Lecture Notes for Algebraic Geometry I

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About the notes

In this notes we mainly focus on the algebraically closed fields.

1 Classical varieties

1.1 Feb 27th: Algebraic sets and morphisms

https://imaginary.org/programs

Recall: $V(I) \subset \mathbb{A}^n = \{x | \forall f \in I, f(x) = 0\}$.

Definition 1.1. *Closed subspaces of $\mathbb{A}^n$ are called **affine algebraic sets** and irreducible algebraic sets are called **affine algebraic varieties***

Definition 1.2. *Given Y an affine algebraic set in $\mathbb{A}^n$, we define the **coordinate ring** $\mathcal{O}(Y)$ as $K[X_1, ..., X_n]/I(Y)$*

Definition 1.3. *Let $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ be affine algebraic sets. A **morphism** $X \longrightarrow Y$ of affine algebraic sets is a map $f : X \longrightarrow Y$ of the underlying sets such that there exist polynomials $f_1, ..., f_n \in k[T_1, ..., T_m]$ with $f(x) = (f_1(x), ..., f_n(x))$ for all $x \in X$.*

We denote the category of affine algebraic sets over K as Alg_K

Theorem 1.4. *Let $Y_1 \subset \mathbb{A}^n, X_1, ..., X_n$, $Y_2 \subset \mathbb{A}^m, T_1, , ..., T_m$ affine algebraic sets. There are bijections*

$$\begin{aligned}
&Hom_{K\text{-}alg}(\mathcal{O}(Y_2), \mathcal{O}(Y_1)) \\
&\overset{(*)}{\longleftrightarrow} \{(f_1, ..., f_m) \in K[X]^m | \forall x \in Y_1, (f_1(x), ..., f_m(x)) \in Y_2)\} \\
&\overset{(**)}{\longleftrightarrow} \{f : Y_1 \longrightarrow Y_2 | \forall \varphi \in \mathcal{O}(Y_2), \varphi \circ f \text{ is in } \mathcal{O}(Y_1)\} \\
&= \ Hom_{Alg_K}(Y_1, Y_2)
\end{aligned}$$

Proof. Key observation

To give $(f_1, ..., f_m) \in K[X]^m$ is "the same " as giving a ring morphism $g_0 : K[T] \longrightarrow K[X] : T_i \mapsto f_i$, which gives by composition $g_1 = \pi_1 \circ g_0$, where $\pi_1 : K[X] \longrightarrow \mathcal{O}(Y_1)$ is the canonical projection.

$$g_1 : K[T] \longrightarrow \mathcal{O}(Y_1)$$

which has a factorization

$$\begin{tikzcd}
K[T] \arrow[r, "g_1"] \arrow[d, "\pi_2"] & \mathcal{O}(Y_1) \\
\mathcal{O}(Y_2) \arrow[ur, "g"] &
\end{tikzcd}$$

iff $g_1(I(Y_2)) = 0$, which means iff

$$g_1(\varphi) = \text{ “replace } T_i \text{ by } f_i \text{ in } \varphi\text{”}$$

belongs to $I(Y_1)$ if $\varphi \in I(Y_2)$, which means if $x \in Y_1$, then $g_1(\varphi)(x) = 0$. That means $\varphi(f_1(x), ..., f_m(x)) = 0$ for $\varphi \in I(Y_2)$, i.e., $(f_1(x), ..., f_m(x)) \in Y_2$. If $x \in Y_1$. In the statement, this gives the $(*)$ bijection. Any k-algebra morphism $\mathcal{O}(Y_1) \longrightarrow \mathcal{O}(Y_2)$ comes from $K[T] \longrightarrow \mathcal{O}(Y_1)$ s.t. it vanishes on $I(Y_2)$.

For the bijection $(**)$, suppose

$$g : Y_1 \xrightarrow{g} Y_2 \xrightarrow{\varphi} K$$

sends $\varphi(Y_2)$ to $\varphi \circ g \in \mathcal{O}(Y_1)$. Then we get

$$\begin{aligned} \mathcal{O}(Y_2) &\longrightarrow \mathcal{O}(Y_1) \\ \varphi &\longmapsto \varphi \circ g, \end{aligned}$$

which is a K-algebra morphism.

As for the reverse direction, given g. From $\mathcal{O}(Y_2) \longrightarrow \mathcal{O}(Y_1)$ to get a $g : Y_1 \longrightarrow Y_2$. We get a $\tilde{g} : Y_1 \longrightarrow Y_2$ in the second set

$$\tilde{g}(x) = (f_1(x), ..., f_m(x))$$

then we have $\varphi \circ g \in \mathcal{O}(Y_1)$ for $\varphi \in \mathcal{O}(Y_2)$. One checks that this shows that the first and third sets are the same. □

Define morphism $Y_1 \longrightarrow Y_2$ by the second(and third) set. Composition in the obvious way and identity is a morphism. $\Longrightarrow$ get a category (Alg_K) of affine algebraic sets over K.

Corollary 1.5. *$Y \mapsto \mathcal{O}(Y)$, $g \mapsto [\varphi \mapsto \varphi \circ g]$ is a functor: $(Alg_K) \longrightarrow (K\text{-}Alg)^{opp}$.*

Facts: The “image” of this functor is the category of finitely generated K-algebras which are reduced.

Proof. A finitely generated reduced K-algebra. ($\exists n \geq 1$, so that $K[X_1, ..., X_n]/I \cong A$). Then “$A$ is reduced”$\Longleftrightarrow$ I is radical ideal. $\Longrightarrow A = \mathcal{O}(V(I))$, where $V(I) \subset \mathbb{A}^n$. □

Corollary 1.6. *There is a equivalence of categories between*

$$(\textit{Algebraic sets over } K) \longleftrightarrow (\textit{finitely generated reduced } K\textit{-Algebras.})$$

Example 1.7.

(1) $\mathbb{A}^1 \longrightarrow V(Y^2 - X^3 - X^2) \subset \mathbb{A}^2$, $t \mapsto (t^2-1, t(t^2-1))$

(2) $\mathbb{A}^1 \longrightarrow V(Y^2 - X^3) \subset \mathbb{A}^2$*:* $t \longmapsto (t^2, t^3)$ *is a bijection but* <u>*Not*</u> *an isomorphism.*

(3) Assume K *with characteristic* $p > 0$*,* $K \supset \mathbb{F}_p$*.* $Y = V(f_1, ..., f_m)$ *where* $f_i \in \mathbb{F}_p[X] \subset K[X]$*. Consider the morphism:*

$$\begin{aligned} Y &\longrightarrow Y \\ (x_1, .., x_n) &\longmapsto (x_1^p, ..., x_n^p). \end{aligned}$$

It is bijective and homeomorphism but not an isomorphism if $dim(Y) \geq 1$*.*

Proposition 1.8. $Y = V(I) \subset \mathbb{A}^n$

(1) The points of Y *are in bijection with maximal ideals* $I \subset \mathcal{O}(Y)$ *by*

$$Y \ni x \longmapsto \{f \in \mathcal{O}(Y) | f(x) = 0\}$$

(2) We have a bijection

$$\mathcal{O}(Y) \longleftrightarrow \ Hom_{Alg_K}(Y, \mathbb{A}^1)$$

Proof. (1) $I_x := Ker(\mathcal{O}(Y) \longrightarrow K)$, $f \mapsto f(x)$, since the evaluation map is surjective $[1 \mapsto 1]$, we get an isomorphism

$$\mathcal{O}(Y)/I_x \xrightarrow{\sim} K,$$

so I_x is maximal in $\mathcal{O}(Y)$.

Conversely, if $I \subset \mathcal{O}(Y)$ is maximal, we get $I = I'/I(Y)$ for $I' \subset K[X]$ maximal.

Nullstellensatz says $\exists (x_1, ..., x_n) \in \mathbb{A}^n$ s.t., $I' = (X_1 - x_1, ..., X_n - x_n)$.

Since $I' \supset I(Y)$, we get $(x_1, .., x_n) \in Y$. Then we check that $\mathcal{O}(Y) \longrightarrow \mathcal{O}(Y)/I \cong K$ is just given by $f \mapsto f(x_1, ..., x_n)$. That means $I = I_x$.

(2) We saw in 1.4, that there is a bijection between sets

$$\mathrm{Hom}_{Alg_k}(Y, \mathbb{A}^1) \longleftrightarrow \ \mathrm{Hom}_{K-alg}(\mathcal{O}(\mathbb{A}^1), \mathcal{O}(Y)).$$

But $\mathrm{Hom}_{K-alg}(\mathcal{O}(\mathbb{A}^1), \mathcal{O}(Y)) = \ \mathrm{Hom}_{K-alg}(K[X], \mathcal{O}(Y)) \cong \mathcal{O}(Y)$ (by $g : \mathcal{O}(\mathbb{A}^1) \longrightarrow \mathcal{O}(Y)$, $g \mapsto g(X)$) $\square$

Projective Algebraic sets

Projective sets can have a good notion of "compactness".

N.B. Any $Y \in (Alg_K)$ is **quasi-compact**(open cover have a finite subcover).

Definition 1.9. $\mathbb{P}^n_K = \mathbb{P}^n$ *can be either defined as*

"the set of lines in $\mathbb{A}^{n+1}$ *that pass through the origin"*

or

"the equivalence classes of points in $K^{n+1}\backslash\{0\}$ *with the equivalence relation* $x \sim y$ *iff* $x = \lambda y$ *for some* $\lambda \in K$*" and we use the notion* $[x_0 : ... : x_n]$ *for the equivalence class of* $(x_0, .., x_n)$

These two definitions are equivalent:

Given a line $l \in \mathbb{A}^1 \longleftrightarrow$ hyperplane in K^{n+1}, corresponds to a equation

$$a_0X_0 + ... + a_nX_n = 0$$

with at least one of a_i non-zero.

Conversely, from $[x_0 : .. : x_n]$, we we get the corresponding hyperplane/line trivially.

Notes the following fact:

$$\mathbb{P}^n = \cup_{0\leq i\leq n} H_i,$$

where $H_i = \{[x_0, ..., x_n] | x_i \neq 0\}$ and there is a bijection

$$\begin{aligned} &H_i \longrightarrow K^n \\ &[x_0 : ... : x_n] \longmapsto \left(\frac{x_0}{x_i}, ..., \widehat{\frac{x_i}{x_i}}, .., \frac{x_n}{x_i}\right) \\ &[y_1 : ... : y_{i-1} : 1 : y_i : ... : y_n] \hookleftarrow (y_1, ..., y_n) \end{aligned}$$

We define from linear algebra some notions in $\mathbb{P}^n$ a line in $\mathbb{P}^n$ is the image by the projection $K^{n+1}\backslash\{0\} \longrightarrow \mathbb{P}^n$ of the two dimensional affine subspace.

Example 1.10. $l_1, l_2 \subset \mathbb{P}^2$ *lines* $l_1 \cap l_2$ *is a line if* l_1 *and* l_2 *are identical and would be a single point otherwise.*

Observation: If $f \in K[X_0, ..., X_{n+1}]$ is homogeneous, then for $x \in \mathbb{P}^n$, it makes no sense to speak of "$f(x) \in K$", but the zero-loci or the set where $f(x) \neq 0$ does make sense.

Definition 1.11. *A* ***projective algebraic set*** $S \subset \mathbb{P}^n$ *is*

$$S = \{x \in \mathbb{P}^n | f_1(x) = ... = f_m(x) = 0\},$$

where $f_1, ..., f_m$ *are homogeneous of some degrees.*

An irreducible projective algebraic set is called a ***projective variety***

Notation: $V(f_1, .., f_n)$

Example 1.12. $V(Y^2Z - X^3 - XZ^2) \subset \mathbb{P}^2$.

Let $0 \leq i \leq n$, then $S \cap H_i = \{[x_0 : ... : x_n] \in S | x_i \neq 0\}$ is , via the bijection $H_i \longrightarrow K^n$, in bijection with an affine algebraic set $S_1 \subset \mathbb{A}^n$ given by $\tilde{f}_1(y) = ... = \tilde{f}_m(y) = 0$, where $\tilde{f}_i(y_1, .., y_n) = f_i(y_1, ..., y_{i-1}, 1, y_i, ..., y_n)$

1.2 Mar 2nd: Projective algebraic sets and regular functions

Recall: $\mathbb{P}^n_K = K^{n-1} - \{0\}/\sim$, and $H_i := \{[x_0 : ... : x_n] | x_i \neq 0\}$ is in bijection with $\mathbb{A}^n$. $V(f_1, ..., f_m) = \{x \in \mathbb{P}^n | \forall i, f_i(x) = 0\}$, where $f_1, ..f_m$ are homogeneous.

More generally, we can define

$$V(I) = V(\text{homogeneous element of } I=) = V(\cup_{d \geq 0} I_d)$$

where I is an homogeneous ideal of $K[X_0, ..., X_n]$ that is $I = \oplus_{d \geq 0} I_d$, I_d the the degree d piece of $K[X_0, ..., X_n]$.

Conversely, given $S \subset \mathbb{P}^n$, we can define

$$I(S) := \text{ideal generated by homogeneous polynomials that vanishes on } S$$

Lemma 1.13. *This is a homogeneous ideal*

Proof. $f \in I(s) \implies f = \sum_{i \in I} g_i f_i$, where f_i is homogeneous and vanishes on S. We can expand each g_i as $\sum_j g_{ij}$, where each g_{ij} is homogeneous in $I(S)$. Then we know $f \in \otimes I(S)_d$ and the converse is clear. □

Lemma 1.14. *The projective sets $V(I)$ where I is homogeneous form the closed sets of a topology. It is called the Zariski topology (same name for the induced topology on projective sets).*

Example 1.15. $H_0 \subset \mathbb{P}^n$ *and* $\sigma : H_0 \cong \mathbb{A}^n$. *Under this bijection, the Zariski topologies correspond σ is a homeomorphism*

$$f \in K[X_0, ..., X_n] \textit{ homogeneous} \rightsquigarrow V(f) \subset \mathbb{P}^n$$

$$\tilde{f} = f(1, X_1, ..., X_n) \in K[X_1, .., X_n] \rightsquigarrow V(\tilde{f}) \subset \mathbb{A}^n$$

and $\sigma(V(f)) = V(\tilde{f})$.

Definition 1.16. *$Y \subset \mathbb{P}^n$ projective $S(Y) = K[X_0, ..., X_n]/I(Y)$,* ***homogeneous coordinate ring***

Note elements in $S(Y)$ are not functions on Y. The geometric meaning of $S(Y)$ will be explained latter with the language of schemes.

We now want to define morphisms of projective algebraic sets. We have to look at it more carefully because we can not simply copy the affine definition.

Definition 1.17. *$Y \subset \mathbb{P}^n$ projective, let $V \subset Y$ be an open subsets of Y.*

(1) $f : V \longrightarrow K$ continuous is called ***regular*** *on Y if $\forall x \in Y$, $\exists U$ open $x \in U$, $\exists f_1, f_2 \in K[X_0, ..., X_n]$ homogeneous of same degree such that $f_2(x) \neq 0$ for all $x \in U$ and $f(x) = \frac{f_1(x)}{f_2(x)}$ for $x \in U \cap Y$*

(2) Y_1, Y_2 are projective sets in $\mathbb{P}^n, \mathbb{P}^m$, $f : Y_1 \longrightarrow Y_2$ is a ***morphism*** *if f is continuous and for any $U \subset Y_1$ open and any $\varphi : U \longrightarrow K$ regular, the composite $\varphi \circ f : f^{-1}(U) \longrightarrow K$ is regular.*

Note: IN (2), one can not restrict to φ regular on Y_2 because often the space of such function is reduced to K

Proposition 1.18. *For $\mathbb{P}^n$, the space of functions regular on $\mathbb{P}^n$ is K.*

Proof. The case $n = 1$ implies the general case: if $f : \mathbb{P}^n \longrightarrow K$ regular, and $x \neq y$ in $\mathbb{P}^n$, the line joining x to y in $\mathbb{P}^n$ is "isomorphic" to $\mathbb{P}^1$ and $f|_L$ is regular so constant, hence $f(x) = f(y)$.

For $n = 1$, suppose x, y are arbitrary points and let $U \ni x$, $V \ni y$ be open neighbourhoods such that $f|_U = f_1(x)/f_2(x)$ and $f|_V = g_1(x)/g_2(x)$ where f_1, f_2, g_1, g_2 are homogeneous polynomials and f_1, f_2 have the same degree as well as g_1, g_2. We may assume that f_1 and f_2 are coprime and also g_1, g_2 are coprime. Hence on $U \cap V$,

$$f_1 g_2 = g_1 f_2.$$

We know that $U \cap V$ is infinite so this implies $f_1 = g_1$ and $f_2 = g_2$. Since x and y were arbitrary points we conclude that $f = f_1(x)/f_2(x)$ on all of $\mathbb{P}^1$ hence f is a constant. □

Concretely: To say that $f : Y_1 \subset \mathbb{P}^n \longrightarrow Y_2 \subset \mathbb{P}^m$ is a morphism of projective algebraic sets. It reduces to $\forall x \in Y_1, \exists U$ open containing x s.t. there exists $f_0, ..., f_m \in K[X_0, ..., X_{n+1}]$ homogeneous of same degree, with no common zero in U, such that $\forall y \in U \cap Y_1$, $f(y) = [f_0(y) : ... : f_m(y)]$. It is easy to see that if f is of this form, then it is a morphism.

The converse is left as an exercise.

