LECTURE 24

Tensor products and a taste of Tor

Welcome to Algebraic Topology II! We have lots to get through this semester, so rather than a long and fancy introduction, this time we'll get straight on to our first topic: *universal coefficients*.

Unfortunately before the exciting topology can start, we need a rather lengthy algebraic prelude on tensor products and the Tor functor, which will take all of today's lecture.

Definition 24.1. Let A and B be two abelian groups. Their **tensor product** is the abelian group $A \otimes B$, which has:

- *Generators:* all ordered pairs (a, b) where $a \in A$ and $b \in B$.
- *Relations* if $a, a' \in A$ and $b, b' \in B$ then

$$(a + a', b) = (a, b) + (a', b), \qquad \text{and} \qquad (a, b + b') = (a, b) + (a, b').$$

More formally, $A \otimes B$ is the quotient F/N where F is the free abelian group with basis $A \times B$ and N is the subgroup of F generated by all the relations. We denote the coset $(a, b) + N$ by $a \otimes b$.

Thus a typical element x of $A \otimes B$ can be written as a sum

$$x = \sum_i m_i \, a_i \otimes b_i, \qquad a_i \in A, \ b_i \in B, \ m_i \in \mathbb{Z}.$$

Actually one can always dispense with the m_i, since for any $m \in \mathbb{Z}$ and $a \in A$, $b \in B$, as element of $A \otimes B$ one has

$$m(a \otimes b) = (ma) \otimes b = a \otimes (mb),$$

as can be easily seen from the relations.

This definition of $A \otimes B$ is rather concrete, but it is presumably not clear to most of you what the point is. We now show that the tensor product can also be specified via a universal property.

Definition 24.2. Let A, B, C be abelian groups. A **bilinear function** $\varphi \colon A \times B \to C$ is a function such that for all $a, a' \in A$ and all $b, b' \in B$,

$$\varphi(a + a', b) = \varphi(a, b) + \varphi(a', b), \qquad \text{and} \qquad \varphi(a, b + b') = \varphi(a, b) + \varphi(a, b').$$

As an example, the natural map $u \colon A \times B \to A \otimes B$ that sends $(a, b) \mapsto a \otimes b$ is bilinear.

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Definition 24.3. Suppose we are given three abelian groups A, B, T and a bilinear map $\eta\colon A \times B \to T$. Consider the following universal property: Then we require that if C is any abelian group and $\varphi\colon A \times B \to C$ is a bilinear map, then there exists a *unique* homomorphism $f\colon T \to C$ such that the following diagram commutes:

$$\begin{tikzcd} A \times B \arrow[rr, "\eta"] \arrow[dr, "\varphi"'] & & T \arrow[dl, dashed, "f"] \\ & C & \end{tikzcd}$$

As with all universal properties[1], if such a pair (T, η) exist, they are unique up to isomorphism. Let us verify that $(A \otimes B, u)$ does indeed solve this universal property.

Lemma 24.4. *The tensor product $A \otimes B$ together with the bilinear map $u\colon A \times B \to A \otimes B$, $u(a, b) = a \otimes b$, satisfies the universal property from Definition 24.3.*

Proof. Let $\varphi\colon A \times B \to C$ be a bilinear function. Recall $A \otimes B = F/N$, where F is free abelian with basis $A \times B$. We first extend $\varphi\colon A \times B \to C$ by linearity to a map $\tilde{\varphi}\colon F \to C$ (cf. Lemma 7.2). The relations that generate N are such that $N \subset \ker \tilde{\varphi}$, and hence $\tilde{\varphi}$ factors to define a homomorphism $f\colon F/N \to C$ such that $(a, b) + N \mapsto \tilde{\varphi}(a, b) = \varphi(a, b)$, that is, $f(a \otimes b) = \varphi(a, b)$. Moreover the map f is unique, since the set of all the $a \otimes b$ generate $A \otimes B$. ■

Being able to use universal properties makes the next result very transparent.

Proposition 24.5. *Let $f\colon A \to A'$ and $g\colon B \to B'$ be two homomorphisms. Then there is a unique homomorphism $A \otimes B \to A' \otimes B'$, denoted by $f \otimes g$, with the property that*

$$(f \otimes g)(a \otimes b) = fa \otimes gb, \qquad \forall\, a \in A,\ b \in B.$$

Moreover if $f'\colon A' \to A''$ and $g'\colon B' \to B''$ are two other homomorphisms then $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$.

