Supergeometry in mathematics and physics

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0 Introduction

Supergeometry is a geometric tool which is supposed to describe supersymmetry ("symmetry between bosons and fermions") in physics. In the impressive arsenal of physico-geometric tools, it occupies a special, somewhat mysterious place.

To an outside observer, it projects a false sense of accessibility, presenting itself as a simple modification of the familiar formalism. However, this is only an appearance. Like a bewitched place, it is protected not by barriers but by something less tangible and therefore much more powerful. It is not even so clear where this place is localized. A naive attempt to "forge ahead" may encounter little resistance but may also end up missing the real point.

Indeed, the physical origin of supergeometry lies in the comparison of the Bose and Fermi statistics for identical particles. So mathematically, supergeometry zooms in on our mental habits related to questions of commutativity and identity. These habits predate quantum physics and therefore are deeply ingrained, but they do not correspond to the ultimate physical reality. The new way of thinking, offered by supergeometry in mathematics and by supersymmetry in physics, requires changes that cannot be reduced to simple recipes (such as "introducing $\pm$ signs here and there" which is, in the popular mind, the best known feature of supergeometry).

Put diferently, if introducing $\pm$ signs can take us so far, then something is happening that we truly don't understand. In addition, one can almost say that mathematicians and physicsts mean different things when they speak about supergeometry. This contributes to the enduring feeling of wonder and mystery surrounding this subject.

The present article (chapter) is subdivided into three parts. Following the specifications for the volume, §1 presents a very short but self-contained exposition of supergeometry as it is generally understood by mathematicians. Not surprisingly, the presentation revolves around the Koszul sign rule (1.1.1) which appears first at the level of elements of a ring, and later again at the level of categories. More systematic expositions can be found in the books [33] [16] and in many articles, esp. [9].

In §2 we discuss the aspects of supergeometry that are used by physicists in relation to supersymmetry. From the mathematical point of view, this amounts to much more than the study of super-manifolds or of the Koszul rule. Some of the basic references are [49] [33] [16] [10]. The entry point for a mathematician here could be found in the idea of taking natural "square roots" of familiar mathematical and physical quantities.

In our presentation, we introduce the abstract concept of a quadratic space (data of Γ-matrices) of which various situations involving spinors form particular cases. This has the advantage of simplifying the general discussion and also of relating the subject to the mathematical theory of intersections of quadrics. In particular, the familiar dichotomy of complete intersections of quadrics vs. non-complete ones has a direct significance for understanding the "constraints" which are usually imposed on super-fields.

Finally, §3 is devoted to an attempt to uncover deeper roots for the mysterious power of the super-geometric formalism, of which the remarkable consistency of its sign rules is just one manifestation. It seems that the right language to speak about such things is given by homotopy theory. Indeed, this theory provides a systematic modern way to talk about the issues of identity: instead of saying that two things are "the same", we say that they are "homotopic", and specify the homotopy (the precise reason why they should be considered the same). We can also consider homotopies between homotopies and so on. From this point of view, the group $\{\pm 1\}$ of signs howering over supergeometry, is nothing but π_1^{st}, the first homotopy group of the *sphere spectrum* $\mathbb{S}$. As emphasized by Grothendieck in a more categorical language, $\mathbb{S}$ can be seen as the most fundamental homotopy commutative object. This suggests that "mining the sphere spectrum" beyond the first level should lead to generalizations of supergeometry (and possibly, supersymmetry) involving not just signs but, for instance, $\sqrt[24]{1}$ to account for $\pi_3^{\mathrm{st}} = \mathbb{Z}/24$.

My understanding of supergeometry owes a lot to lectures and writings of Y. I. Manin. In particular, the idea of square roots provided by the super formalism, was learned from him long time ago. The homotopy-theoretic considerations of §3 were stimulated by the joint work with N. Ganter [17]. I would also like to thank N. Ganter, N. Gurski and Y. I. Manin for remarks and suggestions on the preliminary versions of this text. This work was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

1 Supergeometry as understood by mathematicians

1.1 Commutative superalgebras

For a mathematician, supergeometry is the study of supermanifolds and superschemes: objects whose rings of functions are commutative superalgebras.

We fix a field $\mathbf{k}$ of characteristic $\neq 2$. By an *associative superalgebra* over $\mathbf{k}$ one means simply a $\mathbb{Z}/2$-graded associative algebra $A = A^{\bar{0}} \oplus A^{\bar{1}}$. Elements of $A^{\bar{0}}$ are called *even* (or *bosonic*), elements of $A^{\bar{1}}$ are called *odd* (or *fermionic*).

An associative superalgebra A is called *commutative*, if it satisfies the Koszul sign rule for commutation of homogeneous elements:

$$ab = (-1)^{\deg(a)\cdot\deg(b)} ba. \qquad (1.1.1)$$

The terms "commutative superalgebra" and "supercommutative algebra" are used withut distinction.

Often, one considers $\mathbb{Z}$-graded supercommutative algebras $A = \bigoplus_{i\in\mathbb{Z}} A^i$, with the same commutation rule (1.1.1) imposed on homogeneous elements.

Examples 1.1.2. (a) Any usual commutative algebra A becomes supercommutative, if put in degree $\bar{0}$. The most important example $A = \mathbf{k}[x_1, \cdots, x_m]$ (the polynomial algebra).

(b) The exterior (Grassmann) algebra $\Lambda[\xi_1, \cdots \xi_n]$ over $\mathbf{k}$, generated by the symbols ξ_i of degree $\bar{1}$, subject only to the relations

$$\xi_i^2 = 0, \quad \xi_i\xi_j = -\xi_j\xi_j, \;\; i \neq j,$$

is supercommutative. It can be seen as the *free supercommutative algebra* on the ξ_i: the relations imposed are the minimal ones to ensure supercommutativity.

(c) If A, B are the commutative superalgebras, then the tensor product $A \otimes_{\mathbf{k}} B$ with the standard product grading $\deg(a \otimes b) = \deg(a) + \deg(b)$ and multiplication

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) \quad = \quad (-1)^{\deg(b_1)\cdot \deg(a_2)} (a_1 \cdot b_1) \otimes (a_2 \cdot b_2)$$

(a_i, b_i homegeneous), is again a supercommutative algebra. For example,

$$\mathbf{k}[x_1, \cdots, x_m] \otimes \Lambda[\xi_1, \cdots, \xi_n], \quad \deg(x_i) = \bar{0}, \ \deg(\xi_j) = \bar{1},$$

is a commutative superalgebra. This is a general form of a free commutative superalgebra on a finite set of even and odd generators.

(d) The de Rham algebra $\Omega^\bullet_X$ of differential forms on a C^∞-maniflold X, is a $\mathbb{Z}$-graded supercommutative algebra over $\mathbb{R}$.

(e) The cohomology algebra $H^\bullet(X, \mathbf{k})$ of any CW-complex X, is a $\mathbb{Z}$-graded supercommutative algebra over $\mathbf{k}$.

The importance and appeal of such studies are based on the following heuristic

1.1.3. Principle of Naturality of Supers. *All constructions and features that make commutative algebras special among all algebras, can be extended to supercommutative algebras, and make them just as special.*

The most important of such features is the relation to geometry: a commutative algebra A can be seen as the algebra of functions on a geometric object $\mathrm{Spec}(A)$ which can be constructed from A and used to build more complicated geometric objects by gluing. Supergeometry (as understood by mathematicians) is the study of similar geometric objects for supercommutative algebras.

1.2 The symmetric monoidal category of super-vector spaces

The main guiding principle for extending properties of usual commutative algebras to supercommutative ones is also sometimes called the Koszul sign rule. It goes like this:

1.2.1. *When we move any quantity (vector, tensor, operation) of parity $p \in \mathbb{Z}/2$ past any other quantity of parity q, this move should be accompanied by multiplication with $(-1)^{pq}$.*

An instance is given by the first formula in Example 1.1.2(c). This rule can be formalized as follows.

Let $\mathrm{SVect}_{\mathbf{k}}$ be the category of *super-vector spaces* over $\mathbf{k}$, i.e., $\mathbb{Z}/2$-graded vector spaces $V = V^{\bar{0}} \oplus V^{\bar{1}}$. This category has a monoidal structure $\otimes$, the usual graded tensor product. The operation $\otimes$ is associative up to natural isomorphisms, and has a unit object $\mathbf{1} = \mathbf{k}$ (put in degree $\bar{0}$). Define the symmetry isomorphisms

$$R_{V,W} : V \otimes W \longrightarrow W \otimes V, \quad v \otimes w \mapsto (-1)^{\deg(v)\cdot \deg(w)} w \otimes v. \tag{1.2.2}$$

Proposition 1.2.3. *The family* $R = (R_{V,W})$ *makes* $\mathrm{SVect}_{\mathbf{k}}, \otimes, \mathbf{1}, R)$ *into a symmetric monoidal category.*

For background on symmetric monoidal categories we refer to [32, 7]. A basic example of a symmetric monoidal category is given by the usual category of vector spaces $\mathrm{Vect}_{\mathbf{k}}$, with the usual tensor product, the unit object $\mathbf{1} = \mathbf{k}$, and the symmetry given by $v \otimes w \mapsto w \otimes v$. The meaning of the proposition is that $\mathrm{SVect}_{\mathbf{k}}$ with symmetry (1.2.2) satisfies all the same formal properties as the "familiar" category $\mathrm{Vect}_{\mathbf{k}}$.

If $\dim(V^{\bar{0}}) = m$ and $\dim(V^{\bar{1}}) = n$, we write $\dim(V) = (m|n)$. In particular, we have the standard coordinate superspaces $\mathbf{k}^{m|n}$. We denote by Π the *parity change functor* on $\mathrm{SVect}_{\mathbf{k}}$ given by multiplication with $\mathbf{k}^{0|1}$ on the left.

It is well known that one can develop linear algebra formalism (tensor, symmetric, exterior powers etc.) in any symmetric monoidal $\mathbf{k}$-linear abelian category $(\mathcal{V}, \otimes, \mathbf{1}, R)$. This gives a way to define commutativity. That is, a *commutative algebra in* $\mathcal{V}$ is an object $A \in \mathcal{V}$ together with a morphism $\mu_A : A \otimes A \to A$ satisfying associativity and such that the composition

$$A \otimes A \xrightarrow{R_{A,A}} A \otimes A \xrightarrow{\mu_A} A$$

is equal to μ_A. Given two commutative algebras (A, μ_A) and (B, μ_B), the object $A \otimes B$ is again a commutative algebra with respect to $\mu_{A\otimes B}$ given by the composition

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{Id}\otimes R_{B,A}\otimes \mathrm{Id}} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B.$$

For $\mathcal{V} = \mathrm{SVect}_{\mathbf{k}}$, this gives Example 1.1.2(c).

Remark 1.2.4. Symmetric monoidal categories can be seen as categorical analogs of commutative algebras: instead of the equality $ab = ba$, we now have canonical isomorphisms $V \otimes W \simeq W \otimes V$.