Example 1.19. *(1) Let $g \in Gl_n(K), n \geq 1$. Define*

$$f_g : \mathbb{P}^n \longrightarrow \mathbb{P}^n$$

$$[x_0 : ... : x_n] \longmapsto [g(x_0, ..., x_n)]$$

is a morphism. In fact, it is an isomorphism. $f_g^{-1} = f_{g^{-1}}$. It also has some other properties: $f_g = f_{\lambda g}, \lambda \neq 0$ and we get an induced group morphism

$$\begin{array}{ccc} PGL_{n+1}(K) & = & GL_{n+1}(K)/K^{\times} \\ \downarrow & & \\ Aut_{proj}(\mathbb{P}^n) & & \end{array}$$

which is an isomorphism. A special case is $Aut_{hol}(\mathbb{CP}^1) = PGL_2(\mathbb{C})$

$$g \longmapsto \left[z \mapsto \frac{az+b}{cz+d} \right]$$

(2) $K = \mathbb{C}$. One can do holomorphic geometry (using holomorphic functions instead of polynomials). IN $\mathbb{C}^n$,we get a much more complicated picture [e.g. $V(\sin z)$] is a an infinite sets in $\mathbb{P}^n_{\mathbb{C}}$, however Chow proved that the holomorphic sets and the projective algebraic sets are the same (Serre "GAGA" principle compares many different invariant of both categories.)

(3) Consider the map $S := V(Y^2Z - X^3 - XZ^2) \xrightarrow{f} \mathbb{P}^1$, $[x : y : z] \mapsto [y : z]$.
<u>Claim</u>, this is a morphism of projective sets.
This means that there is no solution to $Y^2Z - X^3 - XZ^2 = 0$ with $Y = Z = 0$. (But $[x : y : z] \mapsto [x : z]$ is not a morphism because $[0 : 1 : 0] \in S$). f is surjective but not injective $[x : y : z]$ and $[x : -y : z]$ have same image. This works in field k with char$k \neq 2$.

(4) $\mathbb{P}^1 \xrightarrow{v} \mathbb{P}^2$, $[x : y] \mapsto [x^2 : xy : y^2]$ (special case of Veronese embedding). This is a morphism. The image of v is equal to $[y_0 : y_1 : y_2], \mathbb{P}^2$. $S = V(Y_1^2 - Y_0Y_2)$. In fact, σ gives an isomorphism $\sigma : \mathbb{P}^1 \longrightarrow S$ with inverse given by

$$\tau : S \longrightarrow \mathbb{P}^1$$

$$[y_0 : y_1 : y_2] \mapsto \begin{cases} [Y_1 : Y_2] & \text{if } Y_2 \neq 0 \\ [Y_0 : Y_1] & \text{if } Y_0 \neq 0 \end{cases}$$

τ is a morphism defined on all of S, because if $[y_0 : y_1 : y_2] \in S$ satisfies $y_0 = y_2 = 0$, it would implie $y_1^2 = y_0y_2 = 0 \Longrightarrow y_1 = 0$

$$\tau \circ \sigma([x : y]) = \tau([x^2 : xy : y^2]) \quad = \begin{cases} [xy : y^2] = [x : y], y \neq 0 \\ [x^2 : xy] = [x : y], x \neq 0 \end{cases}$$

therefore $\tau \circ \sigma = id_{\mathbb{P}^1}$ and $\sigma \circ \tau = id_S$ can proved similarly

One can not find $f_0 : f_1$ in $K[Y_0, Y_1, Y_2]$ s.t. $\tau([y_0 : y_1 : y_2] = [f_0(Y) : f_1(Y)]$ for all $Y \in S$

1.3 Mar 5th: Exercise class

The content covered can be found in Hartshorne, p50ff Proposition 7.4 and Theorem 7.5.

1.4 Mar 6th: Rational/birational maps

$Y \subset \mathbb{A}^n$ algebraic if Y is irreducible, then $\mathcal{O}(Y)$ is an integral domain. Let $K(Y)$ be its quotient field. What is the geometric meaning of $K(Y)$? It is called the **function field** of Y.

We will see

Theorem 1.20. *For Y_1, Y_2 affine varieties (irreducible) $K(Y_1) \cong K(Y_2)$ as fields $\Longleftrightarrow \exists U_1 \subset Y_1$ open dense subset and $\exists U_2 \subset Y_2$ open dense subset such that U_1 and U_2 are isomorphic.*

Definition 1.21. *(Quasi-affine and quasi-projective) varieties)*

1. ***quasi-affine variety** V is an open subset $V \subset Y$, where $Y \subset \mathbb{A}^n$ is an affine variety. $[V \neq \emptyset \Longrightarrow V$ dense in $Y \Longrightarrow V$ irreducible $]$. It is given by the Zariski's topology from Y.*
   *(1') $V \subset Y \subset \mathbb{P}^n$ where V is an open subset of Y is **quasi-projective**, where Y is projective variety.*

2. *A regular function $f : V \longrightarrow K = \mathbb{A}^1$, where V is quasi-affine is an f such that for all $x \in V$, $\exists U \subset V$ open containing x s.t., $\forall x \in V$, $f(x) = \frac{f_1(x)}{f_2(x)}$ where $f_1, f_2 \in \mathcal{O}(\mathbb{A}^n)$ and $f_2(x) \neq 0$ on U.*
   (2') V is quasi-projective variety a regular function f is $\frac{f_1(x)}{f_2(x)}$ f_i homogeneous of same degrees.

3. *If V_1, V_2 are <u>Varieties</u> (of any of the four types), then $f : V_1 \longrightarrow V_2$ is a* ***morphism*** *if for al open $U \subset V_2$ all $\varphi : U \longrightarrow K$ regular, the composition $\varphi \circ f : f^{-1}U \longrightarrow K$ is also regular.*

N.B.

1. This makes sense because if $U \subset V_2$, where V_2 is quasi affine U open, $\Longrightarrow$ $U \subset V_2 \subset Y$ so U is also quasi-affine in $\mathbb{A}^n$

2. <u>Exercise</u> If f is regular on V, then f is continuous $V \longrightarrow \mathbb{A}^1$. (check that $f^{-1}(\{a\})$ is closed, use that closedness is a local condition.)

3. In the (quasi)-affine case, it is enough to check that $\varphi \circ f$ is regular on V_1 for φ regular on V_2.

4. <u>Notation</u>:
$$\mathcal{O}(V) = \{f : V \longrightarrow K | \text{ regular}\}$$
This is a ring of with unity, and because of the condition that for open $V \subset Y$ in a variety Y, either $\mathcal{O}(V) = 0, V \neq \emptyset$ or V is dense in Y, $\Longrightarrow \mathcal{O}(V)$ integral domain.

Example 1.22.

1. *$GL_n(K) = \{x \in M_{n\times n}(K) | \det(x) \neq 0\} \subset \mathbb{A}^{n^2}$ is quasi-affine since $\det : M_{n\times n}(K) \longrightarrow K$ is continuous and not emptyset.*

2. *In fact, for any $0 \neq f \in \mathcal{O}(\mathbb{A}^n)$*
$$U_f = \{x \in \mathbb{A}^n | f(x) \neq 0\}$$
is a quasi-affine variety.

<u>Fact</u>: There is an isomorphism
$$\sigma = \begin{cases} U_f \longrightarrow Y = \{(x, y) \in \mathbb{A}^{n+1} | yf(x) = 1\} \\ \quad x \longmapsto \left(x, \dfrac{1}{f(x)}\right) \end{cases}$$
with inverse $(x, y) \overset{\pi}{\longmapsto} x$. (Indeed, $\pi \circ \sigma = Id_{U_f}$, $\sigma \circ \pi = Id_Y$) and π is a morphism: Consider $\varphi \in \mathcal{O}(U_f)$
$$Y \xrightarrow{u} U_f \xrightarrow{\varphi} K$$

then $\varphi \circ \pi(x,y) = \varphi(x)$.

Indeed, for any $x \in U_f$, $\exists f_1, f_2 \in \mathcal{O}(\mathbb{A}^2)\varphi(x) = \frac{f_1(x)}{f_2(x)}, f_2(x) \neq 0$, *one can show: assume* $f_2(x) = f(x)^d$ *then*

$$\varphi(x) = \frac{f_1(x)}{f(x)^d} = f_1(x)y^d$$

for $(x,y) \in Y$, *so this is regular.*

(2) σ *is a morphism*

$$U_f \xrightarrow{\sigma} Y \xrightarrow{\varphi} K$$

$\varphi \in \mathcal{O}(Y) = K[X_1, ..., X_n, Y]/(Yf(x) = 1)$

$$\varphi \circ \sigma(x) = \varphi(x, 1/f(x)) = \left(\sum_j a_j Y^j\right)|_{Y=1/f(X)}$$
$$= \sum_j a_j(x)/f(x)^j \in \mathcal{O}(U_f)$$

3. $\mathbb{P}^n = \cup_{0 \leq i \leq n} H_i$, *with* $H_i = \{[x_0 : ... : x_n] | x_i \neq 0\}$, $H_i \subset \mathbb{P}^n\}$ *open , so quasi-projective. The map*

$$\begin{cases} H_i \xrightarrow{f_i} \mathbb{A}^n \\ [x_0 : ... : x_n] \longmapsto (\frac{x_0}{x_1}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}x_i}{,}..., \frac{x_n}{x_i}) \end{cases}$$

is an isomorphism.

Definition 1.23. Y *variety,* $K(Y) = \{(U,f) | \emptyset \neq U \subset Y$ *open* $, f \in \mathcal{O}(U)\}/\sim$, *where* $(U_1, f_1) \sim (U_2, f_2)$ *iff* $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$

Fact: $\sim$ is an equivalence relation. We define

$$(U_1, f_1) + (U_2, f_2) = (U_1 \cap U_2, f_1 + f_2)$$

$$0 := (Y, 0), \quad 1 := (Y, 1)$$

Proposition 1.24. Y *is quasi-affine,* $U \subset Y$ *open nonempty.*

1. $\mathcal{O}(Y) \hookrightarrow \mathcal{O}(U) \hookrightarrow K(Y)$
   $f \longmapsto f|_U \mapsto (U, f)$

2. $K(Y)$ *is a field, and identifies with the fraction field of* $\mathcal{O}(Y)$ *and of* $\mathcal{O}(U)$.

3. *if Y is an affine variety, then $\mathcal{O}(Y)$ as defined above coincides with $\mathcal{O}(Y) = K[X_1, ..., X_n]/I(Y)$ as defined in previous sections.*

   (3') If $Y = U_f$ for $0 \neq f$ in $\mathcal{O}(\mathbb{A}^n)$, then $\mathcal{O}(Y) = \{f_1/f^d | f_1 \in \mathcal{O}(\mathbb{A}^n, d \geq 0\} = \mathcal{O}(\mathbb{A}^n)_f$ the localization at f.

Proof. (1), The morphism $\mathcal{O}(Y) \longrightarrow \mathcal{O}(U) \longrightarrow K(Y)$ are injective because any $U \subset Y, \neq \emptyset$ is dense.

(2) Let $(U, f) \neq 0$ in $K(Y)$, then $\exists x_0 \in U$, $f(x_0) \neq 0$ in a $V \subset U$, $x_0 \in V$

$$f(x) = \frac{f_1(x)}{f_2(x)}, f_1, f_2 \in \mathcal{O}(\mathbb{A}^n), f_2 \neq 0 \text{ in } V$$

in particular, $f_1(x_0) \neq 0$ and $(U \cap \{f_1(x) \neq 0\}, \frac{f_2(x)}{f_1(x)}) \in K(Y)$, where $U \cap \{f_1(x) \neq 0\} \neq \emptyset$ is the inverse of (U, f) in $K(Y)$.

By (1), $K(Y) \supset \mathcal{O}(Y)$.

Let $(U, f) \in K(Y)$, pick $x \in Y$ so that around x, $f(x) = \frac{f_1}{f_2}$, $f_i \in \mathcal{O}(\mathbb{A}^n)$, then $(U, f) = \frac{(Y, f_1)}{(Y, f_2)}$, so $K(Y)$ is the fraction field of $\mathcal{O}(Y)$.

(3) Write $\mathcal{O}'(Y) = K[X]/I(Y)$. Note $K[X.., Y]/I(Y)$ identifies to a ring of functions on Y, the claim is that this ring is $\mathcal{O}(Y)$.

Observation: For $x \in Y$, to say that $f : Y \longrightarrow K$ is "regular at x" means precisely that $f \in \mathcal{O}'(Y)_{I_x}$, where $I_x = \{f \in \mathcal{O}'(Y) | f(x) = 0\}$. (Localization at a maximal ideal)

So

$$\begin{aligned} \mathcal{O}(Y) &= \bigcap_{x \in Y} \mathcal{O}'(Y)_{I_x} \\ &= \bigcap_{\mathfrak{m} \subset \mathcal{O}'(Y)} \mathcal{O}'(Y)_{\mathfrak{m}} \\ &= \mathcal{O}'(Y) \end{aligned}$$

the second equality from Nullstellensatz and the third from commutative algebra.

(3') Similarly, using characterization of maximal ideals in $A_f, f \neq 0$ □

Definition 1.25. *K(Y) is called the fraction or function field of Y*

Example 1.26. $K(\mathbb{A}^n) = K(\mathbb{P}^n) = K(X_1, .., X_n)$

Definition 1.27. *(Rational maps) Y_1, Y_2 varieties. A **rational map** $f : Y_1 \dashrightarrow Y_2$ is a pair $(U, \tilde{f})$ where $U \neq \emptyset$ in Y_1 and $\tilde{f} : U \longrightarrow Y_2$ is a morphism with $(U, \tilde{f}) = (U', \tilde{f}')$ iff*

$$\tilde{f}|_{U \cap U'} = \tilde{f}'|_{U \cap U'}$$

[Check: this is coherent, i.e., this is an equivalence relation]

Definition 1.28. *$f : Y_1 \dashrightarrow Y_2$ is a **dominant** if its image $\tilde{f}(U) \subset Y_2$ is dense.*

Example 1.29. *(1) there is a bijection $\{Y \dashrightarrow \mathbb{A}^1\} = K(Y)$*

$$\begin{array}{cc} (U, \tilde{f}) & (U, f) \\ \tilde{f} : U \longrightarrow \mathbb{A}^1 \text{ morphism} & f : U \longrightarrow K \text{ regular} \end{array}$$

So it is enough to check

$$Hom_{Var}(U, \mathbb{A}^1) = \mathcal{O}(U)$$

Left as exercise

(2) $Y, f_1, f_2, f_3 \in \mathcal{O}(Y)$

$$\begin{cases} Y \dashrightarrow \mathbb{P}^2 \\ x \longmapsto [f_1(x) : f_2(x) : f_2(x)] \end{cases}$$

defined on $\{x | f_i(x)$ are not all zero$\}$, which is open if any of the 3 sections is non-zero.

Theorem 1.30. *Y_1, Y_2 varieties*

$\exists\{Y_1 \overset{f}{\dashrightarrow} Y_2 | f$ dominant$\}$

$\overset{bij}{\longleftrightarrow}$

$K(Y_2) \longrightarrow K(Y_1)$

Corollary 1.31. *Y_1, Y_2 varieties. Y_1 and Y_2 are birational*

iff $K(Y_1)$ is isomorphic to $K(Y_2)$

iff $\exists U \subset Y_1$ open $\neq \emptyset$ $\exists V \subset Y_2$, open $\neq \emptyset$ so that U and V are isomorphic as varieties.

Corollary 1.32. *Any variety Y of dimension $d \geq 0$ is birational to a hypersurface $V \subset \mathbb{P}^{d+1}$*

Proof. (1) Given $Y_1 \overset{f}{\dashrightarrow} Y_2$ dominant, we want a morphism $K(Y_2) \longrightarrow K(Y_1)$.

Let $(U, \tilde{f}) = f$, $(V, \varphi), \varphi : V \longrightarrow K$ in $K(Y_2)$

$$\varphi \circ f : \tilde{f}^{-1}(V) \longrightarrow K$$

is in $K(Y_1)$, provided $\tilde{f}^{-1}(V)$ is dense, it is enough that $\tilde{f}^{-1}(V) \neq \emptyset$, $\tilde{f}(U) \cap V \neq \emptyset$, since V is open and $\tilde{f}(U)$ is dense.

(2) Given $i : K(Y_2) \longrightarrow K(Y_1)$. Let $\tilde{Y_2} \subset Y_2 \subset \mathbb{A}^n$ open quasi-affine so that $K(Y_2) = K(\tilde{Y_2}) = Frac(\mathcal{O}(\tilde{Y_2}))$

Let $X_1, .., X_n$ be the coordinates in $\mathbb{A}^n$ as elements of $\mathcal{O}(\tilde{Y_2})$, then let

$$f_j = i(X_j) \in K(Y_1)$$

$f_j \longleftrightarrow (U_j, \tilde{f_j})$ with $U_j \subset Y_1$ dense and $\tilde{f_j} \in \mathcal{O}(U_j)$. Then $f_j \longleftrightarrow (U, \tilde{f_j})$, $U := U_1 \cap ... \cap U_n$ still dense.

