LECTURE 1

# The Brouwer Fixed Point Theorem

Even the simplest problems in topology – for instance, whether two topological spaces $X$ and $Y$ are homeomorphic – are oftentimes very hard to answer. In order to show $X$ and $Y$ are homeomorphic, it suffices to find a single homeomorphism $f\colon X \to Y$. But in order to show that they are *not* homeomorphic, one needs to prove no such homeomorphism can exist. And how on earth are you meant to do that? Even if *you* can't find one, how do you know that tomorrow some really smart mathematician isn't going to magically come up with one? This is where *algebraic topology* comes in. The idea is to associate *algebraic invariants* of a topological space. Here "invariants" means that two homeomorphic spaces should have the same invariants. Thus to show two spaces are *not* homeomorphic, it suffices to show they have different invariants.

So, to summarise the entire course:

- Topology is hard.
- Algebra is easy.
- Algebraic topology converts topological problems into algebraic problems.
- Profit.

We illustrate this philosophy with an example. Let $B^n \subset \mathbb{R}^n$ denote the *closed* n-dimensional unit ball

$$B^n = \{x \in \mathbb{R}^n \mid |x| \le 1\}.$$

The boundary of $B^n$ is the $(n-1)$-dimensional unit sphere $S^{n-1}$:

$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The following famous theorem is due to Brouwer.

Theorem 1.1 (The Brouwer Fixed Point Theorem). *For all $n \ge 1$, every continuous map $f\colon B^n \to B^n$ has a fixed point.*

In the case $n = 1$, this theorem has a simple proof using connectivity:

*Proof of Theorem 1.1 in the case $n = 1$.* Suppose $f(-1) = a$ and $f(1) = b$. If $a = -1$ or $b = 1$ we are done, so assume that $a > -1$ and $b < 1$. Consider the graph of $f$:

$$\mathrm{Gr}(f) := \{(x, f(x)) \mid x \in [-1, 1]\}.$$

A fixed point of $f$ is the same thing as a point of intersection between $\mathrm{Gr}(f)$ and the diagonal

$$\Delta := \{(x, x) \mid x \in [-1, 1]\}.$$

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Since $f$ is continuous, $\mathrm{Gr}(f)$ is connected[1]. Let

$$A := \{(x, f(x)) \mid f(x) > x\}, \qquad B := \{(x, f(x)) \mid f(x) < x\}.$$

Then $(-1, a) \in A$ and $(1, b) \in B$, so in particular $A$ and $B$ are both non-empty. If $\mathrm{Gr}(f) \cap \Delta = \emptyset$ then $\mathrm{Gr}(f) = A \cup B$. Since $f$ is continuous, $A$ and $B$ are open[2] in $\mathrm{Gr}(f)$. This contradicts the fact that $\mathrm{Gr}(f)$ is connected. ■

Interestingly, it is not known how to extend this simple argument to deal with the case $n > 1$. Nevertheless there are several different complicated arguments. For instance, there is an analytical argument that goes as follows: first approximate $f$ by a sequence of *differentiable* functions $g_k$ with the property that $f$ has a fixed point if and only if all the $g_k$ do for large $k$. Then prove directly that any differentiable function must have a fixed point.

The "cutest" proof uses methods from algebraic topology. Later on in the course we will construct a **homology functor** $H_n$ for each $n \geq 0$, which associates to any topological space $X$ an abelian group $H_n(X)$, and to any continuous map $f \colon X \to Y$ a homomorphism

$$H_n(f) : H_n(X) \to H_n(Y).$$

The induced maps $H_n(f)$ have the property that if $f \colon X \to Y$ and $g : Y \to Z$ then

$$H_n(g \circ f) = H_n(g) \circ H_n(f) \colon H_n(X) \to H_n(Z), \tag{1.1}$$

and

$$H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)} \colon H_n(X) \to H_n(X). \tag{1.2}$$

Moreover the homology functor $H_n$ vanishes on the ball $B^{n+1}$ but not on the sphere $S^n$:

$$H_n(B^{n+1}) = 0, \qquad H_n(S^n) \neq 0, \qquad \forall\, n \geq 1. \tag{1.3}$$

The construction of $H_n$ and the verification of (1.1), (1.2) and (1.3) will take some time. Nevertheless, armed with these only these properties, it is easy to prove Theorem 1.1 in all dimensions.

Definition 1.2. Suppose $X$ is a subspace of a topological space $Y$. We say that $X$ is a **retract** of $Y$ if there exists a continuous map $r : Y \to X$ such that $r(x) = x$ for all $x \in X$. Equivalently, denoting by $\imath : X \hookrightarrow Y$ the inclusion, this means that the following diagram *commutes*:

$$\begin{tikzcd} & Y \arrow[dr, "r"] & \\ X \arrow[ur, hook, "\imath"] \arrow[rr, "\mathrm{id}"] & & X \end{tikzcd}$$

[1]It is the image of the continuous map $B^1 \to B^1 \times B^1$ given by $x \mapsto (x, f(x))$.

[2]Consider the map $g : \mathrm{Gr}(f) \to \mathbb{R}$ given by $g(x, f(x)) = x - f(x)$. Then $A = g^{-1}\big((-\infty, 0)\big)$ and $B = g^{-1}\big((0, \infty)\big)$.

LEMMA 1.3. *For all $n \geq 1$, $S^n$ is not a retract of $B^{n+1}$.*

*Proof.* Suppose for contradiction that there exists a retraction $r\colon B^{n+1} \to S^n$, so that the following diagram commutes:

$$\begin{tikzcd} & B^{n+1} \arrow[dr, "r"] & \\ S^n \arrow[ur, hook, "\imath"] \arrow[rr, "\mathrm{id}"] & & S^n \end{tikzcd}$$

Equation (1.1) means that we can "apply the homology functor $H_n$" to this commutative diagram to obtain another one:

$$\begin{tikzcd} & H_n(B^{n+1}) \arrow[dr, "H_n(r)"] & \\ H_n(S^n) \arrow[ur, "H_n(\imath)"] \arrow[rr, "H_n(\mathrm{id})"] & & H_n(S^n) \end{tikzcd}$$

Note this diagram is a commutative diagram of group homomorphisms between abelian groups, rather than a commutative diagram of continuous maps between topological spaces. Since $H_n(B^{n+1}) = 0$ by (1.3) the map $H_n(r)\colon H_n(B^{n+1}) \to H_n(S^n)$ is the zero map. But since $H_n(\mathrm{id}) = \mathrm{id}$ by (1.2) and $H_n(S^n) \neq 0$, this is a contradiction. ∎

REMARK 1.4. In fact, Lemma 1.3 is also true for $n = 0$. The 0-dimensional sphere is just $\{-1, 1\}$, which is disconnected. Since $[-1, 1]$ is connected and the image of a connected subset under a continuous map is connected, it follows there does not exist *any* continuous surjective map $r\colon B^1 \to S^0$ (and thus in particular there does not exist a retraction.)

We now show how Theorem 1.1 follows from Lemma 1.3.

*Proof of Theorem 1.1.* Take $n \geq 0$. Suppose $f\colon B^{n+1} \to B^{n+1}$ has no fixed points. Then for every point $x \in B^{n+1}$, there is a unique line that starts at $f(x)$, goes through $x$, and then hits a point on the boundary $S^n$ of $B^{n+1}$. Let us denote by $r\colon B^{n+1} \to S^n$ the map that sends $x$ to the point on $S^n$ that this line hits. See Figure 1.1. Since $f$ is continuous, the map $r$ is also continuous[3]. If $x \in S^n$ then clearly $r(x) = x$. Thus $r$ is a retraction. This contradicts[4] Lemma 1.3. ∎

Let us now formalise the notion of a "homology functor", by introducing elements of a field of mathematics called **category theory**. In this course, we will only ever use category theory as a convenient "language" to phrase theorems from algebraic topology in—we will never actually use any genuine theorems in category theory.

[3] This is an easy exercise.

[4] If $n = 0$, apply Remark 1.4 instead of Lemma 1.3.

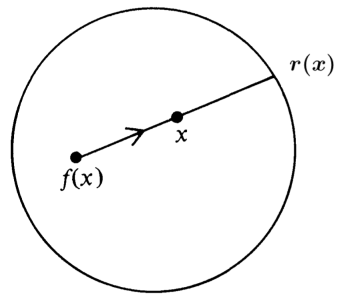


Figure 1.1: The retract $r$.

Remark 1.5. A word of warning: category theory is often (lovingly) referred to as abstract nonsense. But fear not: nothing we will do will ever be *that* abstract!

Definition 1.6. A **category** $\mathsf{C}$ consists of three ingredients. The first is a *class* $\mathrm{obj}(\mathsf{C})$ of **objects**. Secondly, for each ordered pair of objects $(A, B)$ there is a *set* $\mathrm{Hom}(A, B)$ of **morphisms** from $A$ to $B$. Sometimes instead of $f \in \mathrm{Hom}(A, B)$ we write $f : A \to B$ or $A \xrightarrow{f} B$. Finally, there is a rule, called **composition**, which associates to every ordered triple $(A, B, C)$ of objects a map

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C),$$

written

$$(f, g) \mapsto g \circ f,$$

which satisfies the following three axioms:

1. The Hom sets are pairwise disjoint; that is, each $f \in \mathrm{Hom}(A, B)$ has a unique **domain** $A$ and a unique **target** $B$.
2. Composition is associative whenever defined, i.e. given

   $$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

   one has

   $$(h \circ g) \circ f = h \circ (g \circ f).$$

3. For each $A \in \mathrm{obj}(\mathsf{C})$ there is a unique morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ called the *identity* which has the property that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$ for every $f : A \to B$.

Remark 1.7. Note that we said that $\mathrm{obj}(\mathsf{C})$ was a *class* and $\mathrm{Hom}(A, B)$ was a *set*. There is (an important, but technical) difference between a class and a set. If you've ever taken a class on logic/set theory, you'll know that not every "collection" of objects is formally a set. For instance, the collection of all sets is itself not a set! A class is a more general concept (the collection of all sets is a class). Nevertheless, as far as this course is concerned, the distinction is irrelevant, and you are free to ignore this remark!

Here are four examples of categories:

Example 1.8. The category $\mathsf{Sets}$ of **sets**. The objects of $\mathsf{Sets}$ are all the sets, and $\mathrm{Hom}(A, B)$ is just the set of all functions from $A$ to $B$, and composition is just the usual composition of functions.

Example 1.9. The category $\mathsf{Top}$ of **topological spaces**. The objects of $\mathsf{Top}$ are all the topological spaces, and $\mathrm{Hom}(X, Y)$ is just the set $C(X, Y)$ of all *continuous* functions from $X$ to $Y$, and composition is just the usual composition of functions.

Example 1.10. The category $\mathsf{Groups}$ of **groups**. The objects of $\mathsf{Groups}$ are just groups, and $\mathrm{Hom}(G, H)$ is just the set $\mathrm{Hom}(G, H)$ of all *homomorphisms* from $G$ to $H$, and composition is just the usual composition of homomorphisms.

Example 1.11. The category $\mathsf{Ab}$ of **abelian groups**. The objects of $\mathsf{Ab}$ are just abelian groups, and $\mathrm{Hom}(G, H)$ is again just the set $\mathrm{Hom}(G, H)$ of all *homomorphisms* from $G$ to $H$, and composition is just the usual composition of homomorphisms.

Remark 1.12. The fact that we require the morphism sets to be pairwise disjoint has several pedantic consequences. For example, suppose $A \subsetneq B$ are two sets. Then the inclusion $\imath \colon A \hookrightarrow B$ and the identity map $\mathrm{id}_A \colon A \to A$ are different morphisms, since they have different targets. One should be aware that we only allow the composition $g \circ f$ when the range of $f$ is *exactly* the same as the domain of $g$. Suppose $X, Y, Y'$ and $Z$ are topological spaces with $Y \subsetneq Y'$. From the point of view of analysis, say, if $f \colon X \to Y$ and $g \colon Y' \to Z$ are continuous functions then the composition $g \circ f \colon X \to Z$ is clearly a well-defined continuous function. But from the point of view of category theory, the composition $g \circ f$ *does not exist!* Rather, one must first take the inclusion $\imath \colon Y \hookrightarrow Y'$ and then consider the composition $g \circ \imath \circ f$, which is a well-defined element of the morphism space $C(X, Z)$.

A **functor** is a map from one category to another:

Definition 1.13. Suppose $\mathsf{C}$ and $\mathsf{D}$ are two categories. A **functor** $T \colon \mathsf{C} \to \mathsf{D}$ associates to each $A \in \mathrm{obj}(\mathsf{C})$ an object $T(A) \in \mathrm{obj}(\mathsf{D})$, and to each morphism $A \xrightarrow{f} B$ in $\mathsf{C}$ a morphism $T(A) \xrightarrow{T(f)} T(B)$ in $\mathsf{D}$ which satisfies the following two axioms:

1. If $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathsf{C}$ then $T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$ in $\mathsf{D}$ and
$$T(g \circ f) = T(g) \circ T(f).$$
2. $T(\mathrm{id}_A) = \mathrm{id}_{T(A)}$ for every $A \in \mathrm{obj}(\mathsf{C})$.

The easiest example of a functor is a **forgetful functor**:

Example 1.14. The forgetful functor $\mathsf{Top} \to \mathsf{Sets}$ simply "forgets" the topological structure. Thus it assigns to each topological space its underlying set, and to each continuous function it assigns the same function, considered now simply as a map between two *sets* (i.e. it "forgets" the function is continuous).

We can now make sense of the homology functor mentioned earlier.

THEOREM 1.15. *For each* $n \geq 0$ *there exists a functor* $H_n : \mathsf{Top} \to \mathsf{Ab}$ *called a* **homology functor** *with the property that for all* $n \geq 0$,

$$H_n(B^{n+1}) = 0, \qquad H_n(S^n) \neq 0.$$

I say "a" homology functor since $H_n$ is not (quite) unique (we will construct several different ones eventually). In fact, before constructing homology functors we will first construct an "easier" functor called the **fundamental group**. This will (almost[5]) be a functor

$$\pi_1 \colon \mathsf{Top} \to \mathsf{Groups},$$

and its construction will take us up the end of Lecture 4.

[5]Strictly speaking $\pi_1$ will be a functor from the category of *pointed topological spaces*, more on this later.

LECTURE 2

# The notion of homotopy

In this lecture we introduce the notion of "deforming" one function (or one space) into another, which mathematically is known as **homotopy**. Throughout this course, we denote by

$$I := [0,1], \qquad \partial I = \{0,1\}.$$

DEFINITION 2.1. Suppose $X$ and $Y$ are topological spaces and $f_0, f_1 \colon X \to Y$ are two continuous functions. A **homotopy** from $f_0$ to $f_1$ is a continuous function

$$F \colon X \times I \to Y$$

such that

$$F(x,0) = f_0(x), \qquad F(x,1) = f_1(x).$$

We write $F \colon f_0 \simeq f_1$ to indicate $F$ is a homotopy from $f_0$ to $f_1$, and we write $f_0 \simeq f_1$ to indicate there exists such an $F$.

Given a homotopy $F \colon f_0 \simeq f_1$, setting $f_t(x) := F(x,t)$, we obtain a family $f_t$ of continuous functions which deforms $f_0$ at time $t = 0$ into $f_1$ at time $t = 1$. Since $F$ is continuous on $X \times I$, the family $f_t$ depends continously on $t$.

The following lemma will be used time and time again. We will refer to it as "the gluing lemma".

LEMMA 2.2 (The gluing lemma). *Let $X$ be a topological space. Assume $X$ can be written as a finite union*

$$X = \bigcup_{i=1}^{N} X_i,$$

*where each $X_i$ is a closed subspace of $X$. Assume we given a topological space $Y$ and continuous functions*

$$f_i \colon X_i \to Y,$$

*with the property that*

$$f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}, \qquad \forall\, i,j \text{ such that } X_i \cap X_j \neq \emptyset.$$

*Then there exists a unique continuous function $f \colon X \to Y$ such that*

$$f|_{X_i} = f_i, \qquad \forall\, i = 1, \dots, N. \tag{2.1}$$

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*Proof.* We need only check that the function $f$ defined as in (2.1) is continuous (it is clearly unique). Suppose $C \subseteq Y$ is a closed set. Then

$$\begin{aligned} f^{-1}(C) &= \left(\bigcup_{i=1}^{N} X_i\right) \cap f^{-1}(C) \\ &= \bigcup_{i=1}^{N} (X_i \cap f^{-1}(C)) \\ &= \bigcup_{i=1}^{N} (X_i \cap f_i^{-1}(C)) \\ &= \bigcup_{i=1}^{N} f_i^{-1}(C). \end{aligned}$$

Since each $f_i$ is continuous, this is the finite union of closed sets and hence is closed. Since $C$ was arbitrary, $f$ is continuous. ∎

On Problem Sheet A you will enjoy proving the following minor variation of Lemma 2.2.

Lemma 2.3 (Another gluing lemma). *Let $X$ be a topological space. Assume $X$ can be written as an arbitrary union*

$$X = \bigcup_i X_i,$$

*where each $X_i$ is an open subspace of $X$. Assume we given a topological space $Y$ and continuous functions*

$$f_i \colon X_i \to Y,$$

*with the property that*

$$f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}, \qquad \forall\, i, j \text{ such that } X_i \cap X_j \neq \emptyset.$$

*Then there exists a unique continuous function $f \colon X \to Y$ such that*

$$f|_{X_i} = f_i, \qquad \forall\, i \in \mathbb{N}.$$

Our first application of the gluing lemma is to show that homotopy is an equivalence relation on the space of continuous maps.

Proposition 2.4. *Let $X$ and $Y$ denote two topological spaces. Then homotopy is an equivalence relation on the space $C(X, Y)$ of all continuous maps from $X$ to $Y$.*

*Proof.* We check the three properties:

- *Reflexivity*: if $f \in C(X, Y)$ define $F(x, t) := f(x)$. Then clearly $F \colon f \simeq f$.
- *Symmetry*: if $F \colon f \simeq g$ then define $G(x, t) := F(x, 1 - t)$. Then $G \colon g \simeq f$.

- *Transitivity*: if $F\colon f \simeq g$ and $G\colon g \simeq h$, define

  $$H(x,t) := \begin{cases} F(x,2t), & 0 \le t \le \frac{1}{2}, \\ G(x,2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

  Since $F$ and $G$ agree on their overlap $X \times \{\frac{1}{2}\} \subset X \times I$ the gluing lemma implies that $H$ is continuous. Since $H(x,0) = f$ and $H(x,1) = h$, this shows that $f \simeq h$.

This completes the proof. ∎

Definition 2.5. We denote by $[f]$ the equivalence class of $f$ under homotopy, and we denote by $[X,Y]$ the space of equivalence classes.

We now show that composition of equivalence classes makes sense.

Proposition 2.6. *Suppose $f_0, f_1\colon X \to Y$ and $g_0, g_1\colon Y \to Z$ are continuous functions with $f_0 \simeq f_1$ and $g_0 \simeq g_1$. Then $g_0 \circ f_0 \simeq g_1 \circ f_1$, that is*

$$[f_0] = [f_1] \text{ and } [g_0] = [g_1] \qquad \Rightarrow \qquad [g_0 \circ f_0] = [g_1 \circ f_1].$$

*Proof.* Suppose $F\colon f_0 \simeq f_1$ and $G\colon g_0 \simeq g_1$. Define

$$H\colon X \times I \to Z, \qquad H(x,t) = G(f_0(x),t).$$

Then $H$ is continuous and since

$$H(x,0) = G(f_0(x),0) = g_0(f_0(x))$$

and

$$H(x,1) = G(f_0(x),1) = g_1(f_0(x)),$$

this shows that

$$g_0 \circ f_0 \simeq g_1 \circ f_0. \tag{2.2}$$

Next, consider

$$K\colon X \times I \to Z, \qquad K(x,t) = g_1(F(x,t)).$$

Then $K$ is continuous, and since

$$K(x,0) = g_1(F(x,0)) = g_1(f_0(x))$$

and

$$K(x,1) = g_1(F(x,1)) = g_1(f_1(x)),$$

this shows that

$$g_1 \circ f_0 \simeq g_1 \circ f_1. \tag{2.3}$$

Combining (2.2) and (2.3) and using transitivity, we see that $g_0 \circ f_0 \simeq g_1 \circ f_1$ as required. ∎

Now for this lecture's serving of abstract nonsense. Let us explain how to take quotients of categories.

Definition 2.7. Suppose $\mathsf{C}$ is a category. A **congruence** on $\mathsf{C}$ is an equivalence relation $\sim$ on the union

$$\bigcup_{(A,B)\in\mathrm{obj}(\mathsf{C})\times\mathrm{obj}(\mathsf{C})} \mathrm{Hom}(A,B)$$

such that:

1. If $f \in \mathrm{Hom}(A,B)$ and $f \sim g$ then $g \in \mathrm{Hom}(A,B)$.
2. If $f_0 : A \to B$ and $g_0 : B \to C$ and $f_0 \sim f_1$ and $g_0 \sim g_1$ then $g_0 \circ f_0 \sim g_1 \circ f_1$.

A congruence allows us to form the quotient category:

Proposition 2.8. *Suppose $\mathsf{C}$ is a category and $\sim$ is a congruence on $\mathsf{C}$. Denote by $[f]$ the equivalence class of a morphism under $\sim$. Then there is a well-defined* **quotient category** *$\mathsf{C}'$ given as follows: the objects of $\mathsf{C}'$ is simply* $\mathrm{obj}(\mathsf{C})$ *again, and*

$$\mathrm{Hom}_{\mathsf{C}'}(A,B) = \{[f] \mid f \in \mathrm{Hom}(A,B)\},$$

*and composition in $\mathsf{C}'$ is given by*

$$[g] \circ [f] := [g \circ f].$$

*Proof.* Property (1) of Definition 2.7 shows that the morphism sets of $\mathsf{C}'$ are well-defined sets that are pairwise disjoint. Property (2) of Definition 2.7 shows that the composition in $\mathsf{C}'$ is well-defined. It is clear that this composition is associative, and $[\mathrm{id}_A]$ is the identity morphism in $\mathrm{Hom}_{\mathsf{C}'}(A,A)$. This completes the proof. ∎

It will not surprise you to learn we have just constructed a congruence.

Example 2.9. The **homotopy category** $\mathsf{hTop}$ is the category whose objects are topological spaces, with morphism spaces $[X,Y]$ the equivalence class of continuous maps under homotopy. This is the quotient category of $\mathsf{Top}$ under the congruence obtained via homotopy.

Definition 2.10. Let $\mathsf{C}$ be a category. An **isomorphism** in $\mathsf{C}$ is a morphism $f \in \mathrm{Hom}(A,B)$ for which there exists another morphism $g \in \mathrm{Hom}(B,A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Thus in $\mathsf{Sets}$, isomorphisms are just bijections. In $\mathsf{Groups}$, isomorphisms are group isomorphisms, and in $\mathsf{Top}$, isomorphisms are homeomorphims. Let us unravel what an isomorphism in $\mathsf{hTop}$ is.

Definition 2.11. A continuous map $f\colon X \to Y$ between two topological spaces is called a **homotopy equivalence** if there exists a continuous map $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Thus an isomorphism in $\mathsf{hTop}$ is a morphism $[f]$, where $f$ is a homotopy equivalence.

Definition 2.12. We say that two spaces $X$ and $Y$ have the **same homotopy type** if there exists a homotopy equivalence from $f\colon X \to Y$.

Clearly homeomorphic spaces have the same homotopy type, but it is *not* necessarily the case that spaces with the same homotopy type are homeomorphic. We shall see an example of this next lecture. Let us now look at the "opposite" notion to a homotopy equivalence.

Definition 2.13. A continuous map $f\colon X \to Y$ is said to be **nullhomotopic** if there exists a **constant map** $c\colon X \to Y$ (i.e. map such that there exists $q \in Y$ such that $c(x) = q$ for all $x \in X$) such that $f \simeq c$.

When $X = S^n$, there is an easy criterion for deciding whether a map is nullhomotopic. Before stating the result, we need one more definition.

Definition 2.14. Suppose $X$ and $Y$ are topological spaces and $X'$ is a subset of $X$. We say that two continuous map $f_0, f_1\colon X \to Y$ such that $f_0|_{X'} = f_1|_{X'}$ are **homotopic relative to** $X'$ or **homotopic rel** $X'$ for short if there exists a homotopy $F\colon f_0 \simeq f_1$ such that

$$F(x,t) = f_0(x) = f_1(x), \qquad \forall\, x \in X', \forall\, t \in I.$$

If such an $F$ exists we write $F\colon f_0 \simeq f_1 \text{ rel } X'$.

This generalises Definition 2.1, since taking $X' = \emptyset$ recovers our original notion of homotopy. For fixed $X$ being homotopic rel $X'$ is an equivalence relation – you will prove a more general version statement on Problem Sheet B (a special case of this is given in Proposition 3.13 next lecture.) By a slight abuse of notation if $X'$ is a single point $\{p\}$ we will write "rel $p$" instead of "rel $\{p\}$".

Proposition 2.15. *Let $Y$ be a topological space and $n \geq 0$. The following are equivalent for a continuous map $f\colon S^n \to Y$:*

1. *$f$ is nullhomotopic.*
2. *There exists a continuous map $g\colon B^{n+1} \to Y$ such that $g|_{S^n} = f$.*
3. *If $p \in S^n$ and $c\colon S^n \to Y$ is the constant map $c(x) = f(p)$ then $f$ is homotopic to $c$ rel $p$.*

*Proof.* We first show (1) implies (2). Suppose $f$ is nullhomotopic, i.e. there exists $F\colon f \simeq c$, where $c$ is the constant map $c(x) = q$. Define $g\colon B^{n+1} \to Y$ by

$$g(x) := \begin{cases} q, & 0 \leq |x| \leq \frac{1}{2}, \\ F\big(\frac{x}{|x|}, 2 - 2|x|\big), & \frac{1}{2} \leq |x| \leq 1. \end{cases}$$

This makes sense: if $x \neq 0$ then $\frac{x}{|x|}$ belongs to $S^n$, and if $\frac{1}{2} \leq |x| \leq 1$ then $2 - 2|x| \in I$. If $|x| = \frac{1}{2}$ then

$$F\big(\tfrac{x}{|x|}, 1\big) = c\big(\tfrac{x}{|x|}\big) = q.$$

The gluing lemma shows that $g$ is continuous. Moreover $g$ does extend $f$ since if $x \in S^n$ then $|x| = 1$ and hence $g(x) = F(x, 0) = f(x)$.

To show (2) implies (3), suppose $g\colon B^{n+1} \to Y$ extends $f$. Define $F\colon S^n \times I \to Y$ by

$$F(x,t) = g((1-t)x + tp).$$

This makes sense as $(1-t)x + tp$ belongs to $B^{n+1}$. $F$ is clearly continuous, and $F(x,0) = g(x) = f(x)$ (since $g$ extends $f$) and $F(x,1) = g(p) = f(p) = c(x)$. Thus $F\colon f \simeq c$. Moreover $F(p,t) = g(p) = f(p)$ for all $t \in I$, and hence $F\colon f \simeq c$ rel $p$.

Finally, it is obvious that (3) implies (1). ■

DEFINITION 2.16. A space $X$ is said to be **contractible** if the identity map $\mathrm{id}_X$ is nullhomotopic.

COROLLARY 2.17. *For all $n \geq 0$, the sphere $S^n$ is not contractible.*

This proof uses Lemma 1.3, which we have not yet properly proved (we haven't constructed the homology functor yet!). You will be relieved to note that we will not use the proof of Corollary 2.17 in the construction of the homology functor.

*Proof.* Take $Y = S^n$ and $f = \mathrm{id}_{S^n}$. Then by Proposition 2.15, if $f$ is nullhomotopic then there exists a continuous map $g\colon B^{n+1} \to S^n$ which extends $f$. The map $g$ is then a retraction, and this contradicts Lemma 1.3. ■

LECTURE 3

# Paths and the fundamental groupoid

In this lecture we define a rather pathetic functor, called $\pi_0$. We then define the *fundamental groupoid*. In the next lecture we will use the fundamental groupoid to define a much more interesting functor, the *fundamental group* $\pi_1$.

DEFINITION 3.1. A **path** $u$ in a topological space $X$ is a continuous map $u\colon I \to X$. If $u(0) = x$ and $u(1) = y$ we say $u$ is a **path from** $x$ **to** $y$. If $x = y$ then we say that $u$ is a **loop**.

We will always use the letters $u, v$ and $w$ to denote paths (in contrast to $f, g$ and $h$ for arbitrary continuous maps). Moreover we will parametrise a path with the letter $s$, so $u$ is the map $s \mapsto u(s)$, thus keeping the letter $t$ for a homotopy parameter. This will hopefully help to keep the notation clear. Paths gives us a new notion of connectivity.

DEFINITION 3.2. A topological space $X$ is **path connected** if for all $x, y \in X$ there exists a path from $x$ to $y$.

Hopefully you are all easily able to prove the following result[1].

LEMMA 3.3. *Let $X$ and $Y$ be topological spaces. Then:*

1. *If $X$ is path connected then $X$ is connected (but the converse is not necessarily true).*
2. *If $X$ and $Y$ are path connected then so is $X \times Y$.*
3. *If $f\colon X \to Y$ is continuous and $X$ is path connected then so is $f(X)$.*

Here we prove the following equally easy result:

PROPOSITION 3.4. *If $X$ is a topological space then the binary relation $\sim$ on $X$ defined by "$x \sim y$ if there exists a path from $x$ to $y$" is an equivalence relation.*

*Proof.* A constant path based at $x$ shows that $x \sim x$ for all $x \in X$. If $u$ is a path from $x$ to $y$ then the path $\bar{u}(s) := u(1-s)$ is a path from $y$ to $x$, and hence $x \sim y$ implies $y \sim x$. Finally if $u$ is a path from $x$ to $y$ and $v$ is a path from $y$ to $z$ then

$$w(s) := \begin{cases} u(2s), & 0 \le s \le \frac{1}{2}, \\ v(2s-1), & \frac{1}{2} \le s \le 1, \end{cases} \tag{3.1}$$

is a well-defined path from $x$ to $z$ (the gluing lemma shows that $w$ is continuous.) Thus $x \sim y$ and $y \sim z$ implies $x \sim z$. ■

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[1] I debated putting this on Problem Sheet B but decided it was too easy ...

Definition 3.5. The equivalence classes of $X$ under the equivalence relation $\sim$ are called the **path components** of $X$.

We now construct the functor $\pi_0$.

Definition 3.6. Given a topological space $X$, let $\pi_0(X)$ denote the set of path components of $X$. If $f\colon X \to Y$ is a continuous map, define $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ to be the map that send a path component $X'$ of $X$ to the unique path component of $Y$ containing $f(X')$ (this is well-defined due to Lemma 3.3.)

We then have:

Proposition 3.7. *$\pi_0\colon \mathsf{Top} \to \mathsf{Sets}$ is a functor. Moreover if $f \simeq g$ then $\pi_0(f) = \pi_0(g)$.*

*Proof.* The fact that $\pi_0$ is a functor is easy to check (i.e. that $\pi_0$ preserves identities and composition). Let us check that homotopic maps have the same image under $\pi_0$. Suppose $F\colon f \simeq g$. If $X'$ is a path component of $X$ then $X' \times I$ is path connected and hence so is $F(X' \times I)$ (here we are using Proposition 3.3 twice). Since

$$f(X') = F(X' \times \{0\}) \subseteq F(X' \times I)$$

and

$$g(X') = F(X' \times \{1\}) \subseteq F(X' \times I),$$

we see that the unique path component of $Y$ containing $F(X' \times I)$ contains both $f(X')$ and $g(X')$. Thus $\pi_0(f) = \pi_0(g)$. ■

Corollary 3.8. *If $X$ and $Y$ have the same homotopy type then they have the same number of path components.*

Corollary 3.8 can be proved directly, but let us give an "abstract" proof using Problem A.2 and Problem A.3 from Problem Sheet A.

*Proof.* By the last part of Proposition 3.7 and Problem A.3, we may regard $\pi_0$ as a functor $\mathsf{hTop} \to \mathsf{Sets}$. If $X$ and $Y$ have the same homotopy type then there exists a continuous map $f\colon X \to Y$ such that $[f]$ is an isomorphism in $\mathsf{hTop}$. Then by Problem A.2, $\pi_0([f])$ is an isomorphism in $\mathsf{Sets}$. An isomorphism in $\mathsf{Sets}$ is a bijection; thus $\pi_0(X)$ and $\pi_0(Y)$ have the same cardinality. ■

Corollary 3.8 is about as interesting as it gets when it comes to the functor $\pi_0$. This is because $\pi_0$ has the misfortune of taking values in $\mathsf{Sets}$, and there is not much one can with a set other than count it (i.e. the only obstruction to two sets being isomorphic is that they should have the same cardinality) Next lecture we will introduce another functor $\pi_1$ which takes values in $\mathsf{Groups}$. As groups have many obstructions to being isomorphic, this functor will be considerably more interesting.

The basic idea behind $\pi_1$ is that one can "multiply" paths if one ends where the other begins, via (3.1). Let us formalise this as a definition.

Definition 3.9. Let $u$ and $v$ be paths in $X$ with $u(1) = v(0)$. Then we define

$$(u * v)(s) := \begin{cases} u(2s), & 0 \leq s \leq \frac{1}{2}, \\ v(2s-1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Remark 3.10. Note that the ordering here is the *opposite* to composition: $u * v$ means "first do $u$, then do $v$", meanwhile $g \circ f$ means "first do $f$, then do $g$".

Our aim is to construct a group whose elements are certain homotopy classes of paths in $X$ with binary operation given by multiplying paths as above. However by Problem B.1 on Problem Sheet B, if $X$ is path connected then since $I$ is contractible, all paths $u\colon I \to X$ are homotopic, and thus if we tried to construct a group from homotopy classes of paths this group would have precisely one element (and so would be just as uninteresting as $\pi_0(X)$!) To recify this problem we use relative homotopy classes.

Definition 3.11. We define the **path class** of a path $u\colon I \to X$ to be the equivalence class $[u]$ of $u$, where the equivalence relation is being homotopic relative to $\partial I = \{0, 1\}$.

Remark 3.12. There is a potential for confusion here, in that we are using the same notation $[\cdot]$ to denote both the homotopy class and the relative homotopy class. However, it should always be clear from the context which is intended. In particular, for paths we will only ever talk about their path class, not their homotopy class, and thus the notation $[u]$ *always* means the path class.

The next result is similar to Proposition 2.6.

Proposition 3.13. *Suppose $u_0, u_1\colon I \to X$ and $v_0, v_1\colon I \to X$ are paths with*

$$u_0(1) = u_1(1) = v_0(0) = v_1(0).$$

*Assume that*

$$[u_0] = [u_1] \qquad \text{and} \qquad [v_0] = [v_1].$$

*Then*

$$[u_0 * v_0] = [u_1 * v_1].$$

*Proof.* If $U\colon u_0 \simeq u_1$ rel $\partial I$ and $V\colon v_0 \simeq v_1$ rel $\partial I$ then the map $W\colon I \times I \to X$ given by

$$W(s,t) := \begin{cases} U(2s, t), & 0 \leq s \leq \frac{1}{2}, \\ V(2s-1, t), & \frac{1}{2} \leq s \leq 1, \end{cases}$$

is a continuous map (the gluing lemma applies because functions agree on $\{\frac{1}{2}\} \times I$) which determines a homotopy from $u_0 * v_0$ to $u_1 * v_1$ rel $\partial I$. ∎

If $u$ is a path from $x$ to $y$, then running backwards along $u$ gives a path from $y$ to $x$. Let us fix some notation for this:

Definition 3.14. Given a path $u : I \to X$, we denote by $\bar{u} : I \to X$ the path $u$ parametrised backwards:

$$\bar{u}(s) = u(1-s).$$

Next, let us given a name to the constant path:

Definition 3.15. Given a point $p \in X$, we denote by $e_p$ the constant path $e_p(s) = p$. By a slight abuse of notation we denote by $[p]$ the path class $[e_p]$.

We now use this data to define a category. We will phrase this is as "definition" and then prove afterwards that it really is well-defined.

Definition 3.16. Let $X$ be a topological space. We define the **fundamental groupoid** of $X$ to be the category $\Pi(X)$ where:

- $\mathrm{obj}(\Pi(X)) = X$, that is, the objects of $\Pi(X)$ are the points in $X$ themselves,
- $\mathrm{Hom}(x, y)$ is the set of path classes of paths from $x$ to $y$:

$$\mathrm{Hom}(x, y) := \{[u] \mid u \text{ is a path from } x \text{ to } y\},$$

- and finally the composition

$$\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z)$$

  is given by

$$\big([u], [v]\big) \mapsto [u * v]$$

  (note by assumption this concatenation makes sense as $u(1) = y = v(0)$).

Let us prove this really does form a category.

Proposition 3.17. *Let $X$ be a topological space. Then $\Pi(X)$ is a well-defined category. The identity element of $\mathrm{Hom}(p, p)$ is $[p]$.*

*Proof.* From Definition 1.6, there are three things we need to verify:

1. the Hom sets are pairwise disjoint,
2. that composition is associative when defined,
3. that there exists an identity element in each Hom set.

Here (1) is obvious. Let us first prove (3). We claim that $[p]$ (i.e. the path class of $e_p$) is the identity element in $\mathrm{Hom}(p, p)$. For this we must prove that for any path $u$ with $u(0) = p$ we have $e_p * u \simeq u$ rel $\partial I$, and similarly for any path $v$ with $v(1) = p$ we have $v * e_p \simeq v$ rel $\partial I$. We will prove the first statement only, as the second is similar. Consider Figure 3.1. The shaded triangle is the set $\{(s,t) \mid 2s \leq 1 - t\}$. For fixed $t$, consider the horizontal line $L_t$ that runs from the start of the shaded region to the right-hand edge (the point $(1, t)$). The function

$$l_t(s) := \frac{s - \frac{1}{2}(1-t)}{1 - \frac{1}{2}(1-t)}$$

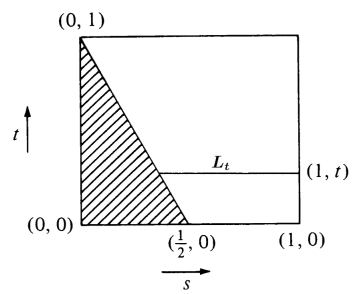


Figure 3.1: Proving $e_p * u \simeq u$ rel $\partial I$.

maps $L_t$ onto $[0,1]$. Now consider the map $U\colon I \times I \to X$ given by

$$U(s,t) := \begin{cases} p, & 2s \leq 1-t, \\ u(l_t(s)), & 2s \geq 1-t. \end{cases}$$

The gluing lemma shows that $U$ is continuous, and by construction $U\colon e_p * u \simeq u$ rel $\partial I$.

Now let us prove associativity. Suppose $u, v$ and $w$ are three paths with $u(1) = v(0)$ and $v(1) = w(0)$. This is a similar but slightly trickier argument, and we will not write out the formulae precisely. Consider Figure 3.2. Draw two slanted lines, one

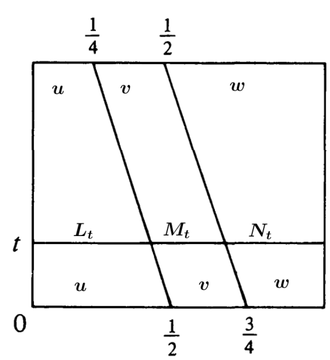


Figure 3.2: Proving $(u * v) * w \simeq u * (v * w)$ rel $\partial I$.

that starts at $(1/4, 1)$ and runs to $(1/2, 0)$, and one that starts at $(1/2, 1)$ and runs to $(3/4, 0)$. Now let $L_t$, $M_t$ and $N_t$ denote the three horizontal lines as marked that come from intersecting the horizontal line with $t$ fixed. Then let $l_t$, $m_t$ and $n_t$ denote reparametrisations that map $L_t$, $M_t$ and $N_t$ onto $[0,1]$ respectively. The desired homotopy is $U$ is obtained by setting $U(t,s) = u(l_t(s))$ on the left-hand region, setting $U(s,t) = v(m_t(s))$ on the middle region and finally setting $U(t,s) = w(n_t(s))$ on the right-hand region. The gluing lemma shows that $U$ is continuous, and by construction we have $U\colon (u * v) * w \simeq u * (v * w)$ rel $\partial I$. This completes the proof. ■

In fact, the category $\Pi(X)$ has an additional special property:

Proposition 3.18. *Every morphism in $\Pi(X)$ is an isomorphism. More precisely, for any path $u$ from $x$ to $y$, one has*

$$[u] * [\bar{u}] = [x], \qquad [\bar{u}] * [u] = [y].$$

*Proof.* We must show $u * \bar{u} \simeq e_x$ rel $\partial I$ and $\bar{u} * u \simeq e_y$ rel $\partial I$. Again, I will prove only the first statement. Moreover this time round, I will give the formulae but not the picture[2]. To this end consider the function $U \colon I \times I \to X$ given by

$$U(s,t) := \begin{cases} u\big(2s(1-t)\big), & 0 \le s \le \frac{1}{2}, \\ u\big(2(1-s)(1-t)\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

The gluing lemma shows that $U$ is continuous, and one checks that $U \colon u * \bar{u} \simeq e_x$ rel $\partial I$. This completes the proof. ∎

Categories with this property have a special name.

Definition 3.19. Let $\mathsf{C}$ be a category. We say that $\mathsf{C}$ is a **groupoid category** if:

- $\mathsf{C}$ is a **small** category[3], which by definition means that $\mathrm{obj}(\mathsf{C})$ is a *set* and not just a class, cf. Remark 1.7.
- Every morphism $f : A \to B$ in $\mathsf{C}$ is an isomorphism.

Thus Proposition 3.18 can alternatively be rephrased as: the fundamental groupoid is a groupoid cateogory.

[2] And thus you should draw the picture out!

[3] Nothing we ever do in this course will ever need to worry about the distinction between a set and a class, so you are free to ignore this part of the definition if you want ...

LECTURE 4

# The fundamental group

In this lecture we define our first genuinely exciting functor, $\pi_1$. The idea is simply to restrict to loops.

Definition 4.1. Let $X$ be a topological space and fix a point $p \in X$, which we call the **basepoint**. The **fundamental group** of $X$ with basepoint $p$ is

$$\pi_1(X,p) := \mathrm{Hom}_{\Pi(X)}(p,p) = \{[u] \mid u \text{ is a loop in } X \text{ based at } p\}.$$

An immediate corollary of Proposition 3.17 is the following result.

Corollary 4.2. *For any topological space $X$ and any $p \in X$, the set $\pi_1(X,p)$ is a group with multiplication given by*

$$[u] * [v] := [u * v]$$

*and identity element $[p]$. The inverse of an element $[u]$ is given by $[\bar{u}]$:*

$$[u]^{-1} = [\bar{u}].$$

Since the fundamental group $\pi_1(X,p)$ involves a choice of basepoint $p$, in order to make $\pi_1$ into a proper functor we need to work with a slightly different category. Before introducing this, let us explain how a smaller category can sit inside a larger one.

Definition 4.3. Suppose $\mathsf{C}$ and $\mathsf{D}$ are two categories. We say that $\mathsf{C}$ is a **subcategory** of $\mathsf{D}$ if:

1. $\mathrm{obj}(\mathsf{C}) \subseteq \mathrm{obj}(\mathsf{D})$,
2. $\mathrm{Hom}_{\mathsf{C}}(A,B) \subseteq \mathrm{Hom}_{\mathsf{D}}(A,B)$ for all $A, B \in \mathrm{obj}(\mathsf{C})$, where we denote Hom sets in $\mathsf{C}$ by $\mathrm{Hom}_{\mathsf{C}}(\square,\square)$,
3. if $f \in \mathrm{Hom}_{\mathsf{C}}(A,B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B,C)$ then the composite $g \circ f \in \mathrm{Hom}_{\mathsf{C}}(A,C)$ is equal to the composite $g \circ f \in \mathrm{Hom}_{\mathsf{D}}(A,C)$,
4. if $C \in \mathrm{obj}(\mathsf{C})$ then $\mathrm{id}_C \in \mathrm{Hom}_{\mathsf{C}}(C,C)$ is equal to $\mathrm{id}_C \in \mathrm{Hom}_{\mathsf{D}}(C,C)$.

If for every pair $A, B \in \mathrm{obj}(\mathsf{C})$ one always has $\mathrm{Hom}_{\mathsf{C}}(A,B) = \mathrm{Hom}_{\mathsf{D}}(A,B)$ then we say that $\mathsf{C}$ is a **full subcategory** of $\mathsf{D}$.

As an example, the category $\mathsf{Ab}$ is a full subcategory of $\mathsf{Groups}$.

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Example 4.4. The category $\mathsf{Top}^2$ has as objects all pairs $(X, X')$ where $X$ is a topological space and $X' \subseteq X$ is a subspace. A morphism $(X, X') \to (Y, Y')$ is a pair $(f, f')$ of maps $f : X \to Y$ and $f' \colon X' \to Y'$ such that the following diagram commutes, where the horizontal maps are inclusions:

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, hook] & Y \end{tikzcd}$$

The composition law is the usual one. Slightly less pedantically, we can think of a morphism in $\mathrm{Hom}((X, X'), (Y, Y'))$ simply as a continuous map $f \colon X \to Y$ with the property that $f(X') \subseteq Y'$.

We can regard $\mathsf{Top}$ as a subcategory of $\mathsf{Top}^2$ if we identify a space $X$ with the pair $(X, \emptyset)$. For us, it is also useful to consider the case where both $X'$ and $Y'$ are a single point in $X$ and $Y$ respectively. This gets its own name.

Example 4.5. The category $\mathsf{Top}_*$ of **pointed topological spaces** has as objects all ordered pairs $(X, p)$ where $X$ is a topological space and $p$ is a point in $X$, referred to as the **basepoint**. Given two objects $(X, p)$ and $(Y, q)$, the morphism space is simply the set of continuous maps $f \colon X \to Y$ which send the basepoint $p$ in $X$ to the basepoint $q \in Y$:

$$\mathrm{Hom}((X, p), (Y, q)) := \{ f \in C(X, Y) \mid f(p) = q \} .$$

We call such a map a **pointed map**.

We will write $f \colon (X, p) \to (Y, q)$ as shorthand to indicate that $f$ is a continuous map from $X$ to $Y$ satisfying $f(p) = q$ (and hence a morphism in $\mathsf{Top}_*$.) Note that $\mathsf{Top}_*$ is again a subcategory of $\mathsf{Top}^2$. Let us now show that $\pi_1$ is a functor from $\mathsf{Top}_*$ to $\mathsf{Groups}$.

Definition 4.6. Suppose $f \colon (X, p) \to (Y, q)$ is a pointed map. Define

$$\pi_1(f) \colon \pi_1(X, p) \to \pi_1(Y, q), \qquad [u] \mapsto [f \circ u].$$

This is well-defined: firstly $f \circ u \colon I \to Y$ is a continuous path that starts and ends at $q$, and hence $[f \circ u]$ is indeed an element of $\pi_1(Y, q)$. Moreover if $u \simeq v$ rel $\partial I$ then $f \circ u \simeq f \circ v$ rel $\partial I$ by Problem B.2 on Problem Sheet B.

Proposition 4.7. *$\pi_1 \colon \mathsf{Top}_* \to \mathsf{Groups}$ is a functor. Moreover if $f, g \colon (X, p) \to (Y, q)$ are continuous maps with $f \simeq g$ rel $p$ then $\pi_1(f) = \pi_1(g)$.*

*Proof.* Suppose $f \colon (X, p) \to (Y, q)$ is a pointed map. To show that $\pi_1(f)$ is a homomorphism, observe that if $u$ and $v$ are closed paths based at $p \in X$ then

$$f \circ (u * v) = (f \circ u) * (f \circ v)$$

(this is an actual pointwise equality, not just homotopy!). The fact that $\pi_1$ preserves composition and identities in $\mathsf{Top}_*$ is clear. Finally, if $f \simeq g$ rel $p$ then by Problem B.2 on Problem Sheet B again, one obtains $f \circ u \simeq g \circ u$ rel $\partial I$ for any closed curve $u$ in $X$ based at $p$. Thus $\pi_1(f) = \pi_1(g)$. ∎

Remark 4.8. It follows from Problem B.2 that being homotopic rel $p$ defines an equivalence relation on pointed maps $(X, p) \to (Y, q)$, and hence a congruence on the category $\mathsf{Top}_*$. The associated quotient category is denoted by $\mathsf{hTop}_*$. By analogy with the category $\mathsf{hTop}$, we write $[(X, p), (Y, q)]$ for the morphism set between two objects $(X, p)$ and $(Y, q)$ in $\mathsf{hTop}_*$. The last statement of Proposition 4.7 together with Problem A.3 implies that $\pi_1$ induces a functor $\pi_1 \colon \mathsf{hTop}_* \to \mathsf{Groups}$.

Remark 4.9. We won't need this (so I won't spell out the relevant definitions), but the association $[u] \mapsto [f \circ u]$ makes perfect sense for arbitrary path classes, and hence given a continuous map $f \colon X \to Y$, we obtain a map $\Pi(f) \colon \Pi(X) \to \Pi(Y)$ defined by

$$\Pi(f) \colon \mathrm{Hom}_{\Pi(X)}(x, y) \to \mathrm{Hom}_{\Pi(Y)}(f(x), f(y)), \qquad [u] \mapsto [f \circ u].$$

This makes the fundamental groupoid into a functor

$$\Pi : \mathsf{Top} \to \mathsf{Groupoids},$$

where $\mathsf{Groupoids}$ is the category of groupoids[1].

The next result shows that when studying fundamental groups, we may as well assume that our spaces are path connected.

Proposition 4.10. *Let $X$ be a topological space, let $p \in X$, and let $X'$ denote the path component containing $p$. Then the inclusion $\imath \colon X' \hookrightarrow X$ induces an isomorphism on $\pi_1$:*

$$\pi_1(X', p) \cong \pi_1(X, p).$$

*Proof.* Suppose $[u] \in \ker \pi_1(\imath)$. This means that $\imath \circ u \simeq e_p$ rel $\partial I$, where as usual $e_p \colon I \to X$ is the constant path at $p$. Let $U \colon I \times I \to X$ denote the homotopy. Then $U(0, 0) = p$. Since $U(I \times I)$ is path connected (cf. Problem B.2) we have $U(I \times I) \subseteq X'$. Thus $u$ is nullhomotopic in $X'$, and hence $\pi_1(\imath)$ is injective. To see $\pi_1(\imath)$ is surjective, suppose $u \colon I \to X$ is a closed path at $p$. Then $u(I) \subseteq X'$. Thus we can pedantically define $u' \colon I \to X'$ by $u'(s) := u(s)$. Then clearly $\imath \circ u' = u$, and surjectivity follows. ■

Now let us investigate what happens when the basepoint is changed.

Proposition 4.11. *Suppose $X$ is path connected and $p_0, p_1 \in X$. Then any path $w$ from $p_0$ to $p_1$ induces an isomorphism*

$$\lambda_w \colon \pi_1(X, p_0) \cong \pi_1(X, p_1)$$

.

*Proof.* Define $\lambda_w \colon \pi_1(X, p_0) \to \pi_1(X, p_1)$ by

$$\lambda_w \colon [v] \mapsto [\bar{w} * v * w] \tag{4.1}$$

(note that the multiplication takes places in the fundamental groupoid $\Pi(X)$.) From Proposition 3.18 one sees that $\lambda_w$ is an isomorphism; the inverse is given by $\lambda_{\bar{w}}$. ■

[1]Since a groupoid is a type of category, this is a category of categories!

Remark 4.12. Thus provided we work with path connected spaces only, we can write simply $\pi_1(X)$ to denote $\pi_1(X,p)$ for any $p \in X$. However should be aware that there is no *canonical* isomorphism between $\pi_1(X,p_0)$ and $\pi_1(X,p_1)$. Thus $\pi_1(X)$ is really a family of isomorphic groups. Moreover, as you will see in Problem Sheet C (cf. Problem C.5), sometimes simply knowing two groups are isomorphic is not enough – one really needs an explicit isomorphism.

We now discuss a subtle (and often tedious) point. To define the fundamental group we were forced to pick a basepoint, and thus the natural category to work with is $\mathsf{Top}_*$. But "most" homotopic maps that crop up "in nature" are *not* pointed maps (i.e. given two homotopic maps, it is typically too much to hope for that they just so happen to preserve the basepoint.) Thus we need to investigate how the fundamental group behaves under a free[2] homotopy.

