to \mathcal{N}$ be a Quillen pair. We say that it is a *Quillen equivalence* if for every cofibrant object $A \in \mathrm{Ob}(\mathcal{M})$ and each fibrant object $X \in \mathrm{Ob}(\mathcal{N})$ a map $f : A \to G(X)$ is a weak equivalence if and only if $\varphi(f) : F(A) \to X$ is a weak equivalence.

Derived functor

Definition 1.5.7. Let $\mathcal{M}$ be a model category and let $F : \mathcal{M} \to \mathcal{C}$ be any functor. We call the right Kan extension of F along $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ the left derived functor of F. We will denote it by $(\mathbf{L}F, t)$. Dually, we call the left Kan extension of F along $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ the right derived functor of F; we will denote it by $(\mathbf{R}F, s)$.

Remark 1.5.8. If $F : \mathcal{M} \to \mathcal{C}$ sends weak equivalences to isomorphisms, then the left derived functor exists because of the universal property of the localization. However, this is not necessary.

Theorem 1.5.9. Let $\mathcal{M}$ be a model category and let $F : \mathcal{M} \to \mathcal{C}$ be any functor. If F sends acyclic cofibrations between cofibrant objects to isomorphisms, then the left derived functor of F exists.

Sketch of the proof. (The details can be found in [Rie12, Theorem 2.2.8]) Introduce a deformation retraction (Q, s) of $\mathcal{M}$ onto $\mathcal{M}_c$ as in the proof of Proposition 1.3.10. Then we can fix a representative for the localization considering

$$\gamma := \gamma_c \circ Q : \mathcal{M} \to \mathcal{M}_c \to \mathrm{Ho}(\mathcal{M}_c)$$

where $\gamma_c : \mathcal{M}_c \to \mathrm{Ho}(\mathcal{M}_c)$ is the localization functor for $\mathcal{M}_c$.

Consider $F \circ j_c \colon \mathcal{M}_c \to \mathcal{C}$; by hypothesis $F \circ j_c$ sends trivial cofibrations to isomorphisms; Ken Brown's Lemma 1.2.20 implies that $F \circ j_c$ sends every weak equivalence to an isomorphism. Universal property of $\mathrm{Ho}(\mathcal{M}_c)$ produces then a factorization of $F \circ j_c$ as $\widetilde{F} \circ \gamma_c$:

$$\begin{tikzcd} \mathcal{M}_c \arrow[r, "\gamma_c"] \arrow[d, shift right, "j_c"'] & \mathrm{Ho}(\mathcal{M}_c) \arrow[d, "\widetilde{F}"] \\ M \arrow[u, shift right, "Q"'] \arrow[ur, "\gamma"] \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

together with a universal natural isomorphism $\varepsilon \colon \widetilde{F} \circ \gamma_c \to F \circ j_c$. Using the natural transformation $s \colon j_c \circ Q \to \mathrm{Id}_{\mathcal{M}}$ we obtain a natural transformation

$$(Fs) \cdot \varepsilon_Q \colon \widetilde{F}\gamma \to F$$

We claim that $(\widetilde{F}, (Fs) \cdot \varepsilon_Q)$ is a left derived functor of F. Let $G \colon \mathrm{Ho}(\mathcal{M}_c) \to \mathcal{C}$ be any functor and let

$$\alpha \colon G\gamma \to F$$

be a natural transformation. Consider $\alpha_{j_c} \colon G\gamma j_c \to F j_c$ and denote s_c the restriction of s to $\mathcal{M}_c$.[1] Since γ sends weak equivalences to isomorphisms, it follows that $G\gamma s_c \colon G\gamma j_c \to G\gamma_c$ is a natural isomorphism. Therefore we have a chain of isomorphisms:

$$\mathrm{Nat}(G\gamma j_c, F j_c) \simeq \mathrm{Nat}(G\gamma_c, F j_c) \simeq \mathrm{Nat}(G, \widetilde{F})$$

which produces a unique natural isomorphism $\beta \colon G \to \widetilde{F}$ such that

$$\varepsilon \cdot \beta_{\gamma_c} = \alpha_{j_c} \cdot (G\gamma s_c)^{-1}$$

We have to check

$$Fs \cdot \varepsilon_Q \cdot \beta_\gamma = \alpha$$

Unravelling the definitions we get

$$\begin{aligned} (Fs) \cdot \varepsilon_Q \cdot (\beta_{\gamma_c})_Q &= (Fs)_Q \cdot (\varepsilon \cdot \beta\gamma_c)_Q \\ &= (Fs)_Q \cdot \alpha_{j_c Q} \cdot (G\gamma s_c)_Q^{-1} \\ &= ((Fs) \cdot \alpha)_Q \cdot (G\gamma s_c)_Q^{-1} \end{aligned}$$

Our thesis is thus equivalent to

$$\alpha \cdot (G\gamma s_c)_Q = ((Fs) \cdot \alpha)_Q$$

which holds by the very definition of natural transformation. Uniqueness of β is similarly proved. $\square$

Example 1.5.10. Let R, S be (commutative) rings; let $F \colon \mathbf{Mod}_R \to \mathbf{Mod}_S$ be an additive functor. This induces an additive functor

$$F \colon \mathbf{Ch}(R) \to \mathbf{Ch}(S)$$

obviously preserving chain homotopies. Since in the (projective) model structure on $\mathbf{Ch}(R)$ every object is fibrant, it follows that every quasi-isomorphism between cofibrant objects (i.e. complexes of projective modules) is a homotopy equivalence; in particular it is preserved by F. This gives rise to a total left derived functor

$$\mathbf{L}F \colon \mathcal{K}(R) \to \mathcal{K}(S)$$

If F is not right exact, the composition $H_0 \circ \mathbf{L}F \circ \deg_0$ doesn't need to be isomorphic to F. However, note that the functors $\{H_i \circ \mathbf{L}F \circ \deg_0\}_{i \in \mathbb{N}}$ do form a homological δ-functor.

[1] Observe that s_c defines a natural transformation $Q \circ j_c \to \mathrm{Id}_{\mathcal{M}_c}$.

Theorem 1.5.11. Let $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$ be a Quillen pair. Then both the left derived functor $\mathbf{L}F$ and the right derived functor $\mathbf{R}G$ exist and they form an adjoint pair

$$\mathbf{L}F\colon \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N})\colon \mathbf{R}G$$

which is an adjoint equivalence if (F,G) is a Quillen equivalence.

Sketch of the proof. The existence of $\mathbf{L}F$ and $\mathbf{R}G$ is implied by Theorem 1.5.9, its dual, and Corollary 1.5.4. Unit and counit pass to the localization; if (F,G) is a Quillen equivalence, unit and counit are weak equivalences, and they induces isomorphisms in the homotopy categories, giving rise to an adjoint equivalence. □

Example 1.5.12. Consider the geometric realization functor $|\cdot|\colon \mathbf{sSet} \to \mathbf{CGHaus}$; we know that $|\cdot|$ is left adjoint to the singular complex functor $\mathrm{Sing}\colon \mathbf{CGHaus} \to \mathbf{sSet}$. Moreover, $|\cdot|$ preserves cofibrations (an inclusion of simplicial sets is sent to a cellular inclusion of CW-complexes) and trivial cofibrations (by definition, a map of simplicial sets is a weak equivalence if and only if its geometric realization is a weak equivalence in **CGHaus**). It follows that $(|\cdot|, \mathrm{Sing})$ is a Quillen pair. Moreover, a map $f: S \to \mathrm{Sing}(X)$ is a weak equivalence if and only if $|f|\colon |S| \to |\mathrm{Sing}(X)|$ is a weak equivalence; however, a classical result states that the counit $|\mathrm{Sing}(X)| \to X$ is always a weak equivalence;[2] it follows that f is a weak equivalence if and only if the adjoint map $|S| \to X$ is a weak equivalence. Therefore $(|\cdot|, \mathrm{Sing})$ is a Quillen equivalence, and induce an equivalence between Ho(**sSet**) and Ho(**CGHaus**).

Homotopy pushout

To conclude this section, I would like to deal with a specific example of great relevance to homotopy theory: homotopy pushout and homotopy pullback (later on we will deal with more general kind of homotopy limits and colimits). Introduce the "pushout category"

$$\mathcal{C} := \{\bullet \longleftarrow \bullet \longrightarrow \bullet\}$$

If $\mathcal{M}$ is a model category, we can consider $\mathcal{M}^{\mathcal{C}}$; in standard category theory we have a functor

$$\mathrm{colim}\colon \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$$

sending a $\mathcal{C}$-diagram to its colimit. However, this functor doesn't behave well with respect to the model structure on $\mathcal{M}$; for example, it doesn't send weak equivalences to weak equivalences:

Example 1.5.13. Let $\mathcal{M} = \mathbf{CGHaus}$ with the standard model structure (fibrations are Serre fibrations). Consider the following diagram:

$$\begin{tikzcd} D^n \arrow[d] & S^{n-1} \arrow[l] \arrow[r] \arrow[d] & D^n \arrow[d] \\ * & S^{n-1} \arrow[l] \arrow[r] & * \end{tikzcd}$$

All the vertical arrows are weak equivalences; however, the induced map on pushouts is $S^n \to *$, which not a weak equivalence.

[2] See for example [May69, Theorem 16.6].

This example is pathological in many ways; for the moment, we can observe that it suggests the subtlety in the relationship between colimits and model structure. To understand better their interaction, we should understand the "homotopical content" of diagrams of shape $\mathcal{C}$; due to the particular choice of $\mathcal{C}$, this is not hard:

Lemma 1.5.14. Define an arrow in $\mathcal{M}^{\mathcal{C}}$ to be:

- a weak equivalence if it is an objectwise weak equivalence;
- a fibration if it is an objectwise fibration;
- a cofibration if it is an objectwise cofibration.

Then there is a model structure on $\mathcal{M}^{\mathcal{C}}$ whose weak equivalences, fibrations and cofibrations are the ones specified above.

Proof. The proof is straightforward. The reader is referred to [DS95, Section 10] for the details. □

Obviously, this suggests to employ Theorem 1.5.9: even though colim doesn't preserve weak equivalences, its derived functor may exists as well. In fact, colim is left adjoint to the diagonal functor

$$\Delta \colon \mathcal{M} \to \mathcal{M}^{\mathcal{C}}$$

Since it's clear that Δ preserves fibrations and trivial fibrations, Theorem 1.5.11 implies that the left derived functor of colim exists.

In practice, it is obviously important to be able to compute $\mathbf{L}\operatorname{colim}$ for given pushout diagrams; thus, we want to recall its construction: Theorem 1.5.11 works via a deformation $Q \colon \mathcal{M} \to \mathcal{M}_c$, and $\mathbf{L}\operatorname{colim}$ is obtained applying the universal property of localization to the functor $\operatorname{colim} \circ Q$. It follows that, in order to do computations, we are essentially interested in $\operatorname{colim} \circ Q$. Recall that we produced explicitly a deformation Q in Proposition 1.3.10; however Q is by no means unique. We can try to exploit this lack of uniqueness to counteract the bad behaviour of colim showed in Example 1.5.13: namely, we can try to choose Q in such a way that $\operatorname{colim} \circ Q$ respects weak equivalences.

Concretely, we can construct Q in the following way: starting with a diagram

$$A_1 \xleftarrow{j} A_0 \xrightarrow{i} A_2$$

we take a cofibrant approximation $w_0 \colon A_0' \to A_0$ and then we factorize $i \circ w$ and $j \circ w$ into cofibrations followed by trivial fibrations. We obtain

$$\begin{tikzcd}
A_1' \arrow[d] & A_0' \arrow[l, "j'"'] \arrow[r, "i'"] \arrow[d, "w"] & A_2' \arrow[d] \\
A_1 & A_0 \arrow[l, "j"'] \arrow[r, "i"] & A_2
\end{tikzcd}$$

and i', j' are cofibrations. Reasoning as in Proposition 1.3.10, we can make functorial this procedure, obtaining a deformation retraction. It can be shown that hocolim is homotopy invariant. We will return to this point later, in greater generality.

Remark 1.5.15. There are even more subtleties. In fact, one may naïvely expect that at the level of $\operatorname{Ho}(\mathcal{M})$ the derived functor $\mathbf{L}\operatorname{colim}$ coincides with the colimit functor. This cannot be true: consider again the situation of Example 1.5.13. It should be more or less clear that in $\operatorname{Ho}(\mathbf{CGHaus})$ the pushout of

$$D^n \longleftarrow S^{n-1} \longrightarrow D^n$$

is $*$. Obviously, this is not what we really want; better, this makes clear that passing to the homotopy category produces a loss of informations. I will try to explain where this loss happens. Instead of working with simple homotopy, let's try to work with "coherent homotopy". If $*$ were, also in this context, the pushout of our diagram, we could consider the natural maps

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \arrow[ddr, bend left] & \\ D^n \arrow[r] \arrow[drr, bend right] & * \arrow[dr, dotted] & \\ & & S^2 \end{tikzcd}$$

The coherence problem is the following: first of all, we are assigned a homotopy

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \arrow[dl, Rightarrow, "h_1"'] \\ D^2 \arrow[r] & S^1 \end{tikzcd}$$

The map $* \to S^n$ must come with other two homotopies

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \arrow[ddr, bend left] & \\ D^2 \arrow[r] \arrow[drr, bend right] & * \arrow[dr, dotted] \arrow[r, phantom, "\overset{h_3}{\Rightarrow}"] \arrow[d, phantom, "h_2\Downarrow"] & {} \\ & {} & S^2 \end{tikzcd}$$

and these three homotopies h_1, h_2, h_3 must be compatible in the sense that $h_1 \circ h_3$ is, up-to-homotopy, equal to h_2. It is important to remark that we are considering homotopies

$$(S^1 \times I) \times I \to Y$$

relatives to

$$Y := (S^1 \times \{0\}) \sqcup (X \times \{1\})$$

(at each fixed time, we must have a homotopy between the two maps of S^1 into S^2). However, such h_1, h_2, h_3 (and the required higher homotopy) cannot exist: $h_2^{-1} \circ h_1 \circ h_3$ defines an element in $\pi_2(S^2) = \mathbb{Z}$, and inspection shows that this element is a generator; if $h_1 \circ h_3$ was homotopic relative to Y to h_2, we would have that, up-to-homotopy, $h_2^{-1} \circ h_1 \circ h_3$ is trivial, which is not the case.

This rough explanation should make clear that in the homotopy category things don't go as expected because we are "inverting too much", producing a loss of (higher homotopical) informations.

Remark 1.5.16. We gave a brief presentation of homotopy pushout in the general context of model categories. The theory developed firstly in the topological context, and then in the simplicial one. In those case, we know several formulas for homotopy limits and colimits. Let's describe the explicit construction of the homotopy pushout in **CGHaus**: if we are given a diagram of spaces

$$Z \xleftarrow{g} X \xrightarrow{f} Y$$

the ordinary colimit is built gluing Z and Y along the images $f(X)$ and $g(X)$. In this process the homotopy type of X can be lost, and this gives rise to the unpleasant situation of Example 1.5.13. It's quite natural, however, to "keep track" of the homotopy type of X using the following trick: replace X with a cylinder object $X \times I$, then glue Z to $X \times I$ along $g(X)$ and $X \times \{1\}$; similarly, glue Y to $X \times I$ along $f(X)$ and $X \times \{0\}$. This gives intuitively the correct result. If we apply this construction to

$$* \longleftarrow S^{n-1} \longrightarrow *$$

we obtain

$$(S^{n-1} \times I)/(S^{n-1} \times \{0\} \sqcup S^{n-1} \times \{1\}) \simeq S^n$$

which is the "correct" result.

1.6 Homotopy limits and colimits

Last section contains a brief summary about the construction of the homotopy pushout. If we want to deal with diagrams of more general shapes, we have to take in account an additional difficulty: the category $\mathcal{M}^{\mathcal{C}}$ doesn't need to come equipped with a natural model structure. There are several ways to overcome this problem: for example, one can restricts to nice enough diagrams; or one can consider only sufficiently well-behaved model categories. In this exposition, we will consider only Reedy diagrams. However, before the technical details, we would like to explain a more general approach to this problem, that doesn't require additional hypothesis.

Recall that Proposition 1.3.9 played a major role in our construction of $\mathrm{Ho}(\mathcal{M})$ for a model category $\mathcal{M}$. Moreover, the existence of a deformation retraction of $\mathcal{M}$ onto $\mathcal{M}_c$ has proven extremely useful also in Theorem 1.5.9. Actually, these constructions can be carried over in the more general context of homotopical categories. Here we state the definition:

Definition 1.6.1. A homotopical category is a category $\mathcal{C}$ equipped with a wide subcategory $\mathcal{W}$ satisfying the 2-out-of-6 axiom: if f, g, h is a composable triple of arrows and hg, gf are in $\mathcal{W}$, then so are f, g, h, hgf.

It can be developed a deformation theory for homotopical categories allowing to show that if $\mathcal{M}$ is a model category and $\mathcal{C}$ is a small category, then $\mathcal{M}^{\mathcal{C}}$ with the componentwise homotopical structure admits a homotopy category. Theorem 1.5.9 about derived functors extends to this context (substituting $\mathcal{M}_c$ with an appropriate deformation retraction of $\mathcal{M}$). From this point of view, we can try seriously to generalize the approach described for homotopy pushouts. Namely, we can prove that the colim functor has a total left derived functor, and we can look for a representative

$$\mathrm{hocolim}\colon \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$$

well behaved with respect to weak equivalences. This approach, appeared firstly in [DHK97], has been further developed in [Shu09] and [Rie12].

Reedy categories

Definition 1.6.2. A Reedy category is a triple $\left(\mathcal{C}, \overleftarrow{\mathcal{C}}, \overrightarrow{\mathcal{C}}\right)$, where $\mathcal{C}$ is a category, $\overleftarrow{\mathcal{C}}$ and $\overrightarrow{\mathcal{C}}$ are subcategories of $\mathcal{C}$ both containing all the objects, and the following requirements are satisfied:

1. $\overleftarrow{\mathcal{C}}$ and $\overrightarrow{\mathcal{C}}$ allows a unique $\left(\overrightarrow{\mathcal{C}}, \overleftarrow{\mathcal{C}}\right)$-factorization;
2. it is possible to define a degree function $\deg\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ in such a way that every morphism in $\overleftarrow{\mathcal{C}}$ increases the degree, and every morphism in $\overrightarrow{\mathcal{C}}$ decreases the degree.

Remark 1.6.3. The degree function is not assigned with the structure of Reedy category.

Example 1.6.4. The category $\mathbf{\Delta}$ is an example of Reedy categories, where $\overrightarrow{\mathbf{\Delta}}$ is the subcategory whose morphisms are the injective functions, $\overleftarrow{\mathbf{\Delta}}$ is the subcategory whose morphisms are the surjective functions. The factorization exists (see Theorem A.1.3), and it is obvious to define a degree function meeting the requirements.

Definition 1.6.5. Let $\mathcal{C}$ be a Reedy category.

1. For each $\alpha \in \mathrm{Ob}(\mathcal{C})$ the latching category at α is the full subcategory $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ of $(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ containing all the objects but the identity of α. We will denote by

$$\pi^l_\alpha \colon \partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \to \mathcal{C}$$

the natural forgetful functor;

2. For each $\alpha \in \mathrm{Ob}(\mathcal{C})$ the matching category at α is the full subcategory $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ of $(\alpha \downarrow \overleftarrow{\mathcal{C}})$ containing all the objects but the identity of α. We will denote by

$$\pi^m_\alpha \colon \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \to \mathcal{C}$$

the natural forgetful functor.

Example 1.6.6. We compute the latching categories of $\mathbf{\Delta}$ at $\mathbf{n}$:

1. $n = 0$. In this case $(\overrightarrow{\mathbf{\Delta}} \downarrow \mathbf{0}) = \emptyset$, the empty category; therefore $\partial(\overrightarrow{\mathbf{\Delta}} \downarrow \mathbf{0}) = \emptyset$.

2. $n = 1$. In this case

$$(\overrightarrow{\mathbf{\Delta}} \downarrow \mathbf{1}) = \{\bullet \to \bullet \leftarrow \bullet\}$$

the pullback category. It follows that

$$\partial(\overrightarrow{\mathbf{\Delta}} \downarrow \mathbf{1}) = \{\bullet, \bullet\}$$

which is the coproduct category.

3. If $n \geq 2$, we claim that the full subcategory of $\mathcal{C} := \partial(\overrightarrow{\mathbf{\Delta}} \downarrow \mathbf{n})$ whose objects are the arrows $\mathbf{n-1} \to \mathbf{n}$ and $\mathbf{n-2} \to \mathbf{n}$ is final in the latching category. In fact, if we are given an arrow $f : \mathbf{m} \to \mathbf{n}$, different from the identity of $\mathbf{n}$ and in $\overrightarrow{\mathbf{\Delta}}$, then necessarily $m < n$. If $m = n-1$ or $m = n-2$, the identity of $\mathbf{m}$ is an initial object for $(f \downarrow \mathcal{C})$; if $m < n-2$, Theorem A.1.3 allows to write

$$f = d^{i_1} \circ \ldots \circ d^{i_s}, \quad s = n - m + 1$$

If $g : \mathbf{m} \to \mathbf{k}$ is an arrow in $(f \downarrow \mathcal{C})$, cosimplicial identities imply that g is connected to

$$d^{i_{s-k}} \circ \ldots \circ d^{i_s}$$

Thus $(f \downarrow \mathcal{C})$ is connected.

Definition 1.6.7. Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a model category. Let $\mathbf{X} \colon \mathcal{C} \to \mathcal{M}$ be a diagram of shape $\mathcal{C}$ in $\mathcal{M}$. Therefore:

1. For each $\alpha \in \mathrm{Ob}(\mathcal{C})$ the latching object of $\mathbf{X}$ at α is

$$\mathrm{L}_\alpha \mathbf{X} := \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)} \mathbf{X} \circ \pi^l_\alpha$$

2. For each $\alpha \in \mathrm{Ob}(\mathcal{C})$ the matching object of $\mathbf{X}$ at α is

$$\mathrm{M}_\alpha \mathbf{X} := \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} \mathbf{X} \circ \pi_\alpha^m$$

We are now ready to introduce the Reedy model structure:

Definition 1.6.8. Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a model category. Let $\mathbf{X}, \mathbf{Y}$ be $\mathcal{C}$-diagrams in $\mathcal{M}$; let $f : \mathbf{X} \to \mathbf{Y}$ be a map of diagrams. We will say that

1. for each $\alpha \in \mathrm{Ob}(\mathcal{C})$, define the relative latching map at α to be the arrow

$$\mathbf{X}_\alpha \sqcup_{\mathrm{L}_\alpha \mathbf{X}} \mathrm{L}_\alpha \mathbf{Y} \to \mathbf{Y}_\alpha$$

2. for each $\alpha \in \mathrm{Ob}(\mathcal{C})$, define the relative matching map at α to be the arrow

$$\mathbf{X}_\alpha \to \mathbf{Y}_\alpha \times_{\mathrm{M}_\alpha \mathbf{Y}} \mathrm{M}_\alpha \mathbf{X}$$

Theorem 1.6.9. Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a model category. Then $\mathcal{M}^{\mathcal{C}}$ has a model structure where:

1. a map is a weak equivalence is a pointwise weak equivalence;
2. a map is a fibration if for each $\alpha \in \mathrm{Ob}(\mathcal{C})$ the relative matching map is a fibration in $\mathcal{M}$;
3. a map is a cofibration if for each $\alpha \in \mathrm{Ob}(\mathcal{C})$ the relative latching map is a cofibration in $\mathcal{M}$.

Proof. See [Hir03, Theorem 15.3.4]. □

Definition 1.6.10. We will refer to the model structure introduced in Theorem 1.6.9 as the *Reedy model structure*.

Proposition 1.6.11. Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a model category. A Reedy fibration in $\mathcal{M}^{\mathcal{C}}$ is always an objectwise fibration. Dually, a Reedy cofibration in $\mathcal{M}^{\mathcal{C}}$ is always an objectwise cofibration.

Proof. We will prove the second statement. Observe that the map $f_\alpha : X_\alpha \to Y_\alpha$ can be factored as

$$\mathbf{X}_\alpha \to \mathbf{X}_\alpha \sqcup_{\mathrm{L}_\alpha \mathbf{X}} \mathrm{L}_\alpha \mathbf{Y} \to \mathbf{Y}_\alpha$$

can be

See [Hir03, Proposition 15.3.11]. □

Recall from our discussion 1.5 about homotopy pushout that the diagonal functor $\Delta : \mathcal{M} \to \mathcal{M}^{\mathcal{C}}$ plays a major role: if it takes fibrant objects to fibrant objects, then it is a right Quillen functor, and $\mathrm{colim} : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$ has a total left derived functor. This situation is (obviously) important; we give it a name:

Definition 1.6.12. Let $\mathcal{C}$ be a Reedy category.

1. we say that $\mathcal{C}$ has cofibrant constants if for every model category $\mathcal{M}$ the diagonal functor $\Delta : \mathcal{M} \to \mathcal{M}^{\mathcal{C}}$ takes cofibrant objects to (Reedy) cofibrant objects;
2. we say that $\mathcal{C}$ has fibrant constants if for every model category $\mathcal{M}$ the diagonal functor $\Delta : \mathcal{M} \to \mathcal{M}^{\mathcal{C}}$ takes fibrant objects to (Reedy) fibrant objects.

Proposition 1.6.13. Let $\mathcal{C}$ be a Reedy category.

1. $\mathcal{C}$ has cofibrant constants if and only if for every $\alpha \in \mathrm{Ob}(\mathcal{C})$ the latching category $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ is either connected or empty;

2. $\mathcal{C}$ has fibrant constants if and only if for every $\alpha \in \mathrm{Ob}(\mathcal{C})$ the matching category $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ is either connected or empty.

Proof. The second statement is the dual of the first one. Assume that every latching category is connected or empty; let $\mathcal{M}$ be a model category and fix $A \in \mathrm{Ob}(\mathcal{M})$. Let $\mathbf{X}_A$ be the constant diagram at A. Then the latching maps at α for $\mathbf{X}_A$ is either $\emptyset \to A$ or the identity of A. In particular $\mathbf{X}_A$ is cofibrant if A is cofibrant.

Conversely, if $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ has λ connected components, and A is a simplicial set, the latching map of $\mathbf{X}_A$ at α is $A^{\sqcup \lambda} \to A$, which is not a monomorphism, hence it is not a cofibration. □

Corollary 1.6.14. Let $\mathcal{C}$ be a Reedy category. Then

1. $\mathcal{C}$ has cofibrant constants if and only if for every model category $\mathcal{M}$ the adjoint pair $(\Delta, \lim)$ is a Quillen pair;

2. $\mathcal{C}$ has fibrant constants if and only if for every model category $\mathcal{M}$ the adjoint pair (colim, Δ) is a Quillen pair.

Proof. We will prove *2.* If (colim, Δ) is a Quillen pair, then Δ takes fibrations to fibrations, and hence fibrant objects to fibrant objects. Thus in this case $\mathcal{C}$ has fibrant constants. Conversely, if $\mathcal{C}$ has cofibrant constants, and $p\colon X \to Y$ is a fibration, then Proposition 1.6.13 implies that the relative matching map of $\Delta(i)$ at each object α is either the identity map of X or isomorphic to p; in both cases it is a fibration. □

holim and hocolim

Using the machinery developed above, we easily obtain the following result:

Proposition 1.6.15. Let $\mathcal{C}$ be a Reedy category with fibrant constants. For every model category $\mathcal{M}$ the colimit functor $\mathrm{colim}\colon \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$ has a total left derived functor.

Proof. This is an immediate consequence of Theorem 1.5.11 and Corollary 1.6.14. □

Previous Proposition holds in particular when $\mathcal{C} = \mathbf{\Delta}$ and $\mathcal{C} = \mathbf{\Delta}^{\mathrm{op}}$, i.e. for simplicial and cosimplicial diagrams. As in the case of homotopy pushouts, the problem becomes to find a representative for $\mathbf{L}\,\mathrm{colim}$ behaving well with respect to weak equivalences. This is absolutely not trivial: for the general case, we will need more theory (in particular, the tool of simplicial and cosimplicial resolutions). However, when the model category $\mathcal{M}$ is simplicial, there are well-known formulas due to Bousfield and Kan mimicking the constructions for topological spaces.

Remark 1.6.16. To conclude this too short introduction to homotopy limits and colimits, we have at least to say something about the $(\infty, 1)$-categorical context. Lurie proposes in [HTT] a definition for homotopy limits and colimits for quasicategories. This definition (due to Joyal, see [Joy05]) has the advantage of being extremely compact and it doesn't require much machinery to be formulated. Moreover, it specializes to our construction when it makes sense (i.e. when the quasicategory is presented by a model category). We refer to successive chapters for a more detailed exposition.

1.7 Mapping spaces

Informal ideas and motivations

As we already said, the homotopy category of a model category $\mathcal{M}$ describes only a small part of the information contained in $\mathcal{M}$; intuitively, we can say that $\mathrm{Ho}(\mathcal{M})$ extracts the 1-homotopical information in $\mathcal{M}$. There are other invariants (under Quillen equivalence, let's say) which represent an attempt to describe the higher homotopical content of $\mathcal{M}$. In this section we will concerned with one of them: the mapping spaces.

If one has some familiarity with the language of $(\infty,1)$-categories in the sense of Lurie (cfr. [HTT]), he will know that the theory of quasicategories is equivalent in an appropriate sense to the theory of topological categories and to the theory of simplicial categories. In these two last cases, for each pair of objects X, Y we are given, by the very definition, a mapping space

$$\mathrm{Map}_{\mathcal{C}}(X,Y)$$

which is a topological space or a simplicial set. However, in the language of quasicategories, the existence of such object is not obvious at all. Lurie propose at least three different models for this space; we will choose one of them for sake of clarity (and leave the details of these constructions to successive seminars): if S is a quasicategory and x, y are vertices, define

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{Map}_S(x,y)) := \mathrm{Hom}^{x,y}_{\mathbf{sSet}}(\Delta^n \times \Delta^1, S) \tag{1.2}$$

where the apex means that we are considering only the maps f such that

$$f|_{\Delta^n \times 0} = x, \qquad f|_{\Delta^n \times 1} = y$$

It's reasonable that this choice produces a "good definition", provided some familiarity with the theory of simplicial sets. In fact, recall that the internal hom of **sSet** is defined

$$\mathbf{Hom}(K,S)_n := \mathrm{Hom}_{\mathbf{sSet}}(K \times \Delta^n, S)$$

Thus (1.2) is nothing but the subcomplex of

$$\mathbf{Hom}(\Delta^1, S)$$

spanned by those elements "starting from x" and "reaching y".

(Co)simplicial resolutions

We can try to explain the main idea behind the notion of (co)simplicial resolution with the following example: consider the model category **CGHaus** (say that fibrations are Serre's fibrations). Each topological space X carries a certain amount of homotopical informations, for example all the homotopy groups $\pi_n(X)$. Since

$$\pi_n(X) = [\mathbb{S}^n, X]$$

it should become clear the fact that to understand the homotopical informations in X we should understand the representable functors $[-,X]$ and $[X,-]$, i.e. the homotopy relations of maps starting from and arriving at X. Now, first order homotopies of maps from X to Y are identified with maps $X \times |\Delta^1| \to Y$; homotopies between homotopies are identified with maps $X \times |\Delta^2| \to Y$ and so on. From this point of view, $X \times |\Delta^n|$ contains the "n-homotopical informations" about X; we could say that $X \times |\Delta^n|$ is a "higher cylinder object" for X.

We wish to abstract these observations to the context of any model category $\mathcal{M}$. This is what we will do with (co)simplicial resolutions: in order to explain the transition, let us observe that for each (compactly generated Hausdorff) space X, the family

$$\mathbf{X}_* := \{X \times |\Delta^n|\}_{n\in\mathbb{N}}$$

is a cosimplicial object in **CGHaus**; moreover, we have an isomorphism

$$\mathbf{X}_*^0 \simeq X$$

But it is true even more: for each n we have a natural map (the projection):

$$\mathbf{X}_*^n = X \times |\Delta^n| \to X$$

which is a weak equivalence (the standard n-simplex $|\Delta^n|$ is contractible). This can be reformulated as follows: if $\mathrm{cc}_* X \in \mathbf{CGHaus}^{\prime}$ denotes the constant cosimplicial object at X, we have a natural map of cosimplicial objects

$$\mathbf{X}_* \to \mathrm{cc}_* X$$

which is an objectwise weak equivalence. Since $\boldsymbol{\Delta}$ is a Reedy category, we can consider the Reedy model structure on $\mathbf{CGHaus}^{\boldsymbol{\Delta}}$; in this way, we can reformulate our previous observation by saying that $\mathbf{X}_* \to \mathrm{cc}_* X$ is a (Reedy) weak equivalence.

This readily allows to generalize our constructions to a generic model category, in order to attain our goal:

Definition 1.7.1. Let $\mathcal{M}$ be a model category and let $A \in \mathrm{Ob}(\mathcal{M})$ be an object. A cosimplicial resolution of A is a cofibrant approximation $\widetilde{\mathbf{A}} \to \mathrm{cc}_* A$ in the Reedy model category $\mathcal{M}^{\boldsymbol{\Delta}}$. Dually a simplicial resolution of A is a fibrant approximation $\mathrm{cs}_* A \to \widehat{\mathbf{A}}$ in the Reedy model category $\mathcal{M}^{\boldsymbol{\Delta}^{\mathrm{op}}}$.

The following proposition shows a first hint of the rightness of this definition: cosimplicial resolutions effectively encode homotopical informations about the objects of $\mathcal{M}$.

Proposition 1.7.2. Let $\mathcal{M}$ be a model category. If $X \in \mathrm{Ob}(\mathcal{M})$ is an object and $\widetilde{\mathbf{X}} \to \mathrm{cc}_* X$ is a cosimplicial resolution, then $\widetilde{\mathbf{X}}^0 \to X$ is a cofibrant approximation and

$$\widetilde{\mathbf{X}}^0 \sqcup \widetilde{\mathbf{X}}^0 \xrightarrow{d^0 \sqcup d^1} \widetilde{\mathbf{X}}^1 \xrightarrow{s^0} \widetilde{\mathbf{X}}^0 \tag{1.3}$$

is a cylinder object for $\widetilde{\mathbf{X}}^0$.

Proof. The first assertion follows from Proposition 1.6.11; let's check the second one. Cosimplicial identities[3] implies that (1.3) gives a factorization of the fold map. The 2-out-of-3 axiom immediately implies that s^0 is a weak equivalence. The computations we did in Example 1.6.6 show that the latching object at $\mathbf{1}$ of $\widetilde{\mathbf{X}}$ is exactly $\mathrm{L}_{\mathbf{1}}\widetilde{\mathbf{X}} = \widetilde{\mathbf{X}}^0 \sqcup \widetilde{\mathbf{X}}^0$, and the map

$$\mathrm{L}_{\mathbf{1}}\widetilde{\mathbf{X}} \to \widetilde{\mathbf{X}}^1$$

is precisely $d^0 \sqcup d^1$. Since $\mathrm{cc}_*\emptyset \to \widetilde{X}$ is a Reedy cofibration, the relative latching map

$$\emptyset \sqcup_{\emptyset} \mathrm{L}_{\mathbf{1}}\widetilde{\mathbf{X}} \to \widetilde{\mathbf{X}}^1$$

is identified with the previous map $d^0 \sqcup d^1$; it follows that this map is a cofibration. □

[3] See (A.3).

Factorization axiom **MC5** allows to produce simplicial and cosimplicial resolutions in a functorial way; Proposition 1.3.24 and its Corollaries shows moreover that they are unique up to (Reedy) weak equivalences. Henceforth, from a homotopical point of view, it doesn't matter the choice of one function complex instead of another one. In any case, it's often useful to fix a specific (co)simplicial resolution functor when dealing with constructions that make a massive use of them. Therefore, we will consider sometimes model categories *equipped* with a given (co)simplicial resolution functor (as, when dealing with colimits, one fix a specified left adjoint to the diagonal functor):

Function complexes

Definitions

We can now use the machinery of (co)simplicial resolutions to define function complexes in every model category. The goal, as we explained in the brief introduction to this section, is to construct for each pair of objects $A, B \in \mathrm{Ob}(\mathcal{M})$,a simplicial set $\mathrm{Map}(A, B)$ encoding the homotopy structure of the maps from A to B. When the model category is simplicial, this is done without efforts; the remarkable fact, is that (co)simplicial resolutions allows to construct a space of morphism for each model category (even though these spaces need not to be "composable"). The present construction will remind to the reader several constructions in homological algebra; however, it's not the best approach to the subject. We will see later on that the theory of simplicial localization developed by Dwyer and Kan in the series of articles [DK80c], [DK80a] and [DK80b] has a natural by-product a natural theory of function complexes, (homotopically) equivalent to the one we are developing *hic et nunc*.

Fix a model category $\mathcal{M}$. Let $\mathbf{X}$ be a cosimplicial object in $\mathcal{M}$, let $Y \in \mathrm{Ob}(\mathcal{M})$. Since $\mathcal{M}(-, Y)$ is a contravariant functor, we obtain a simplicial set $\mathcal{M}(\mathbf{X}, Y)$. Similarly, if $\mathbf{Y}$ is a simplicial object in $\mathcal{M}$ and $X \in \mathrm{Ob}(\mathcal{M})$, then $\mathcal{M}(X, \mathbf{Y})$ is a simplicial set. Finally, if $\mathbf{X}$ is a cosimplicial object in $\mathcal{M}$ and $\mathbf{Y}$ is a simplicial object in $\mathcal{M}$, then we obtain a bifunctor

$$\mathcal{M}(\mathbf{X}, \mathbf{Y}) \colon \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{Set}$$

which is a bisimplicial set. We can extract a simplicial set taking the diagonal:

$$(\mathrm{diag}\, \mathcal{M}(\mathbf{X}, \mathbf{Y}))_n := \mathcal{M}(\mathbf{X}^n, \mathbf{Y}_n)$$

Remark 1.7.3. Fix a cosimplicial object $\mathbf{X}$ in $\mathcal{M}$. Then we obtain a functor

$$\mathcal{M}(\mathbf{X}, -) \colon \mathcal{M} \to \mathbf{sSet}$$

It can be shown that this functor has a left adjoint, the so-called realization. Dually, for each simplicial object $\mathbf{Y}$ in $\mathcal{M}$, the functor

$$\mathcal{M}(-, \mathbf{Y}) \colon \mathcal{M} \to \mathbf{sSet}$$

has a right adjoint, the corealization. We won't need this construction, so we refer the reader to [Hir03, Ch. 16.3] for the details.

We are principally interested in two kind of function complexes: the *left* and *righ* function complexes; however, in order to compare them (and show that they gives rise to the same object up to homotopy) we will need something lying in between.

Definition 1.7.4. Let $\mathcal{M}$ be a model category and let X, Y be objects in $\mathcal{M}$. A *left homotopy function complex* from X to Y is a triple

$$\left(\widetilde{\mathbf{X}}, \widehat{Y}, \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{Y})\right)$$

where

1. $\widetilde{\mathbf{X}} \to \mathrm{cc}_* X$ is a cosimplicial resolution of X;
2. $Y \to \widehat{Y}$ is a fibrant approximation of Y;
3. $\mathcal{M}(\widetilde{\mathbf{X}}, \widehat{Y})$ is the simplicial set defined above.

Definition 1.7.5. Let $\mathcal{M}$ be a model category and let X, Y be objects in $\mathcal{M}$. A *right homotopy function complex* from X to Y is a triple

$$\left(\widetilde{X}, \widehat{\mathbf{Y}}, \mathcal{M}(\widetilde{X}, \widehat{\mathbf{Y}})\right)$$

where

1. $\widetilde{X} \to X$ is a cofibrant approximation of X;
2. $\mathrm{cs}_* Y \to \widehat{\mathbf{Y}}$ is a simplicial resolution of Y;
3. $\mathcal{M}(\widetilde{X}, \widehat{\mathbf{Y}})$ is the simplicial set defined above.

Finally:

Definition 1.7.6. Let $\mathcal{M}$ be amodel category and let X, Y be objects in $\mathcal{M}$. A *two-sided homotopy function complex* from X to Y is a triple

$$\left(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}, \operatorname{diag} \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})\right)$$

where

1. $\widetilde{\mathbf{X}} \to \mathrm{cc}_* X$ is a cosimplicial resolution of X;
2. $\mathrm{cs}_* Y \to \widehat{\mathbf{Y}}$ is a simplicial resolution Y;
3. $\operatorname{diag} \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})$ is the diagonal of the bisimplicial set defined above.

As we were saying in the brief introduction, these definitions remind the construction methods usually employed in homological algebra. However, from our higher homotopical viewpoint, we have some properties that those definitions should satisfy, in order to be *good* definitions. For example, if we think model categories as presentations of a homotopy theory, the maps between two objects A and B should be organized in a "space", and the path-components of this space should be in bijection with homotopy classes of maps from A to B. We will show that both these properties are satisfied; we begin with showing that function complexes gives rise to a space, and we discuss homotopy properties later on.

Remark 1.7.7. The word "space" is among the most overloaded ones in literature. Here, we shall follow Lurie: a space is either a (compactly generated and Hausdorff) topological space or a Kan complex. We don't want to really distinguish between them, so we will use the generic word space.

Proposition 1.7.8. If $\mathcal{M}$ is a model category and X and Y are objects of $\mathcal{M}$ then:

1. each left homotopy function complex from X to Y is a fibrant simplicial set;
2. each right homotopy function complex from X to Y is a fibrant simplicial set;
3. each two-sided homotopy function complex from X to Y is a fibrant simplicial set.

Proof. The proof requires some work about the realization and corealization which we haven't developed here. We refer to [Hir03, Ch. 16.3] for a treatment of those techniques and to [Hir03, Propositions 17.1.3, 17.2.3, 17.3.2] for a proof of our statements. □

Comparing function complexes

Let $\mathcal{M}$ be a model category and let X, Y be objects in $\mathcal{M}$. Let

$$\left(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}, \operatorname{diag} \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})\right) \tag{1.4}$$

be a two-sided homotopy function complex from X to Y. Then Proposition 1.7.2 implies that

$$\widetilde{\mathbf{X}}^0 \to X$$

is a cofibrant approximation to X. Therefore

$$\left(\widetilde{\mathbf{X}}^0, \widehat{\mathbf{Y}}, \mathcal{M}(\widetilde{\mathbf{X}}^0, \widehat{\mathbf{Y}})\right)$$

is a right homotopy function complex from X to Y. Moreover we have a canonical map

$$\widetilde{\mathbf{X}} \to \mathrm{cc}_* \widetilde{\mathbf{X}}^0$$

which induces a morphism

$$\operatorname{diag} \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}) \to \mathcal{M}(\widetilde{\mathbf{X}}^0, \widehat{\mathbf{Y}}) \tag{1.5}$$

One can prove:

Theorem 1.7.9. The map (1.5) is a weak equivalence of simplicial sets.

Proof. See [Hir03, Proposition 17.4.6]. □

Similarly, starting from the function complex (1.4) one see that

$$Y \to \widehat{\mathbf{Y}}_0$$

is a fibrant approximation to Y. Thus we obtain a left homotopy function complex

$$\left(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}_0, \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}_0)\right)$$

together with a map

$$\mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}_0) \to \operatorname{diag} \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}) \tag{1.6}$$

Theorem 1.7.10. The map (1.6) is a weak equivalence of simplicial sets.

Proof. See [Hir03, Proposition 17.4.6]. □

To conclude that all the definitions we gave of function complexes are (weakly) equivalent, one simply need to compare left (resp. right, two-sided) homotopy function complexes among themselves. However, using Corollary 1.3.26 it's easy to show that given two left homotopy function complexes from X to Y:

$$\left(\widetilde{\mathbf{X}}, \widehat{Y}, \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{Y})\right), \quad \left(\widetilde{\mathbf{X}}', \widehat{Y}', \mathcal{M}(\widetilde{\mathbf{X}}', \widehat{Y}')\right)$$

such that $\widetilde{\mathbf{X}}' \to \mathrm{cc}_* X$ and $\mathrm{cs}_* Y \to \widehat{\mathbf{Y}}$ are respectively a (Reedy) trivial fibration and a (Reedy) trivial cofibration, then there are (Reedy weak equivalences)

$$f : \widetilde{\mathbf{X}} \to \widetilde{\mathbf{X}}', \quad g : \widehat{\mathbf{Y}}' \to \widehat{\mathbf{Y}}$$

inducing a morphism

$$\mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}) \to \mathcal{M}(\widetilde{\mathbf{X}}', \widehat{\mathbf{Y}}') \tag{1.7}$$

Using again the machinery of resolutions and coresolutions one can prove

Theorem 1.7.11. The map (1.7) is a weak equivalence of simplicial sets.

Proof. See [Hir03, Proposition 17.1.10]. □

In view of the results of this section, we will adopt the following notation:

Notation. Let $\mathcal{M}$ be a model category and let $X, Y \in \mathrm{Ob}(\mathcal{M})$. We will denote by $\mathrm{map}_{\mathcal{M}}(X, Y)$ a function complex from X to Y. That is, we mean either a left homotopy function complex, or a right homotopy function complex or a two-sided function complex. Since we will be interested only in homotopical properties of such mapping spaces, no harm will arrive from this overloading of the symbol $\mathrm{map}_{\mathcal{M}}(X, Y)$.

Homotopy and Function complexes

We return now to the properties of function complexes. It remains to show that the path-connected components of $\mathrm{map}_{\mathcal{M}}(X, Y)$ are in bijection with the homotopy classes of maps from X to Y.

Lemma 1.7.12. Let $\mathcal{M}$ be a model category.

1. If $\widetilde{\mathbf{A}}$ is a cosimplicial resolution of an object $A \in \mathrm{Ob}(\mathcal{M})$ and X is a fibrant object, then there is a natural bijection between $\pi_0\mathcal{M}(\widetilde{\mathbf{A}}, X)$ and $\pi(\widetilde{\mathbf{A}}^0, X)$.

2. If $\widehat{\mathbf{X}}$ is a simplicial resolution of an object $X \in \mathrm{Ob}(\mathcal{M})$ and A is a cofibrant object, then there is a natural bijection between $\pi_0\mathcal{M}(A, \widehat{\mathbf{X}})$ and $\pi(A, \widehat{\mathbf{X}})$.

Proof. An element in $\pi_0\mathcal{M}(\widetilde{\mathbf{A}}, X)$ is represented by a 0-simplex of $\mathcal{M}(\widetilde{\mathbf{A}}, X)$, i.e. an element

$$f \in \mathcal{M}(\widetilde{\mathbf{A}}^0, X)$$

If g is another element there representing the same class of f, there is a 1-simplex α of $\mathcal{M}(\widetilde{\mathbf{A}}, X)$ such that $d_0(\alpha) = f$ and $d_1(\alpha) = g$. By definition we have:

$$\alpha \in \mathcal{M}(\widetilde{\mathbf{A}}^1, X)$$

and $d_0(\alpha) = \alpha \circ d^0$, $d_1(\alpha) = \alpha \circ d^1$, where d^0, d^1 are the coface maps of $\widetilde{\mathbf{A}}$. Proposition 1.7.2 shows that $\widetilde{\mathbf{A}}^1$ is a cylinder object for $\widetilde{\mathbf{A}}^0$, i.e. the 1-simplex α induces a left homotopy between f and g. Conversely, every left homotopy between f and g defines an element $\beta \in \mathcal{M}(\widetilde{\mathbf{A}}^1, X)$ satisfying $d_0(\beta) = f$ and $d_1(\beta) = g$. Therefore we obtain a well defined map

$$\pi_0\mathcal{M}(\widetilde{\mathbf{A}}, X) \to \pi(\widetilde{\mathbf{A}}^0, X)$$

this map is obviously surjective, and we showed above that it's also injective. Naturality is clear, and the second statement of the lemma is the dual of this one. □

Proposition 1.7.13. Let $\mathcal{M}$ be a model category. If X, Y are objects in $\mathcal{M}$ and $\mathrm{map}_{\mathcal{M}}(X, Y)$ is a function complex from X to Y, $\pi_0\mathrm{map}_{\mathcal{M}}(X, Y)$ is naturally isomorphic to the set of maps from X to Y in $\mathrm{Ho}(\mathcal{M})$.

Proof. We are reduced to the case where X and Y are both fibrant and cofibrant. Now the result follows from Lemma 1.7.12. □

To conclude this brief discussion about function complexes, we want to discuss a lifting criterion and a recognition lemma.

Lemma 1.7.14. Let $\mathcal{M}$ be a model category.

1. if A is a cofibrant object and $p\colon X \to Y$ is a map of fibrant objects that induces a weak equivalence of homotopy function complexes
$$p_*\colon \mathrm{map}_{\mathcal{M}}(A,X) \to \mathrm{map}_{\mathcal{M}}(A,Y)$$
then p induces an isomorphism $p_*\colon \pi(A,X) \to \pi(A,Y)$.
2. if X is a fibrant object and $i\colon A \to B$ is a map of cofibrant objects that induces a weak equivalence of homotopy function complexes
$$i^*\colon \mathrm{map}_{\mathcal{M}}(B,X) \to \mathrm{map}_{\mathcal{M}}(A,X)$$
then i induces an isomorphism $i^*\colon \pi(B,X) \to \pi(A,X)$.

Proof. Let $\widetilde{\mathbf{A}}$ be a cosimplicial resolution of A. Then p induces a weak equivalence
$$p_*\colon \mathcal{M}(\widetilde{\mathbf{A}},X) \to \mathcal{M}(\widetilde{\mathbf{A}},Y)$$
and thus, using Lemma 1.7.12, we obtain an isomorphism
$$p_*\colon \pi(\widetilde{\mathbf{A}}^0,X) \to \pi(\widetilde{\mathbf{A}}^0,Y)$$
Since $\widetilde{\mathbf{A}}^0 \to A$ is a weak equivalence of cofibrant objects, the thesis follows from Lemma 1.3.18. The other statement is dual. □

The following theorem is the reformulation of the lifting criterion in term of function complexes:

Theorem 1.7.15. If A is cofibrant and $p\colon X \to Y$ is a fibration between fibrant objects such that the induced map of function complexes $p_*\colon \mathrm{map}_{\mathcal{M}}(A,X) \to \mathrm{map}_{\mathcal{M}}(A,Y)$ is a weak equivalence, then for any map $f\colon A \to Y$ there is a map $g\colon A \to X$ such that $f = pg$.

Proof. This is a consequence of the standard lifting criterion and Lemma 1.7.14. □

Finally, the recognition result:

Theorem 1.7.16. If $\mathcal{M}$ is a model category and $g\colon X \to Y$ is an arrow, the following statements are equivalent:

1. g is a weak equivalence;
2. for every (cofibrant) object W the map induces a weak equivalence of function complexes $g_*\colon \mathrm{map}_{\mathcal{M}}(W,X) \to \mathrm{map}_{\mathcal{M}}(W,Y)$;
3. for every (fibrant) object Z the map induces a weak equivalence of function complexes $g^*\colon \mathrm{map}_{\mathcal{M}}(X,Z) \to \mathrm{map}_{\mathcal{M}}(Y,Z)$.

Proof. If g is a weak equivalence, results on cosimplicial resolutions imply the thesis (see [Hir03, Theorem 17.6.3]). Conversely, assuming for example *2.*, we can choose $W = X$ and $W = Y$. Let $\widetilde{g}\colon \widetilde{X} \to \widetilde{Y}$ be a cofibrant approximation to g and let $\widehat{g}\colon \widehat{X} \to \widehat{Y}$ be a fibrant approximation to $\widetilde{g}$. Then
$$\widehat{g}_*\colon \mathrm{map}_{\mathcal{C}}(\widehat{X},\widehat{X}) \to \mathrm{map}_{\mathcal{C}}(\widehat{X},\widehat{Y}), \quad \widehat{g}_*\colon \mathrm{map}_{\mathcal{C}}(\widehat{Y},\widehat{X}) \to \mathrm{map}_{\mathcal{C}}(\widehat{Y},\widehat{Y})$$
are isomorphisms. Lemma 1.7.14 implies that $\widehat{g}$ induces isomorphisms
$$\widehat{g}_*\colon \pi(\widehat{X},\widehat{X}) \to \pi(\widehat{X},\widehat{Y}), \quad \widehat{g}_*\colon \pi(\widehat{Y},\widehat{X}) \to \pi(\widehat{Y},\widehat{Y})$$
Therefore $\widehat{g}_*$ is a homotopy equivalence; since $\widehat{g}$ is an arrow between fibrant-cofibrant objects, it is a weak equivalence by Whitehead's theorem 1.3.22. It follows that $\widetilde{g}$ is a weak equivalence, so that g is a weak equivalence too. □

Examples

The main computational tool is given by the following result:

Lemma 1.7.17. If $\mathcal{M}$ is a simplicial model category, then for each cofibrant object X, $\{X \otimes \Delta^n\}_{n\in\mathbb{N}}$ is a cosimplicial resolution of X.

In this way we can say that:

1. for **sSet**, $\mathrm{map}(X,Y)$ is just $\mathbf{hom}(\widetilde{X},\widehat{Y})$, where $\widetilde{X}$ is a cofibrant approximation to X and $\widehat{Y}$ is a fibrant approximation to Y;
2. for $\mathbf{Ch}(R)$, we can exploit the natural simplicial structure: n-simplices of $\mathbf{sSet}(E,F)$ are the chain maps of degree n, i.e. elements of $\mathrm{Hom}(E,F[-i])$. It follows that $\pi_i\mathrm{map}(E,F) = \pi_0\mathrm{map}(E,F[-i]) = \mathrm{Hom}_{D(R)}(E,F[-i])$ (using Proposition 1.7.13).

Hammock localization

Mapping spaces II

1.8 Bousfield localization

Localization of model categories

We remarked several times that our constructions are done accordingly to the philosophy that a model category should be a presentation of a homotopy theory. Again, to understand Bousfield localization, the reader should keep in mind this philosophy.

We try to give an informal explanation of Bousfield localization: let's start with a model category $\mathcal{M}$. We saw in Section 1.3 that we can associate to $\mathcal{M}$ its homotopy category $\mathrm{Ho}(\mathcal{M})$, carrying the first order homotopy informations contained in $\mathcal{M}$. It is sometimes useful to further localize $\mathrm{Ho}(\mathcal{M})$ (see below for a concrete example); however, as we pointed out in our discussion 1.5, working at the level of $\mathrm{Ho}(\mathcal{M})$ can be dangerous, because many higher order informations are forgotten. Bousfield localization is an attempt to solve this problem: localizing the homotopy category, without losing higher homotopical data. This is achieved building another model category, whose homotopy category gives back the desired localization of $\mathrm{Ho}(\mathcal{M})$; the additional data in the new model category represents the higher order informations we didn't want to forget.

If this idea is clear, then the following definition won't surprise the reader: since we agreed that the correct result should be a model category, we will look for a universal property among them; however, we can at least consider two different kind of maps, i.e. left Quillen functors and right Quillen functor. It makes sense to consider them both, and the results will be different, in general; therefore we will make a distinction between left localization and right localization.

Definition 1.8.1. Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of arrows in $\mathcal{M}$.

1. A *left localization of* $\mathcal{M}$ *with respect to* $\mathcal{S}$ is a pair $(\mathrm{L}_{\mathcal{S}}\mathcal{M}, j)$ universal among pairs $(\mathcal{N},\varphi)$, where $\mathcal{N}$ is a model category and $\varphi : \mathcal{M} \to \mathcal{N}$ is a left Quillen functor such that its total left derived functor
$$\mathbf{L}\varphi : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$$
takes the images of elements of $\mathcal{S}$ into isomorphisms in $\mathrm{Ho}(\mathcal{N})$.
2. A *right localization of* $\mathcal{M}$ *with respect to* $\mathcal{S}$ is a pair $(\mathrm{R}_{\mathcal{S}}\mathcal{M}, j)$ universal among pairs $(\mathcal{N},\varphi)$ where $\mathcal{N}$ is a model category and $\varphi : \mathcal{M} \to \mathcal{N}$ is a right Quillen functor such that its total right derived functor
$$\mathbf{L}\varphi : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$$

takes the images of elements of $\mathcal{S}$ into isomorphisms in $\mathrm{Ho}(\mathcal{N})$.

The subtlety hidden in this universal property is that $\mathcal{S}$, being absolutely generic, certainly doesn't need to be saturated with respect to the construction we are interested in. For example, if we are considering left localizations, for each pair $(\mathcal{N}, \varphi)$ satisfying the characteristic property, $\mathbf{L}\varphi$ will take cofibrant approximations to elements in $\mathcal{S}$ to isomorphisms. Therefore the saturation with respect to the localization process of $\mathcal{S}$ contains at least all the cofibrant approximations to the elements of $\mathcal{S}$. To understand better this situation, we would like to characterize abstractly the property of being saturated. We begin with a couple of definitions:

Definition 1.8.2. Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of arrows in $\mathcal{M}$. We will say that an object W is $\mathcal{S}$-local if it is fibrant and for every $f : A \to B$ in $\mathcal{S}$ the induced morphism of function complexes

$$f^*\colon \mathrm{map}_{\mathcal{M}}(B, W) \to \mathrm{map}_{\mathcal{M}}(A, W)$$

is a weak equivalence of simplicial sets.

Remark 1.8.3. We can say that a $\mathcal{S}$-local object is an object that make the arrows in $\mathcal{S}$ "look like weak equivalences". See Theorem 1.7.16 to understand where the intuition comes from.

Definition 1.8.4. Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of arrows in $\mathcal{M}$. We will say that an arrow $f : X \to Y$ is a $\mathcal{S}$-local equivalence if for every $\mathcal{S}$-local object W the induced map of function complexes

$$f^*\colon \mathrm{map}_{\mathcal{M}}(Y, W) \to \mathrm{map}_{\mathcal{M}}(X, W)$$

is a weak equivalence of simplicial sets.

Dually, we introduce:

Definition 1.8.5. Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of arrows in $\mathcal{M}$. We will say that an object W is $\mathcal{S}$-colocal if it is cofibrant and for every $f : A \to B$ the induced map of function complexes

$$f_*\colon \mathrm{map}_{\mathcal{M}}(W, A) \to \mathrm{map}_{\mathcal{M}}(W, B)$$

is a weak equivalence of simplicial sets.

Definition 1.8.6. Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of arrows in $\mathcal{M}$. We will say that an arrow $f : X \to Y$ is a $\mathcal{S}$-colocal equivalence if for every $\mathcal{S}$-colocal object W the induces map of function complexes

$$f_*\colon \mathrm{map}_{\mathcal{M}}(W, X) \to \mathrm{map}_{\mathcal{M}}(W, Y)$$

is a weak equivalence of simplicial sets.

The $\mathcal{S}$-local equivalences characterizes abstractly the saturation with respect to the localization process, as next theorem is going to show:

Theorem 1.8.7. Let $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$ be a Quillen pair. If $\mathcal{S}$ is a class of arrows in $\mathcal{M}$, the following are equivalent:

1. $\mathbf{L}F$ takes the images of elements of $\mathcal{S}$ into isomorphisms of $\mathrm{Ho}(\mathcal{N})$;
2. the functor F takes every cofibrant approximation to an element of $\mathcal{S}$ into a weak equivalence of $\mathcal{N}$;
3. the functor G takes every fibrant object of $\mathcal{N}$ into a $\mathcal{S}$-local object of $\mathcal{M}$;

4. the functor F takes every $\mathcal{S}$-local equivalence between cofibrant objects into a weak equivalence in $\mathcal{N}$.

Sketch of the proof. The construction of $\mathbf{L}F$ shows immediately that *1.* $\iff$ *2.*. Theorem 1.7.16 implies equivalence of *2.* with *3.* and of *3.* with *4.*. For the details, see [Hir03, Theorem 3.1.6]. □

Dually we have:

Theorem 1.8.8. Let $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$ be a Quillen pair. If $\mathcal{S}$ is a class of arrows in $\mathcal{M}$, the following are equivalent:

1. $\mathbf{R}G$ takes the images of elements of $\mathcal{S}$ into isomorphisms of $\mathrm{Ho}(\mathcal{N})$;
2. the functor G takes every fibrant approximation to an element of $\mathcal{S}$ into a weak equivalence of $\mathcal{N}$;
3. the functor F takes every cofibrant object of $\mathcal{M}$ into a $\mathcal{S}$-local object of $\mathcal{N}$;
4. the functor G takes every $\mathcal{S}$-local equivalence between fibrant objects into a weak equivalence in $\mathcal{M}$.

Proof. Dual of the proof of Theorem 1.8.7. □

Bousfield localization

For certain model categories and certain classes of maps, the localization is easier to describe. This is what Bousfield did in his work; in this situation, we will refer to the new model category as the left (resp. right) Bousfield localization. Here is the definition:

Definition 1.8.9. Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of arrows in $\mathcal{M}$. The *left Bousfield localization* of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category $\mathrm{L}_{\mathcal{S}}\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that:

1. the class of weak equivalences of $\mathrm{L}_{\mathcal{S}}\mathcal{M}$ equals the class of $\mathcal{S}$-local equivalences of $\mathcal{M}$;
2. the class of cofibrations of $\mathrm{L}_{\mathcal{S}}\mathcal{M}$ equals the class of cofibrations of $\mathcal{M}$;
3. the class of fibrations of $\mathrm{L}_{\mathcal{S}}\mathcal{M}$ equals the class of arrows with the RLP with respect to all the cofibrations which are also $\mathcal{S}$-local equivalences.

Dually:

Definition 1.8.10. Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of arrows in $\mathcal{M}$. The *right Bousfield localization* of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category $\mathrm{R}_{\mathcal{S}}\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that:

1. the class of weak equivalences of $\mathrm{R}_{\mathcal{S}}\mathcal{M}$ equals the class of $\mathcal{S}$-local equivalences of $\mathcal{M}$;
2. the class of fibrations of $\mathrm{R}_{\mathcal{S}}\mathcal{M}$ equals the class of fibrations of $\mathcal{M}$;
3. the class of cobrations of $\mathrm{R}_{\mathcal{S}}\mathcal{M}$ equals the class of arrows with the LLP with respect to all the fibrations which are also $\mathcal{S}$-local equivalences.

In general, the weak equivalences, fibrations and cofibrations described in the above definitions doesn't form a model structure. However, if they do, they define a left (resp. right) localization in the sense of Definition 1.8.1.

Theorem 1.8.11. Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of maps. If the left Bousfield localization defines a model structure $\mathrm{L}_{\mathcal{S}}\mathcal{M}$ on the underlying category of $\mathcal{M}$, then the identity functor $\mathcal{M} \to \mathrm{L}_{\mathcal{S}}\mathcal{M}$ is a left localization of $\mathcal{M}$ with respect to $\mathcal{S}$. Dually, if the right Bousfield localization defines a model structure $\mathrm{R}_{\mathcal{S}}\mathcal{M}$ on the underlying category of $\mathcal{M}$, then the identity functor $\mathcal{M} \to \mathrm{R}_{\mathcal{S}}\mathcal{M}$ is a right localization of $\mathcal{M}$ with respect to $\mathcal{S}$.

Proof. See [Hir03, Theorem 3.3.19]. □

Finally we state the main existence theorem for Bousfield localizations:

Theorem 1.8.12. Let $\mathcal{M}$ be a left proper cellular model category and let $\mathcal{S}$ be a set of arrows in $\mathcal{M}$. Then:

1. the left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{S}$ exists;
2. the fibrant objects of $\mathrm{L}_{\mathcal{S}}\mathcal{M}$ are the $\mathcal{S}$-local objects of $\mathcal{M}$;
3. $\mathrm{L}_{\mathcal{S}}\mathcal{M}$ is left proper cellular model category.

Proof. See [Hir03, Theorem 4.1.1]. □

A detailed example

To conclude this overview of the theory of Bousfield localizations we work out the details of an example that we will need in the Chapter 3.

First of all we invoke the following result:

Theorem 1.8.13. **sSet** is left proper and cellular

Proof. In **sSet** every object is cofibrant, hence it is left proper. For cellularity, see [Hir03, Proposition 12.1.4]. □

Using Theorem 1.8.12 we conclude the existence of the left Bousfield localization with respect to any set of maps. We consider the set formed only by the inclusion map

$$\alpha\colon \partial\Delta^3 \to \Delta^3$$

Definition 1.8.14. The S^2-nullification of **sSet** is the left Bousfield localization of **sSet** with respect to the map $\alpha\colon \partial\Delta^3 \to \Delta^3$. We will denote this model category as $(S^2)^{-1}\mathbf{sSet}$.

The goal of this example is to characterize the weak equivalences in $(S^2)^{-1}\mathbf{sSet}$, that is, the local equivalences with respect to α. More precisely, we want to prove the following result:

Proposition 1.8.15. The weak equivalences in $(S^2)^{-1}\mathbf{sSet}$ are exactly those maps inducing an isomorphism on π_0 and π_1 at all base points.

We will proceed in several steps. First of all, let's observe that, since we are considering a left Bousfield localization, the cofibrations are unchanged and weak equivalences extends the standard ones. In particular, it follows that for any object $Y \in \mathbf{sSet}$,

$$\{Y \times \Delta^n\}_{n\in\mathbb{N}}$$

is still a cosimplicial resolution of Y; therefore we conclude that for any pair of objects X and Y a left homotopy function complex is (still) given by

$$\mathbf{hom}(X, Y)$$

Lemma 1.8.16. The functor $- \times \Delta^1 : (S^2)^{-1}\mathbf{sSet} \to (S^2)^{-1}\mathbf{sSet}$ preserves weak equivalences.