Define $U \longrightarrow \tilde{Y_2} \hookrightarrow Y_2$ by

$$x \longmapsto (\tilde{f}_1(x), ..., \tilde{f}_n(x)).$$

This is a rational map $Y_1 \dashrightarrow Y_2$

□

1.5 Mar 9th: Continue and Nonsingular varieties

Recall

Theorem 1.33. *Y_1, Y_2 varieties*

$$\{dominant\ Y_1 \dashrightarrow Y_2\} \longleftrightarrow \{K(Y_2 \hookrightarrow K(Y_1)\}$$

Corollary 1.34. *The followings are equivalent:*

- *Y_1 and Y_2 are birational*
- *the function field $K(Y_1)$ and $K(Y_2)$ are isomorphic*
- *$\exists \emptyset \neq U \subset Y_1$, $\emptyset \neq V \subset Y_2$ and isomorphism between U and V*

Proof. The last condition implies the second because $K(Y_1) = Frac(\mathcal{O}(U)) \cong Frac(\mathcal{O}(V)) = K(Y_2)$. Assume we have rational maps

$$Y_1 \overset{f_2}{\dashrightarrow} Y_2 \overset{f_2}{\dashrightarrow} Y_1$$

with $f_2 \circ f_1 = id_{Y_1}, f_2 \circ f_1 = id_{Y_2}$.

Let $f_1 = (U', \tilde{f}_1), f_2 = ()V', \tilde{f}_2$

$$\begin{tikzcd}
Y_1 \arrow[r, dashed, "f_1"] & Y_2 \arrow[r, dashed, "f_2"] & Y_1 \\
U' \arrow[u, hook] \arrow[ur, "\tilde{f}_1"'] & V' \arrow[u, hook] \arrow[ur, "\tilde{f}_2"'] &
\end{tikzcd}$$

$$f_2 \circ f_1 = (Y_1, Id_{Y_1})$$

so $\tilde{f}_2(\tilde{f}_1(x)) = x$ if $\tilde{f}_1(x) \in V'$. Similarly, $f_1 \circ f_1 = (\tilde{f}_2^{-1}(U'), \tilde{f}_1 \circ \tilde{f}_2)$. Define $U = \tilde{f}_1^{-1}(\tilde{f}_2^{-1}(U')) \subset U'$, which is a dense open subset. Also we have $V = \tilde{f}_2^{-1}(\tilde{f}_1^{-1}(V'))$.

Claim: $U \xrightarrow{\tilde{f}_1} V \xrightarrow{\tilde{f}_2} U$ and then $\tilde{f}_1|_U, \tilde{f}_2|_U$ are reciprocal isomorphism.

We check that if $x \in U$, then $\tilde{f}_1(x) \in V$. Let $y = \tilde{f}_1(x) \in V'$ so $\tilde{f}_2(y) = \tilde{f}_2(\tilde{f}_1(x)) = x$ so $\tilde{f}_1(\tilde{f}_2(y)) = \tilde{f}_1(x) \in V' \Longrightarrow y \in V$. Similarly for f_2. □

Definition 1.35. *A **rational variety** Y is a variety Y birational to $\mathbb{P}^n$ for some n (or to $\mathbb{A}^n$). BY the theorem above we know $\exists n, K(Y) \cong K(X_1, ..., X_n)$.*

*A **unirational variety** Y is a variety s.t. there is a dominant $\mathbb{P}^n \dashrightarrow Y$ for some n, by theorem above $\exists n, K(Y) \hookrightarrow K(X_1, ..., X_n)$ We obviously have*

$$\textit{Unirational} \Longleftarrow \textit{rational}$$

but

$$\textit{Unirational} \overset{?}{\Longrightarrow} \textit{rational}$$

For char $= 0$: $\dim Y = 1$ or 2, Luroth and some italian showed that unirational curves or surfaces are rational.

First example in char 0 of non-rational unirational varieties were provided by Clemens-Griffith: certain cubic hypersurfaces in $\dim 3$.

Iskovskih-Manin "general " quantic hypersurfaces of $\dim 3$.

Corollary 1.36. *Any variety Y is birational to a hypersurface in $\mathbb{P}^{\dim(Y)+1}$ or $\mathbb{A}^{\dim(Y)+1}$.*

Proof. Let $d = \dim(Y) = \dim(\mathcal{O}(Y))$. Then a fact in commutative algebra says $K(Y)$ is a finite separable extension of $K(X_1, ..., X_d) =: E$. By the primitive element theorem, there exists $\alpha :\in K(Y)$ such that $K(Y) = E(\alpha)$. Let $f \in E[T]$ be the minimal polynomial of α.

Write

$$f = \sum_{i=0}^{n} a_i T^i = \sum_{i=0}^{n} \frac{b_i}{c_i} T^i,$$

where $a_i \in E$ and $a_i, b_i \in A = K[X_1, ...X_d]$

$\Longrightarrow \tilde{f}(\alpha) = 0$ where $\tilde{f} = (\prod c_i) f \in A[T] = K[X_1, ..., X_d, T]$. Define $\tilde{Y} = V(\tilde{f}) \subset \mathbb{A}^{d+1}$. This is what we wanted.

(1) $\tilde{Y}$ is an irreducible hypersurface.

(2) $\tilde{Y}$ is birational to $Y \iff K(\tilde{Y}) = K(Y)$

Step 1: Need $\tilde{f}_1 \in K[X_1, ..., X_d, T]$ irreducible. Suppose $\tilde{f} = \tilde{f}_1\tilde{f}_2, \tilde{f}_i \in A[T] \implies E \ni f = \frac{\tilde{f}_1}{\prod c_i}\tilde{f}_2$ factors in $E[T]$, since f is irreducible in $E[T]$, one of $\deg(\tilde{f}_1)$ or $\deg(\tilde{f}_2)$ is zero

$\implies \tilde{f}$ is irreducible.

Step (2): $\mathcal{O}(\tilde{Y}) = K[X, T]/(\tilde{f})$. We have an injective morphism

$$\begin{cases} \mathcal{O}(\tilde{Y}) \longrightarrow K(Y) = E(\alpha) \\ \quad X_i \longmapsto X_i \\ \quad T \longmapsto \alpha \end{cases}$$

so the fraction field $K(\tilde{Y})$ injects into $K(Y)$. The image of $K(\tilde{Y})$ contains $X_1, ..., X_d$ and α hence it contains $E(\alpha)$, i.e., $K(\tilde{Y}) = K(Y)$ $\square$

Nonsingular varieties

Concrete geometric definition:

Definition 1.37. *$Y \subset \mathbb{A}^n$ affine variety $\dim Y = d$, $x \in Y$. We say Y is* ***nonsingular*** *at x if for any generating set $\underline{f} := (f_1, ..., f_m)$ of $I(Y)$, the Jacobian matrix at x*

$$J_{\underline{f}}(x) = \left(\frac{\partial f_i(x)}{\partial x_j}\right)_{1 \leq i \leq m, 1 \leq j \leq n} \in M_{m \times n}(K)$$

has rank $n - d$. If this holds for all x, then we say Y is nonsingular.

Key fact: It suffices to check the rank of $J_F(x)$ for some generating set.

Indeed suppose $\underline{h} = (h_1, .., h_k)$ also generate $I(Y)$ so

$$f_i = \sum_{\ell=1}^{k} g_{i\ell} h_\ell,$$

where $g_{i\ell} \in \mathcal{O}(\mathbb{A}^n)$, $\frac{\partial f_i}{\partial x_j} = \sum_{\ell=1}^k \frac{\partial g_{i\ell}}{\partial x_j} h_\ell + \sum_{\ell=1}^k g_{i\ell} \frac{h_\ell}{x_j}$

At x where $h_\ell(x) = 0$, we get

$$\frac{\partial f_i}{x_j}(x) = \sum_{\ell=1}^{k} \frac{\partial g_{i\ell}}{\partial x_j} h_\ell$$

$$\implies J_{\underline{f}}(x) = M J_{\underline{h}}(x)$$

so rank $J_{\underline{f}}(x) \leq$ rank $J_{\underline{h}}(x)$. Exchanging $\underline{f}, \underline{h}$, we get the equality.

Example 1.38.

(1) If $K = \mathbb{C}$, the implicit function theorem says that around a point where $J_{\underline{f}}(x)$ has rank $n-d$, then $V(f_1, ..., f_m)$ is diffeomorphic to $\mathbb{C}^d$

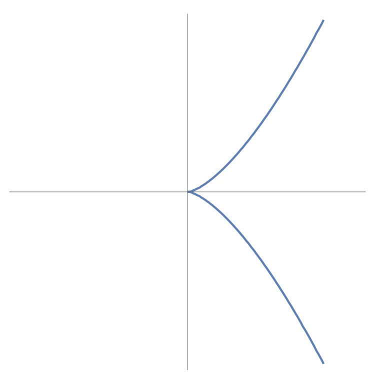
(2) Let $Y = V(f)$, f irreducible in $\mathbb{A}^n$. Then $x \in V(f)$ is nonsingular $\iff$ $(\partial f(x)/\partial x_1, ..., \partial f(x)/\partial x_n) \neq 0$

We have a singular point $\iff$ the system of $n+1$ equations

$$\begin{cases} f(x) = 0 \\ \dfrac{\partial f}{\partial x_1}(x) = 0 \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(x) = 0 \end{cases}$$

has a solution. For instance

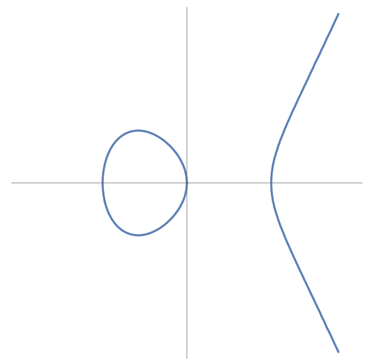
$$Y^2 = X^3$$



$$\begin{cases} f = Y^2 - X^3 \\ \dfrac{\partial f}{\partial X} = -3X^2 \\ \dfrac{\partial f}{\partial Y} = 2Y \end{cases}$$

so $X = Y = 0$ is the only singular point.

$$Y^2 = X^3 - X$$



$$\begin{cases} f = Y^2 - X^3 + X \\ \dfrac{\partial f}{\partial X} = -3X^2 + 1 = 0 \\ \dfrac{\partial f}{\partial Y} = 2Y = 0 \end{cases}$$

If char $k \neq 2$, $\Longrightarrow Y = 0$, $X^3 - X = 0$ $X = 0, -1, 1$ do not satisfy the system of solutions. In the case char $= 2$, $(1, 0) \in Y$ is singular.

The intrinsic characterization was found by Zariski.

Definition 1.39. *$x \in Y$ variety*

*(1) The **local ring** of Y at x*

$$\begin{aligned} \mathcal{O}_{Y,x} &= \{f \in K(Y) | f \text{ defined at } x\} \\ &= \{ \text{ regular functions on some } U \ni x\}/(f_1 \sim f_2 \text{ if they coincide on } U_{f_1} \cap U_{f_2}) \end{aligned}$$

if Y is affine, then $\mathcal{O}_{Y,x} = \{f_1/f_2 \in K(Y) \mid f_i \in \mathcal{O}(Y), f_2(x) \neq 0\} = \mathcal{O}(Y)_{\mathfrak{m}_x}$, where $\mathfrak{m}_x = \{f \in \mathcal{O}(Y) | f(x) = 0\}$ is the maximal ideal corresponding to x.

$$\mathcal{O}(Y) \subset \mathcal{O}_{Y,x} \subset K(Y)$$

Definition 1.40. *$Y \subset \mathbb{A}^n$ affine $x \in Y$. The (Zariski) cotangent spaces of Y at x is the K-vector space*

$$\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2,$$

where $\mathfrak{m}_{Y,x} \subset \mathcal{O}_{Y,x}$ is the maximal ideal

Remark 1.41. *$\mathcal{O}_{Y,x}$ is a local ring, it has a unique maximal ideal $\mathfrak{m}$ which is $\mathcal{O}_x \mathcal{O}_{Y,x}$ in the affine case. Moreover $\mathcal{O}_{Y,x}/\mathfrak{m} = K$ by $f \mapsto f(x)$.*

N.B. Intuitively, the Taylor expansion of $f \in \mathcal{O}_{Y,x}$ about $x \in \mathfrak{m}_{Y,x}$ is

$$f(X) = f(x) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x)(X - x_j) + \dots$$

if $f \in \mathfrak{m}_{Y,x}$ then $f(x) = 0$ and terms of order ≥ 2 belongs to $\mathfrak{m}_{Y,x}^2$, so f has image

$$\sum \frac{\partial f}{\partial x_j} dX_j \in \mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2$$

where $dX_j = X - x_j$.

Definition 1.42. *A local ring $\mathcal{O}$ with maximal ideal $\mathfrak{m}$ is callled* ***regular*** *if*

$$\dim \mathcal{O} = \dim_k \mathfrak{m}/\mathfrak{m}^2$$

where $k = \mathcal{O}/\mathfrak{m}$ is the residue field.

1.6 Mar 13th-A: Continue and proofs

Theorem 1.43. *(Zariski) For $x \in Y \subset \mathbb{A}^n$ the following are equivalent:*

(1) Y is non-singular at x

(2) $\dim(Y) = \dim_K(\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2)$, where $\mathfrak{m}_{Y,x}$ is the maximal ideal in the local ring $\mathcal{O}_{Y,x} := \mathcal{O}(Y)_{\tilde{\mathfrak{m}}_{Y,x}}$ with $\tilde{\mathfrak{m}}_{Y,x} = \{f \in \mathcal{O}(Y) \mid f(x) = 0\}$.

Remark 1.44. *One can show $\dim_K \mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2 \geq \dim(Y)$ so the question is whether it is larger or not.*

Proof. Denote $I := I(Y)$, $d = \dim Y$ and $x := (x_1, ..., x_n) \in \mathbb{A}^n$. Let $I_x := (X_1 - x_1, ..., X_n - x_n) \subset \mathcal{O}(\mathbb{A}^n)$ so that $\tilde{\mathfrak{m}}_{Y,x} = I_x/I$. There is an isomorphism of K-vector spaces

$$\theta : \begin{cases} I_x/I_x^2 \longrightarrow K^n \\ \quad f \longmapsto \left(\dfrac{\partial f}{\partial X_j}(x)\right)_{1 \leq j \leq n} \end{cases}.$$

To see this, note that $f \in I_x^2$ iff $f = \sum_{i,j} h_{ij}(X_i - x_i)(X_j - x_j)$ and thus each $f \in I_x/I_x^2$ can be expressed as

$$f = \sum_{i}^{n} (X_i - x_i) \frac{\partial f}{\partial X_i}(x) + I_x^2.$$

That means each f is uniquely defined by its derivatives and this preserves scalar multiplication.

Let $(f_1, ..., f_m)$ be a generating set of I. Then $(\theta(f_1), ..., \theta(f_m))$ are the columns of $J_{\underline{f}}(x)$ and for any $f \in I$ we can write

$$f = \sum_j g_j f_j$$

for some $g_j \in K[X]$. Thus

$$\frac{\partial f}{\partial X_i}(x) = \sum_{j=1}^{n} g_j(x) \frac{\partial f_j}{\partial X_i}(x).$$

In vector notation this is

$$\theta(f)_i = \sum_{j=1}^{n} g_j(x)\theta(f_j)_i.$$

We conclude that the span of the $\theta(f_j)$ is $\theta((I + I_x^2)/I_x^2)$, so

$$rank\ J_{\underline{f}}(x) = \dim_K \theta((I + I_x^2)/I_x^2) = \dim_K (I + I_x^2)/I_x^2.$$

Consider the short exact sequence

$$0 \longrightarrow (I + I_x^2)/I_x^2 \longrightarrow I_x/I_x^2 \longrightarrow I_x/(I + I_x^2) \longrightarrow 0.$$

From this we see that

$$rank\ J_{\underline{f}}(x) + \dim_K I_x/(I + I_x^2) = \dim_K I_x/I_x^2.$$

We already established that the RHS is n hence x is non-singular iff $d = \dim_K I_x/(I+ I_x^2)$.

Consider

$$\begin{tikzcd} I_x \arrow[r] \arrow[rr, bend right, "\varphi"'] & \tilde{\mathfrak{m}}_{Y,x} \subset \mathfrak{m}_{Y,x} \arrow[r] & \mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2 \end{tikzcd}$$

Note $\varphi(I + I_x^2) = 0$ so we get a K-linear map

$$I_x/(I + I_x^2) \longrightarrow \mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2$$

Claim: This is an isomorphism [$\Longrightarrow$ the theorem]. (a) φ is surjective: $h \in \mathfrak{m}_{Y,x} \subset \mathcal{O}_{Y,x} \subset K(Y)$, $\Longrightarrow h = \frac{h_1}{h_2}$, with $h_1, h_2 \in \mathcal{O}(Y)$ and $h_2(x) \neq 0, h_1(x) = 0$. Then

$$h - \frac{h_1}{h_2(x)} = h_1 \left(\frac{h_2(x) - h_2}{h_2(x) h_2} \right) \in \mathfrak{m}^2_{Y,x}$$

$$\Longrightarrow [h] = \varphi \left(\frac{h_1}{h_2(x)} \right),$$

where $\frac{h_1}{h_2(x)} \in I_x$, so φ is surjective.

(b) $\ker(\varphi) = I + I_x^2 \subset I_x$ (Intuitively, the restriction of f on Y vanishes to order 2 ar x).