Proposition 4.13. *Suppose $f_0, f_1 \colon X \to Y$ are continuous maps and $F \colon f_0 \simeq f_1$ is a free homotopy from $f_0$ to $f_1$. Choose $p \in X$ and let $w$ denote the path in $Y$ given by $w(t) = F(p,t)$. Then there is a commutative diagram:*

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "\pi_1(f_0)"] \arrow[dr, "\pi_1(f_1)"'] & \pi_1(Y, f_0(p)) \arrow[d, "\lambda_w"] \\ & \pi_1(Y, f_1(p)), \end{tikzcd}$$

*where $\lambda_w$ is the isomorphism given in (4.1).*

*Proof.* Take $[u] \in \pi_1(X,p)$. Consider the homotopy[3] $V \colon I \times I \to Y$ given by

$$V(s,t) := \begin{cases} F\big(u\big(2(1-t)s\big), 2st\big), & 0 \le s \le \frac{1}{2}, \\ F\big(u\big(1+2t(s-1)\big), t + (1-t)(2s-1)\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

Then

$$V(s,0) = \begin{cases} F\big(u(2s), 0\big), & 0 \le s \le \frac{1}{2}, \\ F\big(u(1), 2s-1\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since $F(u(2s),0) = f_0(u(2s))$ and $F(u(1), 2s-1) = w(2s-1)$, we have

$$V(s,0) = (f_0 \circ u) * w(s).$$

Similarly

$$V(s,1) = \begin{cases} F\big(u(0), 2s\big), & 0 \le s \le \frac{1}{2}, \\ F\big(u(2s-1), 1\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since $F(u(0), 2s) = w(2s)$ and $F(u(2s-1), 1) = f_1(u(2s-1))$, we have

$$V(s,1) = w * (f_1 \circ u)(s).$$

[2] I will call a homotopy $F \colon f \simeq g$ between two maps a *free* homotopy when it is important to emphasise that it is *not* a relative homotopy.

[3] As ever, I encourage you to draw a picture here!

The gluing lemma shows that $V$ is continuous, and

$$V(0,t) = F(u(0),0) = f_0(p)$$

and

$$V(1,t) = F(u(1),1) = f_1(p).$$

Thus $V$ is a homotopy from $(f_0 \circ u) * w$ to $w * (f_1 \circ u)$ relative to $\partial I$. Thus in the fundamental groupoid $\Pi(Y)$, we have

$$[f_0 \circ u] * [w] = [w] * [f_1 \circ u],$$

or alternatively

$$[f_1 \circ u] = [\bar{w} * (f_0 \circ u) * w],$$

which implies that

$$\pi_1(f_1)[u] = \lambda_w \circ \pi_1(f_0)[u]$$

as maps $\pi_1(X,p) \to \pi_1(Y, f_1(p))$. ■

COROLLARY 4.14. *Suppose $f_0, f_1 \colon X \to Y$ are continuous maps and $F \colon f_0 \simeq f_1$ is a free homotopy from $f_0$ to $f_1$. Suppose $p \in X$ has the property that $f_0(p) = f_1(p)$. Set $q := f_0(p)$. Then $\pi_1(f_0)$ and $\pi_1(f_1)$ are* **conjugate** *group homomorphisms, that is, there exists $[w] \in \pi_1(Y,q)$ such that*

$$\pi_1(f_1)[u] = [w]^{-1} * \pi_1(f_0)[u] * [w], \qquad \forall\, [u] \in \pi_1(X,p). \tag{4.2}$$

*In particular, if $\pi_1(Y,q)$ is abelian then $\pi_1(f_0) = \pi_1(f_1)$.*

*Proof.* Using the notation of Proposition 4.13, the path $w$ is now a *closed* path in $Y$, and hence $[w] \in \pi_1(Y,q)$. Thus the path class $[\bar{w} * (f_0 \circ u) * w]$, which can always be factored in the fundamental groupoid of $Y$ can now be factored in $\pi_1(Y,q)$:

$$[\bar{w} * (f_0 \circ u) * w] = [\bar{w}] * [f_0 \circ u] * [w] = [w]^{-1} * [f_0 \circ u] * [w].$$

Thus (4.2) follows. Finally, the last statement is immediate, since if $\pi_1(Y,q)$ is abelian then we can write

$$[w]^{-1} * \pi_1(f_0)[u] * [w] = [w]^{-1} * [w] * \pi_1(f_0)[u] = \pi_1(f_0)[u].$$

■

We now show that for path connected spaces $X$ and $Y$, having the same homotopy type is enough to ensure that the fundamental groups $\pi_1(X,p)$ and $\pi_1(Y,q)$ coincide for any $p \in X$ and $q \in Y$. This will follow from the following result.

PROPOSITION 4.15. *Suppose $f \colon X \to Y$ is a homotopy equivalence. Then for any $p \in X$ the induced map $\pi_1(f) \colon \pi_1(X,p) \to \pi_1(Y, f(p))$ is an isomorphism.*

*Proof.* Choose a continuous map $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. If $F\colon g \circ f \simeq \mathrm{id}_X$ is a free homotopy, let $w(s) := F(p, s)$, so that $w$ is a path from $g(f(p))$ to $p$. By Proposition 4.13, the lower triangle of the following diagram commutes:

$$\begin{tikzcd}
 & \pi_1(Y, f(p)) \arrow[dr, "\pi_1(g)"] & \\
\pi_1(X, p) \arrow[ur, "\pi_1(f)"] \arrow[rr, "\pi_1(g\circ f)"] \arrow[dr, "\mathrm{id}"'] & & \pi_1(X, g(f(p))) \arrow[dl, "\lambda_w"] \\
 & \pi_1(X, p) &
\end{tikzcd}$$

The top triangle also commutes because $\pi_1$ is a functor. Since $\lambda_w$ is an isomorphism, $\pi_1(g \circ f)$ is also an isomorphism. Thus $\pi_1(f)$ is injective and $\pi_1(g)$ is surjective. A similar diagram starting from $f \circ g \simeq \mathrm{id}_Y$ shows that $\pi_1(g)$ is injective and $\pi_1(f)$ is surjective. ■

An immediate corollary of this result (together with the fact that a space consisting of one point obviously has a trivial fundamental group) we have:

COROLLARY 4.16. *Suppose $X$ has the same homotopy type as a path connected space $Y$. Then for all $p \in X$ and $q \in Y$ one has $\pi_1(X, p) \cong \pi_1(Y, q)$. If $X$ is a contractible space then $\pi_1(X, p) = \{1\}$ for all $p \in X$.*

We conclude by giving the property of having a trivial fundamental group a name:

DEFINITION 4.17. A topological space $X$ is called **simply connected** if it is path connected and has $\pi_1(X, p) = \{1\}$ for some (and hence all) $p \in X$.

LECTURE 5

# The fundamental group of the circle and pushouts

We have yet to give an example of a path-connected space that is *not* simply connected! If it turned out that every such space had trivial fundamental group then as a mathematical tool the fundamental group would be as useful as a dead cat. Luckily this is not the case: many spaces are not simply connected. In this lecture we will perform our very first computation: the fundamental group of the circle $S^1$. At the end of the lecture we flip back into "abstract nonsense" mode and define the notion of a *pushout* in a category. We will need this next lecture when we prove the *Seifert-van Kampen Theorem*.

In this lecture it is convenient to regard $S^1$ as the unit circle in $\mathbb{C}$, i.e

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

This is of course consistent with our previous definition under the identification of $\mathbb{C}$ with $\mathbb{R}^2$.

DEFINITION 5.1. Define $\exp\colon \mathbb{R} \to S^1$ by

$$\exp(s) := e^{2\pi i s}.$$

We will always take 1 as the basepoint in $S^1$. We will use the fact that $S^1$ is itself a group: Given two points $z = e^{2\pi i s}$ and $w = e^{2\pi i t}$ in $S^1$, their product is $z \cdot w = e^{2\pi i(s+t)}$, and the inverse of $z$ is just $e^{-2\pi i s}$. We say $z$ and $w$ are *antipodal* if $z \cdot w^{-1} = -1$.

PROPOSITION 5.2. *Let $n \geq 0$ and let $X$ be a compact convex subset of $\mathbb{R}^n$ and let $p \in X$. Suppose $f\colon (X,p) \to (S^1, 1)$ is a continuous map, and let $m \in \mathbb{Z}$. Then there exists a unique continuous map $\tilde{f}\colon (X,p) \to (\mathbb{R}, m)$ such that $\exp \circ \tilde{f} = f$:*

$$\begin{tikzcd} & (\mathbb{R}, m) \arrow[d, "\exp"] \\ (X,p) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (S^1, 1) \end{tikzcd}$$

REMARK 5.3. We call $\tilde{f}$ a **lift** of $f$. Note that the requirement that $m$ be an integer is forced: if $\tilde{f}(p) = s$ then asking that $\exp(s) = f(p) = 1$ implies that $s$ is an integer.

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*Proof.* Since $X$ is a compact metric space, $f$ is uniformly continuous, and hence there exists $\varepsilon > 0$ such that if $|x-y| < \varepsilon$ then $f(x)$ and $f(y)$ are not antipodal points. Since $X$ is bounded there exists an integer $N$ such that $|x-y| < N\varepsilon$ for all $x, y \in X$. Now for each $x \in X$, subdivide the line segment with endpoints $p$ and $x$ (which is contained in $X$ by convexity) into $N$ intervals of equal length. Call the endpoints of these endpoints $p = l_0(x), l_1(x), \dots, l_N(x) = x$. The functions $l_i \colon X \to X$ are continuous[1] and for each $0 \le i \le N-1$, the points $f(l_i(x))$ and $f(l_{i+1}(x))$ are not antipodal. This means that for each $0 \le i \le N-1$, the map

$$g_i \colon X \to S^1 \setminus \{-1\}, \qquad g_i(x) = f(l_i(x))^{-1} \cdot f(l_{i+1}(x))$$

is continuous (here we are using multiplication in $S^1$ as above). Note that $g_i(p) = 1$ for all $i$. Moreover since $l_N(x) = x$ for all $x$,

$$\begin{aligned} f(x) &= f(p) \cdot f(p)^{-1} \cdot f(l_1(x)) \cdot f(l_1(x))^{-1} \cdots f(l_{N-1}(x))^{-1} \cdot f(l_N(x)) \\ &= \underbrace{f(p)}_{=1} \cdot g_0(x) \cdot g_1(x) \cdots g_{N-1}(x). \end{aligned} \tag{5.1}$$

If we restrict the map exp to $(-\frac{1}{2}, \frac{1}{2})$ then it is a homeomorphsim onto $S^1 \setminus \{-1\}$; let us denote its inverse by $\Lambda$ (actually $\Lambda = \frac{1}{2\pi i} \log$.) Then $\Lambda(1) = 0$. Since $g_i(x) \neq -1$ for all $x$, the function $\Lambda \circ g_i$ is defined and continuous. Now consider the function $\tilde{f} \colon X \to \mathbb{R}$ given by

$$\tilde{f}(x) := m + \sum_{i=0}^{N-1} \Lambda(g_i(x)).$$

Then $\tilde{f}$ is continuous, with $\tilde{f}(p) = m$ and from (5.1) one has

$$\exp \circ \tilde{f} = f.$$

It remains to prove $\tilde{f}$ is unqiue. Suppose $\tilde{g}$ was another such map with $\exp \circ \tilde{g} = f$ and $\tilde{g}(p) = m$. Define $\tilde{h}(x) = \tilde{f}(x) - \tilde{g}(x)$. Then $\tilde{h}$ is continuous and $\exp \circ \tilde{h}$ is identically equal to 1. Since $\exp \colon \mathbb{R} \to S^1$ is a homomorphism (thinking of both $\mathbb{R}$ and $\mathbb{Z}$ as groups) with kernel equal to $\mathbb{Z}$, the function $\tilde{h} \colon X \to \mathbb{R}$ is integer-valued. Since $X$ is connected (as $X$ is convex), $\tilde{h}$ must be constant. Since $\tilde{h}(p) = \tilde{f}(p) - \tilde{g}(p) = m - m = 0$, the constant must be zero. Thus $\tilde{f} = \tilde{g}$ and the proof is complete. ∎

COROLLARY 5.4. *Let $u \colon I \to S^1$ be a loop with $u(0) = u(1) = 1$. Then there exists a unique lift $\tilde{u} \colon I \to \mathbb{R}$ (i.e. $\exp \circ \tilde{u} = u$) with $\tilde{u}(0) = 0$. Moreover if $v \colon (I, \partial I) \to (S^1, 1)$ is another path with $u \simeq v$ rel $\partial I$ then if $\tilde{v}$ is the unique lift of $v$ with $\tilde{v}(0) = 0$ then $\tilde{u} \simeq \tilde{v}$ rel $\partial I$. In particular, $\tilde{u}(1) = \tilde{v}(1)$.*

*Proof.* The first statement follows from Proposition 5.2 since $I$ is a compact convex subset of $\mathbb{R}$. To prove the second statement, suppose $U \colon u \simeq v$ rel $\partial I$. Since $I \times I$ is a compact convex subset of $\mathbb{R}^2$, Proposition 5.2 provides us with a unique map $\tilde{U} \colon I \times I \to \mathbb{R}$ such that $\exp \circ \tilde{U} = U$ with $\tilde{U}(0,0) = 0$. We claim that

$$\tilde{U} \colon \tilde{u} \simeq \tilde{v} \qquad \text{rel } \partial I.$$

[1]Exercise: Check they really are continuous!

For this first note that if $\tilde{u}_1(s) := \tilde{U}(s,0)$ then $\tilde{u}_1$ is a lift of $u$ with $\tilde{u}_1(0) = 0$; thus by the uniqueness part of Proposition 5.2, we must have $\tilde{u}_1 = \tilde{u}$. Similarly $\tilde{U}(s,1) = \tilde{v}(s)$. This shows that $\tilde{U}\colon \tilde{u} \simeq \tilde{v}$, and it remains to show that this homotopy is a homotopy relative to $\partial I$.

For this consider $\tilde{w}(t) := \tilde{U}(0,t)$. Then $\exp(\tilde{w}(t)) = \exp(\tilde{U}(0,t)) = U(0,t) = 1$, and hence arguing as above we see that $\tilde{w}(t) = 0$ for all $t$. Now consider $\tilde{w}_1(t) := \tilde{U}(1,t)$. Then $\exp \circ \tilde{w}_1 = 1$, and thus by uniqueness $\tilde{w}_1$ is a constant function. The constant is equal to $\tilde{U}(1,0) = \tilde{u}(1)$ (and also to $\tilde{U}(1,1) = \tilde{v}(1)$.) This completes the proof. ■

Definition 5.5. Given a loop $u\colon (I, \partial I) \to (S^1, 1)$, we define the **degree** of $u$ to be the integer $\deg(u) = \tilde{u}(1)$, where $\tilde{u}$ is the unique lift of $u$ with $\tilde{u}(0) = 0$.

The last statement of Corollary 5.4 tells us that deg induces a well defined map

$$\deg\colon \pi_1(S^1, 1) \to \mathbb{Z}, \qquad \deg\big([u]\big) := \deg(u),$$

where $u$ is any representative of $[u]$.

We can now prove the main result of this lecture.

Theorem 5.6. *The function* $\deg\colon \pi_1(S^1,1) \to \mathbb{Z}$ *is an isomorphism. In particular,*

$$\deg\big([u] * [v]\big) = \deg\big([u]\big) + \deg\big([v]\big). \tag{5.2}$$

*Proof.* Firstly, deg is surjective since the loop $s \mapsto \exp(ms)$ has degree $m$ (note this is the function $z \mapsto z^m$ if we think of $u$ as a map from $S^1 \subset \mathbb{C}$ to itself). Secondly, if $\deg(u) = 0$ then $\tilde{u}$ is a loop in $\mathbb{R}$ based at 0. Now $\pi_1(\mathbb{R}, 0) = \{1\}$ by the last statement of Corollary 4.16. Since $\pi_1(\exp)\colon \pi_1(\mathbb{R},0) \to \pi_1(S^1,1)$ is a homomorphism and

$$[u] = \pi_1(\exp)[\tilde{u}],$$

we thus have $[u] = 1$ in $\pi_1(S^1, 1)$.

To complete the proof we will show that deg is a homomorphism from $\pi_1(S^1,1)$ to $\mathbb{Z}$, that is, that (5.2) holds. Then from the above its kernel is trivial, whence it follows that deg is injective, and hence an isomorphism. So suppose $u$ and $v$ are loops in $S^1$ based at 1 of degree $m$ and $n$ respectively. To compute $\deg(u*v)$, we must find a path $\tilde{w}\colon I \to \mathbb{R}$ with $\exp \circ \tilde{w} = u * v$. Let $\tilde{u}$ be the unique lift of $u$ with $\tilde{u}(0) = 0$, and similarly for $\tilde{v}$. Then consider the path $\tilde{v}'\colon I \to \mathbb{R}$ given by $\tilde{v}'(s) = m + \tilde{v}(s)$. Then $\tilde{v}$ is a path from $m$ to $m+n$. Then $\tilde{w} := \tilde{u} * \tilde{v}'$ is a path in $\mathbb{R}$ with $\tilde{w}(0) = 0$ and $\tilde{w}(1) = m + n$. Note that

$$\exp(\tilde{w}(s)) = \begin{cases} \exp\big(\tilde{u}(2s)\big) & 0 \le s \le \frac{1}{2}, \\ \exp\big(\tilde{v}'(2s-1)\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since $\exp \circ \tilde{u} = u$ and since

$$\exp\big(\tilde{v}'(s)\big) = \exp\big(m + \tilde{v}(s)\big) = e^{2\pi i m} \exp(\tilde{v}(s)) = v(s),$$

it follows that $\tilde{w}$ is a lift of $u * v$. Since $\tilde{w}(1) = m + n$, we thus have

$$m + n = \deg(u * v) = \deg(u) + \deg(v)$$

as required. This completes the proof. ■

As promised, we conclude this lecture with some more abstract nonsense that we will need next time.

Definition 5.7. Suppose $\mathsf{C}$ is a category and $A, B_1, B_2$ are three objects in $\mathsf{C}$, and $f_1 \colon A \to B_1$ and $f_2 \colon A \to B_2$ are two morphisms. A **diagram** in $\mathsf{C}$ is a picture[2] of the form:

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \\ B_2 & \end{tikzcd} \tag{$\delta$}$$

A **solution** to the diagram ($\delta$) is an object $C$ together with two morphisms $g_1 \colon B_1 \to C$ and $g_2 \colon B_2 \to C$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "g_1"] \\ B_2 \arrow[r, "g_2"'] & C \end{tikzcd}$$

A **pushout**[3] of the diagram ($\delta$) is a solution $(C, g_1, g_2)$ which satisfies the following **universal property**: if $(D, h_1, h_2)$ is any other solution to ($\delta$) then there is a *unique* morphism $k \colon C \to D$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "g_1"'] \arrow[ddr, "h_1"] & \\ B_2 \arrow[r, "g_2"] \arrow[drr, "h_2"'] & C \arrow[dr, dashed, "k"] & \\ & & D \end{tikzcd} \tag{$\Delta$}$$

A pushout may or may not exist (it depends on the category $\mathsf{C}$), but if it does then it is unique up to isomorphism.

Lemma 5.8. *If a pushout exists then it is unique up to isomorphism.*

*Proof.* Suppose $(C, g_1, g_2)$ and $(C', g'_1, g'_2)$ are two pushouts. Then applying diagram

[2] In Lecture 16 we will define a more general notion of a diagram which allows for pictures of a different shapes.

[3] We will generalise this in Lecture 16 when we introduce **colimits**.

($\Delta$) with $D = C'$ gives a morphism $k\colon C \to C'$ such that $k \circ g_1 = g_1'$ and $k \circ g_2 = g_2'$:

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "g_1"'] \arrow[ddr, "g_1'"] & \\
B_2 \arrow[r, "g_2"] \arrow[drr, "g_2'"'] & C \arrow[dr, dashed, "k"] & \\
 & & C'
\end{tikzcd}$$

Reversing the roles of $C$ and $C'$ gives another morphism $k'\colon C' \to C$ such that $k' \circ g_1' = g_1$ and $k' \circ g_2' = g_2$. To complete the proof we claim that $k' \circ k = \mathrm{id}_C$ and $k \circ k' = \mathrm{id}_{C'}$. For this, first note that

$$k' \circ k \circ g_1 = k' \circ g_1' = g_1, \tag{5.3}$$

and similarly

$$k' \circ k \circ g_2 = k' \circ g_2' = g_2. \tag{5.4}$$

Now take $D = C$: then the universal property means there is a unique map $l\colon C \to C$ such that the following commutes:

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "g_1"'] \arrow[ddr, "g_1"] & \\
B_2 \arrow[r, "g_2"] \arrow[drr, "g_2"'] & C \arrow[dr, dashed, "l"] & \\
 & & C
\end{tikzcd}$$

By (5.3) and (5.4) taking $l = k' \circ k$ makes this diagram commute: hence by uniqueness this must be the map $l$:

$$l = k' \circ k.$$

But of course there is another map that also works: take $l = \mathrm{id}_C$! By uniqueness, it thus follows that

$$k' \circ k = \mathrm{id}_C.$$

Now, repeating this but with $C'$ in both the two bottom right slots shows that $k \circ k' = \mathrm{id}_{C'}$. The proof is complete. ∎

Remark 5.9. Morally (we shall see many other examples of this throughout the course), whenever something is defined via a universal property[4] then uniqueness comes "for free". However, one still always needs to prove existence.

[4] Before you ask: Yes, it is possible to give a formal definition of exactly what a "universal property" is, but I'm not going to do so since it requires more category theory than one should use in polite company. Instead, just think of a universal property as meaning "making a diagram commute in the most efficient manner possible".

LECTURE 6

# The Seifert-van Kampen Theorem

In this lecture we prove our first genuinely difficult result, called the *Seifert-van Kampen Theorem*. Roughly speaking, the Seifert-van Kampen Theorem allows us to "decompose" a topological space into smaller pieces, and compute the fundamental group of the full space in terms of the fundamental groups of the smaller pieces.

Let us begin by proving that in the category Groups, a pushout always exists.

DEFINITION 6.1. Let $G$ and $H$ be groups (not necessarily abelian). A **word** of length $n$ in $G$ and $H$ is an expression of the form

$$s_1 s_2 \cdots s_n$$

where each $s_i$ belongs to either $G$ or $H$. A word can be **reduced** in two different ways:

1. If any $s_i$ is equal to the identity element $1_G$ or $1_H$, remove it.
2. If two consecutive elements $s_i$ and $s_{i+1}$ both belong to $G$ (or both belong to $H$), then replace them by their product $s_i \cdot s_{i+1}$ as a single element of $G$ (or $H$). This produces a word of length $n-1$.

After performing these operations as many times as possible, the word is necessarily an alternating product

$$g_1 h_1 g_2 h_2 \cdots g_m h_m,$$

where $g_i \in G$ and $h_i \in H$, and only $g_1$ or $h_m$ is allowed to be the identity element. Such a word is then called a **reduced word**. The **free product** of $G$ and $H$, written $G * H$, is the group whose elements are reduced words, and the product is given by concatenating followed by reduction.

In Problem Sheet C, you will show that the free product can also be characterised by a universal property.

PROPOSITION 6.2. *The pushout exists for the diagram (δ) in* Groups*. Indeed, suppose we are given groups $G, H_1, H_2$ and group homomorphisms $\phi_1, \phi_2$ as in diagram (δ):*

$$\begin{tikzcd} G \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & H_1 \\ H_2 & \end{tikzcd}$$

*Let $N$ denote the normal subgroup of the free product $H_1 * H_2$ generated by all elements of the form $\phi_1(g^{-1}) \cdot \phi_2(g)$ for $g \in G$. Then the quotient group $K := (H_1 * H_2)/N$ is a pushout.*

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*Proof.* Define $\psi_i \colon H_i \to K$ by $\psi_i(h_i) = h_i \cdot N$. We claim that $(K, \psi_1, \psi_2)$ is a solution, i.e. the following commutes:

$$\begin{tikzcd} G \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & H_1 \arrow[d, "\psi_1"] \\ H_2 \arrow[r, "\psi_2"'] & K \end{tikzcd}$$

For this, we must show that for all $g \in G$, as cosets in $K$, one has

$$\phi_1(g) \cdot N = \phi_2(g) \cdot N,$$

or equivalently that

$$\phi_1(g)^{-1} \cdot \phi_2(g) \cdot N = N.$$

Since $\phi_1$ is a homomorphism,

$$\phi_1(g)^{-1} = \phi_1(g^{-1}).$$

Since $\phi_1(g^{-1}) \cdot \phi_2(g) \in N$ for all $g \in G$, the claim follows. Now suppose $(F, \theta_1, \theta_2)$ is another solution. The definition of the free product provides a unique homomorphism[1]

$$\mu \colon H_1 * H_2 \to F$$

such that $\mu|_{H_i} = \theta_i$. Since $\theta_2 \circ \phi_2 = \theta_1 \circ \phi_1$, it follows that $N \leq \ker \mu$, and hence $\mu$ induces a unique homomorphism $\bar{\mu} \colon K \to F$ such that the diagram ($\Delta$) commutes:

$$\begin{tikzcd} G \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & H_1 \arrow[d, "\psi_1"'] \arrow[ddr, "\theta_1"] & \\ H_2 \arrow[r, "\psi_2"] \arrow[drr, "\theta_2"'] & K \arrow[dr, dashed, "\bar{\mu}"] & \\ & & F \end{tikzcd}$$

Thus $K$ is a pushout, as claimed. ∎

Definition 6.3. We call the group $K$ the **free product with amalgamation** of $\phi_1 \colon G \to H_1$ and $\phi_2 \colon G \to H_2$ and write $K = H_1 *_G H_2$. This notation is a little imprecise, since $K$ depends on the homomorphisms $\phi_1$ and $\phi_2$.

Corollary 6.4. *If $H_2 = \{1\}$ is the trivial group then the free product with amalgamation is given by $H_1/N$, where $N$ is the normal subgroup generated by $\phi_1(G)$.*

With these group-theoretic preliminaries out of the way, we can finally state the main result of today's lecture.

[1] If you are confused exactly how $\mu$ is defined, I invite you to look at Problem C.1.

Theorem 6.5 (The Seifert-van Kampen Theorem). *Let $X = X_1 \cup X_2$ with $X_1$ and $X_2$ open subsets. Assume $X_1, X_2$ and $X_0 := X_1 \cap X_2$ are all non-empty and path connected. Let*

$$\imath_i \colon X_0 \hookrightarrow X_i, \qquad \jmath_i \colon X_i \hookrightarrow X$$

*denote inclusions for $i = 1, 2$. Let $p \in X_0$. Then the fundamental group $\pi_1(X, p)$ is the free product with amalgamation of the group homomorphisms $\pi_1(\imath_1) \colon \pi_1(X_0, p) \to \pi_1(X_1, p)$ and $\pi_1(\imath_2) \colon \pi_1(X_0, p) \to \pi_1(X_2, p)$:*

$$\pi_1(X, p) \cong \pi_1(X_1, p) *_{\pi_1(X_0,p)} \pi_1(X_2, p).$$

Before giving the proof, let us note three useful special cases of Theorem 6.5.

Corollary 6.6. *Under the assumptions of Theorem 6.5, one has:*

1. *If $X_2$ is simply connected then*

$$\pi_1(\jmath_1) \colon \pi_1(X_1, p) \to \pi_1(X, p)$$

*is a surjection with kernel the normal subgroup generated by $\pi_1(\imath_1)\big(\pi_1(X_0, p)\big)$.*

2. *If $X_0$ is simply connected then $\pi_1(X, p)$ is the free product of $\pi_1(X_1, p)$ and $\pi_1(X_2, p)$.*
3. *If $X_2$ and $X_0$ are simply connected then*

$$\pi_1(\jmath_1) \colon \pi_1(X_1, p) \to \pi_1(X, p)$$

*is an isomorphism.*

We will need the following piece of point-set topology in the course of the proof of Theorem 6.5.

Lemma 6.7 (The Lebesgue Number Lemma[2]). *Let $(X, d)$ be a compact metric space. Suppose $\mathcal{U}$ is an open cover of $X$. Then there exists $\delta > 0$, called a* **Lebesgue number** *for $\mathcal{U}$ such that every subset $A$ of $X$ with diameter less than $\delta$ is contained in some element of $\mathcal{U}$.*

*Proof of Theorem 6.5.* Consider the diagram:

$$\begin{tikzcd} \pi_1(X_0, p) \arrow[r, "\pi_1(\imath_1)"] \arrow[d, "\pi_1(\imath_2)"'] & \pi_1(X_1, p) \\ \pi_1(X_2, p) & \end{tikzcd}$$

We will show that $(\pi_1(X, p), \pi_1(\jmath_1), \pi_1(\jmath_2))$ is a pushout. Since we already know that a pushout is unique by Lemma 5.8, and since in $\mathsf{Groups}$ the pushout is given by the

[2]See here for a proof.

free product with amalgamation by Proposition 6.2, it thus follows that $\pi_1(X,p)$ must be this product:

$$\pi_1(X,p) \cong \pi_1(X_1,p) *_{\pi_1(X_0,p)} \pi_1(X_2,p).$$

It is clear that $(\pi_1(X,p), \pi_1(\jmath_1), \pi_1(\jmath_2))$ is a solution, i.e. that the following commutes:

$$\begin{tikzcd}
\pi_1(X_0,p) \arrow[r, "\pi_1(\imath_1)"] \arrow[d, "\pi_1(\imath_2)"'] & \pi_1(X_1,p) \arrow[d, "\pi_1(\jmath_1)"] \\
\pi_1(X_2,p) \arrow[r, "\pi_1(\jmath_2)"'] & \pi_1(X,p)
\end{tikzcd}$$

So suppose $(G, \phi_1, \phi_2)$ is another solution:

$$\begin{tikzcd}
\pi_1(X_0,p) \arrow[r, "\pi_1(\imath_1)"] \arrow[d, "\pi_1(\imath_2)"'] & \pi_1(X_1,p) \arrow[d, "\phi_1"] \\
\pi_1(X_2,p) \arrow[r, "\phi_2"'] & G
\end{tikzcd} \tag{6.1}$$

We will construct a unique homomorphism $\psi\colon \pi_1(X,p) \to G$ such that the following diagram commutes.

$$\begin{tikzcd}
\pi_1(X_0,p) \arrow[r, "\pi_1(\imath_1)"] \arrow[d, "\pi_1(\imath_2)"'] & \pi_1(X_1,p) \arrow[d, "\pi_1(\jmath_1)"'] \arrow[ddr, "\phi_1"] & \\
\pi_1(X_2,p) \arrow[r, "\pi_1(\jmath_2)"] \arrow[drr, "\phi_2"'] & \pi_1(X,p) \arrow[dr, dashed, "\psi"] & \\
 & & G
\end{tikzcd} \tag{6.2}$$

Suppose $u$ is a loop in $X$ at $p$. Since $X_1$ and $X_2$ are open, we can find a finite set of points $p = x_0, x_1, \ldots, x_n = p$ along $u$ with the property that each $x_i$ lies in $X_0$ and each segment of $u$ from $x_{i-1}$ to $x_i$ lies in either $X_1$ or $X_2$. Let $v_i\colon I \to X$ denote the path obtained by reparametrising the segment of $u_i$ from $x_{i-1}$ to $x_i$. Explicitly, if $s_{i-1} < s_i$ are such that $u(s_{i-1}) = x_{i-1}$ and $u(s_i) = x_i$, then

$$v_i(s) = u\big((1-s)s_{i-1} + s\, s_i\big).$$

Now, for each $i$, select a path $w_i$ in $X_0$ from $p$ to $x_i$ (take $w_0$ and $w_n$ to be the constant path $e_p$.) Then by concatenating we obtain loops

$$w_{i-1} * v_i * \bar{w}_i$$

which lies entirely in either $X_1$ or $X_2$. See Figure 6.1. Each loop therefore defines an

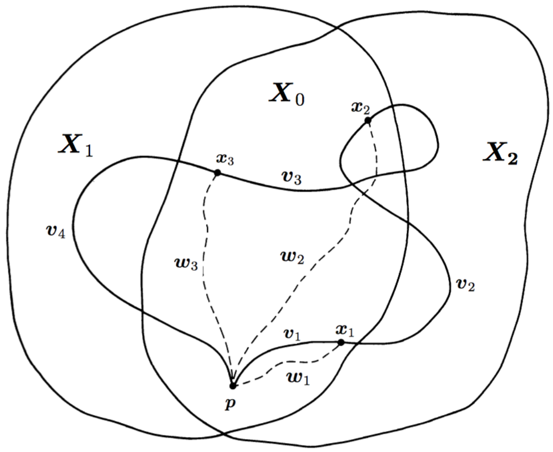


Figure 6.1: Breaking $u$ into smaller pieces.

element either of $\pi_1(X_1, p)$ or $\pi_1(X_2, p)$. Note that

$$u \simeq \prod_{i=1}^{n} \left( w_{i-1} * v_i * \bar{w}_i \right), \qquad \text{rel } \partial I. \tag{6.3}$$

We now define our desired map $\psi$ by

$$\psi[u] := \phi_*[w_0 * v_1 * \bar{w}_1] \cdot \phi_*[w_1 * v_2 * \bar{w}_2] \cdots \phi_*[w_{n-1} * v_n * \bar{w}_n],$$

where $\phi_*$ means either $\phi_1$ or $\phi_2$ depending on whether $[w_{i-1} * v_i * \bar{w}_i]$ belongs to $\pi_1(X_1, p)$ or $\pi_1(X_2, p)$. Note that there is a potential ambiguity if $w_{i-1} * v_i * \bar{w}_i$ lies in both $X_1$ and $X_2$, because we could choose either $\phi_1$ or $\phi_2$. But in this case $[w_{i-1} * v_i * \bar{w}_i]$ lies in $\pi_1(X_0, p)$, and commutativity of the diagram (6.1) shows that we get the same result.

This defines the map $\psi$, but there are still many things we need to check before the proof is complete:

1. Is the definition of $\psi$ independent of the points $x_i$ and the paths $w_i$?
2. Does $\psi$ make sense on $\pi_1(X, p)$? That is, if $u_1$ and $u_2$ are two homotopic loops rel $p$ do we get the same answer if we use $u_1$ and $u_2$ in the definition of $\psi$?
3. Is $\psi$ a homomorphism?
4. Does the diagram (6.2) commute?
5. Is $\psi$ unique with respect to these properties?

Let's pretend for a second that we've already proved (1) and (2). The last three properties are then easy: (3) and (4) follow by construction, and uniqueness comes from the fact that if a loop $v$ is entirely contained in $X_1$ then requiring (6.2) to commutes means we are *forced* to define $\psi[v] = \phi_1[v]$ (and similarly for $X_2$), and thus (6.3) means that if we want $\psi$ to be a group homomorphism we have no choice but to define it as we have done.

So now let's get on with proving the first two parts, starting with (1). Consider a point $x_i$, and suppose $w_i'$ is another path from $p$ to $x_i$. See Figure 6.2. Then we have

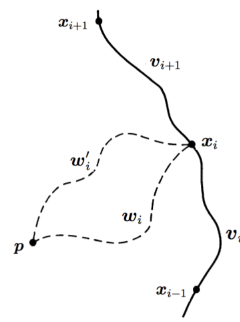


Figure 6.2: $w_i$ and $w_i'$.

$$\phi_*[w_i * v_{i+1} * \bar{w}_{i+1}] = \phi_*[w_i * \bar{w}_i' * w_i' * v_{i+1} * \bar{w}_{i+1}] = \phi_*[w_i * \bar{w}_i'] \cdot \phi_*[w_i' * v_{i+1} * \bar{w}_{i+1}].$$

On the other hand,

$$\begin{aligned}\phi_*[w_{i-1} * v_i * \bar{w}_i] &= \phi_*[w_{i-1} * v_i * \bar{w}_i' * w_i' * \bar{w}_i] \\ &= \phi_*[w_{i-1} * v_i * \bar{w}_i'] \cdot \phi_*[w_i' * \bar{w}_i] \\ &= \phi_*[w_{i-1} * v_i * \bar{w}_i'] \cdot \big(\phi_*[w_i * \bar{w}_i']\big)^{-1}\end{aligned}$$

Since $\bar{w}_i * w_i'$ is a loop based in $X_0$ and since the diagram (6.1) commutes, the value of $\phi_*$ on this loop will be the same whether $\phi_*$ is $\phi_1$ or $\phi_2$. Thus

$$\phi_*[w_{i-1} * v_i * \bar{w}_i] \cdot \phi_*[w_i * v_{i+1} * \bar{w}_{i+1}] = \phi_*[w_{i-1} * v_i * \bar{w}_i'] \cdot \phi_*[w_i' * v_{i+1} * \bar{w}_{i+1}]$$

This means that the product won't change when $w_i'$ is used instead of $w_i$. Repeating this argument at each point $x_i$ shows that $\psi[u]$ does not depend on the choice of the paths $w_i$.

Now we show independence of the points $x_i$. Suppose another point $y \in X_0$ is added along $v_i$ separating the path $v_i$ into two new paths $v_i'$ and $v_{i-1}'$. See Figure 6.3. Let $w'$ denote a path from $p$ to $y$ in $X_0$. Suppose for definiteness that the loop $w_{i-1} * v_i * \bar{w}_i$ is contained in $X_1$. Then the same is true of the two new loops $w_{i-1} * v_{i-1}' * \bar{w}'$ and $w' * v_i' * \bar{w}_i$, and we have

$$\begin{aligned}\phi_1[w_{i-1} * v_{i-1}' * \bar{w}'] \cdot \phi_1[w' * v_i' * \bar{w}_i] &= \phi_1[w_{i-1} * v_{i-1}' * \bar{w}' * w' * v_i' * \bar{w}_i] \\ &= \phi_1[w_{i-1} * v_i * \bar{w}_i].\end{aligned}$$

This shows that adding an additional point to the set of $\{x_i\}$ does not change the value of $\psi[u]$. More generally, the same is true if we add a finite number of points, that is, refining the $\{x_i\}$ leads to the same result. Now suppose we are given two different sets $\{x_i\}$ and $\{y_i\}$ of points. Their union is a common refinement of both the $\{x_i\}$ and the $\{y_i\}$, and we have just shown that the value of $\psi[u]$ doesn't change

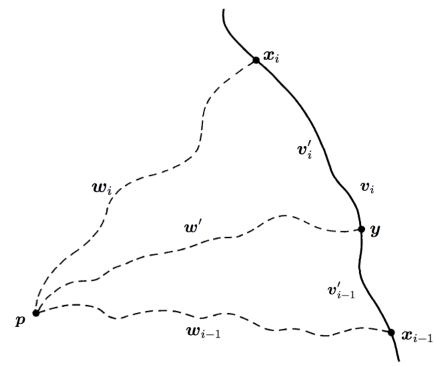


Figure 6.3: The new point $y$.

when refining the set of points. Thus the value of $\psi[u]$ is the same if we use either the $\{x_i\}$ or the $\{y_i\}$. This proves (1).

Now let us move onto the proof of (2). Suppose $u$ and $u'$ are two homotopic loops and $U\colon u \simeq u'$ is a homotopy rel $\partial I$. We subdivide the square $I \times I$ into lots of little squares in such a way that each smaller square is mapped by $U$ into either $X_1$ or $X_2$. Such a decomposition exists by Lemma 6.7, where we take the open cover of $I \times I$ given by the connected components of $U^{-1}(X_1)$ and $U^{-1}(X_2)$. See Figure 6.4. Proceeding one small rectangle at a time, this deforms $u$ into $u'$ through a finite

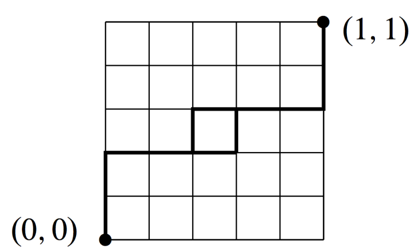


Figure 6.4: Subdividing $I \times I$.

sequence of paths such that each step involves a homotopy in which the only change occurs within either $X_1$ or $X_2$. For such a restricted deformation, the points $\{x_i\}$ may be chosen so that the value of $\psi$ is unchanged. This proves (2), and thus completes the proof.

■

LECTURE 7

# Singular homology

In this lecture we finally get started on defining the *homology functors* $H_n$ referred to in Lecture 1. Let us begin with some preliminaries on free abelian groups.

DEFINITION 7.1. Let $B$ be a subset of an abelian group $F$. We say $F$ is **free abelian** with **basis** $B$ if the subgroup generated by $b$ is infinite cyclic for each $b \in B$ and $F = \bigoplus_{b \in B} \langle b \rangle$ as a direct sum.

Thus a free abelian group is a (possibly uncountable) direct sum of copies of $\mathbb{Z}$. A typical element $x \in F$ has a unique expression

$$x = \sum_{b \in B} m_b \, b, \qquad m_b \in \mathbb{Z}$$

where **almost all** (meaning all but a finite number) of the $m_b$ are zero. The following trivial lemma will be crucial in all that follows.

LEMMA 7.2. *Let $F$ be a free abelian group with basis $B$. If $A$ is an abelian group and $\phi \colon B \to A$ is a function then there exists a unique group homomorphism $\tilde{\phi} \colon F \to A$ such that*

$$\tilde{\phi}(b) = \phi(b), \qquad \forall\, b \in B,$$

*that is, the following diagram commutes:*

$$\begin{tikzcd} F \arrow[dr, dashed, "\tilde{\phi}"] & \\ B \arrow[u, hook] \arrow[r, "\phi"'] & A \end{tikzcd}$$

*Moreover any abelian group $A$ is isomorphic to a quotient group of the form $F/R$, where $F$ is a free abelian group.*

*Proof.* Define $\tilde{\phi}$ by

$$\tilde{\phi}\left( \sum_{b \in B} m_b \, b \right) := \sum_{b \in B} m_b \, \phi(b).$$

Then $\tilde{\phi}$ is well-defined since any element of $F$ has a unique expression of this form, and it is obviously a homomorphism. Moreover $\tilde{\phi}$ is unique since any two homomorphisms that agree on a set of generators (in this case $B$) must coincide. The last statement is on Problem Sheet D. ∎

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We refer to the extension $\phi \mapsto \tilde{\phi}$ given in Lemma 7.2 as **extending by linearity**. By an abuse of notation, we will typically continue to write $\phi$ for the extension, rather than $\tilde{\phi}$.

LEMMA 7.3. *Given any set $B$, there exists a free abelian group having $B$ as basis.*

*Proof.* If $B = \emptyset$, take $F = 0$. Otherwise, for each $b \in B$, let $\mathbb{Z}_b$ be a group whose elements are all symbols $mb$ with $m \in \mathbb{Z}$ and addition defined by $mb + nb = (m+n)b$. Then $\mathbb{Z}_b$ is infinite cyclic with generator $b$. Now set

$$F := \bigoplus_{b \in B} \mathbb{Z}_b.$$

This is a free abelian group with basis given by the set $\{e_b \mid b \in B\}$, where $e_b$ has a zero in each entry apart from the $b$th entry, where it is a 1. Identifying $e_b$ with $b$, we see that $F$ has basis $B$. ■

DEFINITION 7.4. The **rank** of a free abelian group $F$ is the cardinality of any basis of $B$ of $F$.

This is well-defined thanks to Problem D.1. Moreover two free abelian groups are isomorphic if and only if they have the same rank.

REMARK 7.5. One can extend the notion of rank to any abelian group: if $G$ is an arbitrary abelian group then we say $G$ has (possibly infinite) **rank** $r$ if there exists a free abelian subgroup $F$ of $G$ such that $F$ has rank $r$ and $G/F$ is torsion. Such subgroups $F$ always exist (this is also part of Problem D.1). However it is not obvious that this definition is well-defined. Indeed, $F$ is not unique, and it is by no means clear that the rank of $F$ only depends on $G$. At the very end of the course we will develop one way of proving this.

Now let us define the notion of a simplex.

DEFINITION 7.6. An ordered tuple $(z_0, z_1, \dots, z_n)$ of points in $\mathbb{R}^m$ is said to be **affinely independent** if the set $\{z_1 - z_0, z_2 - z_0, \dots, z_n - z_0\}$ is linearly independent (thus necessarily $n \le m$). Given an affinely independent tuple $(z_0, z_1, \dots, z_n)$ of vectors in $\mathbb{R}^m$, we denote by $[z_0, z_1, \dots, z_n]$ the **$n$-simplex spanned by** $(z_0, z_1, \dots, z_n)$, namely the set

$$[z_0, z_1, \dots, z_n] := \left\{ x \in \mathbb{R}^m \mid x = \sum_{i=0}^{n} s_i\, z_i, \text{ where } 0 \le s_i \le 1, \ \sum_{i=0}^{n} s_i = 1 \right\}.$$

We call the points $z_i$ the **vertices** of the $n$-simplex $[z_0, z_1, \dots, z_n]$. The expression $x = \sum_{i=0}^{n} s_i\, z_i$ of any point $x \in [z_0, z_1, \dots, z_n]$ is unique[1]. We call the $(n+1)$-tuple $(s_0, s_1, \dots, s_m)$ the **barycentric coordinates** of $x$. The **barycentre** of the $n$-simplex $[z_0, z_1, \dots, z_n]$ is the unique point where all the $s_i$ are equal, namely

$$\frac{1}{n+1}(z_0 + z_1 + \cdots + z_n). \tag{7.1}$$

[1]Exercise: Why?

Definition 7.7. Let $[z_0, z_1, \dots, z_n]$ be an $n$-simplex. The **face opposite to** $z_i$ is the $(n-1)$-simplex[2] $[z_0, \dots \hat{z}_i, \dots, z_n]$. Here the circumflex ˆ means[3] "delete". Equivalently

$$[z_0, \dots \hat{z}_i, \dots, z_n] := \{x \in [z_0, z_1, \dots, z_n] \mid s_i = 0.\}.$$

An $n$-simplex thus has $n+1$ faces. The **boundary** of an $n$-simplex is the union of its faces.

Definition 7.8. The **standard** $n$**-simplex** in $\mathbb{R}^{n+1}$ is the $n$-simplex $[e_0, e_1, \dots, e_n]$, where $e_i$ is the vector coordinates are all zero, apart from the $i+1$st position, which is 1. We denote the standard $n$-simplex by $\Delta^n$.

So much for a simplex in $\mathbb{R}^{n+1}$. What about in an arbitrary topological space $X$?

Definition 7.9. Let $X$ be a topological space. A **singular** $n$**-simplex in** $X$ is a continuous map $\sigma \colon \Delta^n \to X$.

Since $\Delta^0$ is a point, a 0-simplex in $X$ is simply a point in $X$. Since $\Delta^1$ is a closed interval, a 1-simplex is[4] the same thing as a path in $X$. The adjective "singular" is added to emphasis that the image $\sigma(\Delta^n)$ does not need to "look" anything like $\Delta^n$, i.e. we do *not* require $\sigma$ to be a homeomorphism. In particular, there is nothing stopping $\sigma$ being a constant map.

Definition 7.10. Let $X$ be a topological space and $n \geq 0$. Let $C_n(X)$ denote the free abelian group with basis the singular $n$-simplices in $X$ (cf. Lemma 7.3.) We call an element of $C_n(X)$ a **singular** $n$**-chain**. It is convenient for notational reasons to also define $C_{-1}(X) = 0$.

Note that (as a group), $C_n(X)$ is typically huge: if $X$ is an uncountable set then $C_n(X)$ is itself uncountable for all $n \geq 0$. We will shortly replace $C_n(X)$ with a (usually smaller) abelian group $H_n(X)$. First, let us explain how to obtain a singular $(n-1)$-simplex from a singular $n$-simplex.

Definition 7.11. Let $\sigma \colon \Delta^n \to X$ be a singular $n$-simplex in $X$. If we restrict $\sigma$ to one of the faces of $\Delta^n$, we get a continuous map from an $n-1$-simplex into $X$.

Actually this definition is cheating a little bit; whilst any face of $\Delta^n$ is an $(n-1)$-simplex, it is not the *standard* $(n-1)$-simplex, since the domain is wrong. Thus strictly speaking, the restriction of a $n$-simplex $\sigma$ in $X$ to a face is not actually a singular $(n-1)$-simplex in $X$, since it is not a continuous map from $\Delta^{n-1}$ into $X$. There are two ways round this tedious pedantry:

1. Ignore it. After all, it's clear what we mean.
2. Fix it by making the notation more complicated.

[2] This is clearly an $(n-1)$-simplex as a subset of a linearly independent set is also linearly independent.
[3] This is a convention we will use throughout the course.
[4] Not quite! We will come back to this in Lecture 9.

We shall go[5] for option (2). To this end, let us define the **$i$th face map**

$$\varepsilon_i \colon \Delta^{n-1} \to \Delta^n, \qquad i = 0, 1, \dots, n$$

that maps the standard $(n-1)$-simplex $\Delta^{n-1}$ homeomorphically onto the $i$th face of $\Delta^n$. Explicitly,

$$\varepsilon_0(s_0, s_1, \dots s_{n-1}) = (0, s_0, s_1, \dots s_{n-1}),$$

for $i = 0$, and for $1 \leq i \leq n-1$,

$$\varepsilon_i(s_0, s_1, \dots s_{n-1}) = (s_0, s_1, \dots, s_{i-1}, 0, s_i, \dots, s_{n-1}),$$

and finally

$$\varepsilon_n(s_0, s_1, \dots, s_{n-1}) = (s_0, s_1, \dots, s_{n-1}, 0).$$

Where necessary we will write $\varepsilon_i^n \colon \Delta^{n-1} \to \Delta^n$ (this is needed for instance in (7.2) below).

We can now "improve" Definition 7.11:

Definition 7.12. Let $\sigma \colon \Delta^n \to X$ be a singular $n$-simplex in $X$ and let $0 \leq i \leq n$. The composition $\sigma \circ \varepsilon_i \colon \Delta^{n-1} \to X$ is then a singular $(n-1)$-simplex in $X$, which we call the **restriction of $\sigma$ to the $i$th face**.

We can now define the boundary of a singular $n$-simplex.

Definition 7.13. Let $\sigma \colon \Delta^n \to X$ be a singular $n$-simplex in $X$. The **boundary** of $\sigma$ is the alternating sum of the restriction of $\sigma$ to the faces:

$$\partial \sigma := \sum_{i=0}^{n} (-1)^i \sigma \circ \varepsilon_i.$$

Thus the boundary of $\sigma$ is *not* a singular $(n-1)$-simplex, but rather a formal *sum* of singular $(n-1)$-simplices, and hence (by definition) a singular $(n-1)$-chain: $\partial \sigma \in C_{n-1}(X)$. We define the boundary of a singular 0-simplex to be zero.

Remark 7.14. If we omit the face maps (which we will occasionally do, cf. in Proposition 8.5 next lecture), the formula is slightly more intuitive (albeit formally incorrect):

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_n]}.$$

Applying Lemma 7.2 we obtain a well defined map on the free abelian group $C_n(X)$.

[5] Being pedantic is an important quality for a mathematician to have (or at least, to pretend to have when teaching others ...)

DEFINITION 7.15. The **singular boundary operator**

$$\partial\colon C_n(X) \to C_{n-1}(X)$$

is the unique homomorphism extending the operator from Definition 7.13. Occasionally for clarity we will write $\partial_n\colon C_n(X) \to C_{n-1}(X)$.

Thus for each $n \geq 0$ we have constructed a sequence of free abelian groups and homomorphisms. We illustrate this pictorially as

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial} C_{n-1}(X) \longrightarrow \cdots \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \longrightarrow 0.$$

Anticipating the category $\mathsf{Comp}$ of *chain complexes* that we will introduce in Lecture 10, we will bundle all the groups $C_n(X)$ together and write $(C_\bullet(X), \partial)$ to denote all the groups and maps at once.