1.3 Superschemes and supermanifolds

Given a supercommutative algebra A, the even part $A^{\bar{0}}$ is commutative, and so has the associated affine scheme $\mathrm{Spec}(A^{\bar{0}})$. Explicitly, it is a ringed space $\big(\underline{\mathrm{Spec}}(A^{\bar{0}}), \mathcal{O}_{\mathrm{Spec}(A^{\bar{0}})}\big)$, where $\underline{\mathrm{Spec}}(A^{\bar{0}})$ is the set of prime ideals in $A^{\bar{0}}$ with the Zariski topology and $\mathcal{O}_{\mathrm{Spec}(A^{\bar{0}})}$ is a sheaf of local rings on this space obtained by localization of $A^{\bar{0}}$. That is, the value of $\mathcal{O}_{\mathrm{Spec}(A^{\bar{0}})}$ on a "principal open set"

$$U_f = \{\mathfrak{p} \in \underline{\mathrm{Spec}}(A^{\bar{0}}) : f \in \mathfrak{p}\}, \quad f \in A^{\bar{0}},$$

is given by $\mathcal{O}_{\mathrm{Spec}(A^{\bar{0}})}()U_f) = A^{\bar{0}}[f^{-1}]$.

Further, $A^{\bar{0}}$ is not only commutative but lies in the center of A as an associative algebra. Therefore, associating to U_f the commutative superalgebra $A[f^{-1}]$, we get a sheaf of commutative superalgebras

$$\mathcal{O}_{\mathrm{Spec}(A)} \;=\; \mathcal{O}^{\bar{0}}_{\mathrm{Spec}(A)} \oplus \mathcal{O}^{\bar{1}}_{\mathrm{Spec}(A)}, \quad \mathcal{O}^{\bar{0}}_{\mathrm{Spec}(A)} = \mathcal{O}_{\mathrm{Spec}(A^{\bar{0}})}$$

on the same topological space $\underline{\mathrm{Spec}}(A^{\bar{0}})$ as before. The pair (ringed space)

$$\mathrm{Spec}(A) \;:=\; \big(\underline{\mathrm{Spec}}(A^{\bar{0}}), \mathcal{O}_{\mathrm{Spec}(A)}\big)$$

is the fundamental geometric object associated to A, see [30, 33].

The stalks of $\mathcal{O}_X$ are commutative superalgebras which, considered as ordinary associative algebras, are *local rings*. Indeed, we have:

Proposition 1.3.1. *If B is a commutative superalgebra such that $B^{\bar{0}}$ is a local ring with maximal ldeal $\mathfrak{m}^{\bar{0}}$, then B itself is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{m}^{\bar{0}} \oplus B^{\bar{1}}$.* $\square$

Working with sheaves of local rings is a fundamental technical feature of Grothendieck's theory of schemes. So we introduce the following

Definition 1.3.2. A *super-locally ringed space over* $\mathbf{k}$ is a pair $X = (\underline{X}, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X = \mathcal{O}_X^{\bar{0}} \oplus \mathcal{O}_X^{\bar{1}}$ is a sheaf of commutative $\mathbf{k}$-superalgebras on $\underline{X}$, with stalks being local rings.

A *morphism* of super-locally ringed spaces $f : (\underline{X}, \mathcal{O}_X) \to (\underline{Y}, \mathcal{O}_Y)$ is a pair $f = (f_\sharp, f^\flat)$, where:

- $f_\sharp : \underline{X} \to \underline{Y}$ is a continuous map of topological spaces.
- $f^\flat : f_\sharp^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ is a morphism of sheaves of commutative superalgebras, which, in addition, takes the maximal ideal of each local ring $(f_\sharp^{-1}\mathcal{O}_Y)_x := \mathcal{O}_{Y, f_\sharp(x)}$, $x \in X$ into the maximal ideal of the local ring $\mathcal{O}_{X,x}$.

The resulting category of super-locally ringed spaces over $\mathbf{k}$ will be denoted $\mathcal{SLRS}_{\mathbf{k}}$

Proposition 1.3.3. *For any two commutative superalgebras A, B we have an isomorphism*

$$\mathrm{Hom}_{\mathcal{SA}_{\mathbf{k}}}(A, B) \;\simeq\; \mathrm{Hom}_{\mathcal{SLRS}_{\mathbf{k}}}(\mathrm{Spec}(B), \mathrm{Spec}(A)).$$

Definition 1.3.4. A *superscheme* over $\mathbf{k}$ is a super-locally ringed space over $\mathbf{k}$ locally isomorphic to $\mathrm{Spec}(A)$ for a commutative superalgebra A. The category $\mathcal{SS}ch_{\mathbf{k}}$ of superschemes over $\mathbf{k}$ is defined to be the full subcategory of $\mathcal{SLRS}_{\mathbf{k}}$ formed by superschemes.

Examples 1.3.5. (a) The *affine superspace* of dimension $(m|n)$ is the superscheme

$$\mathbb{A}^{m|n} \;=\; \mathrm{Spec}\big(\mathbf{k}[x_1, \cdots, x_m] \otimes \Lambda[\xi_1, \cdots, \xi_n]\big).$$

(b) The next class of examples is provided by *algebraic supermanifolds* of dimension $(m|n)$. By definition, they are superschemes locally of the form $\mathrm{Spec}\big(A \otimes \Lambda[\xi_1, \cdots, \xi_n]\big)$, where A is the ring of functions on a smooth m-dimensional affine algebraic variety over $\mathbf{k}$.

Each superscheme $X = (\underline{X}, \mathcal{O}_X)$ gives rise to an ordinary scheme $X^{\bar{0}} = (\underline{X}, \mathcal{O}_X^{\bar{0}}/(\mathcal{O}_X^{\bar{1}})^2)$ called the *even part* of X. If X is an algebraic supermanifold of dimension $(m|n)$, then $X^{\bar{0}}$ is a smooth algebraic variety of dimension m.

As with usual schemes, a superscheme X is defined by its *functor of points* on the category of supercommutative algebras

$$A \ \mapsto \ X(A) \ = \ \mathrm{Hom}_{\mathcal{SS}ch_{\mathbf{k}}}(\mathrm{Spec}(A), X).$$

Differentiable or analytic supermanifolds are similarly defined as locally ringed spaces $X = (X^0, \mathcal{O}_X)$ where X^0 is an ordinary differentiable or analytic manifold and $\mathcal{O}_X$ is a sheaf of commutative superalgebras locally isomorphic to $\mathcal{O}_X \otimes_{\mathbf{k}} \Lambda[\xi_1, \cdots, \xi_n]$. Here $\mathcal{O}_X$ is the sheaf of C^∞ or analytic functions on X and $\mathbf{k} = \mathbb{R}$ for differentiable or real analytic and $\mathbb{C}$ for complex analytic manifolds.

1.4 Lie supergroups and superalgebras

Here is an illustration of the Principle of Naturality of Supers: extension of the formalism of differential geometry to the super-case looks totally straightforward.

Given a $\mathbf{k}$-linear symmetric monoidal category $\mathcal{A}$, one can speak about algebras in $\mathcal{A}$ of any given type, for example Lie algebras. In the case $\mathcal{A} = \mathrm{SVect}$, this gives the familiar concept.

Definition 1.4.1. A *Lie superalgebra* over $\mathbf{k}$ is a $\mathbf{k}$-super-vector space $\mathfrak{g} = \mathfrak{g}^{\bar{0}} \oplus \mathfrak{g}^{\bar{1}}$ equipped with a homogeneous (degree $\bar{0}$) operation $(x, y) \mapsto [x, y]$ satisfying

$$[x, y] = -(-1)^{\deg(x)\cdot\deg(y)}[y, x];$$
$$[x, [y, z]] \ = \ [[x, y], z] \ + \ (-1)^{\deg(x)\cdot\deg(y)}[y, [x, z]].$$

The following examples can also be given in any symmetric monoidal category, we use the case $\mathcal{V} = \mathrm{SVect}_{\mathbf{k}}$.

Examples 1.4.2. (a) If R is any associative superalgebra, we can make it into a Lie superalgebra using the *supercommutator*

$$[x, y] \ = \ x \cdot y - (-1)^{\deg(x)\cdot\deg(y)} y \cdot x.$$

(b) If A is an associative superalgebra, then we have the Lie superalgebra $\mathrm{Der}(A) = \mathrm{Der}^{\bar{0}}(A) \oplus \mathrm{Der}^{\bar{1}}(A)$ of *super-derivations* of A. Here $\mathrm{Der}^i(A)$ is the space of $\mathbf{k}$-linear maps $D : A \to A$ satisfying the *super-Leibniz rule*

$$D(a \cdot b) = D(a) \cdot b + (-1)^{i\cdot\deg(a)} a \cdot D(b).$$

The bracket is given by the supercommutator as in (a).

Given a supermanifold $X = (\underline{X}, \mathcal{O}_X)$ of dimension $(m|n)$ in the smooth, analytic or algebraic category, super-derivations of $\mathcal{O}_X$ form a sheaf T_X of Lie superalgebras on X called the *tangent sheaf*. It is also a sheaf of $\mathcal{O}_X$-modules, locally free of rank $(m|n)$. Its fiber at a point $x \in X^0$ will be denoted $T_x X$ and called the *tangent space to X at x*. It is a super-vector space of dimension $(m|n)$.

A *Lie supergroup*, resp. an *algebraic supergroup* over $\mathbf{k}$, is a group object G in the category of smooth supermanifods, resp. algebraic supermanifolds over $\mathbf{k}$. In particular, it has a unit element $e \in G^0$. The tangent space $T_e G$ is a Lie superalgebra denoted by $\mathfrak{g} = \mathrm{Lie}(G)$. Note that $\mathfrak{g}^{\bar{0}} = \mathrm{Lie}(G^{\bar{0}})$ is the usual Lie algebra corresponding to a Lie or algebraic group. Conversely, given a finite-dimensional Lie algebra $\mathfrak{g}$ and a Lie or algebraic group $G^{\bar{0}}$ such that $\mathrm{Lie}(G^{\bar{0}}) = \mathfrak{g}^{\bar{0}}$, one can integrate it to a Lie or algebraic supergroup G with $\mathrm{Lie}(G) = \mathfrak{g}$.

We will specially use the case when $\mathfrak{g}$ is *nilpotent*, that is, the iterated commutators $[x_1, [x_2, \cdots, x_n]...]$ vanish for n greater than some fixed N (degree of nilpotency). We assume $\mathrm{char}(\mathbf{k}) = 0$. In this case we can associate to $\mathfrak{g}$ a canonical algebraic group $G = e^{\mathfrak{g}}$. by means of the classical Hausdorff series

$$x \cdot y \;=\; x + y + \frac{1}{2}[x, y] + \cdots \tag{1.4.3}$$

This means the following. Suppose we want to find the set-theoretic group of Λ-points $e^{\mathfrak{g}}(A) = \mathrm{Hom}(\mathrm{Spec}(A), e^{\mathfrak{g}})$, where A is a super-commutative algebra. As a set, this group is defined to be $(\mathfrak{g} \otimes_{\mathbf{k}} A)^{\bar{0}}$, which has a structure of an *ordinary* (purely even) nilpotent Lie algebra and therefore can be made into a group by means of the *ordinary* Hausdorff series above. This is the group $e^{\mathfrak{g}}(A)$.

Informally, one says that $e^{\mathfrak{g}}$ "is" the Lie algebra $\mathfrak{g}$ considered as a manifold with the multiplication given by (1.4.3) but the formal meaning is that the arguments in commutators in (1.4.3) always belong to some ordinary Lie algebra.

1.4.4. Pfaff systems and Frobenius theorem. Let X be a supermanifold. A *Pfaff system* in X is a subbundle (i.e., a subsheaf of $\mathcal{O}_X$-modules which is locally a direct summand) $C \subset T_X$. For a Pfaff system C the Lie algebra structure in sections of T_X induces the $\mathcal{O}_X$-linear map known as the *Frobenius pairing*

$$\digamma_C : \Lambda^2_{\mathcal{O}_X} C \longrightarrow T_X / C.$$

A Pfaff system C is called *integrable*, if $\digamma_C = 0$, i.e., C is closed under the bracket of vector fields. In this case we a super-analog of the *Frobenius theorem*: in the smooth or analytic case C can be locally represented as the relative tangent bundle to a submersion of supermanifolds $X \to Y$.

