Lecture Notes for Algebraic Geomtry I

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Contents

1 Feb 27th: Algebraic sets and morphisms

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Recall: $V(I) \subset \mathbb{A}^n = \{x | \forall f \in I, f(x) = 0\}$.

Theorem 1.1. *Let $Y_1 \subset \mathbb{A}^n, X_1, ..., X_n$, $Y_2 \subset \mathbb{A}^m, T_1, , ..., T_m$ affine algebraic sets. There are bijections*

$$\begin{aligned}
&Hom_{K-alg}(\mathcal{O}(Y_2), \mathcal{O}(Y_1)) \\
&\xleftrightarrow{(*)} \{(f_1, ..., f_m) \in K[X]^m | \forall x \in Y_1, (f_1(x), ..., f_m(x)) \in Y_2)\} \\
&\xleftrightarrow{(**)} \{f : Y_1 \longrightarrow Y_2 | \forall \varphi \in \mathcal{O}(Y_2), \varphi \circ f \textit{ is in } \mathcal{O}(Y_1)\}
\end{aligned}$$

Proof. Key observation

To give $(f_1, ..., f_m) \in K[X]^m$ is "the same " as giving a ring morphism $g_0 : K[T] \longrightarrow K[X] : T_i \mapsto f_i$, which gives by composition $g_1 = \pi_1 \circ g_0$, where $\pi_1 : K[X] \longrightarrow \mathcal{O}(Y_1)$ is the canonical projection.

$$g_1 : K[T] \longrightarrow \mathcal{O}(Y_1)$$

which has a factorization

$$\begin{tikzcd}
K[T] \arrow[r, "g_1"] \arrow[d, "\pi_2"] & \mathcal{O}(Y_1) \\
\mathcal{O}(Y_2) \arrow[ur, "g"] &
\end{tikzcd}$$

iff $g_1(I(Y_2)) = 0$, which means iff

$$g_1(\varphi) = \text{ “replace } T_i \text{ by } f_i \text{ in } \varphi\text{”}$$

belongs to $I(Y_1)$ if $\varphi \in I(Y_2)$, which means if $x \in Y_1$, then $g_1(\varphi)(x) = 0$. That means $\varphi(f_1(x), ..., f_m(x)) = 0$ for $\varphi \in I(Y_2)$, i.e., $(f_1(x), ..., f_m(x)) \in Y_2$. If $x \in Y_1$. In the statement, this gives the $(*)$ bijection. Any k-algebra morphism $\mathcal{O}(Y_1) \longrightarrow \mathcal{O}(Y_2)$ comes from $K[T] \longrightarrow \mathcal{O}(Y_1)$ s.t. it vanishes on $I(Y_2)$.

For the bijection $(**)$, suppose

$$g : Y_1 \xrightarrow{g} Y_2 \xrightarrow{\varphi} K$$

sends $\varphi(Y_2)$ to $\varphi \circ g \in \mathcal{O}(Y_1)$. Then we get

$$\begin{aligned}
&\mathcal{O}(Y_2) \longrightarrow \mathcal{O}(Y_1) \\
&\varphi \longmapsto \varphi \circ g,
\end{aligned}$$

which is a K-algebra morphism.

As for the reverse direction, given g. From $\mathcal{O}(Y_2) \longrightarrow \mathcal{O}(Y_1)$ to get a $g : Y_1 \longrightarrow Y_2$. We get a $\tilde{g} : Y_1 \longrightarrow Y_2$ in the second set

$$\tilde{g}(x) = (f_1(x), ..., f_m(x))$$

then we have $\varphi \circ g \in \mathcal{O}(Y_1)$ for $\varphi \in \mathcal{O}(Y_2)$. One checks that this shows that the first and third sets are the same. $\square$

Define morphism $Y_1 \longrightarrow Y_2$ by the second(and third) set. Composition in the obvious way and identity is a morphism. $\Longrightarrow$ get a category (Alg_K) of affine algebraic sets over K.

Corollary 1.2. *$Y \mapsto \mathcal{O}(Y)$, $g \mapsto [\varphi \mapsto \varphi \circ g]$ is a functor: $(Alg_K) \longrightarrow (K-Alg)^{opp}$.*

Facts: The "image" of this functor is the category of finitely generated K-algebras which are reduced.

Proof. A finitely generated reduced K-algebra. ($\exists n \geq 1$, so that $K[X_1, ..., X_n]/I \cong A$). Then "$A$ is reduced" $\Longleftrightarrow$ I is radical ideal. $\Longrightarrow A = \mathcal{O}(V(I))$, where $V(I) \subset \mathbb{A}^n$. $\square$

Corollary 1.3. *There is a equivalence of categories between*

$$(Alg_K) \longleftrightarrow (\textit{fin. gen. reduced.}K - Algs.)$$

Example 1.4.

(1) $\mathbb{A}^1 \longrightarrow V(Y^2 - X^3 - X^2) \subset \mathbb{A}^2$, $t \mapsto (t^2 - 1, t(t^2 - 1))$

(2) $\mathbb{A}^1 \longrightarrow V(Y^2 - X^3) \subset \mathbb{A}^2$: $t \longmapsto (t^2, t^3)$ *is a bijection but Not an isomorphism.*

(3) Assume K with characteristic $p > 0$, $K \supset \mathbb{F}_p$. $Y = V(f_1, ..., f_m)$ where $f_i \in \mathbb{F}_p[X] \subset K[X]$. Consider the morphism:

$$\begin{aligned} &Y \longrightarrow Y \\ &(x_1, .., x_n) \longmapsto (x_1^p, ..., x_n^p). \end{aligned}$$

It is bijective and homeomorphism but not an isomorphism if $dim(Y) \geq 1$.

Proposition 1.5. $Y = V(I) \subset \mathbb{A}^n$

(1) The points of Y are in bijection with maximal ideals $I \subset \mathcal{O}(Y)$ by

$$Y \ni x \longmapsto \{f \in \mathcal{O}(Y) | f(x) = 0\}$$

(2) We have a bijection

$$\mathcal{O}(Y) \longleftrightarrow \ Hom_{Alg_K}(Y, \mathbb{A}^1)$$

Proof. (1) $I_x := Ker(\mathcal{O}(Y) \longrightarrow K)$, $f \mapsto f(x)$, since the evaluation map is surjective $[1 \mapsto 1]$, we get an isomorphism

$$\mathcal{O}(Y)/I_x \xrightarrow{\sim} K,$$

so I_x is maximal in $\mathcal{O}(Y)$.

Conversely, if $I \subset \mathcal{O}(Y)$ is maximal, we get $I = I'/I(Y)$ for $I' \subset K[X]$ maximal.

Nullstellensatz says $\exists(x_1, ..., x_n) \in \mathbb{A}^n$ s.t., $I' = (X_1 - x_1, ..., X_n - x_n)$.

Since $I' \supset I(Y)$, we get $(x_1, .., x_n) \in Y$. Then we check that $\mathcal{O}(Y) \longrightarrow \mathcal{O}(Y)/I \cong K$ is just given by $f \mapsto f(x_1, ..., x_n)$. That means $I = I_x$.

(2) We saw in 1.1, that there is a bijection between sets

$$\mathrm{Hom}_{Alg_k}(Y, \mathbb{A}^1) \longleftrightarrow \ \mathrm{Hom}_{K-alg}(\mathcal{O}(\mathbb{A}^1), \mathcal{O}(Y)).$$

But $\mathrm{Hom}_{K-alg}(\mathcal{O}(\mathbb{A}^1), \mathcal{O}(Y)) = \ \mathrm{Hom}_{K-alg}(K[X], \mathcal{O}(Y)) \cong \mathcal{O}(Y)$ (by $g : \mathcal{O}(\mathbb{A}^1) \longrightarrow \mathcal{O}(Y)$, $g \mapsto g(X)$) □

Projective Algebraic sets

Projective sets can have a good notion of "compactness".

N.B. Any $Y \in (Alg_K)$ is **quasi-compact**(open cover have a finite subcover).

Definition 1.6. $\mathbb{P}^n_K = \mathbb{P}^n$ *can be either defined as*

"the set of lines in $\mathbb{A}^{n+1}$ that pass through the origin"

or

"the equivalence classes of points in $K^{n+1} \backslash \{0\}$ with the equivalence relation $x \sim y$ iff $x = \lambda y$ for some $\lambda \in K$" and we use the notion $[x_0 : ... : x_n]$ for the equivalence class of $(x_0, .., x_n)$

These two definitions are equivalent:

Given a line $l \in \mathbb{A}^1 \longleftrightarrow$ hyperplane in K^{n+1}, corresponds to a equation

$$a_0X_0 + ... + a_nX_n = 0$$

with at least one of a_i non-zero.

Conversely, from $[x_0 : .. : x_n]$, we we get the corresponding hyperplane/line trivially.

Notes the following fact:

$$\mathbb{P}^n = \cup_{0\leq i\leq n} H_i,$$

where $H_i = \{[x_0, ..., x_n]|x_i \neq 0\}$ and there is a bijection

$$\begin{aligned}
&H_i \longrightarrow K^n \\
&[x_0 : ... : x_n] \longmapsto \left(\frac{x_0}{x_i}, ..., \frac{\widehat{x_i}}{x_i}, .., \frac{x_n}{x_i}\right) \\
&[y_1 : ... : y_{i-1} : 1 : y_i : ... : y_n] \leftarrow\!\shortmid (y_1, ..., y_n)
\end{aligned}$$

We define from linear algebra some notions in $\mathbb{P}^n$ a line in $\mathbb{P}^n$ is the image by the projection $K^{n+1}\backslash\{0\} \longrightarrow \mathbb{P}^n$ of the two dimensional affine subspace.

Example 1.7. *$l_1, l_2 \subset \mathbb{P}^2$ lines $l_1 \cap l_2$ is a line if l_1 and l_2 are identical and would be a single point otherwise.*

Observation: If $f \in K[X_0, ..., X_{n+1}]$ is homogeneous, then for $x \in \mathbb{P}^n$, it makes no sense to speak of "$f(x) \in K$", but the zero-loci or the set where $f(x) \neq 0$ does make sense.

Definition 1.8. *A **projective algebraic set** $S \subset \mathbb{P}^n$ is*

$$S = \{x \in \mathbb{P}^n | f_1(x) = ... = f_m(x) = 0\},$$

where $f_1, ..., f_m$ are homogeneous of some degrees.

<u>Notation</u>: $V(f_1, .., f_n)$

Example 1.9. *$V(Y^2Z - X^3 - XZ^2) \subset \mathbb{P}^2$.*

Let $0 \leq i \leq n$, then $S \cap H_i = \{[x_0 : ... : x_n] \in S | x_i \neq 0\}$ is , via the bijection $H_i \longrightarrow K^n$, in bijection with an affine algebraic set $S_1 \subset \mathbb{A}^n$ given by $\tilde{f}_1(y) = ... = \tilde{f}_m(y) = 0$, where $\tilde{f}_i(y_1, .., y_n) = f_i(y_1, ..., y_{i-1}, 1, y_i, ..., y_n)$

2 Mar 2nd: Projective algebraic sets and regular functions

Recall: $\mathbb{P}^n_K = K^{n-1} - \{0\}/\sim$, and $H_i := \{[x_0 : ... : x_n] | x_i \neq 0\}$ is in bijection with $\mathbb{A}^n$. $V(f_1, ..., f_m) = \{x \in \mathbb{P}^n | \forall i, f_i(x) = 0\}$, where $f_1, ..f_m$ are homogeneous.

More generally, we can define

$$V(I) = V(\text{homogeneous element of } I =) = V(\cup_{d \geq 0} I_d)$$

where I is an homogeneous ideal of $K[X_0, ..., X_n]$ that is $I = \oplus_{d \geq 0} I_d$, I_d the the degree d piece of $K[X_0, ..., X_n]$.

Conversely, given $S \subset \mathbb{P}^n$, we can define

$$I(S) := \text{ideal generated by homogeneous polynomials that vanishes on } S$$

Lemma 2.1. *This is a homogeneous ideal*

Proof. $f \in I(s) \implies f = \sum_{i \in I} g_i f_i$, where f_i is homogeneous and vanishes on S. We can expand each g_i as $\sum_j g_{ij}$, where each g_{ij} is homogeneous in $I(S)$. Then we know $f \in \otimes I(S)_d$ and the converse is clear. $\square$

Lemma 2.2. *The projective sets $V(I)$ where I is homogeneous form the closed sets of a topology. It is called the Zariski topology (same name for the induced topology on projective sets).*

Example 2.3. *$H_0 \subset \mathbb{P}^n$ and $\sigma : H_0 \cong \mathbb{A}^n$. Under this bijection, the Zariski topologies correspond σ is a homeomorphism*

$$f \in K[X_0, ..., X_n] \text{ homogeneous} \rightsquigarrow V(f) \subset \mathbb{P}^n$$

$$\tilde{f} = f(1, X_1, ..., X_n) \in K[X_1, .., X_n] \rightsquigarrow V(\tilde{f}) \subset \mathbb{A}^n$$

and $\sigma(V(f)) = V(\tilde{f})$.

Definition 2.4. *$Y \subset \mathbb{P}^n$ projective $S(Y) = K[X_0, ..., X_n]/I(Y)$,* ***homogeneous coordinate ring***

<u>Note</u> elements in $S(Y)$ are not functions on Y. The geometric meaning of $S(Y)$ will be explained latter with the language of schemes.

We now want to define morphisms of projective algebraic sets. We have to look at it more carefully because we can not simply copy the affine definition.

Definition 2.5. *$Y \subset \mathbb{P}^n$ projective, let $V \subset Y$ be an open subsets of Y.*

*(1) $f : V \longrightarrow K$ continuous is called **regular** on Y if $\forall x \in Y$, $\exists U$ open $x \in U$, $\exists f_1, f_2 \in K[X_0, ..., X_n]$ homogeneous of same degree such that $f_2(x) \neq 0$ for all $x \in U$ and $f(x) = \frac{f_1(x)}{f_2(x)}$ for $x \in U \cap Y$*

*(2) Y_1, Y_2 are projective sets in $\mathbb{P}^n, \mathbb{P}^m$, $f : Y_1 \longrightarrow Y_2$ is a **morphism** if f is continuous and for any $U \subset Y_1$ open and any $\varphi : U \longrightarrow K$ regular, the composite $\varphi \circ f : f^{-1}(U) \longrightarrow K$ is regular.*

Note: IN (2), one can not restrict to φ regular on Y_2 because often the space of such function is reduced to K

Proposition 2.6. *For $\mathbb{P}^n$, the space of functions regular on $\mathbb{P}^n$ is K.*

Proof. The case $n = 1$ implies the general case: if $f : \mathbb{P}^n \longrightarrow K$ regular, and $x \neq y$ in $\mathbb{P}^n$, the line joining x to y in $\mathbb{P}^n$ is "isomorphic" to $\mathbb{P}^1$ and $f|_L$ is regular so constant, hence $f(x) = f(y)$.

For $n = 1$, $f|_{H_0}$ (respectively $f|_{H_1}$)identifies to a function f, $[x : y] \mapsto \frac{f_0(x,y)}{x^d}$ for f_0 homogeneous of degree d, $x \neq 0$. Respectively, $[x : y] \mapsto \frac{f_1(x,y)}{y^e}$, where f_1 is homogeneous of degree e. According to assumption, f is defined everywhere, we can compatibly glue the two pieces.

For $[x : y] \in H_0 \cap H_1$, we get

$$y^e f_0(x, y) = x^d f_1(x, y)$$
$$Y^e f_0 = X^d f_1 \in K[X, Y].$$

By expanding it to power series and compare term by term, we can check the only possibility is $d = e = 0$, and this means f_0, f_1 are constants, therefore f is a constant. □

Concretely: To say that $f : Y_1 \subset \mathbb{P}^n \longrightarrow Y_2 \subset \mathbb{P}^m$ is a morphism of projective algebraic sets. It reduces to $\forall x \in Y_1, \exists U$ open containing x s.t. there exists $f_0, ..., f_m \in K[X_0, ..., X_{n+1}]$ homogeneous of same degree, with no common zero in U, such that $\forall y \in U \cap Y_1$, $f(y) = [f_0(y) : ... : f_m(y)]$. It is easy to see that if f is of this form, then it is a morphism.

The converse is left as an exercise.

Example 2.7. *(1) Let $g \in Gl_n(K), n \geq 1$. Define*

$$f_g : \mathbb{P}^n \longrightarrow \mathbb{P}^n$$

$$[x_0 : ... : x_n] \longmapsto [g(x_0, ..., x_n)]$$

is a morphism. In fact, it is an isomorphism. $f_g^{-1} = f_{g^{-1}}$*. It also has some other properties:* $f_g = f_{\lambda g}, \lambda \neq 0$ *and we get an induced group morphism*

$$\begin{tikzcd} PGL_{n+1}(K) \arrow[r, equal] \arrow[d] & GL_{n+1}(K)/K^{\times} \\ Aut_{proj}(\mathbb{P}^n) \end{tikzcd}$$

which is an isomorphism. A special case is $Aut_{hol}(\mathbb{CP}^1) = PGL_2(\mathbb{C})$

$$g \longmapsto \left[z \mapsto \frac{az+b}{cz+d} \right]$$

(2) $K = \mathbb{C}$. *One can do holomorphic geometry (using holomorphic functions instead of polynomials). IN* $\mathbb{C}^n$*,we get a much more complicated picture [e.g.* $V(\sin z)$*] is a an infinite sets in* $\mathbb{P}^n_{\mathbb{C}}$*, however Chow proved that the holomorphic sets and the projective algebraic sets are the same (Serre "GAGA" principle compares many different invariant of both categories.)*

(3) *Consider the map* $S := V(Y^2Z - X^3 - XZ^2) \xrightarrow{f} \mathbb{P}^1$, $[x : y : z] \mapsto [y : z]$.
Claim, this is a morphism of projective sets.
This means that there is no solution to $Y^2Z - X^3 - XZ^2 = 0$ *with* $Y = Z = 0$. *(But* $[x : y : z] \mapsto [x : z]$ *is not a morphism because* $[0 : 1 : 0] \in S$*).* f *is surjective but not injective* $[x : y : z]$ *and* $[x : -y : z]$ *have same image. This works in field* k *with* $\text{char} k \neq 2$.

(4) $\mathbb{P}^1 \xrightarrow{v} \mathbb{P}^2$, $[x : y] \mapsto [x^2 : xy : y^2]$ *(special case of Veronese embedding). This is a morphism. The image of* v *is equal to* $[y_0 : y_1 : y_2], \mathbb{P}^2$. $S = V(Y_1^2 - Y_0Y_2)$*. In fact,* σ *gives an isomorphism* $\sigma : \mathbb{P}^1 \longrightarrow S$ *with inverse given by*

$$\tau : S \longrightarrow \mathbb{P}^1$$
$$[y_0 : y_1 : y_2] \mapsto \begin{cases} [Y_1 : Y_2] \text{ if } Y_2 \neq 0 \\ [Y_0 : Y_1] \text{ if } Y_0 \neq 0 \end{cases}$$

τ *is a morphism defined on all of* S*, because if* $[y_0 : y_1 : y_2] \in S$ *satisfies* $y_0 = y_2 = 0$*, it would implie* $y_1^2 = y_0y_2 = 0 \Longrightarrow y_1 = 0$

$$\tau \circ \sigma([x : y]) = \tau([x^2 : xy : y^2]) = \begin{cases} [xy : y^2] = [x : y], y \neq 0 \\ [x^2 : xy] = [x : y], x \neq 0 \end{cases}$$

therefore $\tau \circ \sigma = id_{\mathbb{P}^1}$ *and* $\sigma \circ \tau = id_S$ *can proved similarly*

One can not find $f_0 : f_1$ *in* $K[Y_0, Y_1, Y_2]$ *s.t.* $\tau([y_0 : y_1 : y_2] = [f_0(Y) : f_1(Y)]$ *for all* $Y \in S$

Rational/birational maps

$Y \subset \mathbb{A}^n$ algebraic if Y is irreducible, then $\mathcal{O}(Y)$ is an integral domain. Let $K(Y)$ be its quotient field. What is the geometric meaning of $K(Y)$? It is called the function field of Y.

We will see

Theorem 2.8. *For* Y_1, Y_2 *affine varieties (irreducible)* $K(Y_1) \cong K(Y_2)$ *as fields* $\iff \exists U_1 \subset Y_1$ *open dense subset and* $\exists U_2 \subset Y_2$ *open dense subset such that* U_1 *and* U_2 *are isomorphic.*