PROPOSITION 7.16. *$\partial^2 = 0$, that is, for any $n \geq 0$ the composition*

$$C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X)$$

*is always zero.*

*Proof.* Since $C_{n+1}(X)$ is generated by all the $(n+1)$-simplices, by Lemma 7.2 it suffices to show that if $\sigma\colon \Delta^{n+1} \to X$ is a singular $(n+1)$-simplex then $\partial^2\sigma = 0$. As you can probably guess, the point it that since the boundary operator was defined via an alternating sum, when you apply it twice things cancel. Indeed, if $k < j$ then one has the following *face relation*:

$$\varepsilon_j^{n+1} \circ \varepsilon_k^n = \varepsilon_k^{n+1} \circ \varepsilon_{j-1}^n \colon \Delta^{n-1} \to \Delta^{n+1}. \tag{7.2}$$

To prove (7.2), it suffices to observe that both sides give the same answer when fed a vertex $e_i$ for $i = 0, 1, \ldots, n-1$. Now we compute:

$$\begin{aligned}
\partial^2\sigma &= \partial\left(\sum_{j=0}^{n+1}(-1)^j \sigma \circ \varepsilon_j^{n+1}\right) \\
&= \sum_{k=0}^{n}\sum_{j=0}^{n+1}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n \\
&= \underbrace{\sum_{j\leq k}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n}_{(*)} + \underbrace{\sum_{k<j}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n}_{(\dagger)}.
\end{aligned}$$

We claim that the two terms $(*)$ and $(\dagger)$ cancel. Indeed, to see this first apply (7.2) to $(**)$ and change variables by setting $l = k$ and $m = j - 1$ to obtain:

$$\sum_{k<j}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n = \sum_{k<j}(-1)^{j+k}\sigma \circ \varepsilon_k^{n+1} \circ \varepsilon_{j-1}^n = \sum_{l\leq m}(-1)^{l+m+1}\sigma \circ \varepsilon_l^{n+1} \circ \varepsilon_m^n.$$

The last expression is the same as $(*)$, only every term appears with the opposite sign. This completes the proof. ■

Definition 7.17. A **singular $n$-cycle** in $X$ is a singular $n$-chain that lies in the kernel of $\partial$. We denote by $Z_n(X)$ the set of all singular $n$-cycles. A **singular $n$-boundary** in $X$ is a singular $n$-chain that lies in the image of $\partial$. We denote by $B_n(X)$ the set of all singular $n$-boundaries[6]. Both $Z_n(X)$ and $B_n(X)$ are subgroups of $C_n(X)$. Moreover since $\partial^2 = 0$, we have

$$B_n(X) \subseteq Z_n(X) \subseteq C_n(X).$$

We can therefore form the quotient group. This will be the eponymous *singular homology*.

Definition 7.18. We define the **$n$-singular homology** group of $X$, written $H_n(X)$, to be the quotient group

$$H_n(X) = Z_n(X) \Big/ B_n(X).$$

Thus $H_n(X)$ is an abelian (not free abelian!) group for each $n$. Given a singular $n$-cycle $c$, we denote[7] by $\langle c \rangle$ the coset $c + B_n(X) \in H_n(X)$ and call $\langle c \rangle$ the **homology class** determined by $c$.

We will conclude this lecture by showing that $H_n$ is a functor. This means that we need to associate to each continuous map $f\colon X \to Y$ a homomorphism $H_n(f)\colon H_n(X) \to H_n(Y)$.

Definition 7.19. If $f\colon X \to Y$ is a continuous map and $\sigma\colon \Delta^n \to X$ is a singular $n$-simplex in $X$ then $f \circ \sigma\colon \Delta^n \to Y$ is a singular $n$-simplex in $Y$. We therefore obtain an induced map $f_n^{\#}\colon C_n(X) \to C_n(Y)$ by extending this by linearity (Lemma 7.2:

$$f_n^{\#}\left(\sum m_\sigma \, \sigma\right) := \sum m_\sigma f \circ \sigma.$$

Just as with the entire complex $(C_\bullet(X), \partial)$, we can bundle all the maps $f_n^{\#}$ together and write $f_\bullet^{\#}$. You will not be surprised to learn that $f \mapsto f_\bullet^{\#}$ is a functor. We will study this in Lecture 10 (it's a functor $\mathsf{Top} \to \mathsf{Comp}$.) For now though, let us prove that $f_n^{\#}$ descends to the quotient to define a map on $H_n(X) \to H_n(Y)$. This is the content of the following proposition.

Proposition 7.20. *If $f\colon X \to Y$ is continuous, then the following diagrams commutes for every $n$:*

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \\ {\scriptstyle f_n^{\#}}\downarrow & & \downarrow{\scriptstyle f_{n-1}^{\#}} \\ C_n(Y) & \xrightarrow[\partial]{} & C_{n-1}(Y) \end{array}$$

[6] Boundary begins with a "b", hence the notation $B_n(X)$. Similarly cycle begins with a "c", hence the notation ... wait a second ... Damnit, we already used $C_n(X)$ for the chain groups! Next best option: *Zykel* begins with a "z" ...

[7] We use angle brackets $\langle \cdot \rangle$ rather than square brackets $[\cdot]$ to distinguish between homotopy and homology classes.

*Proof.* It suffice to evaluate both $f_{n-1}^{\#} \circ \partial$ and $\partial \circ f_n^{\#}$ on a singular $n$-simplex $\sigma$ in $X$. Now

$$f_{n-1}^{\#} \circ \partial\sigma = f_{n-1}^{\#}\left(\sum_{i=0}^{n}(-1)^i \sigma \circ \varepsilon_i\right) = \sum_{i=0}^{n}(-1)^i f \circ \sigma \circ \varepsilon_i.$$

Similarly

$$\partial \circ f_n^{\#}\sigma = \partial(f \circ \sigma) = \sum_{i=0}^{n}(-1)^i f \circ \sigma \circ \varepsilon_i.$$

■

Corollary 7.21. *If $f\colon X \to Y$ is continuous then both $f_n^{\#}(Z_n(X)) \subseteq Z_n(Y)$ and $f_n^{\#}(B_n(X)) \subseteq B_n(Y)$. Thus $f_n^{\#}$ induces a map $H_n(f)\colon H_n(X) \to H_n(Y)$.*

*Proof.* If $\partial c = 0$ then $\partial(f_n^{\#} c) = f_{n-1}^{\#}(\partial c) = 0$, so that $f_n^{\#} c \in Z_n(Y)$. Similarly if $b = \partial c$ then $f_n^{\#} b = f_n^{\#}(\partial c) = \partial(f_{n+1}^{\#} c)$, so that $f_n^{\#} b \in B_n(Y)$. ■

Thus $f_n^{\#}$ induces a map $H_n(f)\colon H_n(X) \to H_n(Y)$, given by

$$H_n(f)\langle c\rangle := \langle f_n^{\#} c\rangle.$$

Corollary 7.22. *For each $n \geq 0$, $H_n\colon \mathsf{Top} \to \mathsf{Ab}$ is a functor.*

*Proof.* We need only check that $H_n(g \circ f) = H_n(g) \circ H_n(f)$ and that $H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$. Both of these are immediate from the definitions. ■

Corollary 7.23. *If $X$ and $Y$ are homeomorphic then $H_n(X) \cong H_n(Y)$ for all $n \geq 0$.*

*Proof.* Immediate from Problem A.2. ■

Thinking back to Lecture 1, we have now constructed the singular homology functors from Theorem 1.15. In order for our proof of the Brouwer Fixed Point Theorem 1.1 to be complete, we need to verify that $H_n(B^{n+1}) = 0$ and $H_n(S^n) \neq 0$. We will prove that $H_n(B^{n+1}) = 0$ next lecture; the fact that $H_n(S^n) \neq 0$ will take much longer (Lecture 15).

LECTURE 8

# The homotopy axiom

In this lecture we prove that if $f, g\colon X \to Y$ are homotopic maps then the induced maps $H_n(f)$ and $H_n(g)$ coincide for every $n \geq 0$:

$$[f] = [g] \qquad \Rightarrow \qquad H_n(f) = H_n(g) \qquad \forall n \geq 0. \tag{8.1}$$

we will prove this in Theorem 8.9 below. By Problem A.2, this means that $H_n$ may be regarded as a functor $H_n\colon \mathsf{hTop} \to \mathsf{Ab}$. This should be compared to $\pi_1$: we initially defined $\pi_1$ as a functor $\mathsf{Top}_* \to \mathsf{Groups}$ and then later showed that $\pi_1$ induces a functor from $\mathsf{hTop}_*$ to $\mathsf{Groups}$. The property (8.1) is usually called the **homotopy axiom**. This terminology will be explained at the end of the course when we cover the *Eilenberg-Steenrod axioms*.

We begin by stating two elementary properties of singular homology, both of which appear on Problem Sheet D.

PROPOSITION 8.1 (The dimension axiom). *Let $X$ be a one-point space $\{*\}$. Then $H_n(X) = 0$ for all $n > 0$.*

Just as with the homotopy axiom (8.1), the meaning of the name "dimension axiom" in Proposition 8.1 will get explained later. For the next result, let us recall that if $\{G_\lambda \mid \lambda \in \Lambda\}$ is a collection of groups, an element of $\bigoplus_{\lambda\in\Lambda} G_\lambda$ is a tuple $(g_\lambda)$ where all but finitely many of the $g_\lambda$ are equal to the identity.

PROPOSITION 8.2. *Let $X$ be a topological space. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ denote the path components of $X$. Then for every $n \geq 0$ one has*

$$H_n(X) \cong \bigoplus_{\lambda\in\Lambda} H_n(X_\lambda).$$

Thus, just as with the fundamental group, for computational purposes we may always assume our spaces are path connected. In general, it is very hard to compute $H_n(X)$ for $n > 0$, but it is always possible to compute $H_0(X)$.

PROPOSITION 8.3. *If $X$ is a non-empty path connected space then $H_0(X) = \mathbb{Z}$. A generator is given by $\langle x\rangle$ for any point $x \in X$, and if $x, y \in X$ then $\langle x\rangle = \langle y\rangle$. Moreover, if $\langle c\rangle$ is any generator then $\langle c\rangle = \langle x\rangle$ for some (and hence every) $x \in X$.*

*Proof.* We identify a 0-singular simplex in $X$ with a point in $X$. Since $\partial\colon C_0(X) \to 0$ is the zero map, every point in $X$ is a 0-cycle: $Z_0(X) = C_0(X)$. Thus each point

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$x \in X$ determines a class $\langle x \rangle \in H_0(X)$. Let us now identify $B_0(X)$. A typical element of $c \in C_0(X)$ is of the form

$$c = \sum_{x \in X} m_x\, x, \qquad m_x \in \mathbb{Z},$$

where all but finitely many of the $m_x$ are equal to zero. We claim that:

$$B_0(X) = \left\{ \sum m_x\, x \mid \sum m_x = 0 \right\}. \tag{8.2}$$

Firstly, suppose $c = \sum_{i=1}^n m_i\, x_i$ satisfies $\sum_{i=1}^n m_i = 0$. We wish to build a singular 1-chain $a$ such that $\partial a = c$. For this, choose a point $p \in X$ and for each $i = 1, \dots, n$, let $u_i \colon I \to X$ denote a path starting at $p$ and ending at $x_i$. After identifying $I = [0, 1]$ with $\Delta^1 = [e_0, e_1]$, we may regard each $u_i$ as a singular 1-simplex $\sigma_i \colon \Delta^1 \to X$ such that $\sigma_i(e_0) = p$ and $\sigma_i(e_1) = x_i$. Note that

$$\partial \sigma_i = \sigma_i(e_1) - \sigma_i(e_0) = x_i - p \in C_0(X).$$

Now set $a := \sum_{i=1}^n m_i \sigma_i$. Then

$$\begin{aligned} \partial a &= \partial \left( \sum_{i=1}^n m_i\, \sigma_i \right) \\ &= \sum_{i=1}^n m_i\, \partial \sigma_i \\ &= \sum_{i=1}^n m_i x_i - \left( \sum_{i=1}^n m_i \right) p \\ &= c - 0 = c. \end{aligned}$$

Conversely, suppose $d \in B_0(X)$. Then there exists $b \in C_1(X)$ such that $\partial b = d$. Write $b = \sum_{j=1}^k l_j\, \tau_j$, where $\tau_j \colon \Delta^1 \to X$ and $l_i \in \mathbb{Z}$. Then

$$d = \sum_{j=1}^k l_j \big( \tau_j(e_1) - \tau_j(e_0) \big).$$

Thus in the expansion of $d$, each coefficient $l_j$ appears twice and with the opposite sign. Thus the sum of the coefficients of $d$ is zero. This proves (8.2).

Thus by (8.2), the map

$$\phi \colon Z_0(X) = C_0(X) \to \mathbb{Z}, \qquad \phi \left( \sum_{x \in X} m_x\, x \right) := \sum_{x \in X} m_x \tag{8.3}$$

is a surjection whose kernel is precisely $B_0(X)$. Thus $H_0(X) \cong \operatorname{im} \phi = \mathbb{Z}$.

Now suppose $x, y \in X$. A path from $x$ to $y$ determines a singular 1-simplex $\sigma$ with $\partial \sigma = y - x$. Thus $\langle x \rangle = \langle y \rangle \in H_0(X)$. Finally suppose $a = \sum_i m_i\, x_i$ is a 0-cycle such that $\langle a \rangle$ is a generator of $H_0(X)$. Then we must have $\phi(a) = \pm 1$. Replacing $a$ with $-a$ if necessary, we may assume that $\phi(a) = 1$. Thus $\sum_i m_i = 1$. Then for any point $x \in X$, we have $a = x + (a - x)$. Since $a - x \in B_0(X)$ by (8.2) we therefore have $\langle x \rangle = \langle a \rangle$. This completes the proof. ∎

An immediate corollary of Proposition 8.3 is:

Corollary 8.4. *Let $X$ and $Y$ be path connected spaces and $f\colon X \to Y$ continuous. Then $H_0(f)\colon H_0(X) \to H_0(Y)$ maps a generator of $H_0(X)$ to a generator of $H_0(Y)$.*

The main step in the proof of the homotopy axiom is the following innocuous looking statement.

Proposition 8.5. *Let $X$ be a topological space and define inclusions $\imath, \jmath\colon X \hookrightarrow X \times I$ by*

$$\imath(x) := (x, 0), \qquad \jmath(x) := (x, 1).$$

*Then*

$$H_n(\imath) = H_n(\jmath), \qquad \forall\, n \geq 0.$$

Remark 8.6. There is a very cute three line proof which uses an abstract result in homological algebra called the **Acyclic Models Theorem**. We will prove this right at the end of the course, and we will then come back to Proposition 8.5 and give a second proof. Therefore don't worry if you find the proof of Proposition 8.5 below horrible; we will eventually see a nicer one.

In order to prove Proposition 8.5 we introduce the idea of a **chain homotopy**.

Lemma 8.7. *Let $f, g\colon X \to Y$ be continuous maps. Assume for each*[1] *$n \geq -1$ there is a homomorphism*

$$P\colon C_n(X) \to C_{n+1}(Y)$$

*with*

$$f_n^{\#} - g_n^{\#} = \partial P + P\partial.$$

*Then $H_n(f) = H_n(g)$ for all $n \geq 0$.*

The maps $P_n$ look like this:

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] & C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"] & C_{n-1}(X) \arrow[r] \arrow[dl, "P"] & \dots \\
\dots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"'] & C_n(Y) \arrow[r, "\partial"'] & C_{n-1}(Y) \arrow[r] & \dots
\end{tikzcd}$$

Beware though, this diagram is *not* commutative! Just as with the $\partial$ maps, sometimes for clarity we will include the subscript and write $P_n\colon C_n(X) \to C_{n+1}(Y)$. In Lecture 10 we will define an abstract version of the operators $P$, which will then be called **chain homotopies**.

*Proof.* Take $c \in Z_n(X)$. Then

$$(f_n^{\#} - g_n^{\#})c = (\partial P + P\partial)c = \partial P c \in B_n(Y).$$

Thus $H_n(f)\langle c\rangle = H_n(g)\langle c\rangle$. ■

[1] The map $P_{-1}\colon 0 = C_{-1}(X) \to C_0(Y)$ is necessarily the zero map.

Let us now prove Proposition 8.5. The first step is the following lemma.

LEMMA 8.8. *$\Delta^n \times I$ is the union of $n+1$ copies of $\Delta^{n+1}$.*

*Proof.* For $i = -1, 0, 1, \dots, n-1$, let $g_i \colon \Delta^n \to I$ denote the map

$$g_i(s_0, s_1, \dots, s_n) = \sum_{j>i} s_j.$$

Note $g_i$ is indeed a map to $I$, as $x = \sum s_i\, e_i$ implies $\sum s_i = 1$. Let $G_i \subset \Delta^n \times I$ denote the graph of $g_i$. Then $G_i$ is homeomorphic to $\Delta^n$ via the projection $\Delta^n \times I \to \Delta^n$ onto the first factor. Let us now label the vertices at the "bottom" (i.e. $\Delta^n \times \{0\}$) of $\Delta^n \times I$ by $e'_0, e'_1, \dots, e'_n$ and those at the "top" by $e''_0, e''_1, \dots, e''_n$. Then $G_i$ is the $n$-simplex

$$G_i = [e'_0, \dots, e'_i, e''_{i+1}, \dots, e''_n].$$

Since $G_i$ lies below $G_{i-1}$ as $g_i \le g_{i-1}$, it follows that the region between $G_i$ and $G_{i-1}$ is the $(n+1)$-simplex $[e'_0, \dots, e'_i, e''_i, \dots, e''_n]$; this is indeed an $(n+1)$-simplex as $e''_i$ is not in $G_i$ and hence not in the $n$-simplex $[e'_0, \dots, e'_i, e''_{i+1}, \dots, e''_n]$. Since $0 = g_n \le g_{n-1} \le \cdots \le g_0 \le g_{-1} = 1$, we see that $\Delta^n \times I$ is the union of the regions between the $G_i$, and hence the union of $n+1$ different $(n+1)$-simplices $[e'_0, \dots, e'_i, e''_i, \dots, e''_n]$, each intersecting the next in an $n$-simplex face. ■

We now prove Proposition 8.5.

*Proof of Proposition 8.5.* We will break with our convention here since otherwise the notation becomes too messy. If $\sigma : \Delta^n \to X$ is a singular $n$-simplex, then we denote by

$$(\sigma \times \mathrm{id}_I)|_{[e'_0,\dots,e'_i,e''_i,\dots,e''_n]}$$

the singular $(n+1)$-simplex obtained from the previous lemma by restricting $\sigma \times \mathrm{id}_I \colon \Delta^n \times I \to X \times I$ to the $(n+1)$-simplex $[e'_0, \dots, e'_i, e''_i, \dots, e''_n]$. Of course, we should really precompose with an appropriate face map $\Delta^{n+1} \to [e'_0, \dots, e'_i, e''_i, \dots, e''_n]$ in order to make $\sigma \times \mathrm{id}_I|_{[e'_0,\dots,e'_i,e''_i,\dots,e''_n]}$ into a genuine singular $(n+1)$-simplex, however this is too cumbersome (the notation is already bad enough as it is!)

With this in mind, we define a homomorphism

$$P \colon C_n(X) \to C_{n+1}(X \times I)$$

by requiring that

$$P\sigma = \sum_{i=0}^{n} (-1)^i (\sigma \times \mathrm{id}_I)|_{[e'_0,\dots,e'_i,e''_i,\dots,e''_n]}$$

and then extending by linearity (cf. Lemma 7.2.)

Now let us look at $\partial P\sigma$, remembering that the $\hat{}$ notation over a vertex means "delete" (see also Remark 7.14):

$$\begin{aligned}\partial P\sigma = &\sum_{j \le i} (-1)^{i+j} (\sigma \times \mathrm{id}_I)|_{[e'_0,\dots,\hat{e}'_j,\dots,e'_i,e''_i,\dots,e''_n]} \\ &+ \sum_{i \le j} (-1)^{i+j+1} (\sigma \times \mathrm{id}_I)|_{[e'_0,\dots,e'_i,e''_i,\dots,\hat{e}''_j,\dots e''_n]}.\end{aligned}$$

The terms with $i = j$ all cancel[2] apart from the first and last ones:

$$(\sigma \times \mathrm{id}_I)|_{[\hat{e}'_0, e''_0, \ldots, e''_n]} \qquad \text{and} \qquad -(\sigma \times \mathrm{id}_I)|_{[e'_0, \ldots, e'_n, \hat{e}''_n]},$$

which are precisely $\jmath_n^\# \sigma$ and $-\imath_n^\# \sigma$ respectively. Meanwhile the terms with $i \neq j$ are precisely $-P\partial\sigma$, since

$$\begin{aligned} P\partial\sigma &= \sum_{i=0}^n (-1)^i P\sigma|_{[e_0, \ldots, \hat{e}_i, \ldots, e_n]} \\ &= \sum_{j<i} (-1)^{i+j-1} (\sigma \times \mathrm{id}_I)|_{[e'_0, \ldots, \hat{e}'_j, \ldots, e'_i, e''_i, \ldots, e''_n]} \\ &\quad + \sum_{i<j} (-1)^{i+j} (\sigma \times \mathrm{id}_I)|_{[e'_0, \ldots, e'_i, e''_i, \ldots, \hat{e}''_j, \ldots e''_n]}. \end{aligned}$$

Putting this altogether we obtain

$$\partial P\sigma = \jmath_n^\# \sigma - \imath_n^\# \sigma - P\partial\sigma.$$

The same is true for any chain $c \in C_n(X)$ by Lemma 7.2, and hence Lemma 8.7 completes the proof. ∎

We can now prove (8.1).

THEOREM 8.9 (The homotopy axiom). *Let $f, g \colon X \to Y$ be two homotopic maps. Then $H_n(f) = H_n(g)$ for all $n \geq 0$. Thus for each $n \geq 0$, $H_n \colon \mathsf{hTop} \to \mathsf{Ab}$ is a functor.*

*Proof.* Let $F \colon f \simeq g$ be a homotopy. Then using the maps $\imath$ and $\jmath$ from Proposition 8.5, we have

$$f = F \circ \imath, \qquad g = F \circ \jmath.$$

Thus as $H_n$ is a functor, we have

$$H_n(f) = H_n(F \circ \imath) = H_n(F) \circ H_n(\imath) = H_n(F) \circ H_n(\jmath) = H_n(g).$$

∎

In the same way as Corollary 7.23, we obtain the following result.

COROLLARY 8.10. *If $X$ and $Y$ have the same homotopy type then $H_n(X) \cong H_n(Y)$ for all $n \geq 0$, where the isomorphism is induced by any homotopy equivalence.*

*Proof.* Immediate from Problem A.2. ∎

We also have:

COROLLARY 8.11. *If $X$ is contractible then $H_n(X) = 0$ for all $n > 0$.*

*Proof.* Immediate from the previous corollary and the dimension axiom (Proposition 8.1.) ∎

[2]Remember we are "hiding" the face maps with this notation ...

LECTURE 9

# The Hurewicz Theorem

In this lecture we investigate the relationship between $H_1(X)$ and $\pi_1(X,p)$. It should come as no surprise that there is one, since $\Delta^1$ is homeomorphic to an interval, but at the same time they cannot be identical, since $\pi_1(X,p)$ is not necessarily abelian (cf. Problem C.2 and Problem C.3), meanwhile by definition $H_1(X)$ always is.

We begin by defining a function $h\colon \pi_1(X,p) \to H_1(X)$ called the **Hurewicz map**. We will then prove that $h$ is a surjective homomorphism and identify its kernel.

Remark 9.1. We have already implicitly used the fact that $\Delta^1$ and $I$ are homeomorphic, and thus a singular 1-simplex is the same thing as a path. However in this lecture it is important to keep track of whether we are working with a singular 1-simplex or a path. To this end, let $\theta\colon \Delta^1 \to I$ denote the homeomorphism that sends

$$\Delta^1 \ni \sum_{i=0}^{1} s_i\, e_i \mapsto s_1 \in I.$$

We will use the following slightly imprecise convention: if $u\colon I \to X$ is a path, then $u' := u \circ \theta\colon \Delta^1 \to X$ is a singular 1-simplex. Explicitly,

$$u'(s_0, s_1) = u(s_1).$$

With this convention, if $u$ and $v$ are two paths such that $u(1) = v(0)$ then the concatenated path $u * v$ becomes:

$$(u * v)'(s_0, s_1) = \begin{cases} u'(2s_0 - 1, 2s_1), & 0 \le s_1 \le \frac{1}{2}, \\ v'(2s_0, 2s_1 - 1), & \frac{1}{2} \le s_1 \le 1. \end{cases} \tag{9.1}$$

Conversely if $\sigma\colon \Delta^1 \to X$ is a singular 1-simplex then $\sigma' := \sigma \circ \theta^{-1}\colon I \to X$ is a path. Explicitly

$$\sigma'(s) = \sigma(1 - s, s).$$

The imprecise bit is that the $'$ denotes either composition with $\theta$ or $\theta^{-1}$ depending on whether we start with a path or a simplex). Since we always use $u, v, w$ for paths and $\sigma, \tau$ for simplices, this should not be too confusing.

Proposition 9.2. *Let $p \in X$. There is a well defined function $h\colon \pi_1(X,p) \to H_1(X)$ given by*

$$[u] \mapsto \langle u' \rangle,$$

*where $u$ is a loop in $X$ based at $p$.*

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Where necessary, we will write $h_p$ instead of $h$ to indicate the dependence on $p$.

*Proof.* Clearly $u' = u \circ \theta$ is a singular 1-simplex in $X$, and thus in particular belongs to $C_1(X)$. In fact, $u' \in Z_1(X)$, since

$$\partial u' = u(\theta(e_1)) - u(\theta(e_0)) = u(1) - u(0) = 0.$$

Thus $\langle u' \rangle$ is a well defined element in $H_1(X)$. Now recall the map $\omega : I \to S^1$ from the solution to Problem B.5 given by $\omega(s) = e^{2\pi i s}$. From $u$ we obtain the map $\hat{u} : S^1 \to X$ given by $\hat{u} = u \circ \omega^{-1}$. The map $\hat{u}$ then induces a map $H_1(\hat{u}) : H_1(S^1) \to H_1(X)$. Then using the fact that $H_1$ is a functor, we have

$$\langle u' \rangle = \langle \hat{u} \circ \omega \circ \theta \rangle = H_1(\hat{u}) \langle \omega \circ \theta \rangle,$$

as elements of $H_1(X)$. Here we view $\omega \circ \theta : \Delta^1 \to S^1$ as a singular 1-simplex in $S^1$. Now if $v$ is another closed path in $X$ based at $p$ with $u \simeq v$ rel $\partial I$ then by the solution to Problem B.5 the corresponding maps $\hat{u}$ and $\hat{v}$ are homotopic rel 1. Thus by the homotopy axiom of singular homology, $H_1(\hat{u}) = H_1(\hat{v})$. Thus

$$\langle u' \rangle = H_1(\hat{u}) \langle \omega \circ \theta \rangle = H_1(\hat{v}) \langle \omega \circ \theta \rangle = \langle v' \rangle,$$

which shows that the class $\langle u' \rangle$ only depends on $[u]$. This completes the proof. ■

Now let us prove that $h$ is a homomorphism of groups. Let us emphasise (we have already been doing this, but you may not have noticed) that we are using *additive* notation for homology classes (this makes sense because the homology groups are always abelian).

Proposition 9.3. *The Hurewicz map $h : \pi_1(X, p) \to H_1(X)$ is a group homomorphism: for all $[u], [v] \in \pi_1(X, p)$, we have*

$$h([u]) + h([v]) = h([u * v]).$$

*Proof.* Let $u$ and $v$ be loops in $X$ based at $p$. Define a continuous map $\sigma \colon \Delta^2 \to X$ as indicated by Figure 9.1. Specifically, we define $\sigma$ to be $u', v'$, and $(u * v)'$ on the boundary of $\partial \Delta^2$ as the picture suggests:

$$\sigma(1-s, s, 0) := u(s), \qquad \sigma(0, 1-s, s) := v(s), \qquad \sigma(1-s, 0, s) := (u * v)(s).$$

For fixed $s$, we then define $\sigma$ on the interior of $\Delta^2$ to be constant on the line segments from $a = a(s)$ to $b = b(s)$ and from $c = c(s)$ to $d = d(s)$. Here $a(s) = (1-s, s, 0)$, $b(s) = \left(\frac{1}{2}(2-s), 0, \frac{s}{2}\right)$, $c(s) = (0, 1-s, s)$ and $d(s) = \left(\frac{1}{2}(1-s), 0, \frac{1}{2}(1+s)\right)$. The gluing lemma shows that $\sigma$ is continuous, and hence is a singular 2-simplex in $X$. An explicit formula for $\sigma$ using (9.1) is

$$\sigma(s_0, s_1, s_2) := (u * v)' \left( s_0 + \frac{s_1}{2}, \frac{s_1}{2} + s_2 \right).$$

Now observe that

$$\partial \sigma = \sigma \circ \varepsilon_0 - \sigma \circ \varepsilon_1 + \sigma \circ \varepsilon_2 = v' - (u * v)' + u'.$$

Thus in $H_1(X)$, we have

$$\langle u' \rangle + \langle v' \rangle = \langle (u * v)' \rangle,$$

or equivalently

$$h([u]) + h([v]) = h([u * v]),$$

which shows that $h$ is a homomorphism as desired. ∎

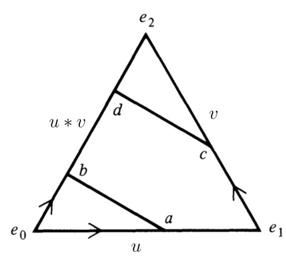


Figure 9.1: Proving $h$ is a homomorphism.

We now present the following simple corollary of Proposition 2.15, which we will need when examining the kernel of $h$.

Proposition 9.4. *Let $\sigma \colon \Delta^2 \to X$ be a singular 2-simplex. Abbreviate $\sigma_i := \sigma \circ \varepsilon_i$ for $i = 0, 1, 2$, so that $\sigma_i$ is a singular 1-simplex in $X$. See Figure 9.2. Then the path $\sigma_0' * \bar{\sigma}_1' * \sigma_2'$ is nullhomotopic rel $\partial I$.*

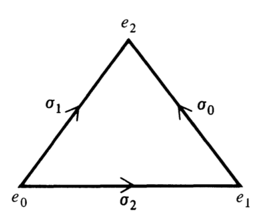


Figure 9.2: $\sigma \colon \Delta^2 \to X$

*Proof.* By Problem D.2, the map $\sigma$ induces a map $g : B^2 \to X$, given by $g(x) = \sigma(h^{-1}(x))$, where $h : \Delta^2 \to B^2$ is the homeomorphism mapping $(\Delta^2, \partial\Delta^2) \to (B^2, S^1)$. Let $f := g|_{S^1}$, then $f$ is a reparametrisation of the loop $\sigma_0' * \bar{\sigma}_1' * \sigma_2'$. By Proposition 2.15, $f$ is nullhomotopic rel 1 in $S^1$. Thus $\sigma_0' * \bar{\sigma}_1' * \sigma_2'$ is nullhomotopic rel $\partial I$. ∎

We will also need the following (trivial) piece of algebra. The lemma is basically a fancy way of saying "if something looks like it should cancel, then it does". The important thing in the following lemma is that the second group $A$ is abelian.

Lemma 9.5 (Substitution Principle). *Let $F$ be a free abelian group with basis $B$. Let $b_0, b_1, \dots, b_k$ be a list (possibly with repetitions) of elements of $B$, and assume $m_0, m_1, \dots, m_k$ are integers such that*

$$m_0\, b_0 = \sum_{i=1}^{k} m_i\, b_i.$$

*Let $A$ be an abelian group, and suppose $a_0, a_1, \dots, a_k$ are elements of $A$ such that*

$$b_i = b_j \qquad \Rightarrow \qquad a_i = a_j.$$

*Then in $A$, one also has*

$$m_0\, a_0 = \sum_{i=1}^{k} m_i\, a_i.$$

*Proof.* Define a function $\varphi : B \to A$ by setting $\varphi(b_i) = a_i$ for $i = 0, 1, \dots, k$ and $\varphi(b) = 0$ for all other elements of $B$. This is well defined by assumption. Then by Lemma 7.2, there exists a unique group homomorphism $\tilde{\varphi} : F \to A$ extending $\varphi$. Then

$$0 = \tilde{\varphi}\left( m_0\, b_0 - \sum_{i=1}^{k} m_i\, b_i \right) = m_0\, a_0 - \sum_{i=1}^{k} m_i\, a_i,$$

■

Now let us recall a standard piece of group theory.

Definition 9.6. Let $G$ be any group (not necessarily abelian). Given $g, h \in G$, we define their **commutator** to be the element

$$[g, h] := ghg^{-1}h^{-1}.$$

We define the **commutator subgroup** of $G$ to be the subgroup $[G, G]$ of $G$ generated by all the commutators. This is a normal abelian subgroup of $G$, and $[G, G] = \{1\}$ (where 1 is the identity element of $G$) if and only if $G$ is abelian. If $N$ is a normal subgroup of $G$, then $G/N$ is abelian if and only if $[G, G] \leq N$. We define the **abelianisation** of $G$ to be the quotient group $G^{\mathrm{ab}} = G/[G, G]$. As you will see on Problem E.1 on Problem Sheet E, the abelianisation $G^{\mathrm{ab}}$ together with the group homomorphism (the projection) $p \colon G \to G^{\mathrm{ab}}$ can be characterised by a universal property: namely that if $A$ is any abelian group and $\varphi \colon G \to A$ is any group homomorphism, there exists a *unique* homomorphism $\tilde{\varphi}$ such that the following commutes:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "p"'] & A \\ G^{\mathrm{ab}} \arrow[ur, dashed, "\tilde{\varphi}"'] & \end{tikzcd}$$

We are now ready to state and prove the main result of today's lecture. As stated, this result is actually due to Poincaré, not Hurewicz, but there is a more general theorem[1] that extends this which is due to Hurewicz, and hence this result is commonly called the "Hurewicz Theorem".

[1] We will discuss the more general version in Algebraic Topology II.

Theorem 9.7 (Hurewicz Theorem). *Let $X$ be a path connected topological space. Then the induced homomorphism $\tilde{h} : \pi_1(X,p)^{\mathrm{ab}} \to H_1(X)$ is a group isomorphism.*

*Proof.* Since $X$ is path connected, choose a path $w_x \colon I \to X$ such that $w_x(0) = p$ and $w_x(1) = x$ for each $x \in X$. Let us insist that $w_p$ is the constant path $e_p$.

We will first show that $h$ is surjective. Suppose

$$c = \sum_{i=1}^{k} m_i\, \sigma_i \in Z_1(X).$$

Since $c$ is a cycle, we have

$$0 = \partial c = \sum_{i=1}^{k} m_i\big(\sigma_i(e_1) - \sigma_i(e_0)\big), \tag{9.2}$$

which we view as an equation among the basis elements (i.e. points in $X$) of the free abelian group $C_0(X)$. Set

$$y_i := \sigma_i(e_1), \qquad z_i := \sigma_i(e_0).$$

See Figure 9.3. We can apply the substitution principle to the list

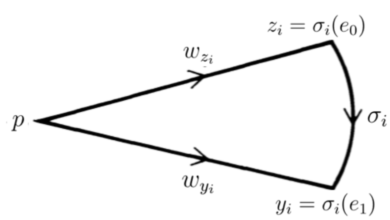


Figure 9.3: Proving $h$ is surjective.

$$y_1, z_1, \ldots, y_k, z_k,$$

in the free abelian group $C_0(X)$ and compare it to the list

$$w'_{y_1}, w'_{z_1}, \ldots, w'_{y_k}, w'_{z_k},$$

in $C_1(X)$. Then (9.2) tells us[2] that

$$\sum_{i=1}^{k} m_i\big(w'_{y_i} - w'_{z_i}\big) = 0$$

[2]Less formally, this is simply the observation that since $c$ is a cycle, the sum of the paths $w_{y_i}$ and $\bar{w}_{z_i}$ cancel.

in $C_1(X)$. Thus

$$c = c - 0 = c - \left(\sum_{i=1}^{k} m_i\big(w'_{y_i} - w'_{z_i}\big)\right) = \sum_{i=1}^{k} m_i\big(w'_{z_i} + \sigma_i - w'_{y_i}\big). \tag{9.3}$$

But now by assumption $w_{z_i} * \sigma_i' * \bar{w}_{y_i}$ is a *loop* in $X$ based at $p$, and hence we can feed it to $h$:

$$\begin{aligned} h\left(\prod_{i=1}^{k}\left[w_{z_i} * \sigma_i' * \bar{w}_{y_i}\right]^{m_i}\right) &\overset{(*)}{=} \sum_{i=1}^{k} m_i h\left[w_{z_i} * \sigma_i' * \bar{w}_{y_i}\right] \\ &\overset{(\dagger)}{=} \sum_{i=1}^{k} m_i \langle w'_{z_i} + \sigma_i + \bar{w}'_{y_i}\rangle \\ &\overset{(\ddagger)}{=} \sum_{i=1}^{k} m_i \langle w'_{z_i} + \sigma_i - w'_{y_i}\rangle \\ &\overset{(\heartsuit)}{=} \langle c\rangle. \end{aligned}$$

where $(*)$ used Proposition 9.3, $(\dagger)$ used Problem E.4, $(\ddagger)$ used Problem E.3 and finally $(\heartsuit)$ used (9.3). This shows that $h$ is surjective.

We now build an inverse $\tilde{\eta}\colon H_1(X) \to \pi_1(X,p)^{\mathrm{ab}}$ to the induced map $\tilde{h}$, which will prove that $\tilde{h}$ is an isomorphism. Suppose $\sigma\colon \Delta^1 \to X$ is a 1-simplex. We associate to $\sigma$ the class in $\pi_1(X,p)^{\mathrm{ab}}$ represented by the loop $w_{\sigma(e_0)} * \sigma' * \bar{w}_{\sigma(e_1)}$. Extend this by linearity to define a map

$$\eta\colon C_1(X) \to \pi_1(X,p)^{\mathrm{ab}}.$$

We claim that $\eta$ vanishes on $B_1(X)$. For this let $\tau : \Delta^2 \to X$ be a singular 2-simplex. We compute $\eta(\partial\tau)$. As in Proposition 9.4, let $\tau_i := \tau \circ \varepsilon_i$ for $i = 0, 1, 2$. See Figure 9.4. Then

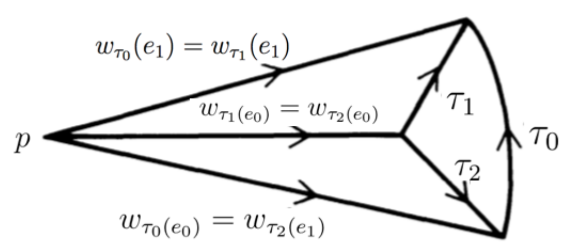


Figure 9.4: Showing $\eta(\partial\tau) = 1$.

$$
\begin{aligned}
\eta(\partial\tau) &= \eta(\tau_0) * \eta(\tau_1)^{-1} * \eta(\tau_2) \\
&= \left[w_{\tau_0(e_0)} * \tau_0' * \bar{w}_{\tau_0(e_1)}\right] * \left[w_{\tau_1(e_0)} * \tau_1' * \bar{w}_{\tau_1(e_1)}\right]^{-1} * \left[w_{\tau_2(e_0)} * \tau_2' * \bar{w}_{\tau_2(e_1)}\right] \\
&= \left[w_{\tau_0(e_0)} * \tau_0' * \bar{\tau}_1' * \tau_2' * \bar{w}_{\tau_2(e_1)}\right] \\
&\overset{(*)}{=} \left[w_{\tau_0(e_0)} * \bar{w}_{\tau_2(e_1)}\right] \\
&= 1,
\end{aligned}
$$

where $(*)$ used Proposition 9.4. Thus $\eta|_{Z_1(X)}$ factors to define a map $\tilde{\eta} : H_1(X) \to \pi_1(X,p)^{\mathrm{ab}}$. By construction $\tilde{\eta} \circ \tilde{h} = \mathrm{id}$. This completes the proof. ■

LECTURE 10

# Chain complexes

In this lecture we introduce a new category, $\mathsf{Comp}$, and a new subject: **homological algebra**.

Definition 10.1. A **chain complex** is a sequence of abelian groups and homomorphisms

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \dots$$

for $n \in \mathbb{Z}$ which satisfies

$$\partial^2 = 0, \qquad \forall\, n \in \mathbb{Z}.$$

We refer[1] to the entire complex as $(C_\bullet, \partial)$ or sometimes just $C_\bullet$.

Of course, we have already met one key example:

Example 10.2. Let $X$ be a topological space. Then the singular chains $(C_\bullet(X), \partial)$ is a chain complex. Note that in this example the abelian groups are all zero for negative subscripts; this however is not part of the definition in general.

Definition 10.3. The fact that $\partial^2 = 0$ means that if we define

$$Z_n = Z_n(C_\bullet) = \ker \partial \colon C_n \to C_{n-1}$$

and

$$B_n = B_n(C_\bullet) = \operatorname{im} \partial \colon C_{n+1} \to C_n$$

then

$$B_n \subseteq Z_n.$$

By analogy with the singular chain complex, we call elements of $Z_n$ $n$**-cycles** and elements of $B_n$ $n$**-boundaries**. We define the $n$**th homology group** of the chain complex $C_\bullet$ to be the quotient group

$$H_n = H_n(C_\bullet) := Z_n(C_\bullet) \Big/ B_n(C_\bullet).$$

We will continue to the use the notation $\langle c \rangle$ to denote the class of an element $c \in Z_n$ in $H_n$.

Now let us introduce another key notion.

Will J. Merry, Algebraic Topology I, Autumn 2017, ETH Zürich. Last modified: November 15, 2017.

[1]Yes, I know it is somewhat illogical to omit the subscript on the $\partial$ and not on the $C$, but in practice it makes things more convenient.

Definition 10.4. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of two homomorphisms of abelian groups is said to be **exact at** $B$ if

$$\operatorname{im} f = \ker g.$$

More generally, a sequence

$$\dots A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \dots, \qquad n \in \mathbb{Z}$$

is said to be **exact** if it is exact at every $A_n$.

Example 10.5. Using the notion of exactness we can rephrase other definitions from basic algebra. Suppose $f\colon A \to B$ is a homomorphism of abelian groups.

- $f$ is injective if and only if $0 \to A \xrightarrow{f} B$ is exact.
- $f$ is surjective if and only if $A \xrightarrow{f} B \to 0$ is exact.
- $f$ is an isomorphism if and only if $0 \to A \xrightarrow{f} B \to 0$ is exact.
- Slightly less obviously, if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is exact then $f$ is surjective if and only if $h$ is injective.

Definition 10.6. A **short exact sequence** of abelian groups is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

In this case, $A \cong \operatorname{im} f$ and $\operatorname{coker} f := B/\operatorname{im} f \cong C$ via $b + \operatorname{im} f \mapsto g(b)$.

In contrast, a **long exact sequence** is one that has (potentially) infinitely many terms.

Definition 10.7. A chain complex $C$ is said to be **acyclic** if $C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1}$ is exact at $C_n$ for all $n$.

We then have trivially:

Proposition 10.8. *A chain complex $C$ is acyclic if and only if $H_n(C) = 0$ for all $n \in \mathbb{Z}$.*

Let us now make the chain complexes into a category.

Definition 10.9. Suppose $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ are two chain complexes. A **chain map** $f_\bullet\colon C_\bullet \to C'_\bullet$ is a sequence of group homomorphisms $f_n\colon C_n \to C'_n$ such that the following diagram commutes for all $n \in \mathbb{Z}$:

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & C'_{n+1} \arrow[r, "\partial'"'] & C'_n \arrow[r, "\partial'"'] & C'_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

that is, $\partial' f_n = f_{n-1}\partial$. Usually we will just write $f\colon C_\bullet \to C'_\bullet$ instead of $f_\bullet\colon C_\bullet \to C'_\bullet$ to minimise the number of $\bullet$s floating about on the page. Composition of two chain maps is defined as one would guess: if $f\colon C_\bullet \to C'_\bullet$ and $g\colon C'_\bullet \to C''_\bullet$ are two chain maps then $g \circ f\colon C_\bullet \to C''_\bullet$ is the chain map given by $(g\circ f)_n = g_n f_n$. This is indeed a valid chain map, i.e. $\partial''(g\circ f)_n = (g\circ f)_{n-1} \circ \partial$, as the following diagram commutes:

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\partial"'] & C'_n \arrow[r, "g_n"] \arrow[d, "\partial'"] & C''_n \arrow[d, "\partial''"] \\ C_{n-1} \arrow[r, "f_{n-1}"'] & C'_{n-1} \arrow[r, "g_{n-1}"'] & C''_{n-1} \end{tikzcd}$$

Example 10.10. If $f\colon X \to Y$ is a continuous map between two topological spaces then $f^\#_\bullet\colon C_\bullet(X) \to C_\bullet(Y)$ is a chain map.

Definition 10.11. The category $\mathsf{Comp}$ has objects the chain complexes $(C_\bullet, \partial)$, morphisms the chain maps $f\colon C_\bullet \to C'_\bullet$, and composition as specified above.

The singular chain complex can now be interpreted as a functor.

Proposition 10.12. *There is a functor* $\mathsf{Top} \to \mathsf{Comp}$ *that associates to a topological space $X$ its singular chain complex $C_\bullet(X)$ and to a continuous map $f\colon X \to Y$ the associated map $f^\#_\bullet\colon C_\bullet(X) \to C_\bullet(Y)$.*

*Proof.* This follows from the results of Lecture 7. ∎

The reason for insisting that $\partial' f = f\partial$ in the definition of a chain map is that it means a chain map induces a map on the respectively homologies:

$$H_n(f)\colon H_n(C_\bullet) \to H_n(C'_\bullet)$$

given by

$$H_n(f)\langle c\rangle := \langle f_n c\rangle.$$

This is well defined as by assumption

$$f_n(Z_n) \subseteq Z'_n, \qquad f_n(B_n) \subseteq B'_n.$$

This means that we can interpret $H_n$ as a functor.

Proposition 10.13. *For each $n \in \mathbb{Z}$, there exists a functor $H_n\colon \mathsf{Comp} \to \mathsf{Ab}$ called the $n$***th homology functor** *that sends $C_\bullet$ to $H_n(C_\bullet)$ and to a chain map $f\colon C_\bullet \to C'_\bullet$ the associated map $H_n(f)\colon H_n(C_\bullet) \to H_n(C'_\bullet)$.*

The proof is immediate. This means we can now see the construction of the singular homology as a two-stage process. The first is *topological*: this is assignment $X \mapsto C_\bullet(X)$. The second is *purely algebraic*: this is the assignment $C_\bullet(X) \mapsto H_n(C_\bullet(X)) = H_n(X)$.

Proposition 10.14. *The functor $H_n\colon \mathsf{Comp} \to \mathsf{Ab}$ is an* **additive functor**[2], *that is $H_n(f+g) = H_n(f) + H_n(g)$.*

[2] I will not give the precise definition of an additive functor, since we do not need it. It's more complicated then you think, since in order to define additive functors one first needs to define additive categories ...

Now let me bombard you with more definitions. They are all straightforward.

Definition 10.15. A **subcomplex** $(C_\bullet, \partial)$ of chain complex $(C'_\bullet, \partial')$ is a chain complex such that $C_n \subseteq C'_n$ for all $n \in \mathbb{Z}$ and such that $\partial_n = \partial'_n|_{C_n}$. Denoting by $i_n \colon C_n \to C'_n$ the inclusion, the second condition is equivalent to saying that $i \colon C_\bullet \to C'_\bullet$ is a chain map.

Definition 10.16. If $(C_\bullet, \partial)$ is a subcomplex of $(C'_\bullet, \partial')$, one can form the **quotient complex** $(\bar{C}_\bullet, \bar{\partial})$ where

$$\bar{C}_n = C'_n / C_n$$

and $\bar{\partial}$ is the induced map.

Definition 10.17. Suppose $f \colon (C_\bullet, \partial) \to (C'_\bullet, \partial')$ is a chain map between two complexes. Then $\ker f_\bullet$ is a subcomplex of $C_\bullet$ and $\operatorname{im} f_\bullet$ is a subcomplex of $C'_\bullet$; the boundary operator of $\ker f_\bullet$ is simply

$$\partial_n|_{\ker f_n} \colon \ker f_n \to \ker f_{n-1},$$

and similarly the boundary operator of $\operatorname{im} f_\bullet$ is the restriction

$$\partial'_n|_{\operatorname{im} f_n} \colon \operatorname{im} f_n \to \operatorname{im} f_{n-1}.$$

Note these only form subcomplexes because $f_\bullet$ is a chain map! The **cokernel** of $f_\bullet$ is the chain complex $\operatorname{coker} f_\bullet$ given by

$$\operatorname{coker} f_n = C'_n / \operatorname{im} f_n,$$

which is itself a quotient complex of $C'_\bullet$ since $\operatorname{im} f_\bullet$ is a subcomplex of $C'_\bullet$.

This allows us to talk about a sequence of complexes being exact.

Definition 10.18. Suppose we are given complexes $(C^m_\bullet, \partial^m)$ for $m \in \mathbb{Z}$ and chain maps $f^m \colon C^m_\bullet \to C^{m-1}_\bullet$. Pictorially, this means we have the following commutating mess:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C^{m+1}_{n+1} \arrow[r, "f^{m+1}_{n+1}"] \arrow[d, "\partial^{m+1}"'] & C^m_{n+1} \arrow[r, "f^m_{n+1}"] \arrow[d, "\partial^m"] & C^{m-1}_{n+1} \arrow[r] \arrow[d, "\partial^{m-1}"] & \cdots \\
\cdots \arrow[r] & C^{m+1}_{n} \arrow[r, "f^{m+1}_{n}"] \arrow[d, "\partial^{m+1}"'] & C^m_{n} \arrow[r, "f^m_{n}"] \arrow[d, "\partial^m"] & C^{m-1}_{n} \arrow[r] \arrow[d, "\partial^{m-1}"] & \cdots \\
\cdots \arrow[r] & C^{m+1}_{n-1} \arrow[r, "f^{m+1}_{n-1}"] \arrow[d] & C^m_{n-1} \arrow[r, "f^m_{n-1}"] \arrow[d] & C^{m-1}_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We say that sequence $(C^m_\bullet, f^m)$ is **exact** if (as complexes) $\ker f^m_\bullet = \operatorname{im} f^{m+1}_\bullet$ for every $m$.

Definition 10.19. A **short exact sequence** of chain complexes is an exact sequence of chain complexes of the form

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0,$$

where 0 denotes the chain complex all of whose entries are zero. By Problem E.5 on Problem Sheet E, the rows of short exact sequences of chain complexes are short exact sequences of abelian groups.

Definition 10.20. Let $C_\bullet$ and $C'_\bullet$ be two subcomplexes of $C''_\bullet$. The **intersection** of $C_\bullet$ and $C'_\bullet$ is the subcomplex $C_\bullet \cap C'_\bullet$ whose $n$th term is $C_n \cap C'_n$, and similarly the **sum** of $C_\bullet$ and $C'_\bullet$ is the subcomplex $C_\bullet + C'_\bullet$ whose $n$th term is $C_n + C'_n$.

One can also form direct sums: if $\{(C^\lambda_\bullet, \partial^\lambda) \mid \lambda \in \Lambda\}$ is a family of complexes indexed by a set $\Lambda$, then their **direct sum** is the complex $\bigoplus_\lambda C^\lambda_\bullet$ equipped with the boundary operator $\sum_\lambda \partial^\lambda$.

Now we define the abstract analogue of Lemma 8.7.

Definition 10.21. Let $f, g\colon (C_\bullet, \partial) \to (C'_\bullet, \partial')$ be two chain maps. We say that $f_\bullet$ and $g_\bullet$ are **chain homotopic**, written $f \simeq g$ if there exists a sequence of homomorphisms

$$P\colon C_n \to C'_{n+1}, \qquad n \in \mathbb{Z},$$

such that

$$\partial' P + P\partial = f_n - g_n, \qquad \forall\, n \in \mathbb{Z}.$$

The sequence $P = P_\bullet$ is called a **chain homotopy**, and we write $P\colon f \simeq g$.

Definition 10.22. We say that $f\colon C_\bullet \to C'_\bullet$ is a **chain equivalence** if there exists $g\colon C'_\bullet \to C_\bullet$ such that $g \circ f \simeq \mathrm{id}_{C_\bullet}$ and $f \circ g \simeq \mathrm{id}_{C'_\bullet}$.

Remark 10.23. The relation of being chain homotopic is a congruence on $\mathsf{Comp}$, and this can be used to define a category $\mathsf{hComp}$.

The next result shows that the homology functor descends to $\mathsf{hComp}$.

Proposition 10.24. *Let $f, g\colon (C_\bullet, \partial) \to (C'_\bullet, \partial')$ be two chain maps with $f \simeq g$. Then for all $n$,*

$$H_n(f) = H_n(g)\colon H_n(C_\bullet) \to H_n(C'_\bullet).$$

*In particular, if $f$ is a chain equivalence then $H_n(f)$ is an isomorphism for each $n$.*

The proof is identical to Lemma 8.7, but let us repeat it anyway.

*Proof.* Take $c \in Z_n$. Then

$$(f_n - g_n)c = (\partial' P + P\partial)c = \partial' P c \in B'_n.$$

Thus $H_n(f)\langle c\rangle = H_n(g)\langle c\rangle$. ∎

A special case of a chain homotopy is where one map is the identity and the other is the zero map. This gets its own name.