Proof. Let $f : X \to Y$ be a S^2-local map. By definition, for any S^2-local object W the map

$$\mathbf{hom}(Y, W) \to \mathbf{hom}(X, W)$$

is a standard weak equivalence of simplicial sets. Using adjunction we obtain a commutative diagram

$$\begin{tikzcd}
\mathbf{hom}(Y \times \Delta^1, W) \arrow[r] \arrow[d] & \mathbf{hom}(X \times \Delta^1, W) \arrow[d] \\
\mathbf{hom}(Y, \mathbf{hom}(\Delta^1, W)) \arrow[r] & \mathbf{hom}(X, \mathbf{hom}(\Delta^1, W))
\end{tikzcd}$$

where the horizontal arrows are the natural ones and the vertical arrows are isomorphisms of simplicial sets. Since f is S^2-local, we will be done proving that $\mathbf{hom}(\Delta^1, W)$ is a S^2-local object, i.e. that

$$\mathbf{hom}(\Delta^3, \mathbf{hom}(\Delta^1, W)) \to \mathbf{hom}(\partial\Delta^3, \mathbf{hom}(\Delta^1, W))$$

is a weak equivalence. Reasoning as above, this is equivalent to show that the natural map

$$\mathbf{hom}(\Delta^3 \times \Delta^1, W) \to \mathbf{hom}(\partial\Delta^3 \times \Delta^1, W)$$

is a weak equivalence. However, the projection maps

$$\Delta^3 \times \Delta^1 \to \Delta^3, \quad \partial\Delta^3 \times \Delta^1 \to \Delta^3$$

are weak equivalences because the geometric realization functor commutes with products and $|\Delta^1|$ is contractible. Now, consider the commutative diagram:

$$\begin{tikzcd}
\mathbf{hom}(\Delta^3 \times \Delta^1, W) \arrow[r] & \mathbf{hom}(\partial\Delta^3 \times \Delta^1, W) \\
\mathbf{hom}(\Delta^3, W) \arrow[r] \arrow[u] & \mathbf{hom}(\partial\Delta^3, W) \arrow[u]
\end{tikzcd}$$

The bottom row is a weak equivalence by hypothesis, and the vertical maps are weak equivalences because they are induced by weak equivalences between cofibrant objects. The 2-out-of-3 axiom implies that also the top row is a weak equivalence, completing the proof. □

Lemma 1.8.17. In $(S^2)^{-1}\mathbf{sSet}$ the inclusion $\partial\Delta^n \to \Delta^n$ is a weak equivalence for every $n > 2$.

Proof. We proceed by induction, the case $n = 3$ being true by definition of left Bousfield localization. Assume that $\partial\Delta^n \to \Delta^n$ is a S^2-local equivalence. Then Lemma 1.8.16 implies that

$$\partial\Delta^n \times \Delta^1 \to \Delta^n \times \Delta^1$$

is a S^2-local equivalence. Consider the commutative diagram

$$\begin{tikzcd}
\partial\Delta^{n+1} \arrow[r] \arrow[d] & \Delta^{n+1} \arrow[d] \\
\partial\Delta^n \times \Delta^1 \arrow[r] & \Delta^n \times \Delta^1
\end{tikzcd}$$

Then the vertical arrows are weak equivalences because $|\Delta^1|$ is contractible, while the bottom row is a weak equivalence by the previous considerations. It follows from the 2-out-of-3 axiom that $\partial\Delta^{n+1} \to \Delta^{n+1}$ is a weak equivalence as well. □

Corollary 1.8.18. If W is a S^2-local object, for any $Y \in \mathbf{sSet}$ we have

$$\pi_n \mathbf{hom}(Y, W) = 0$$

for every $n \geq 2$.

Proof. Using repeatedly the adjuntion between $\mathbf{hom}(X, -)$ and $- \times X$, we see that the map

$$\mathbf{hom}(\Delta^n, \mathbf{hom}(Y, W)) \to \mathbf{hom}(\partial\Delta^n, \mathbf{hom}(Y, W)) \tag{1.8}$$

is isomorphic to the natural map

$$\mathbf{hom}(Y, \mathbf{hom}(\Delta^n, W)) \to \mathbf{hom}(Y, \mathbf{hom}(\partial\Delta^n, W))$$

Since for $n > 2$ the map

$$\mathbf{hom}(\Delta^n, W) \to \mathbf{hom}(\partial\Delta^n, W)$$

is a weak equivalence between fibrant objects, it follows that the map (1.8) is a weak equivalence. This implies that for $n \geq 2$ we have:

$$\pi_n \mathbf{hom}(Y, W) = 0$$

□

Now we are ready to prove the main result:

Proof of Proposition 1.8.15. Assume that $f : X \to Y$ is S^2-local and take a fibrant approximation $\widehat{f} : \widehat{X} \to \widehat{Y}$. Theorem 1.7.16 implies that for every (cofibrant) object A the induced map[4]

$$f_* : \mathbf{hom}(A, \widehat{X}) \to \mathbf{hom}(A, \widehat{Y})$$

is a weak equivalence. In particular, taking $A = \Delta^0$ (since every object in $(S^2)^{-1}\mathbf{sSet}$ is cofibrant), we deduce that $\widehat{f} : \widehat{X} \to \widehat{Y}$ is a standard weak equivalence.

Moreover, functoriality of π_0 and π_1 allows to reduce to the case of a map $f : X \to Y$ with Y fibrant. □

Theorem 1.8.19. The adjoint pair $\pi_f : \mathbf{sSet} \rightleftarrows \mathbf{Grpd} : N$ of Theorem **??** is a Quillen equivalence between **Grpd** and the S^2-nullification of **sSet**.

1.9 Complements to Chapter 1

Uniqueness of the model structure for Grpd

The goal of this section is to provide a proof of the following:[5]

Theorem 1.9.1. On the category of (small) groupoids there is a unique model structure in which the weak equivalences are the equivalences of groupoids.

[4] If X and Y are not fibrant, the assertion is trivially false: just consider $\partial\Delta^3 \to \Delta^3$. However, for fibrant objects we obtain exactly the function complex, hence the stated theorem applies.

[5] I learned this theorem from a post of Chris Schommer-Pries which can be found at `http://sbseminar.wordpress.com/2012/11/16/the-canonical-model-structure-on-cat/`. In this section I simply adapt the proof given there to the case of groupoids, as suggested by Mike Shulman in his answer.

We already described such a model structure for **Grpd** in the example 1.4. From now on, we will assume that **Grpd** is endowed with a generic model structure satisfying the hypothesis of Theorem 1.9.1. We will refer to the model structure already defined as the *canonical model structure* on **Grpd**.

Lemma 1.9.2. The map $\emptyset \to \mathbf{1}$ is a cofibration.

Proof. Choose a cofibrant approximation $(\mathcal{A}, i)$ for $\mathbf{1}$. Then for any functor $F\colon \mathbf{1} \to \mathcal{A}$, the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d] \\ \mathbf{1} \arrow[r, "F"] & \mathcal{A} \arrow[r] & \mathbf{1} \end{tikzcd}$$

commutes and express $\emptyset \to \mathbf{1}$ as a retract of $\emptyset \to \mathcal{A}$. □

Corollary 1.9.3. Acyclic fibrations are surjective on objects.

Proof. This is an easy consequence of the fact that acyclic fibrations have the RLP with respect to all cofibrations, hence in particular with respect to $\emptyset \to \mathbf{1}$. □

Corollary 1.9.4. (Acyclic) fibrations are a subset of canonical (acyclic) fibrations.

Proof. The statement for acyclic fibrations follows from Corollary 1.9.3. The other statement follows from this one because cofibrations must contain canonical cofibrations; hence acyclic cofibrations contain canonical acyclic cofibrations and so fibrations are a subset of canonical fibrations. □

Lemma 1.9.5. Let Δ^1 be the groupoid with two objects and exactly one isomorphism between them. If the cofibrations contain a map which is not a canonical cofibration, then the map $\Delta^1 \to \mathbf{1}$ is a cofibration.

Proof. Assume that the functor $F\colon \mathcal{A} \to \mathcal{B}$ is a cofibration but fails to be an injection on objects. Then we can choose $X, Y \in \mathrm{Ob}(\mathcal{A})$ such that $F(X) = F(Y)$; choose a functor $G\colon \mathcal{A} \to \Delta^1$ separating X and Y and form the pushout

$$\begin{tikzcd} \mathcal{A} \arrow[r, "G"] \arrow[d, "F"'] & \Delta^1 \arrow[d] \\ \mathcal{B} \arrow[r] & \mathcal{C} \end{tikzcd}$$

Then the functor $\Delta^1 \to \mathcal{C}$ is again a cofibration and sends the two objects of Δ^1 to the same object in $\mathcal{C}$. Let now $\mathcal{C}^\delta$ be the free connected groupoid generated by the objects of $\mathcal{C}$; we clearly have a unique functor $\mathcal{C} \to \mathcal{C}^\delta$ which is the identity on objects. This functor is a canonical cofibration, hence it is also a cofibration (cfr. the proof of Lemma 1.9.4). It follows that the map $\Delta^1 \to \mathcal{C}^\delta$ is a cofibration. But now we clearly have a retraction diagram

$$\begin{tikzcd} \Delta^1 \arrow[r, "\mathrm{id}"] \arrow[d] & \Delta^1 \arrow[r, "\mathrm{id}"] \arrow[d] & \Delta^1 \arrow[d] \\ \mathbf{1} \arrow[r] & \mathcal{C}^\delta \arrow[r] & \mathbf{1} \end{tikzcd}$$

which concludes the proof. □

Lemma 1.9.6. If the cofibrations contain the map $\Delta^1 \to \mathbf{1}$, then every fibrant object is gaunt (that is, every isomorphism is an identity).

Proof. The map $\Delta^1 \to \mathbf{1}$ is a weak equivalence. Therefore, the RLP of fibrations with respect to acyclic cofibrations concludes. $\square$

Now we are ready to prove the main theorem:

Proof of Theorem 1.9.1. Thanks to Corollary 1.9.4 we only need to prove that every cofibration is a canonical cofibration. Assume this is not the case and choose a groupoid $\mathcal{G}$ with one object and non-trivial automorphism group. A fibrant replacement for $\mathcal{G}$ will be gaunt by Lemma 1.9.5 and Lemma 1.9.6; but then $\mathcal{G}$ cannot be equivalent to its fibrant replacement, contradiction. The thesis follows. $\square$

The small object argument

Proper model categories

Definition 1.9.7. Let $\mathcal{M}$ be a model category.

1. $\mathcal{M}$ is said to be left proper if every pushout of a weak equivalence along a cofibration is a weak equivalence;

2. $\mathcal{M}$ is right proper if every pullback of a weak equivalence along a fibration is a weak equivalence;

3. $\mathcal{M}$ is proper if it is both left and right proper.

Proposition 1.9.8. Let $\mathcal{M}$ be a model category.

1. every pushout of a weak equivalence between cofibrant objects along a cofibration is a weak equivalence;

2. every pullback of a weak equivalence between fibrant objects along a fibration is a weak equivalence.

Proof. The second statement follows from the first by duality. Let's prove this one: let $f : A \to B$ be a weak equivalence between cofibrant objects and let $i : A \to C$ be a cofibration. $\square$

Corollary 1.9.9. Let $\mathcal{M}$ be a model category.

1. if every object of $\mathcal{M}$ is cofibrant, $\mathcal{M}$ is left proper;

2. if every object of $\mathcal{M}$ is fibrant, $\mathcal{M}$ is right proper;

3. if every object of $\mathcal{M}$ is both fibrant and cofibrant, $\mathcal{M}$ is proper.

Corollary 1.9.10. **sSet** and **Grpd** are left proper; **Top** is right proper.

Cofibrantly generated and combinatorial model categories

Definition 1.9.11. A *cofibrantly generated model category* is a model category $\mathcal{M}$ such that:

1. there exists a set I of maps that permits the small objects argument and such that a map is a trivial fibration if and only if it has the RLP with respect to all the maps in I;

2. there exists a set J of maps that permits the small objects argument and such that a map is a fibration if and only if it has the RLP with respect to all the maps in J.

Definition 1.9.12. A *combinatorial model category* is a cofibrantly generated model caetgory $\mathcal{M}$ which is moreover presentable.

Theorem 1.9.13. Let $\mathcal{M}$ be a combinatorial model category and let $\mathcal{C}$ be a small category. Then there exist two combinatorial model structure over $\mathcal{M}^{\mathcal{C}}$:

- the projective model structure, where weak equivalences and fibrations are defined objectwise;
- the injective model structure, where weak equivalences and cofibrations are defined objectwise.

Proof. See [HTT, Prop. A.2.8.2]. □

Cellular Model Categories

Definition 1.9.14. A *cellular model category* is a cofibrantly generated model category for which there are a set of generating cofibrations I and a set J of generating trivial cofibrations such that

1. both the domains and the codomains of the elements of I are compact;
2. the domains of the elements of J are small relative to I;
3. the cofibrations are effective monomorphisms.

Simplicial Model Categories

Simplicial model categories are well behaved for several reason: they have natural, preferred ways to construct homotopy limits and homotopy colimits, simplicial and cosimplicial resolutions. We could say that all such notions are already "built-in" in a simplicial model category.

The following proposition shows that the situation described in the discussion at the beginning of section 1.7 is common to every simplicial model category:

Proposition 1.9.15. Let $\mathcal{M}$ be a simplicial model category. If X is an object in $\mathcal{M}$ and $W \to X$ is a cofibrant approximation, then the cosimplicial object $\widetilde{\mathbf{W}}$ defined by

$$\widetilde{\mathbf{W}}^n := W \otimes \Delta^n$$

defines a cosimplicial resolution of $\mathbf{X}$.

Comparison for mapping spaces

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EXPOSÉ **2**

Simplicial localization

The goal of this paper is to present the simplicial localization as defined by Dwyer and Kan in their articles [DK80c], [DK80a] and [DK80b]. The reader is supposed to know the theory of model category and homotopy category.

Brice Le Grignou

2.1 Introduction

Localization and higher homotopy

Let M be a model category. The axioms of model categories imply more than just homotopy relations. For example, one can describe homotopy relations between homotopy relations (see [TV02, p. 2.3]). The homotopy localization loose this hidden data as it merges maps which are joined by an homotopy relation. We try here to describe this "higher homotopy structure".

We present first the homotopy function complexes and their relations with homotopy. Then we present the two Dwyer-Kan localizations which is the core of the paper. Finally a small theorem states that the two approach give us the same homotopy.

Notations and terminology

1. If C is a category and $X, Y \in Obj(C)$, then $C(X, Y)$ denotes the set of morphisms from X to Y in C
2. If C is a category, then sC is the category of simplicial objects over C
3. All the categories are small (except when specified). We do not deal of set theory or universes theory issues. This is motivated by the following theorem.
4. Simplicial categories: the usual definition of a simplicial category is an enrichment over the category of simplicial set. As the categories are supposed small, one can use the following presentation:
   a) If A is a simplicial category (with small set of objects); then let $A-cat$ the category whom objects are (small) categories with $Obj(A)$ as set of objects and whom morphisms are functors which sends an object to itself.
   b) Then A is nothing but a simplicial object over the category $A-cat$

5. A weak equivalence of simplicial categories $S : A \to B$ is a functor of simplicial categories such that

   a) If $\pi_0 A$ is the category with the same objects as A and such that $(\pi_0 A)(X,Y) = \pi_0(A(X,Y))$ then S induces an equivalence of (ordinary) categories $\pi_0 A \simeq \pi_0 B$ ($\pi_0 B$ is defined in the same way)

   b) $A(X,Y) \to B(SX,SY)$ is a weak homotopy equivalence.

6. In the previous definition, if S is nothing but a morphism in $sA-cat$, then only the second point holds.

7. In a model category we represent the cofrations and the the fibrations by respectively: $\hookrightarrow$ and $\twoheadrightarrow$. The weak equivalences are noted with a $\sim$.

Theorem 2.1.1 ([DK80b, Thm. 2.3])**.** Let A a simplicial category, and $B \in A$ a small simplicial subcategory. Then there exists a small simplicial subcategory B' such that $B \subseteq B' \subseteq A$ and $B'(X,Y) \to A(X,Y)$ is a weak homotopy equivalence for all $X,Y \in B'$

2.2 Motivation of the simplicial localization

Simplicial model categories

Definition 2.2.1. A simplicial model category M is a model category which is also a simplicial category; this means equivalently:

1. M is a simplicial enriched category, such as we have distinguished three classes of elements of the sets $Map(X,Y)$

2. M is an element of $sM-cat$ such as M_0 has a model structure

Furthermore, M respects the following axioms:

1. For every two objects X,Y of M, and every simplicial set K, there exists two objects $X \otimes K$ and Y^K of M such that we have isomorphisms of simplicial sets:

$$Map(X \otimes K, Y) \simeq Map_{sSet}(K, Map(X,Y)) \simeq Map(X, Y^K) \tag{2.1}$$

2. If $i : A \to B$ is a cofibration in M and $p : X \to Y$ is a fibration, then:

$$Map(B,X) \to Map(A,X) \times_{Map(A,Y)} Map(B,Y) \tag{2.2}$$

   is a fibration which is acyclic if either i or p is acyclic.

The simplicial model category is close to the ideal concept we want to approach from a simple model category in order to get the "higher order information" which is contained in the mapping spaces. We will see how this property happens.

Homotopy function complexes

Let M be a model category, and W a the subcategory of weak equivalence.

Definition 2.2.2. Let Y an object of M. A simplicial resolution of Y is a simplicial object over M noted Y_* together with a weak equivalence $Y \to Y_0$ such that

1. the object Y_0 is fibrant
2. all faces maps in Y_* are acyclic fibrations. Hence, all the objects Y_n are fibrants.
3. Let (d_*, Y_n) the diagram:
   a) for every $0 \le i \le n+1$, a copy $(d_i; Y_n)$ of Y_n
   b) for every $0 \le i < j \le n+1$, a copy $(d_i d_j, Y_{n-1})$ of Y_{n-1}
   c) pair of maps: $(d_j, Y_n) \to (d_i d_j, Y_{n-1}) \leftarrow (d_j, Y_n)$. These arrows are acyclic fibrations.

   Then the map $Y_{n+1} \to (d_*, Y_n)$ is a fibration.

Definition 2.2.3. A simplicial resolution Y_* of Y can be viewed as a simplicial object over M together with a simplicial map $i : Y \to Y_*$ (here Y represents the constant simplicial object over M made from the object Y), and having some further properties. Then one can define a map of simplicial resolutions of Y, $f : Y_* \to Y'_*$ as a simplicial map which makes the following diagram commute:

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Y'_* \\ & Y \arrow[ur] & \end{tikzcd}$$

Definition 2.2.4. The notion of cosimplicial resolution and map between cosimplicial resolutions are defined dually from the notion of simplicial resolution and map of simplicial resolutions.

With these notions, one can try to state the definition:

Definition 2.2.5. $M(X^*, Y_*)$ has an obvious structure of bisimplicial set for X^* a cosimplicial resolution of X and Y_* a simplicial resolution of Y. The homotopy complex of (X, Y) is then defined as $diagM(X^*, Y_*)$.

However, one needs some machinery to make this definitions coherent. Indeed one has to check that the homotopy can be defined for two objects (X, Y) and that it does not depens on the resolutions chosed. The first requirement is answered by:

Proposition 2.2.6. Every object of a model category M has simplicial and a cosimplicial resolutions.

Proof. Let's show the result by induction for a simplicial resolution. One adds a requirement of this simplicial resolution (usefull for the construction): the degeneracies are acyclic cofibrations.

- If $*$ is the final object of M, then $Y \to *$ can be factorized through : $Y \overset{\sim}{\hookrightarrow} Y_0 \twoheadrightarrow *$. Then one got the fibrant element Y_0 and an acyclic fibration from Y to Y_0.

- If one has the elements Y_i for $0 \leq i \leq n$ with the corresponding faces and degeneracies maps which are respectively acyclic fibrations and acyclic cofibrations, then one defines (s_*, Y_n) as the direct limits of the diagram made of copies (s_i, Y_n) of Y_n for $0 \leq i \leq n$ and copies $(s_i s_j, Y_{n-1})$ of Y_{n-1} for $0 \leq i < j \leq n$ together with maps $(s_i, Y_n) \xleftarrow[s_{j-1}]{} (s_i s_j, Y_{n-1}) \xhookrightarrow{s_i} (s_j, Y_n)$. Then one has easily a weak equivalences $(s_*, Y_n) \xrightarrow{\sim} (d_*, Y_n)$ which can be factorized through $(s_*, Y_n) \xhookrightarrow{\sim} Yn+1 \xrightarrow{\sim} (d_*, Y_n)$.

The cosimplicial resolution is created dually. □

Remark 2.2.7. In the proof we had added a property: the map $Y \to Y_0$ and the degeneracies are acyclic cofibrations. Such a simplicial resolution is called cofibrant. We define dually the notion of fibrant cosimplicial resolution. The proof of the previous property shows that every object of M has cofibrant simplicial resolutions and fibrant cosimplicial resolutions.

The next lemma uses this last notion to make links between the various homotopy function complexes one can construct.

Lemma 2.2.8 ([DK80b, 6.9 and 6.10]). 1. Let Y_* and Y'_* be simplicial resolutions of Y. Then, if Y'_* is cofibrant, there exists a map of resolutions $Y_* \to Y'_*$.

2. Let X^* and X'^* be cosimplicial resolutions of X. Then, if X'^* is fibrant, there exists a map of resolutions $X'^* \to X^*$

Sketch of proof. The maps are constructed by induction □

Proposition 2.2.9. [4, 17.3.4] If $f : X'^* \to X^*$ is a map of cosimplicial resolutions of X and $g : Y_* \to Y'_*$ is a map of simplicial resolutions of Y, then they induced a map of simplicial set $diagM(X^*, Y_*) \to diagM(X'^*, Y'_*)$ which is a weak homotopy equivalence

Corollary 2.2.10. One can find a finite string of weak homotopy equivalences between two homotopy function complexes constructed for the pair of objects (X, Y). Hence, the homotopy function complex is unique up to weak homotopy equivalence.

The next proposition allows us to link the homotopy function complexes to the usual notion of homotopy in a model category:

Proposition 2.2.11. 1. If X^* is cosimplicial resolution in M, then $X^0 \sqcup X^0 \to X^1 \to X^0$ is a cylinder object of X^0.

2. If Y_* is simplicial resolution in M, then $Y_0 \to Y_1 \to Y_0 \times Y_0$ is a path object of Y_0.

Then, a cosimplicial resolution of X is a sort of "higher cylinder objects" collection for a cofibrant approxiation of X. Dually, a simplicial resolution of Y is a sort of "higher path objects" collection for a fibrant approximation of Y.

Homotopy function complexes and simplicial model categories

Let M be a simplicial model category

Proposition 2.2.12. 1. If X is an object of M and X' is a cofibrant approximation of X, then the cosimplicial object defined by $X^n = X' \otimes \Delta^n$ is a cosimplicial resolution of X

2. If Y' is a cofibrant approximation of $Y \in M$, then the simplicial object defined by $Y_n = Y'^{\Delta^n}$ is a simplicial resolution of Y

Corollary 2.2.13. If X is cofibrant and Y is fibrant, then the homotopy function complex of (X, Y) is $Map(X, Y)$.

Homotopical limit

This is an other way to define a "mapping space" between two objects in a model category.

We start with the definition of homotopy limits:

Definition 2.2.14. If I is a (small) indexing category, then define C^I the category of functors $I \to C$. Then let W_I the subcategory of C^I spaned with the morphisms of functors (natural transformations) which are objectwise in W. Then, the constant functor $C \to C^I$ leads to a functor $C \to C^I[(W_I)^{-1}]$ which can be factorized through:

$$c : C[W^{-1}] \to C^I[(W_I)^{-1}] \tag{2.3}$$

1. the homotopy limit of $A \in obj(C^I[(W_I)^{-1}])$ is (if it exists) the object $holim_I(A)$ of $C[W^{-1}]$ such that we have a isomorphism of functors (in B) $Hom_{C[W^{-1}]}(holim_I(A), B) \simeq Hom_{C^I[(W_I)^{-1}]}(A, c(B))$. The functoriality in A of the previous isomorphism makes us define the holimit functor (if it exists) as a left adjoint of c

2. The hocolimit functor is a right adjoint of c (if such a functor exists).

Remark 2.2.15. The $holim$ and $hocolim$ are introduced because the usual lim and $colim$ in C is not localized in a good way.

Definition 2.2.16. If Y is an object of C, then:

1. If we consider the object $Y^K = holim(\Delta \to: [n] \mapsto \prod_{K_n} Y)$, this definition induces a functor $Ho(sSet) \to C[W^{-1}] : K \mapsto Y^K$.

2. Then a functor $Map : Ho(sSet) \to Set : K \mapsto Hom(X, Y^K)$.

3. The mapping space is a simplicial set which represents the functor.

2.3 Two definitions of the simplicial localization

For the two first parts of this section, we don't need a model structure on the category but just a category C with a subcategory W. The aim is to understand the localization of C with respect to W through simplicial categories.

Standard simplicial localization

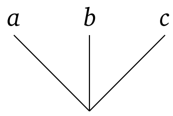
Free categories

Here we develop the machinery which permits to define the standard simplicial localization of a category.

Definition 2.3.1. A free category C is a category such that there is a set of non-identity maps S such that every map in C is written in a unique way as a finite composition of maps of S . If such S exists, then it is unique. The elements of S are called the generators of C.

Definition 2.3.2. If C is a category, the category FC is the free category which has as generators the set of non-identity maps. In other words, FC is such that:

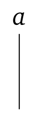
- $Obj(FC) = Obj(C)$
- the morphisms of FC are the trees of heigh 1 with non-identity maps of C as leaves, and such that these leaves can be composed (in the picture it means that one can compose $a \circ b \circ c$):



  Of course, the source of such a map is the source of the first leave and the target is the target of the last leave. The identity maps are represented by the empty trees and the composition consists in joining the roots.

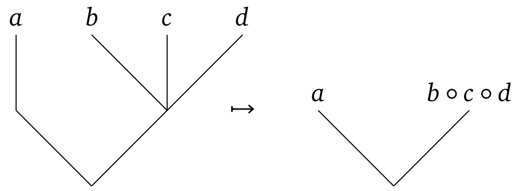
One can repeat the process several times: F^kC is the category with the same objects as C and, as morphisms, the trees of heigh k with maps of C as leaves, which can be composed. The composition consists in joining the roots.

There is an obvious functor ϕ from FC to C: it sends an object to itself ($Obj(FC) = Obj(C)$), and a tree (ie a morphism) to the composition of its leaves (in the previous tree picture, it sends the tree to $a \circ b \circ c$). Define also the obvious functor : $C \to FC$ which sends an object to itself and a morphism a to the corresponding generator in FC, ie:

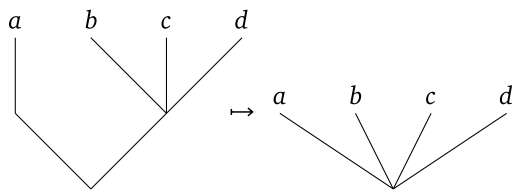


Similarly, there are natural functors between F^2C and FC; one functor from $F2C$ to F^2C and two functors in the opposite sense:

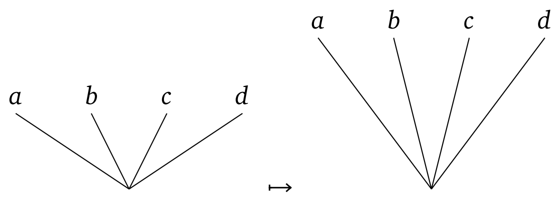
- $\phi F : F^2C \to FC$ reduces the branches between the height 1 and 2, ie:



- $F\phi : F^2C \to FC$ reduces the branches between the height 0 and 1, ie:



- Define also $\psi :: FC \to F^2C$:



One notices for instance that $(F\phi)\psi = (F\phi)\psi$. One can also continue the process to higher order of free category; one then define in the same way n functors from F^nC to $F^{n-1}C$ and n functors

Proposition 2.3.3. $(F^{k+1}C)_k$ together with the functors defined above is a simplicial category.

Furthermore this new category has the same homotopy type as C (if we consider as a constant simplicial category

Proposition 2.3.4. $(F^{k+1}C)_k \to C$ (functor unduced by the reduction of the trees) is a weak equivalence of categories (that means that it induces weak equivalences on the simplicial morphisms sets).

Sketch of the proof. Every tree of height k is equivalent (in the homotopy sense) to the tree of same height and with only one leave which is the composition of the leaves. □

The localization: definitions and first properties

Along those lines, the morphisms of the localized categories $C[w^{-1}]$ are the tree of height 1 such that its leaves are a succession of maps of C and of W (quotiented through an equivalence relation). The underlying idea of the standard simplicial localization is to look at the localization of F^kC by F^kW.

Then, with few changes in the way of thinking with the respect to what we have in the previous sub-part, one can define:

Definition 2.3.5. The (standard) simplicial localization of (C,W) is the simplicial category $L(C,W)$ defined by:

$$L(C,W)_n = F^{n+1}C[(F^{n+1}W)^{-1}] \tag{2.4}$$

The homotopy groups of this localization is expected to encode the higher homotopy data data of C with respect to W. Actually the homotopy relations are basically generated by the relation of two trees which have the same dimension and can be constructed from one same bigger tree. In dimension 0, it gives the relation:

Proposition 2.3.6. $\pi_0(LC(X,Y)) = C[W^{-1}](X,Y)$

Generalization to simplicial categories

One can generalise the notion of standard simplicial localization to simplicial categories.

Definition 2.3.7. If A is a simplicial category, and V a simplicial subcategory. The standard simplicial localization of A with respect to V is the simplicial category:

$$L(A,V) = diag(F^{*+1}A[(F^{*+1}V)^{-1}]) \tag{2.5}$$

We have something which is close of the old localization of an ordinary category if we take the path-components:

Proposition 2.3.8. $\pi_0(LB) = (\pi_0 B)[(im\pi_0 V)^{-1}]$

Finally, let's see one lemma which can be useful:

Lemma 2.3.9 ([DK80c, p. 6.3])**.** Let A and B two simplicial C-categories, U a simplicial subcategory of A and V a simplicial subcategory of B, $S : A \to B$ a functor such that $S(U) \subset V$, and such that $S : A \to B$ and $S : U \to V$ are weak equivalence. Then the induced map $LA \to LB$ is a weak equivalence.

The demonstration of this last lemma is more complicated. One needs some machinery, as for instance a model structure on the category of the simplicial categories with the same objects as C (this last category is not small).

Hammock localization

Definition and first properties

Let C be a (small) category an W a subcategory. Let's define the simplicial set $L^H(C,W)(X,Y)$ (or shortly $L^H C(X,Y)$); the k-simplices are the "reduced hammocks" of width k and any length between X and Y:

$$\begin{tikzcd}
 & C_{0,0} \arrow[r, dash] \arrow[d] & C_{0,1} \arrow[r, dashed, dash] \arrow[d] & C_{0,n-1} \arrow[d, dash] & \\
 & C_{1,0} \arrow[r, dash] \arrow[d] & C_{1,1} \arrow[r, dashed, dash] \arrow[dd, dashed] & C_{1,n-1} \arrow[dd, dashed] & \\
X \arrow[uur, dash] \arrow[ur, dash] \arrow[r, dashed, dash] \arrow[dr, dash] & C_{2,0} \arrow[d, dashed] & & & Y \arrow[uul, dash] \arrow[ul, dash] \arrow[dl, dash] \\
 & C_{k,0} \arrow[r, dash] & C_{k,1} \arrow[r, dashed, dash] & C_{k,n-1} &
\end{tikzcd}$$

such that:

1. n, the length is ≥ 0 (if $n = 0$, the hammock consists in k-times the same morphism from X to Y).

2. all vertical maps are in W.

3. in each column, all maps go to the same direction and if they go to the left, then they are in W.

4. the maps in adjacent columns go in different directions.

5. no column contains only identity maps.

The i^{th} face map d_i consists in omitting the i^{th} row. The j^{th} degeneracy map consists in repeating the j^{th} row. If the resulting hammock is not reduced (it can happen after the application of a face map), then one applies a reduction process:

1. if a column contains only identity maps, then one omits it.

2. one composes two adjacent columns whenever they go in the same direction

Furthermore, there exists a distinguished vertices $id \in L^H C(X,X)$; if X,Y and Z are object of C, then one can also composition law $L^H C(X,Y) \times L^H C(Y,Z) \to L^H C(X,Z)$ which consists in, in each dimension, compose the hammocks and then reduce the resulting hammock. Then:

Proposition 2.3.10. $L^H(C,W)$ is a simplicial category.

The morphisms from X to Y of the localized category $C[W^{-1}]$ are the strings of successive arrows (i.e the elements of $L^H C(X,Y)_0$) quotiented by an equivalence relation. This equivalence is generated by the relation I call $\mathfrak{R}$; if $f,g \in L^H C(X,Y)_0$ then $f \mathfrak{R} g$ if and only if:

$$\exists h \in L^H C(X,Y)_1, d_0(h) = f, d_1(h) = g$$

This just means:

Proposition 2.3.11. For two objects $X,Y \in C$, then:

$$\pi_0(L^H C(X,Y)) = C[W^{-1}](X,Y) \tag{2.6}$$

Therefore, the underlying idea of the hammock localization is to study in a simplicial set the homotopy of string of arrows. Applying the equivalence gives us the usual localization, but makes us loose the higher homotopy data. The hammock localization is the good object to study this higher homotopy data.

Generalisation to simplicial categories and link with the standard simplicial localization

One can generalize the hammock localization to simplicial categories; if B is a simplicial category and V a simplicial subcategory (i.e a subcategory of B together with inclusions of simplicial sets $V(X,Y) \to B(X,Y)$ which commute with the definition of identity and the composition):

1. $L^H(B,V)$ is an obviously defined bisimplicial category

2. the hammock localization of B with respect to V is $diag L^H(B,V)$

Remark 2.3.12. In the above description, one can equivalently use the formalism of simplicial objects over the category of simplicial (small) categories or of bisimplicial enriched categories;

There is an important lemma which comes with the description of the hammock of a simplicial category:

Lemma 2.3.13 ([DK80a, p. 2.4])**.** If $A,B \in sC-cat$ (with the same set objects; see the introduction), and $U \subset A$, $V \subset B$ are simplicial subcategories, if $S : A \to B$ is a morphism in $sC-cat$ such that $S(U) \subset V$, and if $S : A \to B$ and $S : U \to V$ are weak equivalences (of simplicial C-categories), then it induces a weak equivalence of C-simplicial categories $diag L^H A \to diag L^H B$

This lemma is hard to prove. However, it is the main key for the next proposition:

Proposition 2.3.14. If C is a category and W a subcategory (ordinary categories), then the natural functors:

$$L^H C \leftarrow diag L^H F^{*+1} \rightarrow F^{*+1}[(F^{*+1}W)^{-1}] \tag{2.7}$$

are weak homotopy equivalences (in $sC - cat$).

Ideas in the proof. We don't make the proof but give its ingredients:

- the diagonal argument of bisimplicial sets
- the homotopy lemma just above
- the comparison lemma: if a category (ordinary) D is freely generated by subcategories E and W, then $L^H C \rightarrow C[W^{-1}]$ is a weak equivalence.

□

So, we have make the link between the standard simplicial localization and the hammock localization. They are homotopy equivalent. Then one uses more the hammock localization as its definition is simpler and because it is simpler to work with it as we will see.

The II-indexing category and hammock graphs

The II category is defined by:

1. its objects are the pairs (S, T) of ordered subsets of $\mathbb{N}^*$ ($\mathbb{N}-\{0\}$) such that $S \cup T = \{1, ..., |S \cup T|\}$.
2. the morphisms $(S, T) \rightarrow (S', T')$ are the order preserving maps $f : S \cup T \rightarrow S' \cup T'$ such that $f(S) \subseteq S'$ and $f(T) \subseteq T'$

Let C be a category and W a subcategory (ordinary categories).

Definition 2.3.15. 1. The category of C-graphs is defined by:

a) A C-graph is a graph (with directed edges i.e arrows) which set of points is $Obj(C)$.

b) A morphism $G_1 \rightarrow G_2$ of C-graph is the data of maps $G_1(X, Y) \rightarrow G_2(X, Y)$ for all X and Y, objects of C, and where $G_1(X, Y)$ is the set of arrows from X to Y in G_1 (idem for G_2).

2. the category of C-graphs corresponds to the category of C-simplicial categories but without the identity mapas and compositions. Indeed, one has the forgetful functor $C - cat \rightarrow C - graph$.
3. A simplicial C-graph is a simplicial object over the category of C-graph (or equivalently a C-graph with simplicial sets of arrows).
4. One has also the forgetful functor $sC - cat \rightarrow sC - graph$.
5. Weak equivalences between simplicial C-graphs are just morphisms of simplicial graphs which induce weak homotopy equivalences on the simplicial mapping spaces of arrows between two objects of C. This definition is the analogue to the definition of weak equivalence between simplicial C-categories.

We can define a natural functor $\lambda C : II \rightarrow sC - graph$:

- An element of II is equivalent to a word made with the letters C and W^{-1} in the way that transform, for example $(\{1, 2\}, \{3\})$ into $W^{-1}CC$.
- From such a word, one made a simplicial C-graph describe by:

1. The arrows from X to Y (objects of C) are the unreduced hammocks:
2. the length of the hammocks is the length of the word we call n
3. The i^{th} column is made of left-directed arrows of C if the $(n+1-i)^{th}$ is C
4. The i^{th} column is made of right-directed arrows of W if the $(n+1-i)^{th}$ is W^{-1}
5. These hammocks are organized in a simplicial set through the width as in $L^H C$.

- All of this is done functorially (because one can reduce parts of the hammocks and add column with the identity. For instance an injection in II leads to the addition of identity maps in the corresponding C-graph). Therefore we have created a II-diagram in the category $sC - graph$
- This diagram has a direct limit $lim\lambda C$

Proposition 2.3.16. The reduction maps $r_{(S,T)} : \lambda C(S,T) \to L^H C$ (it is a morphism of the category $sC - graph$) induces a map $lim\lambda C \to L^H C$ which is a isomorphism.

So we can see the hammock localization as a limit of graphs with simpler hammocks (of a definite type).

Remark 2.3.17. This theory of II-diagrams is useful to prove the lemma introduced just above and non proved. However, we won't apply it to this purpose.

Homotopy calculi of fractions

Let C be a category and W a subcategory (ordinary categories).

Definition 2.3.18. (C,W) is said to admit:

1. a homotopy calculus of (two-sided) fractions if, for every pairs $i,j > 0$ the obvious maps in $sC - graph$ (adding a column with only identity):
$$W^{-1}C^{i+j}W^{-1} \to W^{-1}C^iW^{-1}C^jW^{-1} \tag{2.8}$$
and
$$W^{-1}W^{i+j}W^{-1} \to W^{-1}W^iW^{-1}W^jW^{-1} \tag{2.9}$$
are both weak equivalences (we consider the words as graph as seen just before).
2. a homotopy calculus of left fractions if, for every pairs $i,j > 0$ the obvious maps in $sC - graph$:
$$W^{-1}C^{i+j} \to W^{-1}C^iW^{-1}C^j \tag{2.10}$$
and
$$W^{-1}W^{i+j} \to W^{-1}W^iW^{-1}W^j \tag{2.11}$$
are both weak equivalences.
3. a homotopy calculus of right fractions is defined in an analogue way.

The usefulness of the previous definition is due to the following proposition:

Proposition 2.3.19 ([DK80a, p. 6.2]).
1. If (C,W) admits a homotopy calculus of (two-sided) fractions, then the reduction maps
$$W^{-1}CW^{-1} \to L^H(C,W), W^{-1}WW^{-1} \to L^H(W,W) \tag{2.12}$$
are weak equivalences.

2. one has the same kind of statement for the homotopy calculus of left or right fractions.

So, if the category admits a homotopy calculus of fractions, then the homotopy of its hammock localization is then much simpler to describe as we have only to consider hammocks of a certain shape.

Case where the homotopy calculus is used

Definition 2.3.20. (C, W) admits a calculus of left fractions if:

1. For each diagram $X' \xleftarrow[u]{} X \xrightarrow{f} Y$ of C with $u \in W$, there exists in C a diagram $X' \xrightarrow{f'} Y' \xleftarrow{v} Y$ with $v \in W$ and such that $v \circ f = f' \circ u$

2. If $f, g : X \to Y$ are morphisms of C and $u : X' \to X$ is in W, and they are such that $f \circ u = g \circ u$, then there exists $v \in W$ such that $v \circ f = v \circ g$.

Note that if (C, W) admits a calculus of left fractions and if it is such that:

if f, g are morphisms of C such that one can define $f \circ g$, and if two of the three morphisms f, g, and $f \circ g$ are in W, then the third is in W

then (W, W) also admits a calculus of left fractions.

We have two important results about the calculus of left fractions:

Proposition 2.3.21. If (C, W) and (W, W) admit a calculus of left fraction, then (C, W) admits a homotopy calculus of left fraction.

Proposition 2.3.22. If (C, W) and (W, W) admit a calculus of left fraction, then the natural map:

$$L^H C \to \pi_0 L^H C = C[W^{-1}] \tag{2.13}$$

is a weak equivalence of simplicial categories.

Now, a final proposition which makes the hammock localization very useful in the frame of model category; here M is a model category, W refers to the weak equivalences, M^c to the cofibrant subcategory, M^f to the fibrant subcategory and M^{cf} to the fibrant-cofibrant subcategory.

Proposition 2.3.23. The pairs (M, W), (M^c, W^c), (M^f, W^f) and (M^{cf}, W^{cf}) admit homotopy calculi of two-sided fractions.

2.4 Homotopy simplicial category and conclusion

We won't prove the following theorem which is the conclusion of our discussion:

Theorem 2.4.1 ([DK80b, Thm. 4.4])**.** Let M be a model category. For any two objects X, Y of M, if X^* is a cosimplicial resolution of X and Y_* is a simplicial resolution of Y, then the simplicial sets $diagM(X^*, Y_*)$ and $L^H(X, Y)$ have the same homotopy type.

2.5 Complements to Chapter 2

Bisimplicial set

Definition 2.5.1. A bisimplicial set is equivalently:

1. A simplicial object over the category $sSet$
2. A functor $(\Delta \times \Delta)^{op} \to Set$
3. A functor $\Delta^{op} \times \Delta^{op} \to Set$
4. A family of sets $(X_{n,m})_{(n,m)\in\mathbb{N}^2}$ together with two types of face maps and degeneracies such that the two simplicial structures commute.

The diagonal of a bisimplicial set is the obvious simplicial set made with the sets $X_{n,n}$, the face maps $X(d_i, d_i)$ and the degeneracies $X(s_j, s_j)$.

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Exposé **3**

Quasi-categories

This chapter is intented to give an intuition of what $(\infty, 1)$-categories are and to present an informal introduction to one of theirs models, namely quasi-categories (sometimes called weak Kan complexes). We show how some basic concepts of ordinary category theory extend to the $(\infty, 1)$-categorical context, and specifically to the model of quasi-categories.

Valerio Melani

3.1 Intuition and first models

We start by trying to give some intuition for what these $(\infty, 1)$-categories are, and how one could search for a suitable formalism to describe them. Next, we will concentrate our efforts in starting to understand one of the different models for $(\infty, 1)$-categories, namely the so-called *quasi-categories*.

Heuristically speaking, an $(\infty, 1)$-category is of course first of all an ∞-category. This means that we have objects, morphisms between objects, 2-morphisms between morphisms, 3-morphisms between 2-morphisms, and so on. Put this way, this is not yet a very precise object, because actually we want to have some kind of compositions between these k-morphisms, and these compositions must of course have some nice properties that fit our general intuition. This is (one of the) main reason why higher category theory has a reputation of being overcomplicated and messy; for a general discussion about the problems of defining higher categories, we recommend for example the lecture of ???.

Luckily we will not have to go into these kind of details, as we just want to describe the special case of $(\infty, 1)$-categories, who are just ∞-categories whose k-morphisms are invertible for $k > 1$. This sensibly simplifies our task : generalizations of classical ideas from (ordinary) category theory becomes much more complicated if we allow the existence of non-invertible k-morphisms (k>2).

That being said, where should we start looking for a possible model of $(\infty, 1)$-categories? We observe that if x and y are two "objects" of a wanna-be $(\infty, 1)$-category, the morphisms between them form an ∞-category whose all morphisms are invertible, even the 1-morphisms ; this is what is called an ∞-groupoid. ∞-groupoids have been extensively studied under the false name of topological spaces. In fact, a general principle of higher category theory (the *homotopy hypothesis*) states exactly that ∞-groupoids and topological spaces are same thing, homotopically speaking. To be more precise, if we are given a topological space X, we can easily associate to it an ∞-groupoid : the classical construction of the fundamental groupoid of a topological space can be extended in a natural way to get an ∞-groupoid $\pi_\infty X$ as follows. The objects of $\pi_\infty X$ are the points of X. If $x, y \in X$, then the morphisms from x to y in $\pi_\infty X$ are the continuous paths in X starting at x and ending at y. The 2-morphisms are homotopies of paths, the 3-morphisms are homotopies between homotopies of paths, and so on all the way to ∞. What we get is an ∞-groupoid who remembers all the homotopical information of X. The homotopy hypothesis tells us the converse : every ∞-groupoid is of the form $\pi_\infty X$ for some topological space X.

So saying that morphisms between two objects of an ∞-category form an ∞-groupoid is the same that saying that they form a topological space. We can therefore try and give the following definition.

Definition 3.1.1. An ∞-category is a *topological category*, i.e. a category which is enriched over the category of topological spaces.

But in homotopy theory, there are many equivalent ways to describe spaces: using one of them, we are led to the following different interpretation of an ∞-category.

Definition 3.1.2. An ∞-category is a *simplicial category*, i.e. a category which is enriched over the category of simplicial sets.

These are two definitions who can be both used as a foundation for higher category theory. They look simple enough, but taking one of these approaches lead to some technical difficulties that we would like to avoid. A first alternative definition of an ∞-category will be given in the next section.

3.2 Quasi-categories

There are two classes of examples we certainly wish to have in any theory of ∞-categories:

- ∞-groupoids (i.e. spaces), as they are of course a special case of ∞-category ;
- ordinary categories, as they can be thought as ∞-categories where for $k > 1$ the only k-morphisms are the identities.

So the question is to find a formalism in containing both these theories. And one possible answer is the theory of simplicial sets. Given a topological space X, the classical construction of his singular complex $\mathrm{Sing}(X)$ is a simplicial set who determines X up to weak homotopy equivalence. The simplicial set $\mathrm{Sing}(X)$ has the important property of being a *Kan complex*

Definition 3.2.1. A simplicial set S is a *Kan complex* if every map $\Lambda^n_k \to S$ extends to a map $\Delta^n \to S$.

It is quite clear that if X is a topological space, then $\mathrm{Sing}(X)$ is a Kan complex : this is due to the fact the horns are retracts of the simplex Δ^n in the world of topological spaces. The converse is also true : Kan complexes are topological spaces, in the sense that just as ∞-groupoids they are models for homotopy types. Now given a category $\mathcal{C}$, we can construct a simplicial space $N(\mathcal{C})$ called the *nerve* of $\mathcal{C}$ as follows. We just let the n-simplices of $N(\mathcal{C})$ be the strings of n composable morphisms of $\mathcal{C}$. If you think about it, this simplicial set knows all about the category $\mathcal{C}$, it just encodes the information in another language. But what kind of simplicial set do we get? The following proposition gives us the answer.

Proposition 3.2.2. A simplicial set S is isomorphic to the nerve of some category if and only if every map $\Lambda^n_k \to S$ with $0 < k < n$ extends uniquely to a map $\Delta^n \to S$.

Knowing that we are trying to generalize spaces and categories, the following definition should not surprise us.

Definition 3.2.3. A *quasi-category* is a simplicial set S in which every map $\Lambda^n_k \to S$ with $0 < k < n$ extends to a map $\Delta^n \to S$.

Part of the literature on the subject refers to quasi-categories as *weak Kan complexes*, to stress the analogy with the precedent notion. We state that quasi-categories are a good model for the theory of $(\infty, 1)$-categories. The rest of this notes is devoted to convincing us of this fact.

Mapping spaces

As we said earlier, we know we should be able to find a space (or a good homotopy types) between any two objects of a $(\infty, 1)$-categories. This is easy in the world of topological (or simplicial) categories. It is less obvious in our setting of quasi-categories.

We should now point out that there are actually explicit adjoint functors that establish an equivalence between the theory of simplicial categories that of quasi-categories. So the problem of finding mapping spaces in a quasi-category could be bypassed by taking a simplicial category which is equivalent to our quasi-category, and take mapping spaces there. Here we try to avoid this point of view, partly because the two adjoint functors are quite complicated, and the resulting mapping spaces could be quite obscure. Details can be found on [HTT].

The question of how to define mapping spaces in quasi-categories is of independent interest, and has been studied in [DS09]. Here we just show one way of defining them.

If x and y are objects of an $(\infty, 1)$-category $\mathcal{C}$ (i.e. they are vertices of $\mathcal{C}$ as a simplicial set) we define a simplicial set $\mathrm{Hom}^R(x, y)$, called the space of right morphisms from x to y. This is done by letting $\mathrm{Hom}_{SSet}(\Delta^n, \mathrm{Hom}^R(x, y))$ be the set of all maps $f : \Delta^{n+1} \to \mathcal{C}$ such that f sends the $(n+1)$-th vertex to y and the first n to x (as a degenerate simplex).

We can prove that is what we were looking for : $\mathrm{Hom}^R(x, y)$ is a Kan complex, and it is equivalent to other more abstract definitions of the mapping space.

The homotopy category

We now define the homotopy category of a quasi-category $\mathcal{C}$. This is to be thought as the underlying ordinary category of $\mathcal{C}$, who remembers only low homotopical information and forgets about the higher one.

Again, if we think in terms of topological (or simplicial) categories, all is easy : we can take the set of morphisms between two objects x, y to be the π_0 of the topological space (or simplicial set) $\mathrm{Hom}(x, y)$. Now that we dispose of mapping spaces in the setting of quasi-categories, we can do the same.

Definition 3.2.4. If $\mathcal{C}$ is a quasi-category, his homotopy category $h\mathcal{C}$ is the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are the path-connected components of the mapping spaces of $\mathcal{C}$.

Another way to describe it is to notice that the nerve functor defined earlier in the text admits a left adjoint $h : \mathbf{sSet} \to \mathbf{Cat}$, doing exactly what we want: it only keeps the 1-categorical information contained in $\mathcal{C}$. See Theorem **??** for a proof of this statement.

Let's get a more explicit understanding of what the homotopy category concretely is: given two vertices $x, y \in \mathcal{C}$ and two edges $f, g : x \to y$ (meaning that the edges start at x and end at y), we say that f and g are homotopic if there is a 2-simplex in $\mathcal{C}$ who has f as the $0 \to 1$ edge, g as the $0 \to 2$ edge, and the degenerate edge at y as $1 \to 2$ edge. (This is very clear if you try and draw the diagram).

The fact that $\mathcal{C}$ is a quasi-category means that "being homotopic" is an equivalence relation. The homotopy category is now just the category whose objects are the the objects of $\mathcal{C}$ and whose morphisms are the equivalence classes of edges of $\mathcal{C}$.

Functors between quasi-categories

We now describe the category of functors between quasi-categories. This is one of the cases in which some technical problem with the topological (and simplicial) categories arise. In any reasonable model, we expect to be able to construct an $(\infty, 1)$-category of functors between two $(\infty, 1)$-categories. The problem is that there is not an obvious way to give a good definition of this category in the setting of topological (simplicial) category.

Here quasi-categories give their best, and the definition are much easier.

Definition 3.2.5. Given two quasi-categories $\mathcal{C}$ and $\mathcal{D}$, the simplicial set $\mathrm{Hom}_{SSet}(\mathcal{C}, \mathcal{D})$ is called the $(\infty, 1)$-*category of functors* from $\mathcal{C}$ to $\mathcal{D}$.

Here we used the fact that the category of simplicial sets has internal Hom objects, and we are defining the maps of quasi-categories as maps of simplicial sets. Notice that $\mathrm{Hom}_{SSet}(\mathcal{C}, \mathcal{D})$ has the right to be called an $(\infty, 1)$-category thanks to the following proposition.

Proposition 3.2.6. If $\mathcal{C}$ is a quasi-category, then the simplicial set $\mathrm{Hom}_{SSet}(S, \mathcal{C})$ is a quasi-category for every simplicial set S.

In particular, a functor between quasi-categories is an equivalence if and only if it is essentially surjective on objects and induces homotopy equivalences between mapping spaces.

Limits and colimits

In this section we explain how to define limits and colimits in the setting of quasi-categories.

In the same way that limits and colimits in an ordinary category can be defined using the definition of limits and colimits in the category of sets, limits and colimits in a quasi-category can be defined using the notion of homotopy limits and colimits in the model category of spaces. If we want to be concise, we could give the following definition: given a functor $I \to \mathcal{C}$ between two $(\infty, 1)$-categories, its limit and colimit (if they exist) are determined by asking that the natural maps

$$\mathrm{Hom}(X, \lim F(i)) \to \mathrm{holim}\ \mathrm{Hom}(X, F(i))$$
$$\mathrm{Hom}(\mathrm{colim}\ F(i), X) \to \mathrm{hocolim}\ \mathrm{Hom}(F(i), X)$$

are weak equivalences of spaces that are natural in X.

Model structure

In order to prove later that the different models of $(\infty, 1)$-categories are equivalent, we now give the definition of a model structure on the category of simplicial sets which is different from the standard one.

We say then that a map of simplicial sets $S \to S'$ is a *weak categorical equivalence* if the induced map $\mathrm{Hom}(S', X) \to \mathrm{Hom}(S, X)$ is an equivalence of quasi-categories for any quasi-category X.

Theorem 3.2.7 (Joyal)**.** There exists a model structure on the category $SSet$ of simplicial sets with the following properties:

- weak equivalences are the weak categorical equivalences ;
- cofibrations are just monomorphisms of simplicial sets ;
- the fibrant objects are precisely the quasi-categories.

In particular, this means that for this model structure weak equivalences between fibrant objects are precisely equivalences of quasi-categories.

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Exposé **4**

Segal spaces and Segal categories

In this talk, we will discuss two models for $(\infty, 1)$-categories in detail, namely complete Segal spaces and Segal categories. These two models are based on the Δ-space construction by Segal. An advantage of these two models over models such as quasi-categories is that we have a simple and explicit construction for the associated homotopy categories. As such, they are used widely in constructions in which we are interested in studying the homotopy. An example which we will construct later shows that they are the appropriate constructions for the $(\infty, 1)$-category of simplicial model categories.

These two models are examples of "weak" $(\infty, 1)$-categories, in that composition (and thus associativity and identity) is only defined up to homotopy. This is in contrast with simplicial categories, which are "strict" $(\infty, 1)$-categories. In the case of $(\infty, 1)$-categories, the notions of weak and strict are equivalent, but they are not for (∞, n)-categories where $n \geq 2$. In line with the philosophy of higher categories, the weak structure is the appropriate object of study. Indeed, there are natural extensions of complete Segal spaces and Segal categories to n-categories (see, for example, [BR12; HS; Lur09]).

Applications of Segal categories are mainly in higher stacks, for example, see the work by Hirschowitz and Simpson [HS]. Complete Segal spaces occur naturally in the construction of bordism categories [Lur09].

Yan Zhao

4.1 Preliminaries

Simplicial space

In this paper, a **space** will always refer to a simplicial set. Let $\mathcal{S}$ be the category of spaces, which we endow with the standard model category structure, i.e. a weak equivalence is a homotopy weak equivalence, a cofibration is an injection and a fibration is a Kan fibration. Let Δ^n, $\partial\Delta^n$ and Λ^n_k be the standard n-simplex, the boundary of the standard n-simplex and the k-th horn of the n-simplex (boundary of the n-simplex with the k-th face removed) respectively. For any $X, Y \in \mathcal{S}$, the mapping space is the function complex $\mathrm{Map}_{\mathcal{S}}(X, Y)$, where the n-simplex can be given by the set of all maps $X \times \Delta^n \to Y$.

Let $\Delta \subset \mathbf{Cat}$ be the category of categories consisting of objects $[n]$, with the usual face and degeneracy maps d^i and s^i. A **simplicial space** is thus a functor $X : \Delta^{op} \to \mathcal{S}$, which sends $[n] \mapsto X_n$, with face and degeneracy maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$. Let $s\mathcal{S}$ be the category of simplicial spaces.

Note that a simplicial space can also be seen as a bisimplicial set $X : \Delta^{op} \times \Delta^{op} \to \mathbf{Set}$ with two sets of arrows d_i and s_i. Let $\mathcal{S}^{(2)}$ denote the category of bisimplicial sets. It is clear that there is an isomorphism of categories between $s\mathcal{S}$ and $\mathcal{S}^{(2)}$. The former notation is more convenient for the discussion of homotopy theory while the latter in the comparison theorems between different models of ∞-categories. We will freely interchange between the two characterisations.

We can identify $\mathcal{S}$ as a full subcategory of $s\mathcal{S}$ by sending each simplicial set K to the constant simplicial space $[n] \mapsto K$ with d_i and s_i being the identity maps. The category $s\mathcal{S}$ can be enriched over simplicial sets in a compatible way with the enrichment of $\mathcal{S}$. For any $X, Y \in s\mathcal{S}$, we have the function complex $\mathrm{Map}_{s\mathcal{S}}(X, Y)$, where each n-simplex is the set of simplicial maps $X \times \Delta^n \to Y$.

Let $F(k)$ be the simplicial space defined by $[n] \mapsto \Delta([n], [k])$. where the set of morphisms $\Delta([n], [k])$ in $\mathcal{S}$ is taken as a discrete simplicial set. $F(k)$ is a k-th space functor, in the sense that there is an isomorphism

$$\mathrm{Map}_{s\mathcal{S}}(F(k), X) \cong X_k$$

of simplicial sets which is natural with respect to $d^i : F(k) \to F(k+1)$ and $s^i : F(k) \to F(k-1)$. Let $\partial F(k)$ be the largest subobject of $F(k)$ not containing the identity map $\iota : [k] \to [k]$. It can easily be seen that $\partial F(k)$ is generated by the faces $d_i \iota \in \Delta([k-1], [k])$. We denote by ∂X_k the mapping space $\mathrm{Map}_{s\mathcal{S}}(\partial F(k), X)$. The inclusion $\partial F(k) \hookrightarrow F(k)$ induces a map $X_k \to \partial X_k$.

Proposition 4.1.1. The category of simplicial spaces $s\mathcal{S}$ is cartesian closed, that is, for $X, Y \in s\mathcal{S}$, there exists an internal hom-object Y^X with a natural isomorphism

$$s\mathcal{S}(X \times Y, Z) \cong s\mathcal{S}(X, Z^Y).$$

Proof. Let $(Y^X)_k = \mathrm{Map}_{s\mathcal{S}}(X \times F(k), Y)$. □

Reedy model category structure on $s\mathcal{S}$

Theorem 4.1.2. There exists a model structure on $s\mathcal{S}$ where $f : X \to Y$ is a

1. weak equivalences if $f_k : X_k \to Y_k$ are degree-wise weak equivalences;
2. cofibrations if f_k are degree-wise cofibrations;
3. fibrations if the induced map
$$X_k \to Y_k \times_{\partial Y_k} \partial X_k \tag{4.1}$$
are fibrations.

Proof. See [GJ99, p. IV.3.2]. □

This is called the **Reedy model structure** on the category of simplicial spaces. It is given by the Reedy model structure construction on the model category $\mathcal{S}$ and the Reedy category Δ. Note that all simplicial spaces are cofibrant and a simplicial space X is Reedy fibrant iff

$$X_k \to \partial X_k, \qquad k \geq 0.$$

As with the standard model structure on the category of simplicial sets, the Reedy model category structure is cofibrantly generated [DHK97], that is, there exist sets of generating cofibrations and generating trivial cofibrations such that trivial fibrations (fibrations, respectively) are characterised by the right lifting property with respect to the set of generating cofibrations (generating trivial cofibrations). The generating cofibrations are

$$\partial F(k) \times \Delta^l \sqcup_{\partial F(k) \times \partial \Delta^l} F(k) \times \partial \Delta^l \to F(k) \times \Delta^l, \qquad k, l \geq 0$$

and the generating trivial cofibrations are

$$\partial F(k)\times\Delta^l \sqcup_{\partial F(k)\times\Lambda^l_t} F(k)\times\Lambda^l_t \to F(k)\times\Delta^l,\qquad k\geq 0, 0\leq t\leq l.$$

Definition 4.1.3. A model category structure on $\mathcal{C}$ is compatible with cartesian closure if for any cofibrations $i : A \to B$, $j : C \to D$ and fibration $p : X \to Y$, either (and hence both) of the equivalent characterisations hold:

1. the induced map $A\times D\sqcup_{A\times C} B\times C \to B\times D$ is a cofibration, which is trivial if either i or j is; or
2. the induced map $X^B \to X^A \times_{Y^A} Y^B$ is a fibration, which is trivial if either i or p is.

Proposition 4.1.4. $s\mathcal{S}$ with the Reedy model structure is compatible with cartesian closure.

Proof. It suffices to check condition (i). This holds since cofibrations and weak equivalences are defined degree-wise and the standard model structure on $\mathcal{S}$ is compatible with cartesian closure (in $\mathcal{S}$, the induced map in (i) is an anodyne extension). □

In particular, this implies that given any cofibration (inclusion) $A\hookrightarrow B$ and X fibrant, we have a fibration $\mathrm{Map}_{s\mathcal{S}}(B,X)\to\mathrm{Map}_{s\mathcal{S}}(A,X)$, which is trivial if $A\hookrightarrow B$ is.

Recall the definition of a proper model category.

Definition 4.1.5. A model category is proper if

1. the pushout of a weak equivalence along a cofibration is a weak equivalence; and
2. the pullback of a weak equivalence along a fibration is a weak equivalence.

Proposition 4.1.6. $s\mathcal{S}$ is a proper model category.

Proof. $\mathcal{S}$ is a proper model category. Since cofibrations and weak equivalences are degree-wise, condition (i) follows trivially. Condition (ii) follows from the fact that under the Reedy model category structure, if all objects are cofibrant, then all fibrations are also degree-wise fibrations. Indeed, since $\partial F(k)$ is cofibrant and the Reedy model structure is compatible with cartesian closure, the cofibration $\emptyset\to\partial F(k)$ and the fibration $X\to Y$ induces a fibration of internal hom-objects

$$X^{\partial F(k)}\to Y^{\partial F(k)}\times_{Y^{\emptyset}} X^{\emptyset}\cong Y^{\partial F(k)}$$

This thus induces a fibration on the 0-spaces $\partial X_k\to\partial Y_k$. Pulling back along $Y_k\to\partial Y_k$ and composing with (4.1), we get a fibration

$$X_k\to\partial X_k\times_{\partial Y_k} Y_k\to Y_k.$$

□

We state a property of proper model categories:

Proposition 4.1.7. Let $\mathcal{C}$ be a proper model category. Then, the pushout along a cofibration is a homotopy pushout and the pullback along a fibration is a homotopy pullback.

Proof. See [HTT, Prop A.2.2.4]. □

4.2 Segal spaces

In this section, we will construct our first model of $(\infty,1)$-categories, the complete Segal spaces. Complete Segal spaces have an explicit homotopy structure, which Rezk described as the study of homotopy theory of homotopy theories. For most of this section, we will give explicit constructions following Rezk's paper [Rez01].

The Segal conditon

The Segal condition is a modification of the Δ-space defined by Graeme Segal, which is a simplicial space X in which X_n is naturally weakly equivalent to $(X_1)^n$. In the Segal condition, we allow the 0-space to be more than a single point.

For $0 \le i < k$, let $\alpha^i : [1] \to [k]$ be the map sending $[0,1] \mapsto [i, i+1]$. Let $G(k) \subset F(k)$ be the simplicial subspace generated by $\alpha^i \in F(k)_1$. Equivalently, α^i induces a map $F(1) \to F(k)$ and we define $G(k)$ to be

$$G(k) = \cup_{i=0}^{k-1} \alpha^i F(1) \subset F(k).$$

The inclusion $\phi^k : G(k) \hookrightarrow F(k)$ induces a map $\phi_k = \mathrm{Map}_{s\mathcal{S}}(\phi^k, X) : \mathrm{Map}_{s\mathcal{S}}(F(k), X) \cong X_k \to \mathrm{Map}_{s\mathcal{S}}(G(k), X)$. We can check that

$$\mathrm{Map}_{s\mathcal{S}}(G(k), X) \cong X_1 \times_{X_0} \cdots \times_{X_0} X_1 = \lim(X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} \cdots \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1)$$

with k copies of X_1.

Definition 4.2.1. We say that a simplicial space X satisfies the Segal condition if

$$\phi_k : X_k \to X_1 \times_{X_0} \cdots \times_{X_0} X_1 \tag{4.2}$$

is a weak equivalence for each $k \ge 2$.

Therefore, two simplicial spaces satisfying the Segal condition are defined up to weak equivalence by their 0-th and 1-st spaces.