Precisely:

$$\mathcal{O}_{Y,x} = (\mathcal{O}(\mathbb{A}^n)/I)_{I_x/I} = \mathcal{O}(\mathbb{A}^n)_{I_x} / I\mathcal{O}(\mathbb{A}^n)_{I_x}$$

the last equality from commutative algebra. $\varphi(f) = 0$ means that $f \mod I$ belongs to $(I_x^2)_{I_x}$ which is an ideal in $\mathcal{O}(\mathbb{A}^n)_{I_x}$ generated by I_x^2

$$f \mod I = \sum_{i,j} (X_i - x_i)(X_j - x_j) h_{ij}$$

$\theta(f \mod I) = 0 \Longrightarrow f \in I + I_x^2$. $\square$

Theorem 1.45. *Let $Y \subset \mathbb{A}^n$ affine variety. Then $Y^\circ = \{x \in Y \mid Y$ non-singular at $x\}$ is dense open subset.*

Corollary 1.46. *Any variety Y is birational to a non-singular variety.*

Proof. (of theorem)
Let $S = Y - Y^\circ = \{$ singular points $\}$. Then we know

(1) S is closed in Y, indeed fixing $(f_1, ..., f_m)$ generating $I(Y)$

$$S = \{x \mid rank\ J_{\underline{f}}(x) \neq n - d\}$$

One can show that $rank\ J_{\underline{f}}(x \leq n - d)$. So

$$\begin{aligned} S &= \{x \mid rank\ J_{\underline{f}}(x) < n - d\} \\ &= \{x \in Y \mid \text{ for all minors } M \text{ of } J_{\underline{f}} \text{ of size } n - d \text{ are degenerate } \det(M) = 0.\} \end{aligned}$$

is a closed algebraic set in $\mathbb{A}^n$.

(b), $S \neq Y (\Longrightarrow Y^\circ \neq \emptyset$ and open, so is dense).

If $S = Y$, then by the theorem of Zariski, the set of non-singular points in an open set of a hypersurface birational to Y would be empty. This means that we may assume $Y = V(f) \subset \mathbb{A}^{d+1}$ with f non-zero irreducible. Then

$$V(f) \supset S = \left\{ x \in \mathbb{A}^{d+1} | 0 = f(x = \frac{\partial f}{x_1}(x) = ... = \frac{\partial f}{x_d}(x) \right\}$$

so if $S = V(f)$, $\frac{\partial f}{\partial x_1} \in I(V(f)) = f\mathcal{O}(\mathbb{A}^{d+1}) = fK[X_1, .., X_{d+1}]$

$$\implies \text{ in char } = 0, \text{ comparing degrees, we have contradiction}$$

$\implies$ in chat $p \neq 0$, we get $\frac{\partial f}{\partial x_i} = 0$ for $1 \leq i \leq d$, $\implies f \in K[x_1^p, .., X_d^p] \implies f = g^p$, contradicting the irreducibility. □

2 Schemes

In this chapter we will mainly follow chap 2 of Hartshorne and chap 1 of Eisenbud-Harris.

2.1 Mar 13th-B: Affine schemes

Motivations

Serious problems with classical approach occur in late 1950's

(1) Intrinsic definitions (Without embeddings in $\mathbb{A}^n$ or $\mathbb{P}^n$)

(2) Construction of various algebraic varieties especially Jacobian variety of a curve, especially w.r.t. base field (is the Jacobian of a curve given by equation with coefficients in the same field?)

(3) Reduction modulo p of a variety given by equation in $\mathbb{Z}[X_1, .., X_n]$

To attack (1), Serre started from

$$\{\text{alg. set } Y \subset \mathbb{A}^n\} \longleftrightarrow \{\text{fin.gen. reduced } K\text{-algebra}\}$$
$$Y \mapsto \mathcal{O}(Y)$$
$$\{\text{maximal ideals in } A\} \leftarrow\!\shortmid A.$$

Grothendieck tried to remove the restriction on the algebras and managed to interpret it geometrically.

$$\{\text{affine schemes }\} \longleftrightarrow \{\text{all commutative rings.}\}$$

To each ring A, we will associate a geometric object called its **spectrum** denoted $\operatorname{Spec}(A)$.

(1) $\operatorname{Spec} A$ is a set. $\operatorname{Spec} A \neq \{$ maximal ideals $\}$ because this choice is not functorial. If $A_1 \xrightarrow{f} A_2$, we want $\operatorname{Spec}(A_2) \overset{f^*}{\operatorname{Spec}}(A_1)$ which would have to be $f^*(\mathfrak{m}) = f^{-1}(\mathfrak{m}) \subset A_1$. But $f^{-1}(\mathfrak{m})$ is NOT necessarily maximal.

Example 2.1. *A is an integral domain*

$$\{0\} \subset A \hookrightarrow Frac(A) \supset \{0\} \text{ maximal}$$

Definition 2.2. *$\operatorname{Spec} A := \{$ prime ideals $\mathfrak{p} \subset A$ $\}$*

Fact: If $f : A_1 \longrightarrow A_2$ is a ring morphism then $\mathfrak{p} \mapsto f^{-1}\mathfrak{p}$ gives map of sets

$$\operatorname{Spec} A_2 \longrightarrow \operatorname{Spec} A_1$$

Proof.

$$A_1 \xrightarrow{f} A_2/\mathfrak{p}$$

$$f^{-1}\mathfrak{p} \mapsto 0$$

leads to an injective map

$$A_1/f^{-1}\mathfrak{p} \hookrightarrow A_2/\mathfrak{p}$$

, then $A/f^{-1}\mathfrak{p}$ is an integral domain and $f^*(\mathfrak{p})$ is therefore a prime ideal. □

Definition 2.3. *If $\mathfrak{p} \in \operatorname{Spec} A$. the fraction field of $A/\mathfrak{p}$ is called the residue field at $\mathfrak{p}$, denoted $\kappa(\mathfrak{p})$.*

If $a \in A$, then a defines a function $\tilde{a} : \operatorname{Spec} A \longrightarrow \coprod_{\mathfrak{p} \in \operatorname{Spec}(A)} \kappa(\mathfrak{p})$, $\mathfrak{p} \mapsto a \bmod \mathfrak{p}$

(2) $\operatorname{Spec} A$ as a topological space

Definition 2.4. *For any set $S \subset A$, let $V(S) = \{\mathfrak{p} \in \operatorname{Spec}(A) | S \subset \mathfrak{p}\}$:*

Note:

(1) $V(S) = V($ ideals generated by S $)$

(2) Not always true that $V(S) = V($ finitely many elements$)$

(3) $V(S) = \{\mathfrak{p} \in \operatorname{Spec} A | \forall x \in S, \tilde{x}(\mathfrak{p}) = 0 \in \kappa(\mathfrak{p})\}$

Lemma 2.5.

(1) The sets $V(I)$, I ideal in A, from the closed set s of a topology on Spec A (called the Zariski topology).

(2) $V(I) \subset V(J) \Longleftrightarrow \sqrt{J} \subset \sqrt{I}$

(3) If $f : A_1 \longrightarrow A_2$ is a ring morphism, then

$$f^* : \ Spec\,(A_2) \longrightarrow \ Spec\,(A_1)$$

is continuous.

Proof. (1) $\emptyset = V(A) = V(\{1\})$ $\operatorname{Spec} A = V(\{0\})$.

$$\begin{aligned} \cap_{i \in X} V(I_i) &= \{\mathfrak{p} \in \ \operatorname{Spec}(A) | I_i \subset \mathfrak{p} \text{ for every } i\} \\ &= \{\mathfrak{p} \in \ \operatorname{Spec}(A) | \sum I_i \subset \mathfrak{p}\} \\ &= V\left(\sum_{i \in X} I_i\right) \end{aligned}$$

$$\begin{aligned} V(I) \cup V(J) &= \{\mathfrak{p} \in \ \operatorname{Spec}(A) | I \subset \mathfrak{p} \text{ or } J \subset \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \ \operatorname{Spec} A | IJ \subset \mathfrak{p}\} (\text{because } \mathfrak{p} \text{ prime}) \\ &= V(IJ) \end{aligned}$$

(2) recall the definition of radicals of an ideal

$$\begin{aligned} \sqrt{I} &:= \{x \in A | \exists k \geq 0, x^k \in I\} \quad = \cap_{I \subset \mathfrak{p}, \mathfrak{p} \in \ \operatorname{Spec} A} \mathfrak{p} \\ &= \cap_{\mathfrak{p} \in V(I)} \mathfrak{p} \end{aligned}$$

then if $V(J) \subset V(I)$, we get $\sqrt{I} \subset \sqrt{J}$.

Conversely, if $\sqrt{I} \subset \sqrt{J}$ then for $\mathfrak{p} \in V(J)$, then $I \subset \sqrt{I} \subset \sqrt{J} \subset \mathfrak{p} \Longrightarrow \mathfrak{p} \in V(I)$.

(3) □

2.2 Mar 16th: affine scheme, example and properties.

Recall

A is a ring with unity $\operatorname{Spec} A = \{$ prime ideals in $A\}$

closed sets: for a subset $S \subset A$, $V(S) = V(I :=$ ideal generated by $S) = \{\mathfrak{p} | I \subset \mathfrak{p}\}$

If $A \xrightarrow{f} B$ is a ring morphism, then $f^* : \ \operatorname{Spec}(B) \longrightarrow \ \operatorname{Spec}(A)$: $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$ is continuous.

Indeed, let $V(I) \subset \operatorname{Spec} A$ be closed,, then $(f^*)^{-1}(V(I)) = \{\mathfrak{p} \in \operatorname{Spec}(B) | f^*(\mathfrak{p}) \in V(I)\} = \{\mathfrak{p} \in \operatorname{Spec} B | I \subset f^{-1}\mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec} B \mid f(I) \subset \mathfrak{p}\}$, therefore

$$(f^*)^{-1} V_A(I) = V_B(f(I))$$

Examples of $\operatorname{Spec} A$

Example 2.6. *$\mathit{Spec}(\{0\}) = \emptyset$*

By definition, this is the only ring with $\mathit{Spec}\, A$ empty.

Example 2.7. *K algebraically closed field, $\emptyset \neq Y \subset K^n$ affine algebraic set. The corresponding affine scheme is*

$$Y^{sc} = \mathit{Spec}(\mathcal{O}(Y))$$

in other words

$$Y^{sc} = \mathit{Spec}(K[X_1, ..., X_n]/I(Y) =: A)$$

Maximal ideals of $\mathcal{O}(Y)$ are in bijection with points of Y by

$$x \mapsto \mathfrak{m}_x = \{f \in \mathcal{O}(Y | f(x) = 0\}$$

so we get an injective map

$$Y \xrightarrow{\varphi} Y^{sc}$$

$$x \longmapsto \mathfrak{m}_x$$

This map φ is continuous.

Let $V(I) \subset Y^{sc}$ be closed and $I \subset \mathcal{O}(Y)$.

$$\begin{aligned} &\varphi^{-1}(V(I)) \\ &= \{x \in Y \mid \mathfrak{m}_x \in V(I)\} \\ &= \{x \in Y \mid I \subset \mathfrak{m}_x\} \\ &= \{x \in Y \mid \forall f \in I, f(x) = 0\} \end{aligned}$$

is a closed algebraic set in K^n.

Observe: for every $x \in Y$, the residue field of $\mathfrak{m}_x$ is $A/\mathfrak{m}_x \cong K$ where the function associated to $f \in A$ is given by

$$\tilde{f}(\mathfrak{m}_x) = f(x).$$

The following are equivalent

1. $Y \xrightarrow{\varphi} Y^{sc}$ *is surjective*

2. *every prime ideal in* $\mathcal{O}(Y)$ *is maximal*

3. $\dim \mathcal{O}(Y) = 0$.

Consider the case $Y = K$ *and* $Y^{sc} =$ *Spec* $(K[X])$ *with* $\dim Y = 1$. $K[X]$ *is a principal ideal domain and* K *is algebraically closed.*

$$Y^{sc} = \{(X - x) | x \in K\} \cup \{0\}$$

where $\eta := \{0\}$ *is called the generic point of* Y^{sc}.

Claim: $\{\eta\}$ *is not closed in* Y^{sc}, *in fact it is dense*

$$\overline{\{\eta\}} = Y^{sc}.$$

Example 2.8. *More generally, Let* A *be an integral domain and* $\eta = \eta_A = \{0\} \in$ *Spec* A.

Claim:

$$\overline{\{\eta\}} = \text{Spec}\, A$$

Let $\mathfrak{p} \in$ *Spec* A.

$$\begin{aligned}\overline{\{\mathfrak{p}\}} &= \bigcap_{\mathfrak{p} \in V(I)} V(I) \\ &= \bigcap_{I \subset \mathfrak{p}} V(I) \\ &= V(\sum_{I \subset \mathfrak{p}} I) = V(\mathfrak{p})\end{aligned}$$

$$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \{Q \in \text{Spec}\,(A) | \mathfrak{p} \in Q\}$$

So:

1. $\overline{\{\eta_A\}} =$ *Spec* A *if* A *is an integral domain.*

2. $\{\mathfrak{p}\}$ *is closed iff* $\mathfrak{p}$ *is maximal.*

Definition 2.9. *If* $\mathfrak{p} \in \overline{\{Q\}}$, *we say that* $\mathfrak{p}$ *is a* ***specialization*** *of* Q, *and that* Q ***specializes to*** $\mathfrak{p}$.

Example 2.10. *Any point is a specialization of* η_A *if* A *is integral domain. What is* $\kappa(\eta_A)$?

$$A/\{0\} = A$$

so $\kappa(\eta_A) = Frac(A)$

Back to the Example 2.7

$Y = K$, $Y^{sc} =$ Spec $(K[X])$, $\eta = \{0\}$ is dense in Y^{sc}, its residue field is $K(X)$.

Remark 2.11. *If $f_1, f_2 \in K[X]$ are such that they coincide at η;*

$$\tilde{f}_1(\eta) = \tilde{f}_2(\eta)$$

then in fact $f_1 = f_2$ in $K[X]$.

We will often encounter situations like "A property holds at $\eta \Longrightarrow$ it holds at for all x in an open set"

Example 2.12. *A is an integral domain. Any $\emptyset \neq U$ open set in Spec A is dense:*

$$U \cap \{\eta\} \neq \emptyset$$

so $\eta \in U$, $\Longrightarrow \overline{\{\eta\}} = \overline{U}$

Example 2.13. *The Zariski topology is **quasi-compact**: any open covering has a finite subcover. Indeed, suppose*

$$\begin{aligned}
&\bigcap_\alpha V(I_\alpha) = \emptyset \\
\Longleftrightarrow & V(\sum_\alpha I_\alpha) = \emptyset = V(A) \\
\Longleftrightarrow & 1 \in \sum_\alpha I_\alpha \\
\Longleftrightarrow & 1 = \sum_{j=1}^m f_{\alpha_j}, f_{\alpha_j} \in I_{\alpha_j} \\
\Longleftrightarrow & V(\sum_j I_{\alpha_j}) = \emptyset \\
\Longleftrightarrow & \bigcap_j V(I_{\alpha_j}) = \emptyset
\end{aligned}$$

Example 2.14. *For any $I \subset A$, $A \xrightarrow{\pi} A/I$ induces*

$$Spec\,(A/I) \xrightarrow{\pi^*} Spec\,A$$

which gives homeomorphism

$$Spec\,(A/I) \cong V(I).$$

Example 2.15. *K is a field, not necessarily algebraically closed. Let $J \subset K[X_1,..,X_n]$ be an ideal and $Y =$ Spec$(K[X_1,...,X_n]/J)$. (Want to understand in particular the relation with the case K is algebraically closed.) Fix $L \supset K$ where L is algebraically closed Then we get an injective ring morphism*

$$K[X]/J \longrightarrow L[X]/JL[X]$$

hence a map

$$Y_L := \text{Spec}\,(L[X]/JL[X]) \longrightarrow Y,$$

where Spec$(L[X]/JL[X])$ is a classical algebraic set (if J is prime).

Take $Y =$ Spec$(KX) = \mathbb{A}^1_K$.

Definition 2.16. *Let A be any ring. The **affine n-space** $\mathbb{A}^n_A$ over A is Spec$A[X_1,...,X_n]$.*

What is $\mathbb{A}^1_L \longrightarrow \mathbb{A}^1_K$?

$$\begin{aligned}\mathbb{A}^1_K &= \{\mathfrak{p} \subset K[X] \text{ prime}\} \\ &= \{0\} \cup \{fK[X] \mid f \text{ irreducible and monic}\}\end{aligned}$$

<u>Check</u>: the Zariski topology has closed sets $\emptyset$, $\mathbb{A}^1_K$, finite sets of closed points. Given $i : K[X] \hookrightarrow L[X]$, what is $\mathbb{A}^1_L \xrightarrow{i^*} \mathbb{A}^1_K$? We have that

$$\begin{aligned}i^*(\eta_L) &= i^{-1}(\{0\}) \\ &= \eta_K\end{aligned}$$

which means the image of i^* us dense.

Let $x \in L$

$$\begin{aligned}&i^*(X-x)L[X] \\ &= i^{-1}((X-x)L[X]) \\ &= \{f \in K[X] \mid (X-x)|f \text{ in } L[X]\} \\ &= \{f \in K[X] \mid f(x) = 0\}\end{aligned}$$

<u>Case 1</u>: x is transcendental over K

$$\Longleftrightarrow i^*(x) = \{0\} = \eta_K$$

<u>Case 2</u>: x is algebraic over K

$$i^*(x) = f_x$$

where f_x is the minimal polynomial x over K.

Observe that i^* is not injective more precisely,

$$(i^*)^{-1}(f) = \{\text{ roots of } f \text{ in } L\}$$

where f is irreducible monic.

Example 2.17. *Given A, B integral domain $A \xrightarrow{f} B$ is injective iff*

$$f^*(\operatorname{Spec} B) \subset \operatorname{Spec} A$$

is dense. The proof is left as an exercise.