DEFINITION 10.25. A **contracting homotopy** $Q$ of a chain complex $C_\bullet$ is a sequence of maps $Q_n \colon C_n \to C_{n+1}$ such that

$$\partial Q + Q\partial = \mathrm{id}_{C_n}, \qquad \forall\, n \in \mathbb{Z}.$$

COROLLARY 10.26. *If a chain complex $C_\bullet$ has a contracting homotopy then it is acyclic.*

*Proof.* By Proposition 10.24, we see that for all $n$, $H_n(\mathrm{id}_{C_\bullet}) = H_n(0) = 0$. Since $H_n$ is a functor, this implies that $H_n(C_\bullet) = 0$. ■

REMARK 10.27. The converse to Corollary 10.26 is false. An example is given by taking

$$C_n := \begin{cases} \mathbb{Z}_2, & n = 0, \\ \mathbb{Z}, & n = 1, 2, \\ 0, & n \neq 0, 1, 2, \end{cases}$$

and defining $\partial \colon C_2 \to C_1$ to be $k \mapsto 2k$ and $\partial \colon C_1 \to C_0$ by $k \mapsto k \bmod 2$. This complex is acyclic but there does not exist a contracting homotopy. Indeed, if such a $Q$ existed, then $Q_0 \colon C_0 \to C_1$ would define a right inverse to $\partial \colon C_1 \to C_0$. However any group homomorphism $\mathbb{Z}_2 \to \mathbb{Z}$ is trivial.

At the end of the course we[3] will show that if $C_\bullet$ is a complex all of whose groups $C_n$ are *free* abelian groups (such a chain complex is called a **free** chain complex), then the converse to Corollary 10.26 *does* hold: a free chain complex is acyclic if and only if it has a contracting homotopy. At the same time we will prove a partial converse to Proposition 10.24: if $f \colon C_\bullet \to C'_\bullet$ is a chain map between two free chain complexes such that $H_n(f)$ is an isomorphism for all $n$, then $f$ is a chain equivalence.

[3]Well, actually you: these claims are all on Problem Sheet L ...

LECTURE 11

# The Snake Lemma and its friends

We first prove the so-called **Snake Lemma**. The reason for the name is the fact that the wiggly arrow in (11.1) below looks like a snake[1].

PROPOSITION 11.1 (The Snake Lemma). *Suppose we are given a commutative diagram of abelian groups where the rows are exact:*

$$\begin{tikzcd} & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' & \end{tikzcd}$$

*Then there is a well defined homomorphism*

$$\delta \colon \ker h \to \operatorname{coker} f$$

*such that there is an exact sequence*

$$\ker f \to \ker g \to \ker h \xrightarrow{\delta} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h.$$

*Explicitly,*

$$\delta(c) = (i')^{-1} g j^{-1}(c) + \operatorname{im} f,$$

*where $(i')^{-1}(\cdot)$ and $j^{-1}(\cdot)$ denote any choice of preimage (the composition is independent of the choices).*

*Proof.* The proof is very easy, but I will be kind and go through it in great detail. We will prove the result in three stages.

Will J. Merry, Algebraic Topology I, Autumn 2017, ETH Zürich. Last modified: January 02, 2018.

[1]Right ... Evidently most mathematicians have never seen snakes.

**1.** We first enlarge our given diagram to include the kernels and cokernels:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker f \arrow[r, "k"] \arrow[d] & \ker g \arrow[r, "l"] \arrow[d] & \ker h \arrow[d] & \\
 & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"'] \arrow[d] & B' \arrow[r, "j'"'] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker} f \arrow[r, "p"'] \arrow[d] & \operatorname{coker} g \arrow[r, "q"'] \arrow[d] & \operatorname{coker} h \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The maps $k, l$ on the top row and the maps $p, q$ are just the induced maps. That is,

$$k := i|_{\ker f}.$$

and

$$p(a' + \operatorname{im} f) := i'(a') + \operatorname{im} g,$$

as cosets, and similarly for $l$ and $q$. We prove that both the top row and the bottom row are exact. Firstly, if $a \in \ker f$ then $gi(a) = i'f(a) = 0$, so that $k(a) := i(a)$ belongs to $\ker g$. Moreover $lk(a) = ji(a) = 0$ and hence $\operatorname{im} k \subseteq \ker l$. Conversely if $l(b) = 0$ then $j(b) = 0$ and hence $b = i(a)$ for some $a$ as $A \xrightarrow{i} B \xrightarrow{j} C$ is exact. Moreover $i'f(a) = gi(a) = g(b) = 0$ as $b \in \ker g$. Since $i'$ is injective, it follows $f(a) = 0$ and thus $a \in \ker(f)$ with so that $k(a) = b$. Thus $\ker l \subseteq \operatorname{im} k$ and we have exactness at $\ker g$.

Let us now prove exactness at $\operatorname{coker} g$. The composition $qp$ is obviously zero since $j'i' = 0$ by exactness at $B'$:

$$qp(a') = j'i'(a') + \operatorname{im} h = 0 + \operatorname{im} h = 0 \in \operatorname{coker} h.$$

Thus $\operatorname{im} p \subseteq \ker q$. Conversely suppose $q(b' + \operatorname{im} g) = 0$. This means that $j'(b') \in \operatorname{im} h$, so there exists $c \in C$ such that $h(c) = j'(b')$. Since $j$ is surjective, there exists $b \in B$ such that $j(b) = c$. Now observe $j'(b' - g(b)) = j'(b') - j'g(b) = h(c) - hj(b) = h(c) - h(c) = 0$. Thus $b' - g(b) \in \operatorname{im} i'$ by exactness at $B'$. If $a'$ is such that $i'(a') = b' - g(b)$ then

$$p(a' + \operatorname{im} f) = i'(a') + \operatorname{im} g = b' - g(b) + \operatorname{im} g = b' + \operatorname{im} g.$$

Thus $\ker q \subseteq \operatorname{im} p$, which proves exactness at $\operatorname{coker} g$.

**2.** Our aim now is to define a map $\delta$ such that the following sequence is exact:

$$\begin{tikzcd}
\ker f \arrow[r, "k"] & \ker g \arrow[r, "l"] & \ker h \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]$(\tikztostart.south)!.5!(\tikztotarget.north)$) node[near end, below]{$\delta$} -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
\operatorname{coker} f \arrow[r, "p"'] & \operatorname{coker} g \arrow[r, "q"'] & \operatorname{coker} h
\end{tikzcd} \tag{11.1}$$

Let's start with an element $c \in \ker h$. Since $j$ is surjective, choose $b$ such that $j(b) = c$. Observe $g(b) \in \ker j'$ since $j'g(b) = hj(b) = h(c) = 0$. By exactness at $B'$, this implies that $g(b) \in \operatorname{im} i'$. Thus there exists $a' \in A'$ such that $i'(a') = g(b)$. In fact, since $i'$ is injective, $a'$ is unique. We define $\delta(c)$ as the coset $a' + \operatorname{im} f$ as an element of $\operatorname{coker} f$.

We made a choice here: $j$ is surjective, not necessarily an isomorphism, and hence we could have chosen a different element, say $b_1$ such that $j(b_1) = c$. This would give rise to a different element $a'_1 \in A'$ such that $i'(a'_1) = g(b_1)$. Nevertheless, we claim that the cosets $a' + \operatorname{im} f$ and $a'_1 + \operatorname{im} f$ coincide. This means we need to find $a \in A$ such that $f(a) = a' - a'_1$. But this is easy: since $j(b) = j(b_1)$ we have $b - b_1 \in \ker j$, and hence by exactness at $B$ there exists $a \in A$ such that $i(a) = b - b_1$. Then by commutativity, $i'f(a) = gi(a) = g(b - b_1) = g(b) - g(b_1)$. Since $i'$ is injective, it follows that $f(a) = a' - a'_1$ as required.

Thus $\delta(c) = (i')^{-1}gj^{-1}(c) + \operatorname{im} f$ is well defined. It is clear that $\delta$ is a homomorphism, i.e. that $\delta(c + c_1) = \delta(c) + \delta(c_1)$, since $i', g$ and $j$ are all homomorphisms.

**3.** Finally, let us check exactness at the two new places: $\ker h$ and $\operatorname{coker} f$. It is clear that $\operatorname{im} l \subseteq \ker \delta$. Indeed, if $c = l(b)$ for some $b$ then we can choose this $b$ in the definition of $\delta$. Then $g(b) = 0$ and hence the unique preimage under $i'$ is $a' = 0$. Then $\delta(c) = 0 + \operatorname{im} f = 0 \in \operatorname{coker} f$.

Now suppose that $\delta(c) = 0$. This means that the element $a'$ we found belongs to the image of $f$, say $a' = f(a)$. The $gi(a) = i'(a') = g(b)$. Thus $b - i(a) \in \ker g$. Moreover $j(b - i(a)) = j(b) - ji(a) = c$ by exactness. This means that $l(b - i(a)) = c$ and thus $c \in \operatorname{im} l$. This proves exactness at $\ker h$.

Now we check exactness at $\operatorname{coker} f$. Again, one direction is immediate: $p\delta(c)$ is just the coset $i'(a') + \operatorname{im} g$ in $\operatorname{coker} g$. But since $i'(a') = g(b)$, this coset is zero, and hence $\operatorname{im} \delta \subseteq \ker p$. Conversely, suppose $p(a' + \operatorname{im} f) = 0$. This means that $i'(a') \in \operatorname{im} g$, so there exists $b \in B$ such that $g(b) = i'(a')$. Set $c = j(b)$. Then $h(c) = hj(b) = j'g(b) = j'i'(a') = 0$ by exactness at $B'$. Then by construction, $\delta(c) = a' + \operatorname{im} f$. Thus $\ker p \subseteq \operatorname{im} \delta$. This finally completes the proof. ∎

Remark 11.2. This proof may look complicated, but in fact there was nothing to it: at every stage we just "did the only thing possible". This type of proof is rather relaxing, and it is usually referred to as **diagram chasing**. The best way to get used to the "yoga" of diagram chasing is to try some examples yourself. Thus you will no doubt be thrilled to discover that the next two results are left as exercises for you to solve on Problem Sheet F.

This first one is slightly less imaginatively named than the Snake Lemma.

Proposition 11.3 (The Five Lemma). *Suppose we have a commutative diagram of abelian groups, where the two rows are exact:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "k"] & E \arrow[d, "l"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

*Then:*

1. *If $g$ and $k$ are injective and $f$ is surjective, $h$ is injective.*
2. *If $g$ and $k$ are surjective and $l$ is injective, $h$ is surjective.*
3. *If $f, g, k, l$ are all isomorphisms then so is $h$.*

This is Problem F.1. The next one has no imagination whatsoever in its name ...

PROPOSITION 11.4 (The Barratt-Whitehead Lemma). *Suppose we have the following commutative diagram of abelian groups, where the two rows are exact:*

$$\begin{tikzcd}
\dots \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"'] & B_n \arrow[r, "j_n"] \arrow[d, "g_n"'] & C_n \arrow[r, "k_n"] \arrow[d, "h_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & A'_n \arrow[r, "i'_n"'] & B'_n \arrow[r, "j'_n"'] & C'_n \arrow[r, "k'_n"'] & A'_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

*Assume each map $h_n\colon C_n \to C'_n$ is an isomorphism. Then there is a long exact sequence:*

$$\cdots \to A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - i'_n} B'_n \xrightarrow{k_n h_n^{-1} j'_n} A_{n-1} \to \dots,$$

*where $(i_n, f_n)\colon A_n \to B_n \oplus A'_n$ is given by $a \mapsto (i_n(a), f_n(a))$ and $g_n - i'_n\colon B_n \oplus A'_n \to B'_n$ is given by $(b, a') \mapsto g_n(b) - i'_n(a')$.*

This is Problem F.2. Let us now use the Snake Lemma to prove the following foundational result in homological algebra.

THEOREM 11.5 (The long exact sequence in homology). *Let*

$$0 \to C_\bullet \xrightarrow{f} C'_\bullet \xrightarrow{g} C''_\bullet \to 0 \tag{11.2}$$

*be a short exact sequence of chain complexes. Then there is a sequence $\delta_n\colon H_n(C''_\bullet) \to H_{n-1}(C_\bullet)$ of homomorphisms such that there is a long exact sequence:*

$$\cdots \to H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(C'_\bullet) \xrightarrow{H_n(g)} H_n(C''_\bullet) \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \to \dots$$

*We call $\delta_\bullet$ the* **connecting homomorphism** *of the short exact sequence (11.2). Explicitly,*

$$\delta_n\langle c\rangle = \langle f_{n-1}^{-1}\partial' g_n^{-1} c\rangle, \qquad \forall c \in Z_n(C''_\bullet), \tag{11.3}$$

*where $\partial'$ is the boundary operator of $C'_\bullet$.*

*Proof.* Write $Z_n = \ker\big(\partial\colon C_n \to C_{n-1}\big)$, $B_n = \operatorname{im}\big(\partial\colon C_{n+1} \to C_n\big)$ and $H_n = Z_n/B_n$, and similarly for the other complexes. Then the following diagram satisfies the requirements of the Snake Lemma, where the written maps are the induced ones:

$$\begin{tikzcd}
 & C_n/B_n \arrow[r, "f_n"] \arrow[d, "\partial"'] & C'_n/B'_n \arrow[r, "g_n"] \arrow[d, "\partial'"] & C''_n/B''_n \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r, "f_{n-1}"'] & Z'_{n-1} \arrow[r, "g_{n-1}"'] & Z''_{n-1} &
\end{tikzcd}$$

Adding in the kernels and cokernels of the vertical maps, we obtain (where now all the maps are omitted for clarity):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H_n \arrow[r] \arrow[d] & H'_n \arrow[r] \arrow[d] & H''_n \arrow[d] & \\
 & C_n/B_n \arrow[r] \arrow[d] & C'_n/B'_n \arrow[r] \arrow[d] & C''_n/B''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & Z'_{n-1} \arrow[r] \arrow[d] & Z''_{n-1} \arrow[d] & \\
 & H_{n-1} \arrow[r] \arrow[d] & H'_{n-1} \arrow[r] \arrow[d] & H''_{n-1} \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The Snake Lemma thus provides us with a map $\delta_n \colon H''_n \to H_{n-1}$, which is the map we are looking for. ■

We now prove that the long exact sequence is *natural*. It won't be until the end of the course that I explain the precise definition of the word "natural". For now, just think of "natural" meaning that whenever you draw a diagram that "ought" to commute, then it does.

Proposition 11.6 (Naturality of the connecting homomorphism). *Suppose we are given a commutative diagram of chain complexes with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "f"] \arrow[d, "i"'] & B_\bullet \arrow[r, "g"] \arrow[d, "j"] & C_\bullet \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & A'_\bullet \arrow[r, "f'"'] & B'_\bullet \arrow[r, "g'"'] & C'_\bullet \arrow[r] & 0
\end{tikzcd}$$

*Then there is a commutative diagram of abelian groups with exact rows:*

$$\begin{tikzcd}
\dots \arrow[r] & H_n(A_\bullet) \arrow[r, "H_n(f)"] \arrow[d, "H_n(i)"'] & H_n(B_\bullet) \arrow[r, "H_n(g)"] \arrow[d, "H_n(j)"] & H_n(C_\bullet) \arrow[r, "\delta_n"] \arrow[d, "H_n(k)"] & H_{n-1}(A_\bullet) \arrow[r] \arrow[d, "H_{n-1}(i)"] & \dots \\
\dots \arrow[r] & H_n(A'_\bullet) \arrow[r, "H_n(f')"'] & H_n(B'_\bullet) \arrow[r, "H_n(g')"'] & H_n(C'_\bullet) \arrow[r, "\delta'_n"'] & H_{n-1}(A'_\bullet) \arrow[r] & \dots
\end{tikzcd}$$

*Proof.* We've already proved most things. That the rows are exact is the content of Theorem 11.5. The first two squares commute because $H_n$ is a functor. Thus we need only check that the right-hand square commutes. So for this, suppose $c \in Z_n(C)$ is a cycle representing a homology class $\langle c \rangle \in H_n(C_\bullet)$. Since $g$ is surjective (as a chain map), the map $g_n \colon B_n \to C_n$ is surjective (cf. Problem E.5). Thus there exists $b \in B_n$ with $g_n b = c$. Then

$$H_{n-1}(i)\delta_n\langle c\rangle = H_{n-1}(i)\delta_n\langle g_n b\rangle.$$

Let $\partial$ denote the boundary operator of $B_\bullet$ and $\partial'$ the boundary operator of $B'_\bullet$ (these are the only two boundary operators we will need in the following). Now we note that from (11.3),

$$H_{n-1}(i)\delta_n\langle g_n b\rangle = H_{n-1}(i)\langle f_{n-1}^{-1}\partial b\rangle = \langle i_{n-1} f_{n-1}^{-1}\partial b\rangle.$$

Now using $j_{n-1} \circ f_{n-1} = f'_{n-1} \circ i_{n-1}$ we have

$$\langle i_{n-1} f_{n-1}^{-1}\partial b\rangle = \langle (f'_{n-1})^{-1} j_{n-1}\partial b\rangle.$$

Since $j$ is a chain map, $\langle (f'_{n-1})^{-1} j_{n-1}\partial b\rangle = \langle (f'_{n-1})^{-1}\partial' j_n b\rangle$. Now using (11.3) for $\delta'_n$, we see that:

$$\langle (f'_{n-1})^{-1}\partial' j_n b\rangle = \delta'_n\langle g'_n j_n b\rangle.$$

Now use $g'_n \circ j_n = k_n \circ g_n$ to obtain

$$\delta'_n\langle g'_n j_n b\rangle = \delta'_n\langle k_n g_n b\rangle.$$

Then as $\langle k_n g_n b\rangle = H_n(k)\langle g_n b\rangle = H_n(k)\langle c\rangle$ we finally have

$$H_{n-1}(i)\delta_n\langle c\rangle = \delta'_n H_n(k)\langle c\rangle,$$

which proves the last square commutes. ∎

LECTURE 12

# Relative homology and reduced homology

In this lecture we extend $H_n$ to a functor $\mathsf{Top}^2 \to \mathsf{Ab}$. To begin with, we need the following piece of pedantry.

LEMMA 12.1. *Let $X'$ be a subspace of $X$ with inclusion $\imath \colon X' \hookrightarrow X$. Then for every $n \geq 0$, the map $\imath_n^{\#} \colon C_n(X') \to C_n(X)$ is an injection.*

*Proof.* Let $c = \sum m_i \sigma_i \in C_n(X')$. We may assume all the $\sigma_i$ are distinct. By definition, $\imath_n^{\#} c = \sum m_i \imath \circ \sigma_i$. Since $\imath \circ \sigma_i$ differs only from $\sigma_i$ only by having its target enlarged, it follows that the $\imath \circ \sigma_i$ are all distinct. Now if $c \in \ker \imath_n^{\#}$ then we have

$$0 = \sum m_i \imath \circ \sigma_i.$$

Since $C_n(X)$ is free abelian with basis all the singular $n$-simplices in $X$, it follows that all the $m_i$ are zero, and hence $c = 0$. ∎

This means that we can unambiguously think of $C_\bullet(X')$ as a *subcomplex* of $C_\bullet(X)$ (i.e. by identifying $C_\bullet(X')$ with $\operatorname{im} \imath_\bullet^{\#}$.) We shall do this without further comment. Thus we have a short exact sequence of complexes:

$$0 \to C_\bullet(X') \to C_\bullet(X) \to C_\bullet(X) \big/ C_\bullet(X') \to 0.$$

DEFINITION 12.2. Let $X' \subseteq X$ be a subspace. We define the **relative homology groups** $H_n(X, X')$ of the pair $(X, X')$ to be the homology of the complex $C_\bullet(X) \big/ C_\bullet(X')$.

The next result is immediate from Theorem 11.5 and Proposition 11.6. Like the dimension axiom (Proposition 8.1) and the homotopy axiom (Theorem 8.9), we call this result an "axiom" since it will turn out to be one of the four axioms of a homology theory.

PROPOSITION 12.3 (The exact sequence axiom). *Let $X'$ be a subspace of $X$. Then there is a long exact sequence*

$$\dots H_n(X') \to H_n(X) \to H_n(X, X') \xrightarrow{\delta} H_{n-1}(X') \to \dots$$

*Moreover if $f\colon (X,X') \to (Y,Y')$ is a map of pairs then there is a commutative diagram:*

$$\begin{tikzcd}
\dots \arrow[r] & H_n(X') \arrow[r] \arrow[d] & H_n(X) \arrow[r] \arrow[d] & H_n(X,X') \arrow[r] \arrow[d] & H_{n-1}(X') \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_n(Y') \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,Y') \arrow[r] & H_{n-1}(Y') \arrow[r] & \dots
\end{tikzcd}$$

*where the vertical maps are all induced by $f$.*

Definition 12.4. This construction also allows us to see homology as a functor $H_n\colon \mathsf{Top}^2 \to \mathsf{Ab}$. Firstly we define the chain complex functor $\mathsf{Top}^2 \to \mathsf{Comp}$ that sends $(X,X')$ to $C_\bullet(X)\big/C_\bullet(X')$ and sends a map $f\colon (X,X') \to (Y,Y')$ to the induced map

$$f_\bullet^\#\colon C_\bullet(X)\big/C_\bullet(X') \to C_\bullet(Y)\big/C_\bullet(Y')$$

(this works as $f_n^\#(C_n(X')) \subset C_n(Y')$.) Then we apply the usual homology functor $H_n\colon \mathsf{Comp} \to \mathsf{Ab}$.

Remark 12.5. Taking $X' = \emptyset$ recovers our original groups:

$$H_n(X,\emptyset) = H_n(X), \qquad \forall\, n \geq 0.$$

Let us now give a slightly more useful way of defining $H_n(X,X')$.

Definition 12.6. Define the group of **relative $n$-cycles mod $X'$** to be

$$Z_n(X,X') := \left\{c \in C_n(X) \mid \partial c \in C_{n-1}(X')\right\},$$

and the group of **relative $n$-boundaries mod $X'$** to be

$$\begin{aligned} B_n(X,X') &:= \left\{c \in C_n(X) \mid c - c' \in B_n(X) \text{ for some } c' \in C_n(X')\right\} \\ &= B_n(X) + C_n(X'). \end{aligned}$$

Then $B_n(X,X') \subseteq Z_n(X,X')$.

We have:

Proposition 12.7. *For all $n \geq 0$,*

$$H_n(X,X') \cong Z_n(X,X')\big/B_n(X,X').$$

*Proof.* By definition, the boundary operator $\bar\partial$ of the quotient complex $C_\bullet(X)\big/C_\bullet(X')$ is given by

$$\bar\partial\colon c + C_n(X') \mapsto \partial c + C_{n-1}(X'), \qquad c \in C_n(X), \qquad n \geq 0.$$

Thus

$$\ker\bar\partial = \left\{c + C_n(X') \mid \partial c \in C_{n-1}(X')\right\} = Z_n(X,X')\big/C_n(X')$$

and

$$\operatorname{im}\bar{\partial} = \{c + C_n(X') \mid c \in B_n(X)\} = B_n(X, X')\big/C_n(X').$$

The claim thus follows from the third isomorphism for groups[1]. ∎

The following result shows that the relative homology groups often vanish in dimension zero, unlike the "absolute" ones which never do.

PROPOSITION 12.8. *Suppose $X$ is path connected and $X'$ is a non-empty subspace. Then $H_0(X, X') = 0$.*

*Proof.* Fix $p \in X'$. Suppose $c = \sum m_x\, x \in Z_0(X, X')$. Choose a path[2] $\sigma_x \colon \Delta^1 \to X$ such that $\sigma_x(e_0) = p$ and $\sigma_x(e_1) = x$. Then $a = \sum m_x\, \sigma_x \in C_1(X)$ and

$$\partial a = c - \left(\sum m_x\right) p.$$

Since $c' := (\sum m_x)\, p \in C_0(X')$, we thus have $c - c' \in B_0(X)$, so that $c \in B_0(X, X')$. Thus $H_0(X, X') = 0$. ∎

Next, we have the following result.

PROPOSITION 12.9. *Let $\{X_\lambda \mid \lambda \in \Lambda\}$ denote the path components of $X$, and let $X' \subseteq X$ denote a subspace. For each $n \geq 0$, one has*

$$H_n(X, X') \cong \bigoplus_{\lambda \in \Lambda} H_n(X_\lambda, X_\lambda \cap X').$$

*Proof.* Immediate from Problem D.4 and Problem E.6. ∎

Let us record a special case of this statement, since it will be useful in Lecture 22 when we discuss the axioms.

COROLLARY 12.10 (The additivity axiom). *Let $(X_\lambda, X'_\lambda)$, $\lambda \in \Lambda$ be a family of pairs of spaces. Denote by*

$$\imath_\lambda \colon (X_\lambda, X'_\lambda) \hookrightarrow \left(\bigsqcup_{\lambda \in \Lambda} X_\lambda, \bigsqcup_{\lambda \in \Lambda} X'_\lambda\right)$$

*the inclusion. Then for all $n \geq 0$, the map*

$$\sum_{\lambda \in \Lambda} H_n(\imath_\lambda) \colon \bigoplus_{\lambda \in \Lambda} H_n(X_\lambda, X'_\lambda) \to H_n\left(\bigsqcup_{\lambda \in \Lambda} X_\lambda, \bigsqcup_{\lambda \in \Lambda} X'_\lambda\right).$$

*is an isomorphism.*

[1] Which states that if $N \leq K \leq G$ are normal subgroups then $(G/N)/(K/N) \cong G/K$, i.e. you can "cancel" the $N$.

[2] We won't bother distinguishing paths and 1-simplices in this lecture.

Corollary 12.11. *Let $\{X_\lambda \mid \lambda \in \Lambda\}$ denote the path components of $X$, and let $X' \subseteq X$ denote a subspace. The group $H_0(X, X')$ is free abelian, with*

$$\operatorname{rank} H_0(X, X') = \#\left\{\lambda \in \Lambda \mid X_\lambda \cap X' = \emptyset\right\}.$$

*Proof.* If $X_\lambda \cap X' \neq \emptyset$ then $H_0(X_\lambda, X_\lambda \cap X') = 0$ by Proposition 12.8, since $X_\lambda$ is path connected for each $\lambda$ by definition.. If $X_\lambda \cap X' = \emptyset$ then $H_0(X_\lambda, X_\lambda \cap X') = H_0(X_\lambda) \cong \mathbb{Z}$ by Proposition 8.3. ∎

Now let us specialise to the case where $X'$ is a single point $\{p\}$ for some $p \in X$.

Corollary 12.12. *If $(X, p)$ is a pointed space then $H_0(X, p)$ is a free abelian group of (possibly infinite) rank $r$, where $X$ has $r + 1$ path components.*

Meanwhile for $n \geq 1$, taking a single point doesn't change the homology:

Proposition 12.13. *Let $(X, p)$ be a pointed space. Then for all $n \geq 1$,*

$$H_n(X, p) \cong H_n(X).$$

*Proof.* By Proposition 12.3 there is an exact sequence

$$\dots H_n(p) \to H_n(X) \to H_n(X, p) \xrightarrow{\delta} H_{n-1}(p) \to \dots$$

If $n \geq 2$ then $H_n(p) = 0$ and $H_{n-1}(p) = 0$ by the dimension axiom, and thus we immediately see $H_n(X, p) \cong H_n(X)$. The case $n = 1$ is slightly more tricky; it can be deduced directly from the long exact sequence by studying what the actual maps do at the tail end:

$$0 \to H_1(X) \to H_1(X, p) \to H_0(p) \to H_0(X) \to H_0(X, p) \to 0.$$

Namely, by the fourth item of Example 10.5, the map $H_1(X) \to H_1(X, p)$ is surjective if and only if the map $H_0(p) \to H_0(X)$ is injective. Since $H_0(p) = \mathbb{Z}$ and $H_0(X)$ is free abelian, either $H_0(p) \to H_0(X)$ is the zero map or it is injective. By exactness, if it was the zero map then $H_0(X) \to H_0(X, p)$ would have to be injective. Thus to complete the proof we need only exhibit an element in the kernel of the map $H_0(X) \to H_0(X, p)$. Such an element is provided by $\langle p \rangle$ (cf. the proof of Proposition 8.3.) ∎

This means that for $n \geq 1$ we can regard $H_n$ as a functor on $\mathsf{Top}_*$. Let us now introduce another algebraic concept.

Definition 12.14. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of abelian groups. We say that the sequence **splits** if there exists a map $h \colon C \to B$ such that $gh = \mathrm{id}_C$. We call $h$ a **splitting map**.

The splitting map $h$ is *not* unique. On Problem Sheet F you will prove that an equivalent definition is asking that $f$ admits a left inverse:

Proposition 12.15. *Suppose* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is a short exact sequence of abelian groups. Then the sequence splits if and only if there exists a map* $k\colon B \to A$ *such that* $kf = \mathrm{id}_A$.

Here we show that if a sequence splits the middle term is a direct sum.

Proposition 12.16. *Suppose* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is a split short exact sequence of abelian group. Then* $B \cong A \oplus C$.

*Proof.* Let $h\colon C \to B$ be such that $gh = \mathrm{id}_C$. We show that $B = \operatorname{im} f \oplus \operatorname{im} h$. If $b \in B$ then $g(b) \in C$ and $b - hg(b) \in \ker g$ because $g(b) - gh(gb) = 0$ as $gh = \mathrm{id}_C$. Thus by exactness, there exists $a \in A$ with $f(a) = b - hg(b)$. Thus $B = \operatorname{im} f + \operatorname{im} h$. It remains to show that $\operatorname{im} f \cap \operatorname{im} h = \{0\}$. If $f(a) = x = h(c)$ then $g(x) = gf(a) = 0$ and also $g(x) = gh(c) = c$, thus $x = h(c) = 0$. ■

Remark 12.17. It is important to realise that the isomorphism $B \cong A \oplus C$ depends on the choice of the splitting map $h$. More formally[3], this means that the splitting is *not* natural. Moreover the converse to Proposition 12.16 is *not* true, as you will show on Problem Sheet F.

The next idea is more important than it looks at first glance.

Definition 12.18. Let $X$ be a non-empty topological space and let $\{*\}$ be a topological space with one point. Let $j\colon X \to \{*\}$ be the unique continuous map that sends every point in $X$ to $*$. For any map $i\colon \{*\} \to X$ we have $j \circ i = \mathrm{id}_{\{*\}}$. Thus the induced map $H_n(j)\colon H_n(X) \to H_n(*)$ is always surjective. We define $\tilde{H}_0(X) := \ker H_0(j)$ and call it the **zeroth reduced homology group**. This gives us a short exact sequence:

$$0 \to \tilde{H}_0(X) \to H_0(X) \xrightarrow{H_0(j)} H_0(*) \to 0.$$

Since this sequence splits (via $H_0(i)$), we have

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z},$$

but this splitting is *not* natural, since it depends on the choice of map $i$.

Remark 12.19. Under the identification $H_0(*) \cong \mathbb{Z}$ given by

$$m\langle * \rangle \mapsto m, \qquad m \in \mathbb{Z},$$

the map $H_0(j)\colon H_0(X) \to H_0(*)$ can be identified with the map $\phi$ from (8.3). Indeed, if $c = \sum m_x\, x \in Z_0(X)$ represents a homology class $\langle c \rangle \in H_0(X)$, then

$$H_0(j)\langle c \rangle = \sum m_x\, j(x) = \left(\sum m_x\right) \langle * \rangle \in H_0(*),$$

and hence the map $H_0(j)$ sends $\sum m_x\, x \mapsto \sum m_x$, which is exactly how $\phi$ was defined in (8.3).

[3]We will define this properly in Lecture 22.

We now extend $\tilde{H}_n$ to all $n$ by simply setting $\tilde{H}_n(X) := H_n(X)$ for $n \geq 1$, and call $\tilde{H}_\bullet(X)$ the **reduced homology groups of** $X$. We then have:

Corollary 12.20. *If $X$ is a non-empty contractible space then $\tilde{H}_n(X) = 0$ for all $n \geq 0$.*

On Problem Sheet F you will show that the long exact sequence for pairs works for reduced homology too:

Proposition 12.21. *Let $\emptyset \neq X' \subseteq X$. There is an exact sequence*

$$\cdots \to \tilde{H}_n(X') \to \tilde{H}_n(X) \to H_n(X, X') \to \tilde{H}_{n-1}(X') \to \dots$$

*which ends with $\tilde{H}_0(X') \to \tilde{H}_0(X) \to H_0(X, X') \to 0$.*

Corollary 12.22. *If $p \in X$ then $\tilde{H}_n(X) \cong H_n(X, p)$ for all $n \geq 0$.*

Remark 12.23. In Corollary 12.22 we can slightly more explicit. Let us take our one point space $\{*\}$ to be $\{p\}$ itself. Then there is an "obvious" choice of map $p \to X$, namely the inclusion. This will be important later.

Remark 12.24. Remark 12.19 allows us to see the reduced homology groups as the homology groups of a chain complex. Let $X$ be a non-empty topological space and define a chain complex $\big(\tilde{C}_\bullet(X), \tilde{\partial}\big)$ by setting:

$$\tilde{C}_n(X) := \begin{cases} C_n(X), & n \geq 0, \\ \mathbb{Z}, & n = -1. \\ 0, & n \leq -2, \end{cases}$$

and for $n \geq 1$, define $\tilde{\partial}\colon \tilde{C}_n(X) \to C_{n-1}(X)$ to be the normal boundary operator, and for $n = 0$, set

$$\tilde{\partial}\colon \tilde{C}_0(X) \to \mathbb{Z} = \tilde{C}_{-1}(X), \qquad \tilde{\partial}\left(\sum_x m_x\, x\right) \mapsto \sum_x m_x.$$

Then by Remark 12.19, one has

$$H_n\big(\tilde{C}_\bullet(X), \tilde{\partial}\big) \cong \tilde{H}_n(X), \qquad \forall\, n \geq 0.$$

This will be important next lecture in Corollary 13.3.

Our first "real" use of the reduced homology groups will come in Lecture 15 when we finally compute the homology of $S^n$ (it will turn out it is more convenient to compute $\tilde{H}_\bullet(S^n)$ using induction.) In Lecture 19, we will prove that if a pair $(X, X')$ is sufficiently "nice" then

$$H_n(X, X') \cong \tilde{H}_n(X/X'), \tag{12.1}$$

where $X/X'$ is the quotient space obtained by collapsing $X'$ to a point (Corollary 12.22 is a special case of (12.1).)

We conclude this lecture by defining the homotopy version of $\mathsf{Top}^2$.

Definition 12.25. If $f, g\colon (X, X') \to (Y, Y')$ are maps of pairs then we say $f \simeq g$ mod $X'$ if there exists a continuous map $F\colon (X \times I, X' \times I) \to (Y, Y')$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$.

This notion is **not** the same as saying $f \simeq g$ rel $X'$, since the definition does **not** require $f|_{X'} = g|_{X'}$ and that $F(x', t)$ is independent of $t$ for all $x' \in X'$. This relation defines a congruence on $\mathsf{Top}^2$ and thus yields a new category, $\mathsf{hTop}^2$. Moreover $H_n$ induces a functor $H_n\colon \mathsf{hTop}^2 \to \mathsf{Ab}$ thanks to the following result.

Theorem 12.26 (The homotopy axiom for pairs). *If $f, g\colon (X, X') \to (Y, Y')$ are maps of pairs such that $f \simeq g$ mod $X'$ then for all $n \geq 0$,*

$$H_n(f) = H_n(g)\colon H_n(X, X') \to H_n(Y, Y').$$

The proof is analogous to the proof of the homotopy axiom (Theorem 8.9) and I won't bore you with it again.

LECTURE 13

# Barycentric subdivision

In this lecture we give a systematic way of chopping up a singular simplex into a bunch of smaller ones. This process, known as *barycentric subdivision*, is interesting in its own right (we will use it to give another proof that a convex space has zero reduced homology), but for us the main application will be in the proof of *excision* that we will carry out next lecture.

DEFINITION 13.1. Let $D$ be a bounded convex subset of some Euclidean space. Fix a point $p \in D$, and suppose $\sigma \colon \Delta^n \to D$ is a singular $n$-simplex. We define the **cone over** $\sigma$ **with vertex** $p$ to be the singular $(n+1)$-simplex $Q(p,\sigma) \colon \Delta^{n+1} \to D$ defined by

$$Q(p,\sigma)(s_0, s_1, \dots, s_{n+1}) := \begin{cases} p, & s_0 = 1, \\ s_0 p + (1-s_0)\sigma\left(\frac{s_1}{1-s_0}, \dots, \frac{s_{n+1}}{1-s_0}\right), & s_0 \neq 1. \end{cases}$$

This is well defined because if $s_0 \neq 1$ then $\frac{1}{1-s_0}\sum_{i=1}^{n+1} s_i = 1$, and it is easy to check that $Q(p,\sigma)$ is continuous and takes values in $D$ by convexity.

We extend this by linearity to a map $C_n(D) \to C_{n+1}(D)$, which we write as $c \mapsto Q(p,c)$.

PROPOSITION 13.2. *If $c = \sum m_i \sigma_i \in C_n(D)$ then*

$$\partial Q(p,c) = \begin{cases} c - Q(p, \partial c), & \text{if } n > 0, \\ c - \left(\sum m_i\right) p, & \text{if } n = 0. \end{cases}$$

*Proof.* If $n \geq 1$ and $i = 0$ then

$$Q(p,\sigma) \circ \varepsilon_0^{n+1}(s_0, \dots, s_n) = Q(p,\sigma)(0, s_0, \dots, s_n) = \sigma(s_0, \dots, s_n),$$

and if $1 \leq i \leq n+1$ then

$$Q(p,\sigma) \circ \varepsilon_i^{n+1}(s_0, \dots, s_n) = Q(p,\sigma)(s_0, \dots, s_{i-1}, 0, s_i, \dots, s_n).$$

If $s_0 = 1$ then this reduces to

$$Q(p,\sigma)(1, 0, \dots, 0) = p,$$

meanwhile if $s_0 \neq 1$ then we have

$$\begin{aligned} Q(p,\sigma) \circ \varepsilon_i^{n+1}(s_0, \dots, s_n) &= s_0 p + (1-s_0)\sigma\left(\frac{s_1}{1-s_0}, \dots, \frac{s_{i-1}}{1-s_0}, 0, \frac{s_i}{1-s_0}, \dots, \frac{s_n}{1-s_0}\right) \\ &= s_0 p + (1-s_0)\sigma \circ \varepsilon_{i-1}^n\left(\frac{s_1}{1-s_0}, \dots, \frac{s_n}{1-s_0}\right) \\ &= Q(p, \sigma \circ \varepsilon_{i-1}^n)(s_0, \dots, s_n). \end{aligned}$$

Thus we see that for $n \geq 1$,

$$Q(p,\sigma) \circ \varepsilon_0^{n+1} = \sigma \tag{13.1}$$

and for $i \geq 1$,

$$Q(p,\sigma) \circ \varepsilon_i^{n+1} = Q(p, \sigma \circ \varepsilon_{i-1}^n). \tag{13.2}$$

Now taking alternating sums, we see that

$$\begin{aligned}
\partial Q(p,\sigma) &= \sum_{i=0}^{n+1} (-1)^i Q(p,\sigma) \circ \varepsilon_i^{n+1} \\
&\overset{(*)}{=} \sigma + \sum_{i=1}^{n+1} (-1)^i Q(p, \sigma \circ \varepsilon_{i-1}^n) \\
&= \sigma - \sum_{j=0}^{n} (-1)^j Q(p, \sigma \circ \varepsilon_j^n) \\
&= \sigma - Q\left(p, \sum_{j=0}^{n} (-1)^j \sigma \circ \varepsilon_j^n\right) \\
&= \sigma - Q(p, \partial\sigma),
\end{aligned}$$

where $(*)$ used (13.1) and (13.2). This proves the result in the case $n \geq 1$. For $n = 0$, it suffices to observe that if $x$ is a point in $X$ then $Q(p,x)$ is a 1-simplex $\sigma$ with $\sigma(e_0) = p$ and $\sigma(e_1) = x$. Thus $\partial\sigma = \sigma(e_1) - \sigma(e_0) = x - p$. ■

This gives us another direct proof of the fact that the reduced homology groups of $D$ vanish. (Of course, since $D$ is contractible this already follows from Corollary 12.22.)

Corollary 13.3. *Let $D$ be a bounded convex subset of some Euclidean space. Then the reduced homology groups vanish: $\tilde{H}_n(D) = 0$ for all $n \geq 0$.*

*Proof.* The operator $c \mapsto Q(p,c)$ is a contracting homotopy in for the chain complex $\tilde{C}_\bullet(D)$ defined in Remark 12.24. Thus by Corollary 10.26 we see that $\tilde{H}_n(D)$ for all $n \geq 0$. ■

Remark 13.4. Note that this proof did *not* use the homotopy axiom. We will use this fact in Lecture 23 when giving an alternative proof of the homotopy axiom using the Acyclic Models Theorem (if Corollary 13.3 used the homotopy axiom then our argument would be circular).

We now introduce the concept of an *affine* simplex.

Definition 13.5. Let $D$ be convex. A singular $n$-simplex $\sigma \colon \Delta^n \to D$ is said to be **affine** if

$$\sigma\left(\sum_{i=0}^{n} s_i\, e_i\right) = \sum_{i=0}^{n} s_i\, \sigma(e_i), \qquad \forall\, (s_0, s_1, \ldots, s_n) \in \Delta^n.$$

If $\sigma$ is affine then $\partial\sigma$ is also affine, and thus the space of affine singular $n$-simplices defines a subcomplex of $C_\bullet(D)$. We write this as $C_\bullet^{\text{affine}}(D)$. Note also that if $\sigma$ is affine then so is $Q(p,\sigma)$ for any $p \in D$.

Let us denote by $b_n := \frac{1}{n+1}(e_0 + e_1 + \cdots + e_n)$ the *barycentre* (cf. (7.1) of the standard simplex $\Delta^n$. We now define the barycentric subdivision operator. We will first define it for affine simplices in convex subsets, and then extend it to arbitrary simplices and spaces.

Definition 13.6. Let $D$ be convex. Define the **convex barycentric subdivision**

$$\mathrm{Sd}_n^{\mathrm{cv}} \colon C_n^{\mathrm{affine}}(D) \to C_n^{\mathrm{affine}}(D)$$

inductively for an affine singular $n$-simplex $\sigma \colon \Delta^n \to D$ by

$$\mathrm{Sd}_n^{\mathrm{cv}}(\sigma) := \begin{cases} \sigma & n = 0, \\ Q\big(\sigma(b_n), \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma)\big), & n \geq 1, \end{cases}$$

and then extending by linearity. See Figure 13.1 for a picture in the case $n = 2$.

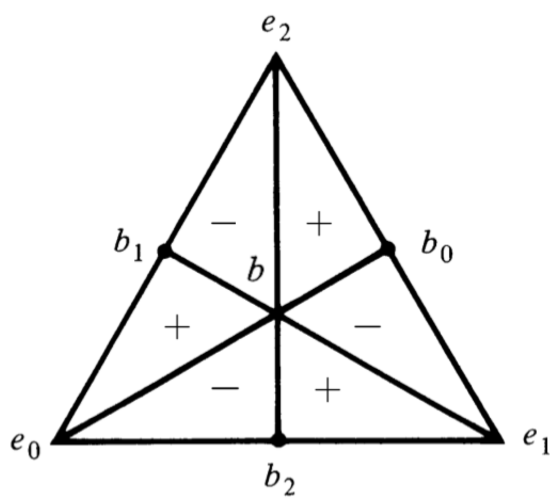


Figure 13.1: The barycentric subdivision of $\Delta^2$.

If $X$ is an arbitrary topological space then this definition doesn't make sense, since we cannot apply the cone construction. Nevertheless, there is an easy way to extend this. In the following, in order to minimise notational confusion, let us denote by $\ell_n \colon \Delta^n \to \Delta^n$ the identity map, *thought of as a singular $n$-simplex in $\Delta^n$*.

Definition 13.7. Let $X$ be an arbitrary topological space. Define the **barycentric subdivision** $\mathrm{Sd}_n \colon C_n(X) \to C_n(X)$ by setting

$$\mathrm{Sd}_n(\sigma) := \sigma_n^{\#}\big(\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n)\big),$$

and then extending by linearity.

This makes sense. The simplex $\ell_n$ is certainly affine, and hence $\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n)$ is well-defined and belongs to $C_n^{\mathrm{affine}}(\Delta^n) \subset C_n(\Delta^n)$. Since $\sigma_n^{\#}$ is by definition a map $C_n(\Delta^n) \to C_n(X)$, we see that $\sigma_n^{\#}\big(\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n)\big)$ does indeed belong to $C_n(X)$.

Lemma 13.8. *If $D$ is a convex bounded subset of some Euclidean space then Definition 13.7 agrees with Definition 13.6 for all affine simplices:*

$$\mathrm{Sd}_n(\sigma) = \mathrm{Sd}_n^{\mathrm{cv}}(\sigma), \qquad \forall\, \sigma\colon \Delta^n \to D \text{ affine.}$$

The proof of Lemma 13.8 is on Problem Sheet G. Moreover on Problem Sheet G you are asked to write out explicit formulae for $\mathrm{Sd}_n$ for $n = 0, 1, 2$.

Proposition 13.9. *The barycentric subdivision is a chain map. Moreover if $f\colon X \to Y$ is continuous then the following diagram commutes for all $n \geq 0$:*

$$\begin{tikzcd} C_n(X) \arrow[r, "f_n^{\#}"] \arrow[d, "\mathrm{Sd}_n"'] & C_n(Y) \arrow[d, "\mathrm{Sd}_n"] \\ C_n(X) \arrow[r, "f_n^{\#}"'] & C_n(Y) \end{tikzcd} \tag{13.3}$$

Remark 13.10. The fact that the diagram (13.3) commutes means that $\mathrm{Sd}_n$ is a **natural** chain map, anticipating terminology will we introduce later on in the course.

*Proof.* We begin by showing that (13.3) commutes:

$$f_n^{\#}\mathrm{Sd}_n(\sigma) = f_n^{\#}\sigma_n^{\#}\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n) = (f\circ\sigma)_n^{\#}\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n) = \mathrm{Sd}_n(f\circ\sigma) = \mathrm{Sd}_n(f_n^{\#}\sigma).$$

Assume now that $X$ is a bounded convex subset of some Euclidean space. Let us prove by induction on $n$ that $\mathrm{Sd}_n$ is a chain map. By Lemma 13.8, it suffices to show that $\mathrm{Sd}_n^{\mathrm{cv}}$ is a chain map. The case $n = 0$ is obvious. For the inductive step we compute

$$\begin{aligned} \partial\mathrm{Sd}_n^{\mathrm{cv}}(\sigma) &= \partial Q\big(\sigma(b_n), \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma)\big) \\ &\overset{(*)}{=} \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma) - Q\big(\sigma(b_n), \partial\mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma)\big) \\ &\overset{(\dagger)}{=} \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma) - Q\big(\sigma(b_n), \mathrm{Sd}_{n-2}^{\mathrm{cv}}(\underbrace{\partial^2\sigma}_{=0})\big) \\ &= \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma), \end{aligned}$$

where $(*)$ used Proposition 13.2 and $(\dagger)$ used the inductive hypothesis[1]. We now prove the general case where $X$ is not necessarily convex. If $\sigma\colon \Delta^n \to X$ we have

$$\begin{aligned} \partial\mathrm{Sd}_n(\sigma) &= \partial\sigma_n^{\#}\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n) \\ &\overset{(\ddagger)}{=} \sigma_{n-1}^{\#}\partial\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n) \\ &\overset{(\spadesuit)}{=} \sigma_{n-1}^{\#}\mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\ell_n) \\ &\overset{(\clubsuit)}{=} \sigma_{n-1}^{\#}\mathrm{Sd}_{n-1}(\partial\ell_n) \\ &\overset{(\heartsuit)}{=} \mathrm{Sd}_{n-1}(\sigma_{n-1}^{\#}\partial(\ell_n)) \\ &\overset{(\ddagger)}{=} \mathrm{Sd}_{n-1}(\partial\sigma_n^{\#}(\ell_n)) \\ &= \mathrm{Sd}_{n-1}(\partial\sigma), \end{aligned}$$

where:

[1] To make the case $n = 1$ work, we can take $\mathrm{Sd}_{-1}^{\mathrm{cv}}$ to be the zero map.

1. (‡) used that $\sigma_n^\#$ is a chain map (both times),
2. (♠) used that we already know that convex barycentric subdivision is a chain map,
3. (♣) used the fact that for the affine simplex $\partial \ell_n$ in the convex set $\Delta^n$ one has $\mathrm{Sd}_{n-1}(\partial \ell_n) = \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial \ell_n)$,
4. (♡) used naturality (13.3),
5. and the last line used the fact that

$$\sigma_n^\#(\ell_n) = \sigma. \tag{13.4}$$

This completes the proof. ■

Now that we know that barycentric subdivision is a chain map, we get an induced map on homology. What could this map be? The answer is about as boring as it can be (it shows we have accomplished nothing!), but this will be crucial in the proof of excision next lecture.

THEOREM 13.11. *For each $n \geq 0$, the induced map*

$$H_n(\mathrm{Sd})\colon H_n(X) \to H_n(X)$$

*is the identity.*

REMARK 13.12. Just as with Proposition 8.5, the fact that barycentric subdivision induces the identity on homology is also an immediate consequence of the **Acyclic Models Theorem** that we will prove later on in the course. However for completeness we will give an independent proof here.

*Proof.* It suffices by Proposition 10.24 to construct a chain homotopy between Sd and the identity. In other words, we need to build map $P_n\colon C_n(X) \to C_{n+1}(X)$ such that

$$\partial P_n + P_{n-1}\partial = \mathrm{id} - \mathrm{Sd}_n. \tag{13.5}$$

We prove the result in two steps.

**1.** We begin in the convex case with affine simplices. Assume $D$ is a bounded convex subset of some Euclidean space. We argue by induction on $n$. If $n = 0$, set $P_0^{\mathrm{cv}} : C_0^{\mathrm{affine}}(D) \to C_1^{\mathrm{affine}}(D)$ to be the zero map[2]. Since $\mathrm{Sd}_0^{\mathrm{cv}}(\sigma) = \sigma$ for an (affine) 0-simplex, both sides of (13.5) are thus zero for $n = 0$. Now suppose $n > 0$. If $c \in C_n^{\mathrm{affine}}(D)$ then we require $P_n^{\mathrm{cv}} c$ to satisfy

$$\partial P_n^{\mathrm{cv}} c = c - \mathrm{Sd}_n^{\mathrm{cv}} c - P_{n-1}^{\mathrm{cv}} \partial c.$$

The right-hand side is a cycle, since by induction,

$$\begin{aligned} \partial\big(c - \mathrm{Sd}_n^{\mathrm{cv}} c - P_{n-1}^{\mathrm{cv}}\partial c\big) &= \partial c - \partial \mathrm{Sd}_n^{\mathrm{cv}} c - (\mathrm{id} - \mathrm{Sd}_{n-1}^{\mathrm{cv}} - P_{n-2}^{\mathrm{cv}}\partial)\partial c \\ &= -\partial \mathrm{Sd}_n^{\mathrm{cv}} c + \mathrm{Sd}_{n-1}^{\mathrm{cv}}\partial c = 0, \end{aligned}$$

[2] $P_{-1}^{\mathrm{cv}}$ is also the zero map; there is no choice about that!

since $\mathrm{Sd}^{\mathrm{cv}}$ is a chain map and $\partial^2 = 0$. Fix $p \in D$, and assume we have already defined $P^{\mathrm{cv}}_{n-1} \colon C^{\mathrm{affine}}_{n-1}(D) \to C^{\mathrm{affine}}_{n}(D)$ to satisfy (13.5). If we define

$$P^{\mathrm{cv}}_n c = Q\big(p, c - \mathrm{Sd}^{\mathrm{cv}}_n c - P^{\mathrm{cv}}_{n-1}\partial c\big)$$

then $P^{\mathrm{cv}}_n(c) \in C^{\mathrm{affine}}_{n+1}(D)$, and moreover since $c - \mathrm{Sd}^{\mathrm{cv}}_n c - P^{\mathrm{cv}}_{n-1}\partial c$ is a cycle, it follows from Proposition 13.2 that $\partial P^{\mathrm{cv}}_n c = c - \mathrm{Sd}^{\mathrm{cv}}_n c - P^{\mathrm{cv}}_{n-1}\partial c$.