We can define a simplicial enriched structure associated to a simplicial space satisfying the Segal condition:

Definition 4.2.2. Let W be a simplicial space. Define $\mathrm{Ob}\, W = (W_0)_0$ to be the vertices of W_0 and for any $x, y \in \mathrm{Ob}\, W$, we define $\mathrm{map}_W(x, y)$ to be the fiber of the map $(d_1, d_0) : W_1 \to W_0 \times W_0$ at the point $(x, y) \in W_0 \times W_0$. The identity map is defined to be $\mathrm{id}_x = s_0 x \in \mathrm{Map}_W(x, x)$.

In general, for a simplicial space W satisfying the Segal condition, $\mathrm{map}_W(x, y)$ is not fibrant, so we cannot define a homotopy equivalence on the simplicial set. However, requiring $\mathrm{map}_W(x, y)$ to be fibrant for all (x, y) (eg. by requiring $(d_1, d_0) : W_1 \to W_0 \times W_0$ to be a fibration) is still not sufficient. We also want the following condition: we want a homotopy relation that is well-defined up to changing of the end points by a path in $(W_0)_1$, that is, if $[x], [y] \in \pi_0(W_0)$ are the path-components of x and y, then we want $\mathrm{map}_W([x], [y])$ to be defined in some way and a homotopy equivalence relation on it.

It turns out that an appropriate condition to impose is Reedy fibrancy.

Definition 4.2.3. A Segal space is a Reedy fibrant simplicial space satisfying the Segal condition.

Note that since a Segal space W is fibrant and $G(k) \subset F(k)$ is a cofibration, the map ϕ_k is a fibration for all k. Similarly, $d_0, d_1 : W_1 \to W_0$ are fibrations, hence $(d_1, d_0) : W_1 \to W_0 \times W_0$ is a fibration and $\mathrm{map}_W(x, y)$ are fibrant. Furthermore, $W_1 \times_{W_0} \cdots \times_{W_0} W_1$ is a homotopy fibre product.

In particular, W_0 is fibrant as W is fibrant, and compositions of the fibrant maps in the previous paragraph gives us that W_k is fibrant for all k. Hence, we can view a Segal space as a Reedy fibrant simplicial object in the category of topological spaces satisfying the Kan condition. This view is more convenient for geometrical examples. For a complete description, see [Lur09].

Example 4.2.4. Every discrete simplicial space (W_k is discrete for each k) is Reedy fibrant. Hence, it is a Segal space iff it satisfies the Segal condition. Indeed, if a discrete simplicial space satisfies the Segal condition, then ϕ_k will be isomorphisms.

Examples of Segal spaces

Classification diagram of categories

We denote by nerve(C) the nerve of a category C, that is, the simplicial set with n-simplices given by a chain of composable morphisms

$$c_0 \to \ldots \to c_n.$$

Proposition 4.2.5. $\mathrm{nerve}([n]) = \Delta^n$. For any categories C and D, there are natural isomorphism $\mathrm{nerve}(C \times D) \cong \mathrm{nerve}(C) \times \mathrm{nerve}(D)$ and $\mathrm{nerve}(D^C) \cong \mathrm{nerve}(D)^{\mathrm{nerve}(C)}$. The nerve functor gives a full embedding $\mathrm{nerve} : \mathcal{C}at \to \mathcal{S}$.

Let C be a category and $W \subset C$ a subcategory such that $\mathrm{Ob}\, W = \mathrm{Ob}\, C$.

Definition 4.2.6. Let (C, W) be a category with its subcategory of weak equivalences. We say that a morphism f is a **weak equivalence** if $f \in W$.

Let D be any other category. For any two functors $f, g \in C^D$, we say that a natural transformation $f \xrightarrow{\alpha} g$, we say that α is a **weak equivalence** if $\alpha d \colon f(d) \to g(d)$ is a weak equivalence for all $d \in \mathrm{Ob}\, D$. Let $\mathrm{we}(C^D) \subset C^D$ be the subcategory of all weak equivalences.

The **classification diagram** of (C, W) is defined to be the simplicial space $N(C, W)$ where

$$N(C, W)_m = \mathrm{nerve}\,\mathrm{we}(C^{[m]}).$$

It is convenient to view an n-simplex of $N(C, W)_m$ as a diagram

$$\begin{array}{ccccc} c_{00} & \to & \cdots & \to & c_{0m} \\ \downarrow & & & & \downarrow \\ \vdots & & & & \vdots \\ \downarrow & & & & \downarrow \\ c_{n0} & \to & \cdots & \to & c_{nm} \end{array} \tag{4.3}$$

where the vertical arrows are weak equivalences.

We consider some special cases:

Example 4.2.7.
1. Let $C_0 \subset C$ be the subcategory consisting of all objects and only the identity morphisms. Then, $\mathrm{discnerve}\, C = N(C, C_0)$ is known as the **discrete nerve**. In particular $\mathrm{discnerve}([n]) = F(n)$. However, note that equivalent categories may not give weakly-equivalent discrete nerves.

2. Let $\mathrm{iso}\, C \subset C$ be the subcategory consisting of all objects and all invertible morphisms in C (i.e. the maximal subgroupoid of C). The **classifying diagram** of C is defined to be $NC = N(C, \mathrm{iso}\, C)$.

Proposition 4.2.8. All classifying diagrams $N(C, W)$ satisfy the Segal condition. The classifying diagrams $\mathrm{discnerve}\, C$ and NC of a category C are Segal spaces.

Proof. Using the representation of an n-simplex in $N(C,W)_m$ by the diagram in (4.3), we easily see that there is a natural isomorphism $N(C,W)_m \cong N(C,W)_1 \times_{N(C,W)_0} \cdots \times_{N(C,W)_0} N(C,W)_1$.

Since all discrete simplicial spaces are Reedy fibrant, thus discnerve C is a Segal space.

To show that NC is Reedy fibrant, we need to show that the maps $l_m : NC_m \to \partial NC_m$ are fibrations for $m \geq 0$. They are easy to check using the representation of n-simplices in NC_m by (4.3). Indeed, l_m is an isomorphism for $m \geq 3$. □

We obtain a result similar to Prop. 4.2.5:

Proposition 4.2.9. Let C and D be categories. There are natural isomorphisms $N(C \times D) \cong NC \times ND$ and $N(D^C) \cong ND^{NC}$. More generally, given $\operatorname{iso} D \subset W \subset D$ a subcategory,

$$N(D^C, \mathrm{we}(D^C)) \cong N(D,W)^{NC} \cong N(D,W)^{\mathrm{discnerve}\,C}. \tag{4.4}$$

The functor $N : \mathcal{C}at \to s\mathcal{S}$ is a full embedding of categories. $F : C \to D$ is an equivalence of categories iff NF is a weak equivalence of simplicial spaces.

Proof. See [Rez01, Thm 3.7, Prop 3.11]. □

Classification space of a closed model category

Let $C = M$ be a closed model category and $W \subset M$ be its subcategory of weak equivalences. As noted above, we have a simplicial space $N(C,W)$. $N(C,W)$ is in general not Reedy fibrant. However, we can take a functorial Reedy fibrant replacement of it, for example, by the small object argument. Note that taking a Reedy fibrant replacement does not change the homotopy type of the spaces in each degree.

Definition 4.2.10. The classification space of a simplicial close model category (M,W) is a functorial Reedy fibrant replacement $N^f(M)$ of $N(M,W)$.

As most of the model categories we are interested in are not small, we need to be take care of some set theoretical considerations. However, we can always work in a larger universe, so $N^f(M)$ may not be in the same universe as M.

Theorem 4.2.11. $N^f(M)$ is a Segal space.

Proof. See [Rez01, Thm 8.3]. □

Note that any category C with finite limits and colimits can be given a model structure in which the weak equivalences are exactly the isomorphisms. In this case, $N(C,W) = N(C, \operatorname{iso} C)$ is already a Segal space, so $N^f(C) = NC$. However, in general, it is difficult to compute the Reedy fibrant replacement of a simplicial space, and thus the classifying space. Later, we will show another way to obtain the classification space through a localisation functor.

Given a small indexing category I, we can define a subcategory of weak equivalence $\mathrm{we}(M^I) \subset M^I$. (4.4) and the Reedy fibrant replacement functor induces a natural map

$$f : N(M^I, \mathrm{we}(M^I)) \cong N(M,W)^{\mathrm{discnerve}\,I} \to N^f(M)^{\mathrm{discnerve}\,I}.$$

In general, the map f is not a weak equivalence, but a result by Dwyer and Kan says that, in some special cases, for example if M is cofibrantly generated, f is a weak equivalence. In that case, we see that the homotopy type of the classification diagram of the functor category M^I is entirely determined by the homotopy type of M. More precisely, we state the following theorem by Dwyer and Kan.

Theorem 4.2.12. Suppose J is a small indexing category and $M = \mathcal{S}^J$, then the map $f : N(M^I, \mathrm{we}(M^I)) \to N^f(M)^{\mathrm{discnerve}\, I} \cong N^f(M)^{NI}$ is a weak equivalence. In particlar, it induces a weak equivalence of Segal spaces $N^f(M^I) \to N^f(M)^{NI}$.

Proof. See [Rez01, Thm 8.11]. □

This is sometimes known as the strictification theorem.

Homotopy theory in Segal spaces and categories

Recall the construction of a simplicial enriched structure associated to a Segal space, given in Def. 4.2.2 (It may not a simplicial enriched category since associativity may not be strict). That construction allows us to construct the homotopy category associated to the Segal space.

We begin by defining homotopic morphisms.

Definition 4.2.13. Let W be a Segal space and $x, y \in \mathrm{Ob}\, W$. Two maps $f, g \in \mathrm{map}(x, y)$ are homotopic if they lie in the same path component of $\mathrm{map}(x, y)$, i.e., $[f] = [g] \in \pi_0 \mathrm{map}(x, y)$.

In general, given $k+1$ objects $x_0, \to, x_k \in \mathrm{Ob}\, W$, we can define $\mathrm{map}_W(x_0, \to, x_k)$ to be the fibre of the fibration $(\alpha_0, \dots, \alpha_n) : W_k \to W_0^k$ at the point $(x_0, \dots, x_k)$, where α_i is the map induced by $\alpha^i : [0] \to [k]$ taking 0 to i. The fibre of the commutative diagram

$$\begin{tikzcd} W_k \arrow[rr, "\sim", "\phi_k"'] \arrow[dr, "(\alpha_0,\dots,\alpha_n)"'] & & W_1 \times_{W_0} \cdots \times_{W_0} W_1 \arrow[dl] \\ & W_0^{k+1} & \end{tikzcd}$$

induces a trivial fibration

$$\phi_k : \mathrm{map}(x_0, \dots, x_k) \to \mathrm{map}(x_0, x_1) \times \cdots \times \mathrm{map}(x_{n-1}, x_n).$$

We can thus define composition of maps up to homotopy.

Definition 4.2.14. If $f \in \mathrm{map}(x, y)$ and $g \in \mathrm{map}(y, z)$, we define $g \circ f = d_1 h$ for some $h \in \mathrm{map}(x, y, z)$ such that $\phi_2(h) \sim (f, g)$ are homotopic.

Proposition 4.2.15. For each Segal space W, we have an associated homotopy category $\mathrm{Ho}\, W$ where $\mathrm{Ob}\, \mathrm{Ho}\, W = \mathrm{Ob}\, W$ and $\mathrm{map}_{\mathrm{Ho}\, W}(x, y) = \pi_0 \mathrm{map}_W(x, y)$.

Proof. It suffices to show that $f \circ (g \circ h) \sim (f \circ g) \circ h$ and $f \circ \mathrm{id} \sim f \sim \mathrm{id} \circ f$. See [Rez01, Prop 5.4] for details. □

Example 4.2.16. Let C be a category. Then, $\mathrm{Ob}\, NC = \mathrm{Ob}\, C$ and $\mathrm{map}_{NC}(x, y) \cong \hom_C(x, y)$ is the discrete simplicial set generated by elements of $\hom_C(x, y)$. Thus, $\mathrm{Ho}\, NC \cong C$.

Now, we define the notion of homotopy equivalence. Let $Z(3) = \mathrm{discnerve}(0 \to 2 \leftarrow 1 \to 3) \subset F(3)$. It induces a fibration $W_3 \to \mathrm{Map}_{s\mathcal{S}}(Z(3), W) \cong W_1 \times_{W_0} W_1 \times_{W_0} W_1$.

Proposition 4.2.17. Let $g \in \mathrm{map}_W(x, y)$. The following statements are equivalent:

1. There exist $f, h \in \mathrm{map}_W(y, x)$ such that $g \circ f \sim \mathrm{id}_y$ and $h \circ g \sim \mathrm{id}_x$.

2. $(\mathrm{id}_x, g, \mathrm{id}_y) \in \mathrm{Map}_{s\mathcal{S}}(Z(3), W)$ admits a lift to some $H \in W_3$.

Proof. Easy check. □

If g satisfies either of the above equivalent statements, we call g a **homotopy equivalence**.

Lemma 4.2.18. If $g \in (W_1)_0$ is a vertex connected by a path in $(W_1)_1$ to a vertex g', then g is a homotopy equivalence iff g' is one.

Proof. See [Rez01, Lemma 5.8]. □

Definition 4.2.19. We can thus define the space of homotopy equivalence to be the components $W_{\text{hoequiv}} \subset W_1$ of homotopy equivalences.

Note that $s_0 : W_0 \to W_1$ factors through W_{hoequiv} since $s_0 x = \mathrm{id}_x \in W_{\text{hoequiv}}$ for all $x \in W_0$.

Complete Segal spaces

We have now defined Segal spaces and their homotopy categories. However, there are too many Segal spaces with respect to other models of $(\infty, 1)$-categories. See Example 4.2.27 below. We define the following:

Definition 4.2.20. A complete Segal space is a Segal space such that the map $s_0 : W_0 \to W_{\text{hoequiv}}$ is a weak equivalence.

We want to find an object in $s\mathcal{S}$ that represents the functor $W \mapsto W_{\text{hoequiv}}$, at least up to weak equivalence. Let $E = \text{discnerve}(I[1])$. We have the following theorem:

Theorem 4.2.21. The map $\mathrm{Map}_{s\mathcal{S}}(E, W) \to W_1$ induced by the inclusion $F(1) \hookrightarrow E$ factors through $W_{\text{hoequiv}} \subset W_1$, and induces a weak equivalence $\mathrm{Map}_{s\mathcal{S}}(E, W) \to W_{\text{hoequiv}}$.

Proof. The proof is technical. See [Rez01, Thm 6.2]. □

Some corollaries of the theorem include

Corollary 4.2.22. Let W be a Segal space. The following are equivalent:

1. W is a complete Segal space.
2. The map $W_0 \to \mathrm{Map}_{s\mathcal{S}}(E, W)$ induced by $E \to F(0)$ is a weak equivalence.
3. For each pair $x, y \in \mathrm{Ob}\, W$, the fibre $\text{hoequiv}(x, y)$ of the fibration $W_{\text{hoequiv}} \xrightarrow{(d_1, d_0)} W_0 \times W_0$ is naturally weak equivalent to the space of paths in W_0 from x to y.

Proof. $(i) \Longleftrightarrow (ii)$ is clear from Thm. 4.2.21.

$(ii) \Longleftrightarrow (iii)$: consider the composition $\Delta : W_0 \xrightarrow{s_0} W_{\text{hoequiv}} \xrightarrow{(d_1, d_0)} W_0 \times W_0$. Since (d_1, d_0) is a fibration, $\text{hoequiv}(x, y)$ is actually a homotopy fibre (Prop. 4.1.7). The spcae of paths in W_0 from x to y is the homotopy fibre of Δ. □

Corollary 4.2.23. Let $\mathrm{Ob}\, W/\sim$ denote the set of homotopy equivalence classes of objects in $\mathrm{Ho}\, W$. If W is a complete Segal space, then $\pi_0 W_0 \cong \mathrm{Ob}\, W/\sim$.

Proof. Follows immediately from (iii) of the previous corollary. □

Example 4.2.24. The classifying diagram $N(C)$ for a category C and the classifying space $N^f(M)$ for a simplicial closed model category M are complete Segal spaces. See [Rez01] for details of the proofs. $\text{discnerve}(C)$ is not complete.

Definition 4.2.25. Let W be a Segal space, we define the completion of W to be a complete Segal space $\hat{W}$ with a map $i_W : W \to \hat{W}$ which is universal among all maps from W to a complete Segal space.

Proposition 4.2.26. There exists a completion functor given by $i_W : W \to \hat{W}$ on the category of Segal spaces.

Proof. Let $E(m) = \text{discnerve}(I[m])$. For each $n \geq 0$, we can define a simplicial set $\tilde{W}_n = \text{diag}([m] \mapsto (W^{E(m)})_n \cong \text{Map}_{s\mathcal{S}}(E(m) \times F(n), W))$ where the diagonal map $\text{diag} : s\mathcal{S} \cong \mathcal{S}^{(2)} \to \mathcal{S}$ is that induced by $[n] \mapsto [n] \times [n]$. The face and degeneracy maps induced from $d^i : F(n) \to F(n+1)$ and $s^i : F(n) \to F(n-1)$ gives us a simplicial space $\tilde{W}$ with a natural map $W \to \tilde{W}$. $\hat{W}$ is defined to be a functorial Reedy fibrant replacement of $\tilde{W}$, thus inducing a map $i_W : W \to \tilde{W} \to \hat{W}$.

For the details of the proof, see [Rez01]. □

In general, the above construction of the completion of a Segal space is not easy to understand.

Example 4.2.27. Given a category C, $\text{discnerve}\, C$ is a Segal space, but not complete. Its completion $\widehat{\text{discnerve}}\, C = NC$. As mentioned in Example 4.2.7, equivalent categories may give rise to non-weakly equivalent discrete nerves, but we see that the completions are weakly equivalent iff the categories are equivalent (Prop. 4.2.9).

Closed model category structures related to Segal spaces

In this section, we will introduce two other model category structures on the category of simplicial spaces $s\mathcal{S}$. The fibrant-cofibrant objects will be the Segal spaces and the complete Segal spaces in the two structures respectively. We will also show the relationship between weak equivalences in these two structures and with Reedy weak equivalences.

Theorem 4.2.28. There exists a closed model category structure on $s\mathcal{S}$ with the following properties:

1. The cofibrations are the monomorphisms.
2. The weak equivalences are maps f such that $\text{Map}_{s\mathcal{S}}(f, W)$ is a weak equivalence for all Segal spaces W.
3. The fibrations are the maps that satisfy the right lifting property with respect to all trivial cofibrations.

This is called the **Segal space model category structure** on $s\mathcal{S}$, an is denoted as $\mathcal{SS}$. This model structure is compatible with the cartesian closed standard model structure on $\mathcal{S}$. All objects are cofibrant and the fibrant objects are precisely the Segal spaces. A Reedy weak equivalence between two objects X, Y is a weak equivalence in $\mathcal{SS}$ and the converse is true if X, Y are Segal spaces.

Proof. $\mathcal{SS}$ is the left Bousfield localisation of the Reedy model category structure with respect to the set of maps $S = \{G(k) \hookrightarrow F(k)\}$. See [Rez01, Thm 7.1] for the proof of the compatibility with cartesian closure. □

Theorem 4.2.29. There exists a closed model category structure on $s\mathcal{S}$ with the following properties:

1. The cofibrations are the monomorphisms.
2. The weak equivalences are maps f such that $\text{Map}_{s\mathcal{S}}(f, W)$ is a weak equivalence for all complete Segal spaces W.

3. The fibrations are the maps that satisfy the right lifting property with respect to all trivial cofibrations.

This is called the **complete Segal space model category structure** on $s\mathcal{S}$, an is denoted as $\mathcal{CSS}$. This model structure is compatible with the cartesian closed standard model structure on $\mathcal{S}$. All objects are cofibrant and the fibrant objects are precisely the complete Segal spaces. A Reedy weak equivalence between two objects X, Y is a weak equivalence in $\mathcal{CSS}$ and the converse is true if X, Y are complete Segal spaces.

Proof. By Cor. 4.2.22, we see that $\mathcal{CSS}$ is the left Bousfield localisation of $\mathcal{SS}$ with respect to the map $E \to F(0)$. See [Rez01, Thm 7.2]for the proof of the compatibility with cartesian closure. □

Let Ho $\mathcal{SS}$ and Ho $\mathcal{CSS}$ be the homotopy categories associated to the two model structures respectively, and Ho $\mathcal{SS}_{cf}$ and Ho $\mathcal{CSS}_{cf}$ denote the respective full subcategories of fibrant-cofibrant objects, namely the Segal spaces and the complete Segal spaces respectively.

Note that for any model category M, the inclusion $M_{cf} \subset M$ induces an equivalence of homotopy categories Ho $M_{cf} \cong$ Ho M. This, in particular, implies that small homotopy limits and colimits exist in the subcategory M_{cf}.

An immediate consequence of the compatibility of these model structures with cartesian closure is that if W is a (complete) Segal space and X is a simplicial space, then W^X is a (complete) Segal space. In particular, given any two complete Segal spaces X and Y, we have a natural $(\infty, 1)$-category structure on the functors from X to Y given by the internal-hom object Y^X.

To understand the relationship between the two model category structures, we introduce the notion of Dwyer-Kan equivalence.

Definition 4.2.30. A map $f : U \to V$ between two Segal spaces is a Dwyer-Kan equivalence if

1. the induced map Ho f : Ho $U \to$ Ho V is an equivalence of categories; and
2. for each pair of objects $x, x' \in U$, the induced function $\mathrm{map}_U(x, x') \to \mathrm{map}_V(fx, fx')$ is a weak equivalence.

An equivalent formulation of condition (i) is

1. the induced map $\mathrm{Ob}\, U/\sim \to \mathrm{Ob}\, V/\sim$ is a bijection on the equivalence classes of objects.

We have the following theorem

Theorem 4.2.31. Let $f : U \to V$ be a map of Segal spaces. Then f is a Dwyer-Kan equivalence iff f is a weak equivalence in $\mathcal{CSS}$. If, in addition, U and V are complete Segal spaces, then f is a Dwyer-Kan equivalence iff f is a Reedy weak equivalence.

Proof. See [Rez01, Thm 7.7]. □

The proof of the above theorem relies on the following proposition, which we will state as it is also of interest on its own.

Proposition 4.2.32. The completion functor $i_W : W \to \hat{W}$ is a Dwyer-Kan equivalence and a weak-equivalence in $\mathcal{CSS}$.

Proof. See [Rez01, Sec. 14]. □

Corollary 4.2.33. If $f : U \to V$ map of Segal spaces is a weak equivalence in $\mathcal{SS}$, then $\hat{f} : \hat{U} \to \hat{V}$ is a weak equivalence in $\mathcal{CSS}$. Thus, the completion functor induces a well-defined functor between the homotopy categories i : Ho $\mathcal{SS}_{cf} \to$ Ho $\mathcal{CSS}_{cf}$.

Proof. This follows immediately from Prop. 4.2.32 and the fact that a weak equivalence in $\mathcal{SS}$ is a weak equivalence in $\mathcal{CSS}$. □

Classifying diagrams: another perspective

To end of this section, we return to the example of the classifying diagram of a category C with weak equivalences W. The ideas for this section are derived from the notes of a talk by Toën [Toe05]. The following diagram of categories

$$\begin{tikzcd} \coprod_{f\in W}[1] \arrow[r, "\sqcup i"] \arrow[d] & \coprod_{f\in W} I[1] \\ C & \end{tikzcd} \tag{4.5}$$

induces a pushout square in $s\mathcal{S}$ of the classifying diagrams

$$\begin{tikzcd} \coprod_{f\in W} F(1) \arrow[r, "\sqcup N(i)"] \arrow[d] & \coprod_{f\in W} N(I[1]) \arrow[d] \\ NC \arrow[r] & \tilde{N}(C,W) \end{tikzcd} .$$

Since $\sqcup i$ is a cofibration and all objects are cofibrant in $\mathcal{CSS}$, the square is also a homotopy pushout (see [HTT, Prop A.2.2.4]). Since $\mathcal{CSS}_{cf}$ is closed under homotopy pushouts, there exists $\hat{N}(C,W)\in\mathcal{CSS}_{cf}$ such that $\hat{N}(C,W)$ is weakly equivalent to $\tilde{N}(C,W)$ and

$$\begin{tikzcd} \coprod_{f\in W} F(1) \arrow[r, "\sqcup N(i)"] \arrow[d] & \coprod_{f\in W} N(I[1]) \arrow[d] \\ NC \arrow[r] & \hat{N}(C,W) \end{tikzcd}$$

is a homotopy pushout square in $\mathcal{CSS}_{cf}$.

If $W\subset \operatorname{iso} C$, there exists a lift

$$\begin{tikzcd} \coprod_{f\in W} F(1) \arrow[r, "\sqcup N(i)"] \arrow[d] & \coprod_{f\in W} N(I[1]) \arrow[d] \arrow[dl, dashed, "\exists"'] \\ NC \arrow[r] & \hat{N}(C,W) \end{tikzcd} ,$$

so $\hat{N}(C,W)=NC$.

We have the following result:

Proposition 4.2.34. Suppose $W\supset \operatorname{iso} C$, then $\hat{N}(C,W)=\widehat{N(C,W)^f}$ is the Segal completion of a functorial fibrant replacement of $N(C,W)$.

Proof. Let $N([1],[1])$ be a classifying diagram of $C=[1]$ with all morphisms regarded as weak equivalences. Then, it is clear that the following is a homotopy pushout square:

$$\begin{tikzcd} \coprod_{f\in W} F(1) \arrow[r, "\sqcup N(i)"] \arrow[d] & \coprod_{f\in W} N([1],[1]) \arrow[d] \\ NC \arrow[r] & N(C,W) \end{tikzcd} .$$

The natural inclusion $N([1],[1]) \subset N(I[1])$ is a Reedy weak equivalence. This is because, for every $m \geq 0$, $\operatorname{nerve}\operatorname{we}([1]^{[m]}) \to \operatorname{nerve}\operatorname{iso}(I[1]^{[m]})$ induces a weak equivalence since their geometric realisations are both contractible.

Hence, the homotopy pushout induces a weak equivalence $N(C,W) \xrightarrow{\sim} \hat{N}(C,W)$. □

Hence, in the case where $C = M$ is a model category and W is the set of weak equivalences, we get $\hat{N}(M,W) = N^f(M)$, the classifying diagram we previously defined.

4.3 Segal Categories

While complete Segal spaces are a good model of weak $(\infty,1)$-categories, a key disadvantage is that it is difficult to construct. In most constructions, we obtain a Segal space that is not complete. The completion functor is abstract and difficult to compute in practice. To overcome this problem, another model, the Segal categories, was developed. Segal categories avoid the cumbersome procedure of completion by imposing an additional condition on the simplicial spaces we are considering (that the 0-space is discrete).

Segal categories were formally defined by Dwyer, Kan and Smith in [DKS89]. They were used extensively and generalised to n-Segal categories by Hirschowitz and Simpson in their studies of n-stacks [HS]. In this section, we will follow the ideas in [HS] but refer to the work of Bergner [Ber07c] for more explicit constructions.

Segal precategories, categories and closed model structure

Definition 4.3.1. A simplicial space X is a Segal precategory if X_0 is discrete. Let $\mathcal{PC}at$ be the category of Segal pre-categories. A Segal precategory X that satisfies the Segal condition (4.2) is called a Segal category.

Example 4.3.2. Let C be any category, then $\operatorname{discnerve} C$ is a Segal category. Note that under the model category structure that we are going to impose on $\mathcal{PC}at$, $f : C \to D$ is an equivalence of categories iff $\operatorname{discnerve} f$ is a weak equivalence (unlike in the Reedy model structure or $\mathcal{SS}$).

Every simplicial enriched category C can be seen as a Segal precategory, which we will also denote by C, by setting $C_0 = \operatorname{Ob} C$ and $C_n = \sqcup_{x,y \in \operatorname{Ob} C}(\operatorname{map}_C(x,y))_{n-1}$ with appropriate face and degeneracy maps. Note that, however, a Segal category may not be a simplicial category since associativity is not strict. To obtain a simplicial category, we will need to consider the category generated by the Segal category, which we will not define here.

Recall that given a simplicial space X, we can define $\operatorname{Ob} X$ and mapping spaces $\operatorname{map}_X(x,y)$ (Def. 4.2.2). We denote $x \sim y$ if $\operatorname{map}_X(x,y) \neq \emptyset$. However, $\sim$ may not be an equivalence relation. We denote by the same symbol the equivalence relation generated by $\sim$.

We have a well-defined notion of equivalence of Segal categories.

Definition 4.3.3. Let $f : A \to B$ be a morphism of Segal categories. We say that f is a Dwyer-Kan equivalence if

1. the induced map $\operatorname{Ob} A/\sim \to \operatorname{Ob} B/\sim$ is surjective (we say that f is essentially surjective); and
2. for any $x, y \in A$, the induced map $\operatorname{map}_A(x,y) \to \operatorname{map}_B(fx, fy)$ is a weak equivalence (we say that f is fully faithful).

Note that injectivity in condition (i) follows from condition (ii).

To define the closed model structure on Segal precategories, we first need to construct a functor $L_C : \mathcal{PC}at \to \mathcal{PC}at$ that sends a precategory X into a Segal category $L_C X$. Hirschowitz and Simpsons proved the existence of such a functor (indeed for Segal n-categories) in [HS] but we shall give the explicit construction given by Bergner [Ber07c].

We want to construct $L_C X$ as a functorial fibrant replacement of X in the Segal space model category structure $\mathcal{SS}$, in such a way that $L_C X$ is still a precategory. We proceed by the small object argument. We have a set of generating trivial cofibrations in $\mathcal{SS}$ (obtained from the generating trivial cofibrations of the Reedy model structure by localisation):

$$F(k) \times \Lambda^l_t \sqcup_{G(k)\times\Lambda^l_t} G(k) \times \Delta^l \to F(k) \times \Delta^l, \qquad k \geq 0, l \geq 1, 0 \leq t \leq l$$

where $G(0) = \emptyset$. The fibrant replacement can be constructed as a colimit of the iterated pushout

$$\begin{tikzcd} \coprod (F(k) \times \Lambda^l_t \sqcup_{G(k)\times\Lambda^l_t} G(k) \times \Delta^l) \arrow[r] \arrow[d] & X_i \arrow[d] \\ \coprod (F(k) \times \Delta^l) \arrow[r] & X_{i+1} \end{tikzcd}$$

where $X = X_0$ and the coproduct is taken over all maps $F(k) \times \Lambda^l_t \sqcup_{G(k)\times\Lambda^l_t} G(k) \times \Delta^l \to X_i$ with $k \geq 0$, $l \geq 1$ and $0 \leq t \leq l$.

Note that the map on the zero space induced by a generating trivial cofibration is given by $[k] \times \Lambda^l_t \sqcup [k] \times \Delta^l \cong [k] \times \Delta^l \xrightarrow{\mathrm{id}} [k] \times \Delta^l$ for $k > 0$ and $\Lambda^l_t \to \Delta^l$ for a $k = 0$.

Thus, the Segal space thus obtained may not have a discrete 0-space, and is thus not a Segal category. To fix this, we exclude the maps with $k = 0$ and consider the iterated pushout with respect to the coproduct over the subset of maps with $k > 0$. Let $L_C X$ be the colimit of this sequence of pushouts.

Proposition 4.3.4. $L_C X$ as defined above is a functorial fibrant replacement of X, so $L_C : \mathcal{PC}at \to \mathcal{PC}at$ is a well-defined functor taking a precategory to a Segal category that is also a Segal space.

By the small object argument, to check that this is indeed a fibrant replacement of X in $\mathcal{SS}$, it suffices to check that $L_C X$ satisfies the RLP with respect to all generating trivial cofibrations with $k = 0$. This is equivalent to checking that the lift exists in the following diagram

$$\begin{tikzcd} \Lambda^l_t \arrow[r] \arrow[d] & \mathrm{Map}_{s\mathcal{S}}(F(0), L_C X) \cong (L_C X)_0 \ . \\ \Delta^l \arrow[ur, dashed] & \end{tikzcd}$$

This is true since $L_C X$ is a discrete simplicial set and hence a Kan complex. □

We remark that in the case where $f : X \to Y$ is a map between two Segal categories which are also Segal spaces, then the two definitions of Dwyer-Kan equivalence are equivalent.

We are now ready to define a closed model category structure on precategories.

Theorem 4.3.5. There exists a closed model category structure on $\mathcal{PC}at$ in which

1. the cofibrations are precisely the monomorphisms;

2. the weak equivalences are precisely the maps $f : X \to Y$ such that $L_C f : L_C X \to L_C Y$ is a Dwyer-Kan equivalence; and

3. the fibrations are maps that satisfy the right lifting property with respect to all trivial cofibrations.

We denote $\mathcal{PC}at$ equipped with this model category structure $\mathcal{SC}$. The fibrant-cofibrant objects of this model structure are precisely the Reedy-fibrant Segal categories.

Proof. See [HS, Thm 2.3] and [Ber07c, Thm 5.1] for two different proofs of the existence of this model structure. See [Ber07c, Cor 5.13] and [Ber07a, Thm 3.2] for the proof of the last statement. □

Let SeCat $\subset \mathcal{SC}$ and Ho SeCat $\subset$ Ho $\mathcal{SC}$ be the full subcategories of Segal categories. Since SeCat contains all fibrant-cofibrant objects, Ho SeCat $\cong$ Ho $\mathcal{SC}$ and in particular contains all small homotopy limits and colimits.

Segal localisation

We want to construct a localisation similar to that in Section 4.2. Let C be a category and $W \subset C$ be a subcategory of weak equivalences. Taking discrete nerves on the diagram of categories (4.5), we obtain a pushout diagram in $\mathcal{SC}$:

$$\begin{tikzcd} \coprod_{f\in W} F(1) \arrow[r] \arrow[d] & \coprod_{f\in W} E \arrow[d] \\ \operatorname{discnerve} C \arrow[r] & \tilde{C} \end{tikzcd} .$$

Since the top arrow is a cofibration and all objects are cofibrant, it is a homotopy pushout square as well. Hence, there exists $L(C,W) \in$ SeCat (indeed, we can even choose $L(C,W) \in \mathcal{SC}_{cf}$ the subcategory of Reedy-fibrant Segal categories) such that $L(C,W)$ is weakly equivalent to $\tilde{C}$ and the following diagram is a homotopy pushout square in SeCat:

$$\begin{tikzcd} \coprod_{f\in W} F(1) \arrow[r] \arrow[d] & \coprod_{f\in W} E \arrow[d] \\ \operatorname{discnerve} C \arrow[r] & L(C,W) \end{tikzcd} .$$

Note that since the completion functor commutes with colimits and $\widehat{\operatorname{discnerve}}\, C \cong NC$, we have that $\widehat{L(C,W)} \cong \hat{N}(C,W)$.

If $W = \operatorname{iso} C$, we get $LC = \operatorname{discnerve} C$ as seen in Section 4.2.

A result of Dwyer and Kan gives us a more explicit construction of $L(C,W)$.

Theorem 4.3.6. Let (C,W) be a category with a subcategory of weak equivalences. Then $L(C,W)$ is given by the hammock localisation $L^H(C,W)$.

Proof. See [DK80a]. □

Now, let M be a simplicial closed model category, with weak equivalences W. Let $M_{cf} \subset M$ be the simplicial subcategory. Dwyer and Kan proved the following result:

Theorem 4.3.7. Let M be a closed simplicial model category. Then, $LM = L(M,W) \cong L^H M \cong M_{cf}$ is a weak equivalence.

Proof. See [DK80b]. □

This theorem allows us to compute the localisation of simplicial model categories easily.

Example 4.3.8. 1. Let $M = \mathcal{S}$ be the category of simplicial sets with the usual model structure, then M_{cf} is the subcategory of Kan complexes. M_{cf} satisfies the Segal condition since homotopy is an equivalence relation on Kan complexes. Hence, $L\mathcal{S}$ gives us the theory of Kan complexes (or equivalently CW-complexes) as an $(\infty, 1)$-category.

2. Let $M = \text{Top}$ be the category of topological spaces, then $L\text{Top} \cong \text{Top}_{cf}$ is the subcategory of CW-complexes. Hence, we see that the localisation of $\mathcal{S}$ and Top give the same Segal category.

3. Let $M = s\mathcal{S}$ with the Reedy model structure. We thus get $Ls\mathcal{S}$ as the Segal category of all Reedy-fibrant simplicial spaces. If we equip the simplicial spaces with the (complete) Segal space model structure, we get $L\mathcal{S}\mathcal{S}$ ($L\mathcal{C}\mathcal{S}\mathcal{S}$) the Segal category of all (complete) Segal spaces.

4. Let $M = \mathcal{S}\mathcal{C}$, then $L\mathcal{S}\mathcal{C}$ is the Segal category of all Segal categories (up to some change in universe).

We see that Segal localisation gives us a $(\infty, 1)$-category of $(\infty, 1)$-categories. As in the case of complete Segal spaces, we have a strictification theorem.

Theorem 4.3.9. Let M be a cofibrantly generated simplicial model category and I a small category, then we have a weak equivalence of Segal categories (Dwyer-Kan equivalence)

$$L(M^I) \cong L(M)^{\text{discnerve}\, I}.$$

Proof. See [HS] and [TV]. □

In particular, this implies that to compute small limits and colimits of (complete) Segal spaces or Segal categories, it suffices to compute the homotopy limits and colimits in the larger categories of simplicial spaces or precategories. The strictification theorem also gives a form of Yoneda's lemma for $(\infty, 1)$-categories. See Toën and Vezzosi's paper [TV] for more details.

4.4 Comparison theorems

In this section, we will show that complete Segal spaces, Reedy-fibrant Segal categories and quasi-categories are equally valid models for $(\infty, 1)$-categories. To this end, we will state a number of comparison theorems between complete Segal spaces, Segal categories and other models of $(\infty, 1)$-categories.

The main mechanism in proving the equivalence of two model categories is Quillen equivalence. Recall the following definitions:

Definition 4.4.1. A pair of adjoint functors $F : C \rightleftarrows D : G$ (with $\phi : \text{Hom}_D(FX, Y) \xrightarrow{\sim} \text{Hom}_C(X, GY)$) is a **Quillen pair** if it satisfies one of the following equivalent conditions:

1. F preserves cofibrations and G preserves fibrations;

2. F preserves cofibrations and trivial cofibrations; or

3. G preserves fibrations and trivial fibrations.

A Quillen pair (F, G) is a **Quillen equivalence** if, in addition, $f : FX \to Y$ is a weak equivalence in D iff $\phi f : X \to GY$ is a weak equivalence in C.

We also recall the theorem:

Theorem 4.4.2. A Quillen pair $F : C \rightleftarrows D : G$ induces an adjoint pair of left and right total derived functors

$$LF : \mathrm{Ho}\ C \rightleftarrows \mathrm{Ho}\ D : RG$$

which is an equivalence of categories if (F, G) is a Quillen equivalence. In this case, we also have an equivalence of full subcategories

$$LF : \mathrm{Ho}\ C_{cf} \rightleftarrows \mathrm{Ho}\ D_{cf} : RG.$$

This implies that if we have a Quillen equivalence between two models of infinity categories, they have the same objects up to weak equivalences. Furthermore, weak equivalences in the full subcategory $M_{cf} \subset M$ of a model category is the same as homotopy equivalence, so they are defined up to homotopy equivalence.

We have the following diagram of Quillen equivalences.

$$\begin{tikzcd}
\mathcal{CSS} \arrow[dr, bend left] & \mathcal{SC} \arrow[l] \arrow[d, bend left] & \mathcal{SC}' \arrow[l] \arrow[r] & s\mathcal{C} \\
& \mathcal{QC} \arrow[ul, bend left] \arrow[u, bend left] \arrow[urr] & &
\end{tikzcd}$$

An arrow in the diagram refers to the direction of the left adjoint. $\mathcal{QC}$ is the category of simplicial sets with the Joyal model category structure, $s\mathcal{C}$ is the category of simplicial enriched categories and $\mathcal{SC}'$ is another model structure on precategories.

We shall only present the Quillen equivalences among $\mathcal{CSS}$, $\mathcal{SC}$ and $\mathcal{QC}$. We refer to Bergner's paper [Ber07a] for the construction of the fibrant model category structure $\mathcal{SC}'$ on precategories (as opposed to the cofibrant structure $\mathcal{SC}$). It was constructed to proved the equivalence with $s\mathcal{C}$. The equivalence between $\mathcal{SC}$ and $\mathcal{SC}'$ is given by the identity functor. For the equivalence between simplicial categories and quasi-categories, we refer to [Joy07].

In this section, we will not present the proofs, but instead just sketch the construction of the functors or model structures.

Some model category structures

First we construct the three model category structures that we have yet to construct.

Definition 4.4.3. Let X be a simplicial set, for any $x, y \in X_0$, we write $x \sim y$ if there exists $w \in X_1$ such that $d_1 w = x$ and $d_0 w = y$. we define $\tau_0 X$ be the set of equivalence class in X_0 under the equivalence relation generated by $\sim$.

Let $X(x, y)$ be the fibre of the projection $X^I \xrightarrow{(d_1, d_0)} X \times X$.

Let $f : X \to Y$ be a map of simplicial sets, we say that f is essentially surjective if the induced map $\tau_0 X \to \tau_0 Y$ is surjective. We say that f is fully faithful if for every pair $x, y \in X_0$, the induced map $X(x, y) \to Y(fx, fy)$ is a weak equivalence (under the standard model category structure). We say that f is a weak categorical equivalence if f is both essentially surjective and fully faithful.

Theorem 4.4.4. There is a closed model category structure on $\mathcal{S}$, which we will denote as $\mathcal{QC}$, the quasi-category model structure, in which

1. the cofibrations are precisely the monomorphisms;

2. the weak equivalences are precisely the weak categorical equivalences; and

3. the fibrations are maps satisfying the right lifting property with respect to trivial cofibrations.

The fibrant-cofibrant objects in this model structure are precisely the quasi-categories.

Proof. See [Joy08]. □

Next we define a closed model category structure on simplical categories.

Theorem 4.4.5. There is a closed model category structure on $s\mathcal{C}$, in which

1. the weak equivalences are the Dwyer-Kan equivalences;
2. the fibrations are maps $F : C \to D$ satisfying:
   - for any $x, y \in C$, the induced map $\mathrm{map}_C(x, y) \to \mathrm{map}_D(Fx, Fy)$ is a fibration of simplicial sets;
   - for any $x_1 \in C$, $y \in D$ and homotopy equivalence $e : Fx_1 \to y$ in D, there exists $x_2 \in C$ and homotopy equivalence $d : x_1 \to x_2$ such that $Fd = e$.
3. the cofibrations are maps satisfying the left lifting property with respect to all trivial fibrations.

Proof. See [Ber07b] □

Quillen equivalences between models of $(\infty, 1)$-categories

First consider the projection and inclusion functors (on the first component) $p_1 : \Delta \times \Delta \to \Delta : ([m], [n]) \mapsto [m]$ and $i_1 : \Delta \to \Delta \times \Delta : [n] \mapsto ([n], 0)$. They induce an adjoint pair of functors

$$p_1^* : \mathcal{S} \rightleftarrows \mathcal{S}^{(2)} : i_1^*$$

between simplicial sets and bisimplicial sets. Under the identification of bisimplicial sets with simplicial spaces, p_1^* sends a simplicial set X into a discrete simplicial space $\tilde{X}$ with $\tilde{X}_m$ being the discrete simplicial set generated by X_m. i_1^* associates a simplicial space Y with the simplicial set with n-simplices given by $(Y_n)_0$.

Theorem 4.4.6. The adjoint pair

$$p_1^* : \mathcal{QC} \rightleftarrows \mathcal{CSS} : i_1^*$$

is a Quillen equivalence.

Proof. See [JT07]. □

Let $\Delta^{|2} = ([0] \times \Delta)^{-1}(\Delta \times \Delta)$ where we formally invert all morphisms in $[0] \times \Delta$. There is a canonical map $\pi : \Delta \times \Delta \to \Delta^{|2}$. Since $p_1 : \Delta \times \Delta \to \Delta$ sends all morphisms in $[0] \times \Delta$ to invertible morphisms, it factors through $q : \Delta^{|2} \to \Delta$ where $q\pi = p_1$. Define $j = \pi i_1 : \Delta \to \Delta^{|2}$. Then q and j restrict p_1 and i_1 to $\Delta^{|2}$ and induce an adjoint pair of functors

$$q^* : \mathcal{S} \rightleftarrows \mathcal{PC}at : j^*.$$

Explicitly, q^* takes a simplicial set X to the discrete bisimplicial space $\tilde{X}$ with $\tilde{X}_m$ being the discrete simplicial set generated by X_m. j^* takes a precategory Y to the simplicial set Y_{*0}.

Theorem 4.4.7. The adjoint pair

$$q^* : \mathcal{QC} \rightleftarrows \mathcal{SC} : j^*$$

is a Quillen equivalence.

Proof. See [JT07]. □

We now show a Quillen equivalence between $\mathcal{CSS}$ and $\mathcal{SC}$. We have a natural inclusion of precategories into simplicial spaces $I : \mathcal{SC} \to \mathcal{CSS}$. We can construct a right adjoint, the discretization functor $R : \mathcal{CSS} \to \mathcal{SC}$ defined by the homotopy pullback square

$$\begin{tikzcd} RW \arrow[r] \arrow[d] & \operatorname{cosk}(W_{0,0}) \arrow[d] \\ W \arrow[r] & \operatorname{cosk}(W_0) \end{tikzcd},$$

that is we take the discretization of the 0-space of W. If W is a complete Segal space, we can explicitly define RW by $RW_0 = W_{0,0}$ is a discrete simplicial set, RW_1 by the homotopy pullback

$$\begin{tikzcd} RW_1 \arrow[r] \arrow[d] & W_{0,0} \times W_{0,0} \arrow[d] \\ W_1 \arrow[r] & W_0 \times W_0 \end{tikzcd}$$

and $RW_k = RW_1 \times_{RW_0} \cdots \times_{RW_0} RW_1$ for $k \geq 2$.

Theorem 4.4.8. The adjoint pair

$$I : \mathcal{SC} \rightleftarrows \mathcal{CSS} : R$$

is a Quillen equivalence.

Proof. See [Ber07a]. □

Note that $I = \pi^*$ as defined above, so we have a commutative triangle $p_1^* = Iq^*$.

With these three Quillen equivalences, we can thus conclude that the categories of quasi-categories, complete Segal spaces and Reedy-fibrant Segal categories are homotopy equivalent.

For the other Quillen equivalences, interested readers can refer to [HS].

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EXPOSÉ **5**

Simplicial presheaves

The goal of this chapter is to explain the transition from the classical theory of stacks to the homotopical one. The main reference is [Hol07]. This chapter is organized as follows:

1. an introductory section about the classical theory of stacks, with some example;
2. the model structure on presheaves in groupoids and the equivalence;
3. the model structure on simplicial presheaves;
4. higher stacks and examples.

Mauro Porta

5.1 Review of sheaf theory

We add this section just for sake of completeness. It wasn't included in the exposition of the 03/29/2013. It gives some major motivation for constructions introduced later, and fixes notations and results.

Grothendieck topologies

The reader is supposed to already have a familiarity with Grothendieck topologies. I strongly recommend the book of MacLane and Moerdijk [MM92, Ch. III] for a clear exposition of this theory. A neat treatment can be also found in the article [Vis04, Ch. II].

A Grothendieck topology is a sort of generalization of the notion of covering. It consists in a family of given covering maps for each object in the category.

Definition 5.1.1. Let $\mathcal{C}$ be a category. A sieve on an object $X \in \mathrm{Ob}(\mathcal{C})$ is a subfunctor $R \subset h_X$.

Notation. Let $\mathcal{C}$ be a category, $X \in \mathrm{Ob}(X)$; if R is a sieve on X and $\varphi : Y \to X$ is any morphism in $\mathcal{C}$ we can form the pullback diagram

$$\begin{tikzcd} h_Y \times_{h_X} R \arrow[r] \arrow[d, "j"] & R \arrow[d, "i"] \\ h_Y \arrow[r, "\varphi^*"] & h_X \end{tikzcd}$$

Since mono are stable under pullback, we see that $h_Y \times_{h_X} R$ is a sieve on Y. We will denote such sieve with the notation $\varphi^* R$.

Remark 5.1.2. A sieve on X can be described alternatively as a set of arrows $\{\varphi_i : U_i \to X\}_{i\in I}$ whose target is X and which is closed for composition on the right. Under identification, we have

$$\varphi^* R := \{f : U \to Y \mid \varphi \circ f \in R\}$$

Definition 5.1.3. Let $\mathcal{C}$ be a category. A (Grothendieck) topology on $\mathcal{C}$ is a family $\mathcal{J} = \{J(X)\}_{X\in \mathrm{Ob}(\mathcal{C})}$ where each $J(X)$ is a collection of sieves on X, called *covering sieves*, satisfying the following conditions:

1. $h_X \in J(X)$ for each $X \in \mathrm{Ob}(X)$;
2. if $R \in J(X)$ and $\varphi : Y \to X$ is any arrow in $\mathcal{C}$, $\varphi^* R \in J(Y)$;
3. if $R \in J(X)$ and S is a sieve on X such that, for each $\varphi : U \to X \in R$, $\varphi^* S \in J(U)$, then $S \in J(X)$.

Example 5.1.4. 1. Let (X, τ) be a topological space...

2. the Ãľtale topology
3. the fpqc topology
4. the fppf topology

- pretopologies, elementary properties.

Grothendieck topoi

- the set of descent data;
- the notion of sheaf: as object with properties;
- the notion of sheaf: as local object
- the notion of point; stalks.

Theorem 5.1.5. Let $(C, \mathcal{J})$ be a site. Let S be the class of arrows formed by all the inclusions of covering sieves $R \subset h_X$, $R \in J(X)$. Then the inclusion $\mathrm{Sh}(\mathcal{C}, \mathcal{J}) \to \mathrm{PSh}(\mathcal{C})$ induces an equivalence

$$\mathrm{Sh}(\mathcal{C}, \mathcal{J}) \to \mathrm{PSh}(\mathcal{C})[S^{-1}]$$

5.2 Fibered categories and stacks

Fibered categories

Definitions and generalities

Our exposition will follow [Vis04, Ch. III], with some integration from [GR71, ExposÃľ VI]. I strongly recommend the reader to think to fiber bundle (vector bundle if he prefers) while reading these notes.

For the whole exposition $\mathcal{C}$ will denote a fixed category.

Notation. If $\mathcal{C}$ is a category and $U \in \mathrm{Ob}(\mathcal{C})$ is an object in $\mathcal{C}$, we will denote by $\kappa(U)$ the subcategory of $\mathcal{C}$ having U as unique object and id_U as unique morphism, i.e. $\kappa(U)$ is the unique morphism $\Delta^0 \to \mathcal{C}$ defined by $* \mapsto U$.

Definition 5.2.1. Let $p\colon \mathcal{F} \to \mathcal{C}$ be a category over $\mathcal{C}$ and let $U \in \mathrm{Ob}(\mathcal{C})$. We define the fiber of $\mathcal{F}$ over U as the subcategory of $\mathcal{F}$ mapping to $\kappa(U)$ via p. We will denote the fiber of $\mathcal{F}$ over U as $\mathcal{F}_U$.

Remark 5.2.2. If $p\colon \mathcal{F} \to \mathcal{C}$ is a category over $\mathcal{C}$ and $U \in \mathrm{Ob}(\mathcal{C})$, then $\mathcal{F}_U$ can be clearly described as the pullback (computed in **Cat**):

$$\begin{tikzcd} \mathcal{F}_U \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "p"] \\ \kappa(U) \arrow[r] & \mathcal{C} \end{tikzcd}$$

Definition 5.2.3. Let $\mathcal{C}$ be a category and let $p_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{C}$ be a category over $\mathcal{C}$. An arrow $\phi : \xi \to \eta$ in $\mathcal{F}$ is said to be *cartesian* if for every other arrow $\psi : \zeta \to \eta$ and any arrow $h\colon p_{\mathcal{F}}\zeta \to p_{\mathcal{F}}\xi$ in $\mathcal{C}$ such that $p_{\mathcal{F}}\phi \circ h = p_{\mathcal{F}}\psi$ there exists a unique arrow $\theta : \zeta \to \xi$ with $p_{\mathcal{F}}\theta = h$ and $\phi \circ \theta = \psi$.

The following lemma contains some trivial but useful properties:

Lemma 5.2.4. Let $p\colon \mathcal{F} \to \mathcal{C}$ be a category over $\mathcal{C}$. Then:

1. the composition of two cartesian arrows is still cartesian;
2. if $\phi : \xi \to \eta$ is a cartesian arrow lying over $\mathrm{id}_{p(\eta)}$, then ϕ is an isomorphism;
3. every isomorphism in $\mathcal{C}$ is cartesian over its image.

Definition 5.2.5. Let $p_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{C}$ be a category over $\mathcal{C}$. We say that $\mathcal{F}$ is fibered over $\mathcal{C}$ if for every arrow $f : y \to x$ in $\mathcal{C}$ and any object $\eta \in \mathrm{Ob}(\mathcal{F})$ such that $p_{\mathcal{F}}(\eta) = x$ there is a cartesian arrow $\phi : \xi \to \eta$ lying over f.

Example 5.2.6. Consider a (Serre) fibration $p\colon X \to Y$ in **CGHaus**. Applying the fundamental groupoid functor $\Pi\colon$ **CGHaus** $\to$ **Grpd** we get a functor $\Pi(p)\colon \Pi(X) \to \Pi(Y)$, and we claim that this functor defines a fibered category. In fact, choose an object $\eta \in \Pi(X)$, set $x = \Pi(p)(\eta)$. For every arrow $[\gamma]\colon y \to x$, represented by a continuous path

$$\gamma\colon I = [0,1] \to Y$$

introduce $\overline{\gamma}\colon I \to Y$, $\overline{\gamma}(t) = \gamma(1-t)$; since p is a fibration we can lift $\overline{\gamma}$ to a path $[0,1] \to X$ sending 0 to η. The lifting is unique up-to-homotopy, hence we obtain (taking again the inverse) an arrow

$$[\phi]\colon \xi \to \eta$$

mapping via $\Pi(p)$ to $[\gamma]$. Since $[\phi]$ is an isomorphism, it is cartesian (Lemma 5.2.4), and so the assertion is proved.

A fibered category has a property of homogeneity of fibers, as we are going to prove. Let's fix some notation. If $p\colon \mathcal{F} \to \mathcal{C}$ is a fibered category and $f : V \to U$ is an arrow in $\mathcal{C}$, denote by

$$(\mathcal{F} \downarrow \mathcal{F}_U)_f$$

the full subcategory of $(\mathcal{F} \downarrow \mathcal{F}_U)_f$ consisting of arrows ϕ such that $p(\phi) = f \circ g$ for some g in $\mathcal{C}$.

Lemma 5.2.7. Let $p\colon \mathcal{F} \to \mathcal{C}$ be a fibered category. For every arrow $f : V \to U$ in $\mathcal{C}$, there exists a functor

$$\Phi_f : \mathcal{F}_U \to (\mathcal{F} \downarrow \mathcal{F}_U)_f$$

sending an object $\eta \in \mathrm{Ob}(\mathcal{F}_U)$ to a *cartesian* arrow $\phi : \xi \to \eta$, with $\xi \in \mathrm{Ob}(\mathcal{F}_V)$. is a cartesian arrow.

Proof. Consider the second projection functor:

$$\Psi_f : (\mathcal{F} \downarrow \mathcal{F}_U)_f \to \mathcal{F}_U$$

For each $\eta \in \mathrm{Ob}(\mathcal{F}_U)$ choose a cartesian arrow $\phi : \xi \to \eta$ lying over f. Then the pair (ϕ, id_η) is a universal arrow from Ψ_f to η.[1] It follows from the standard characterization of adjunctions (cfr. [Mac71, Theorem IV.1.2.(iv)] that Ψ_f has a right adjoint

$$\Phi_f : \mathcal{F}_U \to (\mathcal{F}_V \downarrow \mathcal{F}_U)$$

□

Denote by $(\mathcal{F}_V \downarrow \mathcal{F}_U)_f$ the full subcategory of $(\mathcal{F}_V \downarrow \mathcal{F}_U)_f$ of arrows mapping to f via p. Let

$$\mathbf{d} \colon (\mathcal{F}_V \downarrow \mathcal{F}_U)_f \to \mathcal{F}_V$$

the projection on the first component. Then, consider the functor f^* defined as

$$f^* := \mathbf{d} \circ \Phi_f : \mathcal{F}_U \to \mathcal{F}_V$$

In this construction we are hiding the axiom of choice. We know from [Mac71, Theorem IV.1.2.(iv)] that to construct the adjoint f^* one has only to choose universal arrows for every object in $\mathcal{F}_U$. If we make this choice for every arrow $f : V \to U$ we obtain what is traditionally called a *cleavage*:

Definition 5.2.8. A *cleavage* of a fibered category $p \colon \mathcal{F} \to \mathcal{C}$ consists of a class K of cartesian arrows in $\mathcal{F}$ such that for each arrow $f : U \to V$ in $\mathcal{C}$ and each object η in $\mathcal{F}_V$ there exists a unique arrow in K with target η mapping to f via p.

Lemma 5.2.9. Let $p \colon \mathcal{F} \to \mathcal{C}$ be a fibered category with cleavage K. Then

1. for each object $U \in \mathrm{Ob}(\mathcal{C})$ there is an isomorphism

$$\varepsilon_U : \mathrm{id}_U^* \to \mathrm{Id}_{\mathcal{F}_U}$$

2. for each pair of composable arrows $f : V \to U$ and $g \colon W \to V$ in $\mathcal{C}$ there is a natural isomorphism

$$\alpha_{f,g} : g^* f^* \to (fg)^*$$

3. for each arrow $f : V \to U$ strict equalities

$$\alpha_{\mathrm{id}_V, f} = \varepsilon_V f^*, \qquad \alpha_{f, \mathrm{id}_U} = f^* \varepsilon_U$$

4. for each triple of arrows

$$T \xrightarrow{h} W \xrightarrow{g} V \xrightarrow{f} U$$

a diagram (strictly) commutative

$$\begin{tikzcd}
h^* g^* f^* \arrow[r, "\alpha_{h,g} f^*"] \arrow[d, "h^* \alpha_{h,g}"'] & (gh)^* f^* \arrow[d, "\alpha_{gh,f}"] \\
h^* (fg)^* \arrow[r, "\alpha_{h,fg}"] & (fgh)^*
\end{tikzcd}$$

[1]This is the very definition of cartesian arrow, but remember also that all the arrows in $\mathcal{F}_U$ maps to the identity of U via p.

Proof. The existence of the natural transformations ε_U and $\alpha_{f,g}$ is a trivial consequence of the uniqueness of the uniqueness up-to-natural-isomorphism of the adjoint (thus for example we observe that Φ_{fg} and $\Phi_g \circ \Phi_f$ give right adjoints to Ψ_{fg} in the notation of the proof of Lemma 5.2.7, so that we obtain $\widetilde{\alpha}_{f,g} : \Phi_{fg} \to \Phi_g \circ \Phi_f$, and applying **d** we get the desired $\alpha_{f,g}$). The other checks are still a consequence of the adjointness; the technical details can be found in [Vis04, Prop. 3.11]. □

Accordingly to the traditional definitions, Lemma 5.2.9 says that we can associate to every fibered category $p: \mathcal{F} \to \mathcal{C}$ a pseudo-functor

$$\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$$

Conversely, we can associate to every pseudo-functor a fibered category:

Lemma 5.2.10. Given a pseudo-functor $\Phi: \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ there is a fibered category $p: \mathcal{F} \to \mathcal{C}$ such that $\mathcal{F}_U = \Phi(U)$.

Sketch of the proof. The idea is roughly speaking to mimic the construction of a vector bundle starting from local trivializations. Define a category $\mathcal{F}$ whose objects are

$$\bigcup_{U \in \mathrm{Ob}(\mathcal{C})} \mathrm{Ob}(\Phi(U))$$

If (U,x) and (V,y) are two objects in $\mathcal{F}$, define an arrow

$$(U,x) \to (V,y)$$

to be a pair (f,τ) where $f: U \to V$ is an arrow in $\mathcal{C}$ and $\tau: x \to \Phi(f)(y)$ is an arrow in $\Phi(U)$. The details for the construction can be found in [Vis04, Ch 3.1.3]. □

Finally the two constructions given in Lemma 5.2.9 and 5.2.10 are mutually inverse in a higher categorical sense.

Categories fibered in groupoids and in sets

The equivalence between fibered categories and pseudo-functors with values in **Cat** suggests that fibered categories should be thought of as "presheaves" with values in **Cat**. We will develop in detail this point of view later on. For the moment, we observe that it might be interesting to restrict the attention to categories whose fibers satisfy additional properties. Classically, the main interest is for categories fibered in groupoids.

Definition 5.2.11. A fibered category $p: \mathcal{F} \to \mathcal{C}$ is said to be *fibered in groupoids* if each fiber $\mathcal{F}_U$ is a groupoid.

An useful characterization is the one that follows:

Proposition 5.2.12. A category $p: \mathcal{F} \to \mathcal{C}$ over $\mathcal{C}$ is fibered in groupoids if and only if:

1. every arrow in $\mathcal{F}$ is cartesian;
2. given any arrow $f: V \to U$ in $\mathcal{C}$ and any object $\eta \in \mathcal{F}_U$, there is an arrow $\phi: \xi \to \eta$ such that $p(\phi) = f$.

Proof. Straightforward (for details, see [Vis04, Proposition 3.22]. □

As a particular case, we have categories fibered in sets:

Definition 5.2.13. A fibered category $p\colon \mathcal{F} \to \mathcal{C}$ is said to be *fibered in sets* if each fiber $\mathcal{F}_U$ is a set.

Proposition 5.2.14. A category $p\colon \mathcal{F} \to \mathcal{C}$ over $\mathcal{C}$ is fibered in sets if and only if for any object η of $\mathcal{F}$ and any arrow $f : U \to p\eta$ of $\mathcal{C}$ there is a *unique* arrow $\phi : \xi \to \eta$ of $\mathcal{F}$ with $p(\phi) = f$.

Proof. Straighforward (see [Vis04, Proposition 3.25] for details). □

Corollary 5.2.15. Let $p\colon \mathcal{F} \to C$ be a category fibered in sets. The associated pseudo-functor of Lemma 5.2.9 is a functor that factorizes through **Set** ⊂ **Cat**.

Proof. Proposition 5.2.14 implies that the isomorphisms $\alpha_{f,g}$ and ε_U must be the identities. It follows that Φ is a functor; the factorization property descends from the very definition. □

Remark 5.2.16. Corollary 5.2.15 is saying that categories fibered in sets corresonds, under the equivalence sketched in Lemma 5.2.9 and 5.2.10, to presheaves (of sets).

Building techniques

We propose here a first building technique for fibered categories that eases some of the work required in the examples of next section. Then, we discuss a way to extract a category fibered in groupoids from any fibered category.

We fix a category $\mathcal{C}$ with pullbacks.

Definition 5.2.17. A full subcategory $\mathcal{P}$ of $\mathbf{Arr}(\mathcal{C})$ is said to be stable if:

1. it is closed under isomorphisms in $\mathbf{Arr}(\mathcal{C})$;
2. it is closed under pullback: if
$$\begin{tikzcd} U \times_V Y \arrow[r] \arrow[d, "g"] & Y \arrow[d, "f"] \\ U \arrow[r] & V \end{tikzcd}$$
is a pullback square and $f \in \mathrm{Ob}(\mathcal{P})$, then $g \in \mathrm{Ob}(\mathcal{P})$.

Proposition 5.2.18. Let $\mathcal{C}$ be a category with pullbacks and let $\mathcal{P} \subset \mathbf{Arr}(\mathcal{C})$ be a stable class of maps in $\mathcal{C}$. The restriction of the codomain functor

$$\mathbf{d}_1 : \mathcal{P} \to \mathcal{C}$$

defines a fibered category over $\mathcal{C}$.

Proof. Given an arrow $f : U \to V$ in $\mathcal{C}$ and an object $g : Y \to V$ in $\mathcal{P}$ mapping to V, form the pullback

$$\begin{tikzcd} U \times_V Y \arrow[r, "p"] \arrow[d, "h"] & Y \arrow[d, "g"] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

Then $h \in \mathrm{Ob}(\mathcal{P})$ by hypothesis. The universality coincides exactly with the universal property of pullback, as it is easily seen. □

Now we show how to extract a full subcategory of any fibered category which is fibered in groupoids.

Proposition 5.2.19. Let $\mathcal{F} \to \mathcal{C}$ be a fibered category. Denote by $\mathcal{F}_{\mathrm{cart}}$ the full subcategory of $\mathcal{F}$ formed by cartesian arrows. Then $\mathcal{F}_{\mathrm{cart}} \to \mathcal{C}$ is a category fibered in groupoids.

Proof. This is an immediate consequence of Proposition 5.2.12. □

The 2-category $(\mathbf{Cat} \downarrow \mathcal{C})$

- Definition of **Hom**;
- Definition of $\mathbf{Hom}_{\mathcal{C}}$;
- show that $\mathcal{F}_{\mathrm{cart}} = \mathbf{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{F})$;
- properties of $\mathbf{Hom}_{\mathcal{C}}$ (when it is a groupoid);
- deduce from the previous point that $\mathbf{Grpd}/\mathcal{C}$ is enriched over **Grpd** with tensor and cotensor.