2 Supergeometry as understood by physicists

For a physicist, the really important concept is *supersymmetry*, and supermanifolds per se are of interest only tangentially. For example, they arise as infinite-dimensional spaces of bosonic and fermionic classical fields, over which Feynman integrals are taken. One way (not the only one!) to construct supersymmetric field theories is by using *superspace*, a concept not synonymous with “supermanifold” of mathematicians. In fact, superspace is a supermanifold with a rather special “spinor-conformal" structure.

2.1 Idea of non-observable square roots

To understand the idea of supersymmetry, it is useful to include it into the following more general heuristic principle.

2.1.1. Principle of square roots. *It is useful to represent observable quantities of immediate physical interest (real, positive, bosonic) as bilinear combinations of more fundamental quantities which can be complex, fermionic and not even observable by themselves.*

In other words, it is useful to take “square roots” of familiar objects. Let us give several examples.

Example 2.1.2 (Wave functions and probability density). In elementary quantum mechanics, the wave function $\psi(x)$ of a particle (say, electron), is a complex quantity which can not be measured. But the expression

$$P(x) \ = \ |\psi(x)|^2 \ = \ \overline{\psi}(x) \cdot \psi(x) \ \geq 0$$

represents the probability density of the electron which is real, non-negative and measurable.

Example 2.1.3 (Laplace operator on forms). Let X be a C^∞ Riemannian manifold. The space $\Omega^\bullet(X)$ of all differential forms on X is $\mathbb{Z}$-graded and so can be considered as a super-vector space. The Laplace operator on forms is defined as

$$\Delta \ = \ d \circ d^* + d^* \circ d \ = \ [d, d^*],$$

where d is the exterior derivative and d^* is its adjoint with respect to the Riemannian metric. Thus Δ is non-negative definite, real (self-adjoint) and bosonic, while d and d^* are fermionic.

Example 2.1.4 (Spinors as square roots of vectors). Let V be a d-dimensional vector space over $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, equipped with a non-degenerate quadratic form q. We refer to V as the spacetime, or the Minkowski space. Elements of V (*vectors* in the physical sense of the word) can be represented as bilinear combinations of more fundamental quantities, *spinors*. By definitions, spinors are vectors in minimal (spinor) representations of $\mathrm{Spin}(V)$, a double covering of the group $SO(V)$. There is one spinor representation S, if $d = \dim(V)$ is odd, and two representations, S_+ and S_-, if d is even. Further properties of these representations depends on the residues modulo 8 of d and of the signature of q, if $k = \mathbb{R}$ (Bott periodicity). We refer to [3, 8] for a detailed treatment.

The expression of vectors as bilinear combinations of spinors comes from $\mathrm{Spin}(V)$-equivariant maps γ (Dirac's gamma-matrices) whose nature we indicate in three cases $d = 2, 4, 10$ with Minkowski signature.

$d=2$: S_+, S_- can be defined over $\mathbb{R}$ and have real dimension 2. They are self-dual: $S_\pm^* = S_\pm$. The gamma maps are $\gamma : \mathrm{Sym}^2(S_\pm) \to V$.

$d=4$: S_+, S_- are complex and have complex dimension 2. They are hermitian conjugate of each other: $\overline{S}_\pm^* = S_\mp$. The gamma map is γ" $S_+ \otimes S_- \to V$.

$d=10$: S_+, S_- are real, of dimension 16 and self-dual. The gamma maps are $\gamma : \mathrm{Sym}^2(S_\pm) \to V$.

Needless to say, forming the double covering $\mathrm{Spin}(V) \to SO(V)$ itself can be seen as taking square roots of rotations. The appearance of spinors is a reflection of this procedure at the level of representations of groups.

Example 2.1.5 (Weil conjecture over finite fields). It is tempting to add to this list of "square roots" the following classical example. Let X be a smooth projective curve over a finite field $\mathbb{F}_q$. The étale cohomology group $H^1(X \otimes \overline{\mathbb{F}}_q, \mathbb{Q}_l)$ is acted upon by the Frobenius element Fr, generating $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. It was proved by A. Weil (and motivated the more general Weil conjectures) that each eigenvalue λ of Fr is an algebraic integer whose image in each complex embedding satisfies $|\lambda| = \sqrt{q}$, i.e., $\lambda \cdot \overline{\lambda} = q$. So the first cohomology, a fermionic structure, gives rise to factorizations $q = \lambda \cdot \overline{\lambda}$.

Further, it was suggested by Y. I. Manin that the motive of a supersingular elliptic curve over $\mathbb{F}_q$ can be seen as a "spinorial square root" of the Tate motive. See [37] for developments in this direction.

2.2 Square root of d/dt and theta-functions

The simplest example of supersymmetry can be obtained by looking at the differential operator (super-derivation)

$$Q = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial t} = \begin{pmatrix} 0 & 1 \\ \partial/\partial t & 0 \end{pmatrix}, \quad Q^2 = \frac{\partial}{\partial t} \tag{2.2.1}$$

in the algebra $\mathcal{O}(\mathbb{A}^{1|1}) = \mathbb{C}[t] \otimes \Lambda[\xi]$. One can replace $\mathbb{C}[t]$ by the algebras of smooth or analytic functions of t. The second equality sign in (2.2.1) is an instance of *component analysis*: we view $\mathbb{C}[t] \otimes \Lambda[\xi]$ as a free $\mathbb{C}[t]$-module with basis $1, \xi$ and write Q as a 2×2 matrix differential operator in t alone. One verifies immediately that $Q^2 = \partial/\partial t$, so Q gives a square root of the Hamiltonian (time translation).

Already this example is quite non-trivial: it provides a natural explanation of Sato's approach to theta functions [39, 40]. We do it in two stages. First, consider the exponential

$$e^Q = 1 + Q + \frac{Q^2}{2!} + \frac{Q^3}{3!} + \cdots \tag{2.2.2}$$

as series of operators acting in complex analytic functions in t, ξ.

Proposition 2.2.3. *e^Q converges to a local operator, i.e., to an endomorphism of the sheaf $\mathcal{O}_{\mathbb{C}^{1|1}}$ of analytic functions on $\mathbb{C}^{1|1}$.*

A standard example of a local operator in this sense is a linear differential operator P on the line. An operator with constant coefficients is just a polynomial in d/dt:

$$P \;=\; h(d/dt), \quad h(z) = \sum_{n=0}^{N} a_n z^n.$$

Replacing polynomials by entire analytic functions $h(z) = \sum_{n=0}^{\infty} a_n z^n$, we get expressions which, even when they converge, need not give local operators. For instance, $h(z) = e^z$ gives the shift operator

$$(e^{d/dt} f)(t) \;=\; f(t+1). \tag{2.2.4}$$

However, if $h(z)$ is *sub-exponential*, i.e., the series $k(z) = \sum n! a_n z^n$ still represents an entire function, then $h(d/dt)$ acts on analytic functions in a local way. Indeed, by the Cauchy formula,

$$\big(h(d/dt)f\big)(t) \;=\; \sum_{n=0}^{\infty} a_n f^{(n)}(t) \;=\; \frac{-1}{2\pi i}\oint_{|t'-t|=\varepsilon} f(t') k\left(\frac{1}{t-t'}\right) dt',$$

where ε can be arbitrarily small. So if f is analytic in an open $U \subset \mathbb{C}$, then so is $h(d/dt)f$. For example, $h(z) = \cos\sqrt{z}$ gives a local operator. One can similarly make sense of series $\sum_{n=0}^{\infty} a_n(t)(d/dt)^n$, where $a_n(t)$ are analytic functions in t of sub-exponential growth in n. These series define local operators on analytic functions on $\mathbb{C}$ known simply as *differential operators of infinite order* [40]. They form a sheaf of rings on $\mathbb{C}$, denoted $\mathcal{D}^{\infty}_{\mathbb{C}}$. Similarly for any complex analytic (super-)manifold such as $\mathbb{C}^{1|1}$.

Returning to the situation of Proposition 2.2.3, we see that Q, being a square root of $\partial/\partial t$, has exponential e^Q which is a local operator: a global section of $\mathcal{D}^{\infty}_{\mathbb{C}^{1|1}}$ or, after component analysis, of $\mathrm{Mat}_2(\mathcal{D}^{\infty}_{\mathbb{C}})$.

Remark 2.2.5. Although Q is a super-derivation, it is not a derivation in the usual sense and e^Q is not a ring automorphism. In particular, forming e^Q is not an instance of exponentiating a Lie superalgebra to a Lie supergroup.

We now consider the simplest theta-function

$$\theta(t,x) \;=\; \sum_{n\in\mathbb{Z}} e^{n^2 t + nx}, \quad \Re(t) < 0,\, x \in \mathbb{C}.$$

Its value at $x = 0$ (the Thetanullwert)

$$\theta(x,0) \;=\; \sum_{n\in\mathbb{Z}} q^{n^2}, \quad q = e^t,\; |q| < 1,$$

is a modular form, so there is a relation (modular transformation) relating its values at t and at $1/t$ (in our normalization). The main result of [40] is a characterization of $\theta(t, 0)$ by two differential equations of infinite order in t alone (i.e., by local conditions in t). They are then used to deduce the modular transformation because the system of equations is invariant under it.

This can be done as follows. Note that $\theta(t, x)$, as a function of two variables, satisfies the heat equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}.$$

This means that $\partial/\partial x$ acts on θ as a square root of $\partial/\partial t$. On the other hand, θ has periodicity properties

$$\begin{cases} \theta(t, x + 2\pi i) = \theta(t, x) \\ \theta(t, x + 2t) = e^{-x-t}\theta(t, x). \end{cases}$$

Using (2.2.4) in the x-variable and the heat equation, we can write this formally as

$$\begin{cases} e^{2\pi i\sqrt{\partial/\partial t}}\theta(t, x) = \theta(t, x) \\ e^{2t\sqrt{\partial/\partial t}}\theta(t, x) = e^{-t-x}\theta(t, x), \end{cases}$$

after which we can specialize to $x = 0$. Now, replacing $\sqrt{\partial/\partial t}$ with Q, we get a system of two differential equations of infiite order in t alone, satisfied by $\theta(t, 0)$.

2.3 Square roots of spacetime translations

Representing just the Hamiltonian (the operator of energy, or time translation) as a bilinear combination of fermionic operators (§2.2, or Example 2.1.3), is a non-relativistic procedure. Relativistically, we cannot separate $\partial/\partial t$ from any other constant vector field ∂_v (momentum operator) on the Minkowski space V and so should represent them all. This leads to the following concept.