**2.** Now we prove the general case. The strategy is the same as the definition of Sd and the proof of Proposition 13.9. Using that $\ell_n$ is an affine $n$-simplex, given a singular $n$-simplex $\sigma \colon \Delta^n \to X$, we define

$$P_n\sigma := \sigma^{\#}_{n+1}(P^{\mathrm{cv}}_n(\ell_n)) \in C_{n+1}(X).$$

Just as with Lemma 13.8, if $X$ was already convex and $\sigma$ was affine, we have $P_n(\sigma) = P^{\mathrm{cv}}_n(\sigma)$. We claim that this map $P_n$ is "natural", that is, if $f \colon X \to Y$ is any continuous map, then the following commutes:

$$\begin{tikzcd} C_n(X) \arrow[r, "f^{\#}_n"] \arrow[d, "P_n"'] & C_n(Y) \arrow[d, "P_n"] \\ C_{n+1}(X) \arrow[r, "f^{\#}_{n+1}"'] & C_{n+1}(Y) \end{tikzcd} \tag{13.6}$$

Indeed,

$$f^{\#}_{n+1}P_n\sigma = f^{\#}_{n+1}\sigma^{\#}_{n+1}(P^{\mathrm{cv}}_n(\ell_n)) = (f\circ\sigma)^{\#}_{n+1}(P^{\mathrm{cv}}_n(\ell_n)) = P_n(f\circ\sigma) = P_n(f^{\#}_n\sigma).$$

Then using (13.6) and arguing as in the proof of Proposition 13.9, we have

$$\partial P_n(\sigma) = \sigma^{\#}_n\big(\partial P^{\mathrm{cv}}_n(\ell_n)\big) \tag{13.7}$$

and

$$P_{n-1}(\partial\sigma) = \sigma^{\#}_n\big(P^{\mathrm{cv}}_{n-1}(\partial\ell_n)\big) \tag{13.8}$$

and thus adding (13.7) and (13.8) together and using the fact that we already know that (13.5) holds for $P^{\mathrm{cv}}_n$, we see that

$$\partial P_n(\sigma) + P_{n-1}\partial\sigma = \sigma^{\#}_n\big(\partial P^{\mathrm{cv}}_n(\ell_n) + P^{\mathrm{cv}}_{n-1}(\partial\ell_n)\big) = \sigma^{\#}_n\big(\ell_n - \mathrm{Sd}^{\mathrm{cv}}_n(\ell_n)\big) = \sigma - \mathrm{Sd}_n(\sigma),$$

where we used (13.4) again in the last equality. ■

LECTURE 14

# Excision and the homology of spheres

In this lecture we state and prove the excision axiom. This is the last of the four major axioms needed to define a homology theory. Thus by the end of this lecture we will have proved (using terminology we will introduce in Lecture 22) that *singular homology is a homology theory*. We use this to prove the *Mayer-Vietoris* sequence and to compute the homology of spheres.

Given a set $U \subseteq X$ of a topological space, we denote by $U^\circ$ the interior of $U$.

DEFINITION 14.1. Let $X$ be a topological space and let $\mathfrak{U}$ be a family of subsets of $X$ such that

$$X = \bigcup_{U \in \mathfrak{U}} U^\circ.$$

We say a singular $n$-simplex $\sigma \colon \Delta^n \to X$ is **$\mathfrak{U}$-small** if there exists $U \in \mathfrak{U}$ such that $\sigma(\Delta^n) \subseteq U$. We denote by $C_\bullet^{\mathfrak{U}}(X)$ the subcomplex of $C_\bullet(X)$ generated by $\mathfrak{U}$-small simplices (it is clear this is a subcomplex.) We denote by $H_\bullet^{\mathfrak{U}}(X)$ the homology of this chain complex.

There is an obvious chain map $i_\bullet : C_\bullet^{\mathfrak{U}}(X) \to C_\bullet(X)$ given by inclusion. The main technical result we prove today is that this chain map induces an isomorphism on homology.

THEOREM 14.2. *The inclusion of chain complexes $i_\bullet : C_\bullet^{\mathfrak{U}}(X) \to C_\bullet(X)$ induces an isomorphism $H_n^{\mathfrak{U}}(X) \to H_n(X)$ for all $n \geq 0$.*

We will need a few preliminary results. The next two pertain to genuine simplices (not *singular* simplices!).

PROPOSITION 14.3. *Let $S = [z_0, z_1, \dots, z_n]$ denote an $n$-simplex in some Euclidean space. Then if $x, y \in S$ one has*

$$|x - y| \leq \sup_i |z_i - y|, \tag{14.1}$$

*and hence*

$$\operatorname{diam} S = \max_{i,j} |z_i - z_j|. \tag{14.2}$$

*Moreover if $b$ is the barycentre of $S$ then*

$$|b - z_i| \leq \frac{n}{n+1} \operatorname{diam} S \tag{14.3}$$

*Proof.* Let $x, y \in S$, and write $x = \sum_i s_i\, z_i$ with $\sum_i s_i = 1$. Then

$$|x - y| = \left|\sum_i s_i\, z_i - y\right| \le \sum_i s_i |z_i - y| \le \max_i |z_i - y|$$

This proves (14.1), and (14.2) is an immediate consequence of this. Now since $b = \frac{1}{n+1}\sum_i z_i$ we have

$$\begin{aligned} |b - z_j| &= \left|\left(\sum_i \frac{1}{n+1} z_i\right) - z_j\right| \\ &\overset{(*)}{=} \left|\sum_i \frac{1}{n+1}(z_i - z_j)\right| \\ &\le \frac{1}{n+1}\sum_i |z_i - z_j| \\ &\le \frac{n}{n+1} \max_{i,j} |z_i - z_j| \\ &= \frac{n}{n+1} \operatorname{diam} S, \end{aligned}$$

where $(*)$ used the fact that $\sum_{i=0}^n \frac{1}{n+1} = 1$. This proves (14.3). ∎

We can regard any genuine simplex $[z_0, z_1, \ldots, z_n]$ as a singular $n$-simplex by choosing $\sigma\colon \Delta^n \to [z_0, z_1, \ldots, z_n]$ to be an affine map sending $e_i$ to $z_i$ (cf. Problem D.4). Thus if $S_i$ are genuine $n$-simplices in some convex subset $D$ and $m_i$ are non-zero integers, we can regard $\sum_i m_i\, S_i$ as belonging to $C_n(D)$ (actually, to $C_n^{\text{affine}}(D)$.) We define the **mesh** of such a sum to the maximum diameter of the $S_i$.

In particular, if $S$ is an $n$-simplex then $\operatorname{Sd}_n^{\text{cv}}(S)$ is an element of $C_n(D)$, and we have:

Corollary 14.4. *For any $n$-simplex $S$,*

$$\operatorname{mesh} \operatorname{Sd}_n^{\text{cv}}(S) \le \frac{n}{n+1} \operatorname{diam} S.$$

This allows us to prove the following result, which tells us that we can make any singular simplex into a sum of $\mathfrak{U}$-small simplices by barycentrically subdividing enough times.

Proposition 14.5. *Let $X$ be a topological space and let $\mathfrak{U}$ be a family of subsets of $X$ whose interiors cover $X$. Let $\sigma\colon \Delta^n \to X$ be a singular $n$-simplex. There exists $k \in \mathbb{N}$ such that every simplex in the $n$-chain $\operatorname{Sd}_n^k(\sigma)$ is $\mathfrak{U}$-small.*

*Proof.* Let $\delta > 0$ be a Lebesgue number (cf. Lemma 6.7) for the open covering $\{\sigma^{-1}(U^\circ) \mid U \in \mathfrak{U}\}$ of $\Delta^n$. Choose $k \in \mathbb{N}$ large enough so that

$$\left(\frac{n}{n+1}\right)^k < \delta.$$

The claim now follows from Corollary 14.4 and induction. ∎

With these preliminaries out of the way, we can now prove Theorem 14.2.

*Proof of Theorem 14.2.* Let $n \geq 0$. We first prove $H_n(i)\colon H_n^{\mathfrak{U}}(X) \to H_n(X)$ is injective. Suppose $c \in Z_n^{\mathfrak{U}}(X)$ belongs to the kernel of $H_n(i)$. This means there exists $a \in C_{n+1}(X)$ such that $\partial a = i_n^{\#}c = c$. By applying Proposition 14.5 to each of the finitely many simplices in $a$, we see there exists $k \in \mathbb{N}$ such that $\mathrm{Sd}_{n+1}^k(a) \in C_{n+1}^{\mathfrak{U}}(X)$. Recall Theorem 13.11 gives us a chain homotopy $P$ from Sd to the identity: $\partial P + P\partial = \mathrm{Sd} - \mathrm{id}$. By induction, for any positive integer $k$ we have

$$\partial P \mathrm{Sd}^{k-1} + P \mathrm{Sd}^{k-1}\partial = \mathrm{Sd}^k - \mathrm{Sd}^{k-1}.$$

Thus if we set

$$P^{(k)} := P \circ \left(\mathrm{id} + \mathrm{Sd} + \cdots + \mathrm{Sd}^{k-1}\right)$$

then we have

$$\partial P^{(k)} + P^{(k)}\partial = \mathrm{Sd}^k - \mathrm{id}.$$

Thus

$$\mathrm{Sd}_{n+1}^k(a) - a = \partial P_{n+1}^{(k)}(a) + P_n^{(k)}(\partial a) = \partial P_{n+1}^{(k)}(a) + P_n^{(k)}(c),$$

and hence as $\partial^2 = 0$ we have

$$c = \partial a = \partial\big(\mathrm{Sd}_{n+1}^k(a) - P_n^{(k)}(c)\big).$$

Since $c \in C_n^{\mathfrak{U}}(X)$ we have $P_n^{(k)}(c) \in C_n^{\mathfrak{U}}(X)$ as well; this can be seen from the naturality equation (13.6). Thus as $\mathrm{Sd}_{n+1}^k(a)$ also belongs to $C_{n+1}^{\mathfrak{U}}(X)$, we see that $c \in B_n^{\mathfrak{U}}(X)$. Hence $H_n(i)$ is injective.

Now we prove $H_n(i)$ is surjective. Suppose $d \in Z_n(X)$. Then using Proposition 14.5 for $k$ large enough we have $\mathrm{Sd}_n^k(d) \in C_n^{\mathfrak{U}}(X)$. With $P^{(k)}$ as before we have

$$\mathrm{Sd}_n^k(d) - d = \partial P_n^{(k)}(d) + P_{n-1}^{(k)}(\partial d) = \partial P_n^{(k)}(d).$$

Since $\mathrm{Sd}^k$ is a chain map, $\mathrm{Sd}_n^k(d)$ is also a cycle. The previous equation thus shows that $d$ is homologous to a cycle in $C_n^{\mathfrak{U}}(X)$. This shows that $H_n(i)$ is surjective, and hence completes the proof. ■

Now let $(X, X')$ be a pair of spaces. Write

$$\mathfrak{U} \cap X' := \{U \cap X' \mid U \in \mathfrak{U}\},$$

and define the chain complex

$$C_\bullet^{\mathfrak{U}}(X, X') := C_\bullet^{\mathfrak{U}}(X) \Big/ C_\bullet^{\mathfrak{U}\cap X'}(X').$$

We denote its homology groups by $H_n^{\mathfrak{U}\cap X'}(X, X')$. This gives us a commutative diagram of chain complexes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet^{\mathfrak{U}\cap X'}(X') \arrow[r] \arrow[d] & C_\bullet^{\mathfrak{U}}(X) \arrow[r] \arrow[d] & C_\bullet^{\mathfrak{U}}(X, X') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_\bullet(X') \arrow[r] & C_\bullet(X) \arrow[r] & C_\bullet(X, X') \arrow[r] & 0
\end{tikzcd}$$

Each row has its own long exact sequence, giving the following commutating diagram:

$$\begin{tikzcd} \dots \arrow[r] & H_n^{\mathfrak{U}\cap X'}(X') \arrow[r] \arrow[d] & H_n^{\mathfrak{U}}(X) \arrow[r] \arrow[d] & H_n^{\mathfrak{U}\cap X'}(X,X') \arrow[r] \arrow[d, "j"] & H_{n-1}^{\mathfrak{U}\cap X'}(X') \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & H_n(X') \arrow[r] & H_n(X) \arrow[r] & H_n(X,X') \arrow[r] & H_{n-1}(X') \arrow[r] & \dots \end{tikzcd}$$

All the vertical maps apart from the one marked $j$ are isomorphisms, thanks to Theorem 14.2. But now by the Five Lemma (Proposition 11.3), we see that $j$ is also an isomorphism. This proves:

Proposition 14.6. *The inclusion of chain complexes $C_\bullet^{\mathfrak{U}}(X,X') \to C_\bullet(X,X')$ induces an isomorphism in homology:*

$$H_n^{\mathfrak{U}}(X,X') \cong H_n(X,X'), \qquad \forall\, n \geq 0.$$

We now state two forms of excision.

Theorem 14.7 (The excision axiom). *Assume that $X'' \subset X' \subset X$ are subspaces with $\overline{X''} \subset (X')^\circ$. Then the inclusion $(X \setminus X'', X' \setminus X'') \hookrightarrow (X,X')$ induces an isomorphism in homology:*

$$H_n(X \setminus X'', X' \setminus X'') \cong H_n(X,X'), \qquad \forall\, n \geq 0.$$

Theorem 14.8 (The excision axiom, second form). *Assume that $X_1, X_2$ are subspaces of $X$ such that $X = X_1^\circ \cup X_2^\circ$. Then the inclusion $(X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ induces an isomorphism in homology:*

$$H_n(X_1, X_1 \cap X_2) \cong H_n(X, X_2), \qquad \forall\, n \geq 0.$$

On Problem Sheet G you will show that the two results Theorem 14.7 and Theorem 14.8 are equivalent. Here we will prove the second one.

*Proof of Theorem 14.8.* We take as our covering $\mathfrak{U} = \{X_1, X_2\}$. The hypotheses of Theorem 14.2 are satisfied. By definition we have

$$C_\bullet^{\mathfrak{U}}(X) = C_\bullet(X_1) + C_\bullet(X_2),$$

and hence if we look at long exact sequence in homology associated to the short exact sequence of chain complexes:

$$0 \to \Big(C_\bullet(X_1) + C_\bullet(X_2)\Big) \xrightarrow{f} C_\bullet(X) \to C_\bullet(X) \Big/ \Big(C_\bullet(X_1) + C_\bullet(X_2)\Big) \to 0,$$

every third map

$$H_n(f) \colon H_n\big(C_\bullet(X_1) + C_\bullet(X_2)\big) \to H_n\big(C_\bullet(X)\big)$$

is an isomorphism[1] by Theorem 14.2. Now we consider the short exact sequence of chain complexes:

$$0 \to \frac{C_\bullet(X_1) + C_\bullet(X_2)}{C_\bullet(X_2)} \xrightarrow{g} \frac{C_\bullet(X)}{C_\bullet(X_2)} \to \frac{C_\bullet(X)}{C_\bullet(X_1) + C_\bullet(X_2)} \to 0.$$

The corresponding long exact sequence has every third term zero, so that $H_n(g)$ is an isomorphism for every $n$. Next observe there is an isomorphism of chain complexes:

$$h\colon \frac{C_\bullet(X_1)}{C_\bullet(X_1 \cap X_2)} \cong \frac{C_\bullet(X_1) + C_\bullet(X_2)}{C_\bullet(X_2)}.$$

This is just the fact that

$$C_\bullet(X_1 \cap X_2) = C_\bullet(X_1) \cap C_\bullet(X_2),$$

together with the **second isomorphism theorem** for chain complexes, which is Problem G.6. We thus have a commuting triangle of chain complexes:

$$\begin{tikzcd} \frac{C_\bullet(X_1)}{C_\bullet(X_1 \cap X_2)} \arrow[rr] \arrow[dr, "h"'] & & \frac{C_\bullet(X)}{C_\bullet(X_2)} \\ & \frac{C_\bullet(X_1) + C_\bullet(X_2)}{C_\bullet(X_2)} \arrow[ur, "g"'] & \end{tikzcd}$$

The induced maps $H_n(h)$ and $H_n(g)$ are both isomorphisms, and hence the horizontal map also induces an isomorphism in homology:

$$H_n\left(\frac{C_\bullet(X_1)}{C_\bullet(X_1 \cap X_2)}\right) \cong H_n\left(\frac{C_\bullet(X)}{C_\bullet(X_2)}\right), \qquad \forall\, n \geq 0.$$

This is exactly the statement of the theorem. ■

We now prove the "homology" version of the Seifert-van Kampen Theorem 6.5, which is a simple consequence of excision.

THEOREM 14.9 (Mayer-Vietoris). *Let $X_1$ and $X_2$ be subspaces of $X$ such that $X = X_1^\circ \cup X_2^\circ$. Set $X_0 := X_1 \cap X_2$ and let*

$$\imath_i\colon X_0 \hookrightarrow X_i, \qquad \jmath_i\colon X_i \hookrightarrow X$$

*denote inclusions for $i = 1, 2$. Then there is a long exact sequence*

$$\dots H_n(X_0) \xrightarrow{(H_n(\imath_1), H_n(\imath_2))} H_n(X_1) \oplus H_n(X_2) \xrightarrow{H_n(\jmath_1) - H_n(\jmath_2)} H_n(X) \xrightarrow{D} H_{n-1}(X_0) \to \dots$$

[1] If one has a long exact sequence

$$\dots \to A_n \xrightarrow{f_n} B_n \to C_n \to A_{n-1} \xrightarrow{f_{n-1}} B_{n-1} \to \dots$$

where every third map $f_n\colon A_n \to B_n$ is an isomorphism, then $C_n = 0$ for all $n$ by exactness.

*Proof.* The following diagram of pairs of spaces commutes, where all the maps are inclusions[2]:

$$\begin{tikzcd} (X_0, \emptyset) \arrow[r, "\imath_1"] \arrow[d, "\imath_2"'] & (X_1, \emptyset) \arrow[r, "f"] \arrow[d, "\jmath_1"] & (X_1, X_0) \arrow[d, "h"] \\ (X_2, \emptyset) \arrow[r, "\jmath_2"'] & (X, \emptyset) \arrow[r, "g"'] & (X, X_2) \end{tikzcd}$$

By Problem F.4 we obtain a commutative diagram with exact rows:

$$\begin{tikzcd} \dots \arrow[r] & H_n(X_0) \arrow[r, "H_n(\imath_1)"] \arrow[d, "H_n(\imath_2)"'] & H_n(X_1) \arrow[r, "H_n(f)"] \arrow[d, "H_n(\jmath_1)"] & H_n(X_1, X_0) \arrow[r, "\delta"] \arrow[d, "H_n(h)"] & H_{n-1}(X_0) \arrow[r] \arrow[d, "H_n(\imath_2)"] & \dots \\ \dots \arrow[r] & H_n(X_2) \arrow[r, "H_n(\jmath_2)"'] & H_n(X) \arrow[r, "H_n(g)"'] & H_n(X, X_2) \arrow[r, "\delta"'] & H_{n-1}(X_2) \arrow[r] & \dots \end{tikzcd}$$

Now Theorem 14.8 tells us that the map $H_n(h)$ is an isomorphism for all $n$. This means that we can apply the Barratt-Whitehead Lemma (Proposition 11.4) to obtain the desired long exact sequence. Explicitly, $D$ is induced by the composition $\partial(h_n^{\#})^{-1} g_n^{\#}$. ∎

Provided $X_0$ is non-empty, the Mayer-Vietoris Theorem continuous to hold for reduced homology as well.

Corollary 14.10. *Let $X_1$ and $X_2$ be subspaces of $X$ such that $X = X_1^\circ \cup X_2^\circ$. Set $X_0 := X_1 \cap X_2$ and assume $X_0 \neq \emptyset$. Then there is a long exact sequence*

$$\dots \tilde{H}_n(X_0) \to \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \to \tilde{H}_n(X) \to \tilde{H}_{n-1}(X_0) \to \dots$$

*where the maps are the same as in Theorem 14.9. The sequence ends with*

$$\dots \tilde{H}_0(X_1) \oplus \tilde{H}_0(X_2) \to \tilde{H}_0(X) \to 0.$$

*Proof.* Fix $p \in X_0$ and proceed as before, starting with the commutative diagram

$$\begin{tikzcd} (X_0, p) \arrow[r] \arrow[d] & (X_1, p) \arrow[r] \arrow[d] & (X_1, X_0) \arrow[d] \\ (X_2, p) \arrow[r] & (X, p) \arrow[r] & (X, X_2) \end{tikzcd}$$

This gives the desired long exact sequence, albeit with $H_n(X, p)$ in place of $\tilde{H}_n(X)$, etc. However Corollary 12.22 then completes the proof. ∎

We can now finally compute the homology of the sphere. We will state the result for reduced homology, because this is a neater statement and the proof is shorter.

Theorem 14.11. *For all $n \geq 0$, one has*

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k = n, \\ 0, & k \neq n. \end{cases}$$

[2]We identify eg. $\imath_1 \colon X_0 \hookrightarrow X_1$ with the map of pairs $(X_0, \emptyset) \hookrightarrow (X_1, \emptyset)$.

*Proof.* We induct on $n$. For $n = 0$ this follows from the definition of reduced homology, the dimension axiom (Proposition 8.1), and Proposition 8.3), since $S^0 = \{-1, 1\}$ is a space consisting of two points. For the inductive step, we apply Corollary 14.10. Suppose $n \geq 1$, and let $p$ and $q$ denote the "north pole" and "south pole" of $S^n$ respectively. Set $X_1 = S^n \setminus \{p\}$ and $X_2 := S^n \setminus \{q\}$. Then $X_1$ and $X_2$ are contractible and $X_1 \cap X_2$ is homotopy equivalent to the "equator" $S^{n-1}$. Using Corollary 12.20, the Mayer-Vietoris sequence gives us for all $i \geq 0$ an exact sequence

$$0 \to \tilde{H}_k(S^n) \to \tilde{H}_{k-1}(S^{n-1}) \to 0.$$

The result follows by induction. ∎

Just as with the Seifert-van Kampen Theorem 6.5, the Mayer-Vietoris exact sequence allows us to compute the homology of a number of standard spaces. We will return to this in Lecture 18, when we discuss the idea of *attaching cells*.

REMARK 14.12. In particular, $H_n(S^n) = \mathbb{Z} \neq 0$. Thus we have finally proved Lemma 1.3 from Lecture 1, and thus we have also finally proved the Brouwer Fixed Point Theorem 1.1.

LECTURE 15

# The degree

In this lecture we define the *degree* of a continuous map from a sphere to itself, which extends the earlier one given in Definition 5.5 to higher dimensions.

The definition of the degree starts from the simple observation that any group homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}$ is necessarily multiplication by an integer: $\varphi(n) = mn$ for some $m \in \mathbb{Z}$. Indeed, if $m = \varphi(1)$ then $\varphi(n) = \varphi(1 \cdot n) = \varphi(1) \cdot n = mn$.

In Lecture 5 we defined the degree of a *loop* $u \colon (I, \partial I) \to (X, p)$ for any topological space $X$. But from Problem B.5, we have $\pi_1(X, p) \cong [(S^1, 1), (X, p)]$, and hence our earlier definition can be thought of as a map $[(S^1, 1), (X, p)] \to \mathbb{Z}$. In this lecture we will primarily work with homology, and thus it is convenient to ditch the basepoints. This means we need a stronger version of Problem B.5. This is provided by the next result, which shows we can neglect basepoints possible when $\pi_1(X, p)$ is abelian.

PROPOSITION 15.1. *Let $X$ be path connected and let $\zeta \colon \pi_1(X, p) \to [S^1, X]$ be the function that sends a path class $[u]$ to the free homotopy class of the map $\hat{u} \colon S^1 \to X$ given by*

$$\hat{u}(e^{2\pi i s}) := u(s), \qquad s \in I.$$

*This function is surjective. Moreover if $\zeta([u]) = \zeta([v])$ then there exists $[w] \in \pi_1(X, p)$ such that $[u] = [w] * [v] * [w]^{-1}$. In particular, if $\pi_1(X, p)$ is abelian then $\zeta$ is an isomorphism, and hence $\pi_1(X, p) \cong [S^1, X]$.*

The proof of Proposition 15.1 is on Problem Sheet H as a test to see whether you've already forgotten all the homotopy theory we did ...

Taking $X = S^1$, and using that $\pi_1(S^1) = \mathbb{Z}$ is abelian, this shows that our earlier definition of the degree (Definition 5.5) can be thought of as a map $[S^1, S^1] \to \mathbb{Z}$. We now use take advantage of the fact that $H_n(S^n) = \mathbb{Z}$ to extend this to higher-dimensional spheres using homology.

DEFINITION 15.2. Let $n \geq 1$ and let $f \colon S^n \to S^n$ be continuous. Then $H_n(f) \colon H_n(S^n) \to H_n(S^n)$ is a group homomorphism, and hence is multiplication by an integer. This integer is called the **degree** of $f$ and denoted by $\deg(f)$. Thus

$$H_n(f)\langle c\rangle = \deg(f)\langle c\rangle, \qquad \forall\, \langle c\rangle \in H_n(S^n).$$

The homotopy axiom (Theorem 8.9) implies that $\deg \colon [S^n, S^n] \to \mathbb{Z}$ is a well defined function, since if $f \simeq g$ are homotopic maps from $S^n$ to itself then $H_n(f) = H_n(g)$ and thus in particular $\deg(f) = \deg(g)$.

Let us check that for $S^1$, this definition agrees with the old one. For clarity, since there are now (supposedly) three different definitions of "degree" in play, let us temporarily give them all different names. Firstly we have

$$\deg^{\mathrm{loop}} \colon \pi_1(S^1, 1) \to \mathbb{Z},$$

the one from Theorem 5.6. Secondly, we have

$$\deg^{\mathrm{old}} \colon [S^1, S^1] \to \mathbb{Z},$$

the one obtained from $\deg^{\mathrm{loop}}$ using Proposition 15.1, and finally the function

$$\deg^{\mathrm{new}} \colon [S^1, S^1] \to \mathbb{Z},$$

given from Definition 15.2. Of course $\deg^{\mathrm{loop}}$ and $\deg^{\mathrm{old}}$ are the same map, but in the proof of the next proposition it is convenient to keep the notation distinct.

Proposition 15.3. *Let $f \colon S^1 \to S^1$ be continuous. Then the homomorphism $\pi_1(f)$ is given by mulitplication by $\deg^{\mathrm{old}}(f)$.*

*Proof.* Let $u_k(s) := e^{2\pi i k s}$ for $k \in \mathbb{N}$. Then from the proof of Theorem 5.6, $\deg^{\mathrm{loop}}[u_k] = k$, and thus $[u_1]$ is a generator of $\pi_1(S^1, 1)$. Consider now first the special case where $f = g_m$ for $g_m(z) := z^m$. Since $g_m \circ u_1 = u_m$ we see that $\pi_1(g_m)[u_1] = [u_m]$. Thus $\pi_1(g_m)$ is given by multiplication by $m$ under the identification $\pi_1(S^1, 1) \cong \mathbb{Z}$ given by $\deg^{\mathrm{loop}}$.

For the general case, suppose $f \colon S^1 \to S^1$ has $\deg^{\mathrm{old}}(f) = m$. Then $f \simeq g_m$ by Proposition 15.1. Since $\pi_1(S^1)$ is abelian, Corollary 4.14 shows that $\pi_1(f) = \pi_1(g_m)$. This completes the proof. ∎

We now prove that actually $\deg^{\mathrm{old}} = \deg^{\mathrm{new}}$.

Proposition 15.4. *If $f \colon S^1 \to S^1$ is continuous then $\deg^{\mathrm{old}}(f) = \deg^{\mathrm{new}}(f)$*

*Proof.* By Problem E.2, there is a commutative diagram

$$\begin{array}{ccc} \pi_1(S^1, 1) & \xrightarrow{\pi_1(f)} & \pi_1(S^1, 1) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_1(S^1) & \xrightarrow[H_1(f)]{} & H_1(S^1) \end{array}$$

where $h \colon \pi_1(S^1, 1) \to H_1(S^1)$ is the Hurewicz map. Since $\pi_1(S^1, 1) = \mathbb{Z}$ is abelian, the map $h$ is an isomorphism, and $\deg^{\mathrm{loop}}$ furnishes an explicit isomorphism $\pi_1(S^1, 1) \cong H_1(S^1) \cong \mathbb{Z}$. Now the result follows from Proposition 15.3. ∎

With this out the way, we will go back to just calling all three of the maps deg.

Proposition 15.5. *Let $n \geq 1$ and let $f, g \colon S^n \to S^n$ denote continuous maps. Then:*

1. $\deg(g \circ f) = \deg(g)\deg(f)$,
2. $\deg(\mathrm{id}_{S^n}) = 1$,
3. *if $f$ is a constant map then* $\deg(f) = 0$,

4. *if* $f \simeq g$ *then* $\deg(f) = \deg(g)$,
5. *if* $f$ *is a homotopy equivalence then* $\deg(f) = \pm 1$.

*Proof.* All properties follow from the fact that $H_n$ is a functor. Property (3) follows from the fact that a constant map $f$ can be factored as a composition $S^n \to \{*\} \to S^n$ where $\{*\}$ is a one-point space. ■

We now prove a far less obvious result.

PROPOSITION 15.6. *Let* $n \geq 1$ *and let* $A \in \mathrm{O}(n+1)$ *denote an orthogonal linear transformation. Set* $f := A|_{S^n}$. *Then* $\deg(f) = \det(A)$.

*Proof.* The group $\mathrm{O}(n+1)$ has two connected components, distinguished by $\det\colon \mathrm{O}(n+1) \to \{+1, -1\}$. By homotopy invariance it suffices to check the result for one such $A$ in each component. Since the identity matrix $I_{n+1}$ induces $f = \mathrm{id}_{S^n}$, which has degree 1, it suffices to check the result for a single map $A$ with $\det(A) = -1$. We take $A$ to be reflection in a hyperplane $H \subset \mathbb{R}^{n+1}$. Divide $S^n$ into two hemispheres that are *preserved* by $A$. Then the map $f$ induces a reflection $f'$ in the corresponding hyperplane $H'$ in the equatorial $S^{n-1}$. Now applying the Mayer-Vietoris sequence and using naturality, we obtain the following commutative diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "D"] \arrow[d, "H_n(f)"'] & H_{n-1}(S^{n-1}) \arrow[d, "H_{n-1}(f')"] \\
H_n(S^n) \arrow[r, "D"'] & H_{n-1}(S^{n-1})
\end{tikzcd}$$

Here the maps $D$ are the connecting maps from the Mayer-Vietoris sequence. These maps are isomorphisms, moreover they are the *same* isomorphism. Thus we see that

$$\deg(f) = \deg(f'),$$

and hence by induction it suffices to prove the result for $n = 1$. For this we write $S^1$ as the union of two open intervals $A$ and $B$ which contract onto the two given hemispheres preserved by our reflection. Then $A \cap B$ is homeomorphic to $S^0 = \{p, q\}$. Applying Mayer-Vietoris again, we see that $H_1(S^1)$ is isomorphic to the kernel of the map $j$:

$$0 \to H_1(S^1) \to H_0(S^0) \xrightarrow{j} H_0(A) \oplus H_0(B).$$

We take $p$ and $q$ as generators of $H_0(S^0)$. Both $p$ and $q$ generate $H_0(A)$ and $H_0(B)$, and thus the map $j$ here

$$0 \to H_1(S^1) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j} \mathbb{Z} \oplus \mathbb{Z}$$

is given by $(u, v) \mapsto (u + v, u + v)$. In particular, the kernel of $j$ is generated by $p - q$. Since the reflection $f$ interchanges $p$ and $q$, this shows that $\deg(f) = -1$ as claimed. ■

COROLLARY 15.7. *Let* $n \geq 1$. *The antipodal map* $a\colon S^n \to S^n$ *given by* $a(x) = -x$ *has degree* $(-1)^{n+1}$.

*Proof.* The antipodal map is the composition of the $n+1$ reflections in the coordinates axes of $\mathbb{R}^{n+1}$. These all have degree $-1$ by Proposition 15.6, and degree is multiplicative by part (1) of Proposition 15.5. ■

This immediately gives a proof of the following famous result.

THEOREM 15.8 (The Hairy Ball Theorem). *There exists a nowhere vanishing vector field on $S^n$ if and only if $n$ is odd.*

A vector field can be identified with a continuous map $v\colon S^n \to \mathbb{R}^{n+1}$ such that $\langle x, v(x)\rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^{n+1}$. Rather more visually, suppose we attach a "hair-vector" $v(x)$ at every point $x \in S^n$. If we could successfully "comb" the sphere so that every hair was tangential to $S^n$, we'd have successfully created a vector field on $S^n$. Thus the theorem tells us that if we try this on $S^{2m}$, either there will be a point $x$ where the sphere is bald ($v(x) = 0$), or no matter how hard we try to comb, there will always be a tuft.

*Proof.* If $n = 2m-1$ then define $v\colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ by

$$v(x_1, y_1, \dots x_m, y_m) = (-y_1, x_1, \dots, -y_m, x_m).$$

The restriction of $v$ to $S^n$ is then a nowhere vanishing vector field. Conversely, suppose $v$ is a nowhere vanishing vector field. Let $w\colon S^n \to S^n$ be defined by

$$w(x) = \frac{v(x)}{|v(x)|}.$$

Now define

$$W\colon S^n \times I \to S^n, \qquad W(x,t) := (\cos \pi t)x + (\sin \pi t)w(x),$$

(this does indeed take values in $S^n$ since $\langle x, v(x)\rangle = 0$ for all $x$.) Then $W(x, 0) = x$ and $W(x, 1) = a(x)$, where $a$ is the antipodal map. Thus the degree of the antipodal map is the same as the degree of the identity by part (4) of Proposition 15.5, and thus by Corollary 15.7, we see that $n$ must be odd. ■

Taking the earth to be $S^2$ and our vector field to be the wind, the theorem can also be interpreted as saying: *There is always somewhere on planet where there is no wind.*

LEMMA 15.9. *Let $n \geq 1$. Suppose $f, g\colon S^n \to S^n$ are continuous maps such that $f(x) \neq g(x)$ for all $x \in S^n$. Then $f \simeq a \circ g$. In particular, if $f$ has no fixed points then $f$ is homotopic to the antipodal map.*

*Proof.* If $f(x) \neq g(x)$ for all $x$ then

$$F(x,t) := \frac{(1-t)f(x) - tg(x)}{|(1-t)f(x) - tg(x)|}$$

is a well-defined homotopy from $f$ to $a \circ g$ (since the denominator can never be zero.) ■

We can use this lemma to obtain surprising information on what groups $G$ can act freely on $S^{2n}$. First note that $S^{2n-1}$ can be realised as the unit circle in $\mathbb{C}^n$, and thus carries a free action of $S^1$; namely $z \mapsto e^{i\theta} z$. Inside $S^1$ we then also have free actions of the $m$th roots of unity on $S^{2n-1}$, and thus $\mathbb{Z}_m$ acts on $S^{2n-1}$ for each $m \in \mathbb{N}$. The same however is *not* true for $S^{2n}$.

Corollary 15.10. *If $G$ acts freely on $S^{2n}$ then $G$ is either the trivial group or $G = \mathbb{Z}_2$.*

*Proof.* If $G$ acts freely on $S^{2n}$ then each non-trivial element $g \in G$ has no fixed points. Thus each $g$ has degree $-1$ by Lemma 15.9 and Corollary 15.7. Thus the map $\deg\colon G \to \mathbb{Z}_2 = \{+1, -1\}$ is an injective group homomorphism. ∎

We conclude this lecture with a much deeper result.

Definition 15.11. A continuous map $f\colon S^n \to S^n$ is called an **odd** map if $f(-x) = -f(x)$ for all $x \in S^n$. Equivalently, $f \circ a = a \circ f$, where $a$ is the antipodal map.

Theorem 15.12. *An odd map has odd degree.*

As you will see on Problem Sheet H, Theorem 15.12 implies two classical theorems, the **Borsuk-Ulam Theorem** and the **Lusternik-Schnirelmann Theorem**. There are several ways to prove Theorem 15.12. We will give a proof that (mostly) uses only material that we have covered so far. Alternative approaches use the *Smith Exact Sequence* and *homology with $\mathbb{Z}_2$-coefficients*, or the *ring structure* of the *cohomology $H^\bullet(\mathbb{R}P^n)$*. These are topics we will cover in Algebraic Topology II next semester.

Let us start with some notation. Let $B^n_\pm$ denote the upper and lower hemispheres of $S^n$, so that $B^n_+ \cap B^n_-$ is the equatorial $S^{n-1}$. This process can be iterated, so we can see $S^i$ sitting inside $S^n$ for all $0 \leq i \leq n$. We will need the following result in our proof of Theorem 15.12, which we can't properly prove just yet.

Proposition 15.13. *Let $f\colon S^n \to S^n$ be an odd map. Then there exists an odd map $f'\colon S^n \to S^n$ such that $f'(S^i) \subseteq S^i$ for each $i = 0, \dots, n$ and a homotopy $F\colon f \simeq f'$ with the property that $f_t := F(\cdot, t)$ is an odd map for each $t \in I$.*

*Sketch proof.* Let $p\colon S^n \to \mathbb{R}P^n$ denote the projection map (see Example 18.8 in Lecture 18). Since $f$ is odd, there is an induced map $h\colon \mathbb{R}P^n \to \mathbb{R}P^n$ such that $h \circ p = p \circ f$. Using the **cellular structure** of $\mathbb{R}P^n$ and the **Cellular Approximation Theorem** (for the former, see Example 18.8 again, for the latter see Algebraic Topology II), there exists a homotopy $H\colon h \simeq h'$ such that $h'(\mathbb{R}P^i) \subseteq \mathbb{R}P^i$ for each $i = 0, 1, \dots, n$. Now using the **homotopy lifiting property** (also in Algebraic Topology II), we can **lift** $H$ to a homotopy $F\colon S^n \times I \to S^n$. This means that

$$H(p(x), t)) = p(F(x, t)), \qquad \forall\, (x, t) \in S^n \times I \tag{15.1}$$

(compare this to Proposition 5.2), and such that $F(\cdot, 0)$ is our original map $f$. The map $f_t := F(\cdot, t)$ is odd for each $t$ by (15.1), and $f' := f_1$ has the property that $f'(S^i) \subseteq S^i$ for each $i = 0, 1, \dots, n$. ∎

Proposition 15.13 implies that we can prove Theorem 15.12 via the following result. In the statement, it is convenient to formally define the degree of a map $f\colon S^0 \to S^0$ to be the the integer such that the induced map in *reduced homology* $\tilde{H}_0(S^0) \to \tilde{H}_0(S^0)$ is multiplication by this number.

PROPOSITION 15.14. *Let $n \geq 1$ and let $f\colon S^n \to S^n$ denote an odd map such that $f(S^i) \subseteq S^i$ for each $i = 0, 1, \ldots, n$. Then $\deg(f)$ has the same parity as $\deg(f|_{S^{n-1}})$.*

Indeed, Theorem 15.12 immediately follows from this, since the only odd maps $S^0 \to S^0$ are the identity and the antipodal map itself, and these both have odd degree. Indeed, if $S^0 = \{p, q\}$ then $\langle p - q\rangle$ is a generator of $\tilde{H}_0(S^0)$. The identity map induces $\langle p - q\rangle \mapsto \langle p - q\rangle$, and the antipodal map induces $\langle p - q\rangle \mapsto -\langle p - q\rangle$. Hence the degree is $+1$ or $-1$, and thus in particular is odd. (This also shows that Corollary 15.7 also formally holds in the case $n = 0$.)

*Proof of Proposition 15.14.* Consider the following commuting hexagon (!):

$$\begin{tikzcd}
 & & \tilde{H}_n(S^n) \arrow[dll, "l_+"'] \arrow[drr, "l_-"] \arrow[dd, "h"] & & \\
H_n(S^n, B^n_+) & & & & H_n(S^n, B^n_-) \\
 & & H_n(S^n, S^{n-1}) \arrow[ull, "j_+"] \arrow[urr, "j_-"'] \arrow[dd, "\delta"] & & \\
H_n(B^n_-, S^{n-1}) \arrow[uu, "k_+"] \arrow[urr, "i_-"'] \arrow[drr, "\delta_+"'] & & & & H_n(B^n_+, S^{n-1}) \arrow[uu, "k_-"'] \arrow[ull, "i_+"] \arrow[dll, "\delta_-"] \\
 & & \tilde{H}_{n-1}(S^{n-1}) & &
\end{tikzcd}$$

We use reduced homology at the top and bottom only so the proof still works for $n = 1$. Here the maps $\delta_\pm$ and $\delta$ come from the long exact sequence for reduced homology (Proposition 12.21), and all the other maps are induced by inclusions. Moreover[1]:

1. All the groups apart from the middle one are isomorphic to $\mathbb{Z}$.
2. The maps $k_\pm$, $l_\pm$ and $\delta_\pm$ are all isomorphisms.
3. Exactness holds at $H_n(S^n, S^{n-1})$ for all three diagonals: $\operatorname{im} i_- = \ker j_-$ and $\operatorname{im} i_+ = \ker j_+$ and $\operatorname{im} h = \ker \delta$.

The notation in this proof is rather involved, so to simplify things, we will denote homology classes just by letters $c$ etc. (i.e. no angle brackets). The map $f$ induces maps

$$H_n(f)\colon \tilde{H}_n(S^n) \to \tilde{H}_n(S^n), \qquad H_{n-1}(f|_{S^{n-1}})\colon \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(S^{n-1}),$$

[1] Consider it an exercise to verify all these claims!

and also (by assumption) a map $H_n(S^n, S^{n-1}) \to H_n(S^n, S^{n-1})$. We will denote them *all* by $\varphi$. Similarly we will denote by $\alpha$ all the maps on homology induced by the antipodal map. This should not cause confusion.

Fix a generator $c \in \tilde{H}_{n-1}(S^{n-1})$. This uniquely defines generators $c_\pm \in H_n(B^n_\mp, S^{n-1})$ via the equation $\delta_\pm(c_\pm) = c$. Moreover if $b_\pm := k_\pm(c_\pm)$ then $b_\pm$ are generators of $H_n(S^n, B^n_\pm)$, and there exist $a_\pm \in \tilde{H}_n(S^n)$ such that $l_\pm(a_\pm) = b_\pm$. Now set $u_\pm := i_\mp(c_\pm)$. Some diagram chasing tells us that $\{u_+, u_-\}$ is a basis of $H_n(S^n, S^{n-1})$ (see Problem H.2), and thus in particular

$$H_n(S^n, S^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Let $d := \deg(f)$ and $d' := \deg(f|_{S^{n-1}})$. Then by definition,

$$\varphi a_+ = d a_+, \qquad \varphi c = d' c.$$

Since $\{u_+, u_-\}$ is a basis of $H_n(S^n, S^{n-1})$, there exist integers $p, q$ such that $\varphi(u_+) = p u_+ + q u_-$. To complete the proof, we will show that

$$d = p - q, \qquad d' = p + q.$$

Thus $d' - d = 2q$ which is even. For this note that since $\delta(u_\pm) = c$ by commutativity,

$$\varphi(c) = \varphi\big(\delta(u_+)\big) = \delta\big(\varphi(u_+)\big) = \delta(p u_+ + q u_-) = (p+q)c,$$

where we used naturality (the commuting diagram part of Proposition 12.3) for $\varphi \circ \delta = \delta \circ \varphi$. Thus $d' = p + q$ as claimed.

Now consider $\alpha$. By naturality $\delta_- \circ \alpha = \alpha \circ \delta_+$, and Corollary 15.7 we have $\alpha(c) = (-1)^n c$. Thus also $\alpha(c_+) = (-1)^n c_-$ and $\alpha(u_-) = (-1)^n u_+$. Next, since $f$ is odd, $\varphi \circ \alpha = \alpha \circ \varphi$. Putting this together we see that

$$\varphi(u_-) = (-1)^n \varphi\big(\alpha(u_+)\big) = (-1)^n \alpha\big(\varphi(u_+)\big) = (-1)^n \alpha(p u_+ + q u_-) = p u_- + q u_+.$$

Next, since $\operatorname{im} h = \ker \delta$, the image of $h$ is generated by $u_+ - u_-$. Thus $h(a_+) = r(u_+ - u_-)$ with $r = \pm 1$. In fact, we claim $r = +1$. For this we use that $j_+(u_+) = l_+(a_+)$ by definition, and hence

$$j_+(u_+) = l_+(a_+) = j_+\big(h(a_+)\big) = r j_+(u_+ - u_-) = r j_+(u_+),$$

since $u_- \in \operatorname{im} i_+ = \ker j_+$. Now we observe that by definition of $d$,

$$\begin{aligned} d(u_+ - u_-) &= d h(a_+) \\ &= \varphi\big(h(a_+)\big) \\ &= \varphi(u_+ - u_-) \\ &= (p u_+ + q u_-) - (p u_- + q u_+) \\ &= (p-q)(u_+ - u_-). \end{aligned}$$

Thus $d = p - q$. This completes the proof. ∎

LECTURE 16

# Colimits and filtered colimits

Let us now make good on the promise we made in Lecture 5 and formalise the notion of a pushout.

DEFINITION 16.1. Let $\mathsf{J}$ be a small category[1]. Let $\mathsf{C}$ be another category. A **diagram of shape $\mathsf{J}$ in $\mathsf{C}$** is simply a functor $T\colon \mathsf{J} \to \mathsf{C}$. We call $\mathsf{J}$ an **index category**.

This is easiest to parse with an example.

EXAMPLE 16.2. Let $\mathsf{J}$ be a category with exactly three objects, $\{\heartsuit, \spadesuit, \diamondsuit\}$, and assume that there is unique morphism $\spadesuit \to \heartsuit$ and a unique morphism $\spadesuit \to \diamondsuit$, and that the only other morphisms are the identity morphisms (whose existence is forced). We write this pictorially as

$$\begin{tikzcd} \spadesuit \arrow[r] \arrow[d] & \heartsuit \\ \diamondsuit & \end{tikzcd}$$

A functor $T\colon \mathsf{J} \to \mathsf{C}$ is the same thing as a triple of objects $(A, B_1, B_2)$ in $\mathrm{obj}(\mathsf{C})$ together with a choice of two morphisms $f_1\colon A \to B_1$ and $f_2\colon A \to B_2$.

$$\begin{tikzcd} \spadesuit \arrow[r] \arrow[d] & \heartsuit & & A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \\ \diamondsuit & \arrow[r, squiggly, "\text{apply the functor } T"] & {} & B_2 & \end{tikzcd}$$

This thus recovers what we called a "diagram" in $\mathsf{C}$ in Definition 5.7.

We now generalise the notion of a "solution" to a diagram. To help keep the various objects distinct, we will usually use the letters $\alpha, \beta, \gamma$ to indicate objects of our indexing category $\mathsf{J}$.

DEFINITION 16.3. Let $\mathsf{J}$ be an index category and let $T\colon \mathsf{J} \to \mathsf{C}$ be a diagram in $\mathsf{C}$. A **solution**[2] for $T$ is an object $C$ of $\mathsf{C}$ together with a family of morphisms $c_\alpha\colon T(\alpha) \to C$ in $\mathsf{C}$ for each object $\alpha \in \mathrm{obj}(\mathsf{J})$ such that if $i\colon \alpha \to \beta$ is any morphism in $\mathsf{J}$ then the following commutes:

$$\begin{tikzcd} T(\alpha) \arrow[r, "c_\alpha"] \arrow[d, "T(i)"'] & C \\ T(\beta) \arrow[ur, "c_\beta"'] & \end{tikzcd}$$

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[1] This means that $\mathrm{obj}(\mathsf{J})$ is a set. It does *not* necessarily imply that $\mathsf{J}$ is actually "small" (since sets can be very large!) Nevertheless, in most our examples $\mathsf{J}$ is indeed rather small; for instance our running example (Example 16.2) has $\mathsf{J}$ having three objects and two non-identity morphisms.

[2] This is usually called a "co-cone" but I prefer the name "solution".

We write $(C, \{c_\alpha\})$ to indicate the solution.

Example 16.4. Let us stick with the setup from Example 16.2. In this case a solution is simply an object $D$ of $\mathsf{C}$ together with three morphisms $g = c_\spadesuit \colon A \to D$, $g_1 = c_\heartsuit \colon B_1 \to D$ and $g_2 = c_\diamondsuit \colon B_2 \to D$ such that the following commutes:

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dr, "g"] & B_1 \arrow[d, "g_1"] \\
B_2 \arrow[r, "g_2"'] & D
\end{tikzcd}$$

In fact, we don't need to specify the morphism $g$, since the commutativity requirement means that $g = g_1 \circ f_1 = g_2 \circ f_2$. So a solution is simply an object $D$ of $\mathsf{C}$ together with two morphisms $g_1 \colon B_1 \to D$ and $g_2 \colon B_2 \to D$ such that the following commutes:

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "g_1"] \\
B_2 \arrow[r, "g_2"'] & D
\end{tikzcd}$$

This recovers the notion of solution as given in Definition 5.7.

Now let us appropriately generalise the pushout construction.

Definition 16.5. Let $\mathsf{J}$ be an index category and let $T \colon \mathsf{J} \to \mathsf{C}$ be a diagram in $\mathsf{C}$. A **colimit** is a solution $(L, \{l_\alpha\})$ that satisfies the following *universal property*: if $(C, \{c_\alpha\})$ is any other solution then there exists a *unique* morphism $u \colon L \to C$ such that the following diagram commutes for every morphism $i \colon \alpha \to \beta$ in $\mathsf{J}$:

$$\begin{tikzcd}
 & T(\beta) \arrow[d, "l_\beta"] \arrow[ddr, "c_\beta"] & \\
T(\alpha) \arrow[ur, "T(i)"] \arrow[r, "l_\alpha"'] \arrow[drr, "c_\alpha"'] & L \arrow[dr, dashed, "u"] & \\
 & & C
\end{tikzcd}$$

Example 16.6. Going back to Example 16.2, a colimit is simply a pushout in the sense of Definition 5.7: a solution $(L, l_1, l_2)$ such that for any other solution $(D, g_1, g_2)$ there is a unique map $u \colon L \to D$ such that the following commutes:

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "l_1"] \arrow[ddr, "g_1"] & \\
B_2 \arrow[r, "l_2"'] \arrow[drr, "g_2"'] & L \arrow[dr, dashed, "u"] & \\
 & & D
\end{tikzcd}$$

By now you should be completely happy with proving that colimits are unique if they exist. But since I'm exceedingly generous, I will do it for you.

LEMMA 16.7. *Let* $\mathsf{J}$ *be an index category and let* $T\colon \mathsf{J} \to \mathsf{C}$ *be a diagram in* $\mathsf{C}$. *If a colimit exists then it is unique up to isomorphism.*

*Proof.* If $(L, \{l_\alpha\})$ and $(L', \{l'_\alpha\})$ are two limits then we get unique morphisms $u\colon L \to L'$ and $u'\colon L' \to L$ such that the following both commute for every morphism $i\colon \alpha \to \beta$ in $\mathsf{J}$:

$$\begin{tikzcd} & T(\beta) \arrow[d, "l_\beta"] \arrow[ddr, "l'_\beta"] & \\ T(\alpha) \arrow[ur, "T(i)"] \arrow[r, "l_\alpha"'] \arrow[drr, "l'_\alpha"'] & L \arrow[dr, dashed, "u"] & \\ & & L' \end{tikzcd} \qquad \begin{tikzcd} & T(\beta) \arrow[d, "l'_\beta"] \arrow[ddr, "l_\beta"] & \\ T(\alpha) \arrow[ur, "T(i)"] \arrow[r, "l'_\alpha"'] \arrow[drr, "l_\alpha"'] & L' \arrow[dr, dashed, "u'"] & \\ & & L \end{tikzcd}$$

Then the composition $u' \circ u\colon L \to L$ and $u \circ u'\colon L \to L$ both make the following diagrams commute:

$$\begin{tikzcd} & T(\beta) \arrow[d, "l_\beta"] \arrow[ddr, "l_\beta"] & \\ T(\alpha) \arrow[ur, "T(i)"] \arrow[r, "l_\alpha"'] \arrow[drr, "l_\alpha"'] & L \arrow[dr, dashed, "u' \circ u"] & \\ & & L \end{tikzcd} \qquad \begin{tikzcd} & T(\beta) \arrow[d, "l'_\beta"] \arrow[ddr, "l_\beta"] & \\ T(\alpha) \arrow[ur, "T(i)"] \arrow[r, "l'_\alpha"'] \arrow[drr, "l_\alpha"'] & L' \arrow[dr, dashed, "u \circ u'"] & \\ & & L' \end{tikzcd}$$

But since $\mathrm{id}_L$ and $\mathrm{id}_{L'}$ also make these two diagrams commute respectively, we see by uniqueness that $u' \circ u = \mathrm{id}_L$ and $u \circ u' = \mathrm{id}_{L'}$. This completes the proof. ■

We usually write $\operatorname{colim} T$ to indicate the colimit. This notation is somewhat imprecise (as it should really include the maps $l_\alpha$), but we do it anyway.