Example 2.18. $K = \overline{K}$,

Y^{sc} *for* $Y = \{(x, y) \in K^2 \mid (xy) = 0\}$. *($Y$ is not a variety in this case.)*

$$\mathbb{A}^2_K \supset V(xy) \cong Y^{sc} = \operatorname{Spec}(K[X, Y]/(XY))$$

Check the points of Y^{sc} *are*

$$\begin{aligned} h_x &= \langle (X - x), Y \rangle \subset K[X, Y]/(XY) \\ v_y &= \langle X, (Y - y) \rangle \end{aligned}$$

because $XY = (X - x)Y + xY$. h_x *and* v_y *are closed points with residue field* K. *Let*

$$\begin{aligned} \eta_1 &= XK[X, Y]/(XY) \\ \eta_2 &= YK[X, Y]/(XY). \end{aligned}$$

We have $\{0\} \notin \operatorname{Spec}(K[X, Y]/(XY))$ *because the ring is not an integral domain.*

$$\begin{aligned} \overline{\{\eta_1\}} &= \{\eta_1\} \cup \{\mathfrak{m} \text{ maximal s.t. } X \subset \mathfrak{m}\} \\ &= \{\eta_1\} \cup \{v_y \mid y \in K\}. \end{aligned}$$

Similarly, we have

$$\overline{\{\eta_2\}} = \{\eta_2\} \cup \{h_x \mid x \in K\}.$$

Note $v_0 = h_0$ *is a specialization of both* η_1 *and* η_2.

Example 2.19. *For* $K = \overline{K}$ *consider*

$$\mathbb{A}^2_K = \{(x, y) \mid (x, y) \in K^2\} \cup \{\eta\} \cup \{fK[X, Y] \mid f \text{ irreducible monic}\}$$

where we identify the maximal ideals $(X - x, Y - y)$ *in* $K[X, Y]$ *with points* (x, y). *Note that prime ideals of height* 1 *are principal in a UFD.*

$$\overline{\{fK[X, Y]\}} = \{fK[X, Y]\} \cup \{(x, y) \in K^2 | f(x, y) = 0\}$$

For this reason, we denote $\{fK[X, Y]\}$ *by* η_f *because it is the generic point of* $V(f)$.

$$\begin{aligned} \overline{\{\eta_f\}} &= \eta_f \cup \text{ classical points on } C_f \\ \kappa(\eta_f) &= K[X, Y]/fK[X, Y] = \kappa(C_f) \end{aligned}$$

where C_f *is the classical curve.* η_f *specializes to the point* (x, y) *on* C_f.

2.3 Mar 20th:

Example 2.20. *$A = \mathbb{Z}$, $Spec\, A = \{0\} \cup \{p\mathbb{Z} | p$ prime number$\}$. Recall* $\dim(\mathbb{Z}) = 1$, *with residue fields*

$$\begin{cases} K(\eta) = \mathbb{Q} \\ K(p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p, \text{ finite field} \end{cases}$$

A statement like "property P is true at η" $\Longrightarrow$ "It is true on any open set" means "a property P true for $\mathbb{Q}$ is also true for $\mod p$ *for p large enough."*

*(Topology has closed sets $\emptyset$, $Spec\,\mathbb{Z}, V(n\mathbb{Z}) = \{p\mathbb{Z} : p$ **divides** $n\}$, where $V(n\mathbb{Z})$ is a finite set of closed points.)*

$$\mathbb{Z} \longrightarrow \mathbb{F}_p \longleftrightarrow \begin{array}{c} Spec\,(\mathbb{F}_p) \hookrightarrow \ Spec\,(\mathbb{Z}) \\ \{0\} \in \mathbb{F}_p \mapsto p\mathbb{Z} \end{array}$$

$$\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q} \longleftrightarrow \begin{array}{c} Spec\,(\mathbb{F}_p) \xrightarrow{i^*} \ Spec\,(\mathbb{Z}) \\ \{0\} \in \mathbb{F}_p \mapsto \eta \end{array}.$$

In particular, the image of i^* is dense in $Spec\,\mathbb{Z}$.

structure sheaf

Note recall we want

$$\{\text{affine schemes}\} \longleftrightarrow \{\text{ commutative rings}\}$$

$$\text{Spec}\, A \leftarrow\!\shortmid A$$

$$f^* \leftarrow\!\shortmid f$$

This is functorial but cannot capture the whole category of rings because for instance all rings

$$A = K[X]/(X^n), n \geq 1$$

(K is a field). We have $\text{Spec}\, A = \{XK[X]\}$, independent of K and n. We need to remember what is K and what is n.

We deal with that by defining "regular functions"

Definition 2.21. *A is a ring. For $U \subset$ Spec A open, we define the ring $\mathcal{O}(U)$ of "regular functions on U" by*

$$\mathcal{O}(U) = \left\{ s : U \longrightarrow \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}} \;\middle|\; \begin{array}{l} \text{(1) } s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ for } \mathfrak{p} \in U \\ \text{(2)} \forall \mathfrak{p} \in U, \exists V \text{ open nbhd of } \mathfrak{p} \text{ in } U \\ \text{ and } a \in A, f \in A, \text{ s.t.} \forall \mathfrak{q} \in V, f \notin \mathfrak{q}, \text{ and } s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}} \end{array} \right\}$$

Note: if $V \subset U$ open then $s \mapsto s|_V$ is a ring morphism $res_V^U : \mathcal{O}(U) \longrightarrow \mathcal{O}(V)$ and $res_V^U = id_{\mathcal{O}(U)}$. Then the pair $\big((\mathcal{O}(U))_{U \in \operatorname{Spec} A}, (res_V^U)_{U,V \in \operatorname{Spec} A}\big)$ is a **sheaf of rings** on Spec A.

Definition 2.22. *X is a topological space, $\mathcal{C}$ a category,*

*(1) A $\mathcal{C}$-**presheaf** is sthe data of*

(a) For every open set $U \subset X$, an object $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$ in $\mathcal{C}$.

(b) For every $V \subset U$ opens in X, a $\mathcal{C}$-morphism $res_V^U : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

such that given U opens in X.

(i) $res_U^U = id_{\mathcal{F}(U)}$

(ii) Given $W \subset V \subset\subset U$ opens in X

$$res_W^U = res_W^V \circ res_V^U$$

<u>*Notation:*</u> $res_V^U(s) = s|_V$

*(2) A $\mathcal{C}$-presheaf is a $\mathcal{C}$-**sheaf** if: for any $U \subset X$ open, for every open covering $U = \cup_\alpha V_\alpha$, for any family $(s_\alpha)_\alpha$ with $s_\alpha \in \mathcal{F}(V_\alpha)$ such that $s_\alpha|_{V_\alpha \cap V_\beta} = s_\beta|_{V_\alpha \cap V_\beta}$, there is a unique $s \in \mathcal{F}(U)$ with $s|_{V_\alpha} = s_\alpha$.*

Exercise 2.23. *Check that the sheaf of regular functions is indeed a sheaf.*

Definition 2.24. *A a ring. The **affine scheme** associated to A is (Spec $A, \mathcal{O}$) where the first data is endowed with Zariski's topology and the $\mathcal{O}$ is the structure sheaf.*

Example 2.25. *K a field, Spec $K = \{\eta\}$*

$$\begin{cases} \mathcal{O}(\text{ Spec } K) = \{s : \eta \longrightarrow K_{\{0\}} = K, \text{ (i.e. } s(\eta) \in K)\} \\ \mathcal{O}(\emptyset) = \{0\} \end{cases}$$

Different K gives different affine schemes.

Proposition 2.26. *For $f \in A$, define $U_f = \{\mathfrak{p} \in \text{ Spec}\, A | f \notin \mathfrak{p}\}$*

(1) U_f is a open "basic open sets"

(2) We have a canonical isomorphism

$$\begin{cases} A_f \xrightarrow{\psi} \mathcal{O}(U_f) \\ a/f^m \longmapsto (s : \mathfrak{p} \in U_f \mapsto \dfrac{a}{f^m} \in A_{\mathfrak{p}}) \end{cases}$$

In particular, for $f = 1$, we get a canonical isomorphism

$$A = A_1 \xrightarrow{\sim} \Gamma(\text{ Spec}\, A, \mathcal{O})$$

$\implies$ the affine scheme of A allows you to recover A.

Proof. (injectivity)

Suppose $\psi\left(\frac{a}{f^m}\right) = 0$. This means that

$$\forall \mathfrak{p} \in U_f, \frac{a}{f^m} = \frac{0}{1} \in A_{\mathfrak{p}}$$

$\iff \forall \mathfrak{p} \in U_f$, $\exists h_{\mathfrak{p}} \notin \mathfrak{p}$, $ha = 0$. Let $I = \{x \in A | xa = 0\}$. I is an ideal and $I \not\subset \mathfrak{p}$ for any $\mathfrak{p} \in U_f$

$\implies V(I) \cap U_f = \emptyset$

$\implies V(I) \subset V(f)$

$\sqrt{(f)} \subset \sqrt{I}$

$f \in \sqrt{(f)} \in \sqrt{I}$

$\exists k \geq 0, f^k a = 0 \implies a/f^m = 0 \in A_f$

(Surjectivity): We need the following lemma

Lemma 2.27.

(1) $U_{f_1} \cap U_{f_2} = U_{f_1 f_2}$

(2) $U_{f^n} = U_f, V(f^n) = V(f)$

(3) U_f is quasicompact

(4) The open sets U_f forms a basis of the Zariski topology.

Consider $\psi : A_f \longrightarrow \mathcal{O}(U_f)$, let $s \in \mathcal{O}(U_f)$.

By definition there exists an open covering of U_f, $U_f = \bigcup_\alpha V_\alpha$, and elements a_α, g_α such that $\forall \mathfrak{p} \in V_\alpha$, $s(\mathfrak{p}) = \frac{a_\alpha}{g_\alpha}, g_\alpha \notin \mathfrak{p}$.

Using the above lemma, we may assume there are finitely many V_α and $V_\alpha = U_{h_\alpha}$.

<u>Observe</u>: $\forall \mathfrak{p} \in U_{h_\alpha} = V_\alpha$, $g_\alpha \notin \mathfrak{p} \Longleftrightarrow \mathfrak{p} \in U_{g_\alpha}$

$$\begin{aligned} & U_{h_\alpha} \subset V_{g_\alpha} \\ & \Longrightarrow V(g_\alpha) \subset V(h_\alpha) \\ & \Longrightarrow \sqrt{(h_\alpha)} \subset \sqrt{(g_\alpha)} \\ & \Longrightarrow \exists n_\alpha, h_\alpha^{n_\alpha} \in (g_\alpha) \end{aligned}$$

So $h_\alpha^{n_\alpha} = c_\alpha g_\alpha$, Now for $\mathfrak{p} \in U_{h_\alpha}$

$$\frac{a_\alpha}{g_\alpha} = \frac{a_\alpha c_\alpha}{g_\alpha c_\alpha} = \frac{a_\alpha c_\alpha}{h_\alpha^{n_\alpha}} \in A_\mathfrak{p}$$

Replacing a_α by $a_\alpha c_\alpha$, g_α by $h^{n_\alpha}\alpha$, Using $U_{h_\alpha^{n_\alpha}} = U_{h_\alpha}$, we reduce to the case where $g_\alpha = h_\alpha$ for all α.

On $U_{h_\alpha} \cap U_{h_\beta} = U_{h_\alpha h_\beta}$, we have

$$\forall \mathfrak{p} \in U_{h_\alpha h_\beta}, \frac{a_\alpha}{h_\alpha} = \frac{a_\beta}{h_\beta} \text{ in } A_\mathfrak{p}$$

$$\Longrightarrow \exists n(\alpha,\beta), (h_\alpha h_\beta)^{n(\alpha,\beta)}(a_\alpha h_\beta - h_\alpha a_\beta) = 0$$

Take n to be the largest of the finite many $n(\alpha, \beta)$

$$\Longrightarrow (h_\alpha h_\beta)^n (a_\alpha h_\beta - h_\alpha a_\beta) = 0$$

$$a'_\alpha h'_\beta - a'_\beta h'_\alpha = 0$$

where $a'_\alpha = a_\alpha h_\alpha^n$ and $h'_\alpha = h_\alpha^{n+1}$

Note $\frac{a'_\alpha}{h'_\alpha} = \frac{a_\alpha}{h_\alpha}$ in $A_\mathfrak{p}$ for all $\mathfrak{p} \in U_{h'_\alpha} = U_{h_\alpha}$.

Now

$$\begin{aligned} & \bigcup_\alpha U_{h'_\alpha} = U_f \\ & V(f) = V(\sum (h'_\alpha)) \\ & \Longrightarrow \sqrt{f} = \sqrt{\sum (h'_\alpha)} \\ & \Longrightarrow f^k = \sum_\alpha h'_\alpha c_\alpha \quad \text{for some } k \end{aligned}$$

Define

$$a = \sum_\alpha c_\alpha a'_\alpha \in A$$

Fix β,

$$ah'_\beta = \sum_\alpha c_\alpha a'_\alpha h'_\alpha = \sum_\alpha c_\alpha a'_\beta h'_\alpha = a'_\beta f^k$$

$$\Longrightarrow s(\mathfrak{p}) = \frac{a_\beta}{h_\beta} = \frac{a'_\beta}{h'_\beta} = \frac{a}{f^k}$$

in $A_\mathfrak{p}$ for any $\mathfrak{p} \in U_{h_\beta} = V_\beta$.

So $\psi(\frac{a}{f^k})|_{V_\beta} = s|_{V_\beta}$ for any β. So $\psi(a/f^k)$ and s are elements of $\mathcal{O}(U_f)$ with restrictions equal on open sets forming a covering of U_f, by the uniqueness condition in the definition of sheaf, it follows that $\psi(a/f^k) = s$. □

Proof. (of the lemma)

(1) $U_{f_1} \cap U_{f_2} \stackrel{?}{=} U_{f_1 f_2}$

$$\begin{aligned} V(f_1) \cup V(f_2) &= \{\mathfrak{p} \in \text{ Spec } A | f_1 \in \mathfrak{p} \text{ or } f_2 \in \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \text{ Spec } A | f_1 f_2 \in \mathfrak{p}\} \end{aligned}$$

(2) $f^n \in \mathfrak{p} \Longleftrightarrow f \in \mathfrak{p}, n \geq 1$

(3) Suppose $V(f) \subset \cap_\alpha V(I_\alpha) \Longrightarrow V(\sum I_\alpha) \supset V(f) \Longrightarrow \sqrt{(f)} \subset \sqrt{\sum I_\alpha}$

□

2.4 Mar 23th: Sheaves and stalks

Example 2.28. *(1) Let X be a topological space. Then*

$$\underline{C}(U) = \{f : U \longrightarrow \mathbb{C} \textit{ continuous}\}$$

for $U \subset X$ open is a sheaf. For X a manifold we also have that

$$\underline{C}^\infty(U) = \{f : U \longrightarrow \mathbb{C} \textit{ smooth}\}$$

is a sheaf and lastly for X a complex manifold the following is a sheaf:

$$\mathcal{H}(U) = \{f : U \longrightarrow U \textit{ holomorphic}\}.$$

(2) Let $X = \mathbb{C}^\times$. Then

$$\mathcal{F}(U) = \{f : U \longrightarrow \mathbb{C} \text{ holomorphic and } f = g^2 \text{ for some } g \text{ holomorphic}\}$$

is a pre-sheaf but not a sheaf. A holomorphic function might have a square root locally but not on all of U.

Definition 2.29. *Let $\mathcal{F}_1, \mathcal{F}_2$ be $\mathcal{C}$-pre-sheaves on X. A* ***morphism*** *$\mathcal{C} : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is a collection of morphisms $\varphi_U : \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U)$ such that for any $V \subset U$ open we have a commutative square*

$$\begin{array}{ccc} \mathcal{F}_1(U) & \xrightarrow{\varphi_U} & \mathcal{F}_2(U) \\ {\scriptstyle res_V^U \text{ for } \mathcal{F}_1}\downarrow & & \downarrow{\scriptstyle res_V^U \text{ for } \mathcal{F}_1} \\ \mathcal{F}_1(V) & \xrightarrow{\varphi_V} & \mathcal{F}_2(V) \end{array}$$

A morphism of sheaves we define to be the same as a morphism of pre-sheaves.

Note that $\mathrm{Id}_{\mathcal{F}(U)} : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ gives a morphism and that composition makes sense, i.e. we have that

$$(\varphi \circ \psi)_U = \varphi_U \circ \psi_U.$$

Thus we have now defined the category of $\mathcal{C}$-pre-sheaves and sheaves.

Example 2.30. *(1) If X is a complex manifold, there are morphisms*

$$\mathcal{H} \longrightarrow \underline{C}^\infty \longrightarrow \underline{C}$$

(2) If $X \subset \mathbb{R}^n$ is a manifold, then

$$\begin{cases} \underline{C}^\infty(U) \longrightarrow \underline{C}^\infty(U) \\ \qquad f \longmapsto \partial f/\partial x_1 \end{cases}$$

is a morphism of sheaves from $\underline{C}^\infty \longrightarrow \underline{C}^\infty$.

Proposition 2.31. *Suppose $\mathcal{F}$ is a $\mathcal{C}$-pre-sheaf on X. Then there is a unique, up to unique isomorphism, morphism $\sigma : \mathcal{F} \longrightarrow \mathcal{F}^\sigma$, (the "sheafification" of $\mathcal{F}$) such that for any morphism of presheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{Y}$ there is a unique φ^σ such that $\varphi = \varphi^\sigma \circ \sigma$. In particular $Hom(\mathcal{F}, \mathcal{Y}) = Hom(\mathcal{F}^\sigma, \mathcal{Y})$.*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\sigma"'] & \mathcal{Y} \\ \mathcal{F}^\sigma \arrow[ur, "\varphi^\sigma"'] & \end{tikzcd}$$

If $\mathcal{F}$ is a sheaf then σ is an isomorphism.