Definition 2.3.1. A *quadratic space* over a field $\mathbf{k}$ is a datum of finite-dimensional $\mathbf{k}$-vector spaces V, B and a surjective linear map $\Gamma : \mathrm{Sym}^2(B) \to V$. We will often view Γ as a V-valued scalar product $(b_1, b_2) \mapsto \Gamma(b_1, b_2) \in V$ on B

A quadratic space Γ gives rise to the Lie superalgebra

$$\mathfrak{t} = \mathfrak{t}_\Gamma, \quad \mathfrak{t}^{\bar{0}} = V, \ \ \mathfrak{t}^{\bar{1}} = B, \tag{2.3.2}$$

with the only non-zero component of the bracket being Γ. We call $\mathfrak{t}$ the *supersymmetry algebra* associated to Γ. The abelian central subalgebra $\mathfrak{t}^{\bar{0}} = V$ is the usual Lie algebra of infinitesimal spacetime translations, and we denote by $T^{\bar{0}} = e^{\mathfrak{t}^{\bar{0}}}$ the corresponding algebraic group (i.e., V considered as an algebraic variety and equipped with vector addition). Since $\mathfrak{t}$ is nilpotent: $[x, [y, z]] = 0$ for any homogeneous x, y, z, it is easily integrated to an algebraic supergroup $T = e^{\mathfrak{t}}$ called the *supersymmetry group* using the truncated Hausdorff series (1.4.3).

Examples 2.3.3. In physical applications (see [10, 16] for a detailed exposition), V is the Minkowski space (i.e., a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a non-degenerate quadratic form) and B is a direct sum of (possible several copies of) the spinor bundle(s). Different possible choices of such B are known as different types of *extended supersymmetry*. More precisely:

$N = p$ *supersymmetry (SUSY)* means that $d = \dim(V)$ is odd and $B = S^{\oplus p}$.

$N = (p, q)$ *supersymmetry (SUSY)* means that d is even and $B = S_+^{\oplus p} \oplus S_-^{\oplus q}$.

The map Γ of the quadratic space is constructed out of the gamma-maps γ for spinors. Quadratic spaces obtained in this way will be called *spinorial*. For example:

(a) $d = 10$, $N = (1, 0)$ SUSY: Here $B = S_+$ and $\Gamma = \gamma : \mathrm{Sym}^2(S_+) \to V$. This is the most fundamental example in many respects.

(b) $d = 4$, $N = (1, 1)$ SUSY: Here $B = S_+ \oplus S_-$ and Γ is the composition

$$\mathrm{Sym}^2(S_+ \oplus S_-) \longrightarrow S_+ \otimes S_- \xrightarrow{\gamma} V.$$

(c) $d = 2$, $N = (1, 1)$ SUSY is often written explicitly using two operators H (energy $= \partial/\partial t$) and P (momentum $= \partial/\partial x$). The space S_+ is spanned by two vectors Q_+, Q_+^* and S_- by Q_-, Q_-^*, and the supersymmetric algebra (i.e., the commutation rule in $\mathfrak{t}$) is written as

$$[Q_+, Q_+] = H + P, \quad [Q_-, Q_-] = H - P, \quad [Q_+, Q_-] = 0.$$

For a spinorial quadratic space Γ the group $\mathrm{Spin}(V)$ acts on $\mathfrak{t}$ and on $T = e^{\mathfrak{t}}$. The corresponding semidirect product $\mathcal{P} = \mathrm{Spin}(V) \ltimes T$ is known as the *super-Poincaré group* (corresponding to the type of extended supersymmetry represented by Γ).

2.4 Quadratic spaces and intersections of quadrics

In algebraic geometry, the term *intersection of quadrics* [38, 47] means one of two closely related objects:

(1) A *homogeneous intersection of quadrics*: a subscheme Y in a vector space B, given by homogeneous quadratic equations. That is, the ideal $I(Y) \subset \mathrm{Sym}^\bullet(B^*)$ is generated by $I_2(Y) \subset \mathrm{Sym}^2(B^*)$, its degree 2 homogeneous part.

(2) A *projective intersection of quadrics*: the projectivization $Z = \mathbb{P}(Y) \subset \mathbb{P}(B)$ of Y as above. In this way, projective intersections of quadrics $Z \subset \mathbb{P}(B)$ are in bijection with those homogeneous intersections of quadrics $Y \subset B$ which are not entirely supported at 0.

Each quadratic space $\Gamma : \mathrm{Sym}^2(B) \to V$ gives a homogeneous intersection of quadrics

$$Y_\Gamma \; = \; \{b \mid \Gamma(b, b) = 0\} \; \subset \; B,$$

and we denote by $Z_\Gamma \subset \mathbb{P}(B)$ its projectivization. Alternatively,

$$Y_\Gamma \;=\; \{b \in \mathfrak{t}_\Gamma^{\bar{1}} \mid [b,b]=0\}$$

is the *scheme of Maurer-Cartan elements* of the supersymmetry algebra t_Γ.

Proposition 2.4.1. *Each homogeneous intersection of quadrics $Z \subset B$ can be obtained in this way from some quadratic space $\Gamma : \mathrm{Sym}^2(B) \to V$, defined uniquely up to an isomorphism.*

Proof: Take $V = I_2(Y)^*$, the space dual to the space of quadratic equations of Y and take for Γ the canonical projection.

□

So quadratic spaces are simply data encoding intersections of quadrics. Classically, the simplest intersections of quadrics are as follows.

Definition 2.4.2. (1) A projective intersection of quadrics $Z \subset B$ is called a *complete intersection*, if $\dim I_2(Z)$, the number of quadratic equations of Z, is equal to the codimension of Z in $\mathbb{P}(B)$.

(2) A quadratic space $\Gamma : \mathrm{Sym}^2(B) \to V$ is called of *complete intersection type*, if Z_Γ is a complete intersection, i.e., $\dim(V) = \mathrm{codim}(Z_\Gamma)$.

Examples 2.4.3. (a) For the spinorial quadratic space $\Gamma : \mathrm{Sym}^2(S_+) \to V$ of $d = 10$, $n = (1,0)$ SUSY (Example 2.3.3(a)), X_Γ is the 10-dimensional *space of pure spinors* $\Sigma^{10} \subset \mathbb{P}(S_+) = \mathbb{P}^{15}$. It can be identified with one component of the variety of isotropic 5-planes in V. It is *not* a complete intersection: it has codimension 5 but is given by $d = 10$ equations.

(b) For the quadratic space $\Gamma : \mathrm{Sym}^2(S_+ \oplus S_-) \to V$ of $d = 4$, $N = (1,1)$ SUSY (Example 2.3.3(b)), $X_\Gamma \subset \mathbb{P}(S_+ \oplus S_-) = \mathbb{P}^3$ is the disjoint union of two skew lines $\mathbb{P}(S_+) \sqcup \mathbb{P}(S_-) \simeq \mathbb{P}^1 \sqcup \mathbb{P}^1$. It is *not* a complete intersection: it has codimension 2 but is given by $d = 4$ equations.

(c) The variety $\Sigma^{10} \subset \mathbb{P}^{15}$ is a particular case of the following: a partial flag variety G/P (G reductive algebraic group, $P \subset G$ parabolic), equivariantly embedded into $\mathbb{P}(E)$, where E is an irreducible highest weight representation of G. All such varieties are known to be intersections of quadrics.

So our physical spacetime is really *the space of equations* of an auxiliary intersection (typically, not a complete intersection!) of quadrics.

We will also need families of quadratic spaces parametrized by superschemes.

Definition 2.4.4. Let X be a superscheme. A *quadratic module* over X is a datum of two locally free sheaves B, V of $\mathcal{O}_X$-modules, both of purely even rank, and of an $\mathcal{O}_X$-linear map $\Gamma : \mathrm{Sym}^2_{\mathcal{O}_X}(B) \to V$ with two properties:

(1) q is surjective, and therefore the dual map $\Gamma^\vee : B^\vee \to \mathrm{Sym}^2_{\mathcal{O}_X}(B^\vee)$ is an embedding of a locally direct summand.

(2) The $\mathcal{O}_X$-algebra $\mathrm{Sym}^\bullet_{\mathcal{O}_X}(B^\vee)/(\Gamma^\vee(V^\vee))$ is flat over $\mathcal{O}_X$.

In other words, a quadratic module gives a family of intersections of quadrics, parametrized by X, and the condition (2) means that this family is flat, in particular, its fibers have the same dimension. This is important for non-complete intersections.

2.5 Supersymmetry, superspace and constraints

We start with a quick explanation of some physical terms.

2.5.1. Supersymmetry is a feature of a field theory (say, a collection of fields φ plus a Lagrangian action $S[\varphi]$) defined, a priori, on the usual (non-super!) Minkowski space V. It means that the action of the usual Poincaré group $SO(V) \ltimes V$ on fields by changes of variables (which leaves any relativistic Lagrangian invariant) is extended, *in some way*, to an action of the super-Poincaré group $\mathcal{P}$ so that $S[\varphi]$ is still invariant. Here $\mathcal{P}$ is constructed out of one of the spinorial quadratic spaces $\Gamma : \mathrm{Sym}^2(B) \to V$ (Example 2.3.3).

Thus the new datum in supersymmetry is the extension of the action of V to an action of $\mathfrak{t}_\Gamma$. This means that we need to represent all the momentum operators $\partial_v, v \in V$, as bilinear combinations of fermionic "supercharges" $D_b, b \in B$ so that we have the commutation relations

$$[D_b, D_{b'}] = \partial_{\Gamma(b,b')}, \quad [D_b, \partial_v] = [\partial_v, \partial_{v''}] = 0.$$

For this to be possible, there should be about equally many bosonic and fermionic fields in the theory. This explains why supersymmetry is sometimes called "symmetry between bosons and fermions".

2.5.2. Superspace is a tool to construct supersymmetric theories by replacing the mysterious "in some way" above by a natural construction. More precisely, a superspace is a supermanifold $\mathcal{S}$ extending the spacetime V, (so that $V = \mathcal{S}^{\bar{0}}$ is its even part) and which admits a natural action of $\mathfrak{t}$.

The simplest choice (*flat superspace*) is $\mathcal{S} = T$, the underlying manifold of the supersymmetry group, on which $\mathfrak{t} = \Pi(B) \oplus V$ acts by left-invariant vector fields D_b, ∂_v, see [10, 16]. Any field on $\mathcal{S}$ (referred to as *superfield*) gives an entire multiplet of usual fields on V by *component analysis*: writing $\mathcal{O}_\mathcal{S} = \mathcal{O}_V \otimes \Lambda^\bullet(B^*)$ as a free $\mathcal{O}_V$-module of rank $2^{\dim(B)}$. The Lie algebra $\mathfrak{t}$ acts naturally on superfields, so working only with such fields, we get supersymmetry seemingly "for free"

This construction can, of course, be done for an arbitrary quadratic space $\Gamma : \mathrm{Sym}^2(B) \to V$. If Γ is spinorial, then the action of $\mathfrak{t}$ on superfields extends to an action of the super-Poincaré group $\mathcal{P}$.

Remark 2.5.3. Similarly to §2.2, the exponentials e^{D_b} of the supercharges are local operators on analytic superfields, while the shifts e^{∂_v} are not. It would be interesting to understand the consequences of this phenomenon. The situation of §2.2 corresponds to the simplest example of a quadratic space, when B is 1-dimensional, $V = \mathrm{Sym}^2(B)$ is also 1-dimensional, and $\Gamma = \mathrm{Id}$.

2.5.4. The difficulty with supersymmetry is that it tends to require too many fields (on V) for all of them to make physical sense. The following result [35] is usually intepreted by saying that “supersymmetry in > 11 dimensions is not sensible”.

2.5.5. Nahm's theorem. *Any supersymmetric theory with $d > 11$ contains fields of spin ≥ 2.*

For the superspace construction this is easy to understand. Already the simplest kind of a superfield, a function on $\mathcal{S}$, is a section of $\mathcal{O}_V \otimes \Lambda^\bullet(B^*)$, where B is the direct sum of one or several spinor spaces. As d grows, the decomposition of $\Lambda^\bullet(B^*)$ into $\mathrm{Spin}(V)$-irreducibles, quickly begins to contain higher spin representations such as $\mathrm{Sym}^j(V)$, $j \geq 2$.