Here is another example:

EXAMPLE 16.8. Take $\mathsf{J}$ to have exactly two objects and no morphisms (apart from the identity morphisms).

$$\spadesuit \qquad \heartsuit$$

Let $T\colon \mathsf{J} \to \mathsf{Sets}$ or $T\colon \mathsf{J} \to \mathsf{Groups}$. This type of colimit is called a **coproduct**. In the category $\mathsf{Sets}$, it is simply the disjoint union $T(\spadesuit) \sqcup T(\heartsuit)$. In the category of groups it is the free product $T(\spadesuit) * T(\heartsuit)$ (cf. Problem C.1.)

One can also use colimits to get the category theory analogue of an equivalence relation.

Example 16.9. Let $\mathsf{J}$ have two objects and two morphisms:

$$\begin{tikzcd} \spadesuit \arrow[r, bend left] \arrow[r, bend right] & \heartsuit \end{tikzcd}$$

A colimit for this $\mathsf{J}$ is called a **coequaliser** in $\mathsf{C}$. On Problem Sheet H you get to investigate what coequalisers are in $\mathsf{Sets}$, $\mathsf{Groups}$ and $\mathsf{Top}$.

Remark 16.10. Suppose $\mathsf{J}$ is an index category and $T\colon \mathsf{J} \to \mathsf{C}$ is a functor. Suppose $S\colon \mathsf{C} \to \mathsf{D}$ is another functor. Then $(S \circ T)\colon \mathsf{J} \to \mathsf{D}$ is a diagram in $\mathsf{D}$. Assume that both the colimits $\operatorname{colim} T$ and $\operatorname{colim}(S \circ T)$ exist (as objects of $\mathsf{C}$ and $\mathsf{D}$ respectively). Then we claim there is a natural morphism

$$u\colon \operatorname{colim}(S \circ T) \to S(\operatorname{colim} T).$$

Indeed, if $l_\alpha\colon T(\alpha) \to \operatorname{colim} T$ are the maps from $T$ being a colimit, then applying $S$ we get maps $S(l_\alpha) : ST(\alpha) \to S(\operatorname{colim} T)$. Moreover if $i\colon \alpha \to \beta$ is a morphism in $\mathsf{J}$ then the following commutes:

$$\begin{tikzcd} ST(\alpha) \arrow[r, "S(l_\alpha)"] \arrow[d, "ST(i)"'] & S(\operatorname{colim} T) \\ ST(\beta) \arrow[ur, "S(l_\beta)"'] & \end{tikzcd}$$

This shows that $(S(\operatorname{colim} T), \{S(l_\alpha)\})$ form a solution to the diagram $S \circ T$. Thus by the universal property of the colimit, we get a unique morphism

$$u\colon \operatorname{colim}(S \circ T) \to S(\operatorname{colim} T)$$

as claimed

Let us now discuss a refinement of the idea of a colimit.

Definition 16.11. Let $\mathsf{J}$ be a small category with $\mathsf{obj}(\mathsf{J}) \neq \emptyset$. We say that $\mathsf{J}$ is **filtered** if the following two properties hold:

1. If $\alpha, \beta \in \operatorname{obj}(\mathsf{J})$ then there exists $\gamma \in \operatorname{obj}(\mathsf{J})$ such that $\operatorname{Hom}(\alpha, \gamma) \neq \emptyset$ and $\operatorname{Hom}(\beta, \gamma) \neq \emptyset$.

$$\begin{tikzcd} \alpha \arrow[dr, dashed, "i"] & \\ & \gamma \\ \beta \arrow[ur, dashed, "j"'] & \end{tikzcd}$$

2. If $i, j\colon \alpha \to \beta$ are any two morphisms then there exists an object $\gamma$ of $\mathsf{J}$ and a morphism $k\colon \beta \to \gamma$ such that $k \circ i = k \circ j$.

$$\begin{tikzcd} & \beta \arrow[dr, dashed, "k"] & \\ \alpha \arrow[ur, "i"] \arrow[dr, "j"'] & & \gamma \\ & \beta \arrow[ur, dashed, "k"'] & \end{tikzcd}$$

EXAMPLE 16.12. Let $(\Lambda, \preceq)$ be a **directed set**. This means that $\preceq$ is a binary relation on $\Lambda$ which is *reflexive* and *transitive*, which has the additional property that for all $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. From $(\Lambda, \preceq)$ we can form a filtered category $\mathsf{J}(\Lambda, \preceq)$ as follows:

- Take $\operatorname{obj}(\mathsf{J}) = \Lambda$.
- Set $\operatorname{Hom}(\alpha, \beta) = \emptyset$ if $\alpha \not\preceq \beta$, and if $\alpha \preceq \beta$ let $\operatorname{Hom}(\alpha, \beta)$ consist of a single element $i_{\alpha,\beta}$.
- If $\alpha \preceq \beta \preceq \gamma$ then define $i_{\beta,\gamma} \circ i_{\alpha,\beta} := i_{\alpha,\gamma}$.

DEFINITION 16.13. Let $\mathsf{C}$ be a category. A **filtered diagram** in $\mathsf{C}$ is a functor $T \colon \mathsf{J} \to \mathsf{C}$ where $\mathsf{J}$ is a filtered index category. A **filtered colimit** in $\mathsf{C}$ is a colimit of a filtered diagram $T \colon \mathsf{J} \to \mathsf{C}$. In this case we bung an arrow underneath and write $\underrightarrow{\operatorname{colim}}\, T$ instead of just $\operatorname{colim} T$.

In fact, in this course we will basically only ever need one type of filtered colimit (we will need the general case in Algebraic Topology II).

EXAMPLE 16.14. A **sequential colimit** is a filtered colimit on the directed set $(\mathbb{N}, \leq)$. Let us spell out what this means explicitly, since this is the most important type of filtered colimit. Let $\mathsf{C}$ be a category, and assume we are given a sequence

$$f_n \colon C_n \to C_{n+1}, \qquad n \in \mathbb{N},$$

of morphisms in $\mathsf{C}$. This data is equivalent to a filtered diagram $T \colon \mathsf{J}(\mathbb{N}, \leq) \to \mathsf{C}$: namely define $T(n) := C_n$ and for $n \leq m$ define

$$T(i_{n,m}) := f_{m-1,m} \circ f_{m-2,m-1} \circ \cdots \circ f_{n,n+1} \colon C_n \to C_m.$$

The filtered colimit of $T$ is an object $\underrightarrow{\operatorname{colim}}\, T$ (which we will usually write as $\underrightarrow{\operatorname{colim}}_n C_n$ instead) together with a family of morphisms $l_n \colon C_n \to \underrightarrow{\operatorname{colim}}_n C_n$ for $n \in \mathbb{N}$ such that

$$l_{n+1} \circ f_n = l_n, \qquad \forall\, n \in \mathbb{N}.$$

This satisfies the universal property that if $(D, \{d_n\})$ is object of $\mathsf{C}$ and a family of morphisms $d_n \colon C_n \to D$ for $n \in \mathbb{N}$ such that

$$d_{n+1} \circ f_n = d_n, \qquad \forall\, n \in \mathbb{N},$$

then there exists a *unique* morphism $u \colon \underrightarrow{\operatorname{colim}}_n C_n \to D$ such that the following diagram commutes:

$$\begin{tikzcd}
& C_{n+1} \arrow[d, "l_{n+1}"] \arrow[ddr, "d_{n+1}"] & \\
C_n \arrow[ur, "f_n"] \arrow[r, "l_n"'] \arrow[drr, "d_n"'] & \underrightarrow{\operatorname{colim}}_n C_n \arrow[dr, dashed, "u"] & \\
& & D
\end{tikzcd} \qquad \forall\, n \in \mathbb{N}.$$

Remark 16.15. For the rest of this lecture I will work with a general filtered colimit. If you find the formalism confusing, I strongly urge you to rewrite everything out in the special case of a sequential colimit.

Let us now prove that filtered colimits exist in all our favourite categories. Actually this all works for arbitrary colimits (rather than filtered colimits) but the construction is rather messier. We begin with $\mathsf{Sets}$.

Example 16.16. Let $T\colon \mathsf{J} \to \mathsf{Sets}$ be a filtered diagram. Let us construct the filtered colimit. First, form the disjoint union

$$Z := \bigsqcup_{\alpha \in \mathsf{obj}(\mathsf{J})} T(\alpha).$$

We now define an equivalence relation $\sim$ on $Z$ by declaring that $a \in T(\alpha) \sim b \in T(\beta)$ if and only if there exists $i\colon \alpha \to \gamma$ and $j\colon \beta \to \gamma$ such that $T(i)a = T(j)b$:

$$a \sim b \qquad \Leftrightarrow \qquad T(i)a = T(j)b.$$

It is somewhat tedious to check that $\sim$ really is an equivalence relation on $Z$, and I will leave this you. Given this though, it is easy to see that $X := Z/\sim$ satisfies the universal property, where the maps $l_\alpha\colon T(\alpha) \to X$ are induced from the inclusions $T(\alpha) \to Z$.

Now let us check they exist in $\mathsf{Top}$.

Example 16.17. Colimits always exist in $\mathsf{Top}$. Suppose $T\colon \mathsf{J} \to \mathsf{Top}$ is a diagram. Let $F\colon \mathsf{Top} \to \mathsf{Sets}$ denote the "forgetful" functor (cf. Example 1.14.) Then $F \circ T$ is a diagram in $\mathsf{Sets}$. We just constructed the filtered colimit in $\mathsf{Set}$. Let us call this set $X$. In fact, $X$ will also work for the filtered colimit in $\mathsf{Top}$, once we give it a topology.

For this, note that the construction in the previous example provides us with functions $l_\alpha : F(T(\alpha)) \to X$ (here $F(T(\alpha))$ is just the underlying set of the topological space $T(\alpha)$.) Let us now endow $X$ with a topology by declaring that the functions $l_\alpha$ are continuous. Explicitly, this means that a set $U \subseteq X$ is open if and only if $l_\alpha^{-1}(U)$ is open in $T(\alpha)$ for each $\alpha \in \mathrm{obj}(\mathsf{J})$. Then the functions $l_\alpha\colon T(\alpha) \to X$ are now well-defined morphisms in $\mathsf{Top}$ (i.e. continuous functions), and it is easy to see that $(X, \{l_\alpha\})$ verifies the universal property. Thus $X = \varinjlim T$. Note that this is an example where the natural map from Remark 16.10 is an isomorphism

$$F(\varinjlim T) = \varinjlim(F \circ T)$$

(as they are both the set $X$.)

Remark 16.18. Without additional hypotheses on the types of topological spaces we are working with, the colimit can be rather badly behaved. Indeed, sometimes the topology one gets on $\varinjlim T$ can be "wrong". We will discuss this more in later lectures. Next semester, we will introduce **homotopy colimits** which are much better behaved in $\mathsf{Top}$.

Now we check that filtered colimits exist in $\mathsf{Ab}$.

EXAMPLE 16.19. Let $\mathsf{J}$ be a filtered index category and $T\colon \mathsf{J} \to \mathsf{Ab}$ be a filtered diagram. This time, we consider the forgetful functor $F\colon \mathsf{Ab} \to \mathsf{Sets}$. Our strategy is the same as before. We will endow the set $X = \varinjlim(F \circ T)$ from Example 16.16 with the structure of an abelian group. Given $a \in T(\alpha)$ and $b \in T(\beta)$, choose an object $\gamma$ such that there exist morphisms $i\colon \alpha \to \gamma$ and $j\colon \beta \to \gamma$. Then define

$$[a] + [b] := [T(i)a + T(j)b].$$

It is again slightly tedious to check this operation is well-defined[3], and I will lazily leave this to you. Once this is done though, the functions $l_\alpha\colon T(\alpha) \to X$ are well-defined morphisms in $\mathsf{Ab}$ (i.e. group homomorphisms), and it is easy to see that $(X, \{l_\alpha\})$ verifies the universal property. Thus $X = \varinjlim T$, and once again we have

$$F(\varinjlim T) = \varinjlim(F \circ T).$$

REMARK 16.20. Suppose $T\colon \mathsf{J} \to \mathsf{Ab}$ is a filtered diagram and $[a] \in \varinjlim T$ where $a \in T(\alpha)$. Then $[a] = 0 \in \varinjlim T$ *if and only if* there exists a morphism $i\colon \alpha \to \beta$ in $\mathsf{J}$ such that $T(i)a = 0 \in T(\beta)$. Here "if" is immediate from the definition, but "only if" requires a little bit of thought (which I invite you to do!)

In particular, if we restrict to sequential colimits, say $C = \varinjlim_n C_n$ where each $C_n$ is an abelian group, then an element $c \in C_k$ represents the zero element in $C$ if and only if there exists a *finite* $m \geq k$ such that $c$ is mapped onto the zero element in $C_m$.

Filtered colimits always exist in $\mathsf{Comp}$, too.

EXAMPLE 16.21. The same thing works $\mathsf{Comp}$. Let $\mathsf{J}$ be a filtered index category and let $T\colon \mathsf{J} \to \mathsf{Comp}$ be a filtered diagram. Thus for each $\alpha \in \mathrm{obj}(\mathsf{J})$ we get a chain complex $(T(\alpha)_\bullet, \partial^\alpha)$, and for each morphism $i\colon \alpha \to \beta$, we get a chain map $T(i)_\bullet\colon T(\alpha)_\bullet \to T(\beta)_\bullet$.

For each fixed $n \in \mathbb{Z}$, we get a functor $T_n\colon \mathsf{J} \to \mathsf{Ab}$ given by $T_n(\alpha) = T(\alpha)_n$. Since we already know filtered colimits exist in $\mathsf{Ab}$, this gives us abelian groups $\varinjlim T_n$ for each $n$. Denote by $l_n^\alpha\colon T(\alpha)_n \to \varinjlim T_n$ the map associated to this colimit. The boundary operators $\partial^\alpha$ induce maps

$$\begin{tikzcd} T(\alpha)_n \arrow[r, "\partial^\alpha"] \arrow[d, "l_n^\alpha"'] & T(\alpha)_{n-1} \arrow[d, "l_{n-1}^\alpha"] \\ \varinjlim T_n \arrow[r, "\partial"'] & \varinjlim T_{n-1} \end{tikzcd}$$

These operators square to zero, and hence we get a chain complex $\left(\varinjlim T_\bullet, \partial\right)$. This is the filtered colimit of $T$.

We conclude this lecture by proving the following rather abstract result, which roughly speaking says that the homology functor *commutes* with taking filtered colimits. This result only works for filtered colimits (rather than arbitrary colimits.)

[3]That is, if $a \sim a'$ and $b \sim b'$ then with the obvious notation we need $T(i)a + T(j)b \sim T(i')a' + T(j')b'$.

If $T\colon \mathsf{J} \to \mathsf{Comp}$ is a filtered diagram, for any $n \in \mathbb{Z}$ the composition $H_n \circ T\colon \mathsf{J} \to \mathsf{Ab}$ is a filtered diagram of abelian groups. In order to make the notation a little bit less horrendous, let us abbreviate $C_\bullet^\alpha := T(\alpha)_\bullet$ for $\alpha \in \mathrm{obj}(\mathsf{J})$. Thus $H_n \circ T$ is the diagram where

$$\alpha \mapsto H_n(C_\bullet^\alpha), \qquad \alpha \in \mathrm{obj}(\mathsf{J})$$

and

$$i \in \mathrm{Hom}(\alpha, \beta) \qquad \mapsto \qquad H_n(T(i))\colon H_n(C_\bullet^\alpha) \to H_n(C_\bullet^\beta).$$

We denote by $\varinjlim(H_n \circ T)$ the associated filtered colimit.

THEOREM 16.22. *Let $\mathsf{J}$ be a filtered index category and $T\colon \mathsf{J} \to \mathsf{Comp}$ a filtered diagram. Then for any $n \in \mathbb{Z}$, one has*

$$\varinjlim(H_n \circ T) = H_n\left(\varinjlim T\right).$$

*Proof.* We will prove that the natural map

$$u\colon \varinjlim(H_n \circ T) \to H_n\left(\varinjlim T\right). \tag{16.1}$$

from Remark 16.10 is an isomorphism. Suppose $z \in \varinjlim T$ is a cycle. Choose a representative $a \in C_n^\alpha$ for some $\alpha \in \mathrm{obj}(\mathsf{J})$. We don't necessarily know that $a$ is a cycle in $C_n^\alpha$, but since $z$ is a cycle, $a$ must become a cycle "eventually". In other words, there exists $\beta \in \mathrm{obj}(\mathsf{J})$ and a morphism $i\colon \alpha \to \beta$ such that $\partial^\beta(T(i)a) = 0$ in $C_{n-1}^\beta$. Thus $T(i)a$ defines an element in $\langle T(i)a\rangle \in H_n(C_\bullet^\beta)$. This element represents an element $x \in \varinjlim(H_n \circ T)$, and by construction, $u(x) = \langle z\rangle$. This shows that (16.1) is surjective.

Now let us prove that the map (16.1) is injective. Suppose an element $x$ in $\varinjlim(H_n \circ T)$ goes to zero under the map $u$. Choose a representative $\langle c\rangle \in H_n(C_\bullet^\alpha)$ of $x$. Let $c \in C_n^\alpha$ represent $\langle c\rangle$. Then $c$ also represents an element $z$ of $\varinjlim T$. By assumption $z$ is a boundary $\partial y$; let us now choose a representative $b \in C_{n+1}^\beta$ of $y$. Now since $\mathsf{J}$ is filtered, we can choose an object $\gamma \in \mathrm{obj}(\mathsf{J})$ and morphisms $i\colon \alpha \to \gamma$ and $j\colon \beta \to \gamma$. Then $T(i)c \in C_n^\gamma$ and $T(j)b \in C_{n+1}^\gamma$, and $\partial^\gamma(T(j)b) = T(i)c$. This means that $H_n(T(i))\langle c\rangle = \langle T(i)c\rangle = 0 \in H_n(C_\bullet^\gamma)$, and thus $\langle c\rangle$ represents the zero element in $\varinjlim(H_n \circ T)$. ■

LECTURE 17

# The Jordan-Brouwer Separation Theorem

We begin this lecture with a bit more abstract nonsense. We then use this to prove the famouse *Jordan-Brouwer Separation Theorem*.

Our first result shows that under favourable assumptions, the singular chain complex functor also commutes with colimits. Before proving this, let us talk a little bit about the different *separation axioms* a topological space can have.

DEFINITION 17.1. Let $X$ be a topological space. We say that:

- $X$ is a $T_1$ **space** if the points are closed in $X$.
- a **weakly Hausdorff space** if given every continuous map $f\colon K \to X$ from a compact Hausdorff space, $f(K)$ is closed in $X$.

The next two pieces of point-set topology are on Problem Sheet I.

LEMMA 17.2. *One has*

$$\{\text{Hausdorff spaces}\} \subsetneq \{\text{weakly Hausdorff spaces}\} \subsetneq \{T_1 \text{ spaces}\}.$$

LEMMA 17.3. *Let $X$ be a weakly Hausdorff space. Assume $f\colon K \to X$ is a continuous map from a compact Hausdorff space. Then $f(K)$ is a compact Hausdorff subspace of $X$ with respect to the subspace topology.*

REMARK 17.4. In algebraic topology, the weakly Hausdorff assumption is typically the most useful one to make. In fact, most of modern algebraic topology implicitly always works with the subcategory of $\mathsf{Top}$ of **compactly generated** spaces. A compactly generated space is (by definition) a weakly Hausdorff $k$-space (we will define $k$-spaces in Algebraic Topology II). This category is much more "convenient" than $\mathsf{Top}$ itself: it is large enough that only truly pathological topological spaces (that no self-respecting algebraic topologist would ever care about) fail to lie in it, and behaves nicely under various categorical operations. But we won't talk about this till next semester.

Let us consider sequential colimits of *embeddings* in $\mathsf{Top}$. Thus suppose we are given a family

$$i_n\colon X_n \to X_{n+1}, \qquad n \in \mathbb{N},$$

of continuous maps such that each $i_n$ is an *embedding* (this means that $i_n$ is a homeomorphism onto its image.) Replacing $X_n$ with the homeomorphic space $i_n(X_n)$, we may assume that $X_n \subseteq X_{n+1}$ and that $i_n$ is the inclusion. Then the sequential colimit $X := \varinjlim_n X_n$ is simply the union:

$$X = \bigcup_{n \in \mathbb{N}} X_n,$$

topologised so that a set $C \subseteq X$ is closed if and only if $C \cap X_n$ is closed for each $n$. However the topology on $X$ might be "wrong". Indeed, for each $n$ the colimit gives us a map $l_n \colon X_n \to X$. One would hope that these maps would also be embeddings (i.e. homeomorphisms onto their images), but in general this is not true. In particular, $X_n$ does *not* have to be closed in $X$.

If however we assume $i_n$ is a *closed* embedding (i.e. $i_n(X_n)$ is a closed subspace of $X_{n+1}$ for each $n \in \mathbb{N}$) then the map $l_n \colon X_n \to X$ is itself a closed inclusion[1], and hence also an embedding. In particular, $X_n$ is closed in $X$. If each $X_n$ is weakly Hausdorff then so is $X$: since if $f \colon K \to X$ is a continuous function from a compact Hausdorff space then $f(K) \cap X_n$ is closed for each $n$ as $X_n$ is weakly Hausdorff, and thus also closed in $X$ by definition of the colimit topology.

We now prove the following key result.

Proposition 17.5. *Suppose we are given a family $i_n \colon X_n \to X_{n+1}$ for $n \in \mathbb{N}$ of closed inclusions. Assume in addition that for each $n$ the space $X_n$ is weakly Hausdorff. Then*

$$\varinjlim_n C_\bullet(X_n) = C_\bullet(\varinjlim_n X_n).$$

*Proof.* The main step in the proof is the following claim.

Lemma 17.6. *Let $f \colon K \to X$ be a continuous map from a compact Hausdorff space $K$. Then $f(K)$ is contained in one of the $X_n$.*

*Proof.* Assume for contradiction the claim is false. Then for each $n \in \mathbb{N}$ we may select a point $x_n \in K$ such that $f(x_n) \notin X_n$. Let $S_m := \{f(x_n) \mid n \geq m\}$. Then $S_{m+1} \subset S_m$ for each $m \in \mathbb{N}$ and $\bigcap_m S_m = \emptyset$. Moreover $S_m$ meets $X_n$ in a finite set for each $n$, and thus since each $X_n$ is $T_1$ by Lemma 17.2, it follows that $S_m \cap X_n$ is a closed set in $X_n$ for each $n$. Thus by definition of the colimit topology on $X$, $S_m$ is closed in $X$ for each $m$. This means that if we set $Y_m := X \setminus S_m$ then $Y_m$ is open in $X$ for each $m$ and $\bigcup_m Y_m = X$. In particular, the $Y_m$'s form a cover of $f(K)$. But no finite subcover of them can cover $f(K)$, since any finite subcover is contained in the largest, and by construction $f(K)$ is not a subset of any of the $Y_m$. Thus $f(K)$ is not compact in $X$, which contradicts $K$ being compact and $f$ continuous. ∎

Going back to the proof of Proposition 17.5, it suffices now to observe that any singular simplex $\sigma \colon \Delta^m \to X$ is contained in some $X_n$ by Lemma 17.6. The result now follows immediately from the definition of the colimit in $\mathsf{Comp}$. ∎

The following corollary gives a topological version of Theorem 16.22. In the statement and proof, let us temporarily write $H_k^{\text{sing}}$ for the singular homology functor, which is the composition $H_k^{\text{sing}} = H_k \circ C_\bullet$ where $C_\bullet$ is the singular chain complex functor and $H_k \colon \mathsf{Comp} \to \mathsf{Ab}$ is the usual homology functor.

[1]Exercise: Why?

Corollary 17.7. *Suppose we are given a family $i_n \colon X_n \to X_{n+1}$ for $n \in \mathbb{N}$ of closed inclusions. Assume in addition that for each $n$ the space $X_n$ is weakly Hausdorff. Then for each $k \geq 0$, the singular homology groups satisfy*

$$H_k^{\text{sing}}(\underrightarrow{\text{colim}}_n X_n) = \underrightarrow{\text{colim}}_n H_k^{\text{sing}}(X_n).$$

*Proof.* By Proposition 17.5, $C_\bullet(\underrightarrow{\text{colim}}_n X_n) = \underrightarrow{\text{colim}}_n C_\bullet(X_n)$ (as chain complexes). By Theorem 16.22 we have $H_k(\underrightarrow{\text{colim}}_n C_\bullet(X_n)) = \underrightarrow{\text{colim}}_n H_k(C_\bullet(X_n))$. The latter is by definition the sequential colimit $\underrightarrow{\text{colim}}_n H_k^{\text{sing}}(X_n)$. ■

Remark 17.8. The fundamental group of $X$ is defined by looking at continuous maps $u \colon S^1 \to X$. Since $S^1$ is also a compact Hausdorff space, exactly the same argument as in the proof of Proposition 17.5 shows that under these hypotheses, one also has

$$\pi_1(X, p) = \underrightarrow{\text{colim}}_n \pi_1(X_n, p), \qquad \forall\, p \in X_1$$

(note that the right-hand side is a colimit in the category $\mathsf{Groups}$.)

Remark 17.9. In fact, in this lecture we will use an easier version of Proposition 17.5 and Corollary 17.7. Namely, both statements are still true if we assume instead that each $X_n \subset X$ is an *open* set. In fact, this argument is much easier (and does not require any separation axioms). Indeed, if each $X_n$ is open and $f \colon K \to X$ is a continuous map from any compact space, then $\{X_n\}$ is an open cover of the compact set $f(K)$, and hence there is a finite subcover. Thus $f(K)$ is contained in a single $X_n$, and so the analogue of Lemma 17.6 holds.

Next lecture when we discuss cell complexes we will need the harder version we proved above.

We now move towards proving the Jordan-Brouwer Separation Theorem. The first step is the following proposition.

Proposition 17.10. *Let $X \subset S^n$ be a subset homeomorphic to $I^m := [0,1]^m$ for $0 \leq m \leq n$. Then*

$$\tilde{H}_k(S^n \setminus X) = 0, \qquad \forall k \geq 0.$$

*Proof.* We argue by induction on $m$. If $m = 0$ then $X$ is a point and $S^n \setminus X \cong \mathbb{R}^n$, which has zero reduced homology by Corollary 12.20. Now assume the result is true for $m-1$. Let $f \colon X \to I^m$ be our given homeomorphism. Split the $m$-cube $I^m$ into its upper and lower halves:

$$I^+ = \left\{ (x_1, \dots, x_m) \mid x_1 \geq \frac{1}{2} \right\}, \qquad I^- := \left\{ (x_1, \dots, x_m) \mid x_1 \leq \frac{1}{2} \right\}.$$

Then $I^+ \cap I^-$ is homeomorphic to $I^{m-1}$. Let $X^\pm := f^{-1}(I^\pm)$ and let $Y := X^+ \cap X^-$ so that $Y \cong I^{m-1}$. The set $S^n \setminus Y$ may be written as the union of two sets $(S^n \setminus X^+) \cup (S^n \setminus X^-)$ which satisfy the requirements of the Mayer-Vietoris sequence. Fix $k \geq 0$. We get an exact sequence

$$\dots \tilde{H}_{k+1}(S^n \setminus Y) \to \tilde{H}_k(S^n \setminus X) \to \Big( \tilde{H}_k(S^n \setminus X^+) \oplus \tilde{H}_k(S^n \setminus X^-) \Big) \to \tilde{H}_k(S^n \setminus Y) \to \dots$$

By the inductive hypotheses the end terms are both zero. Thus we have an isomorphism

$$\tilde{H}_k(S^n \setminus X) \xrightarrow{\left(H_k(\imath^+),H_k(\imath^-)\right)} \left(\tilde{H}_k(S^n \setminus X^+) \oplus \tilde{H}_k(S^n \setminus X^-\right)$$

Suppose now we have a non-zero homology class $\langle c \rangle \in \tilde{H}_k(S^n \setminus X)$. Then at least one of $H_k(\imath^+)\langle c \rangle$ and $H_k(\imath^-)\langle c \rangle$ are non-zero, where $\imath^\pm \colon S^n \setminus X \hookrightarrow S^n \setminus X^\pm$ are inclusions. Assume without loss of generality that $H_k(\imath^+)\langle c \rangle \neq 0$. Now we repeat the process, splitting $X^+$ into two pieces whose intersection is homeomorphic to $I^{m-1}$. In this manner a sequence of closed subsets of $S^n$ may be constructed:

$$X = X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$$

having that the property that the inclusion $S^n \setminus X \hookrightarrow S^n \setminus X_j$ induces a homomorphism in homology that takes our non-zero homology class to a non-zero element $\langle c_j \rangle \in \tilde{H}_k(S^n \setminus X_j)$. Set $Z := \bigcap_j X_j$. Since $S^n \setminus X_j \subset S^n \setminus Z$ is open for each $j$, the hypotheses of Remark 17.9 are satisfied[2], and thus

$$\tilde{H}_k(S^n \setminus Z) = \underrightarrow{\operatorname{colim}}_j \tilde{H}_k(S^n \setminus X_j).$$

Moreover since the map $\tilde{H}_k(S^n \setminus X_j) \to \tilde{H}_k(S^n \setminus X_{j+1})$ sends $\langle c_j \rangle$ to $\langle c_{j+1} \rangle$, by definition of the filtered colimit we end up with a non-zero element in $\langle c_\infty \rangle \in \tilde{H}_k(S^n \setminus Z)$ (cf. Remark 16.20.)

Now we play the joker: being an infinite intersection of shrinking $m$-cubes, $Z$ is homeomorphic to $I^{m-1}$! Thus by the inductive hypothesis, $\tilde{H}_k(S^n \setminus Z) = 0$. This contradicts the existence of a non-zero class $\langle c \rangle \in \tilde{H}_k(S^n \setminus X)$, and thus we see that $\tilde{H}_k(S^n \setminus X) = 0$ as required. Since $k \geq 0$ was arbitrary, we are done. ■

We now use this to prove the following corollary.

COROLLARY 17.11. *Let $S \subset S^n$ be a subset which is homeomorphic to $S^m$ for some $0 \leq m \leq n-1$. If $m < n-1$ then*

$$H_k(S^n \setminus S) = \begin{cases} \mathbb{Z}, & k = 0 \text{ and } k = n-m-1, \\ 0, & \text{otherwise.} \end{cases}$$

*Meanwhile if $m = n-1$ then*

$$H_k(S^n \setminus S) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Once again we induct on $m$. For $m = 0$, $S$ is two points, and $S^n \setminus S \simeq S^{n-1}$. Since $S^{n-1}$ has the desired homology, the case $m = 0$ follows. Now assume the result is true for $m-1$. Write $S = B^+ \cup B^-$ where $B^\pm$ are homeomorphic to

[2] Strictly speaking we are working with reduced homology here, but this makes no difference, as the reader is invited to check (alternatively, prove the case $k = 0$ directly!)

closed hemispheres in $S^m$ and $R := B^+ \cap B^-$ is homeomorphic to $S^{m-1}$. Now the Mayer-Vietoris sequence $S^n \setminus R = (S^n \setminus B^+) \cup (S^n \setminus B^-)$ has the form

$$\begin{aligned}\left(H_{k+1}(S^n \setminus B^+) \oplus H_{k+1}(S^n \setminus B^-)\right) &\to H_{k+1}(S^n \setminus R)\\ &\to H_k(S^n \setminus S) \to \left(H_k(S^n \setminus B^+) \oplus H_k(S^n \setminus B^-)\right).\end{aligned}$$

For $k > 0$ the end terms are zero by Proposition 17.10. Thus we obtain an isomorphism $H_{k+1}(S^n \setminus R) \to H_k(S^n \setminus S)$, which allows us to complete the inductive step. ∎

We can now prove the following famous result.

THEOREM 17.12 (The Jordan-Brouwer Separation Theorem). *Suppose $f\colon S^{n-1} \to S^n$ is an embedding. Then $S^n \setminus f(S^{n-1})$ has two components, and $f(S^{n-1})$ is boundary of each component.*

*Proof.* Let $S = f(S^{n-1})$. Then by Corollary 17.11, $H_0(S^n \setminus S) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_k(S^n \setminus S) = 0$ for $k > 0$. Thus by the dimension axiom (Proposition 8.1) and Problem D.4, $S^n \setminus S$ has two path components. Since $S$ is closed, $S^n \setminus S$ is open, and hence in particular locally pathwise connected. Thus the path components agree with the connected components.

Let $X$ and $Y$ be the two components of $S^n \setminus S$. Since $X \cup S$ is closed, the boundary of $\partial X := \overline{X} \setminus X^\circ$ is contained in $S$. We claim that also $S \subseteq \partial X$, whence $S = \partial X$. Let $p \in S$ and let $U$ be a neighbourhood of $p$ in $S^n$. Since $S$ is an embedded copy of $S^{n-1}$, there is a subset $C$ of $U \cap S$ with $p \in C$ and $S \setminus C$ homeomorphic to $B^{n-1}$. See Figure 17.1.

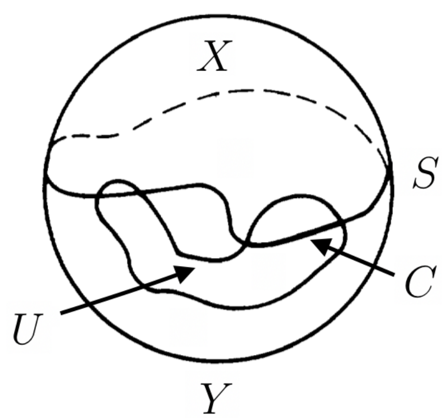


Figure 17.1: Proving the Jordan-Brouwer Separation Theorem.

Now by Proposition 17.10 and the dimension axiom, we see that $S^n \setminus (S \setminus C)$ has one path component. Suppose $x \in X$ and $y \in Y$. Then there is a path $u$ from $x$ to $y$ with image in $S^n \setminus (S \setminus C)$. Since $X$ and $Y$ are distinct path components of $S^n \setminus S$, the path $u$ must intersect $C$, and hence $C$ contains points belonging to both $\overline{X}$ and $\overline{Y}$. Thus $p \in \partial X$. Since $p$ was an arbitrary point of $S$, we have $S \subseteq \partial X$ and so also $S = \partial X$. Similarly $S = \partial Y$. The proof is complete. ∎

On Problem Sheet I, you get to prove the following equally famous result, also due to Brouwer.

Theorem 17.13 (Invariance of Domain Theorem). *Suppose $U$ and $U'$ are two subsets of $S^n$ and $f\colon U \to U'$ is a homeomorphism. If $U$ is open then so is $U'$.*

The theorem is of course obvious if "open" is replaced by "closed". Likewise the theorem is also clear if $f = g|_U$, where $g\colon S^n \to S^n$ is a homeomorphism of the entire sphere $S^n$ and $U' = g(U)$. However the theorem is not true for arbitrary spaces. For example, take $U = (1/2, 1]$ and $U' = (0, 1/2]$. Then $f\colon U \to U'$ given by $x - \frac{1}{2}$ is a homeomorphism and $U$ is open in $I$. But $U'$ is not. The proof of Theorem 17.13 is a simple application of the Jordan-Brouwer Separation Theorem 17.12, and the meaning of the name "invariance of domain" is explained by the following corollary (also on Problem Sheet I.)

Corollary 17.14. *If $\mathbb{R}^n$ contains a subspace homeomorphic to $\mathbb{R}^m$ then $m \leq n$.*

LECTURE 18

# Attaching spaces and cell complexes

In this lecture we introduce a nice class of topological spaces called **cell complexes**[1]. We begin by investigating pushouts in $\mathsf{Top}$.

Definition 18.1. Let $X$ and $Y$ be topological spaces, and let $X' \subseteq X$ be a *closed* subspace. Let $f\colon X' \to Y$ be continuous. We define the **adjunction space** $X \cup_f Y$ to be obtained by taking the disjoint union $X \sqcup Y$ and then identifying $x$ with $f(x)$ for all $x \in X'$. Slightly more formally, $X \cup_f Y$ is the space $(X \sqcup Y)/\sim$, where $\sim$ is the smallest equivalence relation (cf. the definition from the solution to Problem H.5) on $X \sqcup Y$ such that $x \sim f(x)$ for $x \in X'$.

The canonical inclusions $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$ induce maps $g\colon X \to X \cup_f Y$ and $j\colon Y \to X \cup_f Y$. The next piece of point-set topology is on Problem Sheet I.

Lemma 18.2. *The diagram*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, hook, "\imath"'] & Y \arrow[d, "j"] \\ X \arrow[r, "g"'] & X \cup_f Y \end{tikzcd}$$

*is a pushout in* $\mathsf{Top}$. *Moreover*

1. *The map $j$ is a closed embedding,*
2. *The restriction of $g$ to $X \setminus X'$ is an open embedding.*
3. *If $X$ and $Y$ are $T_1$ spaces then so is $X \cup_f Y$.*
4. *The quotient map $X \sqcup Y \to X \cup_f Y$ is closed if and only if $f$ is closed.*
5. *If $X$ and $Y$ are Hausdorff and $X' \subseteq X$ is compact then $X \cup_f Y$ is Hausdorff.*
6. *If $X$ is compact and $X \cup_f Y$ is Hausdorff then $X \mapsto g(X)$ is a quotient map.*

Informally, one should think of $X \cup_f Y$ as being obtained from $Y$ by attaching $X \setminus X'$ to it. We call $f$ the **attaching map** and $g$ the **characteristic map** of the adjunction space.

Example 18.3. Suppose $Y = \{*\}$ is a space with one point. Then there is only one map $f\colon X' \to \{*\}$ (for $X' \neq \emptyset$), and one has $X \cup_f \{*\} \cong X/X'$.

The following lemma is a partial converse to Lemma 18.2.

Will J. Merry, Algebraic Topology I, Autumn 2017, ETH Zürich. Last modified: January 02, 2018.

[1]They are often called *CW complexes* in the literature; the "C" stands for "closure finite" and the "W" stands for "weak topology". But I think the name "cell complex" is catchier.

Lemma 18.4. *Suppose we are given a commutative diagram of continuous maps:*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "j"] \\ X \arrow[r, "g"'] & Z \end{tikzcd}$$

*Assume that*[2]

1. *$i$ and $j$ are closed embeddings,*
2. *$g$ induces a bijection $X \setminus X' \to Z \setminus j(Y)$,*
3. *$g(X)$ is closed,*
4. *$X \to g(X)$ is a quotient map.*

*Then $(Z, j, g)$ is a pushout of $X \xleftarrow{i} X' \xrightarrow{f} Y$.*

*Proof.* We verify the universal property. Suppose we are given a topological space $W$ and continuous maps $h\colon X \to W$ and $k\colon Y \to W$ such that $h \circ i = k \circ f$. We need to build a unique continuous map $u\colon Z \to W$ such that the following commutes:

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "j"] \arrow[ddr, "k"] & \\ X \arrow[r, "g"] \arrow[drr, "h"'] & Z \arrow[dr, dashed, "u"] & \\ & & W \end{tikzcd}$$

It is clear that $Z$ is a *set-theoretic* pushout, so we get a unique map $u\colon Z \to W$ of *sets* that makes the diagram commute. It remains to show that $u$ is continuous. Since $j$ is a closed embedding, $u|_{j(Y)}$ is continuous. Since $g$ is a quotient map, $u|_{g(X)}$ is continuous. Since $g(X)$ and $j(Y)$ are closed sets that cover $Z$, $u$ is continuous. ∎

Definition 18.5. We denote $B^n \setminus S^{n-1}$ by $E^n$, so that $E^n$ is the *open* unit ball[3]. We call $E^n$ the **standard $n$-cell**. If $X$ is a topological space, a set $E \subseteq X$ which is homeomorphic to $E^n$ is called an **$n$-cell in X**. If $f\colon S^{n-1} \to Y$ is continuous, the space $B^n \cup_f Y$ is said to be obtained from $Y$ by **attaching an $n$-cell.**

Proposition 18.6. *Let $Z$ be a Hausdorff space and $Y$ a closed subset. Suppose there exists a continuous map $g\colon B^n \to Z$ which induces a homeomorphism $E^n \to Z \setminus Y$. Then $Z$ is obtained from $Y$ by attaching an $n$-cell.*

[2]Condition (4) is automatic given the others if $X$ is compact and $Z$ is Hausdorff, as in the last part of Lemma 18.2.

[3]If $n = 0$, $E^0$ is just a point since $S^0$ is two points.

*Proof.* It suffices to show that $g(S^{n-1}) \subseteq Y$. Then if we set $f := g|_{S^{n-1}}$ the hypotheses of Lemma 18.4 are satisfied. Then $Z$ is a pushout, and hence $Z \cong B^n \cup_f Y$ by uniqueness of the pushout. So suppose there exists $x \in S^{n-1}$ with $g(x) \in Z \setminus Y$. Since $g|_{E^n} \colon E^n \to Z \setminus Y$ is bijective, there exists a unique $y \in E^n$ with $g(x) = g(y)$. Let $U \subset B^n$ and $V \subset E^n$ be disjoint open neighbourhoods with $x \in U$ and $y \in V$. Then $g(V) \subset Z \setminus Y$ is open in $Z$, since $g|_{E^n} \colon E^n \to Z \setminus Y$ is a homeomorphism. But now using continuity of $g$, there exists an open neighbourhood $U' \subset U$ of $x$ such that $g(U') \subset g(V)$. This contradicts $g|_{E^n}$ being injective. ■

You will no doubt be surprised just how many of the "standard" spaces can be obtained by attaching cells. Let's see some examples, starting with a dumb one.

EXAMPLE 18.7. For all $n \geq 1$, $S^n$ is obtained by attaching an $n$-cell to a point. Indeed, this is simply Example 18.3 together with the observation that $B^n/S^{n-1} \cong S^n$, a fact that we have already used several times.

EXAMPLE 18.8. Recall

$$\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\}/\sim,$$

where $x \sim y$ if $x = ty$ for some $t \neq 0$. Restricting to vectors of length 1, we also see that

$$\mathbb{R}P^n = S^n/(x \sim -x),$$

that is, the sphere with antipodal points identified. But this is the same thing as taking the upper hemisphere and identifying antipodal points on the equator. Since the equator is just the sphere of dimension one less, we see that $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an $n$-cell. Explicitly, if $p \colon S^{n-1} \to \mathbb{R}P^{n-1}$ is the quotient map then

$$\mathbb{R}P^n = B^n \cup_p \mathbb{R}P^{n-1}.$$

This implies that we can write $\mathbb{R}P^n$ as a disjoint union

$$\mathbb{R}P^n = E_0 \cup E_1 \cup \cdots \cup E_n,$$

where each $E_i$ denotes an $i$-cell.

EXAMPLE 18.9. We can play a similar game with complex projective space $\mathbb{C}P^n$. This is the space of lines in $\mathbb{C}^{n+1}$ that go through the origin. Alternatively, $\mathbb{C}P^n = S^{2n+1}/\sim$, where two points $(z_1, \dots, z_{n+1}) \sim (w_1, \dots, w_{n+1})$ in $S^{2n+1} \subset \mathbb{C}^{n+1}$ are equivalent if and only if there exists $\lambda \in S^1$ such that $w_i = \lambda z_i$ for each $i = 1, \dots, n+1$. It follows that $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell, and hence we can write $\mathbb{C}P^n$ as a disjoint union

$$\mathbb{C}P^n = E_0 \cup E_2 \cup \cdots \cup E_{2n},$$

where each $E_{2i}$ denotes a $2i$-cell.

Our next example requires a definition first.

Definition 18.10. Let $X$ and $Y$ be topological spaces, and let $p \in X$ and $q \in Y$. Define the **wedge** of the pointed spaces $(X, p)$ and $(Y, q)$ to be

$$X \vee Y := \big(X \times \{q\}\big) \cup \big(\{p\} \times Y\big) \subseteq X \times Y.$$

This is again a pointed space, where the basepoint is $(p, q)$. This is actually the coproduct (cf. Example 16.8) in the category $\mathsf{Top}_*$.

On Problem Sheet I you get to prove the following result.

Proposition 18.11. *For any $m, n \geq 0$, the space $S^m \times S^n$ can be obtained from $S^m \vee S^n$ by attaching a $(m+n)$-cell.*

Note that one cannot use Corollary 17.7 to compute the homology groups of $X \cup_f Y$, since a pushout is *not* a filtered colimit. Nevertheless, when we are attaching cells (i.e. $X$ is a ball), we can use the Mayer-Vietoris sequence to compute the homology. Before stating the result, let us introduce a strengthening of the notion of a retract from Lecture 1.

Definition 18.12. Let $X' \subseteq X$ be a subspace. Let $\imath \colon X' \hookrightarrow X$ be the inclusion. We say that $X'$ is a **deformation retract** of $X$ if there exists a retract $r \colon X \to X'$ (as defined in Definition 1.2) such that $r \circ \imath = \mathrm{id}_{X'}$ and $\imath \circ r \simeq \mathrm{id}_X$. Equivalently, this means there exists a continuous function $H \colon X \times I \to X$ such that $H(x, 0) = x$ for all $x \in X$, $H(x, 1) \in X'$ for all $x \in X$, and $H(x', 1) = x'$ for all $x' \in X'$ (in this formulation, the retract $r$ is given by $H(\cdot, 1)$.)

If $X'$ is a deformation retract of $X$ then the retract $r$ is a homotopy equivalence, and hence $X'$ and $X$ have the same homotopy type. Next lecture we will introduce an even stronger version[4] called (rather unimaginatively) a *strong deformation retract.*

Proposition 18.13. *Let $Y$ be a Hausdorff topological space. Let $n \geq 1$, and suppose $f \colon S^{n-1} \to Y$ is continuous. Then if $j \colon Y \to B^n \cup_f Y$ denotes the inclusion, there is an exact sequence*

$$\cdots \to H_k(S^{n-1}) \xrightarrow{H_k(f)} H_k(Y) \xrightarrow{H_k(j)} H_k\big(B^n \cup_f Y\big) \to H_{k-1}(S^{n-1}) \to \dots$$

*which ends with*

$$\dots H_0(S^{n-1}) \to \mathbb{Z} \oplus H_0(Y) \to H_0\big(B^n \cup_f Y\big) \to 0.$$

*Proof.* Write $B^n \cup_f Y$ as $U \cup V$, where $U$ is the ball of radius $1/2$ inside $B^n$ and $V$ is $\big(B^n \cup_f Y\big) \setminus 0 \in B^n$. Then $U \cap V$ is homotopy equivalent to $S^{n-1}$ and $U$ is contractible. We claim that $V$ has $Y$ as a deformation retract. Indeed, define $H \colon V \times I \to V$ by

$$H(x,t) := \begin{cases} x, & \text{if } x \in Y, \\ g\big((1-t)z + tz/|z|\big), & \text{if } x = g(z) \in g(E^n). \end{cases} \tag{18.1}$$

[4]**Warning**: There is a discrepancy in the terminology here. Hatcher's textbook calls what we have called a deformation retract a "deformation retract in a weak sense", and he calls what we will call as strong deformation retract next lecture simply a "deformation retract".

$H$ is well-defined and continuous[5]. Denote by $h\colon U \cap V \hookrightarrow V$ the inclusion. The Mayer-Vietoris sequence then gives us a long exact sequence

$$\cdots \to H_k(U \cap V) \xrightarrow{H_k(h)} H_k(V) \to H_k\big(B^n \cup_f Y\big) \to H_{k-1}(U \cap V) \to \dots$$

After replacing $U \cap V$ with $S^{n-1}$ and $V$ with $Y$, we are almost done. It remains to see that we can identify $H_k(h)\colon H_k(U \cap V) \to H_k(V)$ with $H_k(f)\colon H_k(S^{n-1}) \to H_k(Y)$. For this we consider the commutative diagram:

$$\begin{tikzcd}
U \setminus \{0\} \arrow[r, "g"] \arrow[d, hook] & U \cap V \arrow[d, "h"] \\
B^n \setminus \{0\} \arrow[r] & V \\
S^{n-1} \arrow[u, hook] \arrow[r, "f"'] & Y \arrow[u, hook]
\end{tikzcd}$$

The unlabelled vertical maps all induce isomorphisms on homology, since the respective subspaces are deformation retracts. The top horizontal map, which is the restriction of $g$, also induces an isomorphism, since $g|_{U \setminus \{0\}}$ is a homeomorphism. Thus for any $k \geq 0$ we get a commutative diagram in homology where the vertical maps are isomorphisms:

$$\begin{tikzcd}
H_k(U \cap V) \arrow[r, "H_k(h)"] \arrow[d] & H_k(Y) \\
H_k(S^{n-1}) \arrow[r, "H_k(f)"'] & H_k(Y) \arrow[u]
\end{tikzcd}$$

Thus the map $H_k(h)$ can be identified with the map $H_k(f)$. This completes the proof. ■

If $n \geq 2$ we can say a little more:

Corollary 18.14. *Let $Y$ be a Hausdorff topological space. Let $n \geq 2$, and suppose $f\colon S^{n-1} \to Y$ is continuous. Then if $k \neq n-1, n$, one has*

$$H_k(Y) \cong H_k\big(B^n \cup_f Y\big),$$

*and there is an exact sequence*

$$0 \to H_n(Y) \xrightarrow{H_n(j)} H_n\big(B^n \cup_f Y\big) \to H_{n-1}(S^{n-1}) \xrightarrow{H_{n-1}(f)} H_{n-1}(Y) \to H_{n-1}\big(B^n \cup_f Y\big).$$

*If $n \geq 3$ then the last map $H_{n-1}(Y) \to H_{n-1}\big(B^n \cup_f Y\big)$ is a surjection.*

*Proof.* This is immediate from Proposition 18.13 and our computation of the homology of $S^{n-1}$ in Theorem 14.11. ■

[5] Continuity of $H$ is a little exercise using quotient maps.

We can also attach several cells at the same time.

Definition 18.15. Let $Y$ be topological space. We say that a topological space $Z$ is **obtained from $Y$ by attaching $n$-cells** if there exists a pushout

$$\begin{tikzcd} \coprod_{\lambda\in\Lambda} S^{n-1}_\lambda \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, "j"] \\ \coprod_{\lambda\in\Lambda} B^n_\lambda \arrow[r, "g"'] & Z \end{tikzcd}$$

Here the index $\lambda \in \Lambda$ just enumerates different copies of the same space. The map $j\colon Y \to Z$ is again a closed embedding, and $g$ induces a homeomorphism $\coprod_\lambda E^n_\lambda \to Z \setminus Y$. We set $f_\lambda := f|_{S^{n-1}_\lambda}$ and similarly $g_\lambda := g|_{B^n_\lambda}$. We call $g_\lambda$ the **characteristic map** of the $n$-cell $g(E^n_\lambda)$ and we call $f_\lambda$ its **attaching map**. Note that the definition still makes sense if our indexing set $\Lambda$ is empty; then of course $Y = Z$.

We can now define the titular cell complexes.

Definition 18.16. Let $X' \subseteq X$ be topological spaces such that $X'$ is closed in $X$. A **cellular decomposition** of the pair $(X, X')$ consists of a sequence of subspaces:

$$X' = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X$$

such that:

1. $X$ is the colimit of the $(X^n)$ in $\mathsf{Top}$. In other words, $X$ carries the colimit topology: a set $C \subseteq X$ is closed if and only if $C \cap X^n$ is closed for each $n \geq -1$.
2. For each $n \geq 0$, $X^n$ is obtained from $X^{n-1}$ by attaching $n$-cells[6].

We call the pair $(X, X')$ a **relative cell complex**. If $X' = \emptyset$ then $X$ is called a **cell complex** and we just write $X$ instead of $(X, \emptyset)$. The space $X^n$ is called the **$n$-skeleton** of $(X, X')$, and we call the decomposition $(X^n)$ for $n \geq -1$ the **skeleton filtration** of $(X, X')$. We say $(X, X')$ is **finite** (resp. **countable**) if $X \setminus X'$ consists of a finite (resp. countable) number of cells. If $X = X^n$ for some $n$ then the mininal such $n$ is called the **dimension** of $(X, X')$. Note that if $(X, X')$ is a relative cell complex then so is $(X, X^n)$ and $(X^n, X')$ for any $n \geq -1$.

A **subcomplex** of a cell complex $X$ is a subspace $X' \subseteq X$ with the property that for any cell $E$ of $X$, if $X' \cap E \neq \emptyset$ then $\overline{E} \subseteq X'$. In this case $(X, X')$ is a relative cell complex, and $X'$ itself is a cell complex whose cellular decomposition is inherited from $X$.