Proof. The function $\varphi\colon A \times B \to A' \otimes B'$ defined by $\varphi(a, b) := fa \otimes gb$ is bilinear. Thus by Lemma 24.4 there is a unique homomorphism $A \otimes B \to A' \otimes B'$ which maps $a \otimes b \mapsto \varphi(a, b) = fa \otimes gb$. This is our desired homomorphism $f \otimes g$. This proves the first part. For the second part, we define a bilinear map $\varphi\colon A \times B \to A'' \otimes B''$ by

$$\varphi(a, b) = (f'(fa)) \otimes g'(gb)).$$

Then observe that both $(f' \otimes g') \circ (f \otimes g)$ and $(f' \circ f) \otimes (g' \circ g)$ fit on the dashed line to make the following diagram commute:

$$\begin{tikzcd} A \times B \arrow[rr, "u"] \arrow[dr, "\varphi"'] & & A \otimes B \arrow[dl, dashed] \\ & A'' \otimes B'' & \end{tikzcd}$$

Thus by the uniqueness part of Lemma 24.4, we must have $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$. This completes the proof. ■

[1] To check how much you forgot over the Winter Vacation: prove that there is a most one pair (T, f) satisfying the universal property from Definition 24.3.

This means we can view the tensor product as a functor.

Corollary 24.6. *Let A be an abelian group. There is a functor $T\colon \mathsf{Ab} \to \mathsf{Ab}$ such that $T(B) = A \otimes B$ and if $f\colon B \to C$ then $T(f) = \mathrm{id}_A \otimes f\colon A \otimes B \to A \otimes C$.*

Proof. Let $f\colon B \to C$ and $g\colon C \to D$. By the second part of Proposition 24.5, we have

$$(\mathrm{id}_A \otimes g) \circ (\mathrm{id}_A \otimes f) = \mathrm{id}_A \otimes (g \circ f),$$

which shows that T preserves compositions. The fact that $\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A\otimes B}$ is obvious. ■

We normally denote the functor T by $A \otimes \square$. Similarly, given a fixed abelian group B, there is a functor $\square \otimes B\colon \mathsf{Ab} \to \mathsf{Ab}$ that sends $A \mapsto A \otimes B$, and sends morphisms $f\colon A \to C$ to $f \otimes \mathrm{id}_B\colon A \otimes B \to C \otimes B$.

We briefly mentioned additive functors last semester. Let me remind you of the definition.

Definition 24.7. A functor $T\colon \mathsf{Ab} \to \mathsf{Ab}$ is said to be **additive** if $T(f+g) = T(f) + T(g)$ for any two morphisms $f, g\colon A \to B$.

An additive functor has the nice property that $T(0) = 0$, where 0 denotes either the zero group or the zero homomorphism.

The next result summarises some more properties of the tensor product. The proofs of parts (1)-(4) are all trivial. The proofs of (5) and (6) are slightly harder, and they are relegated to Problem Sheet L.

Proposition 24.8.

1. *There is an isomorphism $A \otimes B \cong B \otimes A$ taking $a \otimes b$ to $b \otimes a$. The functors $A \otimes \square$ and $\square \otimes A$ are isomorphic.*
2. *The functors $A \otimes \square$ and $\square \otimes A$ are additive.*
3. *If $f\colon B \to B$ is multiplication by an integer m, $fb = mb$ for all $b \in B$, then $\mathrm{id}_A \otimes f\colon A \otimes B \to A \otimes B$ is also multiplication for m (and similarly for $f \otimes \mathrm{id}_A\colon B \otimes A \to B \otimes A$.)*
4. *For any abelian group A, the map $\mathbb{Z} \otimes A \to A$ given by $m \otimes a \mapsto ma$ is an isomorphism. Denoting this map by $\Phi(A)$, the resulting family Φ defines a natural equivalence $\mathbb{Z} \otimes \square \to \mathrm{id}_{\mathsf{Ab}}$ in Ab.*
5. *If A is an abelian group and $\{B_\lambda \mid \lambda \in \Lambda\}$ is a (possibly uncountable) family of abelian groups then there is an isomorphism*

$$A \otimes \bigoplus_{\lambda\in\Lambda} B_\lambda \cong \bigoplus_{\lambda\in\Lambda} (A \otimes B_\lambda).$$

6. *If F and F' are free abelian groups then so is $F \otimes F'$.*

Now for an algebraic definition.

Definition 24.9. Let $T\colon \mathsf{Ab} \to \mathsf{Ab}$ be an additive functor. We say that T is **exact** if given any exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

the corresponding sequence

$$T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$$

is also exact. We say that T is **left exact** if given any exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C,$$

the corresponding sequence

$$0 \to T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$$

is also exact (That is, a left exact functor T preserves an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ if the first map f is injective). Similarly we say that T is **right exact** if if given any exact sequence of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

the corresponding sequence

$$T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \to 0$$

is also exact. (That is, a right exact functor T preserves an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ if the second map g is surjective).

Thus an exact functor is both left exact and right exact.