However, even in the remaining dimensions $d \leq 11$, the superspace construction typically gives too many component fields. To eliminate some of the components, one usually imposes (in a seemingly *ad hoc* way) some additional restrictions on superfields known as *constraints*. In the next subsection we discuss a conceptual point of view on such constraints.

Supergeometry as understood by physicists, is the study of various versions of (not necessarily flat) superspaces. All the examples that have been considered, fit into the following concept.

Definition 2.5.6. An abstract *superspace* is a supermanifold $\mathcal{S}$ (smooth, analytic or algebraic) of dimension $(m|n)$ together with a Pfaff system $C \subset T_{\mathcal{S}}$ of rank $(0|n)$ satisfying the following properties:

(1) The restriction $C|_{\mathcal{S}^{\bar{0}}}$ coincides with the odd part of $(T_{\mathcal{S}})|_{\mathcal{S}^{\bar{0}}}$.

(2) Denote $B = \Pi(C)$ and $V = (T\mathcal{S})|_C$. Then, the Frobenius pairing $\digamma_C : \Lambda^2 C \to (T_{\mathcal{S}})/C$, written as an $\mathcal{O}_{\mathcal{S}}$-linear map $\Gamma : \mathrm{Sym}^2_{\mathcal{O}_{\mathcal{S}}}(B) \to V$, is a quadratic module over $\mathcal{S}$, see Definition 2.4.4.

Example 2.5.7 (Supercurves, as understood by physicists). For a mathematician, an (algebraic) *supercurve* is an algebraic supermanifold of dimension $1|n$ for some n. For a physicist, a supercurve is a superspace of dimension $(1|n)$, so the Pfaff system C is a necessary part of the structure. See [33, 11, 12]. The geometry of a supercurve of dimension $(1|1)$ is locally modeled on the setting of §2.2.

Example 2.5.8 (Spinorial curved superspaces). Definition 2.5.6 is quite general. It does not require that the fibers of the quadratic module Γ be spinorial quadratic spaces. However, the intersections of quadrics related to spinorial spaces (such as the space of pure spinors) are *rigid* both as abstract algebraic varieties and as intersections of quadrics. This means that a quadratic module whose one fiber is a spinorial quadratic space, has all neighboring fibers spinorial of the same type. Therefore if $\mathcal{S}$ is a superspace (in our sense) with the commutator pairing Γ spinorial at one even point, then we have a similar isomorphic “spinorial structure" in each neighboring tangent space. This amounts to a differential-geometric structure on $\mathcal{S}$ including a conformal structure in the quotient bundle $(T_{\mathcal{S}})/C$ (in particular, on the ordinary manifold $\mathcal{S}^{\bar{0}}$) and a “choice of spinors” for this conformal structure. Cf. [18, 31]

Cobordism categories of such curved spinorial superspaces provide a language for Atiyah-style approach to supersymmetric quantum field theories [45, 46]

2.6 Constraints and complete intersection slices

Various recipes of imposing constrains on superfields can be understood using the idea of *simple plane slices of complicated intersections of quadrics.*

Let $Z \subset \mathbb{P}(B)$ be an intersection of quadrics. A *plane slice* of Z is a scheme of the form $Z \cap M$, where $M \subset \mathbb{P}(B)$ is a projective subspace. It is an intersection of quadrics in M. The two simplest possibilities are as follows:

(0) $Z \cap M = Z$, i.e., M is contained in Z entirely.

(1) $Z \cap M$ is a complete intersection of quadtrics in M.

Let $\Gamma : \mathrm{Sym}^2(B) \to V$ be the quadratic space corresponding to Z. A projective subspace M corresponds to a linear subspace $B' \subset B$, and $Z \cap M$ corresponds to the quadratic space $\Gamma' : \mathrm{Sym}^2(B') \to V'$, where $V' = \Gamma(\mathrm{Sym}^2(B'))$ and Γ' is the restriction of Γ. The supersymmetry algebra $\mathfrak{t}_{\Gamma'}$ is the Lie sub(super) algebra in $\mathfrak{t}$ generated by $B' \subset \mathfrak{t}_\Gamma^{\bar{1}}$. We will call such quadratic spaces *slices of* Γ.

The case (0) above means that $\mathfrak{t}_{\Gamma'} = B'$ is abelian and purely even. In the case (1) (which includes the case (0)) we will say that $\mathfrak{t}_{\Gamma'}$ is a *null-subalgebra.*

Examples 2.6.1. (a) Let $Z = C \sqcup C'$ is the union of two skew lines in $\mathbb{P}^3$ (Example 2.4.3(b)). Clearly, each of the two lines gives an instance of Case (0) above. Other then that, we have a $\mathbb{P}^1 \times \mathbb{P}^1$ worth of complete intersection slices $Z \cap M$, with M being a chord passing through one point on C and one point on C'.

(b) Let $\Gamma : \mathrm{Sym}^2(S_+) \to V$ be the $d = 10$ $N = (1, 0)$ SUSY quadratic space. Each vector $v \in V, q(v) = 0$, gives a linear operator $\Gamma_v : S_+ \to S_+$ given by the Clifford multiplication by v (transpose of the quadratic space map Γ). If v is a non-zero null vector, i.e., $q(v) = 0$, then $\Gamma_v^2 = 0$ and $\mathrm{Ker}(\Gamma_v) = \mathrm{Im}(\Gamma_v)$ is an 8-dimensional subspace in S_+ which we denote L_v. The intersection $\mathbb{P}(L_v) \cap \Sigma^{10}$ is a quadric hypersurface in $\mathbb{P}(L_v)$, and the space of equations of this hypersurface is spanned by v. In other words, $\Gamma(\mathrm{Sym}^2(L_v)) = \mathbb{C} \cdot v$, and

$$\mathfrak{h} \;=\; L_v \oplus \mathbb{C} \cdot v \;\subset\; S_+ \oplus V \;=\; \mathfrak{t}$$

is a (maximal) null subalgebra. The (1|8)-dimensional linear subspaces $L_v \oplus \mathbb{C} \cdot v \subset \mathfrak{t} = \mathbb{C}^{10|16}$ are known as *super-null-geodesics* [51, 28, 34, 10].

It seems that the $\mathbb{P}(L_v) \cap \Sigma^{10}$ are precisely the maximal complete intersection slices of Σ^{10}.

Assuming this, we can formulate the constraints on superfields as follows.

2.6.2. Spin 0 constraints are imposed on scalar superfields which are functions on $\mathcal{S}$ or, more generally maps $\Phi : \mathcal{S} \to X$ where X is a given target manifold. They have the form

$$D_b \Phi = 0, \quad b \in B',$$

where B' is one fixed subspace of B on which Γ vanishes (case (0) above), maximal with this property. Maps Φ satisfying the constraints are known as *chiral superfields.*

2.6.3. Spin 1 constraints are imposed on gauge superfields (connections ∇ in principal bundles on $\mathcal{S}$). They have the form of integrability $(F_\nabla)|_{g\cdot\mathfrak{h}} = 0$, where $\mathfrak{h} = \mathfrak{t}_{\Gamma'}$ runs over all null-subalgebras in $\mathfrak{t}_\Gamma$ and $g \cdot \mathfrak{h}$ is the left translation of $\mathfrak{h}$ in $\mathcal{S}$. In other words,

$$[\nabla_{b_1}, \nabla_{b_2}] = \nabla_{\Gamma'(b_1,b_2)}, \quad [\nabla_b, \nabla_a] = [\nabla_{a_1}, \nabla_{a_2}] = 0, \quad \forall b, b_1, b_2 \in B', \;\; a, a_1, a_2 \in V'$$

where B' runs over all the maximal complete intersection slices of Y (case (1) above).

Remark 2.6.4. When imposing constraints on superfields, it is obviously desirable not to end up restricting their dependence in the usual, even directions of spacetime. In the case 2.6.3 this is ensured by the fact that the null-subalgebras $\mathfrak{h}$ have $\dim(\mathfrak{h}^{\bar{0}}) = 1$ (they are the usual null-lines). In other words, *all the complete intersection slices of* $Z = \Sigma^{10}$ *are quadric hypersurfaces*: $Z \cap M$ is a hypersurface in M. It would be interesting to study other intersections of quadrics Z with this property.

2.6.5. Lie algebra meaning of complete intersections. Given a quadratic space $\Gamma : \mathrm{Sym}^2(B) \to V$, we can associate to it another, $\mathbb{Z}$-graded Lie algebra (i.e., a Lie algebra in $\mathrm{Vect}^{\mathbb{Z}}_{\mathbf{k}}$)

$$\widetilde{\mathfrak{t}}_\Gamma \;=\; \mathrm{FL}(B[-1])\big/(\mathrm{Ker}(\Gamma))$$

Here $\mathrm{FL}(-)$ means the free graded Lie algebra generated by a graded vector space. In our case $B[-1]$ is B put in degree $+1$, so the degree 2 part of $\mathrm{FL}(B[-1])$ is $\mathrm{Sym}^2(B)$. The Lie algebra $\widetilde{\mathfrak{t}}_\Gamma$ is obtained by quotienting $\mathrm{FL}(B[-1])$ by the graded Lie ideal generated by $\mathrm{Ker}(\Gamma) \subset \mathrm{Sym}^2(B)$. Thus

$$\widetilde{\mathfrak{t}}^1_\Gamma = B, \;\; \widetilde{\mathfrak{t}}^2_\Gamma = V,$$

but it is not required that V commutes with the generators and therefore $\widetilde{\mathfrak{t}}_\Gamma$ can be non-trivial in degrees ≥ 3. Cf. [10] §11.3. We note that $\mathfrak{t}_\Gamma$ can be considered as a $\mathbb{Z}$-graded Lie algebra by lifting the degree $\bar{0}$ part to degree 2 and degree $\bar{1}$ part to degree 1 (this is possible is possible since the degree $\bar{0}$ part lies in the center). With this understanding, we have a surjective homomorphism of graded Lie algebras

$$p : \widetilde{\mathfrak{t}}_\Gamma \longrightarrow \mathfrak{t}_\Gamma.$$

Denote by

$$R_\Gamma \;=\; \mathrm{Sym}^\bullet(B^*)\big/I_2(Y_\Gamma)$$

the graded coordinate algebra (commutative in the usual sense) of the homogeneous intersection of quadtics $Y_\Gamma \subset B$. Then, the enveloping algebra $U(\widetilde{\mathfrak{t}}_\Gamma)$ is identified with $R^!_\Gamma$, the quadratic dual of the quadratic algebra R_Γ, see [36] for background. In particular, we have a homomorphism of graded algebras

$$\eta : U(\widetilde{\mathfrak{t}}_\Gamma) \longrightarrow \mathrm{Ext}^\bullet_{R_\Gamma}(\mathbf{k}, \mathbf{k}).$$

The algebra R_Γ is called *Koszul*, if η is an isomorphism. This is the case in all spinorial examples. The role of complete intersections from this point of view is as follows.