Proposition 18.17. *Let $(X, X')$ be a relative cell complex.*

1. *The inclusion $X^n \subset X^{n+1}$ is a closed embedding for all $n \geq -1$.*
2. *If $X'$ is a $T_1$ space then so is $X$, and a compact subset of $X$ only meets finitely many cells of $X$.Thus if $X'$ is $T_1$ then*

$$H_k(X) = \varinjlim_n H_k(X^n), \qquad \forall\, k \geq 0.$$

[6] If $X^{-1} = \emptyset$ then for $n = 0$ read this to mean: $X^0$ is a discrete set of points.

3. *If $X'$ is Hausdorff*[7] *then so is $X$.*
4. *If $X'$ is Hausdorff then $X$ also carries the colimit topology with respect to the family which consists of $X'$ and the closures of all the cells.*

*In particular, taking $X' = \emptyset$ we see that these properties always hold for a cell complex $X$.*

*Proof.* The first three points follow from Lemma 18.2, the proof of Lemma 17.3 and Corollary 17.7. Let us prove that (4). Suppose $C \subseteq X$ has the property that $C \cap X'$ is closed and $C \cap \overline{E}$ is closed for each cell $E$. We prove inductively that $C \cap X^n$ is closed for each $n$. This is true for $n = -1$ by assumption. The space $X^n$ is a quotient of

$$Y^n := X^{n-1} \sqcup \left( \bigsqcup_{\Lambda_n} B^n_\lambda \right),$$

where $\Lambda_n$ is the (possibly empty, possibly uncountable) set indexing the $n$-cells of $X$. Each characteristic map $g_\lambda \colon B^n_\lambda \to \overline{E}_\lambda$ is a quotient map since $X$ is Hausdorff (cf. part (6) of Lemma 18.2.) By assumption $X^n \cap C$ has a closed preimage in $Y^n$. Then by part (4) of Lemma 18.2, we see that $X^n \cap C$ is closed in $X^n$. ■

We conclude this lecture by explaining why cell complexes are so important. Let $X$ be a topological space and $p \in X$. In Algebraic Topology II, we will define the **higher homotopy groups** $\pi_n(X, p)$ for all $n \geq 0$ (we already did $n = 0$ in Lecture 3 and the fundamental group $n = 1$ in Lecture 4.) For $n \geq 2$, $\pi_n$ is a functor $\mathsf{hTop}_* \to \mathsf{Ab}$.

Definition 18.18. A continuous map $f \colon X \to Y$ is called a **weak homotopy equivalence** if the induced map $\pi_n(f) \colon \pi_n(X, p) \to \pi_n(Y, f(p))$ is an isomorphism for all $n \geq 0$ and all $p \in X$.

For now this definition won't mean much to you (since we haven't defined $\pi_n$ yet!) In general a weak homotopy equivalence is strictly weaker than an actual homotopy equivalence[8]. However a weak homotopy equivalence is still strong enough for all homology groups to coincide. Indeed, one of the axioms of a homology theory is that if $f \colon X \to Y$ is a weak homotopy equivalence then $H_n(f) \colon H_n(X) \to H_n(Y)$ is an isomorphism for each $n \geq 0$; more on this in Lecture 22. We will prove the following theorem in Algebraic Topology II.

Theorem 18.19. *Let $Y$ be any (!) topological space. Then there is a cell complex $X$ and a weak homotopy equivalence $f \colon X \to Y$.*

Theorem 18.19 implies that as far as homology is concerned, *all* spaces are cell complexes. If this isn't sufficient motivation to study cell complexes, I don't know what is!

[7] In fact, if $X'$ is *normal* then so is $X$.

[8] Although if $f \colon X \to Y$ is a weak homotopy equivalence between two connected cell complexes then $f$ is automatically a homotopy equivalence. This result is called **Whitehead's Theorem** and will be one of the major results we prove next semester.

LECTURE 19

# The Relative Homeomorphism Theorem

In this lecture we first complete the proof of a claim from Lecture 12: that if $(X, X')$ is a sufficiently "nice" pair then

$$H_n(X, X') \cong \tilde{H}_n(X/X'), \qquad \forall n \geq 0.$$

We will then prove that if $X$ is a cell complex and $X'$ is a subcomplex then the pair $(X, X')$ is always nice in this sense. Let us begin by specifying exactly what we mean by "nice".

DEFINITION 19.1. Let $X$ be a topological space and $X' \subseteq X$ be a subspace. Denote by $\imath \colon X' \hookrightarrow X$ the inclusion. We say that $X'$ is a **strong deformation retract** of $X$ if there exists a continuous map $r \colon X \to X'$ such that $r \circ \imath = \mathrm{id}_{X'}$ and $\imath \circ r \simeq \mathrm{id}_X$ rel $X'$. Thus a strong deformation retract is a deformation retract where the homotopy from $\imath \circ r$ to $\mathrm{id}_X$ can be chosen to be a homotopy which is relative to $X'$. Equivalently, this means there exists a continuous function $H \colon X \times I \to X$ such that $H(x, 0) = x$ for all $x \in X$, $H(x, 1) \in X'$ for all $x \in X$, and $H(x', t) = x'$ for all $x' \in X'$ and $t \in I$.

This really is a stronger condition, as you will see on Problem Sheet J.

THEOREM 19.2. *Let $X' \subset X$ be a closed subspace with the property that there exists a neighbourhood $U$ of $X'$ in $X$ such that $X'$ is a strong deformation retract of $U$. Let $\rho \colon X \to X/X'$ denote the quotient map, and denote by $*$ the point in $X/X'$ corresponding to $X'/X'$. Then for all $n \geq 0$, the map*

$$H_n(\rho) \colon H_n(X, X') \to H_n(X/X', *)$$

*is an isomorphism.*

Here is an example where the theorem is applicable.

EXAMPLE 19.3. Let $Y$ be a Hausdorff space and $f \colon S^{n-1} \to Y$ a continuous map. Let $j \colon Y \to B^n \cup_f Y$ denote the map induced from the inclusion $Y \hookrightarrow B^n \sqcup Y$, so that $j$ is a closed embedding (Lemma 18.2). Then $Y \cong j(Y) \subset B^n \cup_f Y$ satisfies the requirements of Theorem 19.2. Indeed, from the proof of Proposition 18.13, if $V := (B^n \cup_f Y) \setminus 0 \in B^n$, then $V$ is an open neighbourhood of $Y$ and (18.1) shows that $Y$ is a strong deformation retract of $V$.

*Proof of Theorem 19.2.* Let $U$ be as specified in the theorem, and denote by $\jmath\colon X' \hookrightarrow U$ the inclusion. Then we have a commutative diagram, where the horizontal maps come from the long exact sequence for pairs:

$$\begin{tikzcd} H_n(X') \arrow[r] \arrow[d, "H_n(\jmath)"'] & H_n(X) \arrow[r] \arrow[d, "H_n(\mathrm{id})"] & H_n(X,X') \arrow[r] \arrow[d, "H_n(\jmath)"] & H_{n-1}(X') \arrow[r] \arrow[d, "H_n(\jmath)"] & H_{n-1}(X) \arrow[d, "H_{n-1}(\mathrm{id})"] \\ H_n(U) \arrow[r] & H_n(X) \arrow[r] & H_n(X,U) \arrow[r] & H_{n-1}(U) \arrow[r] & H_{n-1}(X) \end{tikzcd}$$

Since $X'$ is a strong deformation retract of $U$, the left-hand $H_n(\jmath)$ and the right-hand $H_n(\jmath)$ are isomorphisms. Since $H_n(\mathrm{id})$ is certainly an isomorphism, the Five Lemma (Proposition 11.3) tells us that the middle $H_n(\jmath)$ is also an isomorphism. Next, since $\{*\}$ is a strong deformation retract of $U/X'$ in $X/X'$ (see Problem J.2), the same argument shows that the induced map $\bar{\jmath}\colon * \hookrightarrow U/X'$ induces an isomorphism

$$H_n(\bar{\jmath})\colon H_n(X/X',*) \to H_n(X/X',U/X')$$

for all $n \geq 0$. Now consider the following diagram:

$$\begin{tikzcd} & H_n(X,U) & \\ H_n(X,X') \arrow[ur, "H_n(\jmath)"] \arrow[d, "H_n(\rho)"'] & & H_n(X \setminus X', U \setminus X') \arrow[ul, "\text{excision}"'] \arrow[d, "H_n(\rho)"] \\ H_n(X/X',*) \arrow[dr, "H_n(\bar{\jmath})"'] & & H_n\big((X/X') \setminus \{*\}, (U/X') \setminus \{*\}\big) \arrow[dl, "\text{excision}"] \\ & H_n(X/X',U/X') & \end{tikzcd}$$

The right-hand side $H_n(\rho)$ is an isomorphism since $\rho$ is a homeomorphism away from $X'$. The two maps labelled "excision" are isomorphisms, and we just proved the two diagonal maps on the left-hand side are isomorphisms. Thus the left-hand $H_n(\rho)$ is also an isomorphism, which is what we wanted to prove. ■

Using Corollary 12.22, we immediately obtain the claim (12.1) made in Lecture 12:

COROLLARY 19.4. *Let $X' \subset X$ be a closed subspace with the property that there exists a neighbourhood $U$ of $X'$ in $X$ such that $X'$ is a strong deformation retract of $U$. Then*

$$H_n(X,X') \cong \tilde{H}_n(X/X'), \qquad \forall\, n \geq 0.$$

Here is another application. An arbitrary wedge sum is defined again as a coproduct[1].

[1]This means that the wedge sum is topologised as a quotient of the disjoint union.

Corollary 19.5. *Suppose $(X_\lambda, x_\lambda)$, $\lambda \in \Lambda$ is a collection of pointed spaces. Assume that each $x_\lambda$ has a neighbourhood $U_\lambda \subseteq X_\lambda$ for which $x_\lambda$ is a strong deformation retract of $U_\lambda$. Consider the wedge sum $\bigvee_{\lambda\in\Lambda} X_\lambda$ along the points $x_\lambda$ (cf. Definition 18.10). Then the inclusions $X_\lambda \hookrightarrow \bigvee_{\lambda\in\Lambda} X_\lambda$ induce an isomorphism*

$$\bigoplus_{\lambda\in\Lambda} \tilde{H}_n(X_\lambda) \cong \tilde{H}_n\left(\bigvee_{\lambda\in\Lambda} X_\lambda\right).$$

*Proof.* By assumption $\left(\bigsqcup_{\lambda\in\Lambda} X_\lambda, \bigsqcup_{\lambda\in\Lambda}\{x_\lambda\}\right)$ satisfies the hypotheses of Theorem 19.2. The claim them following from the definition of the wedge product and Corollary 12.22. ∎

Definition 19.6. Suppose $f\colon (X, X') \to (Y, Y')$ is a map of pairs. We say that $f$ is a **relative homeomorphism** if $f$ restricts to define a homeomorphism $f|_{X\setminus X'}\colon X \setminus X' \to Y \setminus Y'$.

With this terminology, the quotient map $\rho\colon X \to X/X'$ from Theorem 19.2 can also be thought of as a relative homeomorphism $(X, X') \to (X/X', *)$. We now prove a variant of Theorem 19.2.

Theorem 19.7 (The Relative Homeomorphism Theorem). *Let $f\colon (X, X') \to (Y, Y')$ be a relative homeomorphism. Assume that $X$ is compact and that $Y$ is compact Hausdorff, and that $X'$ and $Y'$ are closed in $X$ and $Y$ respectively. Assume further that there exists a neighbourhood $U$ of $X'$ in $X$ such that $X'$ is a strong deformation retract of $U$, and a neighbourhood $V$ of $Y'$ in $Y$ such that $Y'$ is a strong deformation retract of $V$. Then*

$$H_n(f)\colon H_n(X, X') \to H_n(Y, Y') \qquad \text{is an isomorphism for all } n \geq 0.$$

*Proof.* Denote by $\rho\colon X \to X/X'$ and $\rho'\colon Y \to Y/Y'$ the quotient maps. Then there is a well-defined continuous bijective map $f'\colon X/X' \to Y/Y'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho"'] & Y \arrow[d, "\rho'"] \\ X/X' \arrow[r, "f'"'] & Y/Y' \end{tikzcd}$$

Since $X/X'$ is compact and $Y/Y'$ is Hausdorff[2], the map $f'$ is a homeomorphism[3]. Passing to homology, for any $n \geq 0$ we obtain a commutative diagram:

$$\begin{tikzcd} H_n(X, X') \arrow[r, "H_n(f)"] \arrow[d, "H_n(\rho)"'] & H_n(Y, Y') \arrow[d, "H_n(\rho)'"] \\ H_n(X/X', *) \arrow[r, "H_n(f')"'] & H_n(Y/Y', *') \end{tikzcd}$$

[2] $Y/Y'$ is Hausdorff as $Y$ is compact and $Y'$ is closed.

[3] A continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism.

where we denoted by $*$ the point in $X/X'$ corresponding to $X'/X'$ and by $*'$ the point in $Y/Y'$ corresponding to $Y'/Y'$. By Theorem 19.2 the two maps $H_n(\rho)$ and $H_n(\rho')$ are isomorphisms, and $H_n(f')$ is an isomorphism as $f'$ is a homeomorphism. Thus so is the map $H_n(f)\colon H_n(X, X') \to H_n(Y, Y')$. ■

We now prove that Theorem 19.2 is always applicable to a pair $(X, X')$ where $X$ is a cell complex and $X'$ is a subcomplex. The first step is the following technical statement.

Proposition 19.8. *Let $X$ be a cell complex and $X'$ be a subcomplex. For each cell $E_\lambda$ in $X$ which is not in $X'$, choose a single point $x_\lambda \in E_\lambda$. Let*

$$Y^n := \{x_\lambda \mid E_\lambda \text{ is an } n\text{-cell in } X \text{ which is not in } X'\}.$$

*Regard $(X, X')$ as a relative cell complex with skeleton filtration $X' = X^{-1} \subset X^0 \subset X^1 \subseteq \cdots \subseteq X$. Then for every $n \geq 1$, $X^{n-1}$ is a strong deformation retract of $X^n \setminus Y^n$.*

*Proof.* We use the same idea as in (18.1) last lecture. Namely, without loss of generality we may assume that for each $n$-cell not in $X'$, the corresponding characteristic map $g_\lambda\colon B^n_\lambda \to X^n$ satisfies $g_\lambda(0) = x_\lambda$. Then we define $H\colon X^n \setminus Y^n \times I \to X^n \setminus Y^n$ by

$$H(x,t) := \begin{cases} x, & \text{if } x \in X^{n-1}, \\ g_\lambda\big((1-t)z + tz/|z|\big), & \text{if } x = g_\lambda(z) \text{ for } z \in E^n \setminus \{0\}. \end{cases} \tag{19.1}$$

We need only check that $H$ is continuous. Since $X^n$ has the colimit topology from $X'$ and the cells in $X$ not belonging to $X'$ (cf. the last part of Proposition 18.17), $X^n \setminus Y^n$ has the colimit topology determined by the cells in $X'$ and the punctured cells $E_\lambda \setminus \{x_\lambda\}$ for the cells in $X$ that are not in $X'$. It then follows that $X^n \setminus Y^n \times I$ has the colimit topology associated to the sets of the form $E' \times \{0\}$, $E' \times \{1\}$ and $E' \times (0,1)$, where $E'$ is either a cell in $X'$ or a punctured cell $E_\lambda \setminus \{x_\lambda\}$. The restriction of $H$ to any of these subsets is continuous (this is proved in the same way that we proved the map $H$ from (18.1) was continuous), and hence $H$ is continuous by definition of the colimit topology. ■

We now prove the desired result.

Theorem 19.9. *Let $X$ be a cell complex and $X'$ be a subcomplex. Then there exists an open set $U$ in $X$ containing $X'$ such that $X'$ is a strong deformation retract of $U$.*

*Proof.* Using the notation from the previous proposition, let $r_n\colon X^n \setminus Y^n \to X^{n-1}$ denote a strong deformation retract for $n \geq 1$. Set $U_0 = X'$ and set $U_n := r_n^{-1}(U_{n-1})$ for $n \geq 1$. Then each $U_n$ is open in $X^n \setminus Y^n$ and thus $U := \bigcup_n U_n$ is an open set in $X$ containing $X'$. Since the composition of two strong deformation retracts is itself a strong deformation retract, we see that $X'$ is a strong deformation retract of $U_n$ for each $n \geq 1$. This means that there exist continuous maps $F_n\colon U_n \times I \to U_n$ such that

$$F_n(x,0) = x, \qquad F_n(x,1) = r_1 r_2 \cdots r_n(x) \in X', \qquad \forall\, x \in U_n,$$

and such that $F_k(x', t) = x'$ for all $x' \in X'$ and $t \in I$. Moreover by induction we may even require that $F_{n+1}|_{U_n \times I} = F_n$. This means that we can define $F \colon U \times I \to U$ by setting $F = F_n$ on $U_n$. Then $F$ is continuous by definition of the colimit topology, and $F$ exhibits $X'$ as a strong deformation retract of $X$. ∎

Theorem 19.9 implies that for cell complexes, we can use subcomplexes instead of open sets for excision and the Mayer-Vietoris sequence.

COROLLARY 19.10. *Suppose $X$ is a cell complex and $X', X''$ are subcomplexes such that $X = X' \cup X''$. Then the inclusion $(X'', X' \cap X'') \hookrightarrow (X, X')$ induces isomorphisms on homology $H_n(X'', X' \cap X'') \to H_n(X, X')$ for all $n \geq 0$.*

*Proof.* We may assume $X' \cap X'' \neq \emptyset$, otherwise the result trivially follows from Proposition 8.2. Then the quotient spaces $X''/X' \cap X''$ and $X/X'$ are homeomorphic (they both can be identified with the cells in $X''$ that are not in $X'$). Then by Theorem 19.2 and Theorem 19.9, we obtain for any $n \geq 0$

$$H_n(X'', X' \cap X'') \cong \tilde{H}_n(X''/X' \cap X'') \cong \tilde{H}_n(X/X') \cong H_n(X, X').$$

∎

COROLLARY 19.11. *Suppose $X$ is a cell complex and $X', X''$ are subcomplexes such that $X = X' \cup X''$. Then the there is an exact sequence*

$$\dots H_n(X' \cap X'') \to H_n(X') \oplus H_n(X'') \to H_n(X) \to H_{n-1}(X' \cap X'') \to \dots$$

*Proof.* The Mayer-Vietoris sequence is a formal consequence of excision and the Baratt-Whitehead Lemma (Proposition 11.4). In other words, the proof of Theorem 14.9 goes through without any changes. ∎

LECTURE 20

# Cellular homology

In this lecture we introduce a new homology theory which is tailored specifically for cell complexes. *Cellular homology* is much more efficient than singular homology for computational purposes, as the cellular chain complex is much smaller. Indeed, a basis for the $n$th cellular chain group is given by the $n$-cells of the cell complex. For many spaces, the chain groups are then finitely generated abelian group. The resulting homology is the same as singular homology.

The key result that gets the construction of the cellular chain complex going is the following proposition.

PROPOSITION 20.1. *Let $X$ be a cell complex with skeleton filtration $\emptyset = X^{-1} \subset X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X$. Then for all $n \geq 0$,*

$$H_k(X^n, X^{n-1}) = 0, \qquad k \neq n.$$

*Meanwhile $H_n(X^n, X^{n-1})$ is free abelian with a basis in one-to-one correspondence with the $n$-cells of $X$.*

*Proof.* Suppose that $n$-cells of $X$ are given by maps $g_\lambda \colon (B^n, S^{n-1}) \to (X^n, X^{n-1})$, where $\lambda$ ranges over an index set $\Lambda_n$. Since $X^{n-1}$ is a subcomplex of $X^n$ for all $n \geq 1$, by Theorem 19.9 and Theorem 19.2 we see that

$$H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n/X^{n-1}), \qquad \forall\, k \geq 0.$$

But $X^n/X^{n-1}$ is a wedge sum of spheres, one for each of the $n$-cells of $X$. The claim then follows from Corollary 19.5. ■

We now step away from cell complexes for a moment and abstract the conclusion of Proposition 20.1 into a definition.

DEFINITION 20.2. Let $X$ be a topological space. A **cell-like filtration** $\mathcal{F}$ of $X$ is an expanding sequence of weakly Hausdorff closed subspaces $\emptyset = F^{-1} \subseteq F^0 \subseteq F^1 \subseteq F^2 \subseteq \cdots$ such that $X = \bigcup_{n \geq 0} F^n$ carries the colimit topology, and with the property that

$$H_k(F^n, F^{n-1}) = 0, \qquad \forall\, k \neq n.$$

Thus Proposition 20.1 tells us that the skeleton filtration associated to a cell complex is a cell-like filtration. However the notion of a cell-like filtration is much more general, since $F^n$ does not have to be obtained from $F^{n-1}$ by adjoining cells, and the topological space $X$ does not have to be Hausdorff. In this course we will have no need for the extra level of generality afforded by cell-like filtrations; nevertheless it makes[1] the proof of Theorem 20.5 below more transparent to work with cell-like filtrations, since it shows exactly what properties we need.

Our goal is to construct a new homology theory $H_\bullet(X, \mathcal{F})$ associated to a topological space with a cell-like filtration. In the special case where the space is a cell complex and the filtration is the skeleton filtration, this will be called the *cellular homology* of the cell complex. Let us first prove the following statement.

PROPOSITION 20.3. *Let $X$ be a topological space with cell-like filtration $\mathcal{F} = (F^n)$. Then $H_k(F^n) = 0$ for $k > n$ and the inclusion $F^n \hookrightarrow X$ induces an isomorphism $H_k(F^n) \to H_k(X)$ for $k < n$.*

*Proof.* We examine the long exact sequence of the pair $(F^n, F^{n-1})$. It contains the chain

$$H_{k+1}(F^n, F^{n-1}) \to H_k(F^{n-1}) \to H_k(F^n) \to H_k(F^n, F^{n-1}).$$

For $k \neq n, n-1$ the outer two groups are zero by assumption, and hence $H_k(F^{n-1}) \cong H_k(F^n)$ for $k \neq n, n-1$. In particular, if $k > n$ then

$$H_k(F^n) \cong H_k(F^{n-1}) \cong \cdots \cong H_k(F^0) = H_k(F^0, F^{-1}) = 0,$$

The same argument shows that the maps $H_k(F^{k+1}) \to H_k(F^{k+2}) \to H_k(F^{k+3}) \to \dots$ are all isomorphisms. Thus

$$\varinjlim_{n \geq k+1} H_k(F^n) = H_k(F^{k+1}).$$

But by Corollary 17.7, the colimit on the left-hand side is just the homology $H_k(X)$. This completes the proof. ■

We now define our new chain complex.

DEFINITION 20.4. Let $X$ be a topological space with cell-like filtration $\mathcal{F}$. We define a chain complex $\big(C_\bullet(X, \mathcal{F}), \partial^{\mathcal{F}}\big)$ as follows. Firstly, set $C_n(X, \mathcal{F}) = 0$ for $n < 0$. Given $n \geq 0$, define $C_n(X, \mathcal{F})$ to be the abelian group

$$C_n(X, \mathcal{F}) := H_n(F^n, F^{n-1}).$$

Let

$$j_n \colon (F^n, \emptyset) \hookrightarrow (F^n, F^{n-1})$$

denote the inclusion, and abbreviate by

$$\eta_n = H_n(j_n) \colon H_n(F^n) \to H_n(F^n, F^{n-1}).$$

For $n \geq 1$ we define[2] the boundary operator $\partial^{\mathcal{F}} \colon C_n(X, \mathcal{F}) \to C_{n-1}(X, \mathcal{F})$ as the composition

$$H_n(F^n, F^{n-1}) \xrightarrow{\delta_n} H_{n-1}(F^{n-1}) \xrightarrow{\eta_{n-1}} H_{n-1}(F^{n-1}, F^{n-2}),$$

[1] To me at least...

[2] For $n = 0$ the boundary operator is of course the zero map.

where $\delta_n$ is the connecting homomorphism for the long exact sequence of the pair $(F^n, F^{n-1})$. This does indeed define a chain complex as for $n \geq 1$ the composition $\partial^{\mathcal{F}} \circ \partial^{\mathcal{F}} : C_{n+1}(X, \mathcal{F}) \to C_{n-1}(X, \mathcal{F})$ is given by

$$H_{n+1}(F^{n+1}, F^n) \to H_n(F^n) \xrightarrow{\eta_n} H_n(F^n, F^{n-1}) \xrightarrow{\delta_n} H_{n-1}(F^{n-1}) \to H_{n-1}(F^{n-1}, F^{n-2})$$

and this is zero since the composition in the middle

$$H_n(F^n) \xrightarrow{\eta_n} H_n(F^n, F^{n-1}) \xrightarrow{\delta_n} H_{n-1}(F^{n-1})$$

is zero as these are two adjacent maps in the long exact sequence of the pair $(F^n, F^{n-1})$. We denote the associated homology by $H_n(X, \mathcal{F}) := H_n\big(C_\bullet(X, \mathcal{F}), \partial^{\mathcal{F}}\big)$.

Here is main result of today's lecture.

THEOREM 20.5. *Let $X$ be a topological space with cell-like filtration $\mathcal{F} = (F^n)$. Then for every $n \geq 0$, one has*

$$H_n(X) \cong H_n(X, \mathcal{F}).$$

*Proof.* Let us break with out longstanding convention and temporarily give the boundary operator $\partial^{\mathcal{F}}$ a subscript (otherwise the chain of equations below makes no sense.) We have the following commuting diagram, where the row is part of the long exact sequence of the pair $(F^n, F^{n-1})$, the left-hand column is part of the long exact sequence of the pair $(F^{n+1}, F^n)$, and the right-hand column is part of the long exact sequence of the pair $(F^{n-1}, F^{n-2})$. The left-most and the top-right zero entries follow from the Proposition 20.3, and the bottom zero entry is from the definition of a cell-like filtration.

$$\begin{tikzcd}
 & H_{n+1}(F^{n+1}, F^n) \arrow[d, "\delta_{n+1}"'] \arrow[dr, "\partial^{\mathcal{F}}_{n+1}"] & & 0 \arrow[d] \\
0 \arrow[r] & H_n(F^n) \arrow[r, "\eta_n"] \arrow[d] & H_n(F^n, F^{n-1}) \arrow[r, "\delta_n"] \arrow[dr, "\partial^{\mathcal{F}}_n"'] & H_{n-1}(F^{n-1}) \arrow[d, "\eta_{n-1}"] \\
 & H_n(F^{n+1}) \arrow[d] & & H_{n-1}(F^{n-1}, F^{n-2}) \\
 & 0 & &
\end{tikzcd}$$

Using this diagram, we now argue as follows:

$$\begin{aligned}
H_n(X) &\cong H_n(F^{n+1}) && \text{by Proposition 20.3,} \\
&\cong H_n(F^n)\big/\operatorname{im}\delta_{n+1} && \text{by exactness of first column,} \\
&\cong \operatorname{im}\eta_n\big/\operatorname{im}\eta_n\delta_{n+1} && \text{as } \eta_n \text{ is an injection,} \\
&\cong \ker\delta_n\big/\operatorname{im}\eta_n\delta_{n+1} && \text{by exactness of the row,} \\
&\cong \ker\delta_n\big/\operatorname{im}\partial^{\mathcal{F}}_{n+1} && \text{by commutativity of the top triangle,} \\
&\cong \ker\eta_{n-1}\delta_n\big/\operatorname{im}\partial^{\mathcal{F}}_{n+1} && \text{as } \eta_{n-1} \text{ is an injection,} \\
&\cong \ker\partial^{\mathcal{F}}_n\big/\operatorname{im}\partial^{\mathcal{F}}_{n+1} && \text{by commutativity of the bottom triangle,} \\
&= H_n(X, \mathcal{F}) && \text{by the definition of homology.}
\end{aligned}$$

This completes the proof. ■

Remark 20.6. The isomorphism $H_n(X) \cong H_n(X, \mathcal{F})$ can be written down explicitly. This is the subject of Problem J.5 on Problem Sheet J.

Now, with this detour out of the way, let us go back to cell complexes. If $X$ is a cell complex with skeleton filtration $\mathcal{F} = (X^n)$ then we will use the notation $\big(C^{\mathrm{cell}}_\bullet(X), \partial^{\mathrm{cell}}\big)$ instead of $\big(C_\bullet(X, \mathcal{F}), \partial^{\mathcal{F}}\big)$ for the chain complex associated to the skeleton filtration, and by $H^{\mathrm{cell}}_\bullet(X)$ the homology. We call $C^{\mathrm{cell}}_\bullet(X)$ the **cellular chain complex** and we call $H^{\mathrm{cell}}_\bullet(X)$ the **cellular homology** of the cell complex $X$.

Using Proposition 20.1, we have

$$C^{\mathrm{cell}}_n(X) = H_n(X^n, X^{n-1}),$$

and thus we see that the cellular chain group is free abelian with *generators in a one-to-one correspondence with the $n$-cells of $X$.*

Remark 20.7. The chain complex $C^{\mathrm{cell}}_\bullet(X)$ depends on not only the space $X$ but *also* the choice of cellular decomposition. The same topological space can have different cellular decompositions. For example, $S^3$ has a cellular decomposition with one 0-cell and one 3-cell (cf Example 18.7), but it also has a more complicated one (cf. Problem K.2.) However Theorem 20.5 implies that the *homology* $H^{\mathrm{cell}}_\bullet(X)$ does *not* depend on the choice of cellular decomposition (since it agree with the singular homology.)

The following result applies to nearly all of the spaces we will ever meet in this course.

Corollary 20.8. *Let $X$ be a compact cell complex of dimension $n$. Then*

1. *If $X$ has $N_k$ cells of dimension $k$ then $H_k(X)$ has rank at most $N_k$. In particular, $H_k(X)$ is finitely generated for all $k$.*
2. *$H_k(X) = 0$ for all $k > n$.*
3. *$H_n(X)$ is free abelian.*

*Proof.* Since $X$ is compact, it is necessarily a finite cell complex. The first two statements are clear for the cellular homology groups $H^{\mathrm{cell}}_k(X)$, and hence also for the singular homology groups $H_k(X)$ by Theorem 20.5. The last statement follows as $H^{\mathrm{cell}}_n(X) = \ker \partial^{\mathrm{cell}} : C^{\mathrm{cell}}_n(X) \to C^{\mathrm{cell}}_{n-1}(X)$ is a subgroup of $C^{\mathrm{cell}}_n(X)$, and a subgroup of a free abelian group is necessarily free abelian. ■

Here is another immediate corollary.

Corollary 20.9. *Let $X$ be a cell complex. Suppose $X$ has $N$ cells in dimension $n$, and no cells in dimension $n-1$ and $n+1$. Then $H_n(X) \cong \mathbb{Z}^N$.*

*Proof.* We necessarily have $C^{\mathrm{cell}}_{n+1}(X) = C^{\mathrm{cell}}_{n-1} = 0$, and thus $H^{\mathrm{cell}}_n(X) \cong C^{\mathrm{cell}}_n(X)$. ■

Corollary 20.9 gives us a new (and much quicker) way to compute the homology of $S^n$ for $n \geq 2$, of $S^m \times S^n$ for $m, n \geq 2$, and for $\mathbb{C}P^n$ for all $n \geq 0$. Note however we still *cannot* compute the homology of $\mathbb{R}P^n$.

To rectify this, we need to have a way of computing the boundary operator $\partial^{\mathrm{cell}}$. We conclude this lecture by giving an explicit formula for $\partial^{\mathrm{cell}}$.

So let $X$ be a cell complex as before, and fix $n \geq 1$. To help keep the notation transparent, in the following we will use the letter $\lambda$ to index the $n$-cells of $X$ and the letter $\nu$ to index the $(n-1)$-cells. Given an $n$-cell $E_\lambda$, we denote by $e_\lambda$ the corresponding generator in $C_n^{\mathrm{cell}}(X)$, and similarly $e_\nu \in C_{n-1}^{\mathrm{cell}}(X)$ is the generator corresponding to an $(n-1)$-cell.

The idea behind the cellular boundary formula is that we can get an $(n-1)$-sphere from both an $n$-cell and an $(n-1)$-cell. Indeed, if $g_\lambda \colon B^n \to X^n$ is an $n$-cell then $f_\lambda := g_\lambda \colon S_\lambda^{n-1} := \partial B_\lambda^n \to X^{n-1}$ is a map from an $(n-1)$-cell. Meanwhile if $g_\nu \colon B_\nu^{n-1} \to X^{n-1}$ is an $(n-1)$-cell, then if we collapse $f_\nu(\partial B_\nu^{n-1})$ to a point, the quotient space

$$S_\nu^{n-1} := g_\nu(B_\nu^{n-1}) / f_\nu(\partial B_\nu^{n-1})$$

is also an $(n-1)$-sphere.

Let us denote by

$$\rho \colon X^{n-1} \to X^{n-1}/X^{n-2}$$

the quotient map, and denote by

$$q_\nu \colon X^{n-1}/X^{n-2} \to S_\nu^{n-1}$$

the quotient map that collapses all the other $(n-1)$-spheres in $X^{n-1}/X^{n-2}$ to a point.

Definition 20.10. Suppose $e_\lambda$ is an $n$-cell and $e_\nu$ is an $(n-1)$-cell. We define a map

$$h_{\lambda,\nu} \colon S_\lambda^{n-1} \to S_\nu^{n-1}$$

to be the composition:

$$S_\lambda^{n-1} \xrightarrow{f_\lambda} X^{n-1} \xrightarrow{\rho} X^{n-1}/X^{n-2} \xrightarrow{q_\nu} S_\nu^{n-1}.$$

We then define the integer

$$[e_\lambda \colon e_\nu] := \deg(h_{\lambda,\nu}),$$

where we are using Definition 15.2. If $* \in X^{n-1}/X^{n-2}$ denotes the point corresponding to $X^{n-2}$, then since $g_\lambda(B_\lambda^n)$ intersects at most finitely many cells (part (2) of Proposition 18.17), for fixed $e_\lambda$, the map $h_{\lambda,\nu}$ is not the constant map $S_\lambda^{n-1} \to *$ for at most finitely many $e_\nu$.

Theorem 20.11 (The cellular boundary formula). *The boundary operator $\partial^{\mathrm{cell}} \colon C_n^{\mathrm{cell}}(X) \to C_{n-1}^{\mathrm{cell}}(X)$ is given explicitly by the formula*

$$\partial^{\mathrm{cell}} e_\lambda = \sum_\nu [e_\lambda \colon e_\nu] \cdot e_\nu.$$

Note the right-hand side is a well-defined element of $C^{\mathrm{cell}}_{n-1}(X)$ since $h_{\lambda,\nu}$ is non-constant (and hence has non-zero degree) for at most finitely many $e_\nu$.

*Proof.* Let $j\colon (X^{n-1}, \emptyset) \hookrightarrow (X^{n-1}, X^{n-2})$ denote the inclusion. Consider the following diagram, where the two maps labelled $\delta, \delta'$ are the connecting homomorphisms coming from the long exact sequence in reduced homology:

$$\begin{tikzcd}
H_n(B^n_\lambda, S^{n-1}_\lambda) \arrow[r, "\delta'"] \arrow[d, "H_n(g_\lambda)"'] & \tilde{H}_{n-1}(S^{n-1}_\lambda) \arrow[r, "H_{n-1}(h_{\lambda,\nu})"] \arrow[d, "H_{n-1}(f_\lambda)"] & \tilde{H}_{n-1}(S^{n-1}_\nu) \\
H_n(X^n, X^{n-1}) \arrow[r, "\delta"] \arrow[dr, "\partial"] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "H_{n-1}(\rho)"] \arrow[d, "H_{n-1}(j)"] & \tilde{H}_{n-1}\big(X^{n-1}/X^{n-2}\big) \arrow[u, "H_{n-1}(q_\nu)"'] \\
& H_{n-1}(X^{n-1}, X^{n-2}) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

The left-hand square commutes by naturality of the long-exact sequence in reduced homology, the right-hand square commutes by definition of $h_{\lambda,\nu}$ and the fact that $\tilde{H}_{n-1}$ is a functor. The left-hand triangle commutes by definition of $\partial$, and the right-hand triangle commutes from the proof of Theorem 19.2. The map labelled $\cong$ is an isomorphism from Theorem 19.2 as well. Let $\langle c\rangle$ be the generator of $H_n(B^n_\lambda, S^{n-1}_\lambda) \cong \mathbb{Z}$ such that $e_\lambda = H_n(g_\lambda)\langle c\rangle$. Then by commutativity,

$$\partial e_\lambda = H_{n-1}(j) \circ H_{n-1}(f_\lambda) \circ \delta' \langle c\rangle.$$

In terms of the basis of $H_{n-1}(X^{n-1}, X^{n-2})$ given by $(n-1)$-cells, the map $H_{n-1}(q_\nu)$ is the projection of $\tilde{H}_{n-1}\big(X^{n-1}/X^{n-2}\big)$ onto the $\mathbb{Z}$-summand corresponding to $e_\nu$. Thus commutativity of the right-hand square tells us that if $\partial e_\lambda = \sum_\nu m_{\lambda,\nu}\, e_\nu$ for $m_{\lambda,\nu} \in \mathbb{Z}$ then

$$m_{\lambda,\nu} = \deg(h_{\lambda,\nu}).$$

This completes the proof. ■

We will shortly use this to calculate the homology of $\mathbb{R}P^n$. First, however, we need another formula for computing the degree of a map from the sphere to itself. Suppose

$$f\colon (X,p) \to (Z,z), \qquad g\colon (Y,q) \to (Z,z)$$

are two pointed continuous maps. Then there is a well defined pointed continuous map

$$f \vee g\colon X \vee Y \to Z$$

defined by

$$(f \vee g)(x,y) := \begin{cases} f(x), & y = q, \\ g(y), & x = p. \end{cases}$$

Now consider the continuous maps

$$\mathrm{Pinch}\colon S^n \to S^n \vee S^n, \qquad \mathrm{Fold}\colon S^n \vee S^n \to S^n$$

that "pinch" and "fold" the sphere as in Figure 20.1. Explicitly, $\mathrm{Fold} = \mathrm{id} \vee \mathrm{id}$.

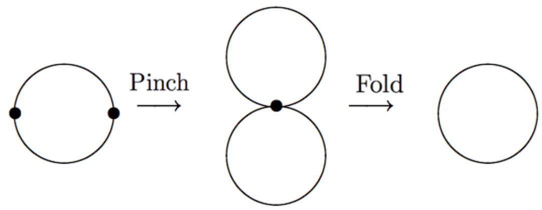


Figure 20.1: Pinching and folding.

LEMMA 20.12. *Let $f, g\colon S^n \to S^n$ be continuous maps. Let $h\colon S^n \to S^n$ denote the composition*

$$h = \text{Fold} \circ (f \vee g) \circ \text{Pinch}.$$

*Then*

$$\deg(h) = \deg(f) + \deg(g).$$

*Proof.* Let $\langle c \rangle, \langle c' \rangle$ denote elements of $H_n(S^n)$. Then one readily checks that

$$H_n(\text{Pinch})\langle c \rangle = (\langle c \rangle, \langle c \rangle),$$

and

$$H_n(f \vee g)\big(\langle c \rangle, \langle c' \rangle\big) = \big(H_n(f)\langle c \rangle, H_n(g)\langle c' \rangle\big),$$

and finally

$$H_n(\text{Fold})\big(\langle c \rangle, \langle c' \rangle\big) = \langle c \rangle + \langle c' \rangle.$$

The claim follows. ■

Here is the promised calculation of the homology of $\mathbb{R}P^n$.

COROLLARY 20.13. *The homology of the real projective space is given by*

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{if } k = 0 \text{ or } k = n \text{ and } n \text{ is odd,} \\ \mathbb{Z}_2, & \text{for odd } 0 < k < n, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* The composition we need to look at is

$$S^{n-1} \xrightarrow{f} \mathbb{R}P^{n-1} \xrightarrow{q} \mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} = S^{n-1}.$$

The map $q \circ f$ is a homeomorphism when restricted to each component of $S^{n-1} \setminus S^{n-2}$, and these two homeomorphisms are obtained from each other by precomposing with the antipodal map $a\colon S^{n-1} \to S^{n-1}$. This has degree $(-1)^n$ by Corollary 15.7. Thus by Lemma 20.12

$$\begin{aligned} \deg(q \circ f) &= \deg(\text{Fold} \circ (\text{id} \vee a) \circ \text{Pinch}) \\ &= \deg(\text{id}) + \deg(a) \\ &= 1 + (-1)^n. \end{aligned}$$

Thus the cellular chain complex for $\mathbb{R}P^n$ is given by

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0, \qquad \text{if } n \text{ is even,}$$
$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0, \qquad \text{if } n \text{ is odd.}$$

The result now follows from the fact that $H_\bullet^{\mathrm{cell}}(\mathbb{R}P^n) = H_\bullet(\mathbb{R}P^n)$. ■

A more involved application of the cellular boundary formula is on Problem Sheet K.

LECTURE 21

# Natural transformations and the Eilenberg-Steenrod Axioms

We begin this lecture by finally define "naturality" properly.

Definition 21.1. Let $\mathsf{C}$ and $\mathsf{D}$ be two categories, and let $S, T\colon \mathsf{C} \to \mathsf{D}$ be two functors. A **natural transformation** $\Phi\colon S \to T$ is a family of morphisms $\Phi(C)\colon S(C) \to T(C)$ for each $C \in \operatorname{obj}(C)$ such that for any morphism $f\colon A \to B$ in $\mathsf{C}$ the following diagram commutes:

$$\begin{array}{ccc} S(A) & \xrightarrow{\Phi(A)} & T(A) \\ {\scriptstyle S(f)}\downarrow & & \downarrow{\scriptstyle T(f)} \\ S(B) & \xrightarrow[\Phi(B)]{} & T(B) \end{array}$$

If each morphism $\Phi(C)$ is an isomorphism then we say that $\Phi$ is a **natural isomorphism**.

If $\Psi\colon R \to S$ and $\Phi\colon S \to T$ are two natural transformations then there it is easy to check that there is a well-defined natural transformation

$$\Phi \circ \Psi\colon R \to T, \qquad (\Phi \circ \Psi)(C) := \Phi(C) \circ \Psi(C).$$

Given any functor $T$, there is a well-defined natural transformation $\mathrm{id}_T\colon T \to T$ given by $\mathrm{id}_T(C) = \mathrm{id}_{T(C)}$ for each object $C \in \mathsf{C}$. The following easy lemma is on Problem Sheet K

Lemma 21.2. *Let $\mathsf{C}$ and $\mathsf{D}$ be two categories and $S, T\colon \mathsf{C} \to \mathsf{D}$ two functors. Suppose $\Phi\colon S \to T$ is a natural transformation. Then $\Phi$ is a natural isomorphism if and only if there is a natural transformation $\Psi\colon T \to S$ such that $\Psi \circ \Phi = \mathrm{id}_S$ and $\Phi \circ \Psi = \mathrm{id}_T$.*

By now you can probably guess what's coming next. Natural transformations "look" like morphisms between functors, and that means time for a new category. Let us suggestively write

$$\operatorname{Nat}(S, T) := \{\text{natural transformations } \Phi\colon S \to T\}.$$

We would like to define a new category called the **functor category** whose objects are all the functors from one category to another, and whose morphisms are the natural transformations between the functors. Unfortunately we run into a set-theoretic bug! Recall that in the definition of a category in Lecture 1, we required the Hom-sets to be actual sets. However sadly $\operatorname{Nat}(S, T)$ need not be a set. Worse, it does not even have to be a class. The following result is not hard to prove, but isn't particularly relevant to our purposes, so I'll just state it.

Will J. Merry, Algebraic Topology I, Autumn 2017, ETH Zürich. Last modified: January 01, 2018.

Proposition 21.3. *Let* $\mathsf{C}$ *be a small category (i.e.* $\mathrm{obj}(C)$ *is a set). Then for any category* $\mathsf{D}$ *and any two functors* $S, T\colon \mathsf{C} \to \mathsf{D}$*,* $\mathrm{Nat}(S, T)$ *is a set.*

This means that we can formally only define the functor category when the domain category is small.

Definition 21.4. Let $\mathsf{C}$ be a small category, and let $\mathsf{D}$ be an arbitrary category. The **functor category** $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is the category with:

- $\mathrm{obj}(\mathsf{Fun}(\mathsf{C}, \mathsf{D}))$ the class of all functors $T\colon \mathsf{C} \to \mathsf{D}$.
- $\mathrm{Hom}_{\mathsf{Fun}(\mathsf{C},\mathsf{D})}(S, T) = \mathrm{Nat}(S, T)$,
- composition is given by composition of natural transformations.

This is well-defined due to the preceding proposition.

Let us give a concrete example of formulating a (non-trivial) result you probably all already know in the language of natural isomorphisms.

Theorem 21.5. *A finite-dimensional vector space is naturally isomorphic to its double dual.*

What exactly does this mean? For simplicity, lets work with real vector spaces. If $V$ is a real vector space then $V^* := \mathrm{Hom}(V, \mathbb{R})$ denotes the set of all linear functionals $V \to \mathbb{R}$. More categorically, if $\mathsf{Vect}$ is the category of all vector spaces, then we can think of $V \mapsto V^* = \mathrm{Hom}(V, \mathbb{R})$ as a functor $\mathrm{Hom}(\square, \mathbb{R})\colon \mathsf{Vect} \to \mathsf{Vect}$ (since $\mathbb{R}$ is itself a real vector space.) Actually this is not quite true: this functor *reverses* the direction of morphisms (compare Problem K.5.) Indeed, a morphism in $\mathsf{Vect}$ is a linear transformation $A\colon V \to W$ between two vector spaces. $A$ induces a map $A^*$ between the dual spaces, but it goes the "wrong" way round: $A^*\colon W^* \to V^*$. Explicitly, if $\lambda \in W^*$ (so $\lambda\colon W \to \mathbb{R}$ is a linear functional) then $A^*\lambda \in V^*$ is defined by

$$A^*\lambda(v) := \lambda(Av), \qquad \forall\, v \in V.$$

Next semester we will study the idea of functors going the "wrong way round" in detail when we study *cohomology*. For now though, let us side-step this issue by applying the functor twice. So let $T\colon \mathsf{Vect} \to \mathsf{Vect}$ denote the functor

$$T(V) := \mathrm{Hom}(\mathsf{Hom}(V, \mathbb{R}), \mathbb{R}).$$

One usually denotes $T(V)$ by $V^{**}$ and calls it the *double dual*. If $A\colon V \to W$ is a linear map then $T(A) : T(V) \to T(W)$ is the linear map usually written as $A^{**}\colon V^{**} \to W^{**}$ and defined by

$$A^{**}(\varphi)(\lambda) = \varphi(A^*\lambda) \qquad \varphi \in V^{**}, \lambda \in W^*.$$

As you probably remember from linear algebra, there is a map $\mathrm{ev}_V\colon V \mapsto V^{**}$ called *evaluation* that simply evaluates a linear functional at a vector:

$$\mathrm{ev}_V(v)(\mu) := \mu(v), \qquad \mu \in V^*.$$

We claim that ev is a natural transformation from the identity functor on $\mathsf{Vect}$ to $T$. This comes down to showing that the following diagram commutes for any pair of vector spaces $V, W$ and any linear map $A\colon V \to W$:

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}_V"] \arrow[d, "A"'] & V^{**} \arrow[d, "A^{**}"] \\ W \arrow[r, "\mathrm{ev}_W"'] & W^{**} \end{tikzcd}$$

This is trivial: take $\lambda \in W^*$ and observe:

$$A^{**}\mathrm{ev}_V(v)(\lambda) = \mathrm{ev}_V(v)(A^*\lambda) = A^*\lambda(v) = \lambda(Av) = \mathrm{ev}_{Av}(\lambda).$$

The map $\mathrm{ev}_V$ is easily seen to be an injection $V \to V^{**}$ but in general if $V$ is infinite-dimensional then $\dim V^* > \dim V$ and hence $\mathrm{ev}_V$ is not an isomorphism. But if $V$ is finite-dimensional then hopefully you all know how to prove that $\dim V = \dim V^*$ and thus in this case $\mathrm{ev}_V\colon V \to V^{**}$ is an isomorphism. Hence by definition ev is a *natural isomorphism* when restricted to the subcategory $\mathsf{FiniteVect}$ of finite-dimensional vector spaces, and this is precisely the statement of Theorem 21.5.

Here are a few more examples of things that we have already proved are natural transformations.

PROPOSITION 21.6. *Let $C_\bullet\colon \mathsf{Top} \to \mathsf{Comp}$ denote the singular chain functor, and given $n \geq 0$ let $C_n\colon \mathsf{Top} \to \mathsf{Ab}$ denote the singular chain functor restricted to a single $C_n$. Then:*

1. *The boundary operator in singular homology is a natural transformation $C_n \to C_{n-1}$.*
2. *The barycentric subdivision operator $\mathrm{Sd}\colon C_\bullet \to C_\bullet$ is a natural transformation.*

*Proof.* The first statement is just Proposition 7.20 in fancy language. The second is is just (13.9). ■

Recall for any topological space $X$ and any $p \in X$, one has $\tilde{H}_1(X) \cong H_1(X, p)$, cf. Corollary 12.22.

PROPOSITION 21.7. *Regard $\pi_1$ and $\tilde{H}_1$ as functors $\mathsf{Top}_* \to \mathsf{Groups}$ (i.e. forget that $\tilde{H}_1(X)$ is abelian). Then the Hurewicz map defines a natural transformation $\pi_1 \to \tilde{H}_1$.*

*Proof.* This is Problem E.2. ■

PROPOSITION 21.8. *Define a functor $R\colon \mathsf{Top}^2 \to \mathsf{Top}^2$ by $R(X, X') = (X', \emptyset)$. Then the connecting homomorphism $\delta$ of the long exact sequence of a pair defines a natural transformation $\delta\colon H_n \to H_{n-1} \circ R$.*

*Proof.* This is just the fact that the right-hand square commutes in the diagram of Proposition 12.3. ■

With the definition of natural transformations out of the way, we can finally introduce the famous *Eilenberg-Steenrod axioms* for a homology theory. Let $R\colon \mathsf{Top}^2 \to \mathsf{Top}^2$ be as in the previous proposition.

Definition 21.9 (The Eilenberg-Steenrod Axioms). A **homology theory** is a sequence $\mathcal{H}_n\colon \mathsf{Top}^2 \to \mathsf{Ab}$ of functors for $n \geq 0$ and a sequence $\delta = \delta_n\colon \mathcal{H}_n \to \mathcal{H}_{n-1} \circ R$ of natural transformations satisfying the following four axioms:

- **The homotopy axiom**: If $f, g\colon (X, X') \to (Y, Y')$ are homotopic mod $X'$ (as in Definition 12.25) then $\mathcal{H}_n(f) = \mathcal{H}_n(g)$ for all $n \geq 0$. Thus $\mathcal{H}_n$ factors to define functors $\mathsf{hTop}^2 \to \mathsf{Ab}$ for each $n \geq 0$.
- **The exact sequence axiom**: For every pair $(X, X')$ with inclusions $\imath\colon (X', \emptyset) \hookrightarrow (X, \emptyset)$ and $\jmath\colon (X, \emptyset) \hookrightarrow (X, X')$, there is an exact sequence
$$\cdots \to \mathcal{H}_n(X') \xrightarrow{\mathcal{H}_n(\imath)} \mathcal{H}_n(X) \xrightarrow{\mathcal{H}_n(\jmath)} \mathcal{H}_n(X, X') \xrightarrow{\delta} \mathcal{H}_{n-1}(X') \to \dots,$$
where we abbreviate $\mathcal{H}_n(X) = \mathcal{H}_n(X, \emptyset)$ etc.
- **The excision axiom**: For every pair $(X, X')$ and every subset $U \subseteq X$ with $\overline{U} \subset (X')^\circ$, the inclusion $(X \setminus U, X' \setminus U) \hookrightarrow (X, X')$ induces an isomorphism
$$\mathcal{H}_n(X \setminus U, X' \setminus U) \cong \mathcal{H}_n(X, X'), \qquad \forall\, n \geq 0.$$
- **The dimension axiom**: If $\{*\}$ is a one-point space then $\mathcal{H}_n(*) = 0$ for all $n > 0$ and $\mathcal{H}_0(*) = \mathbb{Z}$.