Straightening

Descent condition

As we remarked at the end of previous section, fibered categories represents an extension of presheaves of sets. When the base category is endowed with a (Grothendieck) topology we can look for presheaves well-behaved with respect to the topology; classically, this leads to the notion of sheaf. In the more general context of categories fibered in groupoids, we will obtain stacks. The theoretical difficulty in this passage is contained in the fact that **Grpd** is a 2-category, hence certain limits have to be understood in a 2-categorical sense. This suggests that we can, more generally, consider presheaves with values in a category carrying homotopical information; in that context, we will ask for limits and colimits to be understood in the homotopical sense (cfr. Section 1.6).

Descent data

There are several ways to define descent data and descent condition. We will follow the exposition given in [Vis04, Ch. 4].

Let $(\mathcal{C}, J)$ be a site and let $p\colon \mathcal{F} \to \mathcal{C}$ be a category fibered in sets. For each object $U \in \mathrm{Ob}(\mathcal{C})$ and each covering sieve R on U, let

$$\mathcal{F}(U)_R := \varprojlim_{V \to U \in R} \mathcal{F}_V \tag{5.1}$$

It is well-known that this gives a description of the compatible families of objects with respect to the sieve R. A presheaf $\mathcal{F}$ is a sheaf if and only if the natural morphism

$$\mathcal{F}_U \to \mathcal{F}(U)_R \tag{5.2}$$

is an isomorphism. If we want to generalize this construction to the context of categories fibered in groupoids, we have to give the correct meaning to equation (5.1), and we will have to substitute the isomorphism (5.2) with an equivalence of categories. Let's begin with the following observation:

Lemma 5.2.20. With the previous notations and denoting by $\mathcal{F}$ the functor (cfr. Lemma 5.2.16) associated to $p\colon \mathcal{F} \to \mathcal{C}$, we have an isomorphism

$$\mathcal{F}(U)_R \simeq \mathrm{Hom}(R, \mathcal{F})$$

Proof. Let $f : V \to U$ be an arrow in R and define

$$\alpha_f : \mathrm{Nat}(R, \mathcal{F}) \to \mathcal{F}(V)$$

by setting

$$\alpha_f(\varphi) := \varphi_V(V \to U)$$

This gives a cone over $\{\mathcal{F}(V \to U)\}_{V \to U \in R}$ and so we get a morphism

$$\mathrm{Hom}(R, \mathcal{F}) \to \mathcal{F}(U)_R$$

It is straighforward to check that this is a bijection (cfr. [Vis04, Prop. 2.39]). ☐

Motivated by Lemma 5.2.20 we give the following definition:

Definition 5.2.21. Let $p : \mathcal{F} \to \mathcal{C}$ be a fibered category over a site $(\mathcal{C}, J)$. For every covering sieve R over an object $U \in \mathrm{Ob}(\mathcal{C})$, define the descent data of $\mathcal{F}$ with respect to R to be

$$\mathcal{F}(U)_R := \mathbf{Hom}_{\mathcal{C}}(R, \mathcal{F})$$

where R is reviewed as a full subcategory of $(\mathcal{C} \downarrow U)$.

Observe that we have a natural arrow

$$\mathcal{F}_U \to \mathcal{F}(U)_R \tag{5.3}$$

corresponding via the adjunction to the natural morphism

$$\mathcal{F}_U \times_{\mathcal{C}} R \to \mathcal{F}$$

Others description are possible. See [Vis04, Ch. 4.1.2] for a detailed explanation.

Definition 5.2.22. Let $p : \mathcal{F} \to \mathcal{C}$ be a fibered category on a site $(\mathcal{C}, J)$. We will say that:

1. $\mathcal{F}$ is a *prestack* over $\mathcal{C}$ if for every covering sieve the natural functor (5.3) is fully faithful;

2. $\mathcal{F}$ is a *stack* over $\mathcal{C}$ if for every covering sieve the natural functor (5.3) is an equivalence of categories.

Proposition 5.2.23. Let (C, J) be a site and let $p : \mathcal{F} \to \mathcal{C}$ be a category fibered in sets. Then

1. $\mathcal{F}$ is a prestack if and only if it is a separated functor;

2. F is a stack if and only if it is a sheaf.

Proof. This is an immediate consequence of Lemma 5.2.20. ☐

Examples

Quasi-coherent sheaves

Definition 5.2.24. Say that a morphism of schemes $f : X \to Y$ is a fpqc morphism if it is faithfully flat and each quasi-compact open subset of Y is the image of a quasi-compact open subset of X.

Lemma 5.2.25. For any scheme X say that a collection $\{\varphi_i : U_i \to X\}_{i\in I}$ is a fpqc covering it is jointly surjective and the natural morphism $\coprod U_i \to X$ is a fpqc morphism. Then the fpqc covers satisfy the axioms for a pretopology.

Proof. We have to show that a fpqc morphism is stable under base change, but this is clear. □

Definition 5.2.26. Let S be a scheme. The (big) fpqc site over S is the category $\mathbf{Sch}/S$ endowed with the fpqc topology defined in Lemma 5.2.25.

Fix a scheme S and consider the (big) fpqc site over S, $(\mathbf{Sch}/S)_{\mathrm{fpqc}}$. Define a pseudo-functor

$$\Phi\colon (\mathbf{Sch}/S)^{\mathrm{op}}_{\mathrm{fpqc}} \to \mathbf{Cat}$$

on objects as

$$\Phi(X) := \mathrm{QCoh}(X)$$

the category of quasi-coherent $\mathcal{O}_X$-modules. To define the action on arrows, recall the following lemma:

Lemma 5.2.27. Let $f : X \to Y$ be a morphism of schemes. Then if $\mathcal{G}$ is a quasi-coherent module over Y, $f^*\mathcal{G}$ is a quasi-coherent module over X.

Proof. Immediate consequence of the exactness of f^{-1} and the right-exactness of

$$-\otimes_{\mathcal{O}_R} \mathcal{F}\colon \mathbf{Mod}_{\mathcal{O}_R} \to \mathbf{Mod}_{\mathcal{O}_R}$$

where $\mathcal{O}_R$ is a sheaf of rings and $\mathcal{F}$ is a $\mathcal{O}_R$-module (the reader not at ease with Algebraic Geometry might would like to check on [Har77, Prop. II.5.8.(a)]). □

To check that we obtain a pseudo-functor we observe that $\mathrm{QCoh}(X)$ is a full subcategory of $\mathbf{Mod}_{\mathcal{O}_X}$ and that f^* is defined also for the larger category.

The difference, is that the functor

$$f^*\colon \mathbf{Mod}_{\mathcal{O}_Y} \to \mathbf{Mod}_{\mathcal{O}_X}$$

has a right adjoint, namely

$$f_*\colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$$

Remark 5.2.28. Recall that in general f_* doesn't induce a functor

$$f_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$$

at least without additional hypothesis on f (e.g. quasi-compact and separated, see [Har77, Prop. II.5.8.(c)]).

The adjointness allows to construct the required isomorphisms $\alpha_{f,g}$ and ε_X. For the details, see [Vis04, Ch. 3.2.1]. Let now

$$\mathbf{QCoh}_S \to \mathbf{Sch}/S$$

be the fibered category associated to Φ via the construction in Lemma 5.2.10. We can show:

Theorem 5.2.29. $\mathbf{QCoh}_S \to \mathbf{Sch}/S$ is a stack.

Proof. See [Vis04, Thm. 4.23]. □

Elliptic curves

The current goal is twofold: showing that stacks provide an useful enlargement of sheaves and construct an interesting example. More in detail, we want to show that if we want to deal with elliptic curves over a scheme, the naïf approach of taking isomorphism classes fails (we do not get back a sheaf); however, if we don't forget isomorphisms and we consider the category fibered in groupoids, we get a stack.

Definition 5.2.30. Let S be a scheme. An elliptic curve over S is a triple $(E, f, 0)$ where

1. $f : E \to S$ is proper, smooth of finite type and of relative dimension 1;

2. for every $s \in S$ the fiber E_s is a geometrically connected curve of genus 1;

3. $0 \colon S \to E$ is a section of f.

Consider the site $\mathbf{Sch}_{\mathrm{fpqc}}$. Define a presheaf

$$\Phi \colon \mathbf{Sch}_{\mathrm{fpqc}} \to \mathbf{Set}$$

sending a scheme S to the set of isomorphism classes of elliptic curves over S. If $g \colon S' \to S$ is a morphism of scheme and $(E, f, 0)$ is an elliptic curve over S we construct an elliptic curve $(E', f', 0')$ over S' via the fiber product:

$$\begin{tikzcd} E' = E \times_S S' \arrow[r] \arrow[d, "f'"] & E \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Lemma 5.2.31. Let $\mathcal{E}$ be the class of elliptic curves in $\mathbf{Sch}$. Then $\mathcal{E}$ is stable in the sense of Definition 5.2.17.

Proof. First of all, recall the following facts:

1. proper morphisms are stable under base change. See [Liu02, Prop. 3.3.16.(c)];

2. smooth morphisms are stable under base change. See [Liu02, Prop 4.3.38];

3. morphisms of finite type are stable under base change. See [Liu02, Prop 3.2.4.(c)];

4. if $f : X \to Y$ is a smooth morphism and $x \in X$ is a point, the relative dimension of f at x can be computed as the rank of $\Omega^1_{X/Y}$ at x; see [Liu02, Def. 6.4.10].

Now, consider a pullback diagram

$$\begin{tikzcd} E' \arrow[r, "g'"] \arrow[d, "f'"] & E \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

where f is an elliptic curve. Then facts 1. and 2. imply that f' is proper and smooth. Let $x' \in E'$ be any point and set

$$x := g'(x'), \quad y' := f'(x), \quad y := g(y')$$

The computation is local, so we can assume $S = \mathrm{Spec}(A)$, $E = \mathrm{Spec}(B)$, $S' = \mathrm{Spec}(A')$ and $E' = \mathrm{Spec}(B')$. With these notations we have[2]

$$\operatorname{rank} \Omega^1_{E'/S',x'} = \operatorname{rank} \left(\Omega^1_{B'/A'}\right)_{x'} = \dim_{\kappa(x')} \Omega^1_{B'/A'} \otimes_{B'} \kappa(x')$$

However, by construction

$$B' := B \otimes_A A'$$

so that [Liu02, Prop 6.1.8.(a)] implies that

$$\Omega^1_{B'/A'} \simeq \Omega^1_{B/A} \otimes_B B'$$

Therefore

$$\begin{aligned}
\dim_{\kappa(x')} \Omega^1_{B'/A'} \otimes_{B'} \kappa(x') &= \dim_{\kappa(x')} \Omega^1_{B/A} \otimes_B B' \otimes_{B'} \kappa(x') \\
&= \dim_{\kappa(x')} \Omega^1_{B/A} \otimes_B \kappa(x') \\
&= \dim_{\kappa(x')} \left(\Omega^1_{B/A} \otimes_B \kappa(x)\right) \otimes_{\kappa(x)} \kappa(x') \\
&= \dim_{\kappa(x)} \Omega^1_{B/A} \otimes_B \kappa(x) = \operatorname{rank} \left(\Omega^1_{B/A}\right)_x = 1
\end{aligned}$$

Finally, we have to check the properties of the fibers. If $s' \in S$ is any point, let $s = g(s)$; then

$$E'_{s'} = E_s \times_S \kappa(s')$$

and since E_s is geometrically connected, the same is true for $E_s \times_S \kappa(s')$. The condition on genus is still satisfied because we have an isomorphism[3]

$$H^i(X, \mathcal{F}) \otimes_k K \simeq H^i(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K), \mathcal{F} \otimes_k K)$$

for every coherent sheaf $\mathcal{F}$ on X. □

The functoriality of this construction shows that isomorphism classes of elliptic curves over S are sent into isomorphism classes if elliptic curves over S', hence we get a map

$$g^* \colon \Phi(S) \to \Phi(S')$$

It is straightforward to check that the so-defined Φ is a presheaf. We want to show that Φ cannot be a sheaf: let $S = \mathrm{Spec}(\mathbb{F}_3)$ and consider the elliptic curves

$$\begin{aligned}
C &:= \{(x, y) \in \mathbb{A}^2_{\mathbb{F}_3} \mid y^2 = x^3 - x - 1\} \\
D &:= \{(x, y) \in \mathbb{A}^2_{\mathbb{F}_3} \mid y^3 = x^3 - x + 1\}
\end{aligned}$$

They cannot be isomorphic because a simple direct check shows that C doesn't have $\mathbb{F}_3$-rational points, while D has seven $\mathbb{F}_3$-rational points. However, if K is a finite field extension of $\mathbb{F}_3$ containing a square root of 1, then

$$\mathrm{Spec}(K) \to \mathrm{Spec}(\mathbb{F}_3)$$

is a fpqc covering of $\mathrm{Spec}(\mathbb{F}_3)$, and C_K is isomorphic to D_K. Therefore, the canonical map

$$\Phi(\mathbb{F}_3) \longrightarrow \Phi(K) \rightrightarrows \Phi(K \times_{\mathbb{F}_3} K)$$

is not injective; in particular Φ cannot be a sheaf. Under the identification constructed before, we can also say that Φ is not a stack.

[2] The last equality follows from Nakayama lemma and the proof of the following fact: over a local ring, the properties of being projective and finitely generated, free of finite rank, flat and finitely presented are all equivalent.

[3] This is a consequence of the Cech resolution.

Remark 5.2.32. Recall that the fpqc topology is subcanonical. This observation and the previous reasoning imply that the moduli problem $\mathcal{M}_{\mathrm{ell}} \colon \mathbf{Sch} \to \mathbf{Set}$ doesn't have a fine moduli space.

Consider now

$$\Psi \colon \mathbf{Sch} \to \mathbf{Cat}$$

defined on objects sending a scheme S into the category $\mathcal{M}_{\mathrm{ell}}(S)$ whose objects are elliptic curves over S and whose morphisms are morphisms of elliptic curves. Our building technique (Proposition 5.2.18 applies because of Lemma 5.2.31, yielding a fibered category and hence (via Lemma 5.2.9) a pseudo-functor. Proposition 5.2.19 produces then a category fibered in groupoids:

$$\mathcal{M}_{\mathrm{ell}} := \Psi_{\mathrm{cart}}$$

The previous counterexample breaks down because the isomorphism built over K does not satisfy the descent condition. It is indeed easy to check the stack condition for Ãľtale coverings of a field (Weierstrass equation shows that every descent data is effective). However, I do not know a proof for the general statement (for the moment), nor I have a good reference for it.

Higher genus curves

It can be shown that if, instead of consider elliptic curves, we consider curves of a given genus $g \geq 2$, we obtain objects more well-behaved, that do form a stack.

The proof is lengthy, so we will simply sketch the main ideas of it, without going into details. First of all, let us give the correct definition:

Definition 5.2.33. Let S be a scheme. A curve of genus g over S is a scheme $f : C \to S$ such that:

1. f is proper, smooth and of relative dimension 1;

2. for each $s \in S$, the fiber C_s is a geometrically connected curve of genus g.

These morphisms form a local class in $\mathbf{Sch}/S$, hence they define a fibered category in groupoids. To check that they do form a stack if $g \geq 2$, we will employ a general descent technique: the descent via ample invertible sheaves. Let us give a definition, first:

Definition 5.2.34. Let $(\mathcal{C}, J)$ be a site and suppose that $\mathcal{C}$ has pullbacks. Then a class of arrows $\mathcal{P} \subset \mathbf{Arr}(\mathcal{C})$ is said to be *local* if it is stable and the following condition holds: for each covering $\{U_i \to U\}$ in $\mathcal{C}$ and each arrow $X \to U$, if the projections $U_i \times_U X \to U_i$ are in $\mathcal{P}$, then $X \to U$ is also in $\mathcal{P}$.

Theorem 5.2.35. Let S be a scheme, $\mathcal{F}$ be a class of flat, proper morphisms of finite presentation in $\mathbf{Sch}/S$ that is local in the fpqc topology. Suppose that for each object $\xi : X \to U$ of $\mathcal{F}$ is given an invertible sheaf $\mathscr{L}_\xi$ on X, ample relative to $X \to U$, in such a way that each cartesian diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\xi"] & Y \arrow[d, "\eta"] \\ U \arrow[r, "\phi"] & V \end{tikzcd}$$

yields an isomorphism

$$\rho_{f,\phi} : f^*\mathscr{L}_\eta \to \mathscr{L}_\xi$$

of invertible sheaves. Finally, let's assume that for each cartesian diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\xi"] & Y \arrow[r, "g"] \arrow[d, "\eta"] & Z \arrow[d, "\zeta"] \\ U \arrow[r, "\phi"] & V \arrow[r, "\psi"] & W \end{tikzcd}$$

where $\xi, \eta, \zeta \in \mathcal{F}$, the diagram

$$\begin{tikzcd} f^*g^*\mathscr{L}_\zeta \arrow[r, "\alpha_{f,g}(\mathscr{L}_\zeta)"] \arrow[d, "f^*\rho_{g,\psi}"] & (gf)^*\mathscr{L}_\zeta \arrow[d, "\rho_{gf,\psi\phi}"] \\ f^*\mathscr{L}_\eta \arrow[r, "\rho_{f,\phi}"] & \mathscr{L}_\xi \end{tikzcd}$$

commutes. Then $\mathcal{F}$ is a stack in the fpqc topology.

Proof. See [Vis04, Thm. 4.38]. □

If $g \geq 2$, for a proper smooth morphism $\xi : X \to U$ we can take as invertible sheaf the relative cotangent sheaf $\Omega^1_{X/U}$. This satisfies the hypothesis, producing a stack.

5.3 Simplicial presheaves

Motivations

Let $(\mathcal{C}, J)$ be a site and let $\mathcal{F}$ be a presheaf over $\mathcal{C}$. We saw in Lemma 5.2.20 that we can formulate the usual equalizer appearing in the definition of sheaf as

$$\mathrm{Hom}(R, \mathcal{F})$$

where R is a covering sieve over a given object X. We next used this reformulation to introduce the notion of stack (Definition 5.2.22). Now, we would like to reformulate again the notion of stack in a shape more similar to the original definition; we immediately find an obstruction to this: since **Grpd** is a 2-category, the notion of limit should be weakened. This introduce natural complications, which are very fundamental in the whole theory of stacks and make more complicated notion which *should* (and in fact, *do*) exist, like "stackification". Our homotopical point of view, however, can prove really useful in this context.

Roughly speaking, a 2-limit is a lax limit, in the sense that universality is required only up to natural isomorphisms. We know another situation where almost the same problem arose: the problem of finding a good model for homotopy limits and colimits. In fact, there the situation was even more complicated, because we were dealing with homotopies of all order. This suggests that a homotopical point of view can really help in dealing with these constructions, and can also yield a unifying overall vision on the subject.

This is effectively done in [Hol07]. Here, we start with a more general point of view, and we will go back to the original problem only at the end. Before starting, let's sketch what our journey will be: recall from Theorem 1.8.19 that we have an adjoint pair

$$\pi_f : \mathbf{sSet} \rightleftarrows \mathbf{Grpd} : N$$

inducing a Quillen equivalence between **Grpd** and the S^2 nullification of **sSet**. If our goal is to exploit homotopy theory to deal with the definition of stacks, we should begin with endowing a category containing stacks with a model structure. Since, up to straightening, we can consider strict presheaves in groupoids, let's take this point of view (arguably, the arguments will be easier). This can be done intuitively using the model structure on **Grpd**; a good definition, however, should take into account also the topology of the site we are considering.

A quite natural idea at this point, is to replace **Grpd** with the S^2 nullification of **sSet**; since we are planning to work up to homotopy, the final result shouldn't change. And now we can consider more generally presheaves of simplicial sets (or simplicial presheaves). We will analyze with some detail the possible model structures over this category, and then we will show that the S^2 nullification of this model structure yields exactly what we wanted at the beginning. This more general point of view allows to consider, for example, presheaves of quasicategories, which represent a big step toward our notion of ∞-stack.

Definitions and first properties

Definition 5.3.1. Let $\mathcal{C}$ be a category. A presheaf of simplicial sets is a contravariant functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{sSet}$.

Presheaves of simplicial sets can be naturally arranged into a category, the functorial category $\mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{sSet})$. We will denote this category as $\mathrm{sPSh}(\mathcal{C})$. The notation, that reminds "simplicial objects in the category of presheaves" is justified by the following lemma:

Lemma 5.3.2. We have natural isomorphisms

$$\mathrm{sPSh}(\mathcal{C}) \simeq \mathbf{Hom}({}'^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathbf{Set}) \simeq \mathbf{Hom}({}'^{\mathrm{op}}, \mathrm{PSh}(\mathcal{C}))$$

In particular, $\mathrm{sPSh}(\mathcal{C})$ is a topos.

Proof. This is a formal consequence of cartesian closedness of **Cat** and the exponential law:

$$\begin{aligned} \mathrm{sPSh}(\mathcal{C}) &= \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Hom}({}'^{\mathrm{op}}, \mathbf{Set})) \\ &= \mathbf{Hom}(\mathcal{C}^{\mathrm{op}} \times {}'^{\mathrm{op}}, \mathbf{Set}) \\ &= \mathbf{Hom}({}'^{\mathrm{op}}, \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})) \\ &= \mathbf{Hom}({}'^{\mathrm{op}}, \mathrm{PSh}(\mathcal{C})) \end{aligned}$$

□

Corollary 5.3.3. $\mathrm{sPSh}(\mathcal{C})$ is cartesian closed.

We can easily obtain an enrichment over **sSet**:

Proposition 5.3.4. $\mathrm{sPSh}(\mathcal{C})$ is enriched with tensor and cotensor over **sSet**.

Proof. Given a simplicial set K, review it as the constant simplicial presheaf at K. For any other simplicial presheaf F, define

$$F \otimes K := F \times K$$

Since $\mathrm{sPSh}(\mathcal{C})$ is a topos, we can also define

$$F^K := \mathbf{hom}(K, F)$$

where **hom** is the internal hom of $\mathrm{sPSh}(\mathcal{C})$. Given any simplicial presheaf F, we can consider the following cosimplicial object in $\mathrm{sPSh}(\mathcal{C})$:

$$\{F \times \Delta^n\}_{n \in \mathbb{N}}$$

For any other simplicial presheaf G define then

$$\mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(F, G; \mathbf{sSet}) := \mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(F \times \Delta^{\bullet}, G)$$

We clearly obtain a simplicial set. Moreover

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^0, \mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(F, G; \mathbf{sSet})) := \mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(F, G)$$

and so we also have an obvious choice for the enriched identity:

$$\Delta^0 \to \mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(F, F; \mathbf{sSet})$$

It's an exercise in formalism to check that the axioms of Definition C.1.1 are satisfied. The reader can find all the verifications in [GJ99, Lemma II.2.4] in a more general form. □

Finally, we end this introductory section with a result that will be quite useful in what follows: the "simplicial Yoneda lemma". Some notation will be needed. First of all denote by $c \colon \mathbf{Set} \to \mathbf{sSet}$ the diagonal functor sending a set into the constant simplicial set associated. If $\mathcal{C}$ is any category and $X \in \mathrm{Ob}(\mathcal{C})$, we obtain the usual Yoneda representable functor

$$h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

Definition 5.3.5. For an object $X \in \mathrm{Ob}(\mathcal{C})$ in any category $\mathcal{C}$, we define the *simplicial Yoneda functor* to be $sh_X : c \circ h_X$.

Theorem 5.3.6 (Simplicial Yoneda Lemma)**.** Let $\mathcal{C}$ be a category and let $F \in \mathrm{sPSh}(\mathcal{C})$ be a simplicial presheaf. For each object $X \in \mathrm{Ob}(\mathcal{C})$ we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(sh_X, F; \mathbf{sSet}) \simeq F(X)$$

Proof. Using Lemma **??** we get:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(sh_X, F; \mathbf{sSet})) &\simeq \mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(sh_X \otimes \Delta^n, F) \\
&\simeq \mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(sh_X, \mathbf{hom}(\Delta^n, F)) \\
&\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_X, p \circ \mathbf{hom}(\Delta^n, F)) \\
&\simeq (p \circ \mathbf{hom}(\Delta^n, F))(X)
\end{aligned}$$

Moreover, the universal property of cotensor induces the following equality of *functors*:

$$\begin{aligned}
p \circ \mathbf{hom}(\Delta^n, F) &= \mathrm{Hom}_{\mathbf{sSet}}(\Delta^0, \mathbf{hom}(\Delta^n, F)) \\
&= \mathrm{Hom}_{\mathbf{sSet}}(\Delta^0 \times \Delta^n, F) = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, F)
\end{aligned}$$

Thus we obtain a natural isomorphism

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(sh_X, F; \mathbf{sSet})) \simeq F(X)_n$$

Naturality in Δ^n implies that we get an isomorphism of simplicial sets

$$\mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(sh_X, F; \mathbf{sSet}) \simeq F(X)$$

which is what we wanted. □

Model structures

The next step is to produce a model structure over $\mathrm{sPSh}(\mathcal{C})$. Without assumptions on $\mathcal{C}$, one can always define two model structure over $\mathrm{sPSh}(\mathcal{C})$: the injective model structure (objectwise cofibrations) and the projective model structure (objectwise fibrations). In both cases, weak equivalences are objectwise equivalences. This is possible because **sSet** is combinatorial (see Definition 1.9.12 and Theorem 1.9.13). We state it as a separate theorem for future references:

Theorem 5.3.7. There is a left proper, cofibrantly generated, simplicial model structure on $\mathrm{sPSh}(\mathcal{C})$ where:

- cofibrations are the objectwise cofibrations;
- weak equivalences are the objectwise weak equivalences;
- fibrations are the maps with the RLP with respect to the trivial cofibrations.

Proof. This is a corollary of Theorem 1.9.13. □

We will denote by $\mathrm{sPSh}(\mathcal{C})_H$ the model structure on $\mathrm{sPSh}(\mathcal{C})$ of Theorem 5.3.7.

Remark 5.3.8. This model structure for presheaves of simplicial sets goes back to the work of Heller [Hel88].

However, when we consider a site $(\mathcal{C}, J)$, we can define a model structure over $\mathrm{sPSh}(\mathcal{C})$ taking in account the topology J. The idea to do that goes back to Jardine ([Jar]), at least to the best of my knowledge, and relies on the notion of sheaf of homotopy groups.

Sheaves of homotopy groups

From now on, we fix a site $(\mathcal{C}, J)$.

Definition 5.3.9. Let F be a simplicial presheaf over $(\mathcal{C}, J)$. Define $\pi_0(F)$ to be the associated sheaf to the presheaf
$$X \mapsto \pi_n(F(X))$$

Definition 5.3.10. Let F be a simplicial presheaf over $(\mathcal{C}, J)$. For each object $X \in \mathrm{Ob}(\mathcal{C})$ and each 0-simplex in $F(X)$ define $\pi_n(F, x)$ to be the associated sheaf to the presheaf of sets on $\mathcal{C}/X$ defined by
$$(f : U \to X) \mapsto \pi_n(F(U), f^*x)$$

Following Jardine [Jar] we give the following definition:

Definition 5.3.11. A map of simplicial presheaf $f : F \to G$ is said to be a *local weak equivalence* if the following conditions hold:

1. the induced map $\pi_0(F) \to \pi_0(G)$ is an isomorphism of sheaves on $\mathcal{C}$;
2. for each object $X \in \mathrm{Ob}(\mathcal{C})$ and each 0-simplex $x \in F(X)$, the induced map $\pi_n(F, x) \to \pi_n(G, f(x))$ is an isomorphism of sheaves on $\mathcal{C}/X$.

Lemma 5.3.12. Let $f : F \to G$ be an objectwise weak equivalence. Then f is also a local weak equivalence.

Proof. f induces an isomorphisms of *presheaves* of homotopy groups, hence it induces also an isomorphism of sheaves of homotopy groups (sheafification is a functorial operation). □

Then we have the following:

Theorem 5.3.13. There is a model structure on $\mathrm{sPSh}(\mathcal{C})$ where:

- weak equivalences are local weak equivalences;
- cofibrations are objectwise cofibrations;
- fibrations are the maps with the left lifting property with respect to cofibrations which are also local weak equivalences.

Proof. See [Jar, Thm. 2.3]. □

We will refer to this model structure as the *local model structure* on $\mathrm{sPSh}(\mathcal{C})$, and from this moment on we will assume that $\mathrm{sPSh}(\mathcal{C})$ is endowed with this model structure. If confusion may arise, we will denote it as $\mathrm{sPSh}(\mathcal{C})_J$ to make the distinction with the model structure $\mathrm{sPSh}(\mathcal{C})_H$ of Theorem 5.3.7.

Despite being quite intuitive as construction, there is a major drawback: it is not clear at all how to characterize fibrant objects (and more generally fibrations). However, if we want to imitate our construction for presheaves in groupoids and define stacks as fibrant objects with respect to a certain model structure, it is important to have a better understanding of such objects.

Corollary 5.3.14. Let S be the class of local weak equivalences. Then $\mathrm{sPSh}(\mathcal{C})_J$ is the left localization of $\mathrm{sPSh}(\mathcal{C})_H$ with respect to S.

Proof. The identity $\mathrm{Id}\colon \mathrm{sPSh}(\mathcal{C})_H \to \mathrm{sPSh}(\mathcal{C})_L$ takes cofibrations in cofibrations and Lemma 5.3.12 implies that Id preserves also weak equivalences. Thus in the identity adjoint pair $(\mathrm{Id}, \mathrm{Id})$ the left adjoint is also a left Quillen functor. Since we are considering exactly the maps sent to weak equivalences in $\mathrm{sPSh}(\mathcal{C})_J$, we see that this is a left localization. □

Remark 5.3.15. It can be shown that this left localization is exactly the left Bousfield localization. This means that S-local equivalences are exactly the local weak equivalences.

Local lifting properties

Definition 5.3.16. Let $i\colon K \to L$ be a map of simplicial sets and let $f : F \to G$ be a map of simplicial presheaves; for any object $X \in \mathrm{Ob}(\mathcal{C})$ we say that the diagram

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "i"'] & F(X) \arrow[d, "f(X)"] \\ L \arrow[r, "\beta"'] & G(X) \end{tikzcd}$$

has a local filling if there is a covering sieve R on X such that for each $\varphi : U \to X$ in R there exist a commutative diagram

$$\begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "i"'] & F(X) \arrow[r, "\varphi^*"] & F(U) \arrow[d, "f(U)"] \\ L \arrow[r, "\beta"'] \arrow[urr, "\theta_\varphi"] & G(X) \arrow[r, "\varphi^*"'] & G(U) \end{tikzcd}$$

In this case we will also say that $f(X)$ has the local right lifting property with respect to $i\colon K \to L$.

Definition 5.3.17. We say that a map of simplicial presheaves $f : F \to G$ has the local right lifting property with respect to a map $i : K \to L$ of simplicial sets if for every object $X \in \mathrm{Ob}(\mathcal{C})$, the map $f(X)$ has the local right lifting property with respect to i.

There is another way to formulate the local right lifting, which sometimes turns out to be useful. If $X \in \mathrm{Ob}(\mathcal{C})$, consider the Yoneda functor

$$h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

For each $U \in \mathrm{Ob}(\mathcal{C})$ we can think to $h_X(U)$ as a constant simplicial set (concentrated in degree 0). We will denote by

$$sh_X : \mathcal{C}^{\mathrm{op}} \to \mathbf{sSet}$$

the so obtained simplicial presheaf (s stands for "simplicial").

Lemma 5.3.18. A map $f : F \to G$ of simplicial presheaf has the local right lifting property with respect to a map $i : K \to L$ of simplicial sets if and only if for every object $X \in \mathrm{Ob}(\mathcal{C})$, each diagram of simplicial presheaves

$$\begin{tikzcd} sh_X \otimes K \arrow[r] \arrow[d, "\mathrm{id}\otimes i"'] & F \arrow[d, "f"] \\ sh_X \otimes L \arrow[r] \arrow[ur, dotted] & G \end{tikzcd}$$

has a diagonal filling as indicated. Here $\otimes$ denotes the tensor of the enriched structure of $\mathrm{sPSh}(X)$ built in Proposition 5.3.4.

Proof. The universal property of the tensor gives the following isomorphism:

$$\mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(sh_X \otimes K, F; \mathbf{sSet}) \simeq \mathbf{hom}_{\mathbf{sSet}}(K, \mathrm{Hom}_{\mathrm{sPSh}}(sh_X, F; \mathbf{sSet}))$$

However, simplicial Yoneda Lemma (Theorem 5.3.6) implies

$$\mathrm{Hom}_{\mathrm{sPSh}}(sh_X, F; \mathbf{sSet}) = F(X)$$

Thus we obtain a natural isomorphism

$$\mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(sh_X \otimes K, F; \mathbf{sSet}) \simeq F(X)$$

and the conclusion at this point is an exercise in formalism. □

Definition 5.3.19. A map of simplicial presheaves $f : F \to G$ is a *local fibration* if it has the local right lifting property with respect to all the horn inclusions

$$\Lambda^n_k \to \Delta^n, \quad 0 \le k \le n$$

in the sense of Definition 5.3.17.

Lemma 5.3.20. Let $(\mathcal{C}, J)$ be a site. With respect to the local model structure on $\mathrm{sPSh}(\mathcal{C})$, any fibration is a *local fibration* in the sense of Definition 5.3.19.

Proof. In the local model structure, the cofibrations are the objectwise cofibrations. It follows that if we review Λ^n_k and Δ^n as constant simplicial presheaves, the natural inclusion

$$\Lambda^n_k \to \Delta^n$$

is a trivial cofibration (it induces an isomorphism at level of *presheaves* of homotopy groups, hence also at level of sheaves of homotopy groups). For each object $X \in \mathrm{Ob}(\mathcal{C})$, the map

$$sh_X \otimes \Lambda^n_k \to sh_X \otimes \Delta^n$$

is still an objectwise cofibration, hence a cofibration in the local model structure. Moreover, the homotopy groups commutes with the products by trivial arguments, as well as the sheafification functor. It follows that the map

$$sh_X \otimes \Lambda^n_k \to sh_X \otimes \Delta^n$$

is still a local weak equivalence.

Therefore, if $p\colon F \to G$ is a fibration in the local model structure, every diagram

$$\begin{tikzcd} sh_X \otimes \Lambda^n_k \arrow[r] \arrow[d] & F \arrow[d, "p"] \\ sh_X \otimes \Delta^n \arrow[r] \arrow[ur, dotted, "h"] & G \end{tikzcd}$$

has a diagonal filling h. This implies the thesis. □

The following theorem characterizes local acyclic fibrations in term of local lifting properties:

Theorem 5.3.21. A map $p\colon F \to G$ of simplicial presheaves admits local liftings in every square

$$\begin{tikzcd} \partial\Delta^n \otimes X \arrow[r] \arrow[d] & F \arrow[d] \\ \Delta^n \otimes X \arrow[r] & G \end{tikzcd}$$

if and only if it is a local acyclic fibration.

Proof. See [DI02, Prop. 7.2]. □

Characterization of fibrant objects

We want to relate the notion of “being fibrant” and “satisfy descent condition”. First of all, we will need to develop the machinery of hypercovers. Our main reference is [DI03].

Hypercovers

Definition 5.3.22. Let $(\mathcal{C}, J)$ be a site and let $X \in \mathrm{Ob}(\mathcal{C})$. A hypercover of X is a pair (U, ε) where:

1. U is a simplicial presheaf such that each presheaf

$$U_n := \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, U(-))\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

   is a coproduct of representable functors;

2. ε is a map of simplicial presheaf $\varepsilon\colon U \to sh_X$ which is a local acyclic fibration.

Example 5.3.23. Let $(\mathcal{C}, J)$ be a site and let $\{U_i \to X\}_{i\in I}$ be a cover for an object $X \in \mathrm{Ob}(\mathcal{C})$. Denote by $U_{i_1,i_2,\ldots,i_n}$ the fibered product

$$U_{i_1} \times_X U_{i_2} \times_X \ldots \times_X U_{i_n}$$

Then

$$U_\bullet := \left\{ \coprod_{i_1,\ldots,i_n \in I^n} U_{i_1,\ldots,i_n} \right\}_{n\in\mathbb{N}}$$

(where we confused $U_{i_1,\ldots,i_n}$ with the representable functor $\mathrm{Hom}_{\mathcal{C}}(-, U_{i_1,\ldots,i_n})$) has a natural structure of simplicial object. Moreover, we have a natural map $U_\bullet \to sh_X$ which is a local acyclic fibration.

Definition 5.3.24. A collection of hypercovers S is called *dense* if every hypercover $U \to sh_X$ in $\mathrm{sPSh}(\mathcal{C})$ can be refined by a hypercover $V \to sh_X$ belonging to S.

Lemma 5.3.25. Let $(\mathcal{C}, J)$ be a site. The class of all hypercovers in $\mathcal{C}$ contains a subset which is dense.

Proof. See [DI03, Prop. 6.6]. □

Theorem 5.3.26. Let S be a collection of hypercovers which contains a set that is dense. Then the left Bousfield localization of $\mathrm{sPSh}(\mathcal{C})$ at S exists and coincides with Jardine's local model structure $\mathrm{sPSh}(\mathcal{C})_L$.

Proof. See [DI03, Thm. 6.2]. □

Descent with respect to hypercovers

Definition 5.3.27. An objectwise-fibrant simplicial presheaf F satisfies descent for a hypercover $U \to X$ if the natural map from $F(X)$ to the homotopy limit of the diagram

$$\prod_a F(U_0^a) \rightrightarrows \prod_a F(U_1^a) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

is a weak equivalence. If F is not obejctwise-fibrant, we say that it satisfies descent if some object-wise fibrant replacement for F does.

Lemma 5.3.28. A simplicial presheaf F satisfies descent for a hypercover $U \to sh_X$ if and only if the map

$$\mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(sh_X, \widehat{F}; \mathbf{sSet}) \to \mathrm{Hom}_{\mathrm{sPSh}(\mathcal{C})}(U, \widehat{F}; \mathbf{sSet})$$

is a weak equivalence of simplicial sets, where $\widehat{F}$ is a fibrant replacement for F with respect to the local model structure $\mathrm{sPSh}(\mathcal{C})_L$.

Proof. See [DI03, Lemma 4.4]. □

Theorem 5.3.29. Let S be a collection of hypercovers which contains a set that is dense. A simplicial presheaf F is fibrant in $\mathrm{sPSh}(\mathcal{C})_L$ if and only if it is fibrant in $\mathrm{sPSh}(\mathcal{C})_H$ and it satisfies descent for all hypercovers in S.

Proof. See [DI03, Cor. 7.1]. □

Comparison with presheaves of groupoids

We finally finish our initial program going back to presheaves in groupoids, which we will denote by $P(\mathcal{C}, \mathbf{Grpd})$. We begin describing a natural model structure on this category. As for simplicial presheaves, the first model structure we can introduce is a global one:

Theorem 5.3.30. There is a left proper, cofibrantly generated model category structures on $P(\mathcal{C}, \mathbf{Grpd})$ and where

- f is a weak equivalence of a fibration if $\mathbf{Grpd}(X, f)$ is one for all $X \in \mathcal{C}$;
- cofibrations are the maps with the LLP with respect to trivial fibrations.

The maps of the form $X \to X \otimes \Delta^1$, for $X \in \mathcal{C}$ for a set of generating trivial cofibrations. The maps of the form $X \otimes \partial\Delta^i \to X \otimes \Delta^i$ for $X \in \mathcal{C}$ and $i = 0, 1, 2$ form a set of generating cofibrations.

Proof. See [Hol07, Thm. 7.1]. □

Next, we can introduce a local model structure using hypercovers.

Definition 5.3.31. Let $(\mathcal{C}, J)$ be a site. Denote by S the set of maps in $P(\mathcal{C}, \mathbf{Grpd})$ the class of maps

$$S := \{\text{hocolim}\, U_\bullet \to X \mid \{U_i \to X\} \text{ is a cover in } \mathcal{C}\}$$

where $U_\bullet$ denotes the Cech hypercover associated to $\{U_i \to X\}$.

Proposition 5.3.32. The left Bousfield localization of $P(\mathcal{C}, \mathbf{Grpd})$ with respect to the class S of Definition 5.3.31 exists. We will denote it by $P(\mathcal{C}, \mathbf{Grpd})_L$.

Proof. See [Hol07, Prop. 7.5] □

Next, as we announced, we can apply the S^2 nullification to the local model structure on simplicial presheaves:

Theorem 5.3.33. The local model structure on presheaves of groupoids $P(\mathcal{C}, \mathbf{Grpd})_L$ is Quillen equivalent to the S^2 nullification of local model structure sPSh$(\mathcal{C})$.

Proof. See [Hol07, Cor. 8.10]. □

Stacks as fibrant objects

Since the homotopical theory of **Grpd** is a truncation of the model structure of **sSet** (cfr. Proposition 1.8.15), we can expect that homotopy limits and colimits are easier to understand if we consider categories enriched over **Grpd** instead of enriched over the whole **sSet**. For our purpose, the following result is more than enough:

Theorem 5.3.34. Let $\mathcal{M}$ be a simplicial model category whose simplicial structure derives from an enrichment over **Grpd**. Let $X^\bullet$ be a cosimplicial object in $\mathcal{M}$, with each X^i fibrant. A model for the homotopy inverse limit of $X^\bullet$ is given by the equalizer of the natural maps

$$\prod_{i=0}^{2} (X^i)^{\Delta^i} \rightrightarrows \prod_{\mathbf{j}\to\mathbf{i}}^{i\le 2,\, j\le 1} (X^i)^{\Delta^j}$$

Proof. See [Hol07, Thm 4.3]. □

Fix a category $\mathcal{C}$. We showed at the end of Section 5.2 that the 2-category **Grpd**$/\mathcal{C}$ of categories fibered in groupoids over $\mathcal{C}$ is enriched with tensor and cotensor over **Grpd**. The main theorem is the following:

Theorem 5.3.35. A category fibered in groupoids $\mathcal{F} \to \mathcal{C}$ is a stack if and only if for all covers $\mathcal{U} = \{U_i \to X\}$ the natural map

$$\mathbf{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F}) \to \operatorname{holim} \mathbf{Hom}_{\mathcal{C}}(\mathcal{C}/U_\bullet, \mathcal{F})$$

is an equivalence. Here $U_\bullet$ denotes the hypercover associated to $\mathcal{U}$.

Remark 5.3.36. 2-categorical Yoneda Lemma shows in fact that we have an equivalence of categories

$$\mathbf{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F}) \simeq F(X)$$

See [Vis04, Ch. 3.6.2] for a proof.

5.4 Complements to Chapter 5

Cartesian closedness of Cat

The main reference is [GR71, Ch. VI].

Lemma 5.4.1. For each category $\mathcal{C}$ the functor $\mathbf{hom}(\mathcal{C}, -)\colon \mathbf{Cat} \to \mathbf{Cat}$ preserves the adjunctions.

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Exposé **6**

Higher derived stacks

Pietro Vertechi

6.1 Higher stacks

The classical definition of sheaf can be generalised in two different directions: first of all the arrival category of sets may be replaced by the ∞-category of ∞-groupoids (simplicial sets with the Kan-Quillen model structure). Furthermore instead of a site in the classical sense we can take a more general ∞-category with a topology.

Definition and comparison with the classical notion

Definition 6.1.1. A topology on an ∞-category $\mathcal{C}$ is simply a topology on its homotopy category $\mathrm{Ho}(\mathcal{C})$

Remark 6.1.2. This is the natural definition, because it would be absurd if the topology would distinguish between homotopically equivalent sieves: if one is covering, so must be the other.

The ∞-category of prestacks on $(\mathcal{C}, \tau)$ is simply the category of functors from $\mathcal{C}^{op}$ to $sSets$, which we denote $\hat{\mathcal{C}}$. That ∞-category correspond to a model category, whose weak equivalences and fibrations are defined objectwise.

Remark 6.1.3. $\hat{\mathcal{C}}$ possesses limits and colimits. Furthermore $\hat{\mathcal{C}}$ is characterised by a universal property: given an ∞-category $\mathcal{D}$ with colimits, $Fun_{cont}(\hat{\mathcal{C}}, \mathcal{D}) \simeq Fun(\mathcal{C}, \mathcal{D})$, where Fun_{cont} are colimit preserving functors.
It's interesting to remark that this is the same universal property that defines the category of presheaves in the 1-categorical world.

In classical sheaf theory it is possible to define the category of sheaves without actually knowing what a sheaf is, simply by localising the category of presheaves with respect to local equivalences. We will use the same technique in this setting, so the first step is understanding how to generalise the notion of local equivalence.

Definition 6.1.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a map between two ∞-categories. We say that F is essentially k-surjective if for all parallel (k-1)-arrows $\phi, \psi \in \mathcal{C}$ and k-arrow $\chi \in \mathcal{D}$ between $F(\phi)$ and $F(\psi)$ there exist a k-arrow $\eta \in \mathcal{C}$ from ϕ to ψ such that $F(\eta)$ and χ are homotopically equivalent.

A map between two ∞-categories is an equivalence if and only if it is essentially k-surjective for all k. For sets it is equivalent to say that a function between sets is an equivalent if and only if it is 0-essentially surjective (that is to say surjective) and 1-essentially surjective (that is to say injective). It is particularly easy to explain what it means to be locally essentially k-surjective, so we will say that a morphism between prestacks is a local equivalence if it is locally essentially k-surjective for all k.

Definition 6.1.5. Let $f : F \to G$ be a morphism of prestacks.
1) f is locally essentially k-surjective if it has the following property:
for any object $X \in Ob(\mathcal{C})$ and any morphism $h_x \to G$ there exist a covering sieve R of X such that for any morphism $h_u \to h_x$ in R there is an arrow $h_u \to F$ making the following diagram commute up to homotopy:

$$\begin{tikzcd} F^{\Delta_k} \arrow[r] & F^{\partial\Delta_k} \times_{G^{\partial\Delta_k}} G^{\Delta_k} \\ h_u \arrow[u] \arrow[r] & h_x \arrow[u] \end{tikzcd}$$

we are simply saying that any object of G lifts locally and up to homotopy to an object of F.
2) f is a local equivalence if it is locally essentially k-surjective for all k.

Remark 6.1.6. If F and G are 0-presheaves, we are just saying that a map between presheaves is a local equivalence iff:

- for all X and $x \in G(X)$, there is a covering of X in which the restrictions of X come from elements of F
- for all X, if we take $x, y \in F(X)$ which are sent on the same element of G(X) than there is a covering of X in which they are equal.

Definition 6.1.7. A stack is a prestack which is local with respect to local equivalences.
The ∞-category of stacks is the full subcategory of $\hat{\mathcal{C}}$ spanned by stacks.

Proposition 6.1.8 ([TV])**.** The forgetful functor from $St(\mathcal{C})$ to $\hat{\mathcal{C}}$ has an exact left adjoint, called the sheafification functor, which exhibits $St(\mathcal{C})$ as the localisation of $\hat{\mathcal{C}}$ with respect to local equivalences.

sketch. As far as I know it is quite difficult to prove this statement directly in the setting of higher categories. One should rather find a model category which correspond to $\hat{\mathcal{C}}$, for example the category of functors of simplicial categories between a simplicial category equivalent to $\mathcal{C}$ and the simplicial category of simplicial sets. The model structure is the projective model structure (fibrations and weak equivalences are defined objectwise). Then, in this model category, one disposes of the general machinery of Bousfield localisation which produces a Quillen adjunction between the model category of prestacks and the model category of stacks. When passing to ∞ categories the Quillen adjunction induces an adjunction of ∞ categories. □

Proposition 6.1.9 ([HAG-II])**.** We might have started, rather than with weak equivalences, with another (more restrained) class of functions, namely maps of the tipe $|U_*| \to h_X$ where U_* is a split hypercovering of X. In HAG I it is proved that this set of maps and local equivalences induce the same left Bousfield localisation, so one has a more explicit description of stacks: they are prestacks such that, for every object X and split hypercovering $U_* \to X$ the natural morphism

$$F(X) \to \lim_{[n]\in\Delta} F(U_n)$$

is an equivalence.

Remark 6.1.10. Thanks to the previous proposition, we have a nice characterisation of colimit-preserving functors from $St(\mathcal{C})$:
if $\mathcal{D}$ is any ∞-category with colimits, a functor $\mathcal{C} \to \mathcal{D}$ can be extended to a unique colimit preserving functor $\hat{\mathcal{C}} \to \mathcal{D}$ which factorises through a (always colimit preserving) functor $St(\mathcal{C}) \to \mathcal{D}$ iff for all $X \in \mathcal{C}$ and for all hypercovering $U_* \to X$ the natural map

$$\mathrm{col}_{[n]\in\Delta^{op}} F(U_n) \to F(X)$$

is an isomorphism.
Furthermore the adjunction $St(\mathcal{C}) \rightleftarrows \hat{\mathcal{C}}$ assures the existence of limits and colimits in $St(\mathcal{C})$, so we can characterise such category by a universal property.

Remark 6.1.11. Classical sheaves and stacks are a particulare case of this formalism. In general we define an n-stack to be a stack which takes values in the category of n-truncated simplicial sets (that is to say simplicial sets whose homotopy is concentrated in degrees $\leq n$). The inclusion of n-truncated simplicial sets in simplicial sets has a left adjoint, denoted $t_{\leq n}$ which induces an adjunction $n - St \rightleftarrows St$. It can be verified that 0-stacks are sheaves and 1-stacks are stacks in the classical sense.

One might ask why we said we were inverting morphisms that are locally essentially k-surjective for all k and not morphisms that are locally equivalences, which would have been easier to define. The fact is, a morphism which is locally essentially k-surjective for all k is not locally an equivalence, because a priori we can't find a single covering on which it is k-surjective for all k at the same time. Of course for truncated stacks there is no difference.
The problem with the second choice (inverting local equivalences) is that it produces a topos which is not hypercomplete, that is to say a map $X \to Y$ might induce isomorphisms on the π_0 and on all homotopy groups for all base points and still not be an equivalence. Instead, with the definition we've chosen, we have:

Proposition 6.1.12. $St(\mathcal{C})$ is hypercomplete.

Sheaves of model categories

Actually, from these definition it may be rather difficult to prove that a given functor $S^{op} \to sSets$ is a stack. There is, however, a practical way of doing so in the case in which S is the ∞-category associated to a cofibrantly generated model category T.

Definition 6.1.13. A left Quillen presheaf is a functor from T to LModCat, the category of model categories in which arrows are left Quillen functors.
More explicitely to each object $X \in T$ we associate a model category M(X) and to each arrow $f : X \to Y$ we associate a left Quillen functor $f^* : M(Y) \to M(X)$.
Furthermore we demand $(f \circ g)* = f^* \circ g^*$ for any pair of composable arrows.

Given a split hypercovering $U_* \to X$ we can consider the cosimplicial diagram $[n] \mapsto M(U_n)$ (U_n is a coproduct of representables, and we define $M(U_n)$ to be the product of the values of M on those representables).
Let's define the category of weak descent data, denoted M^U. Its objects are families of objects $x_n \in M(U_n)$ with morphisms $\phi_u : u^*(x_m) \to x_n$ for each map $u : U_n \to U_m$ induced by a map $[m] \to [n]$ satisfying the usual cocycle condition. Its morphisms are families of morphisms $f_n : x_n \to y_n$ such that $f_i \circ \phi_u = \phi_u \circ f_j$ for any map $[i] \to [j]$.
There is a model category structure on M^U in which fibrations and weak equivalences are defined levelwise.
An object $x \in M^U$ is called homotopy cartesian if the maps ϕ_u induce isomorphims in the homotopy category $\mathrm{Ho}(M_n)$. We denote M^U_{cart} the full subcategory spanned by homotopy cartesian objects.

Definition 6.1.14. M is a sheaf of model categories if, for any $X \in T$ and any hypercovering $U_* \to X$ the natural map $M(X) \to M^U$ induces an equivalence of categories $\mathrm{Ho}(M(X)) \to \mathrm{Ho}(M^U_{cart})$

Theorem 6.1.15 ([HAG-II])**.** If M is a sheaf of model categories, then $X \to |W(X)^c|$ is a stack.

6.2 Derived algebraic geometry

Classically, it is sometimes useful to substitute a k-scheme with the functor $k-Alg \to Sets$ it represents, the so called functor of points.
The name comes from the fact that if a scheme is given by a system of equations with coefficients in k, than the functor associates to each k-algebra A the set of solutions of those equations in A. This is a rather useful point of view even in the classical setting: for example a group structure on a scheme G is simply the datum of a group structure on each $G(A)$ in a compatible way and if we are given another scheme X, an action of G on X is simply an action of $G(A)$ on $X(A)$ for all A compatibly.
Unfortunately in this setting quotienting by a group action is a rather poorly behaved operation.
A solution to this problem is to enlarge the setting and consider not only sheaves but stacks as well. Besides, to improve intersections as well (classically the schematic intersection loses some information), we will consider the ∞-category of DG k-algebras rather than the category of k-algebras.

Basic definitions

Definition 6.2.1. Let Mod_k denote the monoidal category of unbounded complexes of k-modules with the projective model structure.

Proposition 6.2.2. Mod_k is a simplicial model category in the following way: the space $\mathrm{Map}(A,B)$ is the simplicial k-module associated to the truncation of the function complex between A and B. By abuse of notation we will call Mod_k also the ∞-category spanned by fibrant cofibrant objects in Mod_k

Definition 6.2.3. $DG_kAlg_{\leq 0}$ is the ∞-category of coconnective differential graded algebra.

In order to do some algebraic geometry on this category, we need to generalise the standard definitions of algebraic geometry to this new context (following Toën).

Definition 6.2.4. A morphism $f : A \to B$ is flat (resp. smooth, étale, a Zariski open immersion) if:
1) The induced morphism of affine schemes $\mathrm{Spec}H_0(B) \to \mathrm{Spec}H_0(A)$ is flat (resp. smooth, étale, a Zariski open immersion)
2) For any $i > 0$ the natural map:

$$H_{-i}(A) \otimes_{H_0(A)} H_0(B) \to H_{-i}(B)$$

is an isomorphism

Definition 6.2.5. A finite family of morphisms $\{f_i : A \to B_i\}$ in $\mathbf{dgAlg}_{\leq 0}$ is a ffqc covering if each f_i is flat and the induced morphism of affine schemes

$$\coprod \mathrm{Spec}H_0(B_i) \to \mathrm{Spec}H_0(A)$$

is surjective

Now it is possible to define a topology on the ∞-category $\mathbf{dgAlg}_{\leq 0}^{op}$ saying that a sieve is covering if it contains a ffqc covering.
Using the formalism of the previous setting, we define a derived stack to be a stack for this ∞ site.

Remark 6.2.6. There is an obvious adjunction $t_0 : \mathbf{dgAlg}_{\leq 0} \rightleftarrows k-Alg : i$ which induces an adjunction $i^* : St_k \rightleftarrows Sh(k-Alg) : t_0^*$.
We say that a stack X is a derived enhancement of a sheaf of k-algebras F if $i(X) = F$. Every sheaf possess a trivial enhancement: $t_0(F)$, which is always a 0-stack. However in many cases the best enhancement is not $t_0(F)$, but a more natural one.

Two examples of stacks

Definition 6.2.7 (Stack of quasi-coherent sheaves)**.** Assigning to each DG-algebra A the model category of DG A-modules and to each $\pi : S \to R$ the Quillen functor $S - mod \to R - mod$ corresponding to the extension of scalars is a sheaf of model categories, denoted QC. So if we consider $X \mapsto |W(QC(X))^c|$ we obtain an example of stack, the classifier of quasi-coherent sheaves.

Definition 6.2.8. We can proceed as above, but restraining our attention to perfect A-modules, that is to say with some finitude conditions: namely a DG-module M is perfect if it is a compact object in the homotopy category $\mathrm{Ho}(DG - mod)$. We thus get the stack $Perf$, which classifies perfect quasi coherent sheaves.

6.3 Affinisation

We want to introduce the notion of affine stack (Toën), which is a homotopical generalisation of the notion of affine scheme and will be useful for dealing with CW complexes considered as derived stacks.

Definition 6.3.1. The category of affine stacks is the opposite of the category of DG algebras, without any coconnectivity assumptions: $Aff_k := DGAlg_k^{op}$

Remark 6.3.2. There is an evident map $\mathrm{Spec} : Aff_k \to St_k$ given by the Yoneda functor: $\mathrm{Spec}(R)(A) := \mathrm{Hom}_{DGAlg_k}(R,A)$

As in the classical case, this functor has a left adjoint $\mathcal{O} : St_k \to Aff_k$ which is the only functor sending each derived affine scheme to the corresponding coconnective DG algebra and preserving colimits.
The monad $\mathrm{Aff} := \mathrm{Spec} \circ \mathcal{O}$ is called the affinisation functor. Each stack X comes equipped with a universal arrow $X \to \mathrm{Aff}(X)$ through which every arrow from X to an affine stack factorises.

Now we will try to understand what the affinisation functor does on CW complexes. In order to do that, we need to be able to understand how to interprete CW complexes as derived stacks.

Proposition 6.3.3. There is an adjunction $sSets \rightleftarrows DPrSt_k$ where the left adjoint sends each simplicial set to the corresponding constant prestack and the right adjoint sends each prestack X to the simplicial set X(k)

Proof. $\mathrm{Map}(S, X(k)) = \mathrm{Map}(S, \mathrm{Hom}(\mathrm{Spec}(k), X)) = \mathrm{Hom}(\mathrm{Spec}(k) \otimes S, X)$ and $\mathrm{Spec}(k) \otimes S$ is the constant simplicial prestack associated to S, as colimits are taken objectwise and the prestack $\mathrm{Spec}(k)$ associates to each DG algebra a point. □

We can compose $sSets \rightleftarrows DPrSt_k$ with the adjunction: $DPrSt_k \rightleftarrows DSt_k$ between the forget and the sheafification functor, as in section 1.
We get maps $sSets \rightleftarrows DSt_k$ which allows us to consider simplicial sets as a particular case of derived stacks. In particular, as left adjoints preserve colimits, and finite simplicial sets can be expressed as finite colimits of copies of the point, we can recover finite simplicial sets (or equivalently finite CW complexes) as those stacks which are finite colimits of copies of Spec(k).
For example the circle S^1 is the derived stack:

$$\mathrm{Spec}(k) \coprod_{\mathrm{Spec}(k) \coprod \mathrm{Spec}(k)} \mathrm{Spec}(k)$$

Instead general simplicial sets correspond to arbitrary colimits of copies of Spec(k).

6.4 Cohomology

In classical algebraic topology cohomology is represented by the Eilenberg-MacLane spaces $K(G,n) = B^nG$. We will be interested in cohomology with coefficient in k, that is to say $G = \mathbb{G}_a$ is the additive group. What does $B^n\mathbb{G}_a$ look like?

Proposition 6.4.1. For $n \geq 1$ we have $B^n\mathbb{G}_a \simeq \mathrm{Spec}(S_n)$ where S_n is the free DG algebra on the module $k[-n]$.

Proof. Let us consider the functor $\mathcal{O}$ as a contravariant functor from St_k to $DGMod_k$. It sends pushout squares to pullback squares (which are also pushout squares as the category $DGMod_k$ is stable).
If we consider the pushout diagram:

$$\begin{tikzcd} S^{d-1} \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & S^d \end{tikzcd}$$

It maps to the pushout/pullback diagram:

$$\begin{tikzcd} \mathcal{O}(S^d) \arrow[r] \arrow[d] & k \arrow[d] \\ k \arrow[r] & \mathcal{O}(S^{d-1}) \end{tikzcd}$$

which allows us to calculate $\mathcal{O}(S^d)$ (as a DG module) recursively: $\mathcal{O}(S^d) = k \oplus k[-d]$
It follows from this calculation that

$$[S^d, \mathrm{Spec}S_n] \simeq \pi_0(\mathrm{Hom}_{DGAlg}[S_n, \mathcal{O}(S^d)]) \simeq \pi_0(\mathrm{Hom}_{DGMod}(k[-n], k \oplus k[d])$$

which is k if d=n and 0 otherwise.
On the other hand, $[S^d, B^n\mathbb{G}_a] \simeq H^n(S^d)$ which is k if d=n and 0 otherwise, so Spec(S_n) and $B^n\mathbb{G}_a$ have the same homotopy groups and only one of them is nontrivial, so they are equivalent.

□

Theorem 6.4.2. For X a CW complex, $\mathcal{O}(X) = C^*(X)$ where the right member of the equality is the cochain complex.

Proof. $H^n(X) = \mathrm{Map}(X, B^n\mathbb{G}_a) = \mathrm{Hom}_{DGAlg}(S_n, \mathcal{O}(X)) = H^n(\mathcal{O}(X))$. We have a natural map $C^*(X) \simeq C^*(\mathrm{colSpec}(k)) \to \lim C^*(\mathrm{Spec}(k)) \simeq \lim k \simeq \mathcal{O}(X)$ □

Remark 6.4.3. Actually we have a particularly nice description of the algebra S_1: it is generated by one element of degree one ϵ and the algebra structure is trivial as $\epsilon\epsilon = -\epsilon\epsilon$ (by graded-commutativity) so both sides are equal to 0.
So $S_1 \simeq k[\epsilon]/(\epsilon^2) \simeq \mathcal{O}(S^1)$ so $B\mathbb{G}_a \simeq \mathrm{Aff}(S^1)$

6.5 Groups and group action

Definition 6.5.1. A group stack G is simply a functor $G : \mathbf{dgAlg}_{\leq 0} \to A_\infty - alg$ such that the induced functor $\mathbf{dgAlg}_{\leq 0} \to A_\infty - alg \to sSets$ is a stack.
Similarly, if G is a group stack and X a stack, a G action on X is an objectwise action af G(A) on X(A) for all $A \in \mathbf{dgAlg}_{\leq 0}$ in a compatible way.

Remark 6.5.2. One disposes of the following adjunctions:

$$sSets \rightleftarrows E_1 - alg \rightleftarrows sSets_\bullet$$

where the first is the free/forget adjunction and the second is the B/Ω adjunction. Clearly $B \circ free = \Sigma$. So $\Omega\Sigma = \Omega \circ B \circ free = free$, so if X is a stack the free group on X is simply $\Omega\Sigma X$.

Example 6.5.3. $X = S^1$. $\Omega S^2 = \Omega\Sigma S^1$ is the corresponding free group, so a ΩS^2 action on X is simply a morphism of stacks $S^1 \times X \to X$
It is entertaining to understand what the universal group morphism $\Omega S^2 \to S^1$ correspond to in more traditional terms. We start from the Hopf fibration $S^1 \to S^3 \to S^2$ which is classified by the following pullback square:

$$\begin{tikzcd} S^3 \arrow[r] \arrow[d] & S^2 \arrow[d] \\ pt \arrow[r] & BS^1 \end{tikzcd}$$

As the functor Ω is a right adjoint, it preserves pullback squares, so we have:

$$\begin{tikzcd} \Omega S^3 \arrow[r] \arrow[d] & \Omega S^2 \arrow[d] \\ pt \arrow[r] & S^1 \end{tikzcd}$$

where the vertical map on the right is the universal group morphism $\Omega S^2 \to S^1$ induced by the group structure on S^1

The quotient of a stack by a group stack action is the component-wise homotopical quotient.

Proposition 6.5.4. QC(X/G) is the category of G-equivariant quasi-coherent sheaves over X.

Proof. We only need to prove it for an affine $X = \mathrm{Spec}(A)$. In that case X/G is the colimit of a BG(A) shaped diagram where the only object is Spec(A) and to each arrow we assign the corresponging element of G. As QC sends colimits to limits, the thesis follows. □

In particular, the category of modules with G-action is simply $QC(*/G) = QC(BG)$

Remark 6.5.5. Actually, the preceding reasoning is based on the following rather intresting fact, which is quite interesting per se:
if G is a simplicial group, we can form an ∞-category that has only an object * and $\mathrm{Hom}(*,*) = G$. This category is an ∞-groupoid and so, by the homotopy hypothesis, it corresponds to a topological space (via the Π_∞ construction). Such a topological space is the principal G-bundles classifier BG.
So, a good theory of ∞-categories must include a way of delooping groups. For example, Segal categories (which I personally believe are particularly well suited to doing derived algebraic geometry) correspond to Segal delooping machine: taking the diagonal of the bisimplicial set corresponding to the Segal category we have constructed just above.

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EXPOSÉ **7**

Differential graded categories

Pieter Belmans

7.1 Introduction

The notion of triangulated category lies at the heart of homological algebra. This type of category is essential in the study of derived categories and stable homotopy categories of spectra. But there are some problems with this notion, e.g. the cone construction is not functorial. Already in Verdier's PhD thesis there is the result that if the cones in a (countably) (co)complete triangulated category are functorial that category is necessarily semisimple abelian [Ver67, proposition II.1.2.13]. As not every triangulated category is abelian [CSAM29, exercise 1.4.5] this is a problem. To put it more bluntly:

> "This 'nonfunctoriality of a cone' is the first symptom that something is going wrong in the axioms of a triangulated category."
>
> [GM10, section IV.7]

Two other problems with the axioms of triangulated categories can be summarised as:

1. the fibered product of triangulated categories is no longer triangulated, which makes "glueing" triangulated categories impossible;

2. there is no triangulated structure on the functor category for two triangulated categories.

In a more abstract context: up to now we have seen that there are different models for $(\infty, 1)$-categories. The models we have discussed so far are quasicategories, relative categories using simplicial localisation, Segal categories and complete Segal spaces. These are all models for the *general* theory of $(\infty, 1)$-categories.

One could restrict himself on the other hand to certain *subclasses* of $(\infty, 1)$-categories. An important example of this phenomenon are model categories, which provide a model for certain $(\infty, 1)$-categories that are both complete and cocomplete. Whether they can model *all* complete and cocomplete $(\infty, 1)$-categories is not known. Another possible subclass of $(\infty, 1)$-categories is the one modelled using differential graded categories. They are so called "linear" models, in the same sense that homological algebra is a linear version of homotopical algebra. Under the appropriate conditions we will see that dg categories are equivalent to stable $(\infty, 1)$-categories enriched over the monoidal $(\infty, 1)$-category of k-module spectra, whatever that may mean at this point.