Proposition 24.10. *The functors $A \otimes \square$ and $\square \otimes A$ are both right exact.*

Proof. We will prove $A \otimes \square$ is right exact. The proof for $\square \otimes A$ is almost identical. Suppose $B \xrightarrow{f} C \xrightarrow{g} D \to 0$ is exact. We must show that

$$A \otimes B \xrightarrow{\mathrm{id}_A \otimes f} A \otimes C \xrightarrow{\mathrm{id}_A \otimes g} A \otimes D \to 0$$

is also exact. There are three things to check:

1. $\mathrm{im}(\mathrm{id}_A \otimes f) \subseteq \ker(\mathrm{id}_A \otimes g)$.
2. $\ker(\mathrm{id}_A \otimes g) \subseteq \mathrm{im}(\mathrm{id}_A \otimes f)$.
3. $\mathrm{id}_A \otimes g$ is surjective.

The proof of (1) is easy:

$$(\mathrm{id}_A \otimes g) \circ (\mathrm{id}_A \otimes f) = (\mathrm{id}_A \otimes gf) = (\mathrm{id}_A \otimes 0) = 0,$$

where we used the second statement of Proposition 24.5 and the fact that $A \otimes \square$ is additive.

The proof of (2) is rather trickier. For this, let us denote by $E := \mathrm{im}(\mathrm{id}_A \otimes f)$. Then the map $\mathrm{id}_A \otimes g\colon A \otimes C \to A \otimes D$ induces a map $h\colon (A \otimes C)/E \to A \otimes D$ given by

$$a \otimes c + E \mapsto a \otimes gc$$

(this is well defined by part (1)). By definition, one has $\mathrm{id}_A \otimes g = h \circ p$, where $p\colon A \otimes C \to (A \otimes C)/E$ is the quotient map. We will prove that h is an isomorphism. Then

$$\mathrm{im}(\mathrm{id}_A \otimes f) = E = \ker p = \ker(h \circ p) = \ker(\mathrm{id}_A \otimes g).$$

We will use the universal property of the tensor product to find an inverse to h. Given $d \in D$, since g is surjective there exists $c \in C$ such that $gc = d$. If c' is another such element of C such that $gc' = d$ then $c - c' \in \ker g = \mathrm{im}\, f$, and thus there exists $b \in B$ such that $fb = c - c'$. Thus for any $a \in A$,

$$a \otimes c - a \otimes c' = a \otimes (c - c') = (\mathrm{id}_A \otimes f)(a \otimes b) \in \mathrm{im}(\mathrm{id}_A \otimes f) = E.$$

This means that there is a well defined map $\varphi\colon A \times D \to (A \otimes C)/E$ given by

$$\varphi(a, d) = a \otimes c + E,$$

where c is any element of C such that $gc = d$. The function φ is obviously bilinear, and hence by the universal property there exists a homomorphism $j\colon A \otimes D \to (A \otimes C)/E$ such that $j(a \otimes d) = a \otimes c + E$ (where as before, c is any element of C such that $gc = d$.) Then by definition, $j \circ h = \mathrm{id}_{(A \otimes C)/E}$ and $h \circ j = \mathrm{id}_{A \otimes D}$. This proves (2).

Finally, if $\sum a_i \otimes d_i$ is an element of $A \otimes D$, since g is surjective we can choose $c_i \in C$ such that $gc_i = d_i$. Then

$$(\mathrm{id}_A \otimes g)\left(\sum a_i \otimes c_i\right) = \sum a_i \otimes d_i,$$

which proves (3). ■

Now let us define free resolutions.

Definition 24.11. Suppose A is an abelian group. A **free resolution** of A is an exact sequence of the form

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \to 0,$$

where each F_i is a free abelian group. A **short free resolution**[2] of A is a free resolution where each F_i for $i \geq 2$ is the zero group, that is, a sequence of the form

$$0 \to K \to F \to A \to 0$$

where K and F are both free.

Let us show that short resolutions always exist.

[2] This terminology is non-standard.

Proposition 24.12. *Let A be an abelian group. Then there exists a short free resolution of A.*

Proof. Let F be the free abelian group with basis the elements of A. There is a surjective homomorphism $F \to A$ obtained by extending by linearity (Lemma 7.2) the map that sends each basis element to itself. Let K denote the kernel of this map. Then K is a subgroup of a free abelian group, and hence is also a free abelian group[3], and by construction, $0 \to K \to F \to A \to 0$ is exact. ∎

We now define the Tor functor.

Definition 24.13. Let A be an abelian group. Let $0 \to K \xrightarrow{f} F \to A \to 0$ be a short free resolution of A. Given any other abelian group B, we define

$$\mathrm{Tor}(A, B) := \ker(f \otimes \mathrm{id}_B).$$

Thus $\mathrm{Tor}(A, B)$ measures the failure of $\square \otimes B$ to be left exact on the sequence $0 \to K \to F \to A \to 0$.

You should all immediately be asking: is this well defined? That is, does the value of $\mathrm{Tor}(A, B)$ depend on the choice of short free resolution? The answer (luckily!) is no. We will prove this next lecture using the Comparison Theorem (Proposition 22.7) from last semester.

[3]This is a non-trivial fact!