There are additionally two "optional" axioms:

- **The additivity axiom**: Let $(X_\lambda, X'_\lambda)$, $\lambda \in \Lambda$ be a family of pairs of spaces. Denote by
$$\imath_\lambda\colon (X_\lambda, X'_\lambda) \hookrightarrow \left( \coprod_{\lambda\in\Lambda} X_\lambda, \coprod_{\lambda\in\Lambda} X'_\lambda \right)$$
the inclusion. Then for all $n \geq 0$, the map
$$\sum_{\lambda\in\Lambda} \mathcal{H}_n(\imath_\lambda)\colon \bigoplus_{\lambda\in\Lambda} \mathcal{H}_n(X_\lambda, X'_\lambda) \to \mathcal{H}_n\left( \coprod_{\lambda\in\Lambda} X_\lambda, \coprod_{\lambda\in\Lambda} X'_\lambda \right).$$
is an isomorphism.
- **The weak equivalence axiom**: If $f\colon (X, X') \to (Y, Y')$ is a weak equivalence (cf. Definition 18.18) then $\mathcal{H}_n(f)\colon \mathcal{H}_n(X, X') \to \mathcal{H}_n(Y, Y')$ is an isomorphism for all $n \geq 0$.

Some remarks:

1. The last two axioms are less important, and are both sometimes omitted from the treatment of axiomatic homology theory. The additivity axiom is implied by the excision axiom whenever $\Lambda$ is a finite set (see Problem K.6), and thus is only of interest for infinite disjoint unions.

2. The weak equivalence axiom is used only to ensure that a homology theory is uniquely determined by what it does to cell complexes, cf. Theorem 18.19. We will ignore this axiom for the time being, since we haven't defined the higher homotopy groups. In particular, we currently have no way of proving that singular homology is a homology theory (!) if we insist on this axiom, since we cannot verify that singular homology satisfies the weak equivalence axiom. This will be a major topic in Algebraic Topology II, and we will come back to it then.

3. As currently defined, cellular homology is also *not* a homology theory, since we have only defined cellular homology for cell complexes. Nevertheless, it can be made into a homology theory by arguing as follows. Let us denote by $\mathsf{Cell}$ the category of cell complexes, and denote by $I \colon \mathsf{Cell} \to \mathsf{Top}$ the inclusion functor. Using Theorem 18.19 together with another big result called the **Whitehead Theorem** (see the footnote just before Theorem 18.19), one constructs a functor $\Gamma \colon \mathsf{Top} \to \mathsf{Cell}$ that assigns to any space $X$ a cell complex $\Gamma(X)$, together with a natural transformation $\Phi \colon I \circ \Gamma \to \mathrm{id}_{\mathsf{Top}}$ such that $\Phi(X) \colon \Gamma(X) \to X$ is a weak equivalence. The functor $\Gamma$ is called a **cellular approximation functor**. Extending this to pairs of spaces, one then defines $H_n^{\mathrm{cell}} \colon \mathsf{Top}^2 \to \mathsf{Ab}$ by first applying $\Gamma$: $H_n^{\mathrm{cell}} \circ \Gamma$. The resulting sequence of homology functors is a genuine homology theory. Again, we will go over this construction precisely at the end of Algebraic Topology II next semester.

4. Many of the properties of singular homology continue to hold for an arbitrary homology theory. For instance, if $X$ is contractible then by the homotopy axiom and the dimension axiom one sees $\mathcal{H}_n(X) = 0$ for $n > 0$ and $\mathcal{H}_n(X) = \mathbb{Z}$ for $n = 0$. A more involved fact is that the Relative Homeomorphism Theorem 19.7 continues to hold; more on this later.

5. The exact sequence axiom implies the long exact sequence of a triple: if $X'' \subseteq X' \subseteq X$ are subspaces then there is a long exact sequence

$$\dots \mathcal{H}_n(X', X'') \to \mathcal{H}_n(X, X'') \to \mathcal{H}_n(X, X') \xrightarrow{\delta'} \mathcal{H}_{n-1}(X', X'') \to \dots,$$

where the undecorated maps are induced by inclusion and the map $\delta'$ is the composition

$$\mathcal{H}_n(X, X') \xrightarrow{\delta} \mathcal{H}_{n-1}(X') \to \mathcal{H}_{n-1}(X', X'')$$

Indeed, this follows from the solution of Problem F.4, since the solution given there (using the commutative braid of Problem F.3) used nothing other than the exact sequence axiom.

6. Both of the homology theories (singular and cellular) that we have constructed have arisen from first defining a functor $\mathsf{Top}^2 \to \mathsf{Comp}$ and then composing this with the functor $H_n \colon \mathsf{Comp} \to \mathsf{Ab}$ that takes the homology of the chain complex. However this is *not* part of the axioms, and any proof we gave for singular/cellular homology that used this are therefore *not* valid for an arbitrary homology theory. This is the reason we proved Problem F.4 using the commutative braid (as remarked at the end of the solution to Problem F.4,

there is a much quicker proof that is valid only for homology theories that are the homology of a chain complex.)

7. The only axiom that guarantees non-triviality of $\mathcal{H}_n$ is the dimension axiom, which at least tells us that the zeroth homology of a point is non-zero. Without this axiom, a perfectly valid theory would be $\mathcal{H}_n \equiv 0$. Nevertheless, there are many examples of things that one would like to be a "homology theory" that do *not* satisfy the dimension axiom. Two examples are *topological $K$-theory* and *symplectic homology*[1]. Thus a **generalised homology theory** is a sequence $(\mathcal{H}_\bullet, \delta)$ that satisfies all the axioms apart from the dimension axiom.

Now let us define what it means for two homology theories to be the same.

DEFINITION 21.10. Let $(\mathcal{H}_\bullet, \delta)$ and $\mathcal{K}_\bullet, \varepsilon)$ be two homology theories. A **natural transformation** $\Phi_\bullet \colon (\mathcal{H}_\bullet, \delta) \to (\mathcal{K}_\bullet, \varepsilon)$ is a sequence of natural transformations $\Phi_n \colon \mathcal{H}_n \to \mathcal{K}_n$ for $n \geq 0$ such that the following diagram commutes for all $n \geq 1$ and all pairs $(X, X')$:

$$\begin{tikzcd} \mathcal{H}_n(X,X') \arrow[r, "\delta"] \arrow[d, "\Phi_n(X,X')"'] & \mathcal{H}_{n-1}(X') \arrow[d, "\Phi_{n-1}(X')"] \\ \mathcal{K}_n(X,X') \arrow[r, "\varepsilon"'] & \mathcal{K}_{n-1}(X') \end{tikzcd}$$

If $\Phi_n$ is a natural isomorphism for each $n$ then we say that the two homology theories $(\mathcal{H}_\bullet, \delta)$ and $\mathcal{K}_\bullet, \varepsilon)$ are **naturally isomorphic**.

It is now easy to formulate the big theorem.

THEOREM 21.11 (Existence and uniqueness of a homology theory). *Singular homology is a homology theory. Moreover if $(\mathcal{H}_\bullet, \delta)$ is any homology theory then $(\mathcal{H}_\bullet, \delta)$ is naturally isomorphic to singular homology.*

We can't really come close to proving this as currently stated. Indeed, as remarked above, we cannot even show existence, since we don't know that singular homology satisfies the weak equivalence axiom. If we drop the weak equivalence axiom then we have already shown that singular homology satisfies the other axioms. However the main tool needed to construct a natural isomorphism between singular homology and an arbitrary homology theory is the higher dimensional analogue of the Hurewicz Theorem 9.7, which we won't do until next semester.

Nevertheless, the techniques we have developed thus far in the course allow us to prove the following weaker version.

THEOREM 21.12 (Baby Uniqueness Theorem). *Suppose $(\mathcal{H}_\bullet, \delta)$ and $\mathcal{K}_\bullet, \varepsilon)$ satisfy the first four axioms (homotopy, exact sequence, excision and dimension) and suppose $\Phi_\bullet \colon (\mathcal{H}_\bullet, \delta) \to \mathcal{K}_\bullet, \varepsilon)$ is a sequence of natural transformations such that $\Phi_0(*) \colon \mathcal{H}_0(*) \to \mathcal{K}_0(*)$ is an isomorphism, where $*$ is a one-point space. Then*

$$\Phi_n(X, X') \colon \mathcal{H}_n(X, X') \to \mathcal{K}_n(X, X')$$

[1]The former is the content of next semester's student seminar entitled "Vector Bundles in Algebraic Topology". The latter is my favourite (generalised) homology theory.

*is an isomorphism for all pairs* $(X, X')$ *consisting of a finite cell complex* $X$ *and a subcomplex* $X'$.

In other words, if we *already* have a natural transformation between two homology theories, it suffices to check it's an isomorphism on a one-point space to conclude it's an isomorphism on any finite cell complex. Of course, by assumption one always has $\mathcal{H}_0(*) \cong \mathbb{Z} \cong \mathcal{K}_0(*)$, but the hypotheses of the theorem are asserting much more: that there exists a natural transformation between the two homology theories that realises this isomorphism. This theorem is not too hard to prove, and we will do so next lecture.

LECTURE 22

# Free chain complexes

In this lecture we first prove Theorem 21.12. Our proof will use the Relative Homeomorphism Theorem 19.7, which is valid for an arbitrary homology theory. Let us begin by stating this precisely.

THEOREM 22.1 (The Relative Homeomorphism Theorem Reborn). *Let $(\mathcal{H}_\bullet, \delta)$ denote a homology theory satisfying the first four axioms. Let $f\colon (X, X') \to (Y, Y')$ be a relative homeomorphism. Assume that $X$ is compact and that $Y$ is compact Hausdorff, and that $X'$ and $Y'$ are closed in $X$ and $Y$ respectively. Assume further that there exists a neighbourhood $U$ of $X'$ in $X$ such that $X'$ is a strong deformation retract of $U$, and a neighbourhood $V$ of $Y'$ in $Y$ such that $Y'$ is a strong deformation retract of $V$. Then*

$$\mathcal{H}_n(f)\colon \mathcal{H}_n(X, X') \to \mathcal{H}_n(Y, Y') \qquad \text{is an isomorphism for all } n \geq 0.$$

*Proof.* Go through the proof of the Relative Homeomorphism Theorem 19.7 and check we only used the axioms. ∎

We now prove Theorem 21.12.

*Proof.* We prove the theorem in three steps. In this proof, all vertical maps are induced by $\Phi$, and we won't label them on the diagrams. The moral of the proof is: use the Five Lemma five times.

**1.** We first prove the result for $X = S^0$ and $X' = \emptyset$. We think of $S^0$ as the union of two points $p$ and $q$, and consider the following diagram:

$$\begin{tikzcd} \mathcal{H}_n(q) \arrow[r] \arrow[d] & \mathcal{H}_n(p \cup q, p) \arrow[d] \\ \mathcal{K}_n(q) \arrow[r] & \mathcal{K}_n(p \cup q, p) \end{tikzcd}$$

The two horizontal maps are excision isomorphisms, and the left-hand vertical map is an isomorphism by hypothesis for $n = 0$ and by the dimension axiom for $n > 0$. Thus the right-hand vertical map is also an isomorphism. We now consider the diagram:

$$\begin{tikzcd} \mathcal{H}_{n+1}(S^0, p) \arrow[r, "\delta"] \arrow[d] & \mathcal{H}_n(p) \arrow[r] \arrow[d] & \mathcal{H}_n(S^0) \arrow[r] \arrow[d] & \mathcal{H}_n(S^0, p) \arrow[r] \arrow[d] & \mathcal{H}_{n-1}(p) \arrow[d] \\ \mathcal{K}_{n+1}(S^0, p) \arrow[r, "\varepsilon"'] & \mathcal{K}_n(p) \arrow[r] & \mathcal{K}_n(S^0) \arrow[r] & \mathcal{K}_n(S^0, p) \arrow[r] & \mathcal{K}_{n-1}(p) \end{tikzcd}$$

The rows are exact by the exact sequence axiom. All the vertical maps apart from the middle one are isomorphisms. Thus the middle one is too by the Five Lemma (Proposition 11.3).

**2.** We now prove the result for an arbitrary sphere $S^k$. Let us inductively assume that $\Phi_n(S^{k-1})\colon \mathcal{H}_n(S^{k-1}) \to \mathcal{K}_n(S^{k-1})$ is an isomorphism for all $n \geq 0$. Since $B^k$ is contractible, by the homotopy axiom, the dimension axiom and the hypotheses, the map $\Phi_n(B^k)\colon \mathcal{H}_n(B^k) \to \mathcal{K}_n(B^k)$ is an isomorphism. Then we apply the Five Lemma again to the following diagram:

$$\begin{tikzcd}
\mathcal{H}_n(S^{k-1}) \arrow[r] \arrow[d] & \mathcal{H}_n(B^k) \arrow[r] \arrow[d] & \mathcal{H}_n(B^k, S^{k-1}) \arrow[r, "\delta"] \arrow[d] & \mathcal{H}_{n-1}(S^{k-1}) \arrow[r] \arrow[d] & \mathcal{H}_{n-1}(B^k) \arrow[d] \\
\mathcal{K}_n(S^{k-1}) \arrow[r] & \mathcal{K}_n(B^k) \arrow[r] & \mathcal{K}_n(B^k, S^{k-1}) \arrow[r, "\varepsilon"'] & \mathcal{K}_{n-1}(S^{k-1}) \arrow[r] & \mathcal{K}_{n-1}(B^k)
\end{tikzcd}$$

Again, the rows are exact and all vertical maps apart from the middle one are isomorphisms. Thus the middle one is too. Now consider the following diagram:

$$\begin{tikzcd}
\mathcal{H}_n(B^k, S^{k-1}) \arrow[r, "\mathcal{H}_n(f)"] \arrow[d] & \mathcal{H}_n(S^k, p) \arrow[d] \\
\mathcal{K}_n(B^k, S^{k-1}) \arrow[r, "\mathcal{K}_n(f)"] & \mathcal{K}_n(S^k, p)
\end{tikzcd}$$

The horizontal maps come from a relative homeomorphism $(B^k, S^{k-1}) \to (S^k, p)$, and is thus an isomorphism by the Relative Homeomorphism Theorem. We have just shown that the left-hand map is an isomorphism, and hence the right-hand vertical map is too. Now we apply the Five Lemma again to this diagram:

$$\begin{tikzcd}
\mathcal{H}_{n+1}(S^k, p) \arrow[r, "\delta"] \arrow[d] & \mathcal{H}_n(p) \arrow[r] \arrow[d] & \mathcal{H}_n(S^k) \arrow[r] \arrow[d] & \mathcal{H}_n(S^k, p) \arrow[r] \arrow[d] & \mathcal{H}_{n-1}(p) \arrow[d] \\
\mathcal{K}_{n+1}(S^k, p) \arrow[r, "\varepsilon"'] & \mathcal{K}_n(p) \arrow[r] & \mathcal{K}_n(S^k) \arrow[r] & \mathcal{K}_n(S^k, p) \arrow[r] & \mathcal{K}_{n-1}(p)
\end{tikzcd}$$

**3.** We now prove the theorem for a pair $(X, X')$ by induction on the number of cells of $X$. We have already done the case where $X$ has one cell, so let us assume that $\Phi_n(Y, Y')\colon \mathcal{H}_n(Y, Y') \to \mathcal{K}_n(Y, Y')$ is an isomorphism for all $n \geq 0$ and for any cell complex $Y$ with at most $N-1$ cells. Let $X$ be a cell complex with $N$ cells and $X'$ a subcomplex. If $\dim X = k$, pick a specific $k$-cell of $X$ and let $Z$ denote the complement of this cell. Then $X'$ has $N-1$ cells, and there is a relative homeomorphism

$$f\colon (B^k, S^{k-1}) \to (X, Z).$$

We now consider the diagram:

$$\begin{tikzcd}
\mathcal{H}_n(B^k, S^{k-1}) \arrow[r, "\mathcal{H}_n(f)"] \arrow[d] & \mathcal{H}_n(X, Z) \arrow[d] \\
\mathcal{K}_n(B^k, S^{k-1}) \arrow[r, "\mathcal{K}_n(f)"] & \mathcal{K}_n(X, Z)
\end{tikzcd}$$

The horizontal maps are isomorphisms and the left-hand vertical map is an isomorphism for all $n \geq 0$, and thus the same is true of the right-hand vertical map. Next, apply the Five Lemma to this diagram:

$$\begin{tikzcd}
\mathcal{H}_{n+1}(X,Z) \arrow[r, "\delta"] \arrow[d] & \mathcal{H}_n(Z) \arrow[r] \arrow[d] & \mathcal{H}_n(X) \arrow[r] \arrow[d] & \mathcal{H}_n(X,Z) \arrow[r] \arrow[d] & \mathcal{H}_{n-1}(Z) \arrow[d] \\
\mathcal{K}_{n+1}(X,Z) \arrow[r, "\varepsilon"'] & \mathcal{K}_n(Z) \arrow[r] & \mathcal{K}_n(X) \arrow[r] & \mathcal{K}_n(X,Z) \arrow[r] & \mathcal{K}_{n-1}(Z)
\end{tikzcd}$$

Thus $\Phi_n(X)\colon \mathcal{H}_n(X) \to \mathcal{K}_n(X)$ is an isomorphism for all $n \geq 0$. Similarly $\Phi_n(X')\colon \mathcal{H}_n(X') \to \mathcal{K}_n(X')$ is an isomorphism for every $n \geq 0$. Finally, we apply the Five Lemma a fifth time to the diagram:

$$\begin{tikzcd}
\mathcal{H}_n(X') \arrow[r] \arrow[d] & \mathcal{H}_n(X) \arrow[r] \arrow[d] & \mathcal{H}_n(X,X') \arrow[r, "\delta"] \arrow[d] & \mathcal{H}_{n-1}(X') \arrow[r] \arrow[d] & \mathcal{H}_{n-1}(X) \arrow[d] \\
\mathcal{K}_n(X') \arrow[r] & \mathcal{K}_n(X) \arrow[r] & \mathcal{K}_n(X,X') \arrow[r, "\varepsilon"'] & \mathcal{K}_{n-1}(X') \arrow[r] & \mathcal{K}_{n-1}(X)
\end{tikzcd}$$

Thus $\Phi_n(X,X')\colon \mathcal{H}_n(X,X') \to \mathcal{K}_n(X,X')$ is also an isomorphism. This completes the proof. ∎

Remark 22.2. If we knew our homology theories arose from taking the homology of a chain complex, one could now invoke Theorem 16.22 to deduce the same result for any cell complex (rather than just finite ones). In general, the excision axiom and the additivity axiom allows one to prove (roughly speaking) that a homology theory does indeed commute with filtered colimits, but this is a somewhat involved argument.

We now embark upon some more homological algebra. Our journey will culminate at the end of the next lecture with the famous *Acyclic Models Theorem*, which will allow us to give new and simpler proofs of various statements from the course (eg. the proof that singular homology satisfies the homotopy axiom). We begin with the following lemma, whose proof is similar to the last part of Problem F.6.

Lemma 22.3. *Let $F$ be a free abelian group. Suppose $g\colon B \to C$ is a surjective homomorphism of abelian groups and $h\colon F \to C$ is a homomorphism. Then there exists a homomorphism $f\colon F \to B$ such that $gf = h$.*

$$\begin{tikzcd}
 & F \arrow[dl, dashed, "f"'] \arrow[d, "h"] & \\
B \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

*Proof.* Let $X$ be a basis of $F$. For each $x \in X$, choose $b_x \in B$ such that $g(b_x) = h(x)$. By Lemma 7.2 there exists a unique homomorphism $f\colon F \to B$ with the property that $f(x) = b_x$ for all $x \in X$. Then both $gf$ and $h$ agree on $X$, and hence $gf = h$ as desired. ∎

This result implies the following statement.

Proposition 22.4. *Suppose we have a commutative diagram of abelian groups where the bottom row is exact, that the top row satisfies $ji = 0$, and that $A$ is free abelian.*

$$\begin{tikzcd} A \arrow[r, "i"] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[d, "h"] \\ A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' \end{tikzcd}$$

*Then there exists a homomorphism $f\colon A \to A'$ making the first square commute:*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, dashed, "f"'] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[d, "h"] \\ A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' \end{tikzcd}$$

*Proof.* We claim that $\operatorname{im} gi \subseteq \operatorname{im} i'$. Indeed, by exactness $\operatorname{im} i' = \ker j'$, and thus it suffices to show that $j'gi = 0$. But $j'gi = hji$ by commutativity. Since $ji = 0$ by assumption, the claim follows. This means we have a diagram:

$$\begin{tikzcd} & A \arrow[d, "gi"] & \\ A' \arrow[r, "i'"'] & \operatorname{im} i' \arrow[r] & 0 \end{tikzcd}$$

Now apply Lemma 22.3 to obtain the desired homomorphism $f\colon A \to A'$. ■

Proposition 22.4 has a rather surprising consequence, which we now explain. Suppose $C_\bullet$ and $D_\bullet$ are two chain complexes. A chain map $f_\bullet\colon C_\bullet \to D_\bullet$ induces maps $H_n(f_\bullet)\colon H_n(C_\bullet) \to H_n(D_\bullet)$ for each $n \geq 0$. But when can we go the other way?

Namely, suppose we start with a homomorphism $h\colon H_0(C_\bullet) \to H_0(D_\bullet)$. We are interested in obtaining criterion for the existence of a chain map $f_\bullet\colon C_\bullet \to D_\bullet$ such that

$$H_0(f_\bullet) = h. \tag{22.1}$$

Let us say that $f_\bullet$ is a chain map **over** $h$ if (22.1) holds. Here are two more definitions.

Definition 22.5. A chain complex $C_\bullet$ is called **free** if each group $C_n$ is a free abelian group.

Definition 22.6. A chain complex $C_\bullet$ is **non-negative** if $C_n = 0$ for all $n < 0$. Thus the singular chain complex is always non-negative. A non-negative chain complex $C_\bullet$ is **acyclic in positive degrees** if $H_n(C_\bullet) = 0$ for all $n > 0$.

The next result is sometimes called the *Comparison Theorem* in homological algebra[1].

[1]This is actually a weaker version than the usual "Comparison Theorem" where the top complex is assumed to a complex of *projectives* rather than a free complex.

Theorem 22.7. *Suppose $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ are two non-negative chain complexes. Assume that $C_\bullet$ is free and that $D_\bullet$ is acyclic in positive degrees. Then given any homomorphism $h\colon H_0(C_\bullet) \to H_0(D_\bullet)$, there always exists a chain map $f_\bullet\colon C_\bullet \to D_\bullet$ over $h$. Moreover if $f_\bullet$ and $g_\bullet$ are two chain maps over $h$ then $f_\bullet$ and $g_\bullet$ are chain homotopic.*

Remark 22.8. In particular, if $C_\bullet$ is free and $D_\bullet$ is acyclic in positive degrees, then a chain map $f_\bullet\colon C_\bullet \to D_\bullet$ is determined up to chain homotopy by the map $H_0(f_\bullet)$.

*Proof.* Since $C_n = 0$ for all $n < 0$, $H_0(C_\bullet)$ is the cokernel of $\partial\colon C_1 \to C_0$, and similarly for $H_0(D_\bullet)$. Denote by $\varepsilon\colon C_0 \to H_0(C_\bullet)$ and $\varepsilon'\colon D_0 \to H_0(D_\bullet)$ the two quotient maps. Note that the extended complex

$$\cdots \to D_2 \xrightarrow{\partial'} D_1 \xrightarrow{\partial'} D_0 \xrightarrow{\varepsilon'} H_0(D_\bullet) \to 0$$

is an acyclic complex (its homology is zero in every degree.) Our goal is to find maps $f_n\colon C_n \to D_n$ such that the entire diagram below commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r, "\partial"] \arrow[d, dashed, "f_2"'] & C_1 \arrow[r, "\partial"] \arrow[d, dashed, "f_1"] & C_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "f_0"] & H_0(C_\bullet) \arrow[r] \arrow[d, "h"] & 0 \\
\cdots \arrow[r] & D_2 \arrow[r, "\partial'"'] & D_1 \arrow[r, "\partial'"'] & D_0 \arrow[r, "\varepsilon'"'] & H_0(D_\bullet) \arrow[r] & 0
\end{tikzcd}$$

We argue by induction on $n \geq 0$. For the case $n = 0$, we use Lemma 22.3 with the diagram

$$\begin{tikzcd}
& C_0 \arrow[dl, dashed, "f_0"'] \arrow[d, "h\circ\varepsilon"] & \\
D_0 \arrow[r, "\varepsilon'"'] & H_0(D_\bullet) \arrow[r] & 0
\end{tikzcd}$$

For the inductive step, we apply Proposition 22.4 to the diagram

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] & C_n \arrow[r, "\partial"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_{n+1} \arrow[r, "\partial'"'] & D_n \arrow[r, "\partial'"'] & D_{n-1}
\end{tikzcd}$$

Now suppose we are given two such chain maps $f_\bullet$ and $g_\bullet$ over $h$. We wish to construct maps $P_n\colon C_n \to D_{n+1}$ for $n \geq -1$ such that

$$\partial' P_n + P_{n-1}\partial = f_n - g_n.$$

Define $P_{-1} = 0$. Now consider the diagram:

$$\begin{tikzcd}
C_0 \arrow[r, "\mathrm{id}"] & C_0 \arrow[r] \arrow[d, "f_0-g_0"] & 0 \arrow[d, "0"] \\
D_1 \arrow[r, "\partial'"'] & D_0 \arrow[r] & H_0(D_\bullet)
\end{tikzcd}$$

Proposition 22.4 applies to give us the desired map $P_0\colon C_0 \to D_1$. Abbreviate $k_n = f_n - g_n - P_{n-1}\partial$. For the inductive step we use the diagram

$$\begin{tikzcd} C_n \arrow[r, "\mathrm{id}"] & C_n \arrow[r] \arrow[d, "k_n"] & 0 \arrow[d, "0"] \\ D_{n+1} \arrow[r, "\partial'"'] & D_n \arrow[r, "\partial'"] & D_{n-1} \end{tikzcd}$$

This diagram commutes, since

$$\begin{aligned} \partial' k_n &= \partial'(f_n - g_n) - (\partial' P_{n-1})\partial \\ &= \partial'(f_n - g_n) - (f_{n-1} - g_{n-1} - P_{n-2}\partial)\partial \\ &= \partial'(f_n - g_n) - (f_n - g_n)\partial \\ &= 0 \end{aligned}$$

as $\partial^2 = 0$ and $f_\bullet - g_\bullet$ is a chain map. Thus we can apply Proposition 22.4 again to get a map $P_n\colon C_n \to D_{n+1}$. This completes the proof. ■

Corollary 22.9. *Suppose $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ are two non-negative chain complexes. Assume that $C_\bullet$ and $D_\bullet$ are both free and acyclic in positive degrees. Assume we are given an isomorphism $h\colon H_0(C_\bullet) \to H_0(D_\bullet)$. Then every chain map $f_\bullet$ over $h$ is a chain equivalence.*

*Proof.* We apply Theorem 22.7 twice to obtain chain maps $f_\bullet\colon C_\bullet \to D_\bullet$ over $h$ and $g_\bullet\colon D_\bullet \to C_\bullet$ over $h^{-1}$. Then $g_\bullet \circ f_\bullet\colon C_n \to C_n$ is a chain map over $h^{-1} \circ h = \mathrm{id}_{H_0(C_\bullet)}$. But another obvious chain map over $\mathrm{id}_{H_0(C_\bullet)}$ is $\mathrm{id}_{C_\bullet}$. By the last statement of Theorem 22.7, $g_\bullet \circ f_\bullet$ is chain homotopic to $\mathrm{id}_{C_\bullet}$. Similarly $f_\bullet \circ g_\bullet$ is chain homotopic to $\mathrm{id}_{D_\bullet}$. Thus $f_\bullet$ is a chain equivalence as desired. ■

LECTURE 23

# The Acyclic Models Theorem

In this final lecture we state and prove the *Acyclic Models Theorem*. The main ideas were all contained in Theorem 22.7 and Corollary 22.9 from the last lecture. The formalism below will seem somewhat complicated, but all we are really doing is carrying out the constructions above in a more general setting.

DEFINITION 23.1. Let $\mathsf{C}$ be a category. A family of **models** for $\mathsf{C}$ is simply an indexed *subset* $\mathcal{M} = \{M_\lambda \mid \lambda \in \Lambda\} \subseteq \mathrm{obj}(\mathsf{C})$.

DEFINITION 23.2. Let $\mathsf{C}$ be a category with family of models $\mathcal{M} = \{M_\lambda \mid \lambda \in \Lambda\}$. Suppose $T\colon \mathsf{C} \to \mathsf{Ab}$ is a functor. A **$T$-model set** $\mathcal{X}$ is a choice of element $x_\lambda \in T(M_\lambda)$ for each $\lambda$:

$$\mathcal{X} = \{x_\lambda \in T(M_\lambda) \mid \lambda \in \Lambda\}.$$

DEFINITION 23.3. Let $\mathsf{C}$ be a category with a family of models $\mathcal{M} = \{M_\lambda \mid \lambda \in \Lambda\}$. Suppose $T\colon \mathsf{C} \to \mathsf{Ab}$ is a functor. We say that $T$ is **free with basis in $\mathcal{M}$** if:

1. $T(C)$ is a free abelian group for every $C \in \mathrm{obj}(\mathsf{C})$,
2. There is a $T$-model set $\mathcal{X} = \{x_\lambda \in T(M_\lambda) \mid \lambda \in \Lambda\}$ such that for every object in $\mathsf{C}$ the set

   $$\{T(f)(x_\lambda) \mid f \in \mathrm{Hom}(M_\lambda, C),\ \lambda \in \Lambda\}$$

   is a basis for $T(C)$.

We call $\mathcal{X}$ a **model basis** for $T$.

EXAMPLE 23.4. Fix $n \geq 0$. Consider a family of models for $\mathsf{Top}$ consisting of just one model $\mathcal{M} = \{\Delta^n\}$. Consider now the functor $C_n\colon \mathsf{Top} \to \mathsf{Ab}$ that assigns a topological space $X$ the free abelian group of singular $n$-chains $C_n(X)$. We claim that $C_n$ is free with basis in $\{\Delta^n\}$. The first condition is by definition, so we need only verify the second. For this, recall (see Lecture 13) we denote by $\ell_n\colon \Delta^n \to \Delta^n$ the identity map, thought of as a singular $n$-simplex in $\Delta^n$. Then the set $\{\ell_n\}$ is a $C_n$-model set. We claim that $\{\ell_n\}$ is a model basis. Indeed, if $\sigma\colon \Delta^n \to X$ is any singular $n$-simplex then (thinking of $\sigma$ as a continuous map from $\Delta^n$ to $X$), we have

$$C_n(\sigma)(\ell_n) = \sigma_n^{\#}\ell_n = \sigma$$

(see (13.4)). Since $C_n(X)$ has basis the singular $n$-simplices in $X$, we have that

$$\{C_n(\sigma)(\ell_n) \mid \sigma\colon \Delta^n \to X \text{ continuous}\}$$

is a basis for $C_n(X)$.

Will J. Merry, Algebraic Topology I, Autumn 2017, ETH Zürich. Last modified: January 02, 2018.

Proposition 23.5. *Let* $\mathsf{C}$ *be a category with family of models* $\mathcal{M} = \{M_\lambda \mid \lambda \in \Lambda\}$. *Suppose* $S, T \colon \mathsf{C} \to \mathsf{Ab}$ *are functors, and assume* $T$ *is free with basis in* $\mathcal{M}$. *Let*

$$\{x_\lambda \in T(M_\lambda) \mid \lambda \in \Lambda\}$$

*denote a model basis for* $T$. *Choose elements* $y_\lambda \in S(M_\lambda)$ *for each* $\lambda \in \Lambda$, *and set*

$$\mathcal{Y} := \{y_\lambda \in S(M_\lambda) \mid \lambda \in \Lambda\}$$

*Then there exists a unique natural transformation* $\Phi \colon T \to S$ *such that*

$$\Phi(M_\lambda)(x_\lambda) = y_\lambda, \qquad \forall\, \lambda \in \Lambda.$$

The following picture might help you remember the statement:

$$\begin{tikzcd} T \arrow[r, dashed, "\Phi"] & S \\ \mathcal{X} \arrow[u] \arrow[r, "x_\lambda \mapsto y_\lambda"'] & \mathcal{Y} \arrow[u] \end{tikzcd}$$

*Proof.* Let us first check that $\Phi$ is unique if it exists. For fixed $\lambda \in \Lambda$ and a fixed object $C \in \operatorname{obj}(\mathsf{C})$, we obtain a commutative diagram for every morphism $f \colon M_\lambda \to C$:

$$\begin{tikzcd} T(M_\lambda) \arrow[r, "T(f)"] \arrow[d, "\Phi(M_\lambda)"'] & T(C) \arrow[d, "\Phi(C)"] \\ S(M_\lambda) \arrow[r, "S(f)"'] & S(C) \end{tikzcd}$$

Thus if $x_\lambda \in \mathcal{X}$ we have by the hypothesis on $\Phi$ that

$$\Phi(C) \circ T(f)(x_\lambda) = S(f) \circ \Phi(M_\lambda)(x_\lambda) = S(f)(y_\lambda).$$

Since the family $\{T(f)(x_\lambda)\}$ forms a basis of (and hence generates) $T(C)$, it follows that each homomorphism $\Phi(C)$ is uniquely determined. Since $C$ was an arbitrary object, it follows that $\Phi$ is unique. Now let us construct $\Phi$. Again, fix an object $C$ of $\mathsf{C}$. We first define $\Phi(C)$ on basis elements $\{T(f)(x_\lambda)\}$ by declaring that $\Phi(C)\big(T(f)(x_\lambda)\big) := S(f)(y_\lambda)$. Then since $T(C)$ is free abelian, by Lemma 7.2 there is a unique homomorphism $\Phi(C) \colon T(C) \to S(C)$ that extends this map. It remains to show that $\Phi$ is a natural transformation. For this, take a morphism $g \colon A \to B$ in $\mathsf{C}$. We need to prove the following diagram commutes:

$$\begin{tikzcd} T(A) \arrow[r, "T(g)"] \arrow[d, "\Phi(A)"'] & T(B) \arrow[d, "\Phi(B)"] \\ S(A) \arrow[r, "S(g)"'] & S(B) \end{tikzcd}$$

Since $T(A)$ is free abelian, it suffices to evaluate both sides on a typical basis element $T(f)(x_\lambda)$ for some $\lambda \in \Lambda$ and $f \colon M_\lambda \to A$. Then

$$S(g) \circ \Phi(A)\big(T(f)(x_\lambda)\big) = S(g)S(f)y_\lambda = S(g \circ f)(y_\lambda).$$

But also going the other way round:

$$\Phi(B) \circ T(g)\big(T(f)(x_\lambda)\big) = \Phi(B) \circ T(g \circ f)(x_\lambda) = S(g \circ f)(y_\lambda).$$

Thus $\Phi$ is indeed a natural transformation, and this completes the proof. ∎

We now prove a generalisation of Proposition 22.4 to this setting.

PROPOSITION 23.6. *Let $\mathsf{C}$ be a category with family of models $\mathcal{M}$. Suppose we are given six functors*

$$T_i,\ S_i \colon \mathsf{C} \to \mathsf{Ab}, \qquad i = 1, 2, 3.$$

*together with six natural transformations as pictured below:*

$$\begin{tikzcd} T_1 \arrow[r, "\Phi_1"] & T_2 \arrow[r, "\Phi_2"] \arrow[d, "\Theta_1"] & T_3 \arrow[d, "\Theta_2"] \\ S_1 \arrow[r, "\Psi_1"'] & S_2 \arrow[r, "\Psi_2"'] & S_3 \end{tikzcd}$$

*Assume that:*

1. *Assume that for every object $C \in \operatorname{obj}(\mathsf{C})$ the composition $\Phi_2(C) \circ \Phi_1(C) \colon T_1(C) \to T_3(C)$ is the zero homomorphism.*
2. *The bottom row is exact on $\mathcal{M}$, in the sense that for every model $M \in \mathcal{M}$ one has $\operatorname{im} \Psi_1(M) = \ker \Psi_2(M)$.*
3. *The diagram commutes for every object $C$ of $\mathsf{C}$.*
4. *$T_1$ is free with basis in $\mathcal{M}$.*

*Then there exists a natural transformation $\Upsilon \colon T_1 \to S_1$ such that the first square commutes for every object of $\mathsf{C}$.*

$$\begin{tikzcd} T_1 \arrow[r, "\Phi_1"] \arrow[d, dashed, "\Upsilon"'] & T_2 \arrow[r, "\Phi_2"] \arrow[d, "\Theta_1"] & T_3 \arrow[d, "\Theta_2"] \\ S_1 \arrow[r, "\Psi_1"'] & S_2 \arrow[r, "\Psi_2"'] & S_3 \end{tikzcd}$$

The trickiest thing in Proposition 23.6 is the statement. It is easiest to see this as a direct generalisation of Proposition 22.4 where we use functors instead of maps. Indeed, if $\mathsf{C}$ had exactly one object and only the identity morphism, then Proposition 23.6 would reduce to Proposition 22.4. But in general Proposition 23.6 is *much stronger*: the important bit is that we only require the bottom row to be exact *on* $\mathcal{M}$. If you think back to Example 23.4, this can often be a massive simplification if $\mathcal{M}$ is very small compared to $\operatorname{obj}(\mathsf{C})$.

*Proof.* Let $\mathcal{X} = \{x_\lambda \in T_1(M_\lambda) \mid \lambda \in \Lambda\}$ denote a model basis for $T_1$. Then for each $\lambda \in \Lambda$ we have a commutative diagram in $\mathsf{Ab}$ that satisfies the hypotheses of Proposition 22.4:

$$\begin{tikzcd} T_1(M_\lambda) \arrow[r] & T_2(M_\lambda) \arrow[r] \arrow[d] & T_3(M_\lambda) \arrow[d] \\ S_1(M_\lambda) \arrow[r] & S_2(M_\lambda) \arrow[r] & S_3(M_\lambda) \end{tikzcd}$$

Thus by Proposition 22.4 we obtain a homomorphism $T_1(M_\lambda) \to S_1(M_\lambda)$:

$$\begin{tikzcd}
T_1(M_\lambda) \arrow[r] \arrow[d, dashed] & T_2(M_\lambda) \arrow[r] \arrow[d] & T_3(M_\lambda) \arrow[d] \\
S_1(M_\lambda) \arrow[r] & S_2(M_\lambda) \arrow[r] & S_3(M_\lambda)
\end{tikzcd}$$

Set $y_\lambda \in S_1(M_\lambda)$ denote the image of $x_\lambda$ under this map. By Proposition 23.5 we obtain a natural transformation $\Upsilon\colon T_1 \to S_1$ such that $\Upsilon(M_\lambda)(x_\lambda) = y_\lambda$ for each $\lambda \in \Lambda$. It remains to check that the desired diagram commutes. For this consider

$$z_\lambda := \Psi_1(M_\lambda)(y_\lambda), \qquad \mathscr{Z} := \{z_\lambda \in S_2(M_\lambda) \mid \lambda \in \Lambda\}.$$

Then both $\Theta_1 \circ \Phi_1$ and $\Psi_1 \circ \Upsilon$ are natural transformations $T_1 \to S_2$ that send $x_\lambda$ to $z_\lambda$. The uniqueness part of Proposition 23.5 then implies that $\Theta_1 \circ \Phi_1 = \Psi_1 \circ \Upsilon$. This completes the proof. ■

The main result of today's lecture is basically the "models" version of Theorem 22.7. This means we need to study functors with values in $\mathsf{Comp}$. So suppose $T_\bullet\colon \mathsf{C} \to \mathsf{Comp}$ is a functor. Thus for each $C \in \mathrm{obj}(\mathsf{C})$ we obtain a chain complex $T_\bullet(C)$. Given $n \in \mathbb{Z}$, let $T_n\colon \mathsf{C} \to \mathsf{Ab}$ denote the functor given by $C \mapsto T_n(C)$. As with the case of a single chain complex, we say that $T_\bullet$ is **non-negative** if $T_n(C) = 0$ for all $n < 0$, and we say an object $C \in \mathrm{obj}(\mathsf{C})$ is **$T$-acyclic in positive degrees** if $H_n(T_\bullet(C)) = 0$ for all $n > 0$.

A natural transformation $\Phi_\bullet\colon T_\bullet \to S_\bullet$ between two functors $T_\bullet, S_\bullet\colon \mathsf{C} \to \mathsf{Comp}$ is usually called a **natural chain map**. There is an analogous notion of a natural chain homotopy.

Definition 23.7. Suppose $T_\bullet, S_\bullet\colon \mathsf{C} \to \mathsf{Comp}$ are two functors and $\Phi_\bullet, \Psi_\bullet\colon T_\bullet \to S_\bullet$ are two natural chain maps (i.e. natural transformations). A **natural chain homotopy** is a sequence $\Upsilon_\bullet$ of natural transformations $\Upsilon_n\colon T_n \to S_{n+1}$ such that

$$\partial' \Upsilon_n + \Upsilon_{n-1}\partial = \Phi_n - \Psi_n$$

for every $n$. Here $\partial$ is the boundary operator of $T_\bullet$ and $\partial'$ is the boundary operator of $S_\bullet$. Explicitly, this means for every object $C$ of $\mathsf{C}$, if we denote by $\partial_C$ the boundary operator of $T_\bullet(C)$ and $\partial'_C$ the boundary operator of $S_\bullet(C)$, we have

$$\partial'_C \Upsilon_n(C) + \Upsilon_{n-1}(C)\partial_C = \Phi_n(C) - \Psi_n(C)$$

as homomorphisms $T_n(C) \to S_n(C)$.

Similarly, a **natural chain equivalence** $\Phi_\bullet\colon T_\bullet \to S_\bullet$ is a natural chain map with the property that there exists another natural chain map $\Psi_\bullet\colon S_\bullet \to T_\bullet$ such that $\Psi_\bullet \circ \Phi_\bullet$ is naturally chain homotopic to $\mathrm{id}_{T_\bullet}$ and $\Phi_\bullet \circ \Psi_\bullet$ is naturally chain homotopic to $\mathrm{id}_{S_\bullet}$.

Now we introduce the functor-valued version of (22.1). Suppose $T_\bullet, S_\bullet\colon \mathsf{C} \to \mathsf{Comp}$ are two functors. Then $H_0 \circ T_\bullet$ and $H_0 \circ S_\bullet$ are two functors $\mathsf{C} \to \mathsf{Ab}$, given by

$C \mapsto H_0(T_\bullet(C))$ and $C \mapsto H_0(S_\bullet(C))$ respectively. Suppose we are given a natural transformation

$$\Theta \colon H_0 \circ T_\bullet \to H_0 \circ S_\bullet.$$

We can then ask the question: when does there exist a natural chain map $\Phi_\bullet \colon T_\bullet \to S_\bullet$ such that $H_0(\Phi_\bullet) = \Theta$? The *Acyclic Models Theorem* gives an answer.

THEOREM 23.8 (The Acyclic Models Theorem). *Let $\mathsf{C}$ be a category with models $\mathcal{M}$. Assume that $S_\bullet, T_\bullet \colon \mathsf{C} \to \mathsf{Comp}$ are non-negative functors. Assume that for all $n \geq 0$, $T_n$ is free with basis in $\mathcal{M}_n \subseteq \mathcal{M}$. Assume that each model $M \in \mathcal{M}$ is $S_\bullet$-acyclic in positive degrees. If $\Theta \colon H_0 \circ T_\bullet \to H_0 \circ S_\bullet$ is a natural transformation then there exists a natural chain map $\Phi_\bullet \colon T_\bullet \to S_\bullet$ over $\Theta$. Moreover any two such natural chain maps are naturally chain homotopic.*

COROLLARY 23.9. *If instead we assume that for all $n \geq 0$, both $S_n$ and $T_n$ are free with basis in $\mathcal{M}_n \subseteq \mathcal{M}$, and that each model $M \in \mathcal{M}$ is both $S_\bullet$-acyclic in positive degrees and $T_\bullet$-acyclic in positive degrees. Then if $\Theta \colon H_0 \circ T_\bullet \to H_0 \circ S_\bullet$ is a natural equivalence then every natural chain map $\Phi_\bullet$ over $\Theta$ is a natural chain equivalence.*

Again, by far the hardest part of this theorem is understanding the statement! The proof is basically identical to the proof of Theorem 22.7 and Corollary 22.9.

*Proof of Theorem 23.8 and Corollary 23.9.* We prove both the two results in three steps.

**1.** As in the proof of Theorem 22.7, our goal is construct natural transformations $\Phi_n \colon T_n \to S_n$ such that the following diagram commutes.

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & T_2 \arrow[r, "\partial"] \arrow[d, dashed, "\Phi_2"'] & T_1 \arrow[r, "\partial"] \arrow[d, dashed, "\Phi_1"'] & T_0 \arrow[r] \arrow[d, dashed, "\Phi_0"] & H_0(T_\bullet) \arrow[r] \arrow[d, "\Theta"] & 0 \\
\dots \arrow[r, "\partial'"'] & S_2 \arrow[r, "\partial'"'] & S_1 \arrow[r, "\partial'"'] & S_0 \arrow[r] & H_0(S_\bullet) \arrow[r] & 0
\end{tikzcd}$$

For $n = 0$, we have the following picture:

$$\begin{tikzcd}
T_0 \arrow[r] & H_0(T_\bullet) \arrow[r] \arrow[d, "\Theta"] & 0 \arrow[d, "0"] \\
S_0 \arrow[r] & H_0(S_\bullet) \arrow[r] & 0
\end{tikzcd}$$

The bottom row is exact because $H_0(S_\bullet(C))$ is the cokernel of $S_1(C) \to S_0(C)$. Thus Proposition 23.6 gives us a natural transformation $\Phi_0 \colon T_0 \to S_0$ such that the diagram commutes:

$$\begin{tikzcd}
T_0 \arrow[r] \arrow[d, dashed, "\Phi_0"'] & H_0(T_\bullet) \arrow[r] \arrow[d, "\Theta"] & 0 \arrow[d, "0"] \\
S_0 \arrow[r] & H_0(S_\bullet) \arrow[r] & 0
\end{tikzcd}$$

Now we inductively define $\Phi_n \colon T_n \to S_n$ for $n \geq 1$. Indeed, if we have constructed $\Phi_n$ then we have a diagram

$$\begin{tikzcd} T_{n+1} \arrow[r, "\partial"] & T_n \arrow[r, "\partial"] \arrow[d, "\Phi_n"] & T_{n-1} \arrow[d, "\Phi_{n-1}"] \\ S_{n+1} \arrow[r, "\partial'"'] & S_n \arrow[r, "\partial'"'] & S_{n-1} \end{tikzcd}$$

By assumption the bottom row is exact for every model $M$, and thus as $T_{n+1}$ is free, Proposition 23.6 applies to give us the desired $\Phi_{n+1} \colon T_{n+1} \to S_{n+1}$.

**2.** Now suppose we have two such maps $\Phi_\bullet, \Psi_\bullet \colon T_\bullet \to S_\bullet$. We need to find natural transformations $\Upsilon_n \colon T_n \to S_{n+1}$ for all $n \geq -1$ such that

$$\partial' \Upsilon_n + \Upsilon_{n-1} \partial = \Phi_n - \Psi_n.$$

We define $\Upsilon_{-1} = 0$ and proceed inductively. Let $\Xi_n := \Phi_n - \Psi_n$. Then we have a diagram:

$$\begin{tikzcd} T_0 \arrow[r, "\mathrm{id}"] & T_0 \arrow[r] \arrow[d, "\Xi_0"] & 0 \arrow[d, "0"] \\ S_1 \arrow[r, "\partial'"'] & S_0 \arrow[r] & H_0(S_\bullet) \end{tikzcd}$$

Again, Proposition 23.6 applies to give us the desired map $\Upsilon_0 \colon T_0 \to S_1$. For the inductive step we use the diagram

$$\begin{tikzcd} T_n \arrow[r, "\mathrm{id}"] & T_n \arrow[r] \arrow[d, "\Xi_n - \Upsilon_{n-1} \circ \partial"] & 0 \arrow[d, "0"] \\ S_{n+1} \arrow[r, "\partial'"'] & S_n \arrow[r] & S_{n-1} \end{tikzcd}$$

We need to show this diagram commutes to apply Proposition 23.6. But by induction:

$$\begin{aligned} \partial'(\Xi_n - \Upsilon_{n-1}\partial) &= \partial' \Xi_n - (\partial \Upsilon_{n-1})\partial \\ &= \partial' \Xi_n - (\Xi_{n-1} - \Upsilon_{n-2}\partial)\partial \\ &= \partial' \Xi_n - \Xi_{n-1}\partial \\ &= 0 \end{aligned}$$

as $\partial^2 = 0$ and $\Xi_\bullet$ is a chain map.

**3.** Finally we prove Corollary 23.9. In this case $\Theta$ is a natural isomorphism, and hence there exists a natural transformation $\Pi \colon H_0 \circ S_\bullet \to H_0 \circ T_\bullet$ such that $\Pi \circ \Theta = \mathrm{id}_{H_0 \circ T}$ and $\Theta \circ \Pi = \mathrm{id}_{H_0 \circ S}$. We then have two natural chain maps $\Phi_\bullet : T_\bullet \to S_\bullet$ and $\Psi_\bullet \colon S_\bullet \to T_\bullet$ over from $\Theta$ and $\Pi$ respectively. This gives us two natural chain maps over $\Pi \circ \Theta$: the identity $\mathrm{id}_{T_\bullet} \colon T_\bullet \to T_\bullet$ and $\Psi_\bullet \circ \Phi_\bullet \colon T_\bullet \to T_\bullet$. By what we have already proved, these two natural chain maps are naturally chain homotopic. Similarly $\Phi_\bullet \circ \Psi_\bullet \colon S_\bullet \to S_\bullet$ is naturally chain homotopic to $\mathrm{id}_\bullet \colon S_\bullet \to S_\bullet$. Thus $\Phi_\bullet \colon T_\bullet \to S_\bullet$ is natural chain equivalence. This completes the proof. ∎

With all this heavy lifting out of the way, let us now reap the benefits. Recall Proposition 8.5:

Proposition 23.10. *Let $X$ be a topological space and define inclusions $\imath, \jmath \colon X \hookrightarrow X \times I$ by*

$$\imath(x) := (x, 0), \qquad \jmath(x) := (x, 1).$$

*Then*

$$H_n(\imath) = H_n(\jmath), \qquad \forall\, n \geq 0.$$

We can now give a cute easy proof.

*Proof.* We give $\mathsf{Top}$ models $\mathcal{M} = \{\Delta^n \mid n \geq 0\}$. Then by Example 23.4, for all $n \geq 0$ the singular chain functor $C_\bullet \colon \mathsf{Top} \to \mathsf{Comp}$ has the property that $C_n$ is free with basis in $\mathcal{M}_n := \{\Delta^n\} \subset \mathcal{M}$. Define another functor $S_\bullet \colon \mathsf{Top} \to \mathsf{Comp}$ by $S_\bullet(X) = C_\bullet(X \times I)$. Since $\Delta^n \times I$ is convex, by Corollary 13.3, every model $\Delta^n$ is $S_\bullet$-acyclic in positive degrees (note this argument is not circular, see Remark 13.4.) Now the Acyclic Models Theorem tells us that in order to deduce that $\imath_\bullet^\#, \jmath_\bullet^\# \colon C_\bullet(X) \to C_\bullet(X \times I)$ are naturally chain homotopic (and hence in particular induce the same map on homology, cf. Proposition 10.24) it suffices to show that $H_0(\imath) = H_0(\jmath)$. But this is trivial. ∎

Here is an even simpler application of the Acyclic Models Theorem.

Proposition 23.11. *Let $\Phi_\bullet, \Psi_\bullet \colon C_\bullet \to C_\bullet$ denote natural transformations from the singular chain functor to itself. Assume that $\Phi_0 = \Psi_0$. Then there is a natural chain homotopy from $\Phi_\bullet$ to $\Psi_\bullet$.*

*Proof.* Apply the Acyclic Models Theorem with $S_\bullet = T_\bullet$ both equal to the singular chain functor, and with models $\{\Delta^n \mid n \geq 0\}$ as above and with $\Theta = H_0(\Phi_\bullet) = H_0(\Psi_\bullet)$. ∎

We conclude with a much nicer proof of Theorem 13.11:

Theorem 23.12. *Let $X$ be a topological space and consider the barycentric division operator $\mathrm{Sd}_\bullet \colon C_\bullet(X) \to C_\bullet(X)$. Then $\mathrm{Sd}_\bullet$ is naturally chain homotopic to the identity (and hence induces an isomorphism on homology).*

*Proof.* Immediate from Proposition 23.11 because $\mathrm{Sd}_\bullet$ is a natural (this is (13.3)) and $\mathrm{Sd}_0$ is the identity. ∎

This is the end of Algebraic Topology I. See you next semester!