Proposition 2.6.6. *The following are equivalent, and if they are true, then R_Γ is Koszul:*

(i) Γ *is of complete intersection type, i.e.,* Y_Γ *is a complete intersection of quadrics.*

(ii) We have $\widetilde{\mathfrak{t}}^i_\Gamma = 0$ *for* $i \geq 3$*, i.e., the morphism* $p : \widetilde{\mathfrak{t}}_\Gamma \to \mathfrak{t}_\Gamma$ *is an isomorphism. In particular, the condition of commutativity of* $\widetilde{\mathfrak{t}}^2_\Gamma = V$ *with* $\widetilde{\mathfrak{t}}^1_\Gamma = B$ *already follows from the defining relations of* $\widetilde{\mathfrak{t}}_\Gamma$*.*

Proof: This is a particular case of the general principle in commutative algebra that locally complete intersections are characterized by the cotangent complex being quasi-isomorphic to a 2-term complex. The special case of intersections of quadrics was studied in [27]. □

Further, in many cases (including those related to spinors) the algebra $\widetilde{\mathfrak{t}}_\Gamma$ can be identified with the amalgamated free product of all its null-subalgebras $\widetilde{\mathfrak{t}}_{\Gamma'} = \mathfrak{t}_{\Gamma'}$. This relates the integrability conditions on null-subalgebras with the Koszul duality point of view on constraints for SYM advocated in [34].

3 Homotopy-theoretic underpinnings of supergeometry

3.1 The skeleton of the Koszul sign rule

To understand the nature of the Koszul sign rule 1.2.1, let us "minimize" the symmetric monoidal category $\mathrm{SVect}_\mathbf{k}$ incorporating it.

To account just for the signs, we can disregard all morphisms in $\mathrm{SVect}_\mathbf{k}$ which are not isomorphisms. as well as all objects which have total dimension > 1. Restricted to 1-dimensional super-vector spaces ($\dim(V^{\bar{0}}) + \dim(V^{\bar{1}}) = 1$) and their isomorphisms, we get a symmetric monoidal category $1\text{-}\mathrm{SVect}_\mathbf{k}$. Similarly, to capture the $\mathbb{Z}$-graded sign rule, we can restrict to the category $1\text{-}\mathrm{Vect}^\mathbb{Z}_\mathbf{k}$ of $\mathbb{Z}$-graded 1-dimensional vector spaces. These categories are examples of the following concept.

Definition 3.1.1. A *Picard groupoid* is a symmetric monoidal category $(\mathcal{G}, \otimes, \mathbf{1}, R)$ in which all objects are invertible under $\otimes$ and all morphisms are invertible under composition.

A Picard groupoid $\mathcal{G}$ gives rise to two abelian groups:

- The *Picard group* of $\mathcal{G}$, denoted $\mathrm{Pic}(\mathcal{G})$, or $\pi_0(\mathcal{G})$. It is formed by isomorphism classes of objects, with the operation given by $\otimes$.
- The group $\pi_1(\mathcal{G}) = \mathrm{Aut}_\mathcal{G}(\mathbf{1})$ of automorphisms of the unit object. It is canonically identified with the group of automorphisms of any other object.

In our case

$$1\text{-}\mathrm{SVect}_\mathbf{k} : \quad \pi_0 = \mathbb{Z}/2, \quad \pi_1 = \mathbf{k}^*;$$
$$1\text{-}\mathrm{Vect}^\mathbb{Z}_\mathbf{k} : \quad \pi_0 = \mathbb{Z}, \quad \pi_1 = \mathbf{k}^*.$$

Here, $\mathbf{k}^*$ is still unnecessarily big: to formulate the sign rule. we need only the subgroup $\{\pm 1\} \subset \mathbf{k}^*$. So we cut these Picard groupoids further.

For this, we replace $\mathbf{k}$ with the ring $\mathbb{Z}$, since $\{\pm 1\} = \mathbb{Z}^*$ is precisely its group of invertible elements. Accordingly, we replace 1-dimensional $\mathbf{k}$-vector spaces with free abelian groups of rank 1. This gives Picard groupoids $1\text{-SAb}, 1\text{-Ab}^{\mathbb{Z}}$. Their objects are $\mathbb{Z}/2$- or $\mathbb{Z}$-graded abelian groups which are free of rank 1. As before, the morphisms are isomorphisms,, $\otimes$ is the graded tensor product over $\mathbb{Z}$ and the symmetries are given by the Koszul sign rule. The π_i of these Picard groupoids are now as follows:

$$\begin{aligned} 1\text{-SAb}: &\quad \pi_0 = \mathbb{Z}/2, \quad \pi_1 = \{\pm 1\} = \mathbb{Z}/2, \\ 1\text{-Ab}^{\mathbb{Z}}: &\quad \pi_0 = \mathbb{Z}, \quad \pi_1 = \{\pm 1\} = \mathbb{Z}/2. \end{aligned}$$

We can call 1-SAb and $1\text{-Ab}^{\mathbb{Z}}$ the *sign skeleta* of the Koszul sign rule ($\mathbb{Z}/2$-graded and $\mathbb{Z}$-graded versions). They contain all the data needed to write the sign rule but nothing more.

The following simple but remarkable fact can be seen as a mathematical explanation of the Principle of Naturality of Supers 1.1.3.

Proposition 3.1.2. $1\text{-Ab}^{\mathbb{Z}}$ *is equivalent to $\mathcal{F}_L$, the free Picard groupoid generated by one formal object (symbol) L.*

By definition, $\mathcal{F}_L$ has, as objects, formal tensor powers $L^{\otimes n}, n \in \mathbb{Z}$. It further has only those morphisms that are needed to write the symmetry isomorphisms

$$R_{L^{\otimes m}, L^{\otimes n}} : L^{\otimes m} \otimes L^{\otimes n} = L^{\otimes (m+n)} \longrightarrow L^{\otimes n} \otimes L^{\otimes m} = L^{\otimes (n+m)}$$

satisfying the axioms of a symmetric monoidal category (as well as composition, tensor products etc. of such morphisms).

Sketch of proof: L corresponds to the group $\mathbb{Z}$ placed in degree 1, so $L^{\otimes n}$ is $\mathbb{Z}$ in degree n. Further, $R_{L,L} \in \operatorname{Aut}(L^{\otimes 2}) = \operatorname{Aut}(\mathbf{1})$ corresponds to $(-1) \in \mathbb{Z}^*$ (note that $R_{L,l} \circ R_{L,L} = \operatorname{Id}$ by symmetry. The axioms of a symmetric monoidal category give that $R_{L^{\otimes m}, L^{\otimes n}}$ corresponds to $(-1)^{mn}$, so we recover the Koszul rule. □

In other words, the category $\operatorname{Vect}_{\mathbf{k}}^{\mathbb{Z}}$ which is at the basis of all supergeometry, can be obtained as a kind of $\mathbf{k}$-*linear envelope of a free Picard groupoid*. More precisely, we have the following construction.

Definition 3.1.3. Let $\mathcal{G}$ be a Picard groupoid, and $\chi : \pi_1(\mathcal{G}) \to \mathbf{k}^*$ be a homomorphism. By a $(\mathcal{G}, \chi)$*-graded* $\mathbf{k}$*-vector space* we will mean a functor $V : \mathcal{G} \to \operatorname{Vect}_{\mathbf{k}}$, whose value on objects will be denoted $A \mapsto V^A$, satisfying the following condition. For each object A, the action of each $\lambda \in \pi_1(\mathcal{G}) \simeq \operatorname{Aut}(A)$ on A is taken into the multiplication by $\chi(\lambda)$ on V^A. We denote by $\operatorname{Vect}_{\mathbf{k}}^{(\mathcal{G},\chi)}$ the category of $(\mathcal{G}, \chi)$-graded $\mathbf{k}$-vector spaces.

Since V is a functor, the spaces V^A and $V^{A'}$ for isomorphic objects A, A' are identified, so a $(\mathcal{G}, \chi)$-graded $\mathbf{k}$-vector space V can be viewed as a $\pi_0(\mathcal{G})$-graded vector space in the usual sense.

Proposition 3.1.4. *(a) The category* $\mathrm{Vect}_{\mathbf{k}}^{(\mathcal{G},\chi)}$ *has a structure of a monoidal category with the operation given by*

$$(V\otimes W)^A \;=\; \varinjlim\nolimits_{\{B\otimes C\to A\}}^{\mathrm{Vect}_{\mathbf{k}}} V^B\otimes_{\mathbf{k}} W^C,$$

the colimit taken over the category formed by pairs of objects $B, C\in\mathcal{G}$ *together with an (iso)morphism* $B\otimes C\to A$. *Further, the symmetry in* $\mathcal{G}$ *makes* $\mathrm{Vect}_{\mathbf{k}}^{(\mathcal{G},\chi)}$ *into a symmetric monoidal category.*

(b) If $\mathcal{G}=\mathcal{F}_L$ *and* $\chi:\pi_1(\mathcal{F}_L)=\mathbb{Z}/2\to\mathbf{k}^*$ *is the embedding of* $\{\pm 1\}$, *then* $\mathrm{Vect}_{\mathbf{k}}^{(\mathcal{G},\chi)}$ *is identified with the category* $\mathrm{Vect}_{\mathbf{k}}^{\mathbb{Z}}$ *with the symmetry given by the Koszul sign rule.*

3.2 (Higher) Picard groupoids and spectra

One of the insights of Grothendieck in his manuscript “Pursuing stacks" (cf.[6], p. 114) was the correspondence between Picard groupoids and a particular class of *spectra* in the sense of homotopy topology. See [13] for a discussion and [17] for a slightly more detailed treatment which we follow here. A systematic account can be found in [25].

The concept of a spectrum arises as a result of stabilizing the homotopy category of pointed topological spaces (say CW-complexes) under the two operations (adjoint functors)

$$\Sigma = \text{reduced suspension},\quad \Omega = \text{loop space},$$
$$\mathrm{Hom}(\Sigma X, Y) \;=\; \mathrm{Hom}(X,\Omega Y).$$

For example, the spheres S^n satisfy $\Sigma(S^n)=S^{n+1}$. We always have a canonical map (unit of adjunction)

$$\varepsilon_X : X\longrightarrow \Omega\Sigma X.$$

A spectrum Y can be seen as a topological space $\Omega^\infty Y$ together with a sequence of deloopings: spaces $\Omega^{\infty-j}Y$ equipped with compatible homotopy equivalences $\Omega^j(\Omega^{\infty-j}Y)\sim\Omega^\infty Y$. A spectrum Y has homotopy groups $\pi_i(Y), i\in\mathbb{Z}$ defined by

$$\pi_i(Y) \;=\; \pi_{i+j}(\Omega^{\infty-j}Y),\quad j\gg 0.$$

Example 3.2.1. A topological space X gives the *suspension spectrum* $\Sigma^\infty X$, with $\Omega^\infty\Sigma^\infty X = \varinjlim_n \Omega^n\Sigma^n X$ and $\Omega^{\infty-j}\Sigma^\infty X = \varinjlim_n \Omega^{n-j}\Sigma^n X$ (limits under powers of ϵ). The homotopy groups of $\Sigma^\infty X$ are the *stable homotopy groups* of X:

$$\pi_i(\Sigma^\infty X) \;=\; \pi_i^{\mathrm{st}}(X) \;:=\; \varinjlim_n \pi_{i+n}\Sigma^n X.$$

Spectra form (after inverting homotopy equivalences), a triangulated category SHo known as the *stable homotopy category*. This category has a symmetric monoidal structure (smash product of spectra). Let $m \leq n$ be integers ($m = -\infty$ or $n = \infty$ allowed). By an $[m,n]$-*spectrum* we mean a spectrum Y with $\pi_i(Y) = 0$ with $i \notin [m,n]$, and we denote by $\mathrm{SHo}_{[m,n]} \subset \mathrm{SHo}$ the full subcategory of $[m,n]$-spectra. There is a canonical "truncation" functor

$$\tau_{[m,n]} : \mathrm{SHo} \longrightarrow \mathrm{SHo}_{[m,n]}.$$

Grothendieck's correspondence can be formulated as follows.