$\mathcal{F}^\sigma$ is called the sheaf associated to $\mathcal{F}$. In order to prove the statement, we need another definition.

Definition 2.32. *Suppose $\mathcal{F}$ is a $\mathcal{C}$-presheaf on X. The* ***stalk*** *of $\mathcal{F}$ at $x \in X$ is*

$$\mathcal{F}_x := \{(U, s) \mid x \in U \subset X \text{ open}, s \in \mathcal{F}(U)\}/\sim$$

with

$$(U_1, s_1) \sim (U_2, s_2) \iff \exists V \subset U_1 \cap U_2 \text{ s.t. } s_1|_V = s_2|_V \text{ and } x \in V.$$

This is also called the "germs of sections of $\mathcal{F}$ at x".

Proposition 2.33. *Let A be a ring and consider $\mathcal{O}_A$ on $\operatorname{Spec} A$. Then the following morphism is an isomorphism:*

$$\varphi \begin{cases} \mathcal{O}_{A,p} \longrightarrow A_p \\ (U, s) \longmapsto s(p) \end{cases}$$

Proof. It follows easily from the definitions that φ is well defined. For surjectivity, let $a/f \in A_p$, $f \notin p$. Then we can construct s defined on U_f such that $s(q) = a/f$ in A_q for all $q \in U_f$.

Next, suppose that $s(p) = 0$ for some section (U, s). Then we can write $s(q) = a/f$ for any $q \in V$ where V is an open neighbourhood of p in V. Since $s(p) = 0$, we get $ha = 0$ for some $h \notin p$. But then on $V \cap U_h$, $s \equiv 0$. ☐

Now we get back to the construction of $\mathcal{F}^\sigma$. Define

$$\mathcal{F}^\sigma(U) := \left\{ s : U \longrightarrow \coprod_{x \in U} \mathcal{F}_x \,\middle|\, \begin{array}{l} \forall x \in U, s(x) \in \mathcal{F}_x \text{ and} \\ \forall x \in U, \exists V \subset U \text{ open, s.t. } x \in V, \\ \exists t \in \mathcal{F}(V) \text{ s.t. } \forall y \in V, s(y) = t_y \end{array} \right\}$$

where t_y denotes the equivalence class $[(V, t)] \in \mathcal{F}_y$. Moreover, define $\varphi^\sigma : \mathcal{F} \longrightarrow \mathcal{F}^\sigma$ by

$$\varphi^\sigma_U(t) = (x \longmapsto t_x)$$

and for $s \in \mathcal{F}^\sigma(U)$, $V \subset U$ let

$$\operatorname{res}^U_V(s)(y) = s(y)$$

for $y \in V$. Finally for a given sheaf $\mathcal{Y}$, let $\varphi^\sigma_U : \mathcal{F}^\sigma(U) \longrightarrow \mathcal{Y}(U)$, such that s maps to the unique $\tilde{s} \in \mathcal{Y}(U)$ such that for all $a \in U$, $V \subset U$ and $t \in \mathcal{F}(V)$ such that $s(y) = t_y$ on V, we have $\tilde{s}|_V = \varphi_V(t)$. Using the sheaf property of $\mathcal{Y}$, we see that such an $\tilde{s} \in \mathcal{Y}(U)$ does exist.

Then $\mathcal{F}^\sigma$ is a presheaf and σ is a morphism. One checks that $\mathcal{F}^\sigma$ is a sheaf (because it is defined by local conditions) and that the universal property holds.

Given $\mathcal{F}$ and $\mathcal{F}^\sigma$, we obtain an isomorphism for all x, $\sigma : \mathcal{F}_x \longrightarrow \mathcal{F}_x^\sigma$ by

$$[(U, x)] \longmapsto [(U, \varphi_U^\sigma(s))].$$

Proposition 2.34. *Given sheaves $\mathcal{F}_1$, $\mathcal{F}_2$ on X and a morphism $\varphi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, φ is an isomorphism if and only if for every $x \in X$ the induced $\varphi_x : \mathcal{F}_{1,x} \longrightarrow \mathcal{F}_{2,x}$, $[(U, s)] \longmapsto [(U, \varphi_U(s))]$ is an isomorphism.*

We omit the proof, it can be found in Hartshorne or Eisenbud.

Remark 2.35. *(1) This only holds for sheaves not presheaves.*

(2) The ismorphism φ_x need to come from a "global" map $\varphi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$.

(3) One checks that $\varphi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is and isomorphism iff $\varphi_U : \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U)$ are all isomorphisms. One can define φ to be injective if φ_U is injective for all U. However, the correct definition of surjectivity for φ is not equivalent to saying that φ_U is surjective for all U.

Definition/Theorem 2.36. *Let X, Y be topological spaces and $f_X \longrightarrow Y$ be continuous, $\mathcal{C}$ be a category and $\mathcal{F}$ be a $\mathcal{C}$-presheaf on X. Define $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ and $res_V^U = res_{f^{-1}(V)}^{f^{-1}(U)}$. Then $f_*\mathcal{F}$ is a $\mathcal{C}$-presheaf and is a sheaf is $\mathcal{F}$ is one. $f_*\mathcal{F}$ is called the* ***direct image of $\mathcal{F}$ on*** *Y.*

Proof. We check that $f_*\mathcal{F}$ is a sheaf. Let $U \subset Y$ be open, $U \bigcup U_\alpha$ be and open cover for U and let $s_\alpha \in (f_*\mathcal{F})(U_\alpha)$ be such that

$$s_\alpha | U_\alpha \cap U_\beta = s_\beta | U_\alpha \cap U_\beta.$$

Then $s_\alpha \in \mathcal{F}(f^{-1}(U_\alpha))$ and

$$s_\alpha | f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) = s_\beta | f^{-1}(U_\alpha) \cap f^{-1}(U_\beta).$$

Since $f^{-1}(U) = \bigcup f^{-1}(U_\alpha)$ and $\mathcal{F}$ is a sheaf, there is a unique $s \in \mathcal{F}(f^{-1}(U))$ such that $s|f^{-1}(U_\alpha) = s_\alpha$. Hence $s \in (f_*\mathcal{F})(U)$, $s|U_\alpha = s_\alpha$. The uniqueness follows from the same kind reasoning. $\square$

2.5 Mar 27th: Morphism of schemes

Recall: $\operatorname{Spec} A$, Zariski topology $\mathcal{O}_A$ structure sheaf.

<u>Observe</u>: $f : A \longrightarrow B$ ring morphism and

$$\begin{aligned}\tilde{f} : \operatorname{Spec} B &\longrightarrow \operatorname{Spec} A \\ \mathfrak{p} &\longmapsto f^{-1}(\mathfrak{p})\end{aligned}$$

we also get, (Recall $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$)

$$\mathcal{O}(A) \xrightarrow{f^*} \tilde{f}_*\mathcal{O}_B$$
$$\forall U \subset \operatorname{Spec} A, \mathcal{O}_A(U) \longrightarrow \tilde{f}_*\mathcal{O}_B(U) = \mathcal{O}_B(f^{-1}(U))$$

is defined as follows:

To $s \in \mathcal{O}_A(U), s : U \longrightarrow \sqcup_{\mathfrak{p}\in U} A_{\mathfrak{p}}$, such that....
we associated $t : f^{-1}(U) \longrightarrow \sqcup_{Q\in \tilde{f}^{-1}(U)} B_Q$ such that ...
defined

$$t(Q) = f(s(\tilde{f}(Q))), \quad s(\tilde{f}(Q)) \in A_{\tilde{f}(Q)}$$

where $f(a/b) = f(a)/f(b)$.

$$\begin{aligned}f : A_{f^{-1}(Q)} &\longrightarrow B_Q \\ \frac{a}{b} &\longmapsto \frac{f(a)}{f(b)}\end{aligned}$$

One checks that $t \in \mathcal{O}_B(f^{-1}(U))$ if $s \in \mathcal{O}_A(U)$. IN other words: $f : A \longrightarrow B$ gives

$$(\tilde{f}, f^*) : (\operatorname{Spec} B, \mathcal{O}_B) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_A)$$

Definition 2.37. *A* ***ringed space*** *is $(X\mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A* ***locally ringed space*** *is a ringed space where $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. (e.g. $\mathcal{O}_{A,\mathfrak{p}} = A_{\mathfrak{p}}$, where $A_{\mathfrak{p}}$ is a local ring with unique maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$).*

A morphism of ringed space $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a pair

$$(f, f^*) : \begin{cases} f : X \longrightarrow Y & \text{morphism of topological spaces} \\ f^* : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X & \text{morphism of sheaves}\end{cases}$$

Ringed space form a category $Id_{(X,\mathcal{O}_X)} = (Id_X, Id_{\mathcal{O}_x})$ and

$$\begin{cases} X \xrightarrow{f} \qquad\qquad Y \xrightarrow{g} Z \\ \mathcal{O}_Y \xrightarrow{f^*} f_*\mathcal{O}_X \\ \qquad\qquad \mathcal{O}_Z \xrightarrow{g^*} g_*\mathcal{O}_Y, \end{cases}$$

has composition

$$(g \circ g, g_*(f^*) \circ g^*)$$

where $g_*(f^*)$ is the direct image $\mathcal{F} \longrightarrow g_*\mathcal{F}$ is a functor from (pre)sheaves on Y to sheaves on Z . (Any morphism of sheaves $\varphi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ gives a morphism $g_*\mathcal{F}_1 \longrightarrow g_*\mathcal{F}_2$)

A morphism of locally ringed space $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is (f, f^*) $f : X \longrightarrow Y$ is continuous and $\mathcal{O}_Y \xrightarrow{f^*} f_*\mathcal{O}_X$ such that f^* induces for each $x \in X$ a <u>local</u> morphism $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$.

Recall that A, B are local rings. $f : A \longrightarrow B$ is <u>local</u> iff $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

<u>Note</u>: $f^{-1}(\mathfrak{m}_B) \subset \mathfrak{m}_A$ because if $f(a) \in \mathfrak{m}_B$ $a \notin A^\times \Longrightarrow a \in \mathfrak{m}_A$. So the condition to be local is

$$\mathfrak{m}_A \subset f^{-1}(\mathfrak{m}_B) \Longleftrightarrow f(\mathfrak{m}_A) \subset \mathfrak{m}_B$$

Definition 2.38. *If $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ f^* gives morphisms*

$$\mathcal{O}_Y(U) \xrightarrow{f_U^*} \mathcal{O}_X(f^{-1}(U))$$

for every $y \in Y$

$$\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$$
$$[(U, s)] \longmapsto [(U, f_U^*(s))].$$

Take $y = f(x)$:

$$\begin{aligned} \mathcal{O}_{Y,f(x)} &\longrightarrow (f_*\mathcal{O}_X)_{f(x)} \longrightarrow \mathcal{O}_{X,x} \\ [(U, s)] &\longmapsto [(U, f_U^*(s))] \longmapsto [f^{-1}(U), f_U^*(s)] \end{aligned}$$

is the desired morphism.

Theorem 2.39. *(1) For any $f : A \longrightarrow B$ the pair $(\tilde{f}, \tilde{f}^*)$ is a morphism of locally ringed spaces ($Spec\, B, \mathcal{O}_B) \longrightarrow ($ $Spec\, A, \mathcal{O}_A)$*

(2) Conversely, any morphism of locally ringed spaces ($Spec\, B, \mathcal{O}_B) \longrightarrow ($ $Spec\, A, \mathcal{O}_A)$ is induced by a morphism of rings.

(3) This gives a equivalence of categories

$$(\text{ commutative rings with unity}) \overset{\simeq}{\longleftrightarrow} (\text{affine schemes as locally ringed spaces})$$

$$Hom_{rings}(A, B) = \; Hom_{loc.r.sp}(\; Spec\, B, \; Spec\, A)$$

Proof. Recall $\mathcal{O}_{A,\mathfrak{p}} = A_{\mathfrak{p}}$ and recall $f : A_{f^{-1}(\mathfrak{q})} \longrightarrow B_{\mathfrak{q}}$ i.e.

$$\mathcal{O}_{A,\tilde{f}(\mathfrak{q})} \longrightarrow \mathcal{O}_{B,\mathfrak{q}}$$

Claim: This is exactly the morphism $\mathcal{O}_{A,\tilde{\mathfrak{q}}} \longrightarrow \mathcal{O}_{B,\mathfrak{q}}$ induced by $\tilde{f}^* : \mathcal{O}_A \longrightarrow \tilde{f}^*\mathcal{O}_B$

Claim2: For every $\mathfrak{q} \in \operatorname{Spec} B$,

$$\begin{aligned} A_{f^{-1}(\mathfrak{q})} &\longrightarrow B_{\mathfrak{q}} \\ a/b &\longmapsto f(a)/f(b) \end{aligned}$$

is a local morphism.

We first prove Claim2, it suffices to check $f(\mathfrak{m}_{A_{f^{-1}(\mathfrak{q})}}) \subset \mathfrak{m}_{B_{\mathfrak{q}}}$

$$f(a/b) = \frac{f(a)}{f(b)} \in \mathfrak{q}$$

$$\begin{tikzcd}
\mathcal{O}_{A,\tilde{f}(\mathfrak{q})} \arrow[rrr] \arrow[ddd, "\simeq"'] & & (\tilde{f}^*_*\mathcal{O}_B)_{\tilde{f}(\mathfrak{q})} \arrow[rrr] & & & \mathcal{O}_{B,\mathfrak{q}} \arrow[ddd, "\simeq"] \\
 & (U,s) \arrow[r, mapsto] \arrow[d, mapsto] & (U, \tilde{f}^*_*(s)) \arrow[r, mapsto] & (\tilde{f}^{-1}(U), \tilde{f}^*_*(s)) \arrow[d, mapsto] & & \\
 & s(f^{-1}(\mathfrak{q})) = s(\tilde{f}(\mathfrak{q})) \arrow[rr, mapsto] & & f(s(\tilde{f}(\mathfrak{q}))) = \tilde{f}^*_*(s)(\mathfrak{q}) & & \\
A_{f^{-1}(\mathfrak{q})} \arrow[rrrrr, "f"] & & & & & B_{\mathfrak{q}},
\end{tikzcd}$$

so this indeed works,

(2) Let $(f, f^*) : (\operatorname{Spec} B, \mathcal{O}_B) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_A)$ be a morphism of locally ringed space/

$$f^* : \mathcal{O}_A \longrightarrow f_*\mathcal{O}_B$$

$$\begin{tikzcd}
\mathcal{O}_A(\operatorname{Spec} A) \arrow[r, "f^*_{\operatorname{Spec} A}"] \arrow[dd, equal] & \mathcal{O}_B(f^{-1}(\operatorname{Spec} A)) \arrow[d, equal] \\
 & \mathcal{O}_B(\operatorname{Spec} B) \arrow[d, equal] \\
A \arrow[r] & B.
\end{tikzcd}$$

Let $\varphi = f^*_{\operatorname{Spec} A}$.

Claim: The locally ringed morphism induced by φ is (f, f^*).

To finish the proof of (2), we need to check that the two constructions are reciprocal bijections.

To check the claim, let $\mathfrak{q} \in \operatorname{Spec}(B)$, we have

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ \mathcal{O}_{A,f(\mathfrak{q})} \arrow[r, "f*_{\mathfrak{q}}"] & B_{\mathfrak{q}} = \mathcal{O}_{B,\mathfrak{q}} \end{tikzcd}$$

We know:

(1) $f_{\mathfrak{q}}^*$ is local

$$\iff (f_{\mathfrak{q}}^*)^{01}(\mathfrak{m}_{B_{\mathfrak{q}}}) = \mathfrak{m}_{A_{f(\mathfrak{q})}}$$

(2) The diagram commutes, because f^* is a morphism of sheaves so compatible with restriction. This implies $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) = \tilde{\varphi}(\mathfrak{q})$. (Indeed. let $\alpha \in \varphi^{-1}(\mathfrak{q}), \beta = \varphi(\alpha) \in \mathfrak{q} \Longrightarrow \alpha \in f(\mathfrak{q}) \Longrightarrow \varphi^{-1}(\mathfrak{q}) \subset f(\mathfrak{q})$).

$$\begin{tikzcd} \alpha \arrow[r, mapsto] \arrow[d, mapsto] & \beta \arrow[d, mapsto] \\ (*) \arrow[r, mapsto] & (\bullet \in \mathfrak{m}_{B_{\mathfrak{q}}}), \end{tikzcd}$$

where $(*)$ belongs to $\mathfrak{m}_{A_{f(\mathfrak{q})}}$ (because the morphism is local)

Conversely, let $\alpha \in F(\mathfrak{q})$

$$\begin{tikzcd} \alpha \arrow[r, mapsto] \arrow[d, mapsto] & \beta = \varphi(\alpha) \arrow[d, mapsto] \\ (\bullet \in \mathfrak{m}_{A_{f(\mathfrak{q})}}) \arrow[r, mapsto] & (* \in \mathfrak{m}_{B_{\mathfrak{q}}}) \end{tikzcd}$$

since β maps to an element in $\mathfrak{m}_{B_{\mathfrak{q}}}$, we have $\beta \in \mathfrak{q}$, so $\varphi(f(\mathfrak{q})) \subset \mathfrak{q} \iff f(\mathfrak{q}) \subset \varphi^{-1}(\mathfrak{q})$

Proof of part (3) is left as an exercise. $\square$

Definition 2.40. *$(X, \mathcal{O}_X)$ (locally)-ringed space. $U \subset X$ open set. Define $\mathcal{O}_U(V) = \mathcal{O}_X(V)$ for $V \subset U$ open. Then $(U, \mathcal{O}_U)$ is a (locally) ringed space. (in fact $\forall x \in U$, $\mathcal{O}_{U,x} = \mathcal{O}_{X,x}$)*

Definition 2.41. *A **scheme** S is a locally ringed space $(S, \mathcal{O}_S)$ which is locally isomorphic to affine schemes, i.e. $\forall x \in S \exists U \subset S$ open, $x \in U$, and a ring A such that $(U, \mathcal{O}_U)$ is isomorphic as locally ringed spaces to $(\operatorname{Spec} A, \mathcal{O}_A)$. We view category of schemes as a subcategory of locally ringed spaces.*

Examples of schemes/morphisms

Let K be a field. Let S be a scheme with morphism $f : S \longrightarrow \operatorname{Spec} K$.