Whereas topological or simplicial categories are categories enriched over topological spaces or simplicial sets, a differential graded category is a category enriched over (co)chain complexes of k-modules, for k a commutative ring. So we restrict ourselves from ∞-groupoids to so called abelian and fully strict ∞-groupoids, these are exactly the ∞-groupoids modelled by chain complexes.

The goal of this exposé is to

1. introduce dg categories and related constructions;
2. discuss the model category structures on the category of dg categories;
3. give some applications of this machinery, demonstrating how derived algebraic geometry can be used to generalize results from algebraic geometry;
4. explain how to construct a $(\infty, 1)$-category for every dg category and discuss the properties of this construction.

The main expository references for dg categories are [LNM2008; Kel06]. The results of this exposé are mostly obtained in [Toe07] while, at the end the discussion is based on [HA].

7.2 Differential graded categories

From now on we fix a commutative ring k. Every construction is relative to this base ring. Whenever the ring is required to be a field it will be specified. We moreover use cochain complexes, i.e. the degree of the differential is 1, but the exposition can be done completely the same using chain complexes and morphisms of degree -1.

As already suggested, the definition of a dg category is easy. We will freely use the language of enriched categories, but whenever a down-to-earth interpretation can be made it will be given.

Definition 7.2.1. A *dg category* is a category enriched over cochain complexes of k-modules.

A cochain complex can equivalently be considered as a dg k-module: it is a graded k-module equipped with a differential d of degree 1. So dg categories are categories in which the Hom-sets carry the structure of a cochain complex. The monoidal structure of k-**Mod** implies that the composition in a dg category $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(Y,Z)^{\bullet} \otimes_k \mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet} \to \mathrm{Hom}_{\mathcal{C}}(X,Z)^{\bullet} \tag{7.1}$$

is a morphism of dg k-modules. Remark that the tensor product of graded k-modules $V^{\bullet}$ and $W^{\bullet}$ is given by

$$(V^{\bullet} \otimes_k W^{\bullet})^n := \bigoplus_{p+q=n} V^p \otimes_k W^q \tag{7.2}$$

and tensor products of morphisms are equipped with the Leibniz sign rule.

Recall that we can generalize the notion of a ring by allowing multiple objects. A dg category is the same game, using dg modules over a dg algebra. Hence we get the following example.

Example 7.2.2. A dg k-algebra $A^{\bullet}$ can be interpreted as a dg category with a single object. Such a dg algebra can be considered as a cochain complex and a k-algebra. Examples are Koszul complexes or tensor algebras. Remark that the Koszul sign rule implies that the multiplication in a dg k-algebra satisfies the *graded Leibniz rule*

$$\mathrm{d}_{A^{\bullet}}(ab) = \mathrm{d}_{A^{\bullet}}(a)b + (-1)^p a\mathrm{d}_{A^{\bullet}}(b) \tag{7.3}$$

for $a \in A^p$.

If we equip a (non-dg) k-algebra A with the trivial differential, i.e. $\mathrm{d}_A = 0$ we get an instance of a dg category.

There are also less trivial examples, which generalize the notion of modules over a ring with multiple objects.

Example 7.2.3. Let A again be a k-algebra, consider the category $\mathbf{Ch}(A\text{-}\mathbf{Mod})$ of complexes of (right) A-modules. Instead of taking the usual morphisms of cochain complexes we will introduce the category $\mathbf{Ch}_{\mathrm{dg}}(A\text{-}\mathbf{Mod})$ which has exactly the cochain complexes of A-modules as objects.

But for the morphisms we define the dg k-module $\mathrm{Hom}_{\mathbf{Ch}_{\mathrm{dg}}(A\text{-}\mathbf{Mod})}(M^\bullet, N^\bullet)^\bullet$ for cochain complexes $M^\bullet$ and $N^\bullet$ to have in its nth degree the morphisms of degree n, i.e. for each $p \in \mathbb{Z}$ the map $f^p : M^p \to N^{n+p}$ is a morphism of A-modules, composition being the composition of graded morphisms which clearly is compatible with this structure. The differential between these Hom-structures is defined by setting

$$\mathrm{d}(f) = \mathrm{d}_{N^\bullet} \circ f - (-1)^n f \circ \mathrm{d}_{M^\bullet} \tag{7.4}$$

for f a morphism of degree n and this is where the original structure of cochain complexes is used. To check that this defines a differential, we check that

$$\begin{aligned} \mathrm{d}^2(f) &= \mathrm{d}_{N^\bullet}\left(\mathrm{d}_{N^\bullet} \circ f - (-1)^n f \circ \mathrm{d}_{M^\bullet}\right) - (-1)^{n+1}\left(\mathrm{d}_{N^\bullet} \circ f - (-1)^n f \circ \mathrm{d}_{M^\bullet}\right) \circ \mathrm{d}_{M^\bullet} \\ &= \mathrm{d}^2_{N^\bullet} \circ f - (-1)^n \mathrm{d}_{N^\bullet} \circ f \circ \mathrm{d}_{M^\bullet} - (-1)^{n+1}\mathrm{d}_{N^\bullet} \circ f \circ \mathrm{d}_{M^\bullet} + (-1)^{2n+1} f \circ \mathrm{d}^2_{M^\bullet} \\ &= 0. \end{aligned} \tag{7.5}$$

As we have enriched categories, there is a notion of underlying category.

Definition 7.2.4. The *underlying category* $\mathrm{Z}^0(\mathcal{C})$ is the category with $\mathrm{Ob}(\mathrm{Z}^0(\mathcal{C})) \coloneqq \mathrm{Ob}(\mathcal{C})$ but we take

$$\mathrm{Hom}_{\mathrm{Z}^0(\mathcal{C})}(X, Y) \coloneqq \mathrm{Z}^0\left(\mathrm{Hom}_{\mathcal{C}}(X, Y)^\bullet\right). \tag{7.6}$$

To be more precise, the morphisms in $\mathrm{Z}^0(\mathcal{C})$ are exactly those morphisms who live in the kernel of $\mathrm{d} \colon \mathrm{Hom}_{\mathcal{C}}(X, Y)^0 \to \mathrm{Hom}_{\mathcal{C}}(X, Y)^1$.

This category could also be called the *cocycle category* in this specific context, but it seems I am the only one thinking of doing this. Aside from the underlying category there is another construction of a category, given a dg category.

Definition 7.2.5. The *cohomology category*, or *homotopy category*, $\mathrm{H}^0(\mathcal{C})$ is the category with $\mathrm{Ob}(\mathrm{H}^0(\mathcal{C})) \coloneqq \mathrm{Ob}(\mathcal{C})$ but we take

$$\mathrm{Hom}_{\mathrm{H}^0(\mathcal{C})}(X, Y) \coloneqq \mathrm{H}^0\left(\mathrm{Hom}_{\mathcal{C}}(X, Y)^\bullet\right). \tag{7.7}$$

Now that we have defined these related categories we can see how they might provide a solution to the problem of triangulated categories using this $\mathbf{Ch}(k\text{-}\mathbf{Mod})$-enrichment.

Example 7.2.6. Continuing with the dg category $\mathbf{Ch}_{\mathrm{dg}}(A\text{-}\mathbf{Mod})$ from example 7.2.3 we see that

$$\mathrm{Z}^0(\mathbf{Ch}_{\mathrm{dg}}(A\text{-}\mathbf{Mod})) = \mathbf{Ch}(A\text{-}\mathbf{Mod}) \tag{7.8}$$

because a morphism $f : M^\bullet \to L^\bullet$ (which is at this point not a map of cochain complexes) belongs to the kernel of the differential of $\mathbf{Ch}_{\mathrm{dg}}(A\text{-}\mathbf{Mod})$ if and only if $\mathrm{d}_{M^\bullet} \circ f - f \circ \mathrm{d}_{L^\bullet} = 0$ which is exactly the condition that squares commute in maps of cochain complexes. So we observe that the category $\mathbf{Ch}_{\mathrm{dg}}(A\text{-}\mathbf{Mod})$ is a $\mathbf{Ch}(k\text{-}\mathbf{Mod})$-enrichment of $\mathbf{Ch}(A\text{-}\mathbf{Mod})$.

Similarly we get that

$$\mathrm{H}^0(\mathbf{Ch}_{\mathrm{dg}}(A\text{-}\mathbf{Mod})) = \mathbf{K}(A\text{-}\mathbf{Mod}) \tag{7.9}$$

where $\mathbf{K}(A\text{-}\mathbf{Mod})$ is the *category of complexes up to homotopy*, as it occurs in the (classical) construction of the derived category of the category of chain complexes. So the higher homotopies are in a way contained in $\mathbf{Ch}_{\mathrm{dg}}(A\text{-}\mathbf{Mod})$ and we will be able to use them.

7.3 The category of differential graded categories

Now we will define the category of all small dg categories, which we'll denote $\mathbf{dg\,Cat}_k$. We add the adjective "small" to prevent certain size issues to come up, just like we consider only the category of small categories.

Definition 7.3.1. A *dg functor* $F\colon \mathcal{C} \to \mathcal{C}'$ where $\mathcal{C}$ and $\mathcal{C}'$ are two (small) dg categories is a map $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ on the level of the objects with the data of a morphism $f_{X,Y}$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$ which is a morphism

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet} \to \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y))^{\bullet} \tag{7.10}$$

of dg k-modules. These maps are required to be compatible with composition and units, which implies the commutativity of

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Y,Z)^{\bullet} \otimes_k \mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet} \arrow[r, "-\circ-"] \arrow[d, "F_{Y,Z}\otimes F_{X,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(X,Z)^{\bullet} \arrow[d, "F_{X,Z}"] \\ \mathrm{Hom}_{\mathcal{C}'}(F(Y),F(Z))^{\bullet} \otimes_k \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y))^{\bullet} \arrow[r, "-\circ-"] & \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Z))^{\bullet} \end{tikzcd} \tag{7.11}$$

and

$$\begin{tikzcd} k \arrow[r, "e_X"] \arrow[dr, "e_{F(X)}"'] & \mathrm{Hom}_{\mathcal{C}}(X,X)^{\bullet} \arrow[d, "F_{X,X}"] \\ & \mathrm{Hom}_{\mathcal{C}'}(F(X),F(X))^{\bullet} \end{tikzcd} \tag{7.12}$$

for $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

The *category of small dg categories* $\mathbf{dg\,Cat}_k$ is the category of all small dg categories together with the dg functors as morphisms.

Remark 7.3.2. The category $\mathbf{dg\,Cat}_k$ has the empty dg category as initial object and the dg category with one object $*$, equipped with the zero endomorphism ring, i.e.

$$\mathrm{Hom}_{\mathbf{dg\,Cat}_k}(*,*)^{\bullet} = 0, \tag{7.13}$$

as final object.

We can endow the category $\mathbf{dg\,Cat}_k$ with a tensor product and an internal Hom-functor, hence it will become a closed symmetric tensor category. This is nothing but using the enrichment over the symmetric monoidal category $\mathbf{Ch}(k\text{-}\mathbf{Mod})$ [LNM145, Section 1.4].

Definition 7.3.3. The *tensor product* $\mathcal{C}\otimes\mathcal{D}$ of two dg categories $\mathcal{C}$ and $\mathcal{D}$ is defined by taking $\mathrm{Ob}(\mathcal{C}\otimes\mathcal{D}) := \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C})$ and setting

$$\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}\left((X,Y),(X',Y')\right)^{\bullet} := \mathrm{Hom}_{\mathcal{C}}(X,X')^{\bullet} \otimes_k \mathrm{Hom}_{\mathcal{D}}(Y,Y')^{\bullet}. \tag{7.14}$$

Remark 7.3.4. The unit for the monoidal structure is the dg category k, where k by abuse of notation denotes both the dg category and the dg algebra with k concentrated in degree 0, using example 7.2.2.

To define the internal Hom-functor we need to explain how we will use the **Ch**(k-**Mod**)-enrichment.

Definition 7.3.5. Let $\mathcal{C}$ and $\mathcal{D}$ be small dg categories. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two dg functors. A *natural transformation of degree n* $\phi\colon F \Rightarrow G$ is a family of morphisms $(\phi_X)_{X\in \mathrm{Ob}(\mathcal{C})}$ such that $\phi_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))^n$ for $X \in \mathrm{Ob}(\mathcal{C})$ satisfying $G(f)(\phi_X) = \phi_Y(F(f))$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $Y \in \mathrm{Ob}(\mathcal{C})$. In other words, if f is homogeneous of degree m, we have the commutativity of the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\phi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\phi_Y"] & G(Y) \end{tikzcd} \tag{7.15}$$

up to the sign $(-1)^{nm}$.

The *complex of graded morphisms* $\mathcal{H}\mathrm{om}(F,G)^\bullet$ for two dg functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is the complex of graded morphisms (or natural transformations) such that $\mathcal{H}\mathrm{om}(F,G)^n$ consists of the natural transformations of degree n. The differential in this complex is given for each $X \in \mathrm{Ob}(\mathcal{C})$ by

$$\mathrm{d}^n_{\mathcal{H}\mathrm{om}(F,G)}(\phi)(X) := \mathrm{d}^n_{\mathrm{Hom}_{\mathcal{D}}(F(X),G(X))}(\phi_X) \tag{7.16}$$

which lands in $\mathrm{Hom}_{\mathcal{D}}(F(X), G(X))^{n+1}$.

Example 7.3.6. Just like in example 7.2.6 we get that $\mathrm{Z}^0(\mathcal{H}\mathrm{om}(F,G)^\bullet)$ describes the (classical) natural transformations $F \Rightarrow G$.

Definition 7.3.7. Let $\mathcal{C}$ and $\mathcal{D}$ be two dg categories. The *internal Hom* in $\mathbf{dg\,Cat}_k$ for $\mathcal{C}$ and $\mathcal{D}$ is the dg category $\mathcal{H}\mathrm{om}(\mathcal{C}, \mathcal{D})$ which has the dg functors between $\mathcal{C}$ and $\mathcal{D}$ as objects and the complex of graded morphisms $\mathcal{H}\mathrm{om}(F,G)^\bullet$ between two dg functors $F, G\colon \mathcal{C} \to \mathcal{D}$ as morphism spaces.

If we take the dg category with the single object k as discussed in example 7.2.2 as the unit object, we have that the category $\mathbf{dg\,Cat}_k$ is a symmetric tensor category, i.e. we have the adjunction

$$\mathrm{Hom}_{\mathbf{dg\,Cat}_k}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathrm{Hom}_{\mathbf{dg\,Cat}_k}(\mathcal{A}, \mathcal{H}\mathrm{om}(\mathcal{B}, \mathcal{C})) \tag{7.17}$$

for $\mathcal{A}, \mathcal{B}, \mathcal{C}$ dg categories.

The pair $(\otimes, \mathcal{H}\mathrm{om})$ makes $\mathbf{dg\,Cat}_k$ into a closed symmetric monoidal category. This structure will be important for the remainder of this work. We want our localisations to be compatible with it, but this is the source of an important issue in the obvious model category structure as it will be introduced in what follows.

7.4 Differential graded modules

We've generalized (dg) algebras to (dg) algebras with multiple objects as discussed in examples 7.2.2 and 7.2.3. The same game can be played with dg modules.

Definition 7.4.1. Let $\mathcal{C}$ be a small dg category. We will define a *left dg $\mathcal{C}$-module* to be a dg functor $L\colon \mathcal{C} \to \mathbf{Ch}_{\mathrm{dg}}(k\text{-}\mathbf{Mod})$ while a *right dg $\mathcal{C}$-module* is a dg functor $M\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Ch}_{\mathrm{dg}}(k\text{-}\mathbf{Mod})$.

So a dg $\mathcal{C}$-module could also be defined as a "**Ch**(k-**Mod**)-enriched presheaf on the dg category $\mathcal{C}$". This terminology is not standard though, and we will not use it. But keeping this in mind can help in understanding the philosophy of certain statements and proofs.

As usual we can consider all dg modules and endow them with the structure of a category.

Definition 7.4.2. Let $\mathcal{C}$ be a small dg category. The *category of dg $\mathcal{C}$-modules* $\mathcal{C}\text{-}\mathbf{dg\,Mod}_k$ has all dg $\mathcal{C}$-modules as objects and morphisms of dg functors as morphism spaces.

It is an abelian category, where epi- resp. monomorphisms can be checked degreewise.

We can generalize the notion of a dg $\mathcal{C}$-module which has values in the model category $\mathbf{Ch}(k\text{-}\mathbf{Mod})$ to "dg $\mathcal{C}$-modules with coefficients in a $\mathbf{Ch}(k\text{-}\mathbf{Mod})$-model category $\mathcal{M}$". For this we use the theory of monoidal model categories [MSM63, section 4.2].

Definition 7.4.3. Let $\mathcal{C}$ be a dg category and $\mathcal{M}$ a cofibrantly generated $\mathbf{Ch}(k\text{-}\mathbf{Mod})$-model category. A *dg $\mathcal{C}$-module with values in* $\mathcal{M}$ is a dg functor $\mathcal{C} \to \mathcal{M}$, where $\mathcal{M}$ is a dg category because its Hom-sets are by assumption enriched over $\mathbf{Ch}(k\text{-}\mathbf{Mod})$.

The *category of dg $\mathcal{C}$-modules with values in* $\mathcal{M}$ has as objects the dg functors $\mathcal{C} \to \mathcal{M}$ and its morphisms are given by the complexes of graded morphisms (or natural transformations). It will be denoted $\mathcal{C}\text{-}\mathbf{dg\,Mod}_k(\mathcal{M})$.

For every $\mathbf{Ch}(k\text{-}\mathbf{Mod})$-model category $\mathcal{M}$ we can define its associated "internal dg category" (even without the assumption that it is cofibrantly generated). This provides a dg enrichment of the homotopy category Ho $\mathcal{M}$, as is shown in proposition 7.4.5.

Definition 7.4.4. Let $\mathcal{M}$ be a $\mathbf{Ch}(k\text{-}\mathbf{Mod})$-model category. Its *internal category* $\mathrm{Int}(\mathcal{M})$ is the dg category whose objects are the both fibrant and cofibrant objects of $\mathcal{M}$, i.e. $\mathrm{Ob}(\mathrm{Int}(\mathcal{M})) = \mathrm{Ob}(\mathcal{M}_{\mathrm{cof,fib}})$. Its cochain complexes of morphisms are obtained using the $\mathbf{Ch}(k\text{-}\mathbf{Mod})$-enrichment of $\mathcal{M}$, i.e. we set

$$\mathrm{Hom}_{\mathrm{Int}(\mathcal{M})}(X,Y)^\bullet := \mathrm{Hom}_{\mathcal{M}}(X,Y)^\bullet \tag{7.18}$$

for $X, Y \in \mathrm{Ob}(\mathrm{Int}(\mathcal{M}))$.

This dg category serves as an enrichment for Ho $\mathcal{M}$, as one might already guess from its definition.

Proposition 7.4.5. Let $\mathcal{M}$ be a $\mathbf{Ch}(k\text{-}\mathbf{Mod})$-model category. We have the equivalence

$$\mathrm{Ho}\,\mathcal{M} \cong \mathrm{H}^0(\mathrm{Int}(\mathcal{M})). \tag{7.19}$$

Proof. By the very definition of $\mathrm{H}^0(\mathrm{Int}(\mathcal{M}))$ its objects are the objects of $\mathcal{M}_{\mathrm{cof,F{\small IB}}}$, and we know that this category (after a suitable quotient) is equivalent to Ho $\mathcal{M}$, so we can use this equivalence to define the functor $\mathrm{Ho}(\mathcal{M}) \to \mathrm{H}^0(\mathrm{Int}(\mathcal{M}))$ on the level of objects. Now the essential surjectivity as obtained from this equivalence immediately gives the *essential surjectivity* in this situation.

On the morphisms this functor is defined by sending $f : X \to Y$ in Ho $\mathcal{M}$ first to the corresponding morphism in $\mathcal{M}_{\mathrm{cof,F{\small IB}}}$ and then by using the $\mathbf{Ch}(\mathbf{Mod}k)$-enrichment we can interpret it as a morphism in $\mathrm{Hom}_{\mathrm{Int}(\mathcal{M})}(X,Y)^0$ the zeroth degree of the complex. So after taking the H^0 of this morphism we get a well-defined morphism in $\mathrm{H}^0(\mathrm{Int}(\mathcal{M}))$ which is compatible with the map as we've defined it on the objects. To see that it is *fully faithful* we observe that

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{H}^0(\mathrm{Int}(\mathcal{M}))}(X,Y) \\
&\quad = \mathrm{H}^0(\mathbf{R}\,\mathrm{Hom}_{\mathrm{Int}(\mathcal{M})}(X,Y)^\bullet) && \text{definition} \\
&\quad \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{Ch}(\mathbf{Mod}k))}(k, \mathbf{R}\,\mathrm{Hom}_{\mathrm{Int}(\mathcal{M})}(X,Y)^\bullet) && \text{Yoneda with } k \text{ in degree } 0 \\
&\quad \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(k \otimes^{\mathbf{L}} X, Y) && \text{total derived adjunction} \\
&\quad \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X,Y) && k \text{ is unit for } - \otimes^{\mathbf{L}} -
\end{aligned} \tag{7.20}$$

for X and Y by abuse of notation and the fundamental theorem of model categories both in $\mathrm{H}^0(\mathrm{Int}(\mathcal{M}))$ and $\mathrm{Ho}(\mathcal{M})$. So we can conclude that $\mathrm{H}^0(\mathrm{Int}(\mathcal{M}))$ and $\mathrm{Ho}(\mathcal{M})$ are indeed equivalent categories. □

7.5 Model category structures on $\mathbf{dg\,Cat}_k$

It is possible to put at least two different model category structures on $\mathbf{dg\,Cat}_k$: one with quasi-equivalences as weak equivalences, and the other with Morita equivalences as weak equivalences.

We start with the "canonical" model category structure, where the weak equivalences are the analogue of categorical equivalences.

Definition 7.5.1. Let $f:\mathcal{C}\to\mathcal{D}$ be a morphism in $\mathbf{dg\,Cat}_k$, i.e. a dg functor between dg categories. It is said to be *quasi-fully faithful* if for all $X,Y\in\mathrm{Ob}(\mathcal{C})$ the map

$$f_{X,Y}:\mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet\to\mathrm{Hom}_{\mathcal{D}}(f(X),f(Y))^\bullet \tag{7.21}$$

of cochain complexes is a quasi-isomorphism. It is said to be *quasi-essentially surjective* if the induced functor

$$\mathrm{H}^0(f)\colon\ \mathrm{H}^0(\mathcal{C})\to\mathrm{H}^0(\mathcal{D}) \tag{7.22}$$

on the level of (k-linear) categories is essentially surjective.

These are versions of the two conditions necessary to define the appropriate version of the equivalence of categories in the presence of a dg enrichment, hence we can define the following.

Definition 7.5.2. Let $f:\mathcal{C}\to\mathcal{D}$ be a morphism in $\mathbf{dg\,Cat}_k$. It is said to be a *quasi-equivalence* if it is both quasi-fully faithful and quasi-essentially surjective.

We now introduce the fibrations in this model category structure.

Definition 7.5.3. Let $f:\mathcal{C}\to\mathcal{D}$ be a morphism in $\mathbf{dg\,Cat}_k$. It is said to be a *quasi-fibration* if

1. for all $X,Y\in\mathrm{Ob}(\mathcal{C})$ the map

$$f_{X,Y}:\mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet\to\mathrm{Hom}_{\mathcal{D}}(f(X),f(Y))^\bullet \tag{7.23}$$

   is a fibration in $\mathbf{Ch}(k\text{-}\mathbf{Mod})$ for the projective model category structure on $\mathbf{Ch}(k\text{-}\mathbf{Mod})$, i.e. is it an epimorphism in every degree;

2. for all $X\in\mathrm{Ob}(\mathcal{C})=\mathrm{Ob}(\mathrm{H}^0(\mathcal{C}))$ and for all isomorphisms $v\colon\ \mathrm{H}^0(f)(X)\to Y$ in $\mathrm{H}^0(\mathcal{D})$ we can lift v to an isomorphism $u\colon X\to Y$ in $\mathrm{H}^0(\mathcal{C})$ such that $\mathrm{H}^0(f)(u)=v$.

Theorem 7.5.4 ([Tab05b])**.** If we take the quasi-equivalences as weak equivalences, the quasi-fibrations as fibrations, and the dg functors satisfying the right lifting property with respect to the trivial fibrations as cofibrations we obtain a cofibrantly generated model category structure on $\mathbf{dg\,Cat}_k$.

In the second model category structure we use a bigger class of weak equivalences, the same fibrations and therefore a smaller class of cofibrations. An important use of this model category structure are sheaves on higher stacks [TV09]. To define it we mimick the idea of a Morita equivalence, but now in a (derived) dg context.

Definition 7.5.5. Let $\mathcal{C}$ be a dg category. Its *derived category of dg $\mathcal{C}$-modules*, which we'll denote $\mathbf{D}(\mathcal{C}\text{-}\mathbf{dg\,Mod}_k)$, is the localisation of $\mathcal{C}\text{-}\mathbf{dg\,Mod}_k$ with respect to the quasi-isomorphisms.

Associated to a dg functor $f:\mathcal{C}\to\mathcal{D}$ is the restriction functor

$$f^*\colon\ \mathcal{D}\text{-}\mathbf{dg\,Mod}_k\to\mathcal{C}\text{-}\mathbf{dg\,Mod}_k \tag{7.24}$$

given by composition. This yields the following definition.

Definition 7.5.6. Let $f: \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathbf{dg\,Cat}_k$. It is said to be a *Morita equivalence* if

$$f^*: \mathbf{D}(\mathcal{D}\text{-}\mathbf{dg\,Mod}_k) \to \mathbf{D}(\mathcal{C}\text{-}\mathbf{dg\,Mod}_k) \tag{7.25}$$

is an equivalence of categories.

Theorem 7.5.7 ([Tab05a])**.** If we take take the Morita equivalences as weak equivalences, the quasi-fibrations as fibrations, and the dg functors satisfying the right lifting property with respect to the trivial fibrations as cofibrations we obtain a cofibrantly generated model category structure on $\mathbf{dg\,Cat}_k$.

Remark 7.5.8. From this point on, whenever we denote $\mathrm{Ho}(\mathbf{dg\,Cat}_k)$ it will be the homotopy category of $\mathbf{dg\,Cat}_k$ with respect to the model category structure using quasi-equivalences. This is in line with [Toe07] on which most of the results in this exposé are based.

7.6 Mapping spaces

The following section is a discussion of the main technical result in [Toe07]. Its actual proof is rather long and technical, and there is not enough room here to discuss even the necessary definitions and notations. One needs cosimplicial resolution functors, restrictions to specific subcategories, diagonals of bisimplicial sets and nerves just to write down the statement. So immediately its interpretation in terms of the homotopy category is given.

If we consider $(\mathcal{C} \otimes \mathcal{D}^{\mathrm{op}})$-$\mathbf{dg\,Mod}_k$ we can define for every $X \in \mathrm{Ob}(\mathcal{C})$ a morphism of dg categories $\mathcal{D}^{\mathrm{op}} \to \mathcal{C} \otimes \mathcal{D}^{\mathrm{op}}$ which on the level of objects is defined as $Y \mapsto (X, Y)$ and therefore on the level of the morphisms as

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(Y,Z)^\bullet \mapsto \mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}}}((X,Y),(X,Z))^\bullet \overset{(7.14)}{=} \mathrm{Hom}_{\mathcal{C}}(X,X)^\bullet \otimes_k \mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(Y,Z)^\bullet \tag{7.26}$$

by the mapping $f \mapsto \mathrm{id}_X \otimes f$, for $Y, Z \in \mathrm{Ob}(\mathcal{D}^{\mathrm{op}})$. Because the cofibrant replacement functor of $\mathbf{dg\,Cat}_k$ can be taken to be the identity on the objects [Toe07, proposition 2.3] this defines

$$i_X: \mathcal{D}^{\mathrm{op}} \to \mathrm{Q}(\mathcal{C}) \otimes \mathcal{D}^{\mathrm{op}} =: \mathcal{C} \otimes^{\mathbf{L}} \mathcal{D}^{\mathrm{op}}. \tag{7.27}$$

We can then define the following.

Definition 7.6.1. Let $\mathcal{C}$ and $\mathcal{D}$ be dg categories. A dg $(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})$-module F is *right quasi-representable* if for all $X \in \mathrm{Ob}(\mathcal{C})$ the dg $\mathcal{D}^{\mathrm{op}}$-module $i_X(F)$ is quasi-representable in $\mathcal{D}^{\mathrm{op}}$-$\mathbf{dg\,Mod}_k$, i.e. it is representable in $\mathrm{Ho}(\mathcal{D}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k)$ by an object D of $\mathcal{D}$.

We can now state the theorem.

Theorem 7.6.2. Let $\mathcal{C}$ and $\mathcal{D}$ be small dg categories. We have a functorial bijection

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{dg\,Cat}_k)}(\mathcal{C},\mathcal{D}) \cong \mathrm{Isom}\left(\mathrm{Ho}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-}\mathbf{dg\,Mod}_k^{\mathrm{rqr}}\right)\right). \tag{7.28}$$

Hence every "fraction" of dg functors in the homotopy category can be represented in a unique way by a right quasi-representable bimodule. If one is familiar with Morita theory this might ring a bell. The remainder of [Toe07] is based on the technical result that underlies this theorem.

7.7 Monoidal structure

Recall that there exists a closed monoidal structure on $\mathbf{dg\,Cat}_k$. Unfortunately, this structure does not descend to the homotopy category, as the tensor product of two cofibrant dg categories is not necessarily cofibrant. So in order to prove the existence of a closed monoidal structure on $\mathrm{Ho}(\mathbf{dg\,Cat}_k)$ one needs to do some work.

If on the other hand k is a field, we can use the ideas from [Tab10] if we equip $\mathbf{dg\,Cat}_k$ with the Morita model category structure. In this paper a derived internal Hom is constructed using “localising pairs”. One first proves a Quillen equivalence between $\mathbf{dg\,Cat}_k$ and a second model category, which is a true monoidal model category (i.e. the structures are compatible with eachother). This equivalence is moreover compatible with the monoidal structure, but not with the actual monoidal model structure. Then going to the homotopy categories there is a derived internal Hom for the second category, hence by the equivalence also on the homotopy category (with the Morita equivalences inverted). But we will continue to use the quasi-equivalences.

Theorem 7.7.1. We can equip the monoidal category $(\mathrm{Ho}(\mathbf{dg\,Cat}_k, -\otimes^{\mathbf{L}} -))$ with an internal Hom-object which we'll denote $\mathbf{R}\mathcal{H}\mathrm{om}$, hence it is a closed monoidal category. This internal Hom is moreover described by

$$\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{C},\mathcal{D}) \cong \mathrm{Int}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-}\mathbf{dg\,Mod}_k^{\mathrm{r\,qr}}\right) \tag{7.29}$$

for $\mathcal{C},\mathcal{D}\in\mathrm{Ob}(\mathrm{Ho}(\mathbf{dg\,Cat}_k))$.

So we have the derived tensor-Hom adjunction

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{dg\,Cat}_k)}(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D},\mathcal{E}) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{dg\,Cat}_k)}(\mathcal{C},\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{D},\mathcal{E})) \tag{7.30}$$

for all $\mathcal{C},\mathcal{D}$ and $\mathcal{E}$ dg categories.

Remark that the notation $\mathbf{R}\mathcal{H}\mathrm{om}$ is slightly suggestive: the original proof does not derive the internal Hom of $\mathbf{dg\,Cat}_k$. It is reminiscent of the “exceptional inverse image functor” in algebraic geometry. This functor lives in the world of derived categories of sheaves, and it is related to some derived functors but it is itself *not* the derived functor of some functor between sheaves. Recall that this functor is denoted by both $\mathbf{R}f^!$ and $f^!$, as there is no perfect notation. We will nevertheless $\mathbf{R}\mathcal{H}\mathrm{om}$, sometimes $\mathrm{rep}_{\mathrm{dg}}$ is used.

It is also possible to introduce a $\mathrm{Ho}(\mathbf{sSet})$-structure on $\mathrm{Ho}(\mathbf{dg\,Cat}_k)$. The internal $\mathbf{R}\mathcal{H}\mathrm{om}$ is compatible with this structure.

Corollary 7.7.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two dg categories. Let K be a simplicial set. Then we have the functorial isomorphism

$$K\otimes_{\mathbf{s}}^{\mathbf{L}}(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}) \cong (K\otimes_{\mathbf{s}}^{\mathbf{L}}\mathcal{C})\otimes^{\mathbf{L}}\mathcal{D} \tag{7.31}$$

in $\mathrm{Ho}(\mathbf{dg\,Cat}_k)$.

This corollary yields the following enriched tensor-Hom adjunction.

Corollary 7.7.3. Let $\mathcal{C},\mathcal{D},\mathcal{E}$ be dg categories. Then we have the functorial isomorphism

$$\mathrm{Map}^{\ell}_{\mathrm{Ho}(\mathbf{dg\,Cat}_k)}(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D},\mathcal{E}) \cong \mathrm{Map}^{\ell}_{\mathrm{Ho}(\mathbf{dg\,Cat}_k)}(\mathcal{C},\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{D},\mathcal{E})) \tag{7.32}$$

in $\mathrm{Ho}(\mathbf{sSet})$.

7.8 Applications

Differential graded categories, and the model structures on $\mathbf{dg}\,\mathbf{Cat}_k$ have already found many applications. Two will be discussed here, a myriad others exist. One of them motivates the philosophy that dg categories are the correct tool for studying derived categories, as it fixes the lack of functoriality in the cone construction in the context of triangulated categories. The second application is a generalisation of Orlov's result on Fourier-Mukai transforms [Orl97].

Functoriality of the cone

We will call a dg category $\mathcal{C}$ *pretriangulated* if $\mathrm{H}^0(\mathcal{C})$ is triangulated category. Let Δ_k^1 be the dg category on two objects 0 and 1 with a "unique arrow" between them, i.e. $\mathrm{Hom}_{\Delta_k^1}(0,1)^\bullet = k$ and the endomorphism algebras $\mathrm{Hom}_{\Delta_k^1}(x,x)^\bullet = k$ are generated by the identity element. Then one can prove that there exists a morphism

$$i\colon \mathcal{C} \to \mathbf{R}\mathcal{H}\mathrm{om}(\Delta_k^1, \mathcal{C}) =: \mathrm{Mor}_{\mathrm{dg}}(\mathcal{C}). \tag{7.33}$$

Then one considers its left adjoint

$$c\colon \mathrm{Mor}_{\mathrm{dg}}(\mathcal{C}) \to \mathcal{C}. \tag{7.34}$$

This morphism yields the enriched functorial cone construction, the actual cone construction on the level of triangulated categories is the functor

$$\mathrm{H}^0(\mathrm{Mor}_{\mathrm{dg}}(\mathcal{C})) \to \mathrm{H}^0(\mathcal{C}). \tag{7.35}$$

For a non-dg category $\mathcal{A}$ one has its category of morphisms $\mathrm{Mor}(\mathcal{A})$, and using the dg enriched category of morphisms one can obtain a natural functor

$$\mathrm{H}^0(\mathrm{Mor}_{\mathrm{dg}}(\mathcal{C})) \to \mathrm{Mor}(\mathrm{H}^0(\mathcal{C})) \tag{7.36}$$

which is in general essentially surjective and full. But it is not faithful, which corresponds to the failure of the functoriality of the cone construction for morphisms in $\mathrm{H}^0(\mathcal{C})$. For a more detailed discussion, see [LNM2008, section 5.1].

Integral transforms

Assuming one is familiar with the theory of Fourier-Mukai transforms [Huy06], one is inclined to generalise the theory [Ca05, conjecture 6.4]. Unfortunately, in the context of (triangulated) derived categories this statement is false [Ca05, example 6.5]. If one uses the dg enrichment on the other hand, the statement becomes true for a certain class of schemes and (continuous) morphisms.

Theorem 7.8.1 (Continuous transformations are representable)**.** Let X and Y be quasicompact and separated schemes over a field k such that at least one of them is flat over $\operatorname{Spec} k$. Then we have the isomorphism

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathrm{cont}}\left(\mathrm{Int}(\mathbf{Ch}(\mathbf{Qcoh}_X)), \mathrm{Int}(\mathbf{Ch}(\mathbf{Qcoh}_Y))\right) \cong \mathrm{Int}\left(\mathbf{Ch}(\mathbf{Qcoh}_{X\times_k Y})\right) \tag{7.37}$$

in $\mathrm{Ho}(\mathbf{dg}\,\mathbf{Cat}_k)$.

Remark that the conditions on X and Y originate from the theory of compact generators of derived categories, and are not imposed by the theory of dg categories. And we have started using quasicoherent sheaves. But a similar result can be proven for perfect complexes, which is closer to the original theory which uses coherent sheaves. For a generalisation in the context of derived algebraic geometry, see [BFN10].

7.9 $(\infty,1)$- and dg categories

As discussed in the introduction: dg categories provide a good way to study stable $(\infty,1)$-categories. To do so, one has to associate an $(\infty,1)$-category to a dg category. This will be done using the *dg nerve* construction. This is nothing but a nerve construction that incorporates the dg enrichment, and it is given in definition 7.9.1. Its definition is rather formal, so an interpretation of it in low degrees is given in example 7.9.2.

The important property of the dg nerve construction is that it is equivalent to the "naive" systematic construction of an $(\infty,1)$-category of chain complexes with values in an additive category. This construction is rather involved, and uses right-lax monoidal functors, the Dold-Kan correspondence, the Alexander-Whitney construction, ... But it can be proved that the reasonably down-to-earth construction of the dg nerve yields equivalent (yet not isomorphic) $(\infty,1)$-categories [HA, proposition 1.3.1.17]. So the study of dg categories is important in the context of $(\infty,1)$-categories, it is not just superseded by the generality of $(\infty,1)$-categories.

This section is based on [HA, section 1.3.1]. Observe that in this text the model used for $(\infty,1)$-categories are the quasicategories, and that statements are translated to the cohomological convention in this text.

Definition 7.9.1. Let $\mathcal{C}$ be a dg category over k. Its *dg nerve* $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ is the simplicial set given by taking $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})_n$ to be the set of ordered pairs $(\{X_i\}_{i=0,\dots,n},\{f_I\})$ such that

1. X_i is an object of $\mathcal{C}$ for $i=0,\dots,n$;
2. f_I is an element of the k-module $\mathrm{Hom}_{\mathcal{C}}(X_{i_-},X_{i_+})^{-m}$, for $m\geq 0$ and
$$I=\{i_-<i_m<i_{m-1}<\ldots<i_1<i_+\},\tag{7.38}$$
such that
$$\mathrm{d}(f_I)=\sum_{1\leq j\leq m}(-1)^j\left(f_{I\setminus\{i_j\}}-f_{\{i_j<\ldots<i_1<i_+\}}\circ f_{\{i_-<i_m<\ldots<i_j\}}\right)\tag{7.39}$$

For $\alpha\colon[m]\to[n]$ a non-decreasing function we define $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})_n\to\mathrm{N}_{\mathrm{dg}}(\mathcal{C})_m$ by
$$\left(\{X_i\}_{i=0,\dots,n},\{f_I\}\right)\mapsto\left(\{X_{\alpha(j)}\}_{j=0,\dots,m},\{g_J\}\right)\tag{7.40}$$
where g_J is given by
$$g_J:=\begin{cases}f_{\alpha(J)} & \text{if } \alpha|_J \text{ is injective}\\ \mathrm{id}_{X_i} & \text{if } J=\{j,j'\} \text{ such that } \alpha(j)=\alpha(j')=i\\ 0 & \text{otherwise.}\end{cases}\tag{7.41}$$

To see that this is actually a nerve construction that incorporates the dg information, we interpret this definition for $n=0,1,2$.

Example 7.9.2.

- The 0-simplices are the objects of $\mathcal{C}$, as the condition on f_I is empty.
- The 1-simplices are the morphisms of the underlying category of $\mathcal{C}$, as the pairs consist of objects X,Y of $\mathcal{C}$ and a single map $f\in\mathrm{Hom}_{\mathcal{C}}(X,Y)^0$ such that $\mathrm{d}(f)=0$.

- The 2-simplices are triples (X, Y, Z) of objects in $\mathcal{C}$ together with triples (f, g, h) of morphisms in $\mathcal{C}$ such that
$$\begin{aligned} f &\in \mathrm{Hom}_{\mathcal{C}}(X, Y)^0 \\ g &\in \mathrm{Hom}_{\mathcal{C}}(Y, Z)^0 \\ h &\in \mathrm{Hom}_{\mathcal{C}}(X, Z)^0 \end{aligned} \tag{7.42}$$
such that $\mathrm{d}(f) = \mathrm{d}(g) = \mathrm{d}(h) = 0$ and there is given a morphism $z \in \mathrm{Hom}_{\mathcal{C}}(X, Z)^1$ such that we can write $\mathrm{d}(z) = h - g \circ f$.

And of course, we have to prove that the dg nerve construction actually yields an $(\infty, 1)$-category, and not just an arbitrary simplicial set.

Proposition 7.9.3. Let $\mathcal{C}$ be a dg category. Then the dg nerve $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ is an $(\infty, 1)$-category.

Proof. See [HA, proposition 1.3.1.10]. □

This construction is moreover functorial, and satisfies an important Quillen adjointness property.

Proposition 7.9.4. The dg nerve is a right Quillen functor from $\mathbf{dg\,Cat}_k$ to **sSet**, where we equip **sSet** with the Joyal model structure.

Proof. See [HA, proposition 1.3.1.20]. □

The fibrant objects of **sSet** in the Joyal model structure are exactly the quasicategories, and one can prove that every object in $\mathbf{dg\,Cat}_k$ is fibrant. So the dg nerve being right Quillen yields us again that the image of the dg nerve construction is a $(\infty, 1)$-category. And recall that the weak equivalences in the Joyal model structure are exactly the generalisations of categorical equivalences, so the dg nerve construction sends quasi-equivalences to $(\infty, 1)$-categorical equivalences.

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Exposé **8**

Affine derived schemes

In this exposé we will work in the framework that has been built in the previous ones, in order to define the notion of affine derived scheme. We will try to introduce the main ideas and definitions, and to compare them (when possible) with their classical counterparts.

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8.1 Lifting criteria

8.2 Affine derived schemes

Simplicial objects

In this preliminar section we review the general theory of simplicial objects. We will follow closely Quillen [Qui67, pp. II.1, II.4]. Recall first of all the following definition:

Definition 8.2.1. Let $\mathcal{C}$ be any category. A simplicial object in $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. The (functorial) category of simplicial objects in $\mathcal{C}$ is denoted $\mathrm{s}\mathcal{C}$.

The first main result is that the category $\mathrm{s}\mathcal{C}$ is always enriched over **sSet**:

Theorem 8.2.2. Let $\mathcal{C}$ be any category. Then $\mathrm{s}\mathcal{C}$ is enriched over **sSet**. If moreover $\mathcal{C}$ has coproducts, $\mathrm{s}\mathcal{C}$ is enriched with tensor over **sSet** and if $\mathcal{C}$ has limits, $\mathrm{s}\mathcal{C}$ is enriched with cotensor over **sSet**.

Sketch of the proof. We won't give the full proof, but we will simply give the construction of the tensor product $X \otimes K$ when $\mathcal{C}$ has coproducts. The reference for a complete proof is [Qui67, Prop. II.1.2] (and the discussion preceding this proof).

Given a simplicial set K and a simplicial object $X \in \mathrm{s}\mathcal{C}$, define

$$(X \otimes K)_n := \coprod_{\sigma \in K_n} X_n$$

If $\varphi \colon \mathbf{n} \to \mathbf{m}$ is an arrow in $\mathbf{\Delta}$, define

$$\varphi^*_{X\otimes K} \colon (X \otimes K)_m \to (X \otimes K)_n$$

as the map

$$\coprod_{\sigma \in K_m} \mathrm{in}_{\varphi^*_K(\sigma)} \varphi^*_X$$

This defines a new simplicial object in s$\mathcal{C}$. W $\square$

The second main result concerns, instead, the model structure of s$\mathcal{C}$. First of all recall the following definitions:

Definition 8.2.3. Let $\mathcal{C}$ be a category. A morphism $f : X \to Y$ is said to be an effective epimorphism if it has kernel pair and it is the quotient of its kernel pair, that is if the diagram

$$X \times_Y X \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \xrightarrow{f} Y$$

is a coequalizer.

Definition 8.2.4. Let $\mathcal{C}$ be a category. An object $P \in \mathrm{Ob}(\mathcal{C})$ is said to be projective if $f_* : \mathrm{Hom}_{\mathcal{C}}(P, X) \to \mathrm{Hom}_{\mathcal{C}}(P, Y)$ is surjective for every effective epimorphism $f : X \to Y$.

Definition 8.2.5. A category $\mathcal{C}$ has enough projectives if for each object X there is a projective object P and an effective epimorphism $P \to X$.

Lemma 8.2.6. Let $\mathcal{C}$ be any category with pullbacks and let (T, μ, η) be a monad over $\mathcal{C}$. For any T-algebra (A, h) the map $h \colon T(A) \to A$ is an effective epimorphism.

Proof. Let (A, h) be an algebra for the monad T. We claim that $T(A)$ is projective in $\mathcal{C}^T$ and that $h \colon T(A) \to A$ is an effective epimorphism. First of all observe that Lemma B.1.5 implies that $\mathcal{C}^T$ has pullbacks. Therefore an arrow is an effective epimorphism if and only if it is a regular epimorphism, and as trivial consequence we see that every split epimorphism is effective. However, the unit axiom says that

$$h \circ T\eta_A = \mathrm{id}_A$$

i.e. h is a split epimorphism. $\square$

Theorem 8.2.7. Let $\mathcal{C}$ be a category with finite limits and enough projectives. Define a map f in s$\mathcal{C}$ to be a fibration (resp. a weak equivalence) if $Hom_{s\mathcal{C}}(P, f\,; \mathbf{sSet})$ is a fibration (resp. a weak equivalence) for each projective object P. If moreover every object of s$\mathcal{C}$ is fibrant, this defines a simplicial model structure on s$\mathcal{C}$.

Proof. See [Qui67, Thm. II.4.4]. $\square$

Simplicial abelian groups

Definition 8.2.8. A simplicial abelian group is a simplicial object in **Ab**, the category of abelian groups. The category of simplicial abelian groups will be denoted by **sAb**.

Recall the following standard categorical fact:

Lemma 8.2.9. Let $F \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon G$ be an adjunction between categories and let $\mathcal{B}$ be any other category. Then we have an adjunction

$$F_* \colon \mathbf{Funct}(\mathcal{B}, \mathcal{C}) \rightleftarrows \mathbf{Funct}(\mathcal{B}, \mathcal{D}) \colon G_*$$

Proof. Let, in general, $\varphi : f \to g$ be a natural transformation between functors from $\mathcal{C}$ to $\mathcal{D}$. Then we can define a new natural transformation $\varphi_* : f_* \to g_*$ in the obvious way: if $h : \mathcal{B} \to \mathcal{C}$ is any functor, define

$$(\varphi_*)_h := \varphi_h : f \circ h \to g \circ h$$

It's clear that the construction of φ_* preserves both vertical composition and identities, so that $\mathbf{Funct}(\mathcal{B}, -)$ can be really regarded as a functor. This implies trivially that the adjunction $F \dashv G$ lifts to $F_* \dashv G_*$, since the triangular identities are preserved by $\mathbf{Funct}(\mathcal{B}, -)$. □

Going back to our setting, observe that we have an obvious forgetful functor $U : \mathbf{sAb} \to \mathbf{sSet}$. Denote by $\mathbb{Z}(-) : \mathbf{Set} \to \mathbf{Ab}$ the free abelian group functor. Define the functor

$$\mathrm{s}\mathbb{Z}(-) := (\mathbb{Z}(-))_* : \mathbf{sSet} \to \mathbf{sAb}$$

Previous Lemma implies immediately:

Corollary 8.2.10. There is an adjunction $\mathrm{s}\mathbb{Z}(-) \dashv U$. Moreover, the algebras for the corresponding monad on **sSet** are exactly the simplicial abelian groups.

Proof. Denote by $V : \mathbf{Ab} \to \mathbf{Set}$ the obvious forgetful functor. Then it's clear that $U = V_*$ (i.e. the group structure is forgotten levelwise), so that the first statement descends directly from Lemma 8.2.9. For the second statement, observe that V is monadic (see [Mac71, Thm. VI.8.1]), hence it creates coequalizers for those arrows f, g such that $V(f), V(g)$ have absolute coequalizer. Since coequalizers in **sAb** are computed objectwise, it follows that the same property is shared by U, so that Beck's monadicity theorem ([Mac71, Thm. VI.7.1]) implies the thesis. □

Observe that the following easy categorical fact holds:

Lemma 8.2.11. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. For any other category $\mathcal{D}$, the functorial category $\mathbf{Funct}(\mathcal{D}, \mathcal{C})$ has a monoidal structure where the tensor product is defined objectwise.

Proof. Composing $- \otimes -$ with $F \times G : \mathcal{D} \to \mathcal{C} \times \mathcal{C}$ we obtain a functor $\mathbf{Funct}(\mathcal{D}, \mathcal{C} \times \mathcal{C}) \to \mathbf{Funct}(\mathcal{D}, \mathcal{C})$. Then simply observe that

$$\mathbf{Funct}(\mathcal{D}, \mathcal{C} \times \mathcal{C}) \simeq \mathbf{Funct}(\mathcal{D}, \mathcal{C}) \times \mathbf{Funct}(\mathcal{D}, \mathcal{C})$$

The commutativity of diagrams is therefore checked objectwise, and this allows to conclude immediately that $\mathbf{Funct}(\mathcal{D}, \mathcal{C})$ has a monoidal structure. □

Previous Theorem 8.2.2 shows immediately that **sAb** is a simplicial structure.

Remark 8.2.12. It follows from the direct construction given by Quillen [loc.cit.] that if $G \in \mathbf{sAb}$ and $K \in \mathbf{sSet}$, then

$$G \otimes K = G \otimes \mathrm{s}\mathbb{Z}(K)$$

where on the left hand side the tensor product is derived from the enrichment over **sSet**, while on the right hand side the tensor product is derived from the monoidal structure. Similarly, one can show that Quillen's construction yields:

$$G^K = \mathbf{hom}(K, U(G))$$

where **hom** denotes the internal hom of **sSet** ($\mathbf{hom}(K, U(G))$ is an abelian simplicial group, because every object in **sAb** can be seen as an internal abelian group object in $(\mathbf{sSet}, \times, \Delta^0)$, and moreover $\mathbf{hom}(K, -)$ commutes with products, being a right adjoint).

Finally, we have to deal with the model structure on **sAb**.

Lemma 8.2.13. Each module $s\mathbb{Z}(K)$ is projective in the sense of Definition 8.2.4.

Proof. First of all observe that every morphism in **sAb** has a cokernel, which is computed objectwise. This implies immediately that every epimorphism is surjective. Now that thesis is a trivial consequence of Corollary 8.2.10, the fact that the forgetful functor U preserves surjectivity and the fact that in **sSet** every epimorphism is split. □

Lemma 8.2.14. Let $f : A \to B$ be a morphism in **sAb**. Then

$$f_* : \mathrm{Hom}_{\mathbf{sAb}}(P, A; \mathbf{sSet}) \to \mathrm{Hom}_{\mathbf{sAb}}(P, B; \mathbf{sSet})$$

is a fibration (resp. a weak equivalence) for all projective object P if and only if $U(f) : U(A) \to U(B)$ is a fibration (resp. a weak equivalence).

Proof. Assume that $U(f)$ is a fibration (resp. a weak equivalence). If $P = s\mathbb{Z}(K)$ for some simplicial set $K \in \mathbf{sSet}$, then

$$\mathrm{Hom}_{\mathbf{sAb}}(s\mathbb{Z}(K), A; \mathbf{sSet}) = \mathbf{hom}(K, U(A))$$

(this follows directly from the construction of the enriched hom). Then we know that $\mathbf{hom}(K, U(f))$ is a fibration (resp. a weak equivalence). The general case follows from the fact that every projective module is a direct summand of a free module.

Conversely, if f_* is a fibration (resp. a weak equivalence) for all projective object P, then using the same relation of above we see that $\mathbf{hom}(K, U(f))$ is a fibration (resp. a weak equivalence) for every simplicial set K. This implies that $U(f)$ is a fibration (resp. a weak equivalence. □

Theorem 8.2.15. **sAb** has a simplicial monoidal model structure where

- a map f is a weak equivalence if and only if $U(f)$ is a weak equivalence;
- a map f is a fibration if and only if $U(f)$ is a fibration;
- a map is a cofibration if and only if it has the LLP with respect to trivial fibrations.

Moreover, this category is proper, cofibrantly generated and cellular.

Proof. It can be shown that an object in **sAb** is projective if and only if it is a direct summand of a free simplicial module (of the form $s\mathbb{Z}(K)$). It follows that □

Simplicial commutative algebras

Definition 8.2.16. A simplicial commutative algebra is a commutative monoid in $(\mathbf{sAb}, \otimes, \mathbb{Z})$. We will denote the category of simplicial commutative algebras as **sComm**.

Definition 8.2.17. Let A be a simplicial commutative algebra. A (simplicial) A-module an internal A-module in $(\mathbf{sAb}, \otimes, \mathbb{Z})$. We will denote the category of (simplicial) A-modules by $\mathbf{sMod}_A$.

Corollary 8.2.18. The category $\mathbf{sMod}_A$ has a monoidal model structure where

- an arrow f is a weak equivalence if and only if it is a weak equivalence in **sAb**;
- an arrow f is a fibration if and only if it is a fibration in **sAb**;
- an arrow is a cofibration if and only if it has the LLP with respect to all trivial fibrations.

Proof. It can be shown that this is a consequence of Theorem B.2.13. □

It is more complicated to deal with **sComm**. The idea is to use again Theorem 8.2.7. First of all, observe that an epimorphism in **sComm** is necessarily surjective. Now consider the following lemma:

Lemma 8.2.19. There is an adjunction

$$\mathrm{s}\mathbb{Z}[-]\colon \mathbf{sSet} \rightleftarrows \mathbf{sComm}\colon U$$

where U is the natural forgetful functor and $\mathrm{s}\mathbb{Z}[-] = (\mathbb{Z}[-])_*$, $\mathbb{Z}[-]\colon \mathbf{Set} \to \mathbf{Comm}$ denoting the free polynomial algebra functor.

Proof. This is again a trivial consequence of Lemma 8.2.9. □

As consequence we see that every algebra of the form $\mathrm{s}\mathbb{Z}[K]$ is a projective object in **sComm**, in the sense of Definition 8.2.4. With a little more effort one can show the following:

Theorem 8.2.20. **sComm** has a simplicial model structure where

- a map f is a weak equivalence if and only if $U(f)$ is a weak equivalence;
- a map f is a fibration if and only if $U(f)$ is a fibration;
- a map is a cofibration if and only if it has the LLP with respect to all acyclic fibrations.

Proof. It can be shown that this is a consequence of Theorem 8.2.7. □

Remark 8.2.21. The fact that **sComm** has this model structure is a highly non-trivial result. In fact, the second part of Theorem B.2.13 applies only to *algebras* over a commutative monoid, and not to *commutative algebras*. This is a reason that leads to the introduction of E_∞-algebras.

Let A and B be simplicial commutative algebras and let $f : A \to B$ be a morphism between them.

Lemma 8.2.22. There exists a Quillen adjunction

$$- \otimes_A B\colon \mathbf{sMod}_A \rightleftarrows \mathbf{sMod}_B : f^*$$

where f^* is the forgetful functor along f. This is moreover a Quillen equivalence if f is a weak equivalence.

Proof. The details of the adjunction relation are straightforward and we omit them. It is a Quillen adjunction because f^* preserves both fibrations and trivial fibrations. □

Homotopy groups

Let A be a simplicial commutative algebra. For each $n \in \mathbb{N}$ we define its n-th homotopy group as

$$\pi_n(A) := \pi_n(U(A), 0)$$

where $U\colon \mathbf{sComm} \to \mathbf{sSet}$ is the natural forgetful functor and 0 denotes the 0-simplex corresponding to the zero element of A.

Remark 8.2.23. A being a simplicial group, it is straightforward to check that $\pi_n(U(A), v)$ does not depend on the choice of the 0-simplex v (cfr. for example [CSAM29, Ex. 8.3.1]).

Remark 8.2.24. In $\mathbf{sSet}_*$ let S^1 denote the simplicial set defined as the coequalizer of $d^0, d^1 \colon \Delta^0 \to \Delta^1$. Moreover, define inductively

$$S^{n+1} := S^1 \wedge S^n$$

It can be shown that for any pointed simplicial set (K, v) the relation

$$\pi_n(K, v) = \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sSet}_*)}(S^{n+1}, K)$$

holds.

Lemma 8.2.25. With the notations of Lemma 8.2.19 and for each simplicial commutative algebra A, the relation

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sComm})}(sZ[S^n], A) = \pi_n(A)$$

holds.

Proof. This is a trivial consequence of the adjunction of Lemma 8.2.19 and of previous remark. □

Lemma 8.2.26. For each simplicial commutative algebra A, $\pi_n(A)$ has a natural structure of abelian group.

Sketch of the proof. The structure on $\pi_0(A)$ is easily derived from the sum in A. If $n \geq 1$, the natural group structure on $\pi_n(A)$ is abelian thanks to a (standard) Eckmann-Hilton argument. □

Let A be a simplicial commutative algebra. Define the graded abelian group

$$\pi_*(A) := \bigoplus_{n \in \mathbb{N}} \pi_n(A)$$

It's easy to see that $\pi_*(A)$ has a natural structure of graded commutative algebra. In fact, if $[a] \in \pi_n(A)$ and $[b] \in \pi_m(A)$ are two elements in $\pi_*(A)$ they can be represented by arrows in $\mathbf{sSet}_*$:

$$a \colon S^n \to A, \qquad b \colon S^m \to A$$

Those maps induce

$$a \otimes b \colon S^n \times S^m \to A \times A \to A \otimes A$$

and composing with the multiplication $A \otimes A \to A$ we obtain a morphism

$$a \odot b \colon S^n \times S^m \to A$$

which in fact factorizes through $S^n \wedge S^m \simeq S^{n+m}$. This new map is by definition the multiplication of a and b. It can be shown that this gives a well defined multiplication which is associative, unital and graded commutative.

In a similar way, if M is a simplicial A-module then the graded abelian group

$$\pi_*(M) := \bigoplus_{n \in \mathbb{N}} \pi_n(M)$$

has a natural structure of $\pi_*(A)$-module.

Corollary 8.2.27. Let A be a simplicial commutative algebra. Then $\pi_0(A)$ has a natural structure of commutative algebra. That is, π_0 defines a functor

$$\pi_0 \colon \mathbf{sComm} \to \mathbf{Comm}$$

Proof. This is a consequence of the previous constructions. □

Comparison with Comm

We have a natural functor
$$\widetilde{F}\colon \mathbf{Comm} \to \mathbf{sComm}$$
sending a commutative algebra A to a simplicial commutative algebra A concentrated in degree 0.

Lemma 8.2.28. The induced functor $F\colon \mathbf{Comm} \to \mathrm{Ho}(\mathbf{sComm})$ has as left adjoint the functor of Corollary 8.2.27
$$\pi_0\colon \mathrm{Ho}(\mathbf{sComm}) \to \mathbf{Comm}$$
Moreover, F is fully faithful.

Proof. We omit (for the moment) the details of this adjunction. However, we observe that the counit is an isomorphism. This implies immediately that F is fully faithful. □

Derived schemes

Definition 8.2.29. We define the category of derived affine schemes to be $\mathbf{dAff} := \mathbf{sComm}^{\mathrm{op}}$, endowed with the dual model structure of **sComm**.

Notation. Given $A \in \mathrm{Ob}(\mathbf{dAff}) = \mathrm{Ob}(\mathbf{sComm})$, we will emphasize that we are thinking it as an element to **dAff** by writing it as $\mathrm{Spec}A$.

To justify better the terminology, we end this chapter by introducing also derived schemes. First of all, we look at the category $\mathrm{sPSh}(\mathbf{dAff})$ and we endow it with the global *projective* model structure (cfr. Theorem 5.3.7). As it is explained in more details in Exposé 6, we construct a first localization of $\mathrm{sPSh}(\mathbf{dAff})$: we consider the left Bousfield localization with respect to the maps
$$f_*\colon sh_X \to sh_Y$$
induced by weak equivalences $f: X \to Y$ in **dAff** (here, as in Exposé 5, sh_X denotes the Yoneda functor associated to X reviewed inside the category of simplicial presheaves). We will denote by $\mathbf{dAff}^\wedge$ this new model category.

It is possible to characterize the local objects for this left Bousfield localization: a simplicial presheaf
$$F\colon \mathbf{dAff}^{\mathrm{op}} \to \mathbf{sSet}$$
is fibrant in $\mathbf{dAff}^\wedge$ if an only if the following conditions hold:

1. $F(X)$ is fibrant for all $X \in \mathbf{dAff}$ (that is, F is fibrant for the global model structure on $\mathrm{sPSh}(\mathbf{dAff})$);

2. for any equivalence $X \to Y$ in **dAff**, the induced morphism $F(Y) \to F(X)$ is an equivalence of simplicial sets.

General results about left Bousfield localization show that $\mathrm{Ho}(\mathbf{dAff}^\wedge)$ is equivalent to the full subcategory of
$$\mathrm{Ho}(\mathrm{sPSh}(\mathbf{dAff}))$$
consisting of those objects satisfying the two conditions above. The fibrant replacement functor realizes then a left adjoint to the inclusion
$$\mathrm{Ho}(\mathbf{dAff}^\wedge) \to \mathrm{Ho}(\mathrm{sPSh}(\mathbf{dAff}))$$

Finally, we make **dAff** into a model site, introducing a topology on $\mathrm{Ho}(\mathbf{dAff})$ in the sense of [TV05].

Definition 8.2.30. A family of morphisms $\{A \to A_i\}_{i \in I}$ in $\mathrm{Ho}(\mathbf{sComm}\}_{i \in I}$ is an étale covering if each $A \to A_i$ is étale in the sense previously defined, and there exists a finite subset $J \subset I$ such that the family of functors

$$\{- \otimes^{\mathbb{L}}_A A_i \colon \mathrm{Ho}(\mathbf{sMod}_A) \to \mathrm{Ho}(\mathbf{sMod}_{A_i}\}_{i \in J}$$

is conservative, i.e. a morphism f in $\mathrm{Ho}(\mathbf{sMod}_A$ is an isomorphism if and only if $f \otimes^{\mathbb{L}}_A A_i$ is an isomorphism for every $i \in J$.

In [TV05] it is proved that sPSh($\mathbf{dAff}$) can be endowed with a local model structure taking into account the étale topology on $\mathrm{Ho}(\mathbf{dAff})$ in the sense explained in Exposé 5 and Exposé 6. This new model structure will be denoted $\mathbf{dAff}^{\sim}$. As before, we have a characterization of fibrant objects in $\mathbf{dAff}^{\sim}$: a simplicial presheaf $F \colon \mathbf{dAff}^{\mathrm{op}} \to \mathbf{sSet}$ is fibrant if and only if it is fibrant in $\mathbf{dAff}^{\wedge}$ and furthermore it satisfies

3. for any $X \in \mathrm{Ob}(\mathbf{dAff})$ and any étale hypercovering $H \to X$ the natural morphism

$$F(X) \to \mathrm{holim}_{\mathbf{n} \in \mathbf{\Delta}} F(H_n)$$

is an equivalence of simplicial sets.

Definition 8.2.31. We say that a simplicial presheaf $F \in \mathrm{sPSh}(\mathbf{dAff})$ is a derived stack if it satisfies to conditions 2. and 3. above.

Theorem 8.2.32. Let $\mathrm{Spec}A \in \mathbf{dAff}$ be a derived affine scheme. Then the simplicial presheaf $sh_{\mathrm{Spec}A} = sh_A$ is a derived stack.

Proof. This is proved in [HAG-II]. □

Consider generally the Yoneda embedding

$$sh \colon \mathbf{dAff} \to \mathrm{Ho}(\mathbf{dAff}^{\wedge})$$

This induces by construction a functor

$$\mathrm{Ho}(sh) \colon \mathrm{Ho}(\mathbf{dAff}) \to \mathrm{Ho}(\mathbf{dAff}^{\wedge})$$

We will denote $\mathrm{Ho}(sh)(X)$ by $\mathbb{R}\underline{\mathrm{Spec}}A$. Previous theorem can be restated by saying that $\mathbb{R}\underline{\mathrm{Spec}}A$ is a derived stack.

Definition 8.2.33. A derived stack F is said to be a *derived scheme* if there exists a family of derived affine schemes $\{\mathbb{R}\underline{\mathrm{Spec}}A_i\}_{i \in I}$ and Zariski open immersions $\mathbb{R}\underline{\mathrm{Spec}}A_i \to F$ such that the induced morphism of sheaves

$$\coprod_{i \in I} \mathbb{R}\underline{\mathrm{Spec}}A_i \to F$$

is an epimorphism. Such a family $\{\mathbb{R}\underline{\mathrm{Spec}}A_i \to F\}_{i \in I}$ is called a *Zariski atlas* for F.