Theorem 3.2.2. *There is an equivalence of categories*

$$\mathbb{B} : \big\{ \textit{Picard groupoids} \big\}[\mathrm{eq}^{-1}] \longrightarrow \mathrm{SHo}_{[0,1]},$$

so that $\pi_i(\mathbb{B}(Gc) = \pi_i(\mathcal{G})$, $i = 0, 1$. Here $[\mathrm{eq}^{-1}]$ means that equivalences of Picard groupoids are invertex, similarly to inverting homotopy equivalences in forming SHo.

A more precise result is proved in [25]. The spectrum $\mathbb{B}\mathcal{G}$ corresponding to a Picard groupoid $\mathcal{G}$, is a version of the classifying space of $\mathcal{G}$. That is, the space $\Omega^\infty \mathbb{B}\mathcal{G} = B\mathcal{G}$ is the usual classifying space of $\mathcal{G}$ as a category, and the deloopings are constructed using the symmetric monoidal structure, see [17] (3.1.6) for an explicit construction.

The further point of Grothendieck is that more general $[0,n]$-spectra should have a description in terms of *Picard n-groupoids*, an algebraic concept to be defined, meaning "symmetric monoidal n-categories with all the objects and higher morphisms invertible in all possible senses". Here we can formally allow the case $n = \infty$.

Incredible complexity of the stable homotopy category, well known to topologists, prevents us from hoping for a simple algebraic definition of Picard n-groupoids. Nevertheless, for small values of $n = 2, 3$, this can be accessible and useful. The case $n = 2$ is being treated in the paper [21] building on the theory of symmetric monoidal 2-categories [20, 22].

3.3 The sphere spectrum and the free Picard n-groupoid

The fundamental role in homotopy theory is played by the *sphere spectrum* $\mathbb{S} = \Sigma^\infty S^0$ defined as the suspension spectrum of the 0-sphere. Its homotopy groups are the *stable homotopy groups of spheres*

$$\pi_i(\mathbb{S}) \;=\; \pi_i^{\mathrm{st}} \;:=\; \varinjlim_n \pi_{i+n}(S^n),$$

which vanish for $i < 0$, so $\mathbb{S}$ is a $[0, \infty]$-spectrum. The spectrum $\mathbb{S}$ is the unit object in the symmetric monoidal structure on SHo and for this reason can be considered as a homotopy-theoretic analog of the ring $\mathbb{Z}$ of integers.

This motivates the further installment in Grothendieck's vision of a dictionary between spectra and Picard n-groupoids:

Conjecture 3.3.1. *The Picard n-groupoid corresponding to $\tau_{[0,n]}\mathbb{S}$, shoud be identified with $\mathcal{F}_L^{(n)}$ the free Picard n-groupoid on one formal object L.*

$\pi_0^{\rm st} = \mathbb{Z}$	$\pi_1^{\rm st} = \mathbb{Z}/2$	$\pi_2^{\rm st} = \mathbb{Z}/2$	$\pi_3^{\rm st} = \mathbb{Z}/24$
Cardinality of sets, dimension of vector spaces	Sign of permutation, determinant, Koszul rule	Spin, central extensions of symmetric groups	"String", $\sqrt[24]{1}$ in transformation of $\eta(q)$, $\chi(\mathrm{K3}) = 24$
$K_0(\mathbf{k}) = \mathbb{Z}$	$K_1(\mathbf{k}) = \mathbf{k}^*$	$K_2(\mathbb{R}) \to \pi_1(SL_n(\mathbb{R}))$ $\pi_1(SL_n(\mathbb{R})) = \mathbb{Z}/2$	Dilogarithms etc.

Table 1: The first few $\pi_i^{\rm st}$ and their significance.

The concept of a free Picard n-groupoid presumes that we already have a system of axioms for what a Picard n-groupoid is. If we have such axioms, then $\mathcal{F}_L^{(n)}$ contains as objects, formal tensor powers $L^{\otimes n}$ and only those higher morphisms which are needed to write the necessary "higher symmetry isomorphisms".

As before, for large n this seems unattainable directly, but for small $n \leq 2$ this can be made into a theorem. In particular, the case $n = 1$, proved in [25, Prop. 3.1] has enormous significance for super-geometry. Indeed, combining it with Proposition 3.1.2, we arrive at:

Corollary 3.3.2. *The sign skeleton* $1\text{-SAb}^{\mathbb{Z}}$ *of the* $\mathbb{Z}$*-graded Koszul sign rule, is the Picard groupoid corresponding to the* $[0,1]$*-truncation of* $\mathbb{S}$*. The groups* $\mathbb{Z} = \pi_0(1\text{-Ab}^{\mathbb{Z}})$ *and* $\mathbb{Z}/2 = \pi_1(1\text{-Ab}^{\mathbb{Z}})$ *are the first two stable homotopy groups of spheres.*

In other words, *the entire super-mathematics is obtained by unravelling the first two layers of the sphere spectrum.*

In Table 1 (which expands, somewhat, a table from the online encyclopedia `nLab`) we give the values of the $\pi_i^{\rm st}$ for $i \leq 3$ and indicate mathematical and physical phenomena that these groups govern. We also compare $\mathbb{S}$ with another spectrum, the *algebraic K-theory spectrum* $\mathbb{K}(\mathbf{k})$ of a field $\mathbf{k}$, which has $\pi_i(\mathbb{K}(\mathbf{k})) = K_i(\mathbf{k})$, the Quillen K-groups. These groups are indicated at in the bottom row. We notice that the first two groups are π_i of our intermediate Picard groupoid $1\text{-Vect}_{\mathbf{k}}^{\mathbb{Z}}$ which corresponds to the $[0,1]$-truncation of $\mathbb{K}(\mathbf{k})$, see [6], §4. A philosophy going back to [4] and to Quillen, says that $\mathbb{S}$ can be heuristically considered as the K-theory spectrum of $\mathbb{F}_1$, the (non-existent) field with one element, the symmetric group S_n being the "limit", as $q \to 1$, of the general linear groups $GL_n(\mathbb{F}_q)$.

The first two columns are self-explanatory. In the third column, the phenomenon of spin(ors) is based on the fundamental group $\pi_1(SO_n) = \pi_1(SL_n(\mathbb{R})) = \mathbb{Z}/2$ which is the same as $\pi_1^{\rm st}(SO_n)$ and is identified with $\pi_1^{\rm st} = \mathbb{Z}/2$ via the map $SO_n \to \Omega^{n+1}(S^{n+1})$ (this is known as J-homomorphism). The existence of central extensions of symmetric and alternating groups S_n and A_n (with center $\mathbb{Z}/2$) and of corresponding projective representations [42] [29] is a related phenomenon: A_n embeds into SO_n, and taking the preimage in $\mathrm{Spin}(n)$, we get a $\mathbb{Z}/2$-extension.

One thing is worth noticing. Supergeometry, as understood by mathematicians, tackles only the first two columns of Table 1. A a similar-sounding concept (supersymmetry) used by physicists, dips into the third column as well: fermions are always wedded to spinors in virtue of the Spin-Statistics Theorem. In fact, there is something in the very structure of the sphere spectrum that seems to relate spin (third column) and statistics (the second column). At the most naive level, this is the coincidence of $\pi_1^{\mathrm{st}} = \mathbb{Z}/2$ with $\pi_2^{\mathrm{st}} = \mathbb{Z}/2$.

Further, consider the $[1,2]$-truncated spectrum $\tau_{[1,2]}\mathbb{S}$. Its loop spectrum $\Omega\tau_{[1,2]}\mathbb{S}$ is a $[0,1]$-spectrum with $\pi_0 = \pi_1^{\mathrm{st}} = \mathbb{Z}/2$ and $\pi_1 = \pi_2^{\mathrm{st}} = \mathbb{Z}/2$.

3.3.3. Homotopy-theoretic Spin-Statistics Theorem. *The Picard groupoid corresponding to $\Omega\tau_{[1,2]}\mathbb{S}$, is equivalent to* 1-SAb*, the sign skeleton of the $\mathbb{Z}/2$-graded Koszul sign rule. In other words, $\Omega\tau_{[1,2]}\mathbb{S}$ is homotopy equivalent to $(\tau_{[0,1]}\mathbb{S})/2$, the reduction of the spectrum $\tau_{[0,1]}\mathbb{S}$ by the element* 2 *of its π_0.*

So there is not one, but *two ways* in which the same Koszul sign rule appears out of the sphere spectrum, one through statistics, the other one through spin. Note that the "topological proof" of the usual physical Spin-Statistics Theorem, going back to Feynman [15] (see also [14], §20), is based on the intuitive claim that interchanging two particles is "equivalent" to tracing a non-trivial loop in the rotation group, and this claim needs something like 3.3.3 to be consistent.

Proof of 3.3.3: This is an exercise on known facts in homotopy theory. A $[0,1]$-spectrum Y (or a Picard groupoid) is classified by its π_0, π_1 and the *Postnikov invariant* which is a group homomorphism $k_Y : \pi_0(Y) \to \pi_1(Y)$ satisfying $2k_Y = 0$. Explicitly, k_Y is given by the composition product with the generator $\eta \in \pi_1^{\mathrm{st}}$. For $Y = \tau_{[0,1]}\mathbb{S}$ we have therefore that $k_Y : \mathbb{Z} \to \mathbb{Z}/2$ is the surjection, and for $Y = \tau_{[0,1]}\mathbb{S}/2$ we have $k_Y = \mathrm{Id}_{\mathbb{Z}/2}$.

If both $\pi_0 = \pi_1 = \mathbb{Z}/2$ and $k_Y \neq 0$, then $k_Y = \mathrm{Id}$, like for $Y = (\tau_{[0,1]}\mathbb{S})/2$ and so $Y \sim (\tau_{[0,1]}\mathbb{S})/2$. Now, for $Y = \Omega\tau_{[1,2]}\mathbb{S}$ the Postnikov invariant is the map $\pi_1^{\mathrm{st}} \to \pi_2^{\mathrm{st}}$ given by composition with η. It is known [50] that $\eta^2 \in \pi_2^{\mathrm{st}}$ is the generator and so $k_Y \neq 0$. $\square$

The 4th column of Table 1, headed by $\pi_3^{\mathrm{st}} = \mathbb{Z}/24$, is related to various "string-theoretic" mathematics such as the appearance of 24th roots of 1 in Dedekind's formula for the modular transformation of his η-function [2], the Euler characteristic of a K3 surface being 24, the importance of the central charge modulo 24 in conformal field theory and so on, cf. [23].

3.4 Towards higher supergeometry

Conjecture 3.3.1 means that the sphere spectrum $\mathbb{S}$ is the ultimate source for meaningful twists of commutativity, i.e., for designing truly commutative-like structures, flexible enough to serve as a basis of geometry. The existing super-mathematics uses only the first two levels (0th and 1st) of $\mathbb{S}$, with physical applications exploiting the parallelism between the 1st and the 2nd levels.

This opens up a fantastic possibility of *higher super-mathematics* which would use, as its "sign skeleton", the spectrum $\mathbb{S}$ in its entirety or, at least, the truncations $\tau_{[0,n]}\mathbb{S}$ and the free Picard n-groupoids $\mathcal{F}_L^{(n)}$ for as long as we can make sense of them algebraically. Here we sketch the first step in this direction, the formalism for $n = 2$. For convenience, we adopt a genetic approach.