Affine case: $S = \operatorname{Spec} A$

$$f \longleftrightarrow (K \longrightarrow A)$$

i.e. $f \longleftrightarrow$ structure of K-algebra on A

Example 2.42. *$A = K[X_1, ..., X_m]/I$ has morphism $\operatorname{Spec} A \longrightarrow K$.*

Global case:

$$f \longleftrightarrow \begin{cases} S \overset{continuous}{\longrightarrow} \operatorname{Spec}(K) = \eta = \{0\} \\ \mathcal{O}_{\operatorname{Spec} K} \longrightarrow f_*\mathcal{O}_S \end{cases}$$

where $\mathcal{O}_{\operatorname{Spec} K}$ consists of

$$\emptyset : \mathcal{O}_K(\emptyset) = \{0\} \longrightarrow \{0\}$$

$$\{\eta\} : \mathcal{O}_K(\eta) = K \longrightarrow (f_*\mathcal{O}_S)(\{\eta\}) = \mathcal{O}_S(S)$$

therefore

$$\{S \longrightarrow \operatorname{Spec} K\} \longleftrightarrow \{K \longrightarrow \mathcal{O}_S(S)\}.$$

Definition 2.43. *Let B be a scheme, a scheme **over** B is*

$$f : S \longrightarrow B$$

a morphism of schemes.

A morphism of schemes over B is

$$\begin{tikzcd} S_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & S_2 \arrow[dl, "f_2"] \\ & B & \end{tikzcd}$$

so that $f_2 \circ g = f_1$.

Global case2: K is a field , f : $\operatorname{Spec} K \longrightarrow S$ corresponds to a point $x = f(\eta) \in S$, $\mathcal{O}_S \longrightarrow f_*\mathcal{O}_{\operatorname{Spec} K}$:

$$\forall U, \mathcal{O}_S(U) \longrightarrow \mathcal{O}_{\operatorname{Spec} K}(f^{-1}(U)) = \begin{cases} \{0\} \text{ if } x \notin U \\ K \text{ if } x \in U. \end{cases}$$

Compatibility with restrictions show that this is equivalent to

$$\mathcal{O}_{S,f(\eta)} = \mathcal{O}_{S,x} \xrightarrow{g} \mathcal{O}_{K,\eta} = K$$

such that $g^{-1}(\{0\}) = \mathfrak{m}_{\mathcal{O}_{S,x}}$, i.e. g passes to the quotient

$$K_S(x) \longrightarrow K.$$

Concretely, "the coordinates of x are in K^n"

2.6 Apr 10th:

Recall: A scheme S is a locally ringed space $(S, \mathcal{O}_S)$, $(\mathcal{O}_{S,x})$ are local rings. s.t. $\forall x \in S$, exists an open set $U \in S$ $x \in U$ and a ring A s.t. $(U, \mathcal{O}_S|_U) \simeq \operatorname{Spec} A$.

We will give some examples of morphism of schemes.

Example 2.44. *(1) A, B are rings.*

$$Hom_{Sch}(\operatorname{Spec} A, \operatorname{Spec} B) = Hom_{Rings}(B, A)$$

(2) K is a field.

$$[X \longrightarrow \operatorname{Spec} K - \{\eta\}] \Longleftrightarrow [K \longrightarrow \Gamma(X, \mathcal{O}_X)]$$

(also for any $x \in X$, we get $\mathcal{O}_{K,\eta} = K \longrightarrow \mathcal{O}_{X,x} \longrightarrow \kappa(x)$) so every residue field is an extension of K.

(3)

$$[\operatorname{Spec}(K) \xrightarrow{f} X] \Longleftrightarrow [\text{a point } x = f(\eta) \text{ and } \mathcal{O}_{X,x} \xrightarrow{f^*} K]$$

s.t. $\ker(f^*) = \mathfrak{m}_{X,x}$ *i.e.* $\kappa(x) \hookrightarrow K$.

I.e. $Hom_{Sch}(\operatorname{Spec}(K), X) \cong \{(x,i) | x \in X, i : \kappa(x) \hookrightarrow K\}$.

In particular, take $X = \operatorname{Spec}(K[X_1, ..., X_n]/(f_1, ..., f_m))$ *then using* (1), *we get*

$$Hom_{Sch}(\operatorname{Spec} K, X) \simeq Hom_{Rings}(K[X]/I, K).$$

If we only consider morphism over $\operatorname{Spec} K$,

$$\begin{tikzcd} \operatorname{Spec} K \arrow[rr] \arrow[dr, "id"'] & & X \arrow[dl] \\ & \operatorname{Spec} K & \end{tikzcd},$$

we look ar $Hom_{K-alg}(K[X]/I, K)$

K-linear $K[X]/I \longrightarrow K \Longleftrightarrow$ *giving* $x = (x_1, ..., x_n) \in K^n$ *s.t.* $f_1(x) = ... = f_m(x) = 0$ *so*

$$Hom_{Sch\ over\ K}(\ Spec\, K, X)$$

are the K*-valued solutions of the equation defining* X*.*

Notation: Any $S \xrightarrow{f} X$ *is called an* S*-valued point of* X*.*

(4) *(restriction of morphisms)*

$$U \subset X \xrightarrow{f} Y$$

where U *is open. We want to restrict* f *to* U*, first* $(U, \mathcal{O}_X \mid U)$ *is locally ringed space. (We will see that it is a scheme.)*

Let $f|U : (U, \mathcal{O}_X \mid U) \longrightarrow X$ *be defined by*

$$(f|U)(x) = f(x) \forall x \in U$$

so $f|U$ *is continuous and* $\mathcal{O}_Y \xrightarrow{f|_U^*} (f|U)_*(\mathcal{O}_X|U)$ *defined by* $\forall V \in Y,$ *open*

$$\mathcal{O}_Y(V) \longrightarrow (\mathcal{O}_X|U)((f|U)^{-1}(V)) = \mathcal{O}_X(U \cap f^{-1}(V))$$

obtained by

$$\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V)) \xrightarrow{res} \mathcal{O}_X(f^{-1}(U) \cap V).$$

This is a morphism of ringed spaces. Moreover, we can check that it is a morphism of schemes. On the stalks, the induced morphisms

$$\mathcal{O}_{Y,(f|U)(x)=f(x)} \longrightarrow \mathcal{O}_{X,x}$$

are the same as those from f *it self.*

Check $V \subset U \subset X \Longrightarrow f|V = (f|U)|V$*.*

(5) *Prop. Any* $U \subset X$*, where* X *is a scheme,* U *open , is a scheme. (Note: in general, this is not an affine scheme even if* X *is affine.) Let* $X = \mathbb{A}^2_{\mathbb{C}} = Spec\,(\mathbb{C}X_1, X_2)$*,* $U = X - \{(0,0)\}$ *open, where the point* $(0,0)$ *corresponds*

to the maximal ideal $(X_1, X_2)\mathbb{C}[X]$. U is not an affine scheme because one can check that

$$\begin{tikzcd} \Gamma(U, \mathcal{O}_U) \arrow[r, "\simeq"] \arrow[d, equal] & \mathbb{C}[X_1, X_2] \arrow[d, equal] \\ \Gamma(U, \mathcal{O}_X|U) \arrow[d, equal] & \Gamma(X, \mathcal{O}_X) \\ \Gamma(U, \mathcal{O}_X) & \end{tikzcd}$$

$\boxed{Hartlog'sphenomenon}$ *so if U was affine, we would get $U \overset{\simeq}{\hookrightarrow} X$ which is absurd.*

Proof of Prop. $x \in U$, X is a scheme, $\Longrightarrow \exists x \in V \subset X$ open s.t. $V = \operatorname{Spec} A$ is affine. Then $V \cap U$ is an open nbhd of $x \in U$, and is open in $V = \operatorname{Spec} A$ so it it suffices to check that an open subset of $\operatorname{Spec}(A)$ is a scheme. Recall that the basic open subsets $U_f = \{\mathfrak{p} \in \operatorname{Spec} A | f \notin \mathfrak{p}\}$ form a basis of the topology. So we reduces to showing that U_f is affine. Precisely, U_f is canonically isomorphic to $\operatorname{Spec}(A_f)$. (We already constructed a homeomorphism $U_f \xrightarrow{i} \operatorname{Spec}(A_f) : \mathfrak{p} \mapsto \mathfrak{p}A_f$).

To deduce the Prop, it suffices to have an isomorphism

$$\mathcal{O}_{A_f} \xrightarrow{\simeq} i_* \mathcal{O}_{U_f}$$

i.e. for all $V \subset \operatorname{Spec}(A_f)$ open an isomorphism

$$\mathcal{O}_{A_f}(V) \xrightarrow{\simeq} \mathcal{O}_{U_f}(i^{-1}(V))$$

and compatible with restrictions.

<u>Recall</u>: $\mathcal{O}_{A_f}(U) = \{g : U \longrightarrow \sqcup_{Q \in U} (A_f)_Q | g \text{ "locally" } \frac{a}{b}, a, b \in A_f\}$

$$\mathcal{O}_{U_f}(i^{-1}(V) = \mathcal{O}_A(i^{-1}(V))) = \{\tilde{g} : i^{-1}(V) \longrightarrow \sqcup_{\mathfrak{p} \in A_\mathfrak{p}} | \tilde{g} = \frac{\tilde{a}}{\tilde{b}}, \tilde{a}, \tilde{b} \in A\}$$

The morphism $g \mapsto \tilde{g}$ is given by $\tilde{g}(\mathfrak{p}) = g(\mathfrak{p}A_f) = g(i(\mathfrak{p}))$. This works because $a = \tilde{a}/f^n$ and $b = \tilde{b}/f^m$ so $a.b = f^m\tilde{a}/f^n\tilde{b}$ □

(6) *A discrete valuation ring is a local ring A with maximal ideal $\mathfrak{m}_A \subset A$ being a principal ideal generated by $\varpi \in A$ ("uniformizer")*[1] *$A/\mathfrak{m}_A - k$ is the residue field. (Exercise. $A = \{a/b \in Q | p \nmid b, a \in \mathbb{Z}\}$, $\mathfrak{m}_A = \mathfrak{p}A, \varpi = p$)*

[1] A is a the local ring at a closed point of a non-singular point of a curve

$Spec\,A = \{\eta = \{0\}, s\}$, where s is the "special point" $s = \varpi A = \mathfrak{m}_A$. The open sets are $\emptyset$, $Spec\,A, \{\eta\}$, $\{\eta\}$ is open because $\{s\}$ is closed. Structure sheaf

$$\mathcal{O}_A(\emptyset) = 0, \mathcal{O}_A(\ Spec\,A) = A, \mathcal{O}_A(\{\eta\}) \simeq A_\varpi = K = Frac(A)$$

(since $A_\varpi = \{\frac{a}{\pi^n} | n \geq 0\}$ and any $b \notin \varpi A$ is invertible)

$$\kappa(s) = A/\mathfrak{m}_A = k$$

$$\kappa(\eta) = Frac(A/\{0\}) = K$$

$$res^{Spec\,A}_{\{\eta\}} : A \longrightarrow A_\varpi = K$$

is the inclusion. What is the nature of schemes over A?

$$f : X \longrightarrow\ Spec\,(A)$$

Topologically: $X = X - s \sqcup X_\eta$,

$$f(x) = \begin{cases} s, x \in X_s \\ \eta, x \in X_\eta \end{cases}$$

s.t. $f^{-1}(\{\eta\}) = X_\eta$ is open in X. (topology is determined by an oepn set $X_\eta \subset X$),

sheaf-theoretical point

$$\mathcal{O}_A \longrightarrow F_*\mathcal{O}_X \Longleftrightarrow \begin{array}{ccc} A = \mathcal{O}_A(\ Spec\,A) & \xrightarrow{f_A^*} & \mathcal{O}_X(X) \\ {\scriptstyle res}\downarrow & & \downarrow \\ K = \mathcal{O}_A(\{\eta\}) & \xrightarrow[f_\eta^*]{} & \mathcal{O}_X(X_\eta) \end{array}$$

such that

$$res^X_{X_\eta}(f_A^*(a)) = f_\eta^*(a)(\text{ viewed a elements of } K).$$

It is locally ringed $\forall x, \mathcal{O}_{A,f(x)} \longrightarrow \mathcal{O}_{X,x}$ local $\Longleftrightarrow \forall x \in X_\eta$, $\mathcal{O}_{A,\eta} = A_\eta = K \longrightarrow \mathcal{O}_{X,x}$ (always local) and $\forall s \in X_s$, $\mathcal{O}_A(\ Spec\,A) = A = \mathcal{O}_{A,s} \longrightarrow \mathcal{O}_{X,x}$, where the equality holds because $Spec\,A$ is the only open set that containing s.

Exercise 2.45. *For any scheme X, there is a unique morphism $X \longrightarrow$ Spec$(\mathbb{Z})$. recall* $\dim(\mathbb{Z}) = 1$.

Proof. If $X = \operatorname{Spec} A$, then

$$\operatorname{Hom}_{Sch}(\operatorname{Spec} A, \operatorname{Spec} \mathbb{Z}) = \operatorname{Hom})Rings(\mathbb{Z}, A) = \{1 \mapsto 1\}$$

has a unique element. If X is arbitrary, $X = \cup_i \operatorname{Spec} A_i$ for every i, there is a unique $f_i : \operatorname{Spec}(A_i) \longrightarrow \operatorname{Spec} \mathbb{Z}$. Intuitively, this implies uniqueness $(f, \tilde{f} : x \longrightarrow \operatorname{Spec} \mathbb{Z}) \Longrightarrow f| \operatorname{Spec}(A_i) = f_i = \tilde{f}| \operatorname{Spec}(A_i)$ and thus implies $f = \tilde{f}$ and also the existence since

$$f_i| \operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_j) = f_j| \operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_j)$$

Indeed

Proposition 2.46. *Given X, Y schemes, $X = \cup_i U_i$ open covering. To give $f : X \longrightarrow Y$ is "the same" as giving $f_i|U \longrightarrow Y$ s.t. $f_i|U_i \cap U_j = f_j|U_i \cap U_j$.*

$f \mapsto (f|U_i)_j$ is a bijection.

surjectivity: Given $(f_i), f_i : U_i \longrightarrow Y$ construct f?

$$X \xrightarrow{f} Y$$

Topologically: $f(x) = f_i(x)$ if $x \in U_i$ is well-defined since $f_i(x) = f_j(x)$ if $x \in U_i \cap U_j$. f thus defined is continuous (exercise)

Sheaf-theory we need $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$: $\forall V \subset Y$, $\mathcal{O}_Y(V) \xrightarrow{?} \mathcal{O}_X(f^{-1}(V))$. Given $s \in \mathcal{O}_Y(V)$, we get $s_i \in \mathcal{O}_{U_i}(f_i^{-1}(V)) = \mathcal{O}_X(f_i^{-1}(V))$ and $f^{-1}(V) = \cup_i f_i^{-1}(V)$ and $s_i|f_i^{-1} \cap f_j^{-1}(V) = s_j|f_i^{-1}(V) \cap f_j^{-1}(V)$. By the sheaf condition on $\mathcal{O}_X$, there exists a unique $\tilde{s} \in \mathcal{O}_X(f^{-1}(V))$ s.t.

$$\tilde{s}|f^{-1}(V_i) = s_i, \forall i.$$

The map $s \mapsto \tilde{s}$ is the required $\mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V)) \Longrightarrow$ get $\mathcal{O}_Y \xrightarrow{f^*} f_*\mathcal{O}_X$. It is local because if $x \in U_i \subset X$ the induced □

2.7 Aprl 13th:

Example 2.47.

Proposition 2.48. *For any scheme X and any ring A, there is a bijection*

$$Hom_{Sch}(X,\ Spec\, A) \simeq\ Hom_{Rings}(A, \mathcal{O}_X(X))$$

given by

$$X \xrightarrow{f}\ Spec\, A$$

$$\Longrightarrow \mathcal{O}_A \xrightarrow{f^*} f_*\mathcal{O}_X \Longrightarrow A = \mathcal{O}_{\ Spec\, A}(\ Spec\, A) \longrightarrow \mathcal{O}_X(f^{-1}\ Spec\, A) = \mathcal{O}_X(X)$$

Ex.