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Exposé 9

Derived moduli stacks

Pieter Belmans

9.1 Introduction

Moduli problems are situations in which you try to describe a family of geometric objects using another geometric object. Given a sufficiently nice resulting geometric object one can then prove properties of the family as a whole, or of single objects as one can try to reduce statements object general objects to more tractable ones. Some examples of well-known moduli problems are

- the Hilbert scheme: parametrising closed subschemes of a given scheme;
- the moduli of curves of genus g;
- the moduli of vector bundles of rank n on an algebraic variety.

The goal is to find a scheme, algebraic space or stack which represents the moduli problem. This was (and still is) one of the motivating reasons for the development of algebraic geometry, and whenever a moduli problem is representable it yields an example of a (often highly non-trivial) space. A striking example of this is the moduli of curves of genus g which for various g yields objects of greatly varying complexity.

But there can be certain problems with this approach:

1. the moduli problem at hand is not suited for the language of “classical” algebraic geometry[1]: more difficult objects are involved, we only consider objects up to quasi-isomorphism, . . . ;
2. the result of the moduli problem is not as nice as one would hope for: the *hidden (quasi)smoothness principle* formulated by Beĭlinson–Deligne–Drinfel’d–Kontsevich says singular moduli spaces are the “shadow” of smooth moduli spaces in a more general context [Kon95, sections 1.4 and 5.1].

We will now give examples of these issues. We first state the classical case, and then we remark how derived algebraic geometry will address the problem that exist with the example. The first example is a generalised moduli problem for which the language of classical algebraic geometry is not suited [TV07, section 1].

[1] Often classical means Italian, but in this case it means the geometry from the sixties using commutative rings as building blocks.

Example 9.1.1 (Moduli of objects in a k-linear category)**.** Let $\mathcal{C}$ be a k-linear category. As in exposé 7 we take k a commutative ring, not necessarily a field. We then define the moduli problem

$$\mathcal{M}_{\mathcal{C}}\colon k\text{-}\mathbf{cAlg} \to \mathbf{Grpd} : A \mapsto \mathbf{Fun}_k\left(\mathcal{C}^{\mathrm{op}}, A\text{-}\mathbf{Mod}^{\mathrm{ft,proj}}\right) \tag{9.1}$$

by considering the groupoid of k-linear functors from $\mathcal{C}$ to the category of projective A-modules of finite type. If we take $\mathcal{C} = k$ (the category with a single object and k as its endomorphism ring) this is the category of vector bundles on $\operatorname{Spec} A$, and a possible proof of the geometric properties of $\mathcal{M}_{\mathcal{C}}$ uses this fact. The tensor product

$$B \otimes_A - : A\text{-}\mathbf{Mod}^{\mathrm{ft,proj}} \to B\text{-}\mathbf{Mod}^{\mathrm{ft,proj}} \tag{9.2}$$

yields a morphism $\mathcal{M}_{\mathcal{C}}(A) \to \mathcal{M}_{\mathcal{C}}(B)$, and this gives a stack of the big étale site of affine k-schemes. The question now becomes to find (the necessary conditions for) nice properties of this stack.

If $\mathcal{C}$ is of finite type as a k-linear category, i.e. the category $\mathcal{C}^{\mathrm{op}}$-**Mod** is equivalent to B-**Mod** where B is a finitely presented associative k-algebra[2], then $\mathcal{M}_{\mathcal{C}}$ is an Artin stack, locally of finite presentation.

This gives the mere *existence* of an Artin stack, but we need more conditions on $\mathcal{C}$ to have a good relationship between the k-rational points $\mathcal{M}_{\mathcal{C}}(k)$ and $\mathcal{C}$ itself [TV07, remark 1.2]. These conditions are rather strong, and are for example not satisfied in general for the category $\mathbf{Qcoh}_X$ on a scheme X. We will have to use derived categories to find an analogue of the required property, which is more easily satisfied as discssed in section 9.5.

Remark 9.1.2. We can ask the same thing for dg categories, which are k-linear categories with some extra structure. But whereas we are interested in isomorphisms in the classical case (and where the presence of automorphisms yields the necessity of stacks) we care about quasi-isomorphisms in case of dg categories. Unfortunately the formalism of triangulated categories doesn't glue in the classical situation [LNM2008, example 4 in section 2.2], so we need something else.

Using derived algebraic geometry on the other hand the quasi-isomorphisms become tractable, and by developing good analogues of the conditions in the k-linear case we can try to develop a *derived moduli stack of objects in a dg category*. This instance of a derived moduli stack is the main result of [TV07] and is also the example we will discuss in this exposé.

Recall that the conditions for a good relationship between the k-rational points and the category are more easily satisfied in this case, as will be explained in section 9.5.

The next example is an important motivation for the development of derived algebraic geometry, in the sense that it shows a problem that is already present in the classical case, and is not a different moduli problem [HAG2DAG, section 1.2].

Example 9.1.3 (Vector bundles of rank n)**.** Let C be a smooth projective curve of genus g over a field k. Then we have the moduli problem $\mathrm{Vect}_n(C)$ parametrising vector bundles of rank n on C. We can prove that it is an *algebraic stack*. In a point $\mathcal{E} \in \mathrm{Vect}_n(C)(k)$ we can compute the stacky tangent space $\mathrm{T}_{\mathcal{E}}\mathrm{Vect}_n(C)^\bullet$. By relating it to the Zariski cohomology of $\mathcal{E}\mathrm{nd}(\mathcal{E}) = \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{E}^\vee$ we see that it will be a cochain complex concentrated in degrees $[-1, 0]$

$$\mathrm{T}_{\mathcal{E}}\mathrm{Vect}_n(C)^\bullet \cong \cdots \to 0 \to \mathrm{H}^0(C, \mathcal{E}\mathrm{nd}(\mathcal{E})) \to \mathrm{H}^1(C, \mathcal{E}\mathrm{nd}(\mathcal{E})) \to 0 \to \ldots [1] \tag{9.3}$$

and we conclude that the dimension of the tangent space is equal to $n^2(g-1)$. Hence $\mathrm{Vect}_n(C)$ is both *smooth* and of *constant dimension* $n^2(g-1)$.

Now let S be a smooth projective surface over a field. We can again consider the moduli problem $\mathrm{Vect}_n(S)$ parametrising vector bundles of rank n on S. Again it will be an *algebraic stack*.

[2]We enter the realm of not necessarily commutative objects.

But if we compute the stacky tangent space we no longer get a nice result, because now there is the higher cohomology group $\mathrm{H}^2(S, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ which is not present in the stacky tangent space (9.3). As the Euler characteristic is locally constant, and involves the dimension of this vectorspace, we see that if this dimension changes, we can *no longer have a smooth stack*.

Remark 9.1.4. The problem seems to be that the stacky tangent space is concentrated in $[-1, 1]$, but we have forgotten the object in degree 1. If we can find a *derived moduli stack of vector bundles of rank n*, which we'll denote $\mathbf{R}\mathrm{Vect}_n(S)$, that incorporates this information we could find an object that we expect to be *smooth*, of dimension $-\chi(S, \mathcal{E}\mathrm{nd}(\mathcal{E}))$. Hence we expect to find

$$\mathrm{T}_{\mathcal{E}}\mathbf{R}\mathrm{Vect}_n(C)^\bullet \cong \cdots \to 0 \to \mathrm{H}^0(S, \mathcal{E}\mathrm{nd}(\mathcal{E})) \to \mathrm{H}^1(S, \mathcal{E}\mathrm{nd}(\mathcal{E})) \to \mathrm{H}^2(S, \mathcal{E}\mathrm{nd}(\mathcal{E})) \to 0 \to \ldots [1] \quad (9.4)$$

where no truncation has occurred.

Acknowledgements I would like to thank François Petit for his remarks on this text at the Flumserberg Winter school on derived algebraic geometry.

9.2 Preliminaries

Whenever the terminology "derived stack" is used we refer to the $\mathbf{D}^-$-stacks from [HAG-II]. The notation $\mathbf{D}^-$ is reminiscent of the objects involved: cochain complexes concentrated in non-positive degrees, related to the use of simplicial commutative rings and the Dold-Kan correspondence.

Perfect and pseudoperfect dg modules

In order to get a nice result we need to impose a certain finiteness condition on the codomain category in the generalisation of (9.1) to dg categories. In the k-linear case we considered projective modules of finite type, which corresponds to the notion of perfectness in derived categories. Hence the generalisation to dg categories is motivated by this observation.

Definition 9.2.1. Let $\mathcal{C}$ be a dg category. Let M be a $\mathcal{C}^{\mathrm{op}}$-dg module. We say M is *perfect* (or *compact*) if it is homotopically finitely presented in $\mathcal{C}^{\mathrm{op}}$-$\mathbf{dg\,Mod}_k$, i.e. for every filtered system of $\mathcal{C}^{\mathrm{op}}$-dg modules $(N_i)_{i\in I}$ is the morphism

$$\operatorname*{colim}_{i\in I} \mathrm{Map}_{\mathcal{C}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k}(M, N_i)^* \to \mathrm{Map}_{\mathcal{C}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k}\left(M, \operatorname*{hocolim}_{i\in I} N_i\right)^* \quad (9.5)$$

an isomorphism in $\mathrm{Ho}(\mathbf{sSet})$. The full dg subcategory of perfect objects in $\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k)$ is denoted $\mathrm{Int}_{\mathrm{perf}}(\mathcal{C}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k)$.

Definition 9.2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be dg categories. Let M be an object of $\mathrm{Int}((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D})^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k)$. Let C be an object of $\mathcal{C}$. We can define a morphism of dg categories ev_M which acts as a partial evaluation in the first parameter, by setting

$$\mathrm{ev}_M(C)\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Ch}_{\mathrm{dg}}(k\text{-}\mathbf{Mod}) : D \mapsto M(C, D)^\bullet. \quad (9.6)$$

We consider this morphism in $\mathrm{Ho}(\mathbf{dg\,Cat}_k)$. Using [Toe07, theorem 6.1] (as in the proof of [Toe07, corollary 7.6]) we obtain a natural isomorphism

$$\begin{aligned} \mathrm{Int}((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D})^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k) &\cong \mathbf{R}\mathcal{H}\mathrm{om}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D})^{\mathrm{op}}, \mathrm{Int}(\mathbf{dg\,Mod}_k)\right) \\ &\cong \mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{C}^{\mathrm{op}}, \mathrm{Int}(\mathcal{D}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k)\right) \end{aligned} \quad (9.7)$$

in $\mathrm{Ho}(\mathbf{dg\,Cat}_k)$ hence we see that the object M corresponds to ev_M. We then say that M is *pseudoperfect relative to* $\mathcal{D}$ if we can factorise ev_M in $\mathrm{Ho}(\mathbf{dg\,Cat}_k)$ as

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, "\mathrm{ev}_M"] \arrow[dr] & \mathrm{Int}(\mathcal{D}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k) \\ & \mathrm{Int}_{\mathrm{perf}}(\mathcal{D}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k). \arrow[u, hook] \end{tikzcd} \tag{9.8}$$

If we take $\mathcal{D} = k$ we get the absolute version of this definition, and we say that M is a *pseudoperfect* $\mathcal{C}^{\mathrm{op}}$*-dg module*.

Finiteness conditions for dg categories

We now give several finiteness properties a dg category can satisfy.

Definition 9.2.3. Let $\mathcal{C}$ be a dg category. We say that it

1. is *locally perfect* or *locally proper* if for every C and $D \in \mathrm{Ob}(\mathcal{C})$ the morphism cochain complex $\mathrm{Hom}_{\mathcal{C}}(C,D)^\bullet$ is a perfect complex of k-modules;

2. has *a compact generator* if the triangulated category $\mathrm{H}^0(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k))$ has a compact generator;

3. is *proper* if it is locally proper and it has a compact generator;

4. is *smooth* if it is a perfect object in $\mathrm{Int}\left((\mathcal{C}^{\mathrm{op}} \otimes^{\mathbf{L}} \mathcal{C})^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k\right)$, using the interpretation

$$\mathcal{C}\colon \mathcal{C}^{\mathrm{op}} \otimes^{\mathbf{L}} \mathcal{C} : (C,D) \mapsto \mathrm{Hom}_{\mathcal{C}}(C,D)^\bullet; \tag{9.9}$$

5. is *pretriangulated* if $\mathcal{C} \to \mathrm{Int}_{\mathrm{perf}}(\mathcal{C}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k)$ is a quasi-equivalence, this is equivalent to the notion introduced in section 7.8;

6. is *saturated* if it is proper, smooth and pretriangulated;

7. is *of finite type* if there exists a dg algebra $B^\bullet$ which is homotopically finitely presented in k-$\mathbf{dg\,Alg}$ such that $\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k)$ is quasi-equivalent to $\mathrm{Int}(B^\bullet\text{-}\mathbf{dg\,Mod}_k)$.

Before we define the main object of study we list some of the properties that can be deduced from these finiteness conditions.

Proposition 9.2.4. Let $\mathcal{C}$ be a dg category.

1. If $\mathcal{C}$ is saturated then an object in $\mathrm{Ho}(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-}\mathbf{dg\,Mod}_k))$ is pseudoperfect if and only if it is quasirepresentable [TV07, corollary 2.9(2)].

2. If $\mathcal{C}$ is smooth and proper then it is of finite type [TV07, corollary 2.13].

3. If $\mathcal{C}$ is of finite type then it is smooth [TV07, corollary 2.14].

The derived moduli stack of pseudoperfect dg modules

Definition 9.2.5. Let $\mathcal{C}$ be a dg category. The *derived moduli stack of pseudoperfect* $\mathcal{C}^{\mathrm{op}}$*-dg modules* $\mathcal{M}_{\mathcal{C}}$ is given by the simplicial presheaf

$$\mathcal{M}_{\mathcal{C}} : k\text{-}\mathbf{sc\,Alg} \to \mathbf{sSet} : A^* \mapsto \mathrm{Map}_{\mathbf{dg\,Cat}_k}\left(\mathcal{C}^{\mathrm{op}}, \mathrm{Int}_{\mathrm{perf}}\left(\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right)\right)^*. \tag{9.10}$$

We remark that

1. the opposite dg category is again a dg category, with composition

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y)^p \otimes_k \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y,Z)^q \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Z)^{p+q} : f \otimes g \mapsto (-1)^{pq} g f; \tag{9.11}$$

2. the simplicial structure of $\mathbf{dg\,Cat}_k$ is given by

$$\mathrm{Map}_{\mathbf{dg\,Cat}_k}(\mathcal{C},\mathcal{D})^* := \mathrm{Hom}_{\mathbf{dg\,Cat}_k}(\Gamma^*(\mathcal{C}),\mathcal{D}) \tag{9.12}$$

   where Γ^* is the cosimplicial resolution functor;

3. the subcategory of perfect objects is stable under base change.

Definition 9.2.6. If we take $\mathcal{C} = k$ in definition 9.2.5 we obtain the *derived moduli stack of perfect modules*. It will be denoted $\mathbf{R}$Perf.

In the notation of [TV07] we have $k = \mathbf{1}$ and $\mathbf{R}$Perf is denoted by $\mathcal{M}_{\mathbf{1}}$. To stress the actual nature of this object the notation $\mathbf{R}$Perf is more appropriate, as we will have some specific choices for the category $\mathcal{C}$: these will yield the derived moduli stacks $\mathbf{R}\mathrm{Perf}_X$ and $\mathbf{R}\mathrm{Perf}_Q$ [TV07, section 3.5]. The derived moduli stack $\mathbf{R}\mathrm{Perf}_X$ is moreover discussed in section 9.5.

Remark 9.2.7. In case the dg category is saturated, we see that by proposition 9.2.4 every quasirepresentable object is pseudoperfect (and vice versa). Hence in this case $\mathcal{M}_{\mathcal{C}}$ is truly a moduli space of objects in $\mathcal{C}$!

The derived moduli stack $\mathbf{R}$Perf serves as a "base point" for all other constructions. One first proves the properties for $\mathbf{R}$Perf (see theorem 9.3.9), then one can try to lift these to derived moduli stacks for arbitrary dg categories, under some appropriate finiteness conditions (see theorem 9.3.10). But first we have to show that it is indeed a derived stack [TV07, lemma 3.1]! Recall again that "derived stack" should be interpreted in terms of [HAG-II, definition 1.3.2.1].

Theorem 9.2.8. The simplicial presheaf $\mathcal{M}_{\mathcal{C}}$ is a derived stack.

Proof. As discussed in [HAG-II, corollary 1.3.2.4] the proof is reduced to showing the following three properties:

1. for every weak equivalence $A^* \xrightarrow{\sim} B^*$ in k-$\mathbf{sc\,Alg}$ is the natural morphism

$$\mathrm{Int}_{\mathrm{perf}}\left(\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right) \xrightarrow{\sim} \mathrm{Int}_{\mathrm{perf}}\left(\mathrm{N}(B^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right) \tag{9.13}$$

   a quasi-equivalence;

2. for every A^* and B^* in k-$\mathbf{sc\,Alg}$ is the natural morphism

$$\mathrm{Int}_{\mathrm{perf}}\left(\mathrm{N}\left(A^* \times B^*\right)^\bullet\text{-}\mathbf{dg\,Mod}_k\right) \xrightarrow{\sim} \mathrm{Int}_{\mathrm{perf}}\left(\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right) \times \mathrm{Int}_{\mathrm{perf}}\left(\mathrm{N}(B^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right) \tag{9.14}$$

   a quasi-equivalence;

3. for every étale hypercovering $X_* \to Y$ in $\mathbf{D}^-\mathbf{Aff}$, or equivalently a co-augmented cosimplicial object $A^* \to B^*_*$ in k-$\mathbf{sc\,Alg}$ is

$$\mathrm{Int}_{\mathrm{perf}}\left(\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right) \xrightarrow{\sim} \operatorname*{holim}_{[n]\in\mathrm{Ob}(\Delta)} \mathrm{Int}_{\mathrm{perf}}\left(B^*_n\text{-}\mathbf{dg\,Mod}_k\right) \tag{9.15}$$

a quasi-equivalence.

Let $A^* \xrightarrow{\sim} B^*$ be a weak equivalence. The base change $B^* \otimes_{A^*} -$ is a Quillen equivalence by lemma 8.2.22, so we obtain a quasi-equivalence $\mathrm{Int}(\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k) \to \mathrm{Int}(\mathrm{N}(B^*)^\bullet\text{-}\mathbf{dg\,Mod}_k)$ [Toe07, proposition 3.2]. This then descends to the full dg subcategories of perfect objects, hence (9.13) is a quasi-equivalence and $\mathbf{R}\mathrm{Perf}$ satisfies property 1.

By [HAG-II, corollary 1.3.7.4] we have that perfectness is étale local, hence we can drop the perfectness condition when proving properties 2 and 3. We then observe that

$$\mathrm{Int}\left(\mathrm{N}(A^*\times B^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right) \to \mathrm{Int}\left(\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right) \times \mathrm{Int}\left(\mathrm{N}(B^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right) \tag{9.16}$$

is a quasi-equivalence because the nerve construction respects products, just like the internal dg category, which proves property 2.

The last property is slightly more involved, and requires the results from [Toe07] discussed in section 7.6. First of all, we have reduced (9.15) to

$$\mathrm{Int}\left(\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right) \to \operatorname*{holim}_{[n]\in\mathrm{Ob}(\Delta)} \mathrm{Int}\left(B^*_n\text{-}\mathbf{dg\,Mod}_k\right) \tag{9.17}$$

by the previous remark. By applying Yoneda we reduce this to proving that

$$\mathrm{Map}\left(\mathcal{C}, \mathrm{Int}\left(\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right)\right) \to \mathrm{Map}\left(\mathcal{C}, \operatorname*{holim}_{[n]\in\mathrm{Ob}(\Delta)} \mathrm{Int}\left(B^*_n\text{-}\mathbf{dg\,Mod}_k\right)\right) \tag{9.18}$$

is a weak equivalence of simplicial sets, for every dg category $\mathcal{C}$. By [Toe07, theorem 4.2] we have a weak equivalence

$$\mathrm{Map}\left(\mathcal{C}, \mathrm{Int}\left(\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k\right)\right) \xrightarrow{\sim} \mathrm{N}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathrm{N}(A^*)^\bullet)\text{-}\mathbf{dg\,Mod}_k^{\mathrm{cof,weq}}\right) \tag{9.19}$$

and weak equivalences

$$\mathrm{Map}\left(\mathcal{C}, \mathrm{Int}\left(\mathrm{N}(B^*_n)^\bullet\text{-}\mathbf{dg\,Mod}_k\right)\right) \xrightarrow{\sim} \mathrm{N}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathrm{N}(B^*_n)^\bullet)\text{-}\mathbf{dg\,Mod}_k^{\mathrm{cof,weq}}\right) \tag{9.20}$$

for each $n \geq 0$. Hence we have to prove that

$$\mathrm{N}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathrm{N}(A^*)^\bullet)\text{-}\mathbf{dg\,Mod}_k^{\mathrm{cof,weq}}\right) \to \operatorname*{holim}_{[n]\in\mathrm{Ob}(\Delta)} \mathrm{N}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathrm{N}(B^*_n)^\bullet)\text{-}\mathbf{dg\,Mod}_k^{\mathrm{cof,weq}}\right) \tag{9.21}$$

is a weak equivalence. But this follows from [HAG-II, corollary B.0.8]. □

Example 9.2.9. If we evaluate $\mathcal{M}_{\mathcal{C}}$ in k we get

$$\mathrm{Map}_{\mathbf{dg\,Cat}_k}\left(\mathcal{C}^{\mathrm{op}}, \mathrm{Int}_{\mathrm{perf}}\left(\mathbf{dg\,Mod}_k\right)\right) \tag{9.22}$$

and this simplicial set is weakly equivalent to the category of weak equivalences of $\mathcal{C}^{\mathrm{op}}$-dg modules M such that $M(C)^\bullet$ is a perfect complex of k-modules for every $C \in \mathrm{Ob}(\mathcal{C})$ [Toe07, corollary 7.6].

9.3 Properties of derived moduli stacks

Tor amplitude for simplicial commutative algebras

Before we describe the properties of **R**Perf and more generally $\mathcal{M}_{\mathcal{C}}$ we introduce some ideas that are necessary for its proof. A reference for this is [SGA6, exposés I–III].

Definition 9.3.1. Let A^* be a simplicial commutative k-algebra. Let $P^\bullet$ be a $\mathrm{N}(A^*)^\bullet$-dg module. We say it is of *Tor amplitude contained in* $[a,b]$ if for every $\pi_0(A^*)$-module $M \in \mathrm{Ob}(\pi_0(A^*)\text{-}\mathbf{Mod})$ we have

$$\mathrm{H}^i\left(P^\bullet \otimes^{\mathbf{L}}_{\mathrm{N}(A^*)^\bullet} M\right) = 0 \tag{9.23}$$

if $i \notin [a,b]$.

The following properties give some idea on how this notion behaves [TV07, proposition 2.22]. See also [Stacks, tag 0651] for these properties in a more classical context.

Proposition 9.3.2. Let A^* be a simplicial commutative k-algebra. Let $P^\bullet$ and $Q^\bullet$ be two perfect $\mathrm{N}(A^*)^\bullet$-dg modules.

1. Let the Tor amplitude of $P^\bullet$ (resp. $Q^\bullet$) be contained in $[a,b]$ (resp. $[c,d]$), then $P^\bullet \otimes^{\mathbf{L}}_{\mathrm{N}(A^*)^\bullet} Q^\bullet$ is a perfect $\mathrm{N}(A^*)^\bullet$-dg module whose Tor amplitude is contained in $[a+c, b+d]$.

2. Let the Tor amplitude of $P^\bullet$ and $Q^\bullet$ be contained in $[a,b]$, then the homotopy fiber of $f : P^\bullet \to Q^\bullet$ has Tor amplitude contained in $[a, b+1]$.

3. The Tor amplitude of a $\mathrm{N}(A^*)^\bullet$-dg module $P^\bullet$ is contained in $[a,b]$ if and only if the base change $P^\bullet \otimes^{\mathbf{L}}_{\mathrm{N}(A^*)^\bullet} \pi_0(A^*)$ is a perfect complex of $\pi_0(A^*)$-modules with Tor amplitude contained in $[a,b]$.

4. Let $A^* \to B^*$ be a morphism in k-**scAlg**. Let the Tor amplitude of $P^\bullet$ be contained in $[a,b]$. Then $P^\bullet \otimes^{\mathbf{L}}_{\mathrm{N}(A^*)^\bullet} \mathrm{N}(B^*)^\bullet$ is a perfect $\mathrm{N}(B^*)^\bullet$-dg module whose Tor amplitude is contained in $[a,b]$.

5. We can find $a \leq b$ such that the Tor amplitude of $P^\bullet$ is contained in $[a,b]$.

6. Let the Tor amplitude of $P^\bullet$ be $[a,a]$. Then $P^\bullet$ is isomorphic in $\mathrm{Ho}(\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k)$ to the shift with a positions of a projective $\mathrm{N}(A^*)^\bullet$-module of finite type.

7. Let the Tor amplitude of $P^\bullet$ be $[a,b]$. We can find a projective $\mathrm{N}(A^*)^\bullet$-dg module of finite type E and a morphism $E[-b] \to P^\bullet$ such that the homotopy cofiber has Tor amplitude contained in $[a, b-1]$.

Properties of higher and derived stacks

Now we list the higher and derived properties a derived stack can satisfy, before proving that **R**Perf and $\mathcal{M}_{\mathcal{C}}$ satisfy them. The ideas of higher n-stacks are used [Sim96], and these definitions are based on an induction on n. As these definitions depend on eachother for the induction step it is impossible to give them in a logically independent order.

Definition 9.3.3. Let F be a derived stack, i.e. $F \in \mathrm{Ob}(\mathbf{D}^-\mathbf{St}_k)$. It is an *n-geometric* derived stack if

1. the morphism $F \to F \times^{\mathrm{h}} F$ is $(n-1)$-representable;

2. there exists a family of affine derived stacks $(X_i)_{i\in I}$ (or *n-atlas*) such that $\coprod_{i\in I} X_i \to F$ is a covering and for every $i \in I$ the map $X_i \to F$ is smooth.

An affine derived stack is (-1)-geometric.

Definition 9.3.4. Let $f : F \to G$ be a morphism between derived stacks. It is *n-representable* if for every affine derived stack X and every morphism $X \to G$ the derived stack $F \times^{\mathrm{h}}_G X$ is n-geometric

Definition 9.3.5. Let $f : F \to G$ be a morphism between derived stacks. It is *smooth* if for every affine derived stack X and every morphism $X \to G$ there exists an n-atlas $(Y_i)_{i\in I}$ of $F \times^{\mathrm{h}}_G X$ such that $Y_i \to X$ is a smooth morphism of affine derived stacks as in definition 6.2.4.

The next two definitions are a finiteness condition on a derived stack.

Definition 9.3.6. Let $X = \mathbf{R}\mathrm{Spec} A^*$ be an affine derived stack. It is *finitely presented* if the morphism

$$\operatorname*{colim}_{i\in I} \mathrm{Map}_{k\text{-}\mathbf{sc\,Alg}}(A^*, B_i^*)^* \to \mathrm{Map}_{k\text{-}\mathbf{sc\,Alg}}\left(A^*, \operatorname*{colim}_{i\in I} B_i^*\right)^* \tag{9.24}$$

is an isomorphism in $\mathrm{Ho}(\mathbf{sSet})$ for every filtered[3] system $(B_i^*)_{i\in I}$ in k-$\mathbf{sc\,Alg}$.

Definition 9.3.7. Let F be an n-geometric derived stack. It is *locally of finite presentation* if there exists an n-atlas $(X_i)_{i\in I}$ such that each X_i is finitely presented.

The last property we define for derived stacks is its geometricity.

Definition 9.3.8. Let F be a derived stack. We say it is *locally geometric* if it can be written as a filtered homotopy colimit, i.e. $F \cong \mathrm{hocolim}_{i\in I} F_i$, such that

1. each derived stack F_i is n_i-geometric;
2. each morphism $F_i \to F$ is a monomorphism.

If moreover every F_i can be chosen to be locally of finite presentation, then we say F is *locally of finite presentation*.

Properties of $\mathbf{R}$Perf and $\mathcal{M}_{\mathcal{C}}$

We first prove that the base derived stack $\mathbf{R}$Perf is locally geometric and locally of finite presentation [TV07, proposition 3.7], then we lift this in theorem 9.3.10 to dg categories satisfying a certain finiteness condition [TV07, theorem 3.6]. The full proof of these statements is about 10 pages long, so we have to restrict ourselves to a mere sketch.

Theorem 9.3.9. The derived moduli stack $\mathbf{R}$Perf is locally geometric and locally of finite presentation.

Sketch of a proof. We first make the following reduction: let $a \le b$ be integers, then there exists a full derived substack $\mathbf{R}\mathrm{Perf}^{[a,b]}$ of $\mathbf{R}$Perf.

For $A^* \in \mathrm{Ob}(k\text{-}\mathbf{sc\,Alg})$ we see that $\pi_0(\mathbf{R}\mathrm{Perf}(A^*))$ consists of the quasi-isomorphism classes of perfect $\mathrm{N}(A^*)^\bullet$-dg modules. Hence we can define $\mathbf{R}\mathrm{Perf}^{[a,b]}(A^*)$ to be the full subsimplicial set of $\mathbf{R}\mathrm{Perf}(A^*)$ of connected components that correspond to perfect $\mathrm{N}(A^*)^\bullet$-dg modules whose Tor amplitude is contained in $[a,b]$. We can then write

$$\mathbf{R}\mathrm{Perf} = \bigcup_{a\le b} \mathbf{R}\mathrm{Perf}^{[a,b]} \tag{9.25}$$

[3] Remark that it doesn't matter whether we consider colimits or homotopy colimits, they agree in these cases.

and we have reduced the proof to showing that $\mathbf{R}\mathrm{Perf}^{[a,b]}$ is locally geometric and locally of finite presentation. We can moreover prove that $\mathbf{R}\mathrm{Perf}^{[a,b]}$ is n-geometric, where $n = b - a + 1$.

To prove that $\mathbf{R}\mathrm{Perf}^{[a,b]}$ is n-geometric we use [TV07, lemma 2.18] to reduce this proof to a statement on the diagonal. Hence, let $X = \mathbf{R}\mathrm{Spec} A^*$ be a representable derived stack and let $f, g\colon X \to \mathbf{R}\mathrm{Perf}^{[a,b]}$ be X-valued points. By taking $f \times g$ one obtains the diagonal case. Then one considers the homotopy fiber product

$$\begin{tikzcd} X \times^{\mathrm{h}}_{\mathbf{R}\mathrm{Perf}^{[a,b]}} X \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ X \arrow[r, "f"'] & \mathbf{R}\mathrm{Perf}^{[a,b]}. \end{tikzcd} \tag{9.26}$$

We then have to prove that this homotopy fiber product is $n-1$-geometric. This is done by induction on $n - 1 = b - a$. If $a = b$, we have that $\mathbf{R}\mathrm{Perf}^{[a,a]}$ is the derived stack of vector bundles $\mathbf{R}\mathrm{Vect} = \bigcup_{n\geq 0} \mathbf{R}\mathrm{Vect}_n$ [HAG-II, corollary 1.3.7.4], which finishes this case.

Now let $a < b$. As X is affine we have a simplicial commutative k-algebra A^* representing it, and f and g are two A^*-rational points of $\mathbf{R}\mathrm{Perf}^{[a,b]}$, hence they correspond to two perfect $\mathrm{N}(A^*)^\bullet$-dg modules whose Tor amplitude is contained in $[a, b]$. By the strictification theorem [HAG-II, appendix B] we can rewrite $\mathbf{R}\mathrm{Spec} A^* \times_{\mathbf{R}\mathrm{Perf}^{[a,b]}} \mathbf{R}\mathrm{Spec} A^*$ as

$$k\text{-}\mathbf{sc\,Alg}_{/A^*} \to \mathbf{sSet} : B^* \mapsto \mathrm{Map}^{\mathrm{weq}}_{\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k}\left(P^\bullet, Q^\bullet \otimes^{\mathbf{L}}_{\mathrm{N}(A^*)^\bullet} \mathrm{N}(B^*)^\bullet\right) \tag{9.27}$$

and we can then consider the (bigger) derived stack

$$F : k\text{-}\mathbf{sc\,Alg}_{/A^*} \to \mathbf{sSet} : B^* \mapsto \mathrm{Map}_{\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k}\left(P^\bullet, Q^\bullet \otimes^{\mathbf{L}}_{\mathrm{N}(A^*)^\bullet} \mathrm{N}(B^*)^\bullet\right) \tag{9.28}$$

together with the morphism

$$j : X \times^{\mathrm{h}}_{\mathbf{R}\mathrm{Perf}^{[a,b]}} X \hookrightarrow F. \tag{9.29}$$

This morphism is proven to be 0-representable, hence its fibers are 0-geometric derived stacks. One then shows that F is a $(n-1)$-geometric derived stack. To do so one first writes

$$F : k\text{-}\mathbf{sc\,Alg}_{/A^*} \to \mathbf{sSet} : B^* \mapsto \mathrm{Map}_{\mathrm{N}(A^*)^\bullet\text{-}\mathbf{dg\,Mod}_k}\left(P^\bullet \otimes^{\mathbf{L}}_{\mathrm{N}(A^*)^\bullet} Q^{\bullet,\vee}, \mathrm{N}(B^*)^\bullet\right) \tag{9.30}$$

where the first parameter has Tor amplitude $[a - b, b - a]$ by proposition 9.3.2(1). This is the content of [TV07, sublemma 3.9], which uses proposition 9.3.2(7).

The last thing to prove is that $\mathbf{R}\mathrm{Perf}^{[a,b]}$ has an n-atlas $U \to \mathbf{R}\mathrm{Perf}^{[a,b]}$, where $n = b - a + 1$ and U is locally of finite presentation. This is again done by induction on the amplitude. If $a = b$ we get the derived stack of vector bundles, i.e.

$$\mathbf{R}\mathrm{Perf}^{[a,a]} = \bigcup_{n\geq 0} \mathbf{R}\mathrm{Vect}_n \tag{9.31}$$

and by [HAG-II, corollary 1.3.7.4] we obtain that it is a 1-geometric derived stack, locally of finite presentation.

Now let $a < b$, or equivalently $n > 1$. We have the $(n-1)$-geometric derived stack $\mathbf{R}\mathrm{Perf}^{[a,b-1]}$ which is locally of finite presentation. The construction of U is done by considering the model category of morphisms in $\mathrm{N}(A^*)^\bullet$-$\mathbf{dg\,Mod}_k$. This category of morphisms is equipped with its projective model category structure, restricting oneself to cofibrant objects $u\colon Q^\bullet \to R^\bullet$ such that $Q^\bullet$ has Tor amplitude contained in $[a, b-1]$ and $R^\bullet$ has Tor amplitude $[b-1, b-1]$. This yields a $(n-1)$-geometric stack, locally of finite presentation, by considering the morphism

$$p\colon U \to \mathbf{R}\mathrm{Perf}^{[a,b-1]} \times^{\mathrm{h}} \mathbf{R}\mathrm{Vect} : (u\colon Q^\bullet \to R^\bullet) \mapsto (Q^\bullet, R^\bullet[b-1]). \tag{9.32}$$

Given the derived stack U we construct a morphism

$$\pi\colon U \to \mathbf{R}\mathrm{Perf}^{[a,b]} \tag{9.33}$$

by setting

$$\pi_{A^*} : U(A^*) \to \mathbf{R}\mathrm{Perf}(A^*) : (u\colon Q^\bullet \to R^\bullet) \mapsto \mathrm{hofib}(u) \tag{9.34}$$

for A^* a simplicial commutative k-algebra. One then proves that π is a $(n-1)$-representable smooth covering. To check that it is a covering we use proposition 9.3.2(7). To check that it is smooth one reduces things to the cotangent complex [HAG-II, corollary 2.2.5.3]. □

Theorem 9.3.10. Let $\mathcal{C}$ be a dg category of finite type. Then the derived moduli stack $\mathcal{M}_\mathcal{C}$ is locally geometric and locally of finite presentation.

Sketch of a proof. We can again make the decomposition

$$\mathcal{M}_\mathcal{C} = \bigcup_{a\leq b} \mathcal{M}_\mathcal{C}^{[a,b]} \tag{9.35}$$

hence it suffices to prove that $\mathcal{M}_\mathcal{C}^{[a,b]}$ is locally geometric and locally of finite presentation. We can actually proof that it is n-geometric for some value of n, which cannot be expressed in terms of a and b alone, but also depends on the number of generators used in the homotopically finite presentation of the dg algebra $B^\bullet$ that represents $\mathcal{C}$ (as it is assumed to be of finite type) by [TV07, corollary 2.12].

Hence to lift the result on $\mathbf{R}\mathrm{Perf}$ we have to prove that the morphism

$$\pi\colon \mathcal{M}_\mathcal{C}^{[a,b]} \to \mathbf{R}\mathrm{Perf}^{[a,b]} \tag{9.36}$$

is n-representable (for some n) and strongly of finite presentation [TV07, lemma 2.20]. This then suffices (after some reductions) to prove the result by [TV07, lemma 2.15]. □

9.4 The tangent complex of a derived moduli stack

Moduli problems and deformation theory are closely related: a solution to a moduli problem is a *global* thing, while deformation theory parametrises *local* information. The ideas of derived algebraic geometry lead to a natural notion of a (co)tangent complex and associated obstruction theory [HAG-II, section 2.2]. One can moreover argue that the cotangent complex with its origins in the 1960s is already a manifestation of derived algebraic geometry.

We can prove the following theorem, which interprets this general phenomenon [HAG-II, section 1.4] in case of derived moduli stacks.

Theorem 9.4.1. Let $\mathcal{C}$ be a pretriangulated dg category of finite type. Let E be an object of $\mathrm{H}^0(\mathcal{C})$, or equivalently a morphism

$$E\colon \operatorname{Spec} k \to \mathcal{M}_\mathcal{C}. \tag{9.37}$$

Then we can describe the tangent complex of $\mathcal{M}_\mathcal{C}$ at E by

$$\mathrm{T}^\bullet_{\mathcal{M}_\mathcal{C},E} \cong \mathrm{Hom}_\mathcal{C}(E,E)^\bullet[1]. \tag{9.38}$$

Remark 9.4.2. This result indicates how we can derive a moduli problem: a priori there are different possible extensions of a moduli space to the context of derived algebraic geometry. The easiest being the trivial extension, one just embeds the classical scheme, algebraic space or stack in the category of derived stacks. But this is not always the "correct" derived moduli stack, as seen in the case of $\mathbf{R}\mathrm{Vect}_n$ for a surface (or a higher-dimensional object). The description of the tangent complex might give a clue as to which of the possible extensions is the most natural one.

9.5 Example: derived moduli stack of perfect complexes on a scheme

Consider a smooth and proper scheme X over a field k. By $\mathbf{Qcoh}_X$ we denote its category of quasicoherent sheaves. There exists a model category structure on $\mathbf{Ch}(\mathbf{Qcoh}_X)$ where the cofibrations are the monomorphisms and weak equivalences are quasi-isomorphisms. There is moreover an obvious $\mathbf{Ch}(k\text{-}\mathbf{Mod})$-enrichment, so $\mathbf{Ch}(\mathbf{Qcoh}_X)$ is a $\mathbf{Ch}(k\text{-}\mathbf{Mod})$-model category. Using the internal dg category as seen in definition 7.4.4 we can define

$$\mathbf{L}_{\mathrm{qcoh}}(X) := \mathrm{Int}\left(\mathbf{Ch}(\mathbf{Qcoh}_X)\right). \tag{9.39}$$

By proposition 7.4.5 this category serves as a dg enrichment of the "standard" unbounded derived category of quasicoherent sheaves, in the sense that

$$\mathbf{D}_{\mathrm{qcoh}}(X) \cong \mathrm{Ho}\left(\mathbf{L}_{\mathrm{qcoh}}(X)\right). \tag{9.40}$$

If we restrict ourselves to the perfect objects in $\mathbf{L}_{\mathrm{qcoh}}(X)$ we obtain the equivalence

$$\mathbf{D}_{\mathrm{perf}}(X) \cong \mathrm{Ho}\left(\mathbf{L}_{\mathrm{perf}}(X)\right), \tag{9.41}$$

where now the smoothness comes into play for this to have any meaning.

In order to apply the theory of derived moduli stacks certain finiteness conditions on the categories must be satisfied. Crucial in this is the result from [BV03]. One could paraphrase the main result as

> Quasicompact and quasiseparated[4] schemes are *derived affine*.
>
> [BV03, corollary 3.1.8]

This means that $\mathbf{D}_{\mathrm{qcoh}}(X)$ is equivalent to $\mathbf{D}(A^\bullet)$, where $A^\bullet$ is a dg algebra with bounded cohomology. The algebra $B^\bullet$ is moreover obtained by looking at $\mathrm{Ext}^\bullet(E,E)$ where E is a compact generator for $\mathbf{D}_{\mathrm{qcoh}}(X)$.

We can then define the desired derived moduli stack.

Definition 9.5.1. Let X be as before. We define the *derived moduli stack of perfect complexes on X* as

$$\mathbf{R}\mathrm{Perf}_X := \mathcal{M}_{\mathbf{L}_{\mathrm{perf}}(X)} \cong \mathcal{H}\mathrm{om}_{\mathbf{D}^-\text{-}\mathbf{St}_k}(X, \mathbf{R}\mathrm{Perf}) \tag{9.42}$$

Remark 9.5.2. We can prove that the dg category $\mathbf{L}_{\mathrm{perf}}(X)$ is saturated, so by remark 9.2.7 we get that $\mathbf{R}\mathrm{Perf}_X$ really classifies objects in $\mathbf{L}_{\mathrm{perf}}(X)$.

Remark 9.5.3. Given this derived stack we can also consider its truncation, or underived part. We will denote this stack by

$$\mathrm{Perf}_X := \mathrm{t}_0(\mathbf{R}\mathrm{Perf}_X). \tag{9.43}$$

If one moreover restricts himself to the substack of 1-rigid objects[5] we can obtain the classical result of the existence of an Artin stack Coh_X that serves as a moduli space for coherent sheaves on X [LM00, théorème 4.6.2.1] [Stacks, tag 08KA]. The proof goes by constructing a morphism $\mathrm{Coh}_X \to \mathrm{Perf}_X$, and uses in an essential way higher stacks, while avoiding Artin's representability theorem!

[4] The intersection of two affine opens is again affine. This is always the case for separated schemes.

[5] These are objects satisfying the vanishing of all higher homotopy groups for all base changes.

References

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EXPOSÉ **10**

Algebraic structures in higher categories

10.1 Infinity operads

The main idea behind Moerdijk ∞-operads is the following: an ∞-operad, whatever that may be, should be characterised by the way in which some elementary operads (namely trees) can be mapped into it. The category of ∞-operads should therefore be a particular case of presheaves over $\mathbf{\Omega}$, the category of trees. More precisely:

Definition 10.1.1. $\mathbf{\Omega}$ is the category whose objects are finite, rooted, non planar trees. Arrows between two trees are morphisms of the corresponding operads. See [MW07] for more details.
A dendroidal sets is a functor $\Omega \to \mathbf{Set}$. The category of dendroidal sets will be denoted **dSet**.
Given a tree T, we denote by $\Omega[T]$ the corresponding representable dendroidal set.

Definition 10.1.2 ([MW07])**.** A monomorphism of dendroidal sets $X \to Y$ is normal if for any tree T, any non degenerate dendrex $y \in Y(T)$ which does not belong to the image of $X(T)$ has a trivial stabilizer $\mathrm{Aut}(T)_y \subseteq \mathrm{Aut}(T)$. A dendroidal set X is normal if the map $\emptyset \to X$ is normal.

Given an inner edge e in a tree T, we get an inclusion

$$\Lambda^e[T] \to \Omega[T]$$

Such maps are called inner horn inclusions.

Definition 10.1.3. A dendroidal set is an ∞-operad if it has the right lifting property with respect to inner horn inclusions.

Remark 10.1.4. Let O be an operad in the traditional sense (we mean a symmetric multicategory): it can be interpreted as a dendroidal set $N(O) := \mathrm{Hom}_{\mathrm{Operads}}(-, O)$. Such a dendroidal set is an ∞-operad. The nerve functor $\mathrm{N}\colon \mathrm{Operads} \to \infty\text{-Operads}$ admits a left adjoint Ho. In analogy with quasi-categories, given an ∞-operad $\mathcal{M}$ we call $\mathrm{Ho}(\mathcal{M})$ its homotopy operad.

Classically, one may interpret an operad O as an "algebraic structure". Given a symmetric monoidal category $\mathcal{C}^{\otimes}$ one can build a corresponding operad $\hat{\mathcal{C}}$in the following way: the objects of $\hat{\mathcal{C}}$ are the objects of $\mathcal{C}^{\otimes}$ and

$$\mathrm{Arr}_{\hat{\mathcal{C}}}(A_1, \ldots, A_n; B) := \mathrm{Arr}_{\mathcal{C}^{\otimes}}(A_1 \otimes \cdots \otimes A_n; B)$$

Then O-objects in $\mathcal{C}^{\otimes}$ are simply functors from O to $\hat{\mathcal{C}}$. Furthermore one disposes of a tensor product on operads (called the Boardman-Vogt tensor product) such that A-objects in B-objects are the same as $A \otimes B$-objects. This is the situation we wish to generalise.

The classical Boardman-Vogt tensor product $\otimes_{BV}$ on operads extends to dendroidal sets in the following way:

- on trees it is the classical one: $\Omega[S] \otimes \Omega[T] := N(S \otimes_{BV} T)$;
- it is preserves colimits separately in each variable.

Proposition 10.1.5 ([MW07])**.** There is a closed monoidal model structure on dendroidal sets such that:

- cofibrations are normal monomorphisms;
- fibrant objects are ∞-operads.

One should think of weak equivalences between fibrant objects as equivalences of ∞-operads.

Remark 10.1.6. There is a Quillen adjunction $i_! \colon \mathbf{sSet} \leftrightarrows \mathbf{dSet} \colon i^*$ induced by the forgetful functor $i \colon \mathbf{\Delta} \to \mathbf{\Omega}$. Given an ∞-operad O we will refer to $i^*(O)$ as its underlying ∞-category.

10.2 Weak algebraic structures

Definition 10.2.1. A weak algebraic structure is simply an ∞-operad.

Let $\mathcal{M}$ be a weak algebraic structure, let $\mathcal{C}^\otimes$ be a ∞-operad. Then the category of $\mathcal{M}$-objects in $\mathcal{C}^\otimes$ is $i^*\mathrm{Fun}(\mathbb{R}\mathcal{M}, \mathcal{C}^\otimes)$ where $\mathbb{R}$ is a cofibrant replacement of $\mathcal{M}$. It will be denoted $\mathcal{M}$-$\mathcal{C}^\otimes$

The notation $\mathcal{C}^\otimes$ is appropriate because the definition is particularly interesting when $\mathcal{C}^\otimes$ is the ∞-operad associated to a "symmetric monoidal ∞-category" (the inverted commas are due to the fact that there isn't much theory of symmetric monoidal ∞-categories in the literature up to now).

Indeed one could think that the definition of $\mathcal{M}$ object is completely useless as monoidal categories in nature do not appear as dendroidal sets. There are several possible solutions to this problem:

- Given a "symmetric monoidal ∞-category" the way they arise in nature, for instance as simplicial monoidal model categories, one could simply construct the corresponding simplicial operad and then, using the equivalence between simplicial operads and ∞-operads (see [CM11]), obtain the corresponding dendroidal set.
- Another possibility is simply to translate all the results in another equivalent setting, e.g. simplicial or Segal operads.
- One could also try to understand what is a symmetric monoidal quasi-category and how to associate a dendroidal set to it.

Definition 10.2.2.

- E_1 is the dendroidal nerve of the 1-operad corresponding to monoids. E_∞ is the nerve of the 1-operad corresponding to abelian monoids. It is interesting to remark that $E_\infty(T)$ is a point for all tree T.
- $\mathbf{Cat}_\infty^\times$ is the ∞-operad corresponding to quasi-categories with the direct product.
- A monoidal ∞-category is a E_1-∞-category. A symmetric monoidal ∞-category is a E_∞-∞-category.

Proposition 10.2.3. Each tree T can be seen as a symmetric monoidal 1-category $\mathrm{tens}(T)$ in the following way:

- objects of tens(T) are the free abelian monoid on edges of T;
- Arrows are freely generated by operations on T;
- The tensor produc is simply the composition in the free abelian monoid.

This functor tens: $\mathbf{\Omega} \to \mathbf{Cat}_{\infty}^{\text{sym.mon.}}$ induces a functor

$$\text{tens}^* : \mathbf{Cat}_{\infty}^{\text{sym.mon.}} \to \mathbf{dSet}$$

It can be checked that its image is contained in ∞-operads.

We will often omit the notation tens* and refer to a symmetric monoidal ∞-category or to the corresponding ∞-operad in the same way.

Remark 10.2.4. As we promised in the introduction, the notion of weak algebraic structure that we get is manifestly ∞-categorical, so if there is an equivalence between two objects, than a weak structure on the first induces a weak structure on the second in a compatible way. This will be of great importance in the applications in derived algebraic geometry.

10.3 Looping and delooping

Before studying E_1 or E_∞ ∞-categories, it is already interesting to consider E_1 or E_∞ spaces. The first interesting thing to remark is that E_1-spaces are the same as quasi-categories with only one 0-simplex, so that it is particularly easy to construct the looping/delooping adjunction.

Definition 10.3.1. There is a map $i\colon \mathbf{\Delta} \to E_1$-**sSet** sending $[n]$ to the discrete free monoid on n elements. It induces a functor E_1-**sSet** $\to$ **sSet** in the following way:

$$M \mapsto ([n] \mapsto \text{Hom}(i(n), M)_0)$$

We call such functor the delooping functor, denoted B.

Proposition 10.3.2. It can be show that if M is a weak monoid, then BM is an ∞-category with one element. We say that a monoid M is *grouplike* if BM is a Kan complex.

Remark 10.3.3. So far we have a functor B: E_1-**sSet** $\to \mathbf{Cat}_{\infty,\bullet}$. The easiest way to build its adjoint Ω is to observe that $\mathbf{Cat}_{\infty,\bullet}$ is equivalent to the model category $\mathbf{sCat}_\bullet$ of pointed simplicial categories. We have the obvious functor Ω : $\mathbf{sCat}_\bullet \to E_1$-**sSet** sending a pointed simplicial category $\mathcal{C}$ to the (strict) monoid $\text{Hom}_{\mathcal{C}}(*,*)$.

The looping/delooping adjunction identifies E_1-**sSet** with ∞-categories with one object and furthermore provides a strictification result for weak simplicial monoids. It is also clear that $\Omega \circ B \simeq Id$, whereas B$(\Omega X) \simeq X$ only if X is connected.

Proposition 10.3.4. Let $\mathcal{M}$ be a weak algebraic structure. If G is a $E_1 \otimes \mathcal{M}$-simplicial set, then BM is an $\mathcal{M}$-simplicial set in a natural way.

Proof. By Freyd's adjoint theorem, B has a left adjoint L: **sSet** $\to E_1$-**sSet**: it is the only colimits preserving functor that extends i. Therefore B is a monoidal functor (as both monoidal structures are cartesian) and so sends $\mathcal{M}$-E_1-**sSet** to $\mathcal{M}$-**sSet**. □

Corollary 10.3.5. Let E_∞ be the operad of abelian monoids. As $E_\infty \simeq E_\infty \otimes E_1$, the functor B induces a functor E_∞-**sSet** $\to E_\infty$-**sSet**.

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Exposé **11**

Derived loop spaces and de Rham cohomology

Pietro Vertechi

11.1 Topological spaces and groups in derived algebraic geometry

In this section we briefly discuss the concept of affinisation in derived algebraic geometry as it will play a rather important role in studying the loop space of a stack. We work over the site of graded commutative dg algebras concentrated in nonpositive degrees (which we denote $\mathbf{dgAlg}_{\leq 0}$) with the étale topology. We work over a fixed base field k of characteristic 0 which is fixed for all time and often implicit. Whenever we say stack, we actually mean derived stack.

Affinisation

We want to introduce the notion of affine stack (Toën), which is a homotopical generalisation of the notion of affine scheme and will be useful for dealing with CW complexes considered as derived stacks.

Definition 11.1.1. The category of affine stacks is the opposite of the category of dg algebras, without any coconnectivity assumptions: $Aff := dgAlg^{op}$

Remark 11.1.2. There is an evident map Spec : $Aff \to St$ given by the Yoneda functor: $\mathrm{Spec}(R)(A) := \mathrm{Hom}_{cdga}(R, A)$

As in the classical case, this functor has a left adjoint $\mathcal{O} : St \to Aff$ which is the only functor sending each derived affine scheme to the corresponding coconnective dg algebra and preserving colimits. The monad $\mathbf{Aff} := \mathrm{Spec} \circ \mathcal{O}$ is called the affinisation functor. Each stack X comes equipped with a universal arrow $X \to \mathbf{Aff}(X)$ through which every arrow from X to an affine stack factorises.

Now we will try to understand what the affinisation functor does on CW complexes. In order to do that, we need to be able to understand how to interprete CW complexes as derived stacks.

Proposition 11.1.3. There is an adjunction $sSets \rightleftarrows PrSt$ where the left adjoint sends each simplicial set to the corresponding constant prestack and the right adjoint sends each prestack X to the simplicial set X(k)

Proof. $\mathrm{Map}(S, X(k)) = \mathrm{Map}(S, \mathrm{Hom}(\mathrm{Spec}(k), X)) = \mathrm{Hom}(\mathrm{Spec}(k) \otimes S, X)$ and $\mathrm{Spec}(k) \otimes S$ is the constant simplicial prestack associated to S, as colimits are taken objectwise and the prestack $\mathrm{Spec}(k)$ associates to each dg algebra a point. □

We can compose $sSets \rightleftarrows PrSt$ with the adjunction: $PrSt \rightleftarrows St$ between the forget and the sheafification functor.
We get maps $sSets \rightleftarrows St$ which allows us to consider simplicial sets as a particular case of derived stacks. In particular, as left adjoints preserve colimits, and finite simplicial sets can be expressed as finite colimits of copies of the point, we can recover finite simplicial sets (or equivalently finite CW complexes) as those stacks which are finite colimits of copies of $\mathrm{Spec}(k)$.
For example the circle S^1 is the derived stack:

$$\mathrm{Spec}(k) \coprod_{\mathrm{Spec}(k) \coprod \mathrm{Spec}(k)} \mathrm{Spec}(k)$$

Instead general simplicial sets correspond to arbitrary colimits of copies of $\mathrm{Spec}(k)$.

Cohomology

In classical algebraic topology cohomology is represented by the Eilenberg-MacLane spaces $K(G, n) = B^n G$. We will be interested in cohomology with coefficient in k, that is to say $G = \mathbb{G}_a$ is the additive group. What does $B^n\mathbb{G}_a$ look like?

Proposition 11.1.4. $B^n\mathbb{G}_a \simeq \mathrm{Spec}(\mathrm{Sym}k[-n])$

Proof. Let us consider the functor $\mathcal{O}$ as a contravariant functor from St to $dgMod$. It sends pushout squares to pullback squares (which are also pushout squares as the category $dgMod$ is stable).
If we consider the pushout diagram:

$$\begin{tikzcd} S^{d-1} \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & S^d \end{tikzcd}$$

It maps to the pushout/pullback diagram:

$$\begin{tikzcd} \mathcal{O}(S^d) \arrow[r] \arrow[d] & k \arrow[d] \\ k \arrow[r] & \mathcal{O}(S^{d-1}) \end{tikzcd}$$

which allows us to calculate $\mathcal{O}(S^d)$ (as a dg module) recursively: $\mathcal{O}(S^d) = k \oplus k[-d]$. Therefore, for any $A \in \mathbf{dgAlg}_{\leq 0}$

$$\mathrm{Hom}(S^d \times \mathrm{Spec}(A), \mathrm{Spec}\mathrm{Sym}k[-n]) \simeq \mathrm{Hom}_{dgAlg}(\mathrm{Sym}k[-n], \mathcal{O}(S^d) \otimes A]) \simeq \mathrm{Hom}_{dgMod}(k[-n], A \oplus A[-d])$$

So it is clear that, if one points $\mathrm{Spec}\mathrm{Sym}k[-n]$ with the zero map $\mathrm{Sym}k[-n] \to k$, one has: $\Omega^d \mathrm{Spec}\mathrm{Sym}k[-n] \simeq \mathrm{Spec}\mathrm{Sym}k[-n+d]$ so $\pi_d(\mathrm{Spec}\mathrm{Sym}k[-n])$ is trivial for $d < n$ and $\Omega^n(\mathrm{Spec}\mathrm{Sym}k[-n]) \simeq \mathbb{G}_a$. As the first n-1 homotopy groups are trivial $\mathrm{Spec}\mathrm{Sym}k[-n] \simeq B^n\Omega^n\mathrm{Sym}k[-n] \simeq B^n\mathbb{G}_a$ □

Theorem 11.1.5. For X a CW complex, $\mathcal{O}(X) \simeq C^*(X)$ where the right member of the equality is the cochain complex.

Proof. $H^n(X) = \mathrm{Map}(X, B^n\mathbb{G}_a) = \mathrm{Hom}_{dgAlg}(\mathrm{Sym}k[-n], \mathcal{O}(X)) = H^n(\mathcal{O}(X))$. The natural map $C^*(X) \simeq C^*(\mathrm{colSpec}(k)) \to \lim C^*(\mathrm{Spec}(k)) \simeq \lim k \simeq \mathcal{O}(X)$, induces those isomorphisms on cohomology.

□

Remark 11.1.6. Actually we have a particularly nice description of the algebra $\mathrm{Sym}k[-1]$: it is generated by one element of degree one ϵ and the algebra structure is trivial as $\epsilon\epsilon = -\epsilon\epsilon$ (by graded-commutativity) so both sides are equal to 0. So $\mathrm{Sym}k[-1] \simeq k[\epsilon]/(\epsilon^2)$

Remark 11.1.7. So far we've seen that $B^n\mathbb{G}_a \simeq \mathrm{SpecSym}k[-n]$ as stacks. As $\mathbb{G}_a$ is an abelian monoid, $B^n\mathbb{G}_a$ also has a monoid structure which induces the sum structure in the n-th cohomology group. There is a comonoid structure on $\mathrm{Sym}k[-n]$ corresponding to the group structure on the H^n for dg-algebras: the comultiplication is given by $\epsilon \mapsto 1 \otimes \epsilon + \epsilon \otimes 1$ and the counit by $\epsilon \mapsto 0$. As Spec is a monoidal functor (it sends colimits to limits, so in particular the tensor product is sent to the direct product), $\mathrm{SpecSym}k[-n]$ is also a monoid. I and it is tautological that this monoid structure induces the sum in the n-th cohomology group.
It is legitimate to ask whether $B^n\mathbb{G}_a$ and $\mathrm{SpecSym}k[-n]$ are isomorphic as monoids. Fortunately, to prove that it is possible to use the same reasoning as above on $B\mathrm{SpecSym}k[-n]$: its first n homotopy groups are trivial and $\Omega^{n+1}B\mathrm{SpecSym}k[-n] \simeq \mathbb{G}_a$, so $B\mathrm{SpecSym}k[-n] \simeq B^{n+1}\mathbb{G}_a$. By applying the Ω functor, one gets a monoid equivalence between $B^n\mathbb{G}_a$ and $\mathrm{SpecSym}k[-n]$

Actions of CW complexes

There is of course a two-coloured operad which axiomatises an object and a monoid acting over it. Using the formalism developed in section 10, we can easily define the ∞-category of monoid stacks acting on a stack. If we take the sull subcategory of those functors for which the first object goes to a given monoid G, we obtain the category of G-modules. We shall now consider the case of finite CW-complexes which are monoids, show that their affinisation is also a monoid in a canonical way and that the categories of G-modules and Aff(G)-modules are canonically equivalent.

Lemma 11.1.8. Let G be a finite CW complexes and X a stack, then $\mathcal{O}(G \times X) \simeq \mathcal{O}(G) \otimes \mathcal{O}(X)$

Proof. Let us consider the two functors: $\mathcal{O}(- \times X)$ and $\mathcal{O}(-) \otimes \mathcal{O}(X)$. The first sends colimits to limits, so in particular pushout squares to pullback squares. If we apply $\mathcal{O}(-)$ to a pushout square we get a pullback square which is also pushout as Mod is stable. If then we apply $- \otimes \mathcal{O}(X)$ we get a pushout square which is also pullback.
The two functors are equal on $\mathrm{Spec}(k)$ and, since finite CW complexes can be obtained starting from $\mathrm{Spec}(k)$ with iterated pushouts, the two functors agree on finite CW complexes. □

Corollary 11.1.9. A monoid structure over a CW complex G induces canonically a comonoid structure on $\mathcal{O}(G)$ and a monoid structure over $\mathbf{Aff}(G)$. Furthermore whenever we are given a G action on a stack X, we get a $\mathcal{O}(G)$ coaction on $\mathcal{O}(X)$ and an $\mathbf{Aff}(G)$ actions on $\mathbf{Aff}(X)$.
In particual, G actions on affine derived schemes are the same as $\mathcal{O}(G)$ coaction on derived algebras or $\mathbf{Aff}(G)$ actions on affine derived schemes.

Example 11.1.10. $G = S^1 = B\mathbb{Z}$. There is an obvious group morphism $\mathbb{Z} \to \mathbb{G}_a$ sending $1 \in \mathbb{Z}$ to the multiplicative unit of k. Such a morphism induces a monoid morphism $S^1 \simeq B\mathbb{Z} \to B\mathbb{G}_a \simeq \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$. By adjunction, we obtain a map $\mathcal{O}(S^1) \to k[\epsilon]/(\epsilon^2)$ It is easy to verify that such map is a quasi-isomorphism. Moreover, as the adjunction (restricted to finite CW complexes) is monoidal and such map comes from a monoid morphism, it is also a comonoid morphism. Concretely, this means that:

- The dg Hopf algebra $k[\epsilon]/(\epsilon^2)$ is a strict model for the comonoid $\mathcal{O}(S^1)$
- $\mathbf{Aff}(S^1)$ is isomorphic to $B\mathbb{G}_a$ as monoid stacks.

11.2 Loop space and the algebra of differential forms

Definition 11.2.1. The loop space $\mathcal{L}X$ of a derived stack X is the mapping stack $\mathrm{Map}(S^1, X)$

Remark 11.2.2. If X is affine, then all map $S^1 \to X$ factors through $S^1 \to Aff(S^1) \simeq BG_a$, so $\mathcal{L}X \simeq \mathrm{Map}(BG_a, X)$

Remark 11.2.3. As $S^1 \simeq * \coprod_{*\coprod *} *$, it follows immediately that $\mathcal{L}X \simeq X \times_{X\times X} X$.
Similarly, if $X = \mathrm{Spec}(R)$ is an affine stack, $\mathcal{L}X \simeq \mathrm{Spec}(R \otimes_{R\otimes R} R)$

Hochschild homology and the multiplicative HKR isomorphims

For A a derived algebra, $A \otimes_{A\otimes A} A$ corresponds to what is classically known as its Hochschild chain complex. In this setting it is much easier to prove the HKR isomorphism:

$$A \otimes_{A\otimes A} A \simeq \Omega_A^{-\bullet}$$

where $\Omega_A^{-\bullet} = \mathrm{Sym}^\bullet \Omega_A^1[1]$ is the algebra of differential forms.

To understand this isomorphism, it's important to consider the equivalence of categories

$$S^1 - Aff^{op} \simeq k[\epsilon]/(\epsilon^2) - \mathbf{dgAlg}_{\leq 0}$$

given by the Spec, $\mathcal{O}$ adjunction (and by the formality of cochains of S^1, see example 11.1.10).

There are the obvious functors: $\mathcal{O} : S^1 - Aff^{op} \to \mathbf{dgAlg}_{\leq 0}$ and $forget : \mathcal{O}(S^1) - \mathbf{dgAlg}_{\leq 0} \to \mathbf{dgAlg}_{\leq 0}$, which commute with the Spec, $\mathcal{O}$ adjunction. Moreover both functors have a left adjoint, giving the following commutative diagram:

$$\begin{tikzcd}
S^1 - Aff^{op} \arrow[rr, shift left, "\simeq"] \arrow[dr, shift left] & & k[\epsilon]/(\epsilon^2) - \mathbf{dgAlg}_{\leq 0} \arrow[ll, shift left] \arrow[dl, shift left] \\
& \mathbf{dgAlg}_{\leq 0} \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

It is almost tautological that the first left adjoint is the loop space functor $A \mapsto \mathcal{L}\mathrm{Spec}A \simeq \mathrm{Spec}(A \otimes_{A\otimes A} A)$. Indeed, given a map $A \to O(X)$ one gets a map $X \to \mathrm{Spec}(A)$ and, by consequence, a map $\mathcal{L}X \to \mathcal{L}\mathrm{Spec}(A)$ and, if we precompose it by the action map $X \to \mathcal{L}X$ we get the desired morphism $X \to \mathcal{L}\mathrm{Spec}(A)$. If instead we are given a map $X \to \mathcal{L}\mathrm{Spec}(A)$ we must simply compose with map $\mathcal{L}\mathrm{Spec}(A) \to \mathrm{Spec}(A)$ sending a loop to its value in the identity to get a map $X \to \mathrm{Spec}(A)$ and so $A \to O(X)$.