3.4.1. Idea of supersymmetric monoidal categories. Super-mathematics begins with replacing commutatibity $ab = ba$ with supercommutativity (1.1.1). Categorical analogs of commutative algebras are symmetric monoidal categories, where we have coherent isomorphisms $V \otimes W \simeq W \otimes V$. So we introduce "categorical minus signs" into these isomorphisms as well.

More precisely, by a $\mathbf{k}$-*superlinear category* we mean a module category $\mathcal{V}$ (a category tensored over) the symmetric monoidal category $\mathrm{SVect}_{\mathbf{k}}$. In such a category we have the *parity change functor* Π given by tensoring with the super-vector space $\mathbf{k}^{0|1}$. We take Π as the categorical analog of the minus sign. In doing so, we use the identification of $\pi_0(\mathbb{S}/2) = \mathbb{Z}/2$ (the $\mathbb{Z}/2$-grading) with $\pi_1^{\mathrm{st}} = \mathbb{Z}/2$ (the ± 1 signs).

We now consider $\mathbf{k}$-superlinear categories $\mathcal{A}$ which are $\mathbb{Z}$- or $\mathbb{Z}/2$-*graded*, i.e., split into a categorical direct sum $\mathcal{A} = \boxplus_i \mathcal{A}^i$, where $i \in \mathbb{Z}$ or $\mathbb{Z}/2$. We assume that $\mathcal{A}$ is equipped with graded $\mathrm{SVect}_{\mathbf{k}}$-bilinear bifunctors

$$\otimes \; = \; \otimes_{i,j} : \mathcal{A}^i \times \mathcal{A}^j \longrightarrow \mathcal{A}^{i+j}$$

subjects to associativity isomorphisms of the usual kind and want to impose *twisted commutativity isomorphisms*

$$R_{V,W} : V \otimes W \longrightarrow \Pi^{\deg(V)\cdot\deg(W)}(W \otimes V)$$

subjects to natural axioms, in which, further, various numerical minus signs will be introduced.

3.4.2. Definition of supersymmetric monoidal categories. For simplicity consider the $\mathbb{Z}$-graded version. We use Proposition 3.1.4 as a guideline, and start with $\mathcal{F} = \mathcal{F}_L^{(2)}$, the free Picard 2-groupoid on one object L. It corresponds to the truncation $\tau_{[0,2]}\mathbb{S}$. Thus $\pi_0(\mathcal{F})$, the group of equivalence classes of objects, is identified with $\pi_0^{\mathrm{st}} = \mathbb{Z}$ and will account for the grading.

The category $\mathrm{Aut}_{\mathcal{F}}(\mathbf{1})$ formed by automorphisms of the unit object and 2-morphisms between them, is a usual Picard groupoid which corresponds to $\Omega\tau_{[1,2]}\mathbb{S}$ and so, by the "Spin-Statistics Theorem" 3.3.3, it is identified with 1-SAb, the sign skeleton of $\mathrm{SVect}_{\mathbf{k}}$.

We denote by $\mathrm{SCat}_{\mathbf{k}}$ the 2-category of $\mathbf{k}$-superlinear categories. It serves as a categorical analog of the category of ordinary vector spaces. We will denote by $\mathcal{V} \boxtimes_{\mathrm{SVect}_{\mathbf{k}}} \mathcal{W}$ the categorical tensor product [19] of two $\mathbf{k}$-superlinear categories $\mathcal{V}$ and $\mathcal{W}$

The Picard groupoid 1-$\mathrm{SVect}_{\mathbf{k}}$ plays the role of the multiplicative group $\mathbf{k}^*$ for $\mathrm{SCat}_{\mathbf{k}}$: it acts on each object by equivalences. The monoidal functor (embedding) $\boldsymbol{\chi}$: 1-SAb $\to$ 1-$\mathrm{SVect}_{\mathbf{k}}$ is therefore an analog of the homomorphism χ from Definition 3.1.3.

We now consider the 2-category $\mathrm{SCat}_{\mathbf{k}}^{(\mathcal{F},\boldsymbol{\chi})}$ formed by all 2-functors $\mathcal{V} : \mathcal{F} \to \mathrm{SCat}_{\mathbf{k}}$ which take the action of $\mathrm{Aut}_{\mathcal{F}}(\mathbf{1})$ on each object A into the action on $\mathcal{V}^A$ given by $\boldsymbol{\chi}$. As before, a datum of such $\mathcal{V}$ is the same as a datum of a family of superlinear categories $\mathcal{V}^i = \mathcal{V}^{L^{\otimes i}}$, $i \in \mathbb{Z}$, one for each equivalence class of objects of $\mathcal{F}$. Now, the formula

$$(\mathcal{V} \boxtimes \mathcal{W})^A \;=\; 2\varinjlim_{\{B\otimes C\to A\}} \mathcal{V}^B \boxtimes_{\mathrm{SVect}_{\mathbf{k}}} \mathcal{W}^C$$

makes $\mathrm{SCat}_{\mathbf{k}}^{(\mathcal{F},\boldsymbol{\chi})}$ into a symmetric monoidal 2-category. It can be seen as the categorical analog of the category of super-vector spaces.

By definition, a *supersymmetric monoidal* $\mathbf{k}$*-category* is a symmetric monoidal object $\mathcal{A}$ in $\mathrm{SCat}_{\mathbf{k}}^{(\mathcal{F},\boldsymbol{\chi})}$. An explicit algebraic model for $\mathcal{F}$ was proposed in [5], Ex. 5.2, see also [41], Ex. 2.30. Taking this model for $\mathcal{F}$, we can unravel the data involved in $\mathcal{A}$. These data in particular, contain superlinear categories $\mathcal{A}^i$, bifunctors $\otimes_{i,j}$ and isomorphisms $R_{V,W}$ as outlined in 3.4.1.

The definition of a $\mathbb{Z}/2$-graded supersymmetric monoidal category is similar, using the Picard 2-groupoid $\mathcal{F}/2$.

If V is an object of a symmetric monoidal category, then there is an action of the symmetric group S_n on $V^{\otimes n}$. If, instead $V \in \mathcal{A}^{2m+1}$ is an odd object of a supersymmetric monoidal category $\mathcal{A}$, then $V^{\otimes n} \oplus \Pi(V^{\otimes n})$ has an action of the (spin) central extension of S_n, first discovered by Schur [42].

Examples 3.4.3. (a) The exterior algebra of a superlinear category. Similarly to usual linear algebra (Examples 1.1.2), an example of a supersymmetric monoidal category can be extracted from the *categorical version of the exterior power construction* developed in [17]. This construction is based on the *categorical sign character* which is a functor of monoidal categories

$$\mathrm{sgn}_2 : S_n \longrightarrow 1\text{-SAb}$$

(S_n is the symmetric group considered as a discrete monoidal category). It combines the usual sign character $\mathrm{sgn} : S_n \to \mathbb{Z}/2$ at the level of π_0 and the "spin-cocycle" $c \in H^2(S_n, \mathbb{Z}/2)$ at the level of π_1. The exterior power $\Lambda^n\mathcal{V}$ of a superlinear category $\mathcal{V}$ is obtained from the tensor power $\mathcal{V}^{\boxtimes n}$ by considering objects equipped with sgn_2-twisted S_n-equivariance structure (see [17] for precise context and details). The analog of the wedge product is given by the functors

$$\wedge_{m,n} : \Lambda^m\mathcal{V} \times \Lambda^n\mathcal{V} \longrightarrow \Lambda^{m+n}\mathcal{V}$$

given by partial Π-antisymmetrization, as in [17] §4.2.

(b) Superalgebras of types M and Q and the half-tensor product of Sergeev. Let $\mathbf{k}$ be algebraically closed. It is known since C.T.C. Wall [48] that simple finite-dimensional associative superalgebras over $\mathbf{k}$ are of two types: *type M*, formed by the matrix superalgebras $M_{p|q} = \mathrm{End}(\mathbf{k}^{p|q})$ and *type Q*, formed by the so-called *queer superalgebras* $Q_n \subset M_{n|n}$, see [26] and [29]. The simplest nontrivial queer algebra is the Clifford algebra Cliff_1 on one generator

$$Q_1 = \mathrm{Cliff}_1, \quad \mathrm{Cliff}_n \;=\; \mathbf{k}[\xi_1, \cdots \xi_n]/\big(\xi_i^2 = 1,\ \xi_i\xi_j = -\xi_j\xi_i\big),\ \deg(\xi_i) = \bar{1}.$$

Their behavior under tensor multiplication is

$$M_{p|q} \otimes M_{m|n} \simeq M_{pm+qn|pn+qm}, \quad M_{p|q} \otimes Q_n \simeq Q_{(p+q)n}, \quad Q_m \otimes Q_n \simeq M_{mn|mn}. \tag{3.4.4}$$

This means that the *super-Brauer group* formed by Morita equivalence classes of these algebras, is identified with $\mathbb{Z}/2$, with type M mapping to $\bar{0}$ and type Q mapping to $\bar{1}$. This $\mathbb{Z}/2$ is nothing but π_2^{st}, responsible for spin.

As a consequence, irreducible objects of any semisimple $\mathbf{k}$-superlinear category $\mathcal{V}$ also split into two types M and Q, according to their endomorphism algebras being $\mathbf{k}$ or Q_1. Denoting $\mathcal{V}^{\bar{0}}$ the subcategory formed by direct sum of objects of type M and $\mathcal{V}^{\bar{1}}$ the subcategory formed by sums of objects of type Q, we get an intrinsic $\mathbb{Z}/2$-grading on $\mathcal{V}$. By (3.4.4), any exact monoidal structure $\otimes$ on $\mathcal{V}$ preserves this grading.

Further, if V, W are irreducible objects of type Q, then $V \otimes W$ is acted upon by $Q_1 \otimes Q_1 \simeq \mathrm{End}(\mathbf{k}^{1|1})$ and so is identified with the direct sum of some object with its shift:

$$V \otimes W \;\simeq\; (2^{-1}V \otimes W) \oplus \Pi(2^{-1}V \otimes W), \quad 2^{-1}V \otimes W := I \otimes_{Q_1 \otimes Q_1} (V \otimes W)$$

where $I \simeq (\mathbf{k}^{1|1})^*$ is an irreducible right module over $Q_1 \otimes Q_1$. We get in this way a new monoidal operation

$$\otimes_{\bar{1},\bar{1}} : \mathcal{V}^{\bar{1}} \times \mathcal{V}^{\bar{1}} \longrightarrow \mathcal{V}^{\bar{0}}, \quad V \otimes_{\bar{1},\bar{1}} W := 2^{-1}V \otimes W.$$

This operation was introduced by A. Sergeev [43], see also [29], p.163 for more discussion. If $\otimes$ is symmetric, then $\otimes_{\bar{1},\bar{1}}$ satisfies

$$V \otimes_{\bar{1},\bar{1}} W \;\simeq\; \Pi(W \otimes_{\bar{1},\bar{1}} V).$$

Indeed, the interchange of the factors in $V \otimes W$ corresponds to the interchange of ξ_1 and ξ_2 in $\mathrm{Cliff}_2 = Q_1 \otimes Q_1$, and the pullback of I under this interchange is isomorphic to $\Pi(I)$ (which, as a right Cliff_2-module, is not isomorphic to I). The operation $\otimes_{\bar{1},\bar{1}}$ can be used as a source of examples of supersymmetric monoidal structures.

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