(1) $Hom_{Sch}(X,\ Spec\, K) \longleftrightarrow K \hookrightarrow c\mathcal{O}_X(X)$

(2) $Hom_{Sch}(X,\ Spec\, \mathbb{Z}) \simeq\ Hom(\mathbb{Z}, \mathcal{O}_X(X))$ *has a unique element. ($Spec\, \mathbb{Z}$ is the final object in* Sch*)*

(3) $Hom_{Sch}(X, \mathbb{A}^1_{\mathbb{Z}}) \simeq\ Hom(\mathbb{Z}[T], \mathcal{O}_X(X))$

Proof.

$$X = \cup_i U_i m U_i =\ \text{Spec}\,(A_i) \text{ open in } X$$

$$\begin{aligned}
\text{Hom}_{Sch}(X,\ \text{Spec}\, A) &= \{(f_i) : f_i : U_i \longrightarrow\ \text{Spec}\, A \text{ s.t. } f_i|U_i \cap U_j = f_j|U_i \cap U_j\} \\
&\cong \{(g_i) : g_i : A \longrightarrow A_i,\ \text{which are compatible on intersections}\} \\
&\cong \{(g_i\ g_i A \longrightarrow \Gamma(U_i \mathcal{O}_X), \forall a \in A, g_i(a)|U_i \cap U_j = g_j(a)|U_i \cap U_j\} \\
&\simeq \{g | g : A \longrightarrow \mathcal{O}_X(X)\} >>>>>>>>>> 1
\end{aligned}$$

□

Note in general,

$$Hom_{Sch}(\ Spec\, A.X) \neq\ Hom_{rings}(\mathcal{O}_X(X), A)$$

Ex. $X = \mathbb{P}^1_K, \Longrightarrow \mathcal{O}_X(X) = K$. *If* $A = K$*, then* $Hom_{rings}(K, K) = \{id\}$ *but* $Hom(\ Spec\, \mathbb{Q}, \mathbb{P}^1_{\mathbb{Q}})$ *has infinitely many elements.*

3 Fiber product

This is a notion that makes sense in any category.

Definition 3.1. *$\mathcal{C}$ a category, X, Y objects of $\mathcal{C}$, S an object of $\mathcal{C}$. Assume given*

$$\begin{tikzcd} & Y \arrow[d, "f_2"] \\ X \arrow[r, "f_1"] & S \end{tikzcd}$$

We say that an object Z of $\mathcal{C}$ with morphisms

$$\begin{tikzcd}
Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "f_2"] \\
X \arrow[r, "f_1"] & S
\end{tikzcd}$$

makes the diagram commutes is a ***fibre product*** *of X, Y over S if it has the universal property*

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "f_2"] \\
& X \arrow[r, "f_1"] & S
\end{tikzcd}$$

Notation: $Z = X \times_S Y$

N.B. This notation is ambiguous because the fibre product depends on f_1, f_2. The fibre product is only suitably unique when it is specified with its two projections π_1, π_2.

Example 3.2. *In* Sets *fibre products exist and*

$$X \times_S Y = \{(x, y) \in X \times Y | f_1(x) = f_2(y)\}$$

with $\pi_1(x, y) = x$ *and* $\pi_2(x, y) = y$

Proof.

(1) $f_2 \circ \pi_2(x, y) = f_2(y) = f_1(x) = f_1 \circ \pi_1(x, y)$ for $(x, y) \in Z$.

(2) Let T be a set with $p_1 : T \longrightarrow X$ and $p_2 : T \longrightarrow Y$ s.t.

$$\begin{tikzcd}
T \arrow[dr, "f"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "f_2"] \\
& X \arrow[r, "f_1"] & S
\end{tikzcd}$$

define $f(t) = (p_1(t), p_2(t))$, $f_1(p_1(t)) = f_2(p_2(t))$, therefore $f(t) \in Z$. So f is a map makes the above diagram commute. Uniqueness of f is obvious. If there is another $\tilde{f}(t) = (\tilde{f}_1(t), \tilde{f}_2(t))$ making the above diagram commute, then $\tilde{f}_1(t) = \pi_1 \circ \tilde{f}(t) = p_1(t)$ and $\tilde{f}_2(t) = \pi \circ \tilde{f}(t) = p_2(t)$.

Note: This construction/definition is a an example of "universal" object in the categorical sense. It is universal in the following sense.

Given $X \xleftarrow{\pi_1} Z_1 \xrightarrow{\pi_2} Y$, $X \xleftarrow{\tilde{\pi}_1} Z_2 \xrightarrow{\tilde{\pi}_2} Y$ both fibre products over S there is a unique isomorphism $j : Z_1 \longrightarrow Z_2$, s.t. $\tilde{\pi}_1 = \pi_1 \circ j^{-1}$ and $\tilde{\pi}_2 = \pi_2 \circ j^{-1}$. $\square$

Example 3.3. *If $\mathcal{C} = \mathsf{Sets}$, $S = \{*\}$ any 1 element set, the fibre product over S is just the Cartesian product*

$$X \times_S Y = \{(x, yinX \times Y | f_1(x) = f_2(y)\}$$

But the restriction on f_i is just vacuous, the fibre product contains the usual Cartesian product.

(2) Let $X \overset{f_1}{\hookrightarrow} S \overset{f_2}{\hookleftarrow} Y$ (inclusion of subsets) We can see that the fibre product is isomorphic to the intersection of X, Y

$$\begin{tikzcd} X \cap Y \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \\ X \arrow[r, hook] & S \end{tikzcd}$$

(3)

$$\begin{tikzcd} f_2^{-1}(X) \arrow[r, hook] \arrow[d, "f_2|_{f_2^{-1}(X)}"'] & Y \arrow[d, "f_2"] \\ X \arrow[r, hook, "f_1"] & S \end{tikzcd}$$

Theorem 3.4. *In the category Sch of schemes, arbitrary fibre product exists.*

Note This is false in the category of affine algebraic sets over K, with K algebraically closed.

Proof of theorem. Step 1 We prove this for affine schemes.

Assume $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} R$. Given a diagram

$$\begin{tikzcd} X \arrow[r] & S \\ & Y \arrow[u] \end{tikzcd}$$

in $\mathsf{AffnSch}$, we have a reversed diagram in Rings

$$\begin{tikzcd} A & \\ R \arrow[u] \arrow[r] & B \end{tikzcd}$$

Define $Z = \operatorname{Spec}(A \otimes_R B)$, and set $A \otimes_R B =: C$. We have

$$\begin{tikzcd}
A \arrow[rr] & & C & 1 \otimes b \\
& a \arrow[r, mapsto] & a \otimes 1 & b \arrow[u, mapsto] \\
R \arrow[uu] \arrow[rr] & & B \arrow[uu] &
\end{tikzcd}$$

This diagram is commutative, which guarantees a diagram in AffSch

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

$$\begin{tikzcd}
T \arrow[drr] \arrow[ddr] & & \\
& Z \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

<u>N.B.</u> $\operatorname{Spec}(A \otimes_R B)$ is not easy to describe as a set.

<u>Step 2</u> Uniqueness of $X \times_S Y$, when it exists, is formal.

<u>Step 3.</u> If $X \times_S Y$ exists, for any open subset $U \subset X$, $U \times_S Y$ exists and is $\pi_1^{-1}(U)$

$$\begin{tikzcd}
\pi_1^{-1}(U) \arrow[r] \arrow[d] & X \times_S Y \arrow[r] \arrow[d, "\pi_1"] & Y \arrow[d] \\
U \arrow[r, hook] & X \arrow[r] & S
\end{tikzcd}$$

<u>Step 4</u> U_i affine. If for each i, $U - i \times_S Y$ exists, then so does $X \times_S Y$

$$\begin{tikzcd}
& Y \arrow[d, "f_2"] \\
\cup_i U_i = X \arrow[r, "f_1"] & S
\end{tikzcd}$$

If for each i, $U_i \times_S Y$ exists, then so does $X \times_S Y$.

Define $V_{i,j} = \pi_{1,i}^{-1}(U_i \cap U_j) \subset U_i \times_S Y$ open. Check that $V_{i,j} = (U_i \cap U_j) \times_S Y$ ¿¿¿¿¿¿¿¿¿2

One can glue the $U_i \times_S Y$ along the isomorphisms.

Check This scheme is $X \times_S Y$

Step 5.

step 1+ step 4.

Y, S affine $\Longrightarrow X \times_S Y$ exists. □

3.1 17th Aprl: Examples and Applications of the Fiber Product

Recall. If we have maps $X \longrightarrow S$, $Y \longrightarrow S$, a space Z with maps $Z \longrightarrow X, Z \longrightarrow Y$ if the fiber product of $X \longrightarrow S \longleftarrow Y$ if it has the universal property

$$\begin{tikzcd} T \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Z \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

$X = \operatorname{Spec} A, Y = \operatorname{Spec} B, S = \operatorname{Spec} R$. The contravariant functor Spec would invert the fiber coproduct of rings to fiber product of schemes.

$$X \times_S Y = \operatorname{Spec} A \otimes_R B$$

Define: $X \times_R Y := X \times_{\operatorname{Spec} R} Y$.

Example 3.5. *Why not product? For X, Y and schemes, each have a unique map to $\operatorname{Spec} \mathbb{Z}$. The fiber product $X \times_{\mathbb{Z}} Y$ depends only on X and Y. ($\operatorname{Spec} \mathbb{Z}$ is the final object in* Schemes*)*

$X = \operatorname{Spec} \mathbb{Z}[T]$, Krull dimension 2, $Y = \operatorname{Spec} \mathbb{Z}[V]$, dimension 2. $X \times_{\mathbb{Z}} Y \neq \operatorname{Spec} \mathbb{Z}[T, V]$ Krull dimension 3.

$$\dim X \times_{\mathbb{Z}} Y \neq \dim X + \dim Y.$$

Example 3.6. *K a field, X, Y schemes over K $\dim X \times_K Y = \dim X + \dim Y$. But hte set of points of $X \times_K Y$ is not simply a topological product. For example, $X = \mathbb{A}_K^1, Y = \mathbb{A}_K^1$, $X \times_K Y = \mathbb{A}_K^2$, but it true that $X(K) = \operatorname{Hom}(\operatorname{Spec} K, X)$*

$$X \times_K Y(K) = X(K) \times Y(K)$$

$$\operatorname{Hom}_{\operatorname{Spec} K}(\operatorname{Spec} K, X \times_K Y) = \operatorname{Hom}_{\operatorname{Spec} K}(\operatorname{Spec} K, X) \times \operatorname{Hom}_{\operatorname{Spec} K}(\operatorname{Spec} K, Y)$$

by the universal product of fiber products.

If X is a scheme over S, T is a scheme over S. We can define $X(T)$ as $\mathrm{Hom}_S(T, X)$ and call it the T-valued points of X.

$$T = \text{ Spec}\, R, X(R)$$

$$X \times_S Y(T) = X(T) \times Y(T).$$

If we have the following morphism of schemes

$$\begin{tikzcd}
X \arrow[dr] & & Y \arrow[dl] \\
& S' \arrow[d] & \\
& S &
\end{tikzcd},$$

we have

$$X \times_{S'} Y(T) = X(T) \times_{S'(T)} Y(T).$$

$\mathrm{Hom}_S(T, X \times_{S'} Y)$ ={ pair (f_1, f_2) of elements $f_1 \in$ $\mathrm{Hom}_S(T, X)$ and $f_2 \in$ $\mathrm{Hom}_S(T, Y)$ such that $p_1 \circ f_1 = p_2 \circ f_2$}={ pair of elements $f_1 \in X(T)$ and $f_2 \in Y(T)$, $p_1 \circ f_1 = p_2 \circ f_2 \in S'(T)$}

$$\begin{tikzcd}
T \arrow[dr] \arrow[drr, "f_2"] \arrow[ddr, "f_1"'] \arrow[dddrr, dashed, bend right] & & & \\
& X \times_{S'} Y \arrow[r] \arrow[d] & Y \arrow[d, "p_2"] & \\
& X \arrow[r, "p_1"] & S' \arrow[dr] & \\
& & & S
\end{tikzcd}$$

Example 3.7. *Consider $GL_n(K)$. There exists a scheme $\mathcal{GL}_n$ with $\mathcal{GL}_n(K) = GL_n(K)$. $\mathbb{A}^{n^2} \supset V(det)$, $\mathcal{GL}_n$ gives the "open complement of $V(det)$". What are the R-valued points of $\mathcal{GL}_n$?*

$$\mathcal{GL}_n(R) \neq \{n \times n \textit{ matrices over } R \textit{ with } det \neq 0\}$$

But

$$\begin{aligned}
\mathcal{GL}_n(R) &= \{n \times n \textit{ matrices over } R \textit{ s.t. } \textit{Spec}\, R \longrightarrow \mathbb{A}^{n^2} \textit{ does not intersect } V(det)\} \\
&= \{n \times n \textit{ matrices } M \textit{ over } R \textit{ s.t.} det(M) \notin \textit{ any prime ideal of } R\} \\
&= \{n \times n \textit{ matrices } M \textit{ over } R \textit{ where } det(M) \textit{ is invertible}\}
\end{aligned}$$

Example 3.8. *Equation*

$$x^3 + y^3 + z^2 = 0,$$

find all solutions in $\mathbb{Z}$*.* $X = \mathit{Spec}\,\mathbb{Z}[x, y, z]/(x^3 + y^3 + z^2)$*. The set of solutions is* $X(\mathbb{Z})$

Example 3.9. $f : X \longrightarrow S$*,* $\mathfrak{p}$ *a point of* S $K(\mathfrak{p})$ *residue field of* $\mathfrak{p}$*.* $\mathit{Spec}\,K(\mathfrak{p}) \longrightarrow S$*. Define*$K(\mathfrak{p}) \times_S X$ *as the fiber of* f *over* $\mathfrak{p}$

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ \mathit{Spec}\,K(\mathfrak{p}) & \longrightarrow & S \end{array}$$

Lemma 3.10. *The set of points of the fiber is the inverse image of* $\mathfrak{p}$ *where* f *is the set of points of* X*. The underlying set of* $K(\mathfrak{p}) \times_S X$ *maps to the underlying set of* X*.*

The relative point of view "A parametrized family of varieties" $y^2 = x^3 - 3x - t$ viewed as a family of algebraic sets in $\mathbb{A}^2$ with coordinates X, Y parameter t. For each , we get an equation in X, Y, this defines a curve in $\mathbb{A}^2$. Consider the morphism

$$f : \operatorname{Spec} k[x, y, t]/(y^2 - x^3 - 3x + t) \longrightarrow \operatorname{Spec} K[t].$$

The fibers of f over closed points are curves in the family.

Idea from Grothendieck: view any morphism as a family where elements are the fiber. $\mathbb{A}^3 \cup pt \Longrightarrow \mathbb{A}^1$, $\mathbb{A}^3$ to 0 pt to 1. Not all maps make nice families but this point of view is helpfull in general, why?

- Fibers of a family are often simper (e.g.)
- Full family are simpler than individual fibers.

Example 3.11. *Reduction* $\mod p$*.* X *is a scheme over* $\mathbb{Z}$*.* $X \times_{\mathbb{Z}} \mathit{Spec}\,\mathbb{F}_p$ *is a scheme over* $\mathbb{F}_p$*. "reduction* $\mod p$*" of* X $X = \mathit{Spec}\,\mathbb{Z}[x_1, ..., x_n]/(f_1, ..., f_m)$ $X \times_{\mathbb{Z}} \mathit{Spec}\,\mathbb{F}_p = \mathit{Spec}\,\mathbb{F}_p[x_1, ..., x_n]/(f_1, .., f_m)$*. This can be tricky, for example* $\mathit{Spec}\,\mathbb{Z}[T]/T(T+2)$ *has no nilpotents (is "reduced" scheme) but* $\mathit{Spec}\,\mathbb{F}_2[T]/Y(T+2) = \mathit{Spec}\,\mathbb{F}_2[T]/T^2$ *has nilpotent.*

Example 3.12. X *a scheme over* $\mathbb{Z}$ $X \times_{\mathbb{Z}} \mathit{Spec}\,\mathbb{Q}$ *(or* $X \times_{\mathbb{Z}} \mathit{Spec}\,\overline{\mathbb{Q}}$*). Given* Y *over* $\mathit{Spec}\,\mathbb{Q}$*, can we find* X *over* $\mathit{Spec}\,\mathbb{Z}$ *with* $X \otimes_{\mathbb{Z}} \mathit{Spec}\,\mathbb{Q} = Y$*? If so* $X \times_{\mathit{Spec}\,\mathbb{Z}} \mathit{Spec}\,\mathbb{F}_p$ *will give some perspective on* Y*.*

Lemma 3.13. *If Y in $\mathbb{F}^n_{\mathbb{Q}}$ is the vanishing scheme of $f_1, .., f_m$ then this is possible.*

Proof. $Y = \operatorname{Spec} \mathbb{Q}[T_1, .., T_n]/(f_1, .., f_m)$, where f_i are polynomials with rational coefficients. We can find $c_1, .., c_m$ positive integes where $c_i f_i$ has integer coefficients for all i. $X = \operatorname{Spec} \mathbb{Z}[T_1, ..., T_n]/(c_1 f_1, ..., c_m f_m)$. Because c_i^{-1} exists in $\mathbb{Q}$, it produces the same scheme over $\mathbb{Q}$.

□

However, this is not unique. For example, $Y = \operatorname{Spec} \mathbb{Q}[T]/(T^2/2 + 1)$. We can take $c = 2$ to get $\mathbb{Z}[T]/T^2 + 2$ and $c = 4$ to get $\mathbb{Z}[T]/2T^2 + 4$. These are not isomorphic fibers over 2, in fact, they are distinct $\mathbb{F}_2[T]/T^2$ v.s. $\mathbb{F}_2[T]$. Worse $T = 2U$, $Y = \operatorname{Spec} \mathbb{Q}[U]/(2U^2 + 1), X = \operatorname{Spec} \mathbb{Z}[U]/(2U^2 + 1)$. Reducetion over 2 gives $\operatorname{Spec} \mathbb{F}_2[U]/1 = \emptyset$

Example 3.14. *(Base change) $f : X \longrightarrow S$ a morphism, family of schemes.*

$$\begin{tikzcd} X \times_S T \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

we can think of