The adjoint of the second forget functor is more interesting: it sends a derived algebra A to its algebra of differential forms. To see why it is so, we need the following:

Lemma 11.2.4. If R is a derived algebra together with a square-zero derivation $d : R \to R[-1]$ then the map $R \to R \oplus R[-1]$ which is the identity on the first component and d on the second is a $k[\epsilon]/(\epsilon^2)$ coaction map on R.

What the lemma is saying is that derived algebras with a square-zero derivation $d : R \to R[-1]$ (the so-called ϵ dg algebras) correspond to algebras with a strict $k[\epsilon]/(\epsilon^2)$ coaction, so in particular $\Omega_A^{-\bullet}$ is a $k[\epsilon]/(\epsilon^2)$-comodule in a natural way.

Proposition 11.2.5. $\mathcal{L}\mathrm{Spec}A \simeq \mathrm{Spec}\Omega_A^{-\bullet}$. $\mathrm{Spec}\Omega_A^{-\bullet}$ is also called the odd tangent complex of A and denoted $\mathbb{T}_A[-1]$.

Proof. $\Omega_A^{-\bullet}$ is a $k[\epsilon]/(\epsilon^2)$-comodule, so $\mathrm{Spec}\Omega_A^{-\bullet}$ is a S^1-module, so by the universal property we have a map of S^1-stacks $\mathrm{Spec}\Omega_A^{-\bullet} \to \mathcal{L}\mathrm{Spec}A$. We want to check that it is an equivalence.

Let's consider any S point of $\mathrm{Spec}A$: $x : \mathrm{Spec}S \to \mathrm{Spec}A$ and write the universal property of the cotangent complex:

$$\begin{tikzcd} \mathrm{Hom}_S(x^*\Omega_A, M) \arrow[r] \arrow[d] & \mathrm{Hom}(A, S \oplus M) \arrow[d] \\ pt \arrow[r] & \mathrm{Hom}(A, S) \end{tikzcd}$$

where M is any S-module. Indeed, it can be proved [HA] that the universal property of the cotangent complex holds without any connectivity assumptions on M.
Let's apply this universal property to the case $M = S[-1]$:

$$\begin{tikzcd} \mathrm{Hom}_S(x^*\Omega_A, S[-1]) \arrow[r] \arrow[d] & \mathrm{Hom}(A, S \oplus S[-1]) \arrow[d] \\ pt \arrow[r] & \mathrm{Hom}(A, S) \end{tikzcd}$$

The key point is that $\mathcal{L}\mathrm{Spec}(A)(S) = \mathrm{Map}(B\mathbb{G}_a, \mathrm{Spec}A)(S) = \mathrm{Hom}(\mathrm{Spec}(k[\epsilon]/(\epsilon^2) \otimes S), \mathrm{Spec}A) = \mathrm{Hom}(A, S \oplus S[-1])$.
Furthermore $\mathrm{Hom}_S(x^*\Omega_A, S[-1]) \simeq \mathrm{Hom}_S(x^*\Omega_A[1], S) \simeq \mathrm{Hom}_{S-alg}(x^*\Omega_A^{-\bullet}, S) = x^*\mathbb{T}_X[-1](S)$
So the diagram we have found is actually:

$$\begin{tikzcd} x^*\mathbb{T}_A[-1](S) \arrow[r] \arrow[d] & \mathcal{L}\mathrm{Spec}(A)(S) \arrow[d] \\ pt \arrow[r] & \mathrm{Spec}(A)(S) \end{tikzcd}$$

and so the map we constructed is an isomorphism. □

Corollary 11.2.6. The left adjoint of the forgetful functor from $k[\epsilon]/(\epsilon^2) - \mathbf{dgAlg}_{\leq 0}$ to $\mathbf{dgAlg}_{\leq 0}$ is the algebra of differential forms.

Remark 11.2.7. This is to be read as a strictification result, because if we started with algebra with a strict $k[\epsilon]/(\epsilon^2)$-coaction (that is to say ϵ dg algebras) the thesis would be obvious.

So, to summarise our results:

Theorem 11.2.8. Given A a derived algebra, we have:

- $A \otimes_{A\otimes A} A \simeq \Omega_A^{-\bullet}$ as derived algebras.
- The map $Id \oplus d : \Omega_A^{-\bullet} \to \Omega_A^{-\bullet} \oplus \Omega_A^{-\bullet}[-1]$ is a $k[\epsilon]/(\epsilon^2)$-coaction.
- Applying the Spec functor to everything we get the map

$$S^1 \times \mathcal{L}\mathrm{Spec}(A) \to B\mathbb{G}_a \times \mathcal{L}\mathrm{Spec}(A) \to \mathcal{L}\mathrm{Spec}(A)$$

  which is isomorphic to the rotation action of S^1 on $\mathcal{L}\mathrm{Spec}(A)$.

Actually, what we've done can be generalised, with very little effort, to derived schemes, thanks to the following:

Proposition 11.2.9. [BN10] For X a derived scheme and $U \to X$ a Zariski open, the induced map $\mathcal{L}U \to \mathcal{L}X$ is also Zariski open. Furthermore the square:

$$\begin{tikzcd} \mathcal{L}U \arrow[r] \arrow[d] & \mathcal{L}X \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

is a pullback

Therefore the isomorphism $\mathcal{L}X \simeq \mathbb{T}_X[-1]$ holds, more generally, for derived schemes.

Remark 11.2.10. It is important to notice that in the HKR isomorphism, we do not consider the de Rham differential as a differential: the differential on $\Omega_A^{-\bullet}$ is the one given by the differential of the cotangent complex. Fortunately the information of the differential is not lost: it is encoded in the S^1 action on $\mathcal{L}X$, so to find for example negative cyclic homology (see [Con94] or [Pan+11]) one simply takes S^1 homotopy invariants functions on $\mathcal{L}X$ that is to say $\mathcal{O}_{\mathcal{L}X}^{S^1}$

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EXPOSÉ **12**

Derived formal deformation theory

Brice Le Grignou and Mauro Porta

12.1 Introduction

We already discussed several non-basic aspects of the derived algebraic geometry in the previous exposés; in this one, the main subject is derived deformation theory, with a particular attention to the formal one. The classical deformation theory is a really rich and interesting subject and it is tightly related to the obstruction theory (for infinitesimal extensions). Even though everything works fine for schemes, there have been problems in generalizing in a satisfying way obstruction theory to the world of 1-stacks; in particular, the constructions of B. Fantechi and K. Beherend produce obstruction spaces which are *not* functorial. This is somehow related to the "hidden smoothness principle": the lack of functoriality is due to the fact that we are considering only "shadows" of more natural objects, for which functoriality holds. We could take this as a foundational principle for the DAG; to put it with the words of B. Toën:

> *Derived algebraic geometry is a generalization of algebraic geometry for which obstruction theory becomes natural.*
>
> B. Toën in *Higher and derived stacks*

The first part of this exposé will be devoted to the foundations of derived deformation theory in the simplicial setting; in particular, we will sketch how we can interpret the space of obstructions as the space of deformation over non-classical rings. In a second moment, we will turn toward a less foundational result: it has been known for a very long time that a formal deformation problem should be controlled by (higher) Maurer-Cartan equations living in a differential graded Lie algebra which is well-determined (up to quasi-isomorphism) from the problem itself. If we agree that a formal deformation problem is nothing but a deformation functor (in the sense of [Man05, Definition V.61]), then a possible way to formalize that idea is the following:

> the category of deformation functors has a fully faithful embedding into the category of differential graded Lie algebras, in such a way that first order deformations up to isomorphism corresponds to the solution of the Maurer-Cartan equation.

Example 12.1.1. Fix a base field k, and consider an associative k-algebra A. It is well known that the Hochschild cohomology $\mathrm{HH}^1(A,A)$ can be used to parametrize the deformations of the product on A as an associative product. The involved differential graded Lie algebra in this case is the (shifted) Hochschild cochain complex, with the bracket defined by Gerstenhaber.

From a derived point of view, the interpretation of this phenomenon becomes easier: in fact, if we work with a formal *derived* moduli problem F, then we expect the tangent complex of F at a point x to be a complex which is not concentrated in degree zero (or, if one prefers the approach of J. Lurie, we expect it to be a non-discrete spectrum). If we work with formal moduli problems, then the assignment

$$F \mapsto \mathbb{T}F$$

defines a functor which preserves finite limits. It follows that

$$\mathbb{T}F[1] \simeq \Omega\mathbb{T}F \simeq \mathbb{T}(\Omega F)$$

and ΩF carries a natural group structure (up to homotopy). It is therefore reasonable to expect the complex $\mathbb{T}F[1]$ to carry a Lie algebra structure. Moreover, in characteristic zero, this Lie algebra structure allows to define the higher Maurer-Cartan equations, which completely describe the solutions to the formal moduli problem F, and therefore determine also the problem F. This heuristic argument can be formalized as follows:

Theorem 12.1.2. Let k be a field of characteristic 0. There exists a functor of ∞-categories $\Psi\colon \mathrm{Moduli}_k \to \mathrm{Lie}_k$ informally given by $\Psi(F) = \mathbb{T}F[1]$ which is moreover an equivalence of ∞-categories.

12.2 An overview of classical deformation theory

We fix throughout this section a field k (not necessarily of characteristic 0), and a scheme over $\mathrm{Spec}(k)$, say X_0. A deformation of X_0 over a base scheme S is a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \mathrm{Spec}(k) \arrow[r, "s"] & S \end{tikzcd}$$

where the map $\pi\colon X \to S$ is flat. The other fibers of S over k-rational points are said to be *deformations* of X_0. The basic problem of deformation theory is to understand the deformations of a given scheme X_0 over an arbitrary base field S; in order to formulate correctly this problem, one should adopt the functor of points philosophy: we are simply considering the contravariant functor

$$\mathrm{Def}_{X_0}\colon \mathbf{Sch}_{k,*} \to \mathbf{Set}$$

associating to a given k-pointed scheme (S,s) over k the isomorphism classes of deformations (X,π) of X_0 over S (such that $X_s \simeq X_0$). The functor Def_{X_0} collects (almost) all the informations about the deformations of X_0; to understand them, one should therefore understand the properties of Def_{X_0}. For example one might ask whether this functor is representable, and in such case he might be interested in studying the geometry of the corresponding moduli space, etc. More generally, one might be interested in considering different deformation problems. Examples include deformations of coherent sheaves over a scheme, or deformation of subscheme in a given ambient scheme.

In order to have a flexible enough theory allowing to deal with general deformation problems, one is essentially led to consider moduli problems satisfying some additional structural property. However, even with the assumption that are usually made, the difficulty of the general problem is often overwhelming. In order to make some progress, it is convenient to reduce the size of the problem; to better explain how this is done in practice, let us assume for a moment that the moduli problem

$$F : \mathbf{Sch}_{k,*} \to \mathbf{Set}$$

is representable by a scheme $\mathcal{M}$. In general, the geometry of $\mathcal{M}$ can be hard to understand; but if we fix a point η in $\mathcal{M}$, then the task of studying the formal completion of $\mathcal{M}$ at η might be more treatable. Let $(A, \mathfrak{m}_A)$ be the complete local ring representing such completion; then the rings $A/\mathfrak{m}_A^n$ are local artinian and A can be recovered as the inverse limit

$$A \simeq \varprojlim A/\mathfrak{m}_A^n$$

This can be restated by saying that A corresponds to a pro-object in the category of local artinian rings.

The key observation in formal deformation theory is that we can repeat the above constructions even without assuming that the problem F is representable. In fact, if we fix a point $\eta \in F(k)$ we can construct a new moduli problem, $\widetilde{F}$, defined as

$$\widetilde{F}(X) := F(X) \times_{F(k)} \{\eta\}$$

If we further restrict this functor to the category of local artinian rings with residue field isomorphic to k, Art_k we obtain a functor

$$\widehat{F}(\mathrm{Spec}(R)) := \widetilde{F}(\mathrm{Spec}(R))$$

which is referred to as the "completion" of F at the point η. It becomes then interesting to understand whether $\widehat{F}$ is a pro-object in the category Art_k.

This problem is addressed by Schlessinger's criterion, which gives conditions in term of the tangent space of the moduli problem F guaranteeing the pro-representability of F. To understand the definition of tangent space, it will be sufficient for the reader to think to the simpler case where F is representable by a scheme:

Definition 12.2.1. Let $F : \mathbf{Sch}_{k,*} \to \mathbf{Set}$ be a moduli problem and let $\eta \in F(k)$ be a given point. The tangent space of F at η is defined to be $T_F := F(k[\varepsilon]/(\varepsilon^2))$.

Theorem 12.2.2 (Schlessinger)**.** Let $F : \mathrm{Art}_k \to \mathbf{Set}$ be a functor of Artin rings such that $F(k)$ is just one element. Given a pullback in Art_k

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$$

let

$$\alpha : F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

be the natural map. Then F is prorepresentable if and only if it satisfies the following conditions:

1. the map α is bijective whenever $B = C$ and $B \to A$ is a square-zero extension;
2. if $C \to A$ is a square-zero extension, then α is surjective;
3. if $A = k$ and $C = k[\varepsilon]/(\varepsilon^2)$ the map α is bijective;
4. $\dim_k(T_F) < \infty$.

Proof. See [Ser06, Theorem 2.3.2] □

The first condition in the previous theorem is a mild assumption on the formal moduli problem F. Indeed, whenever F arises in a "sufficiently geometric" way, it is satisfied (for example, the completion of the functor Def_{X_0} at every point satisfies this condition).

12.3 Derived deformation theory

As we saw in the above discussion, there is an easy procedure allowing to extract a formal deformation problem out of a classical one at a point:

1. one first chooses a point $\eta \in F(k)$ and defines the new moduli problem $\widetilde{F}$ defined by
$$\widetilde{F}(X) := F(X) \times_{F(k)} \{\eta\}$$
which carries the same amount of information of F "near" the point η;
2. next one "completes" the new moduli problem $\widetilde{F}$ by restricting it to the category of local artinian algebras with fixed residue field.

To extend this procedure to the derived setting, one has first of all to introduce the analogue of the notion of Artinian algebra. However, one has first to choose an algebraic model for the derived geometry; since we will be working in characteristic 0, we have at least two possibilities: simplicial algebras and $\mathbf{cdga}_k^{\leq 0}$. Although we feel the language of simplicial algebras being more natural, we choose the dg formalism; this is due to the fact that actual computations will be involved from a certain point on, and the best way of dealing with them is to use the machinery of Koszul duality.

Definition 12.3.1. A connective differential graded k-algebra A is said to be *artinian* if the following conditions are met:

1. A is homotopically of finite presentation;
2. $\mathrm{H}^0(A)$ is a local artinian algebra whose residue field is isomorphic to k.

We will denote by $\mathbf{dgArt}_k^{\leq 0}$ the full subcategory of augmented cdga $\mathbf{cdga}_{k,*}^{\leq 0}$ spanned by artinian algebras.

In principle, a formal moduli problem could be simply a functor
$$F : \mathbf{cdga}_{k,*}^{\leq 0} \to \mathbf{sSet}$$
However, as in similar situations, this definition includes functors which are too wild. Following [DAGX], we will restrict ourselves to the study of a milder subcategory, whose objects will be functors behaving similarly to the way prescribed by the first three conditions in the Theorem 12.2.2. In order to give a precise definition, we first need to generalize the notion of square-zero extension. A possibility is to use [PV13, Definition 1.1] (and we refer there for a more detailed discussion):

Definition 12.3.2. Let A be a cdga k-algebra, M be a dg A-module and
$$\overline{d} \in \pi_0(\mathrm{Map}_{\mathbf{cdga}_k^{\leq 0}/A}(A, A \oplus M[1]))$$
be a derived derivation from A to $M[1]$, represented by a map $d : A \to A \oplus M[1]$ in $\mathbf{cdga}_k^{\leq 0}/A$. If we denote by $\varphi_d : \mathbb{L}_{A/k} \to M[1]$ the map of A-modules corresponding to d, the *infinitesimal extension* $\psi_d : A \oplus_d M \to A$ *of* A *by* M *along* d is the map in $\mathrm{Ho}(\mathbf{cdga}_k^{\leq 0}/A)$ defined by the following homotopy cartesian diagram in $\mathbf{cdga}_k^{\leq 0}$
$$\begin{tikzcd} A \oplus_d M \arrow[r] \arrow[d, "\psi_d"'] & A \arrow[d, "\varphi_0"] \\ A \arrow[r, "d"'] & A \oplus M[1] \end{tikzcd}$$
where φ_0 denotes the section corresponding to the trivial derived derivation $0 : \mathbb{L}_{A/k} \to M[1]$.

This calls for an explanation, as it is not clear at all that this is a real generalization of the classical notion of square-zero extension. The previous definition is philosophically motivated by the following result:

Proposition 12.3.3. Let $0 \to I \to B \to A \to 0$ be a square-zero extension of A by I in the category of (discrete) commutative k-algebras. Then there exists a (derived) derivation $A \to A \oplus I[1]$ and a homotopy pullback

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A \oplus I[1] \end{tikzcd}$$

Proof. The proof is by no means straightforward. We refer the reader to [PV13, Theorem 3.1] for the details. ☐

Let us now remark that it is possible to introduce a notion of infinitesimal extension in the category of *augmented* simplicial k-algebras: one will simply require that all the maps appearing in Definition 12.3.2 are maps augmented simplicial k-algebras. In this case, we can consider also a different class of diagrams:

Lemma 12.3.4. A map $B \to A$ in $\mathbf{cdga}_{k,*}^{\leq 0}$ is an infinitesimal extension of A by an A-module M if and only if there exists a map $A \to k \oplus M[1]$ and a homotopy pullback diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & k \arrow[d, "f_0"] \\ A \arrow[r] & k \oplus M[1] \end{tikzcd} \tag{12.1}$$

where $f_0 \colon k \to k \oplus M[1]$ corresponds to the trivial derivation.

Proof. Suppose first that $B \to A$ is an infinitesimal extension. First of all, observe that we have a natural map of simplicial algebras

$$A \oplus M[1] \to k \oplus M[1]$$

given by $(a, m) \mapsto (\phi(a), m)$, where $\phi : A \to k$ is the augmentation. It is easy to check then that the following square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & k \arrow[d] \\ A \oplus M[1] \arrow[r] & k \oplus M[1] \end{tikzcd}$$

is a strict pullback. Finally,

$$A \oplus M[1] \to k \oplus M[1]$$

is clearly a fibration. It follows that the previous square is a homotopy pullback, so that the square (12.1) is a homotopy pullback as well.

Conversely, if $\delta : A \to k \oplus M[1]$ is a map of augmented cdga we can write it as

$$a \mapsto (\phi(a), d(a))$$

where $d : A \to M[1]$ is easily checked to be a derivation. It follows that we can factor δ as

$$A \to A \oplus M[1] \to k \oplus M[1]$$

Since the homotopy pullback of the map $A \oplus M[1] \to k \oplus M[1]$ along $f_0 \colon k \to k \oplus M[1]$ is isomorphic to A, we completely proved the lemma. □

Following [DAGX, Definition 1.1.5], we are led to introduce the following:

Definition 12.3.5. A morphism $B \to A$ in $\mathbf{cdga}_{k,*}^{\leq 0}$ is said to be *elementary* if it is an infinitesimal extension of A by $k[n]$ for some $n \geq 0$. A morphism $B \to A$ is said to be *small* if it can be written as a finite composition of elementary morphisms. Finally, an object A in $\mathbf{cdga}_{k,*}^{\leq 0}$ is said to be *small* if the morphism $A \to k$ is small.

Example 12.3.6. Each morphism $k \to k \oplus k[n]$ is elementary, in a trivial way. Slightly more interesting is the fact that each morphism $k \oplus k[n] \to k$. In fact, it is an useful exercise to check that

$$\begin{tikzcd} k \oplus k[n] \arrow[r] \arrow[d] & k \arrow[d] \\ k \arrow[r] & k \oplus k[n+1] \end{tikzcd}$$

is a homotopy pullback. Observe that this is equivalent to say that $k \oplus k[n] \simeq \Omega(k \oplus k[n+1])$ in the pointed category $\mathbf{cdga}_{k,*}^{\leq 0}$.

It follows from the definition that the category of small objects is essentially controlled (via pullbacks) by the objects $k \oplus k[n]$. It is more remarkable that the full subcategory $\mathbf{cdga}_{k,\mathrm{sm}}^{\leq 0}$ of $\mathbf{cdga}_{k,*}^{\leq 0}$ spanned by small objects *coincides with* $\mathbf{dgArt}_k$:

Proposition 12.3.7. For an object $A \in \mathbf{cdga}_k^{\leq 0}$ the following conditions are equivalent

1. A is artinian;
2. there exists a finite sequence of maps

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq k$$

where each map $A_i \to A_{i+1}$ exhibits A_i as a square-zero extension of A_{i+1} by $k[m_i]$ for some $m_i \geq 0$.

Proof. See [DAGX, Proposition 1.1.11]. □

At this point we can introduce the notion of formal moduli problem:

Definition 12.3.8. A *formal moduli problem* is a functor $F \colon \mathbf{dgArt}_k^{\leq 0} \to \mathbf{sSet}$ such that:

1. $F(k)$ is weakly contractible;
2. whenever we are given a homotopy pullback square η in $\mathbf{dgArt}_k^{\leq 0}$:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "\phi"] \\ C \arrow[r] & D \end{tikzcd}$$

and ϕ is small, then $F(\eta)$ is a pullback square.

We will denote by Moduli_k the full subcategory of $\mathrm{Fun}(\mathbf{dgArt}_k^{\leq 0}, \mathbf{sSet})$ spanned by formal moduli problems.

Remark 12.3.9. It can be shown that the category Moduli_k arises as a left Bousfield localization of $\mathrm{Fun}(\mathbf{dgArt}_k^{\leq 0}, \mathbf{sSet})$. An indirect proof can be obtained by passing to quasicategories and then apply [HTT, Propositions 5.5.4.18 and 5.5.4.19].

12.4 The tangent spectrum of a formal moduli problem

We already remarked in Example 12.3.6 that the family of objects $\{k \oplus k[n]\}_{n\geq 0}$ can be endowed with equivalences

$$k \oplus k[n] \simeq \Omega(k \oplus k[n+1])$$

where the loop is taken in the pointed category $\mathbf{cdga}_{k,*}^{\leq 0}$. In the language of [HA], this is a *spectrum* in $\mathbf{cdga}_{k,*}^{\leq 0}$, and the reader should think of it as the analogue of the sphere spectrum in the category of (augmented) simplicial k-algebras.

Let $F\colon \mathbf{dgArt}_k^{\leq 0} \to \mathbf{sSet}$ be a formal moduli problem. Since k is an initial object for $\mathbf{dgArt}_k^{\leq 0}$ and $F(k)$ is weakly contractible by definition, we see that F extends naturally to a functor landing in pointed simplicial sets, which we will denote again by F. Moreover, given any formal moduli problem $F\colon \mathbf{dgArt}_k^{\leq 0} \to \mathbf{sSet}_*$ we have a family of pointed spaces

$$\{F(k \oplus k[n])\}_{n\geq 0}$$

endowed with structural morphisms

$$(F(k \oplus k[n])) \to \Omega F(k \oplus k[n+1])$$

which are in fact equivalences. Therefore, we can associate to every formal moduli problem F a spectrum in the category of spaces:

Definition 12.4.1. Let F be a formal moduli problem. The *tangent spectrum* of F is the spectrum:

$$\mathbb{T}F := \{F(k \oplus k[n])\}_{n\geq 0}$$

Remark 12.4.2. In particular the 0-th space of this spectrum is $F(k[\varepsilon]/(\varepsilon^2))$. Since this is part of a spectrum, it is an infinite loop space. This can be seen as a deep motivation for the structure of k-vector space of the tangent space of a (underived) formal moduli space.

Proposition 12.3.7 implies that every artinian object in $\mathbf{cdga}_{k,*}^{\leq 0}$ can be obtained via a finite number of base changes along maps

$$k \to k \oplus k[n]$$

for various $n \geq 0$. Since we know the behaviour of a formal moduli problem over similar pullback squares, we deduce that a formal moduli problem F is completely determined by its tangent spectrum $\mathbb{T}F$. As consequence, we can use the tangent spectrum to detect equivalences between formal moduli problem; this is a sort of "linearization process" and therefore it is quite interesting:

Proposition 12.4.3. A morphism of formal moduli problem $f : F \to G$ is an equivalence if and only if the induced morphism $\mathbb{T}(f)\colon \mathbb{T}F \to \mathbb{T}G$ is an equivalence of spectra.

Proof. This is an exercise, but the reader can find the details in [DAGX, Proposition 1.2.10]. □

We already have evidence of the fact that $\mathbb{T}F$ completely determines the formal moduli problem F. To complete our discussion of the main theorem of [DAGX], we are left to explain the following facts:

1. $\mathbb{T}F[1]$ can be identified with a differential graded Lie algebra;

2. every differential graded Lie algebra arises in this way.

Let us make the following easy remark:

Lemma 12.4.4. The assignment $F \mapsto \mathbb{T}F$ defines a functor

$$\mathbb{T}\colon \mathrm{Moduli}_k \to \mathrm{Sp}$$

where Sp denotes the category of spectra in the category of spaces. Moreover, the functor $\mathbb{T}$ commutes with homotopy limits.

Proof. This is essentially trivial, because homotopy limits can be computed objectwise. □

This implies that $\mathbb{T}F[1] \simeq \mathbb{T}(\Omega F)$, and therefore it is at least reasonable to expect that $\mathbb{T}F[1]$ carries a Lie algebra structure.

12.5 The relationship between spectra and complexes

In characteristic 0 the two things can be identified... an explanation should be added...

Example 12.5.1. If F is the formal moduli problem associated to a cdga A via $\mathrm{Map}(A, -)$, then its tangent complex corresponds to André - Quillen cochains of A.

12.6 Review of cdga and dgla

We are left to understand why we should expect that every dgla arises as the tangent space of a formal moduli problem. To understand this, we should take a step back to Example 12.1.1; in that example we had a concrete deformation problem and we observed that first-order deformation were controlled by the Maurer-Cartan equation. Basically, what one can attempt to do is to build a formal moduli problem out of a differential graded Lie algebra $\mathfrak{g}_*$ by considering a kind of “functor of points” which associates to a pointed cdga $(A, \mathfrak{m}_A)$ the set of solutions of the (higher) Maurer-Cartan equations in $\mathfrak{m}_A \otimes_k \mathfrak{g}_*$. Stated more formally, a natural choice for the moduli problem associated to $\mathfrak{g}_*$ would be

$$(A, \mathfrak{m}_A) \mapsto \mathrm{MC}(\mathfrak{m}_A \otimes_k \mathfrak{g}_*)$$

However, it is not possible to do this on the nose, because this functor doesn’t behave well with homotopies. If $A^\vee = \mathrm{Hom}(A, k)$ were a coalgebra, we would have

$$\mathrm{MC}(\mathfrak{m}_A \otimes_k \mathfrak{g}_*) = \mathrm{MC}(\mathrm{Hom}(A^\vee, \mathfrak{g}_*)) = \mathrm{Tw}(A^\vee, \mathfrak{g}_*)$$

and we could apply classical Koszul duality theory in order to rewrite

$$\mathrm{Tw}(A^\vee, \mathfrak{g}_*) \simeq \mathrm{Hom}(A^\vee, C_*(\mathfrak{g}_*)) \simeq \mathrm{Hom}(\mathcal{L}(A^\vee), \mathfrak{g}_*)$$

where $C_*(\mathfrak{g}_*)$ is the Chevalley-Eilenberg chain complex associated to $\mathfrak{g}_*$. Since the dual of a coalgebra is always an algebra, we still have a functor

$$C^*\colon \mathbf{dgla}_k \to (\mathbf{cdga}_k)^{\mathrm{op}}$$

This functor doesn’t have an adjoint on the nose, but it does when passing to the ∞-categories associated (if one wants to construct a functor the other way around on the level of model categories, he has necessarily to work with dg Lie algebras up-to-homotopy, that is with L_∞ algebras).

12.7 The formal moduli problem associated to a dgla

Let $\mathfrak{g}_*$ be a differential graded Lie algebra. We define a formal moduli problem by the formula

$$\mathrm{Map}(\mathfrak{D}(-), \mathfrak{g}_*)\colon \mathbf{dgArt}_k^{\mathrm{aug}} \to \mathbf{sSet}$$

Lemma 12.7.1. This is a formal moduli problem.

Theorem 12.7.2. The two functors $\mathrm{Map}(\mathfrak{D}(-), \mathfrak{g}_*)$ and $\mathbb{T}(-)[1]$ define an equivalence of ∞-categories.

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Exposé **13**

Noncommutative motives and non-connective K-theory of dg-categories

Marco Robalo

In these notes we explain a new approach to the theory of noncommutative motives suggested by B. Toën, M. Vaquié and G. Vezzosi, which I have been studying in my ongoing doctoral thesis with B. Toën (see [Rob12; Rob13]). The idea of a motivic theory for noncommutative spaces was originally proposed by M. Kontsevich [Kona; Konb] and in the later years G.Tabuada and D-C. Cisinski [Tab08; CT12; CT11; Tab12] developped a formal setting that makes this idea precise. The new approach we want to explain in this talk is independent of the approach by Cisinski-Tabuada (as all the proofs) and the motivation for it is simple: to have a natural theory of noncommutative motives that is naturally comparable to the motivic stable $\mathbb{A}^1$-homotopy theory of Morel-Voevodsky. At the end of this talk we dedicate a special section to explain how our theory relates to methods of Cisinski-Tabuada.

We assume the reader to be familiar with the theory of $(\infty,1)$-categories as developed by J. Lurie in [HTT; HA].

Throughout these notes we fix k a commutative base ring.

13.1 Noncommutative Algebraic Geometry

The original guiding principle of noncommutative geometry is that we can replace the study of a given space X by the study of its ring of functions. This principle appears in the very foundations of algebraic geometry. Grothendieck took it to a new level by understanding that the abelian category $\mathbf{Qcoh}(X)$ of quasi-coherent sheaves of an algebraic varitiety X contains most of the information encoded by X. For instance, everything that at that time was considered as a “cohomology theory” can be obtained by means of the two steps [1]

$$\text{Schemes} \xrightarrow{\mathbf{Qcoh}(-)} \text{Abelian Categories} \xrightarrow{\text{cohomology}} \text{Abelian Groups}$$

Grothendieck was also the first one to realize that the second factorization (aka, homological algebra) factors through a more fundamental step

[1] In fact, the replacement $X \to \mathbf{Qcoh}(X)$ is so strong that in some cases it allow us to reconstruct the whole scheme (see the thesis of P. Gabriel for the notion of spectrum of an abelian category [Gab]).

$$\begin{tikzcd}
\text{Abelian Categories} \arrow[r, "\text{cohomology}"] \arrow[d, dashed] & \text{Abelian Groups} \\
\text{(k-linear) homotopy theories} \arrow[ur, dashed, "\text{homotopy groups}"] &
\end{tikzcd}$$

where the vertical arrow sends a k-abelian category A to the k-linear homotopy theory of complexes of objects in A studied up quasi-isomorphisms. The problem was that at that time homotopy theories were studied by means of the strict formal (categorical) inversion of those morphisms we wish to consider as isomorphism. The resulting object (the classical derived category) is very poorly behaved and many of the expected information of the homotopy theory is lost in the process. In the 90's, dg-categories[2] appeared as a technological enhancement of this classical derived category (see [BK; BK89; BK90] and [Kel06; Bau+11] for an introduction) and Kontsevich envisioned them as the perfect objects for noncommutative geometry (the reason will become clear below). Nowadays, Kontsevich vision adds to the understanding that:

$$\text{“homotopy theories”} = (\infty, 1)\text{-categories}$$

$$\text{“}k\text{-linear homotopy theories”} = k\text{-linear } (\infty, 1)\text{-categories} =^{3} k\text{-dg-categories}$$

Under this vision, the assignement $X \mapsto L_{\mathrm{qcoh}}(X)$ realizes the dotted map first foreseen by Grothendieck. Here $L_{\mathrm{qcoh}}(X)$ is the big k-dg-category obtained from the dg-enhanced category of complexes of sheaves with quasi-coherent cohomology by inverting the quasi-isomorphisms, this time not in the world of categories but within dg-categories. This big dg-category is generated by its full sub dg-category $L_{\mathrm{pe}}(X)$ spanned by the (homotopy) compact objects - so-called perfect complexes. We will return to this below.

We should also emphasize that dg-categories are important not only for algebraic geometry. They appear out of many different mathematical contexts

$$\begin{tikzcd}
\text{Algebraic Geometry} \arrow[dr, "L_{\mathrm{qcoh}}"] & & \text{Symplectic Geometry} \arrow[dl, "\text{Fukaya cat.}"'] \\
& \mathrm{dg}-\mathrm{categories} & \\
\text{complex geometry} \arrow[ur, "DQ\text{-modules}"] & & \text{Matrix Factorizations} \arrow[ul] \\
& \mathrm{dg}-\mathrm{algebras} \arrow[uu] &
\end{tikzcd}$$

and can be used as a unifying language. The map from dg-algebras to dg-categories is particularly important. It sends a dg-algebra A to the dg-category with one object and A as a complex of endomorphisms with the composition given by the associative product on A. The theory of modules makes

[2]See Exposé 7 for a treatment.

[3]More precisely there is an equivalence of $(\infty, 1)$-categories between the $(\infty, 1)$-category underlying the Morita model structure on small k-dg-categries and the $(\infty, 1)$-category of stable presentable compactly generated $(\infty, 1)$-categories endowed with an action of the ∞-derived category of k. This was recently proved in [Coh13].

sense for any dg-category. If T is a small dg-category, we define its big dg-category of dg-modules as the dg-category of dg-enriched functors from T to the category of complexes of k-modules with its natural dg-enrichement. This carries a natural k-linear homotopy theory and we will write $\widehat{T}$ for the dg-category that codifies it. If T the dg-category associated to a dg-algebra A, $\widehat{T}$ is a dg-enhancement of the classical derived category of A. The fact that $L_{\text{pe}}(X)$ generates the whole $L_{\text{qcoh}}(X)$ can now be made precise by means of an equivalence $\widehat{L_{\text{pe}}(X)} \simeq L_{\text{qcoh}}(X)$.

Let us now review some standard invariants that we can extract from a small dg-category:

- The Hoschschild Homology of a dg-category T, $\text{HH}(T)$ (see [Bla] for a precise construction of this invariant with values in spectra). The cyclic homology and the periodic cyclic homology of a dg-category are defined by an appropriate base-change of $\text{HH}(T)$. By the Hochschild-Konstant-Rosenberg theorem, the periodic cyclic homology of the small dg-category $L_{\text{pe}}(X)$ is isomorphic to the de Rham cohomology of X.

- More recentely, Kaledin introduced a noncommutative version of the cristalline cohomology.

- Connective K-theory, Non-connective K-theory and $\mathbb{A}^1$-invariant K-theory. We will come back to this later in the talk (Section 4). At this point we merely want to emphasise that all these K-theory flavours can be defined directly at the level of dg-categories and that their values at $L_{\text{pe}}(X)$ recover the values associated to X;

- More recently, A. Blanc in his thesis [Bla], introduce a noncommutative version of topological K-theory. If X is a scheme defined over $\mathbb{C}$, $\text{K}^{\text{top}}(L_{\text{pe}}(X))$ recovers the topological K-theory of the underlying space of complex points in X;

Comparison results like the ones above enhance the understanding of dg-categories as good bodies for noncommutative spaces.

An important feature of all these theories is the invariance along morphisms $T \to T'$ such that the map induced between the big dg-categories of modules $\widehat{T} \to \widehat{T'}$ is an equivalence. Such morphims are called Morita equivalences. The study of dg-categories up to Morita equivalence admits a (combinatorial) model structure (see [Tab05]) and throughout this notes we will let $\mathcal{D}g(k)^{\text{idem}}$ denote its underlying $(\infty, 1)$-category. It admits a natural monoidal structure given by the derived tensor product of dg-categories and by the results of [Toe07] it admits an internal-hom (which can be explicitely described).

We now introduce a bit of notation. Let A be a dg-algebra and $\widehat{A}$ its dg-category of modules. We let $\widehat{A}_{\text{pe}}$ denote the full sub dg-category of $\widehat{A}$ spanned by the (homotopy) compact objects. Of course, we have $\widehat{\widehat{A}_{\text{pe}}} \simeq \widehat{A}$. The following is a crucial result in noncommutative algebraic geometry:

Theorem 13.1.1. (Bondal-Van den Bergh [BV03]) Let X be a scheme over k. Then, if X is quasi-compact and quasi-separated there is a dg-algebra A_X such that $L_{\text{pe}}(X)$ is Morita equivalent to the small dg-category $\widehat{A_X}_{\text{pe}}$.

In other words, every quasi-compact and quasi-separated schemes becomes affine in the non-commutative world.

Definition 13.1.2. (Toen-Vaquie [TV07]) A dg-category $T \in \mathcal{D}g(k)^{\text{idem}}$ is said to be of finite type if it is a compact object in the $(\infty, 1)$-category $\mathcal{D}g(k)^{\text{idem}}$. We say that T is smooth and proper if it is a dualizable object. In particular, smooth and proper dg-categories are of finite type.

We will denote $\mathcal{D}g(k)^{\mathrm{ft}}$ the full subcategory of $\mathcal{D}g(k)^{\mathrm{idem}}$ spanned by the dg-categories of finite type. We can easily see that it is closed under the derived tensor product of dg-categories. Moreover, we can prove that a small dg-category T is of finite type if and only if it is Morita equivalent to a small dg-category of the form $\widehat{A}_{\mathrm{pe}}$ for a dg-algebra A which is (homotopy) compact in the homotopy theory of dg-algebras (see [TV07]). This description, together with the result of Bondal- Van den Bergh motivates the following definition:

Definition 13.1.3. The $(\infty,1)$-category of smooth non-commutative spaces $\mathcal{N}cS(k)$ is the opposite of $\mathcal{D}g(k)^{\mathrm{ft}}$

and we have

Proposition 13.1.4. (Toen-Vaquie [TV07, p. 3.27]) Let X be a smooth and proper variety over k. Then $L_{\mathrm{pe}}(X)$ is a smooth and proper dg-category. In particular, it is of finite type.

The assignement $X \to L_{\mathrm{pe}}(X)$ is known not to be fully faithful (see [Ueh04] for an explicit example and [Rou06] for a more complete discussion about the behavior of this assignement). In his program [Kona; Konb] Kontsevich envisioned also that similarly to schemes, noncommutative spaces should also admit a motivic theory and one of the interesting questions is how better L_{pe} behaves at the motivic level. Our goal in this talk is to explain a possible approach to this idea. Motives are generated by affine objects so that any comparison between the commutative and noncommutative motivic theories will require a proper definition of L_{pe} as a functor at the affine level. This is the content of the following result:

Proposition 13.1.5. (see [Rob12, p. 6.38]) The assignement $X \mapsto L_{\mathrm{pe}}(X)$ defines a monoidal ∞-functor

$$\mathrm{AffSm}^{\mathrm{ft}}(k)^{\times} \to \mathcal{N}cS(k)^{\otimes}$$

where $\mathrm{AffSm}^{\mathrm{ft}}(k)^{\times}$ denotes the category of smooth affine schemes of finite type over k, equipped with the cartesian product.

13.2 Motives

Before introducing noncommutative motives we should say some words about motives. In the original program envisioned by Grothendieck, the motif of a geometric object X (eg. X a projective smooth variety) was something like "the arithmetical content of X" [4]. More precisely, in the sixties, Grothendieck and his collaborators started a quest to construct examples of the so-called Weil cohomology theories, designed to capture different arithmetic information about X. In the presence of many such theories he envisioned the existence of a universal one, which would gather all the arithmetic information. Such a theory is not yet known to exist. It relies on the standard conjectures. See the books [And04; 55] and the course notes by B. Kahn [Kah13] for a introduction to this arithmetic program.

In the late 90's, Morel and Voevodsky [MV99] developped a more general theory of motives. In their theory, the motif of X can be the described as the cohomological skeleton of X, not only in the eyes of a Weil cohomology theory, but for all the generalized cohomology theories for schemes (like K-theory, algebraic cobordism and motivic cohomology) at once. The inspiration comes from the stable homotopy theory of spaces where all generalized cohomology theories (of spaces) become representable. Their construction can be summarized as follows:

1. Start from the category of smooth schemes $\mathrm{Sm}^{\mathrm{ft}}(S)$ (over a base scheme S) and freely complete it with homotopy colimits;

[4] like L-functions or Z-functions

2. Force descent with respect to the Nisnevich topology on schemes [5].

3. Force the affine line $\mathbb{A}^1_S$ to become contractible;

4. Point the theory;

5. Stabilize the theory (as to fabricate the stable homotopy theory of spaces) by tensor-inverting the topological circle S^1 pointed at 1;

6. Stabilize the theory with respect to the algebraic circle G_m pointed at 1;

Notice that after the first four steps the product of the (pointed) topological circle with the (pointed) algebraic circle becomes equivalent to the projective space $\mathbb{P}^1$ pointed at ∞ (any choice of base point here is $\mathbb{A}^1$-homotopic). In particular the last two steps can be performed all at once by stabilizing with respect to $\mathbb{P}^1$ pointed at ∞.

Their original construction was achieved using the theory of model categories. Nowadays we know that model categories are strict presentations of more fundamental objects, namely, $(\infty,1)$-categories. For a model category $\mathcal{M}$ with weak-equivalences W, we will write $\infty(\mathcal{M})$ to denote its underlying $(\infty,1)$-category [6]. The results of J. Lurie in [HTT] allow us to give characterize the underlying $(\infty,1)$-categories associated to the first three steps of Morel-Voevodsky: the first step corresponds to take presheaves of homotopy types [HTT, p. 4.2.4.4] and the second and third step correspond to accessible reflexive localizations (recall that an $(\infty,1)$-category $\mathcal{C}$ is presentable if and only if it is the underlying $(\infty,1)$-category of a combinatorial model category $\mathcal{M}$ and that accessible reflexive localizations of $\mathcal{C}$ correspond bijectively to Bousfield localizations of $\mathcal{M}$ [HTT, A.3.7.4, A.3.7.6, A.3.7.8]). The following result is part of my thesis work and provides a way to understand the two stabilization steps at the underlying ∞-categorical level:

Theorem 13.2.1. ([Rob12, p. 4.29]) Let $\mathcal{M}$ be a combinatorial symmetric monoidal model category and let X be a cofibrant object in $\mathcal{M}$. Let $\mathrm{Sp}^{\Sigma}(\mathcal{M},X)$ be the combinatorial symmetric monoidal monoidal category of (symmetric) spectrum objects in $\mathcal{M}$ with respect to X introduced by Hovey in [Hov01] corresponding to the stabilization of $\mathcal{M}$ with respect to X. This comes naturally equipped with a left Quillen monoidal map $\mathcal{M} \to \mathrm{Sp}^{\Sigma}(\mathcal{M},X)$ sending X to a tensor-invertible object. Assume that X satisfies the following condition:

$(*)$ The ciclic permutation of factors $X \otimes X \otimes X \to X \otimes X \otimes X$ is the identity map in the homotopy category of $\mathcal{M}$.

Then, the presentable symmetric monoidal $(\infty,1)$-category $\infty(\mathrm{Sp}^{\Sigma}(\mathcal{M},X))^{\otimes}$ underlying the model category $\mathrm{Sp}^{\Sigma}(\mathcal{M},X)$ has the following universal property: for any presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$ the composition along the monoidal functor $\infty(\mathcal{M})^{\otimes} \to \infty(\mathrm{Sp}^{\Sigma}(\mathcal{M},X))^{\otimes}$

$$\mathrm{Fun}^{\otimes,L}(\infty(\mathrm{Sp}^{\Sigma}(\mathcal{M},X))^{\otimes},\mathcal{D}^{\otimes}) \to \mathrm{Fun}^{\otimes,L}(\infty(\mathcal{M})^{\otimes},\mathcal{D}^{\otimes})$$

is fully-faithful and its image is the full subcategory spanned by those monoidal functors $\infty(\mathcal{M})^{\otimes} \to \mathcal{D}^{\otimes}$ sending X to a tensor-invertible object [7].

[5] most of the interesting generalized cohomology theories for schemes satisfy Nisnevich descent

[6] To be more precise, if we use simplicial categories as models for $(\infty,1)$-categories, $\infty(\mathcal{M})$ is the Dwyer-Kan localization of $\mathcal{M}$ along the class of weak-equivalences. If we use quasi-categories, $\infty(\mathcal{M})$ can be identified with a fibrant replacement for the marked simplicial set $(N(\mathcal{M}),W)$ in the Lurie's upgraded Joyal model structure for marked simplicial sets, where $N(\mathcal{M})$ is the nerve of $\mathcal{M}$.

[7] Both on the left and right we have the $(\infty,1)$-categories of colimit preserving monoidal functors

More generally, we can prove that given a presentable symmetric monoidal $(\infty,1)$-category $\mathcal{C}^{\otimes}$ together with an object $X \in \mathcal{C}$ we can fabricate a new presentable symmetric monoidal $(\infty,1)$-category together with a monoidal map from $\mathcal{C}$ sending X to a tensor-invertible object and universal in this sense amongst presentable symmetric monoidal $(\infty,1)$-categories (see [Rob12, p. 4.10]). With this in mind we can rephrase our result above as saying that whenever X satisfies the cyclic condition, the model category $\mathrm{Sp}^{\Sigma}(\mathcal{M},X)$ with its convolution product is a strict model for the universal tensor-inversion of X.

Example 13.2.2. If $\mathcal{C}^{\otimes}$ is a pointed presentable symmetric monoidal $(\infty,1)$-category category then it is a commutative algebra (in the $(\infty,1)$-category of presentable $(\infty,1)$-categories) over the $(\infty,1)$-category of pointed spaces with the smash product. We can check that $\mathcal{C}$ is stable if and only if the image of the topological circle S^1 under the canonical morphism $\mathcal{S}_* \to \mathcal{C}$ is tensor-invertible in $\mathcal{C}^{\otimes}$. For the complete details see [Rob12, p. 4.28]

As an application we get the following characterization for the theory of Morel-Voevodsky

Corollary 13.2.3. ([Rob12, p. 5.11]) Let S be a base scheme and let $\mathrm{Sm}^{\mathrm{ft}}(S)^{\times}$ denote the category of smooth schemes over S considered as a trivial $(\infty,1)$-category, together with the cartesian product of schemes. Let $\mathcal{SH}(S)^{\otimes}$ denote the presentable stable symmetric monoidal $(\infty,1)$-category underlying the motivic stable model category of Morel-Voevodsky. Since all the steps in the list above are monoidal, we end up with a natural monoidal map $\mathrm{Sm}^{\mathrm{ft}}(S)^{\times} \to \mathcal{SH}(S)^{\otimes}$. This map has the following universal property: for any pointed presentable symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$, the composition map

$$\mathrm{Fun}^{\otimes,L}(\mathcal{SH}(S)^{\otimes},\mathcal{D}^{\otimes}) \to \mathrm{Fun}^{\otimes}(\mathrm{Sm}^{\mathrm{ft}}(S)^{\times},\mathcal{D}^{\otimes})$$

is fully-faithful and its image consists of those monoidal functors F such that:

- F satisfies Nisnevich descent;
- for any smooth scheme X over S we have $F(X \times \mathbb{A}^1_S) \simeq F(X)$;
- the cofiber of the map $F(S) \to F(\mathbb{P}^1_S)$ induced by the inclusion of the point at infinity, is a tensor-invertible object in $\mathcal{D}^{\otimes}$.

In particular, since $(\mathbb{P}^1_S,\infty)$ is equivalent to $S^1 \wedge G_m$ in $\mathcal{SH}(S)$, we find that any pointed symmetric monoidal $(\infty,1)$-category $\mathcal{D}^{\otimes}$ admitting a monoidal map from $\mathrm{Sm}^{\mathrm{ft}}(S)^{\times}$ satisfying these conditions is necessarily stable because S^1 becomes tensor-invertible (see the example above).

Remark 13.2.4. An important result by Morel [Mor99] is that the construction of $\mathcal{SH}(S)^{\otimes}$ does not required the category of all smooth schemes as a basic ingredient. It is enough to start with smooth affine schemes.

This corollary is also helpful in the construction of motivic realizations (see [Dre13]).

We should also remark that this theorem would be impossible to prove only with the techniques of model category theory due to its bad functoriality properties.

To conclude this section we recall that this theory of Morel-Voevodsky allows us to recover the so-called theory of Voevodsky Motives (which is an attempt to formalize the initial vision of Grothendieck). More precisely, in $\mathcal{SH}(k)$ we have homotopy commutative ring objects $M\mathbb{Q}$ and $M\mathbb{Z}$

representing, respectively, motivic cohomology with rational coefficients and with integer coefficients. For the last, it is known that $\mathrm{Mod}_{M\mathbb{Q}}(\mathcal{SH}(k))$ is equivalent to the stable $(\infty, 1)$-category $\mathrm{DM}(k)_{\mathbb{Q}}$ encoding Voevodsky Motives (see [MVW06; Rs06])so that, in particular, its homotopy category contains Chow motives as full subcategory.

13.3 Noncommutative Motives

Our goal now is to explain our new approach to the theory of noncommutative motives. The basic idea is to mimic the construction of Morel-Voevodsky in the most direct way, by introducing a noncommutative version of the Nisnevich topology for our noncommutative spaces. To give an explicit description of this new notion, we investigate the image of the Nisnevich topology for schemes under the functor $L_{\mathrm{pe}} : \mathrm{AffSm}^{\mathrm{ft}}(k) \to \mathcal{D}g(k)^{\mathrm{ft}}$. The first main observation is that the Nisnevich topology is a cd-topology in the sense that to verify that something has Nisnevich descent it is enough to test the descent property with respect to the certain Nisnevich coverings given by squares, in this case of the form

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

where i is an open immersion and p is an etale map such that the canonical map $p^{-1}(Z) \to Z$ with $Z := X \setminus U$ equipped with the reduced structure, is an isomorphism. We now describe the image of these basic squares under L_{pe}:

- The image of an elementary Nisnevich square under L_{pe} is a pullback square of dg-categories of finite type in the morita theory of dg-categories. This follows from a crucial result of A. Hirschowitz and C. Simpson in [SH01], namely that L_{pe} satisfies Nisnevich descent;

- If $U \hookrightarrow X$ is an open immersion of schemes with closed complementary Z, then the sequences of dg-categories $L_{\mathrm{pe}}(X)_Z \to L_{\mathrm{pe}}(X) \to L_{\mathrm{pe}}(U)$ is an exact sequence of dg-categories (meaning, a cofiber sequence in the Morita theory of dg-categories where the first map is fully faithful). Here $L_{\mathrm{pe}}(X)_Z$ denotes the full sub dg-category of $L_{\mathrm{pe}}(X)$ spanned by those perfect complexes on X with support on Z. This result was originally proven by Verdier [Ver67] in the context of triangulated categories and then upgraded by Thomason [TT90] to the theory of perfect complexes and more recently by B. Keller in the setting of dg-categories [Kel99].

- The dg-categories $L_{\mathrm{pe}}(X)_Z$ and $L_{\mathrm{pe}}(V)_{V-p^{-1}(U)}$ do not have to be of finite type (The kernel of things of finite type does not have to be of finite type). However, by the results of Neeman [Nee96] and Bondal-Van den Bergh [BV03], they admit compact generators. Moreover, the condition that the map $p^{-1}(Z) \to Z$ is an isomorphism forces the induced map $L_{\mathrm{pe}}(V)_{V-p^{-1}(U)} \to L_{\mathrm{pe}}(X)_Z$ to be an equivalence;

With these observations in mind we can give the following definition, whose originality we should attribute to B. Toën, M.Vaquié and G. Vezzosi.

Definition 13.3.1. (see [Rob12, p. 6.44]) An elementary Nisnevich square of noncommutative spaces is a commutative square in $\mathcal{N}cS(k)$ corresponding to a commutative square of dg-categories of finite type

$$\begin{tikzcd} T_{\mathcal{X}} \arrow[r] \arrow[d] & T_{\mathcal{U}} \arrow[d] \\ T_{\mathcal{V}} \arrow[r] & T_{W} \end{tikzcd}$$

such that:

- The square is a pullback in the Morita theory of dg-categories (and therefore corresponds to a pushout between the associated noncommutative spaces).
- Both rows fit in exact sequences of dg-categories (as defined above) $K_{\mathcal{X}-\mathcal{U}} \hookrightarrow T_{\mathcal{X}} \to T_{\mathcal{U}}$ and $K_{\mathcal{V}-\mathcal{W}} \hookrightarrow T_{\mathcal{V}} \to T_{W}$ where both $K_{\mathcal{X}-\mathcal{U}}$ and $K_{\mathcal{V}-\mathcal{W}}$ are dg-categories with a compact generator and such that the induced map $K_{\mathcal{X}-\mathcal{U}} \to K_{\mathcal{V}-\mathcal{W}}$ is an equivalence in $\mathcal{D}g(k)^{\mathrm{idem}}$.

Let us give some examples:

Example 13.3.2.

- By definition, and because of the preliminary observations above, every Nisnevich square of smooth affine schemes provides a Nisnevich square of noncommutative spaces under the functor L_{pe};
- If $A \to B \to C$ is an exact sequence of dg-categories (as defined above) with A a dg-category of finite type then the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

  provides a Nisnevich square of dg-categories (see [Rob12, p. 6.47]).
- Recall that a semi-orthogonal decomposition of a dg-category B is an exact sequence $A \to B \to C$ that splits [8]. If both B, A, and C are of finite type, by the previous example we have a Nisnevich square. If we have a split exact sequence in a non-stable context, then it is not necessarily true that the middle term is the direct sum of the extreme terms. However, by mapping such a sequence to a stable context this becomes true. This is exactly what happens when we use the canonical map $\mathcal{N}cS(k) \to \mathcal{SH}_{nc}(k)$. An important example is $B = L_{\mathrm{pe}}(\mathbb{P}^1)$ and the fact it admits an exceptional collection with two generators, meaning,a semi-orthogonal decomposition where both factors A and C are equivalent to $L_{\mathrm{pe}}(k)$

With this notion, we can now construct a direct analogue for the Morel-Voevodsky theory in the noncommutative world (over a commutative ring k). Namely, we perform exactly the same steps replacing the category of affine smooth schemes over k by the $(\infty,1)$-category $\mathcal{N}cS(k)$, the elementary Nisnevich squares of schemes by the elementary Nisnevich squares of noncommutative spaces over k and the affine line and the projective space by their noncommutative incarnations, respectively given by the dg-categories $L_{\mathrm{pe}}(\mathbb{A}^1_k)$ and $L_{\mathrm{pe}}(\mathbb{P}^1_k)$. The result is a new stable presentable symmetric monoidal $(\infty,1)$-category $\mathcal{SH}_{nc}(k)^{\otimes}$ together with a universal monoidal map

$$\mathcal{N}cS(k)^{\otimes} \to \mathcal{SH}_{nc}(k)^{\otimes}$$

[8] meaning that the first map admits a left inverse and the second map admits a right-inverse.

analogue to the universal map characterizing the theory of Morel-Voevodsky. An important feature of the noncommutative world is that we don't need to invert the noncommutative image of $(\mathbb{P}^1,\infty)$ because it is already invertible (see [Rob12, p. 6.55]). This follows from the existence of an exceptional collection on $L_{\mathrm{pe}}(\mathbb{P}^1)$ with two generators and the fact that exceptional collections provide (noncommutative) Nisnevich coverings.

It follows from definition that Nisnevich squares of schemes are sent to Nisnevich squares of noncommutative spaces and that images of the affine line and the projective space satisfy the necessary conditions for our universal description of $\mathcal{SH}(k)^{\otimes}$ to ensure the existence of a unique monoidal colimit preserving map rendering the diagram commutative

$$\begin{tikzcd} \mathrm{AffSm}^{\mathrm{ft}}(k)^{\times} \arrow[r, "L_{\mathrm{pe}}^{\otimes}"] \arrow[d] & \mathcal{N}cS(k)^{\otimes} \arrow[d] \\ \mathcal{SH}(k)^{\otimes} \arrow[r, dashed, "\mathcal{L}^{\otimes}"] & \mathcal{SH}_{nc}(k)^{\otimes} \end{tikzcd} \tag{13.1}$$

13.4 K-Theory and Noncommutative Motives

In the previous section we introduced a new theory of noncommutative motives. In this section we will explore this new theory and how it allow us to explain some constructions in algebraic K-theory. Before giving our main result in this section let us remark that the construction of both $\mathcal{SH}(k)^{\otimes}$ and $\mathcal{SH}_{nc}(k)^{\otimes}$ can be achieved by taking directly presheaves of spectra instead of presheaves of spaces. Essentially this corresponds to performing the first stabilization immediately after the first step in the list. The fact that the two procedures give the same final result follows from the final universal properties obtained. With this in mind, the construction of the comparison map in the diagram (13.1) can be explained in a sequence of steps

$$\begin{tikzcd} \mathrm{AffSm}^{\mathrm{ft}}(k)^{\times} \arrow[r, "L_{\mathrm{pe}}^{\otimes}"] \arrow[d, hook, "(\Sigma^{\infty}_{+}\circ j)^{\otimes}"] & \mathcal{N}cS(k)^{\otimes} \arrow[d, hook, "(\Sigma^{\infty}_{+}\circ j_{\mathrm{nc}})^{\otimes}"] \\ \mathrm{Fun}(\mathrm{AffSm}^{\mathrm{ft}}(k)^{\mathrm{op}},\mathrm{Sp})^{\otimes} \arrow[r, dashed] \arrow[d, "l_{\mathrm{Nis}}^{\otimes}"] & \mathrm{Fun}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp})^{\otimes} \arrow[d, "l_{\mathrm{Nis}}^{nc,\otimes}"'] \\ \mathrm{Fun}_{\mathrm{Nis}}(\mathrm{AffSm}^{\mathrm{ft}}(k)^{\mathrm{op}},\mathrm{Sp})^{\otimes} \arrow[r, dashed] \arrow[d, "l_{\mathbb{A}^1}^{\otimes}"] & \mathrm{Fun}_{\mathrm{Nis}}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp})^{\otimes} \arrow[d, "l_{\mathbb{A}^1}^{nc,\otimes}"'] \\ \mathrm{Fun}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{AffSm}^{\mathrm{ft}}(k)^{\mathrm{op}},\mathrm{Sp})^{\otimes} \arrow[r, dashed] \arrow[d, "\Sigma_{\mathbb{G}_m}^{\otimes}"] & \mathrm{Fun}_{\mathrm{Nis},L_{\mathrm{pe}}(\mathbb{A}^1)}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp})^{\otimes} \arrow[d, "\sim"] \\ \mathcal{SH}(k)^{\otimes} \arrow[r, dashed, "\mathcal{L}^{\otimes}"] & \mathcal{SH}_{nc}(k)^{\otimes} \end{tikzcd} \tag{13.2}$$

where each dotted map is induced by the corresponding universal property and the vertical arrows correspond to the localization functors. By [**lurie-ha**] these functors admit lax monoidal right-adjoints.

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{AffSm}^{\mathrm{ft}}(k)^{\mathrm{op}}, \mathrm{Sp})^{\otimes} & \mathrm{Fun}(\mathcal{D}g(k)^{\mathrm{ft}}, \mathrm{Sp})^{\otimes} \arrow[l, "\mathcal{M}_1^{\otimes}"'] \\
\mathrm{Fun}_{\mathrm{Nis}}(\mathrm{AffSm}^{\mathrm{ft}}(k)^{\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[u, hook] & \mathrm{Fun}_{\mathrm{Nis}}(\mathcal{D}g(k)^{\mathrm{ft}}, \mathrm{Sp})^{\otimes} \arrow[l, "\mathcal{M}_2^{\otimes}"'] \arrow[u, hook] \\
\mathrm{Fun}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{AffSm}^{\mathrm{ft}}(k)^{\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[u, hook] & \mathrm{Fun}_{\mathrm{Nis},L_{\mathrm{pe}}(\mathbb{A}^1)}(\mathcal{D}g(k)^{\mathrm{ft}}, \mathrm{Sp})^{\otimes} \arrow[l, "\mathcal{M}_3^{\otimes}"'] \arrow[u, hook] \\
\mathcal{SH}(k)^{\otimes} \arrow[u, "\Omega^{\infty,\otimes}_{\mathbb{G}_m}"] & \mathcal{SH}_{nc}(k)^{\otimes} \arrow[l, "\mathcal{M}^{\otimes}"'] \arrow[u, "\sim"']
\end{tikzcd} \tag{13.3}$$

Let me now describe the second main result in my thesis work. For that I need to recall some facts about algebraic K-theory.

K-theory was already discovered as an invariant of categorical nature. Quillen promoted this vision by explaining how to a given exact category we can associate a space whose homotopy groups encode the information of K-theory [Qui73]. This space cames naturally equipped with a homotopy commutative group-law so that, by the equivalence of Segal [Seg74], it is the same as a connective spectra. Later on, Waldhausen [Wal85] enlarged the domain of K-theory from exact categories to categorical structures which today we know as a Waldhausen categories. More recentely, it became clear that the natural domain of K-theory is in fact the world of $(\infty,1)$-categories [TV04; Bar12]. In particular, any dg-category $T \in \mathcal{D}g(k)^{\mathrm{idem}}$ has an associated K-theory (connective) spectrum which we will denote as $K^c(T)$.

The story of non-connective algebraic K-theory goes back to Bass [Bas68] and Karoubi [Kar68]. If $U \hookrightarrow X$ is an open immersion of schemes with closed complementary Z, the associated cofiber sequence of dg-categories produces a fiber sequence of connective spectra

$$K^c(L_{\mathrm{pe}}(X)_Z) \to K^c(L_{\mathrm{pe}}(X)) \to K^c(L_{\mathrm{pe}}(U))$$

but as the inclusion $\mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}$ does not preserve limits, we don't have a cofiber/fiber sequence of spectra and in particular, we don't get a long exact sequence relating the K-theory groups. Non-connective K-theory K^S was designed to solve this problem: its connective part is equivalent to connective K-theory but this time the image of an exact sequence of dg-categories is sent to a cofiber/fiber sequence of spectra so that these new K-groups are related by a long exact sequence. This version of K-theory was first introduced for schemes by Bass in [Bas68] and Thomason in [TT90] and more recently Schlichting [Sch06] gave a general framework that allows us to define it for dg-categories, so that by taking $K^S(L_{\mathrm{pe}}(X))$ we recover the spectrum of Bass-Thomason.

The important point we want to stress in this discussion is that both K^c and K^S belong naturally to the noncommutative world. More precisely, K^c can be seen as an object in $\mathrm{Fun}(\mathcal{D}g(k)^{\mathrm{ft}}, \mathrm{Sp}_{\geq 0}) \subseteq \mathrm{Fun}(\mathcal{D}g(k)^{\mathrm{ft}}, \mathrm{Sp})$ and concerning K^S, since it sends exact sequences of dg-categories to cofiber/fiber sequences in spectra, we can prove that it satisfies Nisnevich descent, so that it is an object in $\mathrm{Fun}_{\mathrm{Nis}}(\mathcal{D}g(k)^{\mathrm{ft}}, \mathrm{Sp})$. Moreover, the right-adjoints take K^c to the spectral presheaf encoding connective algebraic K-theory of schemes (this is just because of the Yoneda lemma) and K^S to the spectral presheaf encoding the non-connective K-theory of schemes introduced by Bass and Thomason (this follows from a comparison result by Schlichting in [Sch06]).

A second main result of my thesis work is the following:

Theorem 13.4.1.

- ([Rob13, p. 1.9]) The canonical morphism $K^c \to K^S$ presents non-connective K-theory as the (noncommutative) Nisnevich sheafification of connective K-theory;
- ([Rob13, p. 1.12]) The further (noncommutative) $\mathbb{A}^1$-localization $l^{\mathrm{nc}}_{\mathbb{A}^1}(K^S)$ is a unit 1_{nc} for the monoidal structure in $\mathcal{SH}_{nc}(k)^{\otimes}$;
- ([Rob13, p. 1.11]) The image of $l^{\mathrm{nc}}_{\mathbb{A}^1}(K^S)$ along the right-adjoint $\mathcal{M}$ in the diagram (13.3) recovers the object KH in $\mathcal{SH}(k)$ representing $\mathbb{A}^1$-invariant algebraic K-theory of Weibel (also known as homotopy invariant K-theory). In particular, since $\mathcal{M}$ is lax monoidal (it is right-adjoint to a monoidal functor) it sends the trivial algebra structure in 1_{nc} to a commutative algebra structure in KH so that the monoidal map $\mathcal{L}^{\otimes}$ factors as

$$\mathcal{SH}(k)^{\otimes} \xrightarrow{-\otimes KH} \mathrm{Mod}_{KH}(\mathcal{SH}(k))^{\otimes} \dashrightarrow \mathcal{SH}_{nc}(k)^{\otimes}$$

The first part of this theorem is not true if we restrict ourselves to the non-connective of schemes. The phenomenon that makes it possible in the noncommutative world is the fact the new notion of Nisnevich squares of noncommutative spaces combines at the same time coverings of geometrical origin (namely, those coming via L_{pe} from classical Nisnevich squares) and coverings of categorical origin, namely, the ones induced by exceptional collections.

The first part of this theorem is proved by showing that the Bass-construction $(-)^B$ given in Thomason's paper [TT90] is an explicit model for the Nisnevich localization of presheaves with values in connective spectra and sending Nisnevich squares of noncommutative spaces to pull-back squares in connective spectra. Recall that the inclusion of connective spectra in all spectra does not preserve pullbacks. More generally, we prove that the connective truncation functor induces an equivalence of $(\infty, 1)$-categories between the $(\infty, 1)$-category of Nisnevich local spectral presheaves and the $(\infty, 1)$-category of spectral presheaves with values in connective spectra and satisfying connective Nisnevich descent. The $(-)^B$-contruction is an explicit inverse to this truncation.

The second part of this theorem relies on the following fundamental result of A. Blanc in his Phd Thesis.

Proposition 13.4.2. (A.Blanc [Bla, Prop. 4.6]) The splitted version of the Waldhausen S-construction (meaning, using only those cofibrations that split) is $\mathbb{A}^1$-homotopic to the full Waldhausen S-construction.

To prove the last item we show that the commutative and noncommutative versions of the $\mathbb{A}^1$-localizations are compatible with the right-adjoints.

The following corollary provides a new formalization of something understood by Kontsevich [Kona; Konb] long ago and also already satisfied by the formalism of Cisinski-Tabuada:

Corollary 13.4.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be two noncommutative smooth spaces and assume that $\mathcal{Y}$ is smooth and proper. Then we have an equivalence of spectra

$$\mathrm{Map}_{\mathcal{SH}_{nc}(k)}(\mathcal{X}, \mathcal{Y}) \simeq l^{\mathrm{nc}}_{\mathbb{A}^1}(K^S)(\mathcal{X} \otimes \mathcal{Y}^{\mathrm{op}})$$

where $\mathcal{Y}^{\mathrm{op}}$ is the dual of $\mathcal{Y}$ and we have identified $\mathcal{X}$ and $\mathcal{Y}$ with their images in $\mathcal{SH}_{nc}(k)$.

The next result concludes the main content of these notes. It is a corollary of the previous theorem together with the results of J. Riou describing the compact generators in $\mathcal{SH}(k)$ over a field with resolutions of singularites (see [Rio05]).

Corollary 13.4.4. If k is a field admitting resolutions of singularities then the canonical factorization

$$\mathrm{Mod}_{KH}(\mathcal{SH}(k))^{\otimes} \dashrightarrow \mathcal{SH}_{nc}(k)^{\otimes}$$

is fully faithful.

Let me say that this result as been known by some people after some time already. I think particularly of B. Toën, M. Vaquié and G.Vezzosi and also D-C. Cisinski and G. Tabuada.

13.5 Relation with the work of Cisinski-Tabuada

In this section we explain how the approach of Cisinski-Tabuada relates to ours. For that purpose let us focus again on the diagram (13.3). The theory developed by G.Tabuada [Tab08] in his thesis and later in joint work with D-C. Cisinski [CT12; CT11] concerns the study of localizing invariants of dg-categories. More precisely, inside the $(\infty,1)$-category $\mathrm{Fun}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp}) \simeq \mathrm{Fun}_{flt}(\mathcal{D}g(k)^{\mathrm{idem}},\mathrm{Sp})$ we can isolate the full subcategory $\mathrm{Fun}_{\mathrm{Loc}}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp})$ spanned by those functors sending exact sequence of dg-categories to cofiber/fiber sequences in Sp. We can easily check that any functor with this property satisfies also Nisnevich descent in our sense, so that we have a canonical inclusion $\mathrm{Fun}_{\mathrm{Loc}}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp}) \subseteq \mathrm{Fun}_{\mathrm{Nis}}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp})$. The main theorem of Cisinski-Tabuada's approach is the existence of stable presentable $(\infty,1)$-category $\mathcal{M}^{\mathrm{Tab}}_{\mathrm{Loc}}$ together with a natural equivalence

$$\mathrm{Fun}_{\mathrm{Loc}}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp}) \simeq \mathrm{Fun}^{L}(\mathcal{M}^{\mathrm{Tab}}_{\mathrm{Loc}},\mathrm{Sp}) \tag{13.4}$$

such that along this equivalence, K^S is sent to a co-representable functor. The first visible advantage of our approach is that it explains how K^S appears out of K^c by means of the Nisnevich sheafification. In Cisinski-Tabuada's approach K^S is taken as basic input. The second important observation is that the construction $\mathcal{M}^{\mathrm{Tab}}_{\mathrm{Loc}}$ of Tabuada admits analogues adapted to each of the full subcategories in the diagram

$$\begin{tikzcd}
 & \mathrm{Fun}(\mathcal{D}g(k)^{\mathrm{idem}},\mathrm{Sp}) & \\
\mathrm{Fun}_{\mathrm{Nis}}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp}) \arrow[ur, hook] & & \mathrm{Fun}_{\mathrm{Loc}}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp}) \simeq \mathrm{Fun}^{L}(\mathcal{M}^{\mathrm{Tab}}_{\mathrm{Loc}},\mathrm{Sp}) \arrow[ul, hook] \arrow[ll, hook] \\
\mathrm{Fun}_{\mathrm{Nis},\mathbb{A}^1}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp}) =: \mathcal{SH}_{nc}(k) \arrow[u, hook] & & \mathrm{Fun}_{\mathrm{Loc},\mathbb{A}^1}(\mathcal{D}g(k)^{\mathrm{ft}},\mathrm{Sp}) \arrow[u, hook] \arrow[ll, hook]
\end{tikzcd} \tag{13.5}$$

More precisely one can easily show the existence of new stable presentable symmetric monoidal $(\infty,1)$-categories $\mathcal{M}^{\mathrm{Tab}}_{\mathrm{Nis}}$, $\mathcal{M}^{\mathrm{Tab}}_{\mathrm{Nis},\mathbb{A}^1}$, $\mathcal{M}^{\mathrm{Tab}}_{\mathrm{Loc},\mathbb{A}^1}$ providing analogues for the formula in (13.4). In particular we find an equivalence

$$\mathcal{SH}_{nc}(k) \simeq \mathrm{Fun}^{L}(\mathcal{M}^{\mathrm{Tab}}_{\mathrm{Nis},\mathbb{A}^1},\mathrm{Sp}) \tag{13.6}$$

exhibiting the duality between our approach and the corresponding Nisnevich-$\mathbb{A}^1$-version of Tabuada's construction (recall that the very big $(\infty,1)$-category of big stable presentable $(\infty,1)$-categories has a natural symmetric monoidal structure [HA, 6.3.2.10, 6.3.2.18 and 6.3.1.17] where the big $(\infty,1)$-category of spectra Sp is a unit and $\mathrm{Fun}^L(-,-)$ is the internal-hom).

One can show that all the vertical inclusions in the diagram admit left adjoints and therefore $\mathcal{SH}_{nc}(k)^{Loc}$ is endowed with an obvious universal property concerning Localizing descent. As the last forces Nisnevich descent, the universal properties involved provide a monoidal left adjoint to the inclusion $\mathcal{SH}_{nc}(k)^{Loc} \subseteq \mathcal{SH}_{nc}(k)$, relating our theory to the dual of localizing theory of Tabuada. As emphasized before, the main advantage (in fact, la *raison-d'être*) of our approach to noncommutative motives is the easy comparison with the motivic stable homotopy theory of schemes. The duality here presented explains why the original approach of Cisinski-Tabuada is not directly comparable.

13.6 Future Works

In the continuation of my thesis I will study the existence of a formalism of six-operations in this new approach to noncommutative motives $\mathcal{SH}_{nc}(k)^{\otimes}$. This formalism will allow us to extend the fully faithfulness of the map $\mathrm{Mod}_{KH}(\mathcal{SH}(k)) \to \mathcal{SH}_{nc}(k)$ to any base.

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APPENDIX **A**

Simplicial sets

In this appendix we will quickly overview the homotopy theory of simplicial sets. The exposition given here is by no means complete, and the interested reader is referred to [May69] and to [GJ99, Chapter 1] for a more comprehensive treatment.

Simplicial sets have been introduced in order to manipulate in a easier way the theory of CW-complexes, in the sense that they provide a combinatorial model for the same homotopy theory. Using the language introduced in the first Exposé, we can make precise this statement by saying that there is a Quillen equivalence

$$|\cdot| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$$

The functors involved in this equivalence are the geometric realization and the singular complex. We assume that the reader has some experience with Sing coming from the Algebraic Topology; the first important task of this appendix is, on the other side, to describe the geometric realization and to analyse its most important properties.

Later, we will describe the homotopy theory of **sSet** using the internal combinatorics: a particular attention is paid to the formalism of anodyne extensions and to its applications. We will conclude describing the notion of minimal simplicial set and a sketch of the proof that **sSet** has a model structure.

A.1 The category $\boldsymbol{\Delta}$ and (co)simplicial objects

Notation. For every $n \in \mathbb{N}$ let $\mathbf{n}$ be the category associated to the (unique) linearly ordered set with n elements.

Definition A.1.1. Let $\boldsymbol{\Delta}$ be the full subcategory of **Cat** spanned by the categories $\mathbf{n}$ as n varies in $\mathbb{N}$.

Remark A.1.2. $\boldsymbol{\Delta}$ is thus the category whose objects are finite totally ordered sets and whose arrows are (weakly) increasing functions.

The combinatorial nature of $\boldsymbol{\Delta}$ allows to produce a simple factorization system[1] in this category. In order to do so, we introduce two particular classes of arrows: the *coface* maps $d^i : \mathbf{n} \to \mathbf{n+1}$ ($i \in \{0, 1, \dots, n+1\}$) defined by

$$d^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

These are the only strictly increasing arrows from $\mathbf{n}$ to $\mathbf{n+1}$.

[1] Remember that a factorization system in a category $\mathcal{C}$ is the datum of two subcategories $\mathcal{C}_0$ and $\mathcal{C}_1$ such that every arrow f can be written as $f = gh$, where g is in $\mathcal{C}_0$ and h is in $\mathcal{C}_1$.

Similarly, we can construct $n+1$ arrows $s^i : \mathbf{n+1} \to \mathbf{n}$ by requiring that exactly two successive numbers have the same image:

$$s^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

We will call the maps s^i the codegeneracy maps.

Theorem A.1.3. Let $f : \mathbf{n} \to \mathbf{m}$ be any arrow in $\mathbf{\Delta}$. Then there are uniquely determined maps $d, s \in \mathrm{Ar}(\mathbf{\Delta})$ such that

$$f = d \circ s$$

where

$$d = d^{i_1} \circ \ldots \circ d^{i_s} \qquad 0 \leq i_s \leq \ldots \leq i_1 \leq m \tag{A.1}$$

and

$$s = s^{j_1} \circ \ldots \circ s^{j_r} \qquad 0 \leq j_1 < \ldots < j_t < n \tag{A.2}$$

Remark A.1.4. Before starting the proof, let's observe that if $f : \mathbf{n} \to \mathbf{m}$ is injective (surjective) as map of sets, then it is monic (epi) in $\mathbf{\Delta}$. This is essentially obvious, and has as simple consequence the fact that any iterated composition of the face maps d^i is monic.

Proof. Let $i_s < \ldots < i_1$ be the elements in $\mathbf{m}$ not in the image of f, and let $j_1 < \ldots < j_t$ be the elements in $\mathbf{n}$ such that $f(j) = f(j+1)$. Write also

$$p = n - t = m - s$$

We claim that with this choice of indexes, equations (A.1) and (A.2) gives the desired factorization:

$$\mathbf{n} \xrightarrow{d} \mathbf{p} \xrightarrow{s} \mathbf{m}$$

In fact, fix $k \in \{0, 1, \ldots, n\}$. Assume that $j_h \leq k < j_{h+1}$. Then necessarily $f(k)$ is the $(k-h)$-th element in $\mathbf{m} \setminus \{i_1, \ldots, i_s\}$; therefore

$$f(k) = d(k-h) = d(s(k))$$

The factorization is unique because if f miss $i_s, \ldots, i_1$, then f must factor through d, hence $f = ds'$, and since d is mono, $s' = s$. □

Theorem A.1.3 gives a set of generators for the arrows of the category $\mathbf{\Delta}$. The next Theorem gives the relations between them:

Theorem A.1.5. Let $\mathcal{D}$ be the graph whose objects are natural numbers and such that for each $n \in \mathbb{N}$ there are exactly $n+2$ arrows δ^i_n $(i \in \{0 \ldots, n+1\})$ with source n and target $n+1$, and exactly $n+1$ arrows σ^i_n $(i \in \{0, \ldots, n\})$ with source n and target $n-1$. Then $\mathbf{\Delta}$ is obtained as the quotient of the free category $F\mathcal{D}$ via the relations

$$\begin{gathered} d^j d^i = d^i d^{j-1} \quad \text{if } i < j \\ s^j s^i = s^i s^{j+1} \quad \text{if } i \leq j \\ s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1 \\ d^{i-1} s^j & \text{if } i > j+1 \end{cases} \end{gathered} \tag{A.3}$$

Proof. It is an exercise to check that the coface and codegeneracy maps satisfy these relations. We have now to check the universal property of the quotient. Let $G \colon F\mathcal{D} \to \mathcal{C}$ be a functor such that the maps $G(\delta_n^i)$ and $G(\sigma_n^i)$ satisfy the cosimplicial identities. Then define $\overline{G} \colon \mathbf{\Delta} \to \mathcal{C}$ using the factorization epi-mono provided in Theorem A.1.3; the simplicial identities allows to check functoriality of this definition. The uniqueness is, finally, clear. □

Definition A.1.6. Let $\mathcal{C}$ be any category. A cosimplicial object in $\mathcal{C}$ is a functor $F \colon \mathbf{\Delta} \to \mathcal{C}$; a simplicial object is $\mathcal{C}$ is a functor $F \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$.

Corollary A.1.7. Let $\mathcal{C}$ be any category. To give a cosimplicial object A in $\mathcal{C}$, it is necessary and sufficient to give a sequence of objects $\{A^n\}_{n\in\mathbb{N}}$ together with coface operators $\partial^i \colon A^{n-1} \to A^n$ and codegeneracy operators $\sigma^i \colon A^{n+1} \to A^n$ $(i = 0, \dots, n)$ satisfying the cosimplicial identities (A.3).

Corollary A.1.8. Let $\mathcal{C}$ be any category. To give a simplicial object in $\mathcal{C}$ it is necessary and sufficient to give a sequence of objects $\{A_n\}_{n\in\mathbb{N}}$ together with face operators $\partial_i \colon A^{n+1} \to A^n$ and degeneracy operators $\sigma_i \colon A_{n-1} \to A_n$ $(i = 0, \dots, n)$ satisfying the simplicial identities

$$\partial_i \partial_j = \partial_{j-1}\partial_i \quad \text{if } i < j$$
$$\sigma_i \sigma_j = \sigma_{j+1}\sigma_i \quad \text{if } i < j$$
$$\partial_i \sigma_j = \begin{cases} \sigma_{j-1}\partial_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \partial_{i-1} & \text{if } j > j+1 \end{cases}$$

Proof. Just observe that these simplicial identities are exactly the dual of the cosimplicial ones. □

A.2 Simplicial sets

Definitions and examples

According to Definition A.1.6, a simplicial set is just a presheaf over $\mathbf{\Delta}$. We will denote this category by **sSet**. The reader should know that this abstract definition won't be of any help in understanding the rich theory of simplicial sets. However, it allows to develop quickly a large amount of theory.

Definition A.2.1. Let $K \in \mathbf{sSet}$. For each $n \in \mathbb{N}$, the set $K_n := K(\mathbf{n})$ is called the set of n-simplices in K; an element $\alpha \in K_n$ is said to be an n-simplex of K.

For example, our experience in classical category theory leads us to consider immediately special simplicial sets, namely those corresponding to representable functors.

Definition A.2.2. Let $n \in \mathbb{N}$. The standard n-simplex Δ^n is by definition the representable functor

$$\Delta^n := \mathrm{Hom}_{\mathbf{\Delta}}(-, \mathbf{n})$$

Definition A.2.3. Let $n \in \mathbb{N}$. For every $0 \le i \le n$, the i-th face of Δ^n is defined to be the subobject represented by the map $d_i := (d^i)^* \colon \Delta^{n-1} \to \Delta^n$.

Proposition A.2.4. Let $K \in \mathbf{sSet}$ be a simplicial set. Then

$$K_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, K)$$

Moreover, if $f : \mathbf{m} \to \mathbf{n}$ is any map in $\mathbf{\Delta}$ and $x \in K_n$ corresponds to $\omega \colon \Delta^n \to K$, then $K(f)(x)$ corresponds to $\omega \circ f_*$.

Proof. It's simply Yoneda lemma. □

Definition A.2.5. Let $K \in \mathbf{sSet}$ be a simplicial set and let ω be a simplex in K. We say that ω is degenerate if it can be written as $K(s)(\omega')$ where $s: \mathbf{m} \to \mathbf{n}$ is a composition of degeneracy maps. We say that ω is non-degenerate if it is not degenerate.

Definition A.2.6. Let $K \in \mathbf{sSet}$ be a simplicial set. We say that K is *finite dimensional* (or of *finite dimension*) if there exists a non-negative integer $N \in \mathbb{N}$ such that every non-degenerate simplex of K is of degree $\leq N$.

Sub-simplicial sets, skeleta

Usually, in presence of some kind of algebraic structure, the notions of generator and sub-structure play an important role. Simplicial sets aren't an exception to this rule.

Let's begin with the notion of sub-simplicial set. If $\mathbf{A}$ is an algebraic category of some sort (e.g. abelian groups) and we consider the category of presheaves $\mathbf{A}^{\mathcal{C}^{\mathrm{op}}}$, we inherit in a natural way a notion of sub-structure directly from the category of $\mathbf{A}$: if $\{F_i\}_{i\in I}$ are sub-presheaves of a given presheaf F it makes sense to consider the intersection of this family. It follows that if we assign (generalized) elements

$$\{m_k: A_k \to F(C_k)\}_{k\in J}$$

it also makes sense to consider the smallest sub-presheaf G of F such that for each $k \in J$ it holds a factorization

$$m_k: A_k \to G(C_k) \to F(C_k)$$

This G will be called the sub-presheaf G of F generated by the elements m_k:

Definition A.2.7. Let $\mathcal{A}$ be a category with pullbacks and let $F: \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$ be an $\mathcal{A}$-valued presheaf. Let $\{m_k: A_k \to F(C_k)\}_{k\in J}$ be a family of generalized elements of F; we define the $\mathcal{A}$-valued presheaf generated by m_k to be the intersection of all the sub-presheaves of F containing the elements m_k.

For simplicial sets, the Theorem A.1.5 allows a simple description of the sub-simplicial set generated by a number of simplexes of a given simplicial set.

Proposition A.2.8. Let X be a simplicial set and let $\{\omega_i\}_{i\in I}$ be a family of simplices of X. The sub-simplicial set Y generated by those simplices is characterized as follows:

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, Y) = \{\omega \in X_n \mid \exists f: \mathbf{n} \to \mathbf{m},\ \exists\ \omega_i \in X_m \text{ such that } \omega = X(f)(\omega_i)\}$$

Proof. It's the standard model-theoretic proof: the right-hand side is contained in all the sub-presheaves of X containing the family $\{\omega_i\}_{i\in I}$; since it is a sub-presheaf on its own, it is exactly the sub-presheaf generated by those elements by definition. □

Definition A.2.9. The *boundary of the standard n-simplex* is by definition the sub-simplicial set $\partial\Delta^n$ of Δ^n generated by the $(n-1)$-simplexes of Δ^n.

Definition A.2.10. Let $n \in \mathbb{N}$ and let $0 \leq i \leq n$ be an integer. Define Λ_i^n to be the sub-simplicial set of Δ^n generated by all the $(n-1)$-simplexes of Δ^n but the i-th face (cfr. Definition A.2.3).

Definition A.2.11. Let K be a simplicial set and let $N \in \mathbb{N}$ be a non-negative integer. We define the N-th skeleton of K to be the sub-simplicial set of K generated by the simplexes of K of degree $\leq N$. We will denote the N-th skeleton of K by $\mathrm{sk}_N(K)$.

Remark A.2.12. $\mathrm{sk}_N(K)$ is also the sub-simplicial set generated by K_N. In particular we have the identity:

$$\partial\Delta^n = \mathrm{sk}_{n-1}(\Delta^n)$$

A.3 Geometric realization

So far we looked at simplicial sets as purely combinatorial objects. In this section, we explain what is the geometric intuition behind them using the geometric realization functor $|\cdot|\colon \mathbf{sSet} \to \mathbf{Top}$. The idea is that this functor uses the combinatorial data encoded by a simplicial set K in order to build a CW-complex $|K|$ whose n-cells are in bijection with non-degenerates n-simplices of K; moreover, this functor has several nice properties which can be often used in order to prove statements about simplicial sets avoiding the hard combinatorics involved.

Remember first of all that in our conventions, **Top** denotes the cartesian closed category of compactly generated and Hausdorff topological spaces. Observe then that we have a cosimplicial object

$$S\colon \mathbf{\Delta} \to \mathbf{Top}$$

sending $\mathbf{n}$ to the standard n-simplex Δ_n in $\mathbb{R}^n$. We can define the geometric realization as the left Kan extension

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, "S"] \arrow[d, "\mathbf{y}"'] & \mathbf{Top} \\ \mathbf{sSet} \arrow[ur, "|\cdot|"'] & \end{tikzcd}$$

Explicitly, this is the unique functor defined by the condition of sending $\mathbf{n}$ in Δ_n and commuting with colimits. As usual, in presence of a left Kan extension, we can produce a right adjoint to $|\cdot|$:

$$\mathrm{Sing}\colon \mathbf{Top} \to \mathbf{sSet}$$

simply by setting

$$\mathrm{Sing}(X) := \mathbf{Top}(\Delta^\bullet, X)$$

Lemma A.3.1. The adjunction $|\cdot| \dashv \mathrm{Sing}$ holds.

Proof. This is a straightforward exercise. □

The following theorem summarizes the main properties of the geometric realization functor $|\cdot|$:

Theorem A.3.2. The geometric realization functor $|\cdot|\colon \mathbf{sSet} \to \mathbf{CGHaus}$ commutes with finite limits and with colimits. Moreover it reflects isomorphisms.

Proof. The previous lemma implies readily that $|\cdot|$ commutes with colimits. A formal argument can be used to show that it commutes with finite products as well, so that we are left to check that it commutes with equalizers. We refer to [GZ67, Ch. III.3] for the proof of this statement, as well as for the proof of the last one. □

It is particularly useful that $|\cdot|$ *reflects* isomorphism. This can be used to show that certain colimits are pushouts in the category **sSet**, without the need for hard combinatorial arguments. The following lemma explains how to do this:

Lemma A.3.3. Let $\mathcal{C}$ be a cocomplete category and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor preserving colimits. If in addition F reflects isomorphisms, then F reflects colimits.

Proof. Straightforward. □

Corollary A.3.4. The geometric realization functor $|\cdot|\colon \mathbf{sSet} \to \mathbf{CGHaus}$ reflects colimits.

Proof. It is a consequence of Theorem A.3.2 and Lemma A.3.3, since **sSet** is cocomplete. □

Exercise A.3.5. Show that there exists a coequalizer diagram as the following:

$$\coprod_{i=0}^{\frac{n(n+1)}{2}} \Delta^{n-2} \rightrightarrows \coprod_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n$$

Corollary A.3.6. For every simplicial set K, its geometric realization $|K|$ is a CW-complex.

Proof. See [GJ99, Proposition I.2.3]. □

A.4 Kan fibrations

As it was claimed in the introduction, the category **sSet** has a homotopical content which is equivalent to the one carried by **Top**; however, the existence of this equivalence probably wouldn't be so interesting if it weren't possible to describe a huge part of this homotopical information in a combinatorial way.

In this section we describe the first bit of the internal homotopy theory of **sSet**: we introduce the class of maps which will become the fibrations of a model structure.

Definition A.4.1. A map of simplicial sets $f : F \to G$ is said to be a *Kan fibration* if it has the right lifting property with respect to every map $\Lambda^n_i \to \Delta^n$, for every $n \in \mathbb{N}$ and every $0 \le i \le n$.

This combinatorial definition is often not too hard to check in practice. For example, a basic fact that is used over and over through these notes is the following:

Lemma A.4.2. Let $f : G \to H$ be a morphism of simplicial groups. If f is a surjection, then f is a Kan fibration (forgetting the group structure).

Proof. See [CSAM29, Lemma 8.2.8]. □

A.5 Anodyne extensions

The very definition of simplicial sets makes clear that **sSet** is a cartesian closed category (because it is a category of presheaves). The internal hom can be described quite explicitly:

$$\mathbf{hom}(X, Y) := \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta^\bullet, Y)$$

Exercise A.5.1. Verify that the adjunction $- \times X \dashv \mathbf{hom}(X, -)$ holds for every X. (*Hint*: every simplicial set can be written as colimit of standard simplices).

Notation. We will denote $\mathbf{hom}(X, Y)$ with the lighter notation Y^X.

A key point in proving that **sSet** has a model structure is to show that **hom** behaves well with respect to the class of Kan fibrations. The main result of this section will imply in particular that whenever K is a Kan complex and L is any simplicial set, then $K^L := \mathbf{hom}(L, K)$ is again a Kan complex. This result has been known since the very beginning of the theory of simplicial sets, and proofs can be found in [GZ67] and in [May69]; however, these proofs use sophisticated combinatorial arguments based on a careful analysis of the non-degenerate simplices of $X \times Y$ via (p, q)-shuffles. Nowadays, the theory of anodyne extensions makes these arguments to appear unnecessarily complicated.

Definition A.5.2. A class of monomorphisms S in **sSet** is said to be *saturated* if it satisfies the following requirements:

1. S is closed under isomorphisms;

2. S is closed under retracts;

3. S is closed under pushouts;

4. S is closed under countable compositions and arbitrary disjoint unions.

Lemma A.5.3. 1. Let $f : F \to G$ be a map between simplicial sets. The class of monomorphisms with the LLP with respect to f is saturated;

2. the intersection of saturated classes in **sSet** is again saturated;

3. if A is a saturated class of monomorphisms, $A = \mathrm{LLP}(\mathrm{RLP}(A))$.

Proof. The first two statement are easy exercises left to the reader. We prove the last one: the inclusion $A \subset \mathrm{LLP}(\mathrm{RLP}(A))$ holds by definition; the other one is a consequence of the small object argument. □

In particular, if we are given a class S of monomorphisms we can consider the saturated class generated by S as the intersection of all the saturated classes containing S (the reader will verify that there is always at least one saturated class containing S).

Theorem A.5.4. The following three classes of monomorphisms have the same saturation:

1. $\mathbf{B}_1$, the class of all inclusions $\Lambda^n_k \to \Delta^n$ for $n \in \mathbb{N}$, $n > 0$ and $0 \leq k \leq n$;

2. $\mathbf{B}_2$, the class of all inclusions

$$(\partial \Delta^n \times \Delta^1) \cup (\Delta^n \times \{e\}) \to \Delta^n \times \Delta^1$$

for $e \in \{0, 1\}$;

3. $\mathbf{B}_3$, the class of all inclusions

$$(Y \times \Delta^1) \cup (X \times \{e\}) \to X \times \Delta^1$$

induced by inclusions $Y \to X$.

Proof. See [GJ99, Proposition I.4.2]. □

Corollary A.5.5. A map is a fibration if and only if it has the RLP with respect to all anodyne extensions. Similarly, a monomorphism is anodyne if and only if it has the LLP with respect to every fibration.

Proof. If f has the RLP with respect to all anodyne extensions, it has in particular the RLP with respect to all the horn inclusions $\Lambda^n_k \subset \Delta^n$, $n \in \mathbb{N}$, $n > 0$ and $0 \leq k \leq n$; hence f is a fibration. Conversely, if f is a fibration, then the class of monomorphisms S_f having the LLP with respect to f is a saturated class thanks to Lemma A.5.3.1; since it contains $\mathbf{B}_1$, it follows that S_f contains all the anodyne extensions, hence the thesis.

For the second statement, denote by A the class of anodyne extensions. Then we have that A is saturated by definition and moreover if we denote by F the class of Kan fibrations, we have

$$F := \mathrm{RLP}(A)$$

so that

$$A = \mathrm{LLP}(\mathrm{RLP}(A)) = \mathrm{LLP}(F)$$

because of Lemma A.5.3.3. □

Corollary A.5.6. If $i\colon K \to L$ is an anodyne extension and $j\colon X \to Y$ is an arbitrary inclusion then the induced map

$$i * j\colon (K \times Y) \coprod_{K \times X} (L \times X) \to L \times Y$$

is anodyne.

Proof. We fix j and consider the class T_j of monomorphisms $f : K' \to L'$ such that $f * j$ is anodyne. We claim that T_j is saturated; indeed, adjoint nonsense shows that a monomorphism $f : K' \to L'$ is in T_j if and only if it has the LLP with respect to

$$F^L \to F^K \times_{G^K} G^L$$

for every Kan fibration $F \to G$. Lemma A.5.3.1 and .2 imply then that T_j is saturated. It can be checked that it contains the class $\mathbf{B}_2$ (see [GJ99, Corollary I.4.6] for this), hence we conclude from Theorem A.5.4 that T_j contains all the anodyne extensions. □

Corollary A.5.7. If $i\colon K \to L$ is an inclusion and $p\colon X \to Y$ is a Kan fibration, then the natural map

$$i \star p\colon X^L \to X^K \times_{Y^K} Y^L$$

is again a Kan fibration.

Proof. Corollary A.5.5 implies that the map $i \star p$ is a Kan fibration if and only if it has the RLP with respect to every anodyne extension $j\colon A \to B$. Adjoint nonsense shows that this is equivalent to say that p has the RLP with respect to every map

$$i * j\colon (K \times B) \coprod_{K \times A} (L \times A) \to L \times B$$

Since $A \to B$ is anodyne, Corollary A.5.6 implies that $i * j$ is anodyne, so that Corollary A.5.5 again shows that $p \in \mathrm{RLP}(i * j)$, completing the proof. □

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Appendix **B**

Algebras and modules in category theory

The goal of this chapter is to review the classical categorical tools developed in order to deal with notions of "algebra" and "module" in full generality. To the author's best knowledge there are, in the classical setting, two ways of approaching the problem: via the notion of monad and via the notion of internal monoid (module, algebra, etc.). The two approaches are not equivalent and they are both useful to get a complete understanding of the subject.

A monad can be thought, roughly speaking, as an endofunctor detecting objects allowing a certain kind of algebraic structure. This structure is highly variable and depend on the monad itself. More precisely, if we fix a monad T, we can form a category of T-algebras, and this category comes equipped with a forgetful functor to the starting category; moreover, this forgetful functor has a left adjoint, playing the role of a "free T-algebra functor".

On the other side, the internalization is a process that allows to associate to a monoidal category various algebraic categories, corresponding to the notion of monoid, module over a monoid, algebra over a monoid and so on. In this case, the kind of algebraic structure is determined by the data we choose and the diagrams we require to be commutative. These algebraic categories come with a forgetful functor over the starting category, and this forgetful functor often has good categorical properties (it often creates limits and colimits, it may have a left adjoint and so on).

This chapter comes with a natural subdivision in two part. In each of them, we will stick to the following pattern:

1. definition of the general framework (i.e. monads, monoidal categories);
2. definition of algebraic objects of a certain kind; these will form a category $\mathcal{A}$ equipped with a forgetful functor $U\colon \mathcal{A} \to \mathcal{C}$, where $\mathcal{C}$ is the category we started with;
3. categorical properties of the forgetful functor $U\colon \mathcal{A} \to \mathcal{C}$;
4. homotopical properties of the forgetful functor $U\colon \mathcal{A} \to \mathcal{C}$.

B.1 The theory of monads

Definitions and first properties

Definition B.1.1. Let $\mathcal{C}$ be a category. A monad over $\mathcal{C}$ is a triple (T, μ, η) where

$$T : \mathcal{C} \to \mathcal{C}$$

is a functor and

$$\mu\colon T^2 \to T, \qquad \eta\colon \mathrm{Id}_{\mathcal{C}} \to T$$

are natural transformations, making the following diagrams to commute:

$$
\begin{tikzcd}
T^3 \arrow[r, "\mu_T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta_T"'] \arrow[dl, equal] \\
& T &
\end{tikzcd}
$$

Lemma B.1.2. Let $(\mathcal{C}, \mathcal{D}, F, G, \eta, \varepsilon)$ be an adjunction. Then $(GF, G\varepsilon_F, \eta)$ is a monad over $\mathcal{C}$.

Proof. Essentially, this is a consequence of the triangular identities for an adjunction. □

An interesting fact is that Lemma B.1.2 has a converse: every monad arises from an adjunction. More precisely, there is a whole category of adjunctions on $\mathcal{C}$ giving rise to the same monad T. The classical constructions of Eilenberg-Moore and Kleisli identify final and initial objects of such category.

Eilenberg-Moore algebras for a monad

Definition B.1.3. Let $\mathcal{C}$ be a category and let (T, μ, η) be a monad over $\mathcal{C}$. A *(Eilenberg – Moore)* algebra for T (or a (*Eilenberg – Moore*) T-algebra) is a pair (A, φ) where $A \in \mathrm{Ob}(\mathcal{C})$ and $\varphi : T(A) \to A$ is a morphism making the following diagrams to commute:

$$
\begin{tikzcd}
T^2(A) \arrow[r, "\mu_A"] \arrow[d, "T\varphi"'] & T(A) \arrow[d, "\varphi"] \\
T(A) \arrow[r, "\varphi"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "T\eta_A"] \arrow[dr, equal] & T(A) \arrow[d, "\varphi"] \\
& A
\end{tikzcd}
$$

Definition B.1.4. Let $\mathcal{C}$ be a category and let (T, μ, η) be a monad over $\mathcal{C}$. Let (A, φ) and (B, ψ) be T-algebras. A T-morphism from A to B is an arrow $f : A \to B$ in $\mathcal{C}$ such that the diagram

$$
\begin{tikzcd}
T(A) \arrow[r, "\varphi"] \arrow[d, "T(f)"'] & A \arrow[d, "f"] \\
T(B) \arrow[r, "\psi"] & B
\end{tikzcd}
$$

is commutative.

Lemma B.1.5. Let $\mathcal{C}$ be a category and let (T, μ, η) be a monad over $\mathcal{C}$. The T-algebras form a category $\mathcal{C}^T$ and this category is endowed with a forgetful functor $G^T : \mathcal{C}^T \to \mathcal{C}$ which creates limits.

Proof. The proof is entirely straightforward. □

Theorem B.1.6. Let $\mathcal{C}$ be a category and let (T, μ, η) be a monad over $\mathcal{C}$. The functor G^T of Lemma B.1.5 has a left adjoint $F^T : \mathcal{C} \to \mathcal{C}^T$ and the adjunction $F^T \dashv G^T$ gives rise (via the construction of Lemma B.1.2) to the same monad (T, μ, η).

Proof. See [Mac71, Thm. VI.2.1]. □

Remark B.1.7. As we were suggesting before, it is possible to show that the Eilenberg – Moore construction $F^T : \mathcal{C} \rightleftarrows \mathcal{C}^T : G^T$ is final among all the adjunction on $\mathcal{C}$ giving rise to the monad T (cfr. [Mac71, Thm VI.3.1]). This subject is particularly rich, in any case: Lemma B.1.5 shows an interesting property: the forgetful functor G^T creates limits. It becomes interesting, to a certain extent, to be able to decide when an adjunction inducing a given monad T is isomorphic to the Eilenberg – Moore construction. This is achieved by Beck's monadicity theorem. We refer to [Mac71, Ch. VI.7] for more details.

Remark B.1.8. From this moment on, when a monad T over a category $\mathcal{C}$ is given, we shall reserve the word "T-algebra" for Eilenberg – Moore T-algebras.

We end this review recalling several well-known examples:

Example B.1.9.

1. Let $F : \mathbf{Set} \rightleftarrows \mathbf{Grp} : G$ be the adjunction where F is the free group functor and G is the forgetful one. Then the resulting monad is called the *free group monad*; its algebras are exactly the groups, and the Eilenberg – Moore construction gives back exactly the free group adjunction;

2. let $F : \mathbf{Set} \rightleftarrows \mathbf{CRing} : G$ be the adjunction where F is the free polynomial ring (with coefficient in $\mathbb{Z}$) functor and G is the forgetful one. Then the algebras for the corresponding monad are exactly commutative rings.

3. More generally, fix a commutative ring (with unit) A and consider the adjunction $F : \mathbf{Set} \rightleftarrows \mathbf{Alg}_A : G$, where F sends a set X to the free polynomial ring $A[X]$, and G is the forgetful functor. Then the algebras for the corresponding adjunction are exactly the A-algebras.

Lifting of model structures (I)

Definition B.1.10. Let $\mathcal{C}$ be a model category and let (T, μ, η) be a monad over $\mathcal{C}$. A T-path object for a T-algebra A is a path object

$$A \xrightarrow{\sim} A^I \to A \times A$$

where A^I is again a T-algebra.

Theorem B.1.11. Let $\mathcal{C}$ be a cofibrantly generated model category where every object is fibrant; let moreover I (resp. J) be a set of generating cofibrations (resp. generating acyclic cofibrations). Assume that (T, μ, η) is a monad over $\mathcal{C}$ and write $I_T := F^T(I)$, $J_T := F^T(J)$ (notations being as in Theorem B.1.6). Then if T commutes with filtered direct limits and each T-algebra has a T-path object, $\mathcal{C}^T$ has a cofibrantly generated model structure where:

- a map f in $\mathcal{C}^T$ is a weak equivalence if and only if $G^T(f)$ is a weak equivalence;
- a map f in $\mathcal{C}^T$ is a fibration if and only if $G^T(f)$ is a fibration;
- a map f in $\mathcal{C}^T$ is a cofibration if and only if it has the LLP with respect to all acyclic fibrations.

Moreover, I_T is a set of generating cofibrations and J_T is a set of generating acyclic cofibrations.

Proof. See [SS00, Lemma 2.3]. □

B.2 Monoidal categories

Definitions

Another way of producing a categorical analogue of modules and algebras is to use the framework of monoidal categories.

The reader is referred for example to [Mac71, Ch. VII] for a detailed exposition of the theory of monoidal categories.

Definition B.2.1. A *monoidal category* is a 6-tuple $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$, where $\mathcal{C}$ is a category,

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

is a bifunctor, $I \in \mathrm{Ob}(\mathcal{C})$ is an object of $\mathcal{C}$ and α, λ, ρ are natural isomorphisms:

$$\begin{aligned} \alpha \colon & - \otimes (- \otimes -) \simeq (- \otimes -) \otimes - \\ \lambda \colon & I \otimes - \simeq \mathrm{Id}_{\mathcal{C}} \\ \rho \colon & - \otimes I \simeq \mathrm{Id}_{\mathcal{C}} \end{aligned}$$

making commutative the coherence diagrams

$$\begin{tikzcd} A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "\mathrm{Id}_{\mathcal{C}} \otimes \alpha"] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \\ A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha \otimes \mathrm{Id}_{\mathcal{C}}"'] \end{tikzcd}$$

and

$$\begin{tikzcd} A \otimes (I \otimes C) \arrow[rr, "\alpha"] \arrow[dr, "\mathrm{Id}_{\mathcal{C}} \otimes \lambda"'] & & (A \otimes I) \otimes C \arrow[dl, "\rho \otimes \mathrm{Id}_{\mathcal{C}}"] \\ & A \otimes C & \end{tikzcd}$$

Finally we require that $\lambda_I = \rho_I \colon I \otimes I \to I$.

These conditions imply that a very large class of diagrams will commute in every monoidal category. These diagrams are those that can be obtained by specializing commutative diagrams in the free monoidal category on one generator. For more details, see [Mac71, Ch. VII.2].

Remark B.2.2. We will systematically abuse notations and refer to a monoidal category simply as a triple $(\mathcal{C}, \otimes, I)$, omitting the natural isomorphisms.

Definition B.2.3. A monoidal category $(\mathcal{C}, \otimes, I)$ is said to by symmetric if it is equipped with natural isomorphisms

$$\gamma_{A,B} \colon A \otimes B \to B \otimes A$$

such that the diagrams

$$\gamma_{A,B} \circ \gamma_{B,A} = \mathrm{Id}_{\mathcal{C}}, \qquad \rho_B = \lambda_B \gamma_{B,I}$$

$$\begin{tikzcd} A \otimes (B \otimes C) \arrow[r, "\alpha"] \arrow[d, "\mathrm{Id}_{\mathcal{C}} \otimes \gamma"] & (A \otimes B) \otimes C \arrow[r, "\gamma"] & C \otimes (A \otimes B) \arrow[d, "\alpha"] \\ A \otimes (C \otimes B) \arrow[r, "\alpha"] & (A \otimes C) \otimes B \arrow[r, "\gamma \otimes \mathrm{Id}_{\mathcal{C}}"] & (C \otimes A) \otimes B \end{tikzcd}$$

commute.

Definition B.2.4. A monoidal category $(\mathcal{C}, \otimes, I)$ is said to be *closed* if for every $X \in \mathrm{Ob}(\mathcal{C})$ the functor

$$- \otimes X : \mathcal{C} \to \mathcal{C}$$

has a *specified* right adjoint.

Definition B.2.5. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. An internal monoid in $\mathcal{C}$ is a triple (M, μ, η) where $\mu: M \otimes M \to M$ and $\eta: I \to M$ are morphisms in $\mathcal{C}$ making the following diagrams to commute:

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[r, "\alpha"] \arrow[d, "\mu\otimes 1"] & M \otimes (M \otimes M) \arrow[r, "1\otimes\mu"] & M \otimes M \arrow[d, "\mu"] \\
M \otimes M \arrow[rr, "\mu"] & & M
\end{tikzcd}$$

$$\begin{tikzcd}
M \otimes I \arrow[r, "\mathrm{id}_M\otimes\eta"] \arrow[dr, "\lambda_M"'] & M \otimes M \arrow[d, "\mu"] & I \otimes M \arrow[l, "\eta\otimes\mathrm{id}_M"'] \arrow[dl, "\rho_M"] \\
& M &
\end{tikzcd}$$

Monoids and modules

Definition B.2.6. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. Let (M_1, μ_1, η_1) and (M_2, μ_2, η_2) be two internal monoids in $\mathcal{C}$. A morphism of monoid from M_1 to M_2 is an arrow $f : M_1 \to M_2$ in $\mathcal{C}$ making commutative the following diagrams

$$\begin{tikzcd}
M_1 \otimes M_1 \arrow[r, "\mu_1"] \arrow[d, "f\otimes f"] & M_1 \arrow[d, "f"] \\
M_2 \otimes M_2 \arrow[r, "\mu_2"] & M_2
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\eta_1"] \arrow[dr, "\eta_2"'] & M_1 \arrow[d, "f"] \\
& M_2
\end{tikzcd}$$

Definition B.2.7. Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category and fix an internal monoid (R, μ, η). An internal left R-module is a pair (M, φ), where $M \in \mathrm{Ob}(\mathcal{C})$ and

$$\varphi : R \times M \to M$$

is a morphism in $\mathcal{C}$ making the following diagrams commutative:

$$\begin{tikzcd}
(R \otimes R) \otimes M \arrow[r, "\alpha"] \arrow[d, "\mu\otimes\mathrm{id}_M"] & R \otimes (R \otimes M) \arrow[r, "\mathrm{id}_R\otimes\varphi"] & R \otimes M \arrow[d, "\varphi"] \\
R \otimes M \arrow[rr, "\varphi"] & & M
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes M \arrow[r, "\eta\otimes\mathrm{id}_M"] \arrow[dr, "\rho_M"'] & R \otimes M \arrow[d, "\varphi"] \\
& M
\end{tikzcd}$$

Definition B.2.8. A map of internal modules is ....

A common feature of all these constructions is a certain well-behavior with respect to limits and colimits. We briefly recall the main results:

Theorem B.2.9. If the monoidal category $(\mathcal{C}, \otimes, I)$ has (countable) coproducts and the functors $- \otimes A$, $A \otimes -$ preserve such coproducts for every $A \in \mathrm{Ob}(\mathcal{C})$, the forgetful functor $\mathbf{Mon}_{\mathcal{C}} \to \mathcal{C}$ has a left adjoint. If moreover the category is symmetric monoidal and has coequalizers, the forgetful functor $\mathbf{CMon}_{\mathcal{C}} \to \mathcal{C}$ from commutative monoids has a left adjoint.

Proof. For the first part we refer to [Mac71, Thm. VII.3.2]. The second part is similar to the first, in an obvious way. □

We give a more detailed proof of the following:

Theorem B.2.10. Let $(\mathbf{C}, \otimes, 1)$ be a closed symmetric monoidal category. Let (R, μ, η) be a monoid in $\mathbf{C}$ and consider the category $\mathbf{Mod}_R$ of R-modules. The forgetful functor $\mathcal{U}\colon \mathbf{Mod}_R \to \mathbf{C}$ creates limits and colimits.

Proof. Let $\mathbf{I}$ a small category and let $\mathcal{F}\colon \mathbf{I} \to \mathbf{Mod}_R$ be a diagram of type $\mathbf{I}$ in $\mathbf{Mod}_R$. Write $(M_i, \rho_i) := \mathcal{F}(i)$ for any $i \in \mathbf{I}$ and if $u\colon i \to j$ is an arrow in $\mathbf{I}$, then let $\varphi_u := \mathcal{F}(u)\colon M_i \to M_j$. Let $\mathcal{U}\colon \mathbf{Mod}_R \to \mathbf{C}$ be the obvious forgetful functor and assume that $\mathcal{U} \circ \mathcal{F}$ has colimit in $\mathbf{C}$. Let (M, ψ_i) be the universal co-cone in $\mathbf{C}$. Since $R \otimes - \cong - \otimes R \dashv [R, -]$ (the internal hom), it follows that $R \otimes -$ preserves colimits and so we can consider the following diagram:

$$\begin{tikzcd}
 & R \otimes M_i \arrow[dl, "1\otimes\varphi_u"'] \arrow[dr, "1\otimes\psi_i"] \arrow[dd, "\rho_i"] & \\
R \otimes M_j \arrow[rr, "1\otimes\psi_j"] \arrow[dd, "\rho_j"'] & & R \otimes M \arrow[dd, dotted, "\rho"] \\
 & M_i \arrow[dl, "\varphi_u"] \arrow[dr, "\psi_i"] & \\
M_j \arrow[rr, "\psi_j"'] & & M
\end{tikzcd}$$

Now the maps $\psi_i \circ \rho_i \colon R \otimes M_i \to M$ produces a co-cone over M and so universal property of $(R \otimes M, 1 \otimes \psi_i)$ give a unique map $\rho\colon R \otimes M \to M$. We claim that this map induce an R-module structure over M. In fact, we have the following diagram:

$$\begin{tikzcd}
R \otimes (R \otimes M_i) \arrow[rr, "\alpha"] \arrow[dr, "1\otimes(1\otimes\psi_i)"] \arrow[ddd, "1\otimes\rho_i"'] & & (R \otimes R) \otimes M_i \arrow[rr, "\mu\otimes 1"] \arrow[d, "(1\otimes 1)\otimes\psi_i"] & & R \otimes M_i \arrow[dl, "1\otimes\psi_i"] \arrow[ddd, "\rho_i"] \\
 & R \otimes (R \otimes M) \arrow[r, "\alpha"] \arrow[d, "1\otimes\rho"] & (R \otimes R) \otimes A \arrow[r, "\mu\otimes 1"] & R \otimes A \arrow[d, "\rho"] & \\
 & R \otimes M \arrow[rr, "\rho"] & & M & \\
R \otimes M_i \arrow[ur, "1\otimes\psi_i"] \arrow[rrrr, "\rho_i"] & & & & M_i \arrow[ul, "\psi_i"']
\end{tikzcd}$$

We know that every square commutes but the central one. Thus a simple chase reveals that

$$\begin{aligned}\rho \circ (\mu \otimes 1) \circ \alpha \circ (1 \otimes (1 \otimes \psi_i)) &= \psi_i \circ \rho_i \circ (\mu \otimes 1) \circ \alpha \\ &= \psi_i \circ \rho_i \circ (1 \otimes \rho_i) \\ &= \rho \circ (1 \otimes \rho) \circ (1 \otimes (1 \otimes \psi_i))\end{aligned}$$

Since $R \otimes -$ is left adjoint, $R \otimes (R \otimes M)$ is the colimit of $R \otimes (R \otimes M_i)$ and so universal property of this object shows now that

$$\rho \circ (\mu \otimes 1) \circ \alpha = \rho \circ (1 \otimes \rho)$$

Similarly we have

$$\begin{tikzcd}
 & R \otimes M_i \arrow[dr, "1\otimes\psi_i"] \arrow[dd, "1\otimes\psi_i"'] & \\
1 \otimes M_i \arrow[ur, "\eta\otimes 1"] \arrow[rr, "\lambda"] \arrow[dd, "1\otimes\psi_i"'] & & M_i \arrow[dd, "\psi_i"] \\
 & R \otimes M \arrow[dr, "\rho"] & \\
1 \otimes M \arrow[ur, "\eta\otimes 1"] \arrow[rr, "\lambda"] & & M
\end{tikzcd}$$

Now $1 \otimes M$ is the colimit of $1 \otimes M_i$ and so

$$\begin{aligned}\lambda \circ (1 \otimes \psi_i) &= \psi_i \circ (1 \otimes \psi_i) \circ (\eta \circ 1) \\ &= \rho \circ (\eta \otimes 1) \circ (1 \otimes \psi_i)\end{aligned}$$

implies, via universal property, that

$$\rho \circ (\eta \times 1) = \lambda$$

The other identity is proven in the same way. Therefore we showed that (M, ρ) is an R-module and that the maps ψ_i are morphisms of R-modules. We check that (M, ρ) together with the maps ψ_i form a universal co-cone in $\mathbf{Mod}_R$. Let N be any other R-module and let $\vartheta_i \colon M_i \to N$ be a co-cone over N in $\mathbf{Mod}_R$. Let $\vartheta \colon M \to N$ be the unique map in $\mathbf{C}$ such that $\vartheta \circ \psi_i = \vartheta_i$ for all $i \in \mathbf{I}$. Consider the following diagram:

$$\begin{tikzcd}
R \otimes M_i \arrow[rr, "1\otimes\psi_i"] \arrow[dr, "1\otimes\vartheta_i"] \arrow[dd, "\rho_i"'] & & R \otimes M \arrow[dl, "1\otimes\vartheta"'] \arrow[dd, "\rho"] \arrow[ddl, dotted, "\exists!\beta"] \\
 & R \otimes N \arrow[dd, "\rho_N"] & \\
M_i \arrow[rr, "\psi_i"] \arrow[dr, "\vartheta_i"'] & & M \arrow[dl, "\vartheta"] \\
 & N &
\end{tikzcd}$$

Now $R \otimes M$ is the colimit of $R \otimes M_i$ and moreover we have that the maps

$$\rho_N \circ (1 \otimes \vartheta_i) = \vartheta_i \circ \rho_i$$

form a co-cone over N, so that universal property of colimits shows that there is a unique $\beta : R\otimes M \to N$ such that $\beta \circ (1 \otimes \psi_i) = \vartheta_i \circ \rho_i$. Since

$$\begin{aligned} \rho_N \circ (1 \otimes \vartheta) \circ (1 \otimes \psi_i) &= \vartheta \circ \rho \circ (1 \otimes \psi_i) \\ &= \vartheta \circ \psi_i \circ \rho_i \\ &= \vartheta_i \circ \rho_i \end{aligned}$$

and

$$\begin{aligned} \vartheta \circ \rho \circ (1 \otimes \psi_i) &= \vartheta \circ \psi_i \circ \rho_i \\ &= \vartheta_i \circ \rho_i \end{aligned}$$

uniqueness of β implies

$$\vartheta \circ \rho = \beta = \rho_N \circ 1 \otimes \vartheta$$

Therefore ϑ is a map of R-modules and so (M, ψ_i) is the universal co-cone of $\mathcal{F}$. As consequence, if $\mathbf{C}$ is co-complete, also $\mathbf{Mod}_R$ is co-complete.

Next, we show that the same thing holds for limits instead of colimits. Therefore, with the same notations as before, assume that $\mathcal{U} \circ \mathcal{F}$ has limit in $\mathbf{C}$ and let (M, τ_i) be the universal cone in $\mathbf{C}$. Consider the following diagram:

$$\begin{tikzcd}
& & R\otimes M_j \arrow[rr, "\rho_j"] & & M_j \\
& R\otimes M_i \arrow[ur, "1\otimes\varphi_u"] \arrow[rr, "\rho_i"] & & M_i \arrow[ur, "\varphi_u"] & \\
R\otimes M \arrow[uurr, bend left, "1\otimes\tau_j"] \arrow[ur, "1\otimes\tau_i"] \arrow[rr, dotted, "\exists! l"] & & \varprojlim R\otimes M_i \arrow[ul, "\sigma_i"] \arrow[uu, "\sigma_j"] \arrow[rr, dotted, "\exists!\sigma"] & & M \arrow[ul, "\tau_i"] \arrow[uu, "\tau_j"']
\end{tikzcd}$$

We applied the obvious universal properties to get the maps σ and l. Set $\rho := \sigma \circ l$. We claim that $\rho : R \otimes M \to M$ determines a structure of R-module over M. Let $P := \varprojlim R \otimes M_i$ and consider the following diagram:

$$\begin{tikzcd}
R\otimes(R\otimes M_i) \arrow[rr, "\alpha"] \arrow[dddd, "1\otimes\rho_i"] & & (R\otimes R)\otimes M_i \arrow[rr, "\mu\otimes 1"] & & R\otimes M_i \arrow[dddd, "\rho_i"] \\
& R\otimes(R\otimes M) \arrow[ul, "1\otimes(1\otimes\tau_i)"] \arrow[r, "\alpha"] \arrow[dd, "1\otimes\rho"] & (R\otimes R)\otimes M \arrow[u, "(1\otimes 1)\otimes\tau_i"] \arrow[r, "\mu\otimes 1"] & R\otimes M \arrow[ur, "1\otimes\tau_i"] \arrow[d, "l"] & \\
& & & P \arrow[uur, "\sigma_i"'] \arrow[d, "\sigma"] & \\
& R\otimes M \arrow[r, "l"] \arrow[dl, "1\otimes\tau_i"'] & P \arrow[r, "\sigma"] \arrow[dll, "\sigma_i"] & M \arrow[dr, "\tau_i"] & \\
R\otimes M_i \arrow[rrrr, "\rho_i"] & & & & M_i
\end{tikzcd}$$

By universal property of M, in order to check that

$$\rho \circ (\mu \otimes 1) \circ \alpha = \rho \circ (1 \otimes \rho)$$

we only need to check that

$$\tau_i \circ \rho \circ (\mu \otimes 1) \circ \alpha = \tau_i \circ \rho \circ (1 \otimes \rho)$$

for all $i \in \mathbf{I}$. However

$$\begin{aligned} \tau_i \circ \rho &= \tau_i \circ \sigma \circ l = \rho_i \circ \sigma_i \circ l \\ &= \rho_i \circ (1 \otimes \tau_i) \end{aligned}$$

and so

$$\begin{aligned} \tau_i \circ \rho \circ (\mu \otimes 1) \circ \alpha &= \rho_i \circ (1 \otimes \tau_i) \circ (\mu \otimes 1) \circ \alpha \\ &= \rho_i \circ (1 \otimes \rho_i) \circ (1 \otimes (1 \otimes \tau_i)) \\ &= \rho_i \circ (1 \otimes \tau_i) \circ (1 \otimes \rho) \\ &= \rho_i \circ \sigma_i \circ l \circ (1 \otimes \rho) \\ &= \tau_i \circ \sigma \circ l \circ (1 \otimes \rho) \\ &= \tau_i \circ \rho \circ (1 \otimes \rho) \end{aligned}$$

showing that associativity holds. To check the unit axioms, one proceeds exactly in the same way, but we omit the details (we already used this technique twice, even though in the limit situation it requires one more step).

Therefore this shows that (M, ρ) is an R-module and that the maps $\tau_i : M \to M_i$ are morphisms of R-modules. We only need to show that (M, τ_i) is a universal cone in $\mathbf{Mod}_R$. Let (N, γ_i) a cone in $\mathbf{Mod}_R$ and let $\gamma : N \to M$ be the unique map in $\mathbf{C}$ such that $\tau_i \circ \gamma = \gamma_i$. We want to show that γ is a map of R-modules. Consider the following diagram:

$$\begin{tikzcd}
 & R \otimes N \arrow[dl, "1\otimes\gamma"'] \arrow[dr, "1\otimes\gamma_i"] \arrow[dd, "\rho_N"] & \\
R \otimes M \arrow[rr, "1\otimes\tau_i"] \arrow[d, "l"] & & R \otimes M_i \arrow[dd, "\rho_i"] \\
P \arrow[urr, "\sigma_i"] \arrow[d, "\sigma"] & N \arrow[dl, "\gamma"'] \arrow[dr, "\gamma_i"] & \\
M \arrow[rr, "\tau_i"] & & M_i
\end{tikzcd}$$

Now we have $\rho_i \circ (1 \otimes \gamma_i) = \gamma_i \circ \rho_N$ and clearly

$$\varphi_u \circ \rho_i \circ (1 \otimes \gamma_i) = \rho_j \circ (1 \otimes \varphi_u) \circ (1 \otimes \gamma_i) = \rho_j \circ (1 \otimes \gamma_j)$$

so that we obtain a cone on $R \otimes N$; universal property of M produces a unique map $\beta : R \otimes N \to M$ such that $\tau_i \circ \beta = \gamma_i \circ \rho_N$. However

$$\begin{aligned} \tau_i \circ \rho \circ (1 \otimes \gamma) &= \rho_i \circ \sigma_i \circ l \circ (1 \otimes \gamma) \\ &= \rho_i \circ (1 \otimes \tau_i) \circ (1 \otimes \gamma) \\ &= \gamma_i \circ \rho_N \end{aligned}$$

and similarly

$$\begin{aligned} \tau_i \circ \gamma \circ \rho_N &= \rho_i \circ \sigma_i \circ l \circ (1 \otimes \gamma) \\ &= \rho_i \circ (1 \otimes \tau_i) \circ (1 \otimes \gamma) \\ &= \rho_i \circ (1 \otimes \gamma_i) = \gamma_i \circ \rho_N \end{aligned}$$

and so uniqueness of β yields

$$\rho \circ (1 \otimes \gamma) = \beta = \gamma \circ (1 \otimes \tau_i)$$

concluding this part of proof. As consequence, we see that if **C** is complete, then $\mathbf{Mod}_R$ is complete. We can state what ☐

Lifting of model structures (II)

Until this point we emphasized that taking internal monoids, modules or algebras is usually a well-behaved operation with respect to limits and colimits. We turn now to a more homotopical problem: namely, we will assume to work in a monoidal model category (the definition is recalled); then we ask whether it is possible to lift the model structure to the category of monoids, or to the category of modules over a given monoid. The main reference here is the article [SS00], and we will simply recall the main results.

Definition B.2.11. Let $(\mathcal{M}, \otimes, I)$ be a monoidal model category; denote by $S := \textsc{Cofib} \cap W$ the set of trivial cofibrations and let

$$S' := \{f \otimes \mathrm{id}_Z : A \otimes Z \to B \otimes Z \mid f \in S',\ Z \in \mathrm{Ob}(\mathcal{M})\}$$

$\mathcal{M}$ is said to satisfy the *monoid axiom* if every arrow in $S'-\mathrm{cof}_{\mathrm{reg}}$ is a weak equivalence.

Lemma B.2.12. Let $(\mathcal{M}, \otimes, I)$ be a monoidal model category. If every object of $\mathcal{M}$ is cofibrant, the monoid axiom is satisfied.

Proof. Since acyclic cofibrations are closed under transfinite composition and pushout, it will be sufficient to show that $f \otimes \mathrm{id}_Z$ is an acyclic cofibration if f is so. However, $\emptyset \otimes A \simeq \emptyset$ because $- \otimes A$ preserves colimits, and so the pushout product axiom implies immediately the thesis. ☐

Theorem B.2.13. Let $(\mathcal{M}, \otimes, I)$ be a cofibrantly generated monoidal model category. Assume that every object in $\mathcal{M}$ is small relative to the whole category and that $\mathcal{M}$ satisfies the monoid axiom. Let R be a commutative monoid in $\mathcal{M}$, let $\mathbf{Mod}_R$ be the category of internal R-modules and let $U : \mathbf{Mod}_R \to \mathcal{M}$ be the natural forgetful functor. Then the followings hold:

1. the category of R-modules is a cofibrantly generated monoidal model category satisfying the monoid axiom, where a map f is a weak equivalence (resp. a fibration) if and only if $U(f)$ is a weak equivalence (resp. a fibration), and a map is a cofibration if and only if it has the LLP with respect to all the acyclic fibrations.

2. The category of R-algebras is a cofibrantly generated model category where a map f is a weak equivalence (resp. a fibration) if and only if $U(f)$ is a weak equivalence (resp. a fibration), and a map is a cofibration if and only if it has the LLP with respect to all the acyclic fibrations.

References

[Mac71] S. MacLane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics 5, New York: Springer-Verlag, 1971.

[SS00] S. Schwede and B. E. Shipley, “Algebras and modules in monoidal model categories”, in: *Proceedings of London Mathematical Society* 80 (2000).

APPENDIX **C**

Enriched category theory

This appendix is not meant to prove anything in detail. We will collect here the definitions commonly used in Enriched Category Theory in order to recall them in an easier way. However, we will give appropriate references to every statement.

C.1 Enriched categories

We will follow the exposition given in [Kel82].

Definition C.1.1. Let $\mathbb{V} = (\mathcal{V}, \otimes, I)$ be a monoidal category (see Definition B.2.1). A $\mathbb{V}$-category consists of the following data:

1. a class of *objects* $\mathcal{A}$;
2. for each pair of objects $A, B \in \mathcal{A}$, the given of a *hom-object* $\mathcal{A}(A, B; \mathbb{V}) \in \mathcal{V}$;
3. for each triple of objects $A, B, C \in \mathcal{A}$ a composition law
$$\mu_{A,B,C} : \mathcal{A}(A, B; \mathbb{V}) \otimes \mathcal{A}(B, C; \mathbb{V}) \to \mathcal{A}(A, C; \mathbb{V})$$
4. for each object $A \in \mathcal{A}$ an identity
$$j_A : I \to \mathcal{A}(A, A; \mathbb{V})$$

Moreover, we require this data to satisfy the following compatibility relations:

1. for each 4-tuple of objects $A, B, C, D \in \mathcal{A}$ the pentagonal diagram
$$\begin{tikzcd}
(\mathcal{A}(A,B;\mathbb{V}) \otimes \mathcal{A}(B,C;\mathbb{V})) \otimes \mathcal{A}(C,D;\mathbb{V}) \arrow[r, "\mu_{A,B,C}\otimes 1"] \arrow[d, "\alpha"'] & \mathcal{A}(A,C;\mathbb{V}) \otimes \mathcal{A}(C,D;\mathbb{V}) \arrow[dd, "\mu_{A,C,D}"] \\
\mathcal{A}(A,B;\mathbb{V}) \otimes (\mathcal{A}(B,C;\mathbb{V}) \otimes \mathcal{A}(C,D;\mathbb{V})) \arrow[d, "1\otimes\mu_{B,C,D}"'] & \\
\mathcal{A}(A,B;\mathbb{V}) \otimes \mathcal{A}(B,D;\mathbb{V}) \arrow[r, "\mu_{A,B,D}"] & \mathcal{A}(A,D;\mathbb{V})
\end{tikzcd}$$
commutes;

2. for each pair of objects $A, B \in \mathcal{A}$ the diagram

$$\begin{tikzcd}
\mathcal{A}(A,A) \otimes \mathcal{A}(A,B) \arrow[rr, "\mu_{A,A,B}"] & & \mathcal{A}(A,B) & & \mathcal{A}(A,B) \otimes \mathcal{A}(B,B) \arrow[ll, "\mu_{A,B,B}"'] \\
I \otimes \mathcal{A}(A,B) \arrow[u, "j_A \otimes 1"] \arrow[urr, "\lambda"'] & & & & \mathcal{A}(A,B) \otimes I \arrow[u, "1 \otimes j_B"'] \arrow[ull, "\rho"]
\end{tikzcd}$$

commutes.

C.2 Tensor and cotensor

C.3 Induced structures

Theorem C.3.1. Let $\mathbb{V}_1 = (\mathbf{V}_1, \otimes_1, I_1)$ and $\mathbb{V}_2 = (\mathbf{V}_2, \otimes_2, I_2)$ be two monoidal categories. Let $F\colon \mathbb{V}_1 \to \mathbb{V}_2$ be a strong monoidal functor. If $\mathbf{A}$ is a $\mathbb{V}_1$-category, then setting for every $X, Y \in \mathrm{Ob}(\mathbf{A})$

$$\mathbf{V}_2(X,Y) := F(\mathbf{V}_1(X,Y))$$

gives to $\mathbf{A}$ a $\mathbb{V}_2$-enriched structure. Moreover, if F is right adjoint to G and $\mathbf{A}$ is enriched with tensor and cotensor over $\mathbb{V}_1$, it is enriched with tensor and cotensor over $\mathbb{V}_2$:

$$A \otimes_{\mathbb{V}_2} V := A \otimes_{\mathbb{V}_1} G(V), \qquad A^V_{\mathbb{V}_2} := A^{G(V)}_{\mathbb{V}_1}$$

References

[Kel82] G. M. Kelly, *Basic Concepts of Enriched Category Theory*, London Mathematical Society Lecture Note Series 64, Cambridge University Press, 1982.