A SYNTHESIS OF CLASSICAL BOUNDARY THEOREMS

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ABSTRACT

A Synthesis of Classical Boundary Theorems

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Bounded analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ are a frequent area of study in complex function theory. While it is easy to understand the behavior of analytic functions on sequences with limit points inside $\mathbb{D}$, the theory becomes much more complicated as sequences converge to the boundary, $\partial\mathbb{D}$. In this thesis, we will explore boundary theorems, which can guarantee specific desired behavior of these analytic functions. The thesis describes an elementary approach to proving Fatou's Non-Tangential Limit Theorem, as well as proofs and discussion of the subsequent classical boundary theorems for specific points, Julia's Theorem and the Julia-Carathéodory Theorem. This thesis serves as a synthesis of these boundary theorems in order to fill a gap in the overarching literature.

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INTRODUCTION

1.1 Motivation

The open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is one of the most fundamental sets to study in all of Complex Analysis. Analytic functions defined on $\mathbb{D}$ have very "nice" behavior inside $\mathbb{D}$, but the extent to which we can confidently say this about analytic functions ends on the boundary, $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$. A great example of this is the function and its convergent power series

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$

on regions that do not include $z = 1$. Notably, this function is analytic on $\mathbb{D}$, yet has a pole of order 1 at $z = 1 \in \partial\mathbb{D}$. But what can be said about the behavior of f near $z = 1$?

Another thing to mention is the idea of what is called a non-tangential limit. A non-tangential limit is one where the limit points converge within a sector of the circle (for $\partial\mathbb{D}$, we will define this as a Stolz region), rather than approaching close to the boundary. A great example that demonstrates this is the function

$$f(z) = e^{-\frac{1+z}{1-z}},$$

which has an incredibly complicated singularity at $z = 1$.

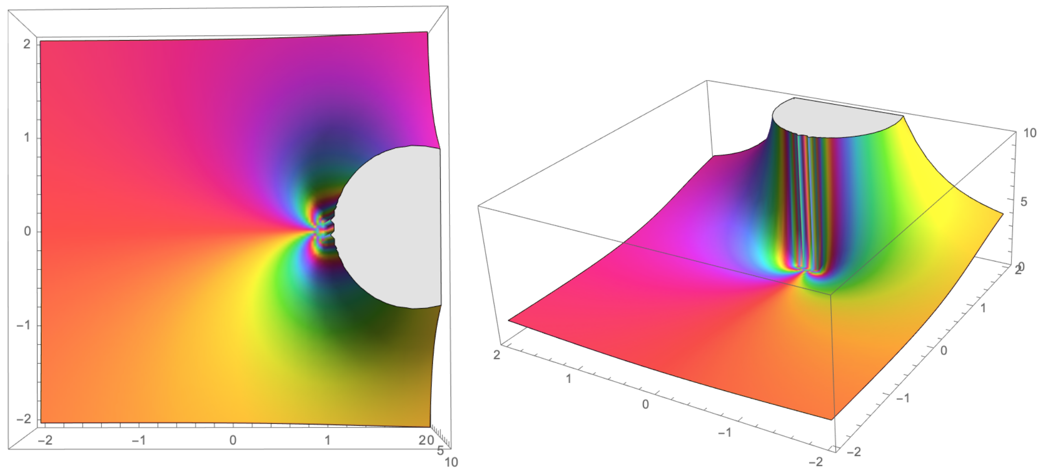

Figure 1.1: Function With Complicated Boundary Behavior

Figure 1.1 represents the function $f(z)$ using in a way that is perceivable. Normally, one would need four dimensions to grasp complex function behavior point-by-point. In the top-down view of Figure 1.1, the x-axis and y-axis correspond to the real and imaginary components of the input, respectively. For the output, by mapping $|f(z)|$ to the z-axis and using a color wheel to represent $\arg(f(z))$ (with a full cycle through the colors corresponding to a full rotation about 0), we are able to visualize the output in terms of its polar coordinates. The result is a plot that looks similar to a level curve depiction of a function in two variables.

Another thing to mention is that $f(z) = e^{-\frac{1+z}{1-z}}$ is bounded. We can see this from how its exponent acts on $\mathbb{D}$. Using a conformal mapping process, it is clear that $w = \frac{1+z}{1-z}$ takes

$$1 \to \infty$$

$$-1 \to 0$$

$$i \to i$$

$$-i \to -i$$

$$0 \to 1.$$

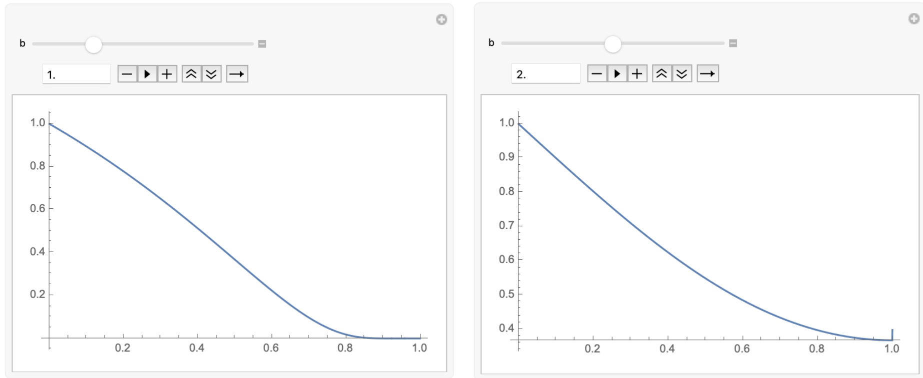

Figure 1.2: Limit Along Tangential Paths of Order $b \leq 2$

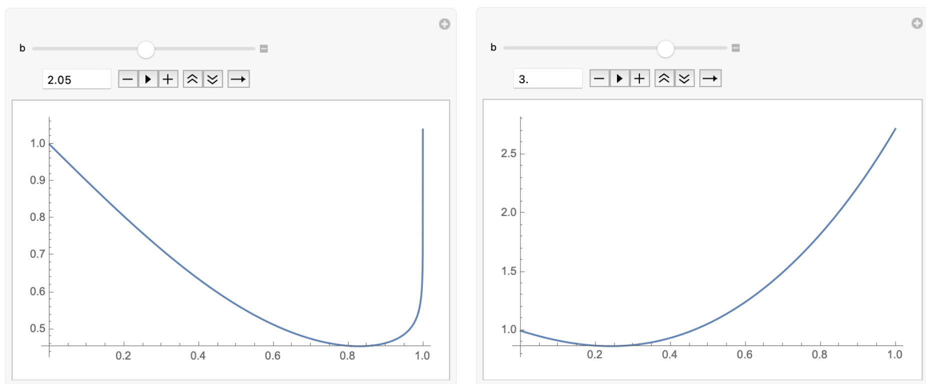

Figure 1.3: Limit Along Tangential Paths of Order $b > 2$

The result is that the image of $\mathbb{D}$ under $w = \frac{1+z}{1-z}$ is just the right half-plane. Then, $w' = -\frac{1+z}{1-z}$ must map $\mathbb{D}$ to the left half-plane. The last thing to consider is how $f(z) = e^{-\frac{1+z}{1-z}}$ acts on the left half-plane. It turns out, since elements of the imaginary axis are of the form αi for some $\alpha \in \mathbb{R}$ and $e^{\alpha i} \in \partial\mathbb{D}$, f maps the imaginary axis to $\partial\mathbb{D}$. Additionally, since $f(-1) = 0$, the interior of the left half plane is mapped to the interior of $\mathbb{D}$, namely itself. This shows that for $z \in \mathbb{D}$, $|f(z)| < 1$.

Around the singularity at $z = 1$, the frequent interchange of colors in Figure 1.1 shows that f is winding around the origin rapidly. But, the complex magnitude of points approaching radially approaches 0, meaning that we expect f to have a non-tangential limit. What we will see is that for paths approaching $z = 1$ tangentially, we cannot guarantee the limit exists.

Figures 1.2 and 1.3 depict the behavior of the output of $f(z)$ for points along curves that approach $z = 1$. The value, b, corresponds to the order of approach, which is similar to the order of a polynomial. When $b = 1$, our paths are straight lines that approach 1 non-tangentially. For $b > 1$, we have a tangential path, which means that the there is, in a sense, a defined tangent at 1. When $b = 2$, the path resembles the real valued graph of $y = x^2$ with 1 as its vertex.

What becomes clear is that the more closely the path approaches along $\partial\mathbb{D}$ (increasing b), f goes from approaching the expected value, 0, to another value e. But this graph is only capturing the magnitude of the output. In fact, from Figure 1.1, we see that $\arg(f(z))$ is changing rapidly around $z = 1$. This means that although $|f(z)|$ appears to converge to e, $f(z)$ does not converge at all and just spins endlessly along a circle of radius e.

In this discussion, we will answer many questions about what can be said about functions analytic on a domain, yet with potentially difficult behavior on the boundary of that domain. Does the function have a limit that exists at points along the boundary? Could it also be possible for the function to have a tangible derivative or even be analytically continued on the boundary? For our purposes, the domain we will be considering is $\mathbb{D}$, so we will direct our attention to the behavior of analytic functions on $\partial\mathbb{D}$.

Another tool of ours that will help in describing the behavior on $\mathbb{D}$ is the ability to conformally map sets in $\mathbb{C}$. Notably, it is possible to conformally map $\mathbb{D}$ to the upper half-plane $H_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ or to the right half-plane $\Pi = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. By mapping back and forth between these regions, we may more easily study the desired behavior of analytic functions in $\mathbb{D}$.

1.2 History of the Field

Much of the work in this field started with Pierre Fatou, where his study of non-tangential limits found that non-tangential limits exist almost everywhere along $\partial\mathbb{D}$. While this result was useful when looking generally from the function perspective, the problem arise that if one started with a specific point, there was no way to determine whether there existed a non-tangential limit.

Following Fatou's discovery, Gaston Julia found a solution to the problem. He found that at for some arbitrary boundary point, if certain growth conditions were met, then we can conclude the existence of a non-tangential limit. The proof of this came from studying the Schwarz Lemma with conformal automorphisms. Once there was a way to show the existence of limits, the discussion shifted to the existence of derivatives.

From the 1920's to the 1940's, Constantin Carathéodory and Julius Wolff were studying one-variable interpretations of boundary behavior. They found that Julia's Theorem was a useful building block for the existence of derivatives, since satisfying the same growth condition ended up also proving the existence of derivatives.

The early pioneers created a foundation for boundary theorems that could be built upon. Later on, other methods of understanding these problems were uncovered. Sarason provided an alternative approach using Hilbert spaces and operator theory. Other mathematicians such as Agler, McCarthy, and Young generalized boundary theorems related to the classical work of Julia and Carathéodory to two variables.

This paper serves as a synthesis of the boundary theorems developed by Fatou, Julia and Carathéodory, while bringing their achievements into modern language and terminology. Though, it should be noted that generalizations of the theory is still currently an active area of research.

Chapter 2

INTEGRAL REPRESENTATIONS OF PICK FUNCTIONS

2.1 The Herglotz Theorem and the Dirichlet Problem

Definition 2.1.1. A Pick function is a complex analytic function defined on $H_+ = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ with range inside the closure of H_+. All functions with this property make up P, called the class of Pick functions.

One additional thing to note when studying Pick functions is the fact that any nonconstant complex analytic function is an open mapping. In other words, the image of an open set is also an open set. We can use this fact to say that any nonconstant Pick function $f \in P$ should satisfy $range(f) \subseteq H_+$, rather than the closure of H_+. We can also see that the composition of two nonconstant Pick functions is also a Pick function by the open mapping property.

Pick functions are useful in that they take elements in the upper-half plane and map them to complex numbers with at least nonnegative imaginary components. With the use of conformal maps, we will be able to map the upper-half plane H_+ to the open unit disk $\mathbb{D}$ and vice versa. In many of the upcoming topics, we will be using this conformal mapping process to take problems from inside $\mathbb{D}$ and study them more easily inside H_+.

Inside $\mathbb{D}$, we will want to understand how some functions behave by using an integral representation. The following theorem, called the Herglotz Theorem, is used to describe how nonnegative harmonic functions can be represented inside $\mathbb{D}$, which is used in the proof of the Dirichlet problem. Before stating the theorem, we should refresh ourselves with the Poisson kernel P_r, which is defined by

$$P_r(\theta) = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)}.$$

What we will see from the Herglotz Theorem is that every nonnegative harmonic function defined on $\mathbb{D}$ is just the Poisson kernel of a positive measure. The statement and proof of this theorem follows closely from Chapter 4 of Bhatia's *Matrix Analysis* [3, Chapter 4].

Theorem 2.1.1 (Herglotz Theorem)**.** *Let $u(z)$ be a nonnegative harmonic function on the open unit disk $\mathbb{D}$. Then there exists a finite measure m on $[0, 2\pi]$ such that*

$$u(re^{i\theta}) = \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} dm(t).$$

Conversely, any function of this form is strictly positive and harmonic on $\mathbb{D}$.

Proof. Let u be a continous real-valued function on the closed unit disk that is nonnegative and harmonic on $\mathbb{D}$. Note that any complex number z_0 can be written in the form $z_0 = re^{i\theta}$ for some $r > 0$ and $0 \leq \theta < 2\pi$. This gives us a representation for u using the Poisson kernel, $P_r(\theta)$:

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} u(e^{it}) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) u(e^{it}) dt. \end{aligned}$$

Given that u is nonnegative, we can define a positive measure $dm(t) = \frac{1}{2\pi} u(e^{it})$ on $[0, 2\pi]$. Also, from the mean value property of harmonic functions, the total mass of $dm(t)$ is given by

$$\int_0^{2\pi} \frac{1}{2\pi} u(e^{it}) dt = u(0).$$

At this point, we would be done if not for the assumption that u is continuous on the closed unit disk. To prove the statement in the theorem, we must remove this additional hypothesis.

Let $\varepsilon > 0$ and define $u_\varepsilon(z) = u(\frac{z}{1+\varepsilon})$. We choose this definition so that u_ε converges to u uniformly on compact subsets of $\mathbb{D}$. This u_ε is positive and harmonic on the disk $\{z \mid |z| < 1+\varepsilon\}$. From our result, u_ε can be represented in the form using the Poisson kernel with a measure $m_\varepsilon(t)$ with finite total mass $u_\varepsilon(0) = u(0)$. As we have convergence of u_ε and all m_ε have the same mass, there must be a positive measure m such that

$$u(re^{i\theta}) = \lim_{\varepsilon \to 0} u_\varepsilon(re^{i\theta}) = \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} dm(t).$$

Conversely, since P_r is nonnegative, any such u is also nonnegative. $\square$

Another consequence of the Herglotz Theorem is a an integral representation for any analytic function on $\mathbb{D}$ given by the Dirichlet problem.

Theorem 2.1.2 (Dirichlet problem)**.** *Let $f(z) = u(z) + iv(z)$ be analytic on the open unit disk $\mathbb{D}$. If $u(z) \geq 0$ for all $z \in \mathbb{D}$, then there exists a finite positive measure m on $[0, 2\pi]$ such that*

$$f(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} dm(t) + iv(0).$$

Conversely, every function of this form is analytic with positive real part on $\mathbb{D}$.

Although the details of the proof are not shown, the theorem can be quickly proven by representing an analytic function in $\mathbb{D}$ with harmonic conjugate real and imaginary components and then applying the Herglotz Theorem to each component [3, Chapter 4]. The Dirichlet problem is a useful result that will be used to prove Fatou's Non-Tangential Limit Theorem (3.1.5).

2.2 Nevanlinna's Theorem

The next step is to prove Nevanlinna's Theorem. But, before we can do that, we must establish some preliminary steps. We note that the disk $\mathbb{D}$ and the upper-half plane H_+ are conformally equivalent. This is achieved using the conformal maps

$$\zeta(z) = \frac{1}{i}\frac{z+1}{z-1}$$

from $\mathbb{D}$ to H_+, and its inverse

$$z(\zeta) = \frac{\zeta - i}{\zeta + i}$$

from H_+ to $\mathbb{D}$. We then can establish an equivalence between the Pick class P and the class of analytic functions on $\mathbb{D}$ with a positive real part. If f is an analytic function on $\mathbb{D}$ with a postive real part, then let

$$\varphi(\zeta) = if(z(\zeta)).$$

This gives us that $\varphi \in P$ and an inverse transformation

$$f(z) = -i\varphi(\zeta(z)).$$

With these preliminary steps, we can now prove Nevanlinna's Theorem using an argument found in Bhatia [3, Chapter 4].

Theorem 2.2.1 (Nevanlinna's Theorem)**.** *A function φ is in the Pick class P if and only if it can be represented as*

$$\varphi(\zeta) = \alpha + \beta\zeta + \int_{-\infty}^{\infty} \frac{1+\lambda\zeta}{\lambda - \zeta} d\nu(\lambda),$$

where $\alpha, \beta \in \mathbb{R}$, $\beta \geq 0$, and ν is a positive finite measure on $\mathbb{R}$.

Proof. Let f be defined on $\mathbb{D}$ as before using $\varphi \in P$. From the Dirichlet problem, there exists a finite positive measure m on $[0, 2\pi]$ such that

$$f(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} dm(t) - i\alpha,$$

where if $f(z) = u(z) + iv(z)$, then $\alpha = -v(0)$ and the total mass of m is $u(0)$. If the mass of m is positive at $t = 0$, then denote it β. We then get the representation

$$f(z) = \int_{(0,2\pi)} \frac{e^{it} + z}{e^{it} - z} dm(t) + \beta \frac{1+z}{1-z} - i\alpha.$$

If we then use the transformations from $\mathbb{D}$ to H_+ and vice-versa, the equation becomes

$$-i\varphi(\zeta) = \int_{(0,2\pi)} \frac{e^{it} + \frac{\zeta - i}{\zeta + i}}{e^{it} - \frac{\zeta - i}{\zeta + i}} dm(t) + -i\beta\zeta - i\alpha.$$

After multiplying both sides by i, we get

$$\varphi(\zeta) = \alpha + \beta\zeta + \int_{(0,2\pi)} \frac{e^{it} + \frac{\zeta - i}{\zeta + i}}{e^{it} - \frac{\zeta - i}{\zeta + i}} dm(t).$$

Also, note that

$$\frac{e^{it} + \frac{\zeta - i}{\zeta + i}}{e^{it} - \frac{\zeta - i}{\zeta + i}} = \frac{e^{it}(\zeta + i) + (\zeta - i)}{e^{it}(\zeta + i) - (\zeta - i)},$$

which has a trigonometric representation given by $\frac{\zeta \cos\frac{t}{2} - \sin\frac{t}{2}}{\zeta \cos\frac{t}{2} + \sin\frac{t}{2}}$. We now have our representation of φ given by

$$\varphi(\zeta) = \alpha + \beta\zeta + \int_{(0,2\pi)} \frac{\zeta \cos\frac{t}{2} - \sin\frac{t}{2}}{\zeta \cos\frac{t}{2} + \sin\frac{t}{2}}.$$

If we do a change of variables $\lambda = -\cot\frac{t}{2}$, our measure m on $(0, 2\pi)$ is transformed into a new measure ν on $(-\infty, \infty)$. The following substitutions are also made:

$$\frac{\zeta\cos z - \sin z}{\zeta\sin z + \cos z} = \frac{\zeta\cot z\sin z - \sin z}{\zeta\sin z + \cot z\sin z} = \frac{-\sin z(1 + \zeta\lambda)}{-\sin z(\lambda - \zeta)} = \frac{1 + \zeta\lambda}{\lambda - \zeta},$$

giving us the desired representation of φ:

$$\varphi(\zeta) = \alpha + \beta\zeta + \int_{-\infty}^{\infty} \frac{1 + \lambda\zeta}{\lambda - \zeta} d\nu(\lambda).$$

For the reverse direction of this theorem, our representation of φ makes it so that for any $\zeta \in H_+$, $\varphi(\zeta) \in \{z \mid \operatorname{Im}(z) \geq 0\}$. This completes the proof. $\square$

The result of Nevanlinna's Theorem is almost an equivalence between Pick functions and integral representations of this form. The lack of a true equivalence falls from the fact that we have not yet shown why this representation should be unique. Despite that, it gives us a concrete way to describe any Pick function using conformal maps to go back and forth between H_+ and $\mathbb{D}$.

While the representation from Nevanlinna's Theorem is sufficient, another form used to express Pick functions uses the Cauchy transform

$$f(\zeta) = \frac{1}{\lambda - \zeta}.$$

More information about the Cauchy transform can be found in Ross [8]. This is achieved by rewriting our integrand as

$$\frac{1 + \lambda\zeta}{\lambda - \zeta} = (\frac{1}{\lambda - \zeta} - \frac{\lambda}{\lambda^2 + 1})(\lambda^2 + 1)$$

and then substituting our measure ν for another positive finite measure μ given by the change of variables formula

$$d\mu(\lambda) = (\lambda^2 + 1)d\nu.$$

We then have the pieces to rewrite our equation from Nevanlinna's Theorem:

$$\varphi(\zeta) = \alpha + \beta\zeta + \int_{-\infty}^{\infty} \frac{1}{\lambda - \zeta} - \frac{\lambda}{\lambda^2 + 1} d\mu(\lambda).$$

Interestingly, evaluating the latter term in the integral comes out finitely, so we may use this representation to more easily study Pick functions by focusing our attention on the Cauchy transform.

2.3 Uniqueness of Integral Representations

Our final task in our study of Pick functions is to ensure that our representation from Nevanlinna's Theorem is unique, so it falls to the real values α, β as well as the measure.

As it turns out, $\alpha = \mathrm{Re}(\varphi(i))$, which makes it uniquely determined.

For β, the task is more complicated. Consider any positive $\eta \in \mathbb{R}$. We can see that by using Nevanlinna's Theorem, we get

$$\begin{aligned}
\frac{\varphi(i\eta)}{i\eta} &= \frac{\alpha}{i\eta} + \beta + \int_{-\infty}^{\infty} \frac{1 + i\lambda\eta}{\lambda - i\eta} d\nu(\lambda) \\
&= \frac{\alpha}{i\eta} + \beta + \int_{-\infty}^{\infty} \frac{(1 + i\lambda\eta)(\lambda + i\eta)}{(\lambda - i\eta)(\lambda + i\eta)} d\nu(\lambda) \\
&= \frac{\alpha}{i\eta} + \beta + \int_{-\infty}^{\infty} \frac{1 + \lambda^2 + i\lambda(\eta - \eta^{-1})}{\lambda^2 + \eta^2} d\nu(\lambda).
\end{aligned}$$

As $\eta \to \infty$, $\frac{\alpha}{i\eta} \to 0$. Also, since for $\eta > 1$, the integrand is bounded uniformly by 1, the entire integral tends to 0 by the Dominated Convergence Theorem. Thus, we have that

$$\beta = \lim_{\eta \to \infty} \frac{\varphi(i\eta)}{i\eta},$$

so β must be uniquely determined by φ.

The last component to prove as unique is the measure. In this instance, we will use $d\mu$ as seen in the Cauchy transform version of Nevanlinna's Theorem due to the fact that it is monotonically increasing on $\mathbb{R}$. We will also suppose that $\mu(0) = 0$ and $\mu((a, b] = \mu(b) - \mu(a)$ for any interval $(a, b]$. The uniqueness of μ is accomplished by the Stieltjes inversion formula.

Theorem 2.3.1 (Stieltjes inversion formula)**.** *For any finite interval $a < x < b$,*

$$\mu(b) - \mu(a) = \lim_{\eta \to 0} \frac{1}{\pi} \int_a^b Im(\varphi(x + i\eta))dx.$$

This proof is very involved, as it uses the integral representation of arctan. A great proof of the theorem can be found in Donoghue's *Monotone matrix functions and analytic continuation* [6, Chapter 2].

The major result of the Stieltjes inversion formula is that we are able to confirm that our integral representations for Pick functions are unique, giving us the equivalence that we had been searching for.

Chapter 3

THE JULIA-CARATHÉODORY THEOREM

3.1 Fatou's Non-Tangential Theorem

The next area of study we would like to focus on is how to show existence of limits and derivatives along $\partial\mathbb{D}$. The first step is to work towards the existence of non-tangential limits of sequences approaching the boundary. What we will find is Fatou's Non-Tangential Theorem (3.1.5), which states that these non-tangential limits exist almost everywhere along the boundary.

Notably, Fatou's Non-Tangential Theorem is used frequently in arguments pertaining to boundary behavior in subsets of $\mathbb{C}$, but there is a lack of literature on how it is proved. Conway writes a proof of the theorem using bounded variation, and then states the non-tangential theorem as a corollary [5]. In this paper, we will walk through an approach using Fourier series and Fatou's Radial Theorem (3.1.3) to prove the non-tangential theorem. Much of the information and techniques gathered for this proof come from Stein's *Boundary behavior of holomorphic functions*, [11]. Before proving anything, it should be noted what it means for sequences to converge non-tangentially.

Definition 3.1.1. Let $\mathcal{D} \subset \mathbb{C}$, $\sigma \in \partial\mathcal{D}$. The Stolz region in $\mathcal{D}$ at σ with aperture M is denoted $S(\sigma, M)$. For $\sigma \in \partial\mathbb{D}$, the Stolz region of vertex σ and amplitude M is defined by

$$S(\sigma, M) = \left\{\zeta \in \mathbb{D} \mid \frac{|\sigma - \zeta|}{1 - |\zeta|} < M\right\}.$$

Note that for $S(\sigma, M)$ to be nonempty, we have $M \leq 1$.

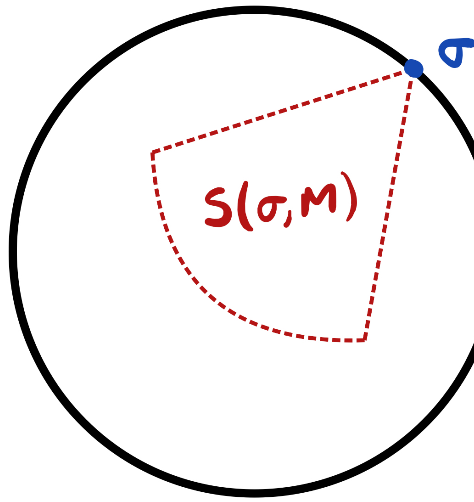

Figure 3.1: Stolz Region

Definition 3.1.2. A function $f : \mathbb{D} \to \mathbb{C}$ has non-tangential limit $L \in \mathbb{C}$ at $\sigma \in \partial\mathbb{D}$ if $f(\zeta) \to L$ as $\zeta \to \sigma$ entirely within any Stolz region $S(\sigma, M)$.

The steps to achieving this include a few other pieces that we will prove.

Before we begin, it is important to mention the space of functions that we are working in. Let $L^2[-\pi, \pi]$ be the Hilbert space of Fourier series with basis $\{e^{inx}\}_{n=-\infty}^{\infty}$. Note that $L^2 \subset L^1$, which means that anything in L^2 is integrable. In other words,

$$\int_{-\pi}^{\pi} |f| \, d\mu < \infty,$$

where μ is Lebesgue measure. Formally,

$$f \sim \sum_{n=-\infty}^{\infty} a_n e^{inx},$$

with

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

The first claim will be stated as a theorem to be used in the proof of Fatou's Theorem, but it will not be proved rigorously in this paper.

Theorem 3.1.1. *Suppose $f \in L^1[-\pi, \pi]$ with Fourier coefficients a_n. Then,*

$$\sum_{n=-\infty}^{\infty} a_n |r|^n e^{inx}$$

converges to $f(x)$ for almost every (a.e.) x as $r \to 1$ from below.

The proof of this theorem requires use of the Poisson kernel, which can be found in Stein's *Boundary behavior of holomorphic functions* [11]. Its utility comes from its use in moving the interior of $\mathbb{D}$ to $\partial\mathbb{D}$. The next result, which we will prove, establishes a connection between holomorphic functions and Fourier series.

Theorem 3.1.2. *Suppose f is holomorphic on $D(z_0, R)$, the disk of center z_0 and radius R. Then for $0 < r < R$,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

in the disk and

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta$$

for all $n \geq 0$. Also, if $n < 0$, then $a_n = 0$.

Proof. Note that since f holomorphic, it can be represented by a convergent power series in $D(z_0, R)$. From that power series, we see that $f^{(n)}(z_0) = n! a_n$. Also, from the Cauchy integral formula, we have that

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

where γ is the circle of radius r centered at z_0, oriented counter-clockwise.

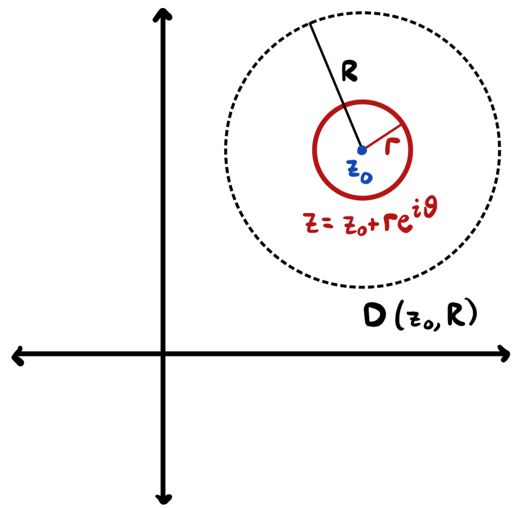

Figure 3.2: The Disk $D(z_0, R)$

Using the change of variables $\zeta = z_0 + re^{i\theta}$ in the equation above, we get that $d\zeta = rie^{i\theta}d\theta$ and so

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} rie^{i\theta} d\theta \\ &= \frac{1}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta} d\theta. \end{aligned}$$

It should also be clear to see that when $n < 0$, the integrand above is analytic in $D(z_0, R)$. From Cauchy's Theorem, the evaluation of a analytic function on and inside a simply connected path is 0 [5, Chapter 4]. Thus, when $n < 0$, $a_n = 0$. □

After this result, we are now equipped to prove Fatou's Radial Theorem (3.1.3). One thing to mention is that the proof of Theorem 3.1.3 uses Parseval's identity, which states that

$$||f||_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{i\theta})\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n| \, r^{2n}$$

for $0 < r < 1$. This fact will not be proven in this paper, but can be found in Stein [11].

Theorem 3.1.3 (Fatou's Radial Theorem)**.** *Suppose that f is holomorphic and bounded on $\mathbb{D}$. Then, $\lim_{r\to 1^-} f(re^{i\theta})$ exists for almost every angle θ.*

Proof. Note that a holomorphic f in $\mathbb{C}$ is also analytic. Thus, f has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

that converges absolutely and uniformly for $z = re^{i\theta} \in \mathbb{D}$. From Theorem 3.1.2, we have that for $z_0 = 0$ and $R = 1$,

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta$$

for $0 < r < 1$. Also, note that f has the Fourier series representation given by $f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$. Since f is bounded on $\mathbb{D}$, let $M \geq |f(z)|$ for all $z \in \mathbb{D}$. Using Parseval's identity, we see that for each r,

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} M^2 d\theta = M^2.$$

By the monotone convergence test, we may let $r \to 1$ and still have the series converge. Thus,

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

This means that $\{a_n\} \in l^2$ and $f(z) = f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta} \in L^2(-\pi, \pi)$. From Theorem 3.1.1,

$$\sum_{n=0}^{\infty} a_n r^n e^{in\theta} \to \sum_{n=0}^{\infty} a_n e^{in\theta}$$

for almost every θ. This concludes the proof. □

With all of these pieces, we may begin our proof of Fatou's Non-Tangential Theorem. The idea behind this proof is to analyze the centered Stolz region at τ, and be able to track the behavior of sequences that remain within the Stolz region.

In the proof, a couple assertions must be made. The first follows from the solution to the Dirichlet problem (Theorem 2.1.2).

Fact 3.1.1. If F is a holomorphic function on $\mathbb{D}$, it can be written as the convolution of some L^2 function f with the Poisson kernel. So we have the formula

$$F(re^{i\theta}) = f \star P_r(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta - \varphi)P_r(\varphi)d\varphi$$

with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ using the Fourier coefficients of F on $\partial\mathbb{D}$.

The second is about integrating the Poisson kernel.

Fact 3.1.2. Integrating the Poisson kernel with respect to a measure defined in terms of θ is given by

$$\int_0^{2\pi} P_r(\theta)dm(\theta) = 1.$$

Another vital theorem in the upcoming proof is that of Luzin's Theorem, stated from Axler's *Measure, Integration & Real Analysis* [2, Chapter 2]

Theorem 3.1.4 (Luzin's Theorem)**.** *Suppose $g : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function. Then for every $\varepsilon > 0$, there exists a closed set $F \subset \mathbb{R}$ such that $|\mathbb{R} - F| < \varepsilon$ and $g\big|_F$ is a continuous function on F.*

A proof of this theorem can be found in Axler.

Theorem 3.1.5 (Fatou's Non-Tangential Theorem)**.** *Let $F : \mathbb{D} \to \mathbb{D}$ be a bounded holomorphic function. Then, f has a non-tangential limit at almost every point of $\partial\mathbb{D}$.*

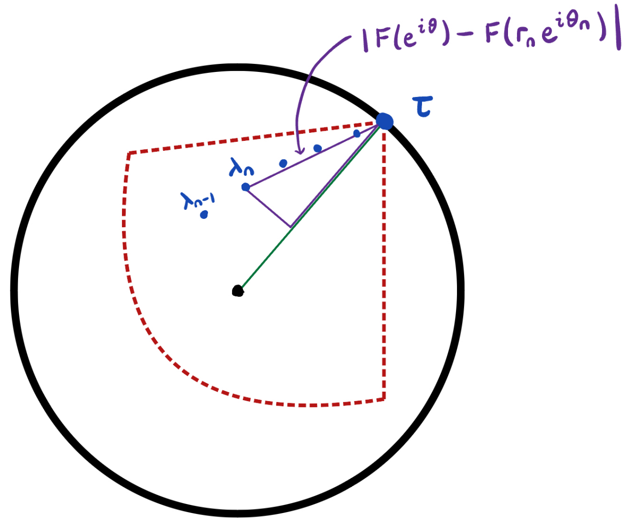

Figure 3.3: Sequence Approaching τ Non-Tangentially

In other words, the set of points in $\partial\mathbb{D}$ for which F does not have a non-tangential limit has measure zero.

Proof. Let F be a bounded and holomorphic function on $\mathbb{D}$. Let us consider $\tau = e^{i\theta}$ and the sequence defined by $\lambda_n = r_n e^{i\theta_n}$ with $r_n \to 1$ and $\theta_n \to \theta$. From Figure 3.3, we can see how $F(\lambda_n)$ behaves on the sequence: What we would like to show is that $\left|F(e^{i\theta}) - F(r_n e^{i\theta_n})\right|$ is arbitrarily small, giving us the non-tangential limit. To achieve this, we will bound the expression using the triangle inequality:

$$\left|F(e^{i\theta}) - F(r_n e^{i\theta_n})\right| \leq \left|F(e^{i\theta}) - F(r_n e^{i\theta})\right| + \left|F(r_n e^{i\theta}) - F(r_n e^{i\theta_n})\right|.$$

The first term in the split expression converges to 0 by Theorem 3.1.3. The second term in the expression will require some more work.

Using Fact 3.1.1,

$$\begin{aligned}\left|F(r_ne^{i\theta}) - F(r_ne^{i\theta_n})\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(\theta-\varphi) - f(\theta_n-\varphi)P_{r_n}(\varphi)d\varphi\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(\theta-\varphi) - f(\theta_n-\varphi)\right| P_{r_n}(\varphi)d\varphi.\end{aligned}$$

Now, using the change of variables given by $\varphi' = \theta - \varphi$ and $\phi = \theta - \theta_n$, the final expression above becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(\varphi') - f(\varphi'-\phi)\right| P_{r_n}(\theta-\varphi')d\varphi'.$$

The only major issue left to fix in order for this quantity to be small is for f to be continuous. Fortunately, by Luzin's Theorem, we can remove a set of arbitrarily small measure such that f is continuous on the remaining set.

Let $G \subseteq [0, 2\pi]$ be the set removed from $[0, 2\pi]$ from Luzin's Theorem (3.1.4). Then, our integral can be split up by $[0, 2\pi] \setminus G$. Let f^* be the continuous approximation of f in $[0, 2\pi]$ (computed by connecting the gaps in f on $[0, 2\pi] \setminus G$ with straight lines). Now, we may bound $\left|F(r_ne^{i\theta}) - F(r_ne^{i\theta_n})\right|$ by two integrals:

$$\begin{aligned}\left|F(r_ne^{i\theta}) - F(r_ne^{i\theta_n})\right| \leq& \frac{1}{2\pi}\int_0^{2\pi} \left|f^*(\varphi') - f^*(\varphi'-\phi)\right| P_{r_n}(\theta-\varphi')d\varphi' \\ &+ \frac{1}{2\pi}\int_{[0,2\pi]\setminus G} \left|f(\varphi') - f(\varphi'-\phi)\right| P_{r_n}(\theta-\varphi')d\varphi'.\end{aligned}$$

For the first integral f^* is continuous and $f^*(\varphi'-\phi) \to f^*(\varphi')$ as $\phi \to 0$, it can be made arbitrarily small. For the second interval, we get convergence is due to the fact that under Lebesgue integration, evaluation of integrals on sets of arbitrarily small measure, like $[0, 2\pi] \setminus G$, are also arbitrarily small.

Since both of these quantities can be made arbitrarily small, we have shown that for all $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that

$$|F(\tau) - F(\lambda_n)| = \left|F(e^{i\theta}) - F(r_n e^{i\theta_n})\right| < \varepsilon.$$

Thus, we have that F has a non-tangential limit for almost every $\tau \in \partial\mathbb{D}$. $\square$

The result of Fatou's Non-Tangential Theorem provides the existence of non-tangential limits almost everywhere, which means that the boundary points with difficult behavior can be narrowed down to a set of measure zero. The next logical question to ponder is about what we can say about any particular boundary point of our choosing. This leads us to prove another result.

3.2 Julia's Theorem

Our next step in reaching the Julia-Carathéodory Theorem is to arrive at a preliminary result, Julia's Theorem. The significance of Julia's Theorem comes from it ability to improve upon Fatou's Theorem. In Fatou's Theorem, we are able to conclude results about boundary behavior almost everywhere, yet this does not provide insight into what we can say about an arbitrary point. Julia's Theorem provides this detail so long as the Julia quotient, which measures growth, remains bounded.

Our method of proving Julia's Theorem will come from the Invariant Schwarz Lemma, which demonstrates the Schwarz Lemma on $\mathbb{D}$ mapped by a conformal automorphism. This will bring us to the discussion of horocycles, which is the context that we will prove Julia's Theorem in. Much of this discussion largely follows from Shapiro's *Composition operators and classical function theory* [10, Chapter 4].

To begin, we should first state the more familiar Schwarz Lemma, from Conway [5, Chapter 6].

Lemma 3.2.1 (Schwarz Lemma)**.** Suppose that $f : \mathbb{D} \to \mathbb{D}$ is analytic and $f(0) = 0$. Then

$$|f(z)| \leq |z|$$

for all $z \in \mathbb{D}$.

While this is a useful fact, we want to use it to understand more about the boundary of $\mathbb{D}$. This is where the invariant form comes in. We will pose a general conformal automorphism α_p of $\mathbb{D}$ defined by

$$\alpha_p(z) = \frac{p - z}{1 - \overline{p}z}.$$

It is important to recognize that our notions of distance needs to be altered to account for the conformal automorphism α_p. For this, we will define a pseudo-hyperbolic metric d defined by

$$d(p, q) = |\alpha_p(q)| = \left| \frac{p - q}{1 - \overline{p}q} \right|.$$

Let us note that this metric has a few useful properties other than the definitive properties of being a metric. Notably, the distance between any two points is always strictly less than 1. Another property is that for each compact subset K of $\mathbb{D}$,

$$\lim_{|q| \to 1^-} \inf_{p \in K} d(p, q) = 1.$$

This means that the limit of the distance of a point in $\mathbb{D}$ to a fixed compact set tends to 1 as that point approaches the boundary. We can use this fact to show that

$$1 - d(p, q)^2 = 1 - \frac{|p - q|^2}{|1 - \overline{p}q|^2} = \frac{(1 - |p|^2)(1 - |q|^2)}{|1 - \overline{p}q|^2},$$

which is a formula that we will use later in the proof of Julia's Theorem.

With our metric equipped, we are now ready to state and prove the following lemma.

Lemma 3.2.2 (Invariant Schwarz Lemma)**.** Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic. Then, for any $p, q \in \mathbb{D}$,

$$d(\varphi(p), \varphi(q)) \leq d(p, q).$$

This inequality becomes equality precisely when φ is an automorphism.

Proof. Let $b = \varphi(p)$, and let α_b be defined as before

$$\alpha_b(z) = \frac{b - z}{1 - \bar{b}z}$$

for $z = \alpha_p(q)$. Let us also consider a new map $\hat{\varphi} = \alpha_b \circ \varphi \circ \alpha_p$. Note that due to our composition of automorphisms, we have that $\hat{\varphi}(0) = 0$.

Now, we can apply the original Schwarz Lemma to $\hat{\varphi}$ and get

$$|\hat{\varphi}(z)| \leq |z|$$

for all $z \in \mathbb{D}$. Replacing z with $\alpha_p(q)$, we see that

$$|(\alpha_b \circ \varphi)(q)| \leq |\alpha_p(q)|$$

which implies that

$$\left| \frac{\varphi(p) - \varphi(q)}{1 - \overline{\varphi(p)}\varphi(q)} \right| \leq \left| \frac{p - q}{1 - \overline{p}q} \right|.$$

The above inequality is the statement $d(\varphi(p), \varphi(q)) \leq d(p, q)$, which concludes the proof. $\square$

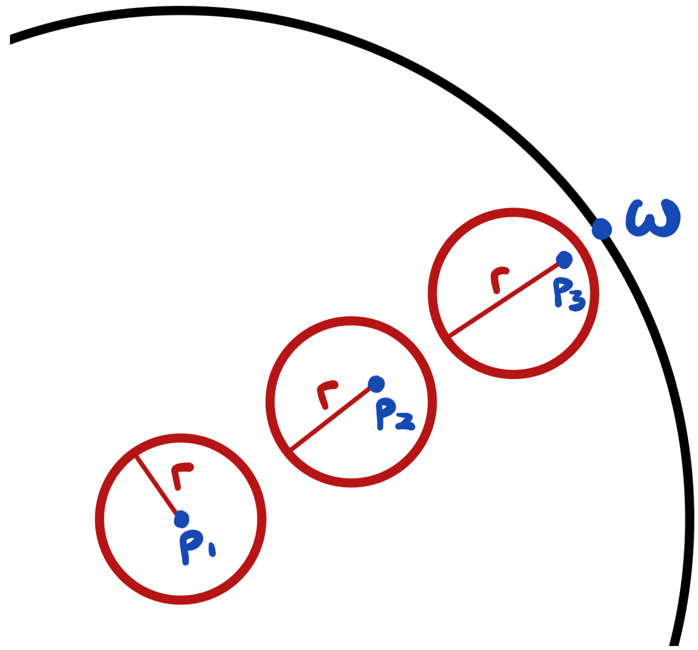

Figure 3.4: r-Disks With Centers p_n Converging to ω

Next, what we would like to explore are what disks with the pseudo-hyperbolic metric look like as we approach $\partial\mathbb{D}$. For $r < 1$, let us denote the r-ball centered at p as

$$\delta_r(p) = \{z \in \mathbb{D} \mid d(p, z) < r\}.$$

Since we are using the pseudo-hyperbolic metric, these r-balls appear to have their centers approach the edge of the circle as they centers approach $\partial\mathbb{D}$.

In the limit as $p \to \omega \in \partial\mathbb{D}$, we can imagine the r-ball with a center on the boundary.

Definition 3.2.1. Let $\sigma \in \partial\mathbb{D}$ and let $\rho > 0$. The set $H(\sigma, \rho)$ defined by

$$H(\sigma, \rho) = \left\{\zeta \in \mathbb{D} \mid \frac{|\sigma - \zeta|^2}{1 - |\zeta|^2} < \rho\right\}$$

is called the horocycle contained in $\mathbb{D}$ of center σ and radius ρ.

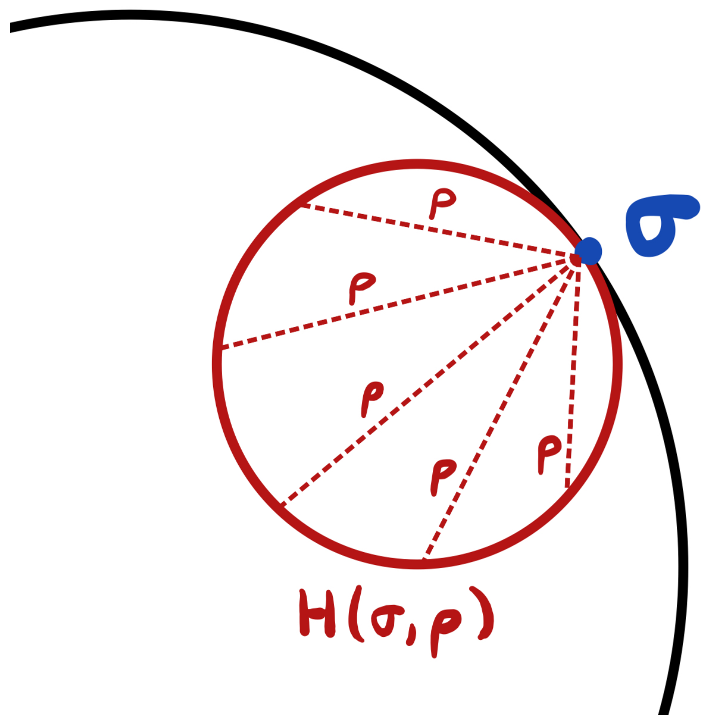

Figure 3.5: Horocycle With Center σ and Radius ρ

A neat result from this is that of the Disk Convergence Lemma, which shows that you can create a sequence of r_n-balls that converge to a horocycle.

Lemma 3.2.3 (Disk Convergence Lemma)**.** Suppose that $\sigma \in \partial\mathbb{D}$, $\lambda_n \to \sigma$ and $0 < r_n \to 1^-$ such that

$$\rho = \lim_{n\to\infty} \frac{1 - |\lambda_n|}{1 - r_n}$$

exists. Then

$$H(\sigma, \rho) \subset \liminf_{n\to\infty} \delta_{r_n}(\lambda_n) \subset \limsup_{n\to\infty} \delta_{r_n}(\lambda_n) \subset \overline{H(\sigma, \rho)}.$$

Although the details are not shown, the interpretation is that as the balls converge towards the horocycle, eventually the balls become contained inside the closure of the horocycle.

Definition 3.2.2. Let $\mathcal{D}_1, \mathcal{D}_2$ be open proper subsets of $\mathbb{C}$ and let $f : \mathcal{D}_1 \to \mathcal{D}_2$ be an analytic function. The Julia quotient of f at $z \in \mathcal{D}_1$ is the ratio

$$J_f(z) = \frac{\operatorname{dist}(f(z), \partial\mathcal{D}_2)}{\operatorname{dist}(z, \partial\mathcal{D}_1)}.$$

What becomes more interesting is that when $\mathcal{D}_1 = \mathcal{D}_2 = H_+$, the Julia quotient is simplified to the formula

$$J_f(z) = \frac{\operatorname{Im}(f(z))}{\operatorname{Im}(z)}.$$

Inside the unit disk, the Julia quotient becomes

$$J_f(z) = \frac{1 - |f(z)|}{1 - |z|},$$

which will be the more familiar version of the Julia quotient that we will see in the statement of Julia's Theorem and throughout the rest of the paper.

At this point, we have the pieces required to prove Julia's Theorem.

Theorem 3.2.4 (Julia's Theorem)**.** *Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic and nonconstant and let $\tau, \omega \in \partial\mathbb{D}$. If $\lambda_n \to \tau$ so that*

$$\varphi(\lambda_n) \to \omega \qquad \textit{and} \qquad \frac{1 - |\varphi(\lambda_n)|}{1 - |\lambda_n|} \to \delta < \infty,$$

then

1. $\delta > 0$
2. $\varphi(H(\tau, \rho)) \subseteq H(\omega, \rho\delta)$ *(called horocyclic continuity)*
3. $\lim_{z \to \tau} \varphi(z) = \varphi(\tau) = \omega$ *non-tangentially*

Proof. Firstly we must show that $\delta > 0$. In the case where φ is just a rotation, Lemma 3.2.1 (Schwarz) would immediately give us $\delta \geq 1 > 0$. For φ where $\varphi(0) \neq 0$, we will use Lemma 3.2.2 (Invariant Schwarz) with $q = 0$. Thus, we have that for all $p \in \mathbb{D}$,

$$d(\varphi(p), \varphi(0)) \leq d(p, 0) = |p|.$$

Using the formula resulting from our properties of d, the above inequality becomes

$$\begin{aligned}1 - |p|^2 &\leq 1 - d(\varphi(p), \varphi(q))^2 \\ &\leq \frac{(1 - |\varphi(p)|^2)(1 - |\varphi(0)|^2)}{\left|1 - \overline{\varphi(p)}(\varphi(0))\right|^2}.\end{aligned}$$

Rearranging the terms in the inequality gives us

$$\frac{\left|1 - \overline{\varphi(p)}(\varphi(0))\right|^2}{1 - |\varphi(0)|^2} \leq \frac{1 - |\varphi(p)|^2}{1 - |p|^2}.$$

By the triangle inequality,

$$\frac{1 - |\varphi(0)|}{1 + |\varphi(0)|} \leq \frac{\left|1 - \overline{\varphi(p)}(\varphi(0))\right|^2}{1 - |\varphi(0)|^2}.$$

This statement holds true for any $p \in \mathbb{D}$. Now, let us consider a sequence $\{\lambda_n\}$ in $\mathbb{D}$. We may use the previous two formulas with the fact that $\lambda_n \to \tau$ to show that

$$\begin{aligned}\frac{1 - |\varphi(0)|}{1 + |\varphi(0)|} &\leq \frac{1 - |\varphi(\lambda_n)|^2}{1 - |\lambda_n|^2} \\ &= \left(\frac{1 - |\varphi(\lambda_n)|}{1 - |\lambda_n|}\right)\left(\frac{1 + |\varphi(\lambda_n)|}{1 + |\lambda_n|}\right) \\ &\to \delta\left(\frac{2}{2}\right) \qquad \text{as } n \to \infty \\ &= \delta.\end{aligned}$$

This gives us a lower bound for δ, which forces it to be strictly positive.

For the second part of the theorem, we want to show $\varphi(H(\tau,\rho)) \subseteq H(\omega,\rho\delta)$. Let $\rho > 0$. We can assume that $\rho > 1 - |\lambda_n| \to 0$ for all but finitely many n. Then, we can define a new sequence r_n given by

$$r_n = 1 - \frac{1 - |\lambda_n|}{\rho}.$$

Note that $r_n \in [0, 1]$ for all but finitely many n, meaning that $r_n \to 1$. To satisfy the condition of the Disk Convergence Lemma (Theorem 3.2.3), let us rewrite the equation above as follows:

$$\rho = \frac{1 - |\lambda_n|}{1 - r_n}.$$

Now, we can use Theorem 3.2.3 on the sequence of r_n-balls centered at λ_n combined with the hypothesis

$$\frac{1 - |\varphi(\lambda_n)|}{1 - |\lambda_n|} \to \rho\delta$$

to conclude that

$$\varphi(H(\tau,\rho)) \subset \overline{H(\omega,\rho\delta))}.$$

But, since φ is nonconstant, it is an open mapping. This means that we can restrict the set inclusion further than the closure, proving that

$$\varphi(H(\tau,\rho)) \subset H(\omega,\rho\delta)).$$

For the third and final part, let us consider a Stolz region S in $\mathbb{D}$ at τ of some arbitrary aperture. We would like to show that $\varphi(\lambda_n) \to \omega$ with $\lambda_n \in S$.

For a given $\varepsilon > 0$, let us choose a ρ such that $H(\omega, \rho\delta) \subset \delta_\varepsilon(\omega)$. Then, there must exist some $\beta > 0$ such that

$$(S \cap \delta_\beta(\tau)) \subset H(\tau, \rho).$$

Thus, from our result in part 2 of the theorem, $\lambda_n \in S$ and $|\lambda_n - \tau| < \rho$, which implies that $|\varphi(\lambda_n) - \omega| < \varepsilon$. It follows that $\lim_{z \to \tau} \varphi(z) = \varphi(\tau) = \omega$ non-tangentially.

With all three parts shown, this concludes the proof of Julia's Theorem. $\square$

3.3 The Julia-Carathéodory Theorem

While Julia's Theorem provides some major conclusions about horocyclic continuity and the existence of non-tangential limits of analytic functions approaching $\partial\mathbb{D}$, nothing is said about whether functions can be differentiable at those limit points. For this, we will use the angular derivative, which is defined as follows.

Definition 3.3.1. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. Then φ has an angular derivative at τ if for some $\omega \in \partial\mathbb{D}$, the non-tangential limit

$$\lim_{z \to \tau} \frac{\omega - \varphi(z)}{\tau - z}$$

exists.

The Julia-Carathéodory Theorem is written as an equivalence of three statements, so the proof will require multiple implication arguments. What we will uncover is that the forward directions in the statement are much more difficult than the backward directions. For this reason, we will begin the proof with the easier parts and then progress to the harder parts. The latter two directions of this proof follow largely from the proof found in Shapiro [10, Chapter 4].

Theorem 3.3.1 (Julia-Carathéodory Theorem)**.** *Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic and $\omega \in \partial\mathbb{D}$. The following statements are equivalent:*

1. $\displaystyle\lim_{z\to\omega} \frac{1-|\varphi(z)|}{1-|z|} = \delta < \infty$,
2. $\displaystyle\lim_{z\to\omega} \frac{\eta-\varphi(z)}{\omega-z}$ *exists for some* $\eta \in \partial\mathbb{D}$,
3. $\displaystyle\lim_{z\to\omega} \varphi'(z)$ *exists, and* $\displaystyle\lim_{z\to\omega} \varphi(z) = \eta \in \partial\mathbb{D}$.

Moreover, $\delta > 0$ in (1), η is the same in (2) and (3), and the limit of the angular derivative in (2) and the limit of derivatives in (3) are equal to $\omega\overline{\eta}\delta$.

Proof. (3) $\implies$ (2) : Assume that $\lim_{z\to\omega} \varphi'(z) = \alpha$ and $\lim_{z\to\omega} \varphi(z) = \eta \in \partial\mathbb{D}$.

Then we can see that integrating the derivative yields the expression

$$\lim_{z\to\omega} \frac{\varphi(\omega)-\varphi(z)}{\omega-z},$$

which exists from our assumption. Due to the other assumption that $\lim_{z\to\omega} \varphi(z) = \eta$, we may at the formula for (2) while also preserving equality of the limits of the derivative and the angular derivative expressions.

(2) $\implies$ (1) : Assume that $\lim_{z\to\omega} \frac{\eta-\varphi(z)}{\omega-z}$ exists for some $\eta \in \partial\mathbb{D}$. If we consider $z \to \omega$ along the radius, then we can see that

$$\left|\lim_{z\to\omega} \frac{\eta-\varphi(z)}{\omega-z}\right| \geq \lim_{z\to\omega} \frac{|\eta|-|\varphi(z)|}{|\omega|-|z|} = \lim_{z\to\omega} \frac{1-|\varphi(z)|}{1-|z|} > 0.$$

This shows that our expression for (1) converges since it is bounded by our convergent expression for (2). Thus, $\lim_{z\to\omega} \frac{1-|\varphi(z)|}{1-|z|} = \delta$ with $0 < \delta < \infty$.

(2) $\implies$ (3) : Assume that $\lim_{z\to\omega} \frac{\eta - \varphi(z)}{\omega - z}$ exists for some $\eta \in \partial\mathbb{D}$. By the previous implication, we have that

$$\lim_{z\to\omega} \frac{1-|\varphi(z)|}{1-|z|} = \delta.$$

To simplify the problem, let us consider a mapping that takes $\omega = \eta = 1$. Let us also consider the Stolz regions $S(1,\alpha)$ and $S'(1,\beta)$ such that $\alpha < \beta < \pi/2$.

From the hypothesis,

$$\varphi(z) - 1 = \delta(z-1) + \psi(z),$$

where

$$\frac{\psi(z)}{1-(z)} \to 0$$

as $z \to 1$ in S'. Let $z_0 \in S$ and let $C(z_0)$ be the circle centered at z_0 tangent to the boundary of S'. Denote $r(z_0)$ to be the (Euclidean) radius of $C(z_0)$. Let θ be the angle made between the real axis and the line segment connecting z_0 and 1. We then get the following relation

$$\frac{r(z)}{|1-z|} = \sin(\beta - \theta) \geq \sin(\alpha - \theta).$$

Using the Cauchy Integral Formula, we see that

$$\begin{aligned}
\varphi'(z) &= \frac{1}{2\pi i}\int_{C(z)} \frac{\varphi(\zeta)-1}{(\zeta - z)^2} d\zeta \\
&= \frac{1}{2\pi i}\int_{C(z)} \frac{\delta(\zeta-1)+\psi(\zeta)}{(\zeta - z)^2} d\zeta \\
&= \frac{\delta}{2\pi i}\int_{C(z)} \frac{d\zeta}{\zeta - z} + \frac{\delta}{2\pi i}\int_{C(z)} \frac{\delta(\zeta-1)+\psi(\zeta)}{(\zeta - z)^2} d\zeta.
\end{aligned}$$

which can then be simplified to the following formula

$$\varphi'(z) = \delta + \frac{1}{2\pi i}\int_{C(z)} \frac{\psi(\zeta)}{(\zeta - z)^2} d\zeta.$$

All we need is for the integral to tend to 0 as $z \to 1$. Let $\varepsilon > 0$. By the definition of ψ, there exists z close enough to 1 such that $|\psi(z)| < \varepsilon\,|1 - z|$. Applying this fact to the integral, we get

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{C(z)} \frac{\psi(\zeta)}{(\zeta - z)^2} d\zeta\right| &\leq \frac{\varepsilon}{2\pi}\int_{C(z)} \frac{|1-\zeta|}{(\zeta - z)^2}\,|d\zeta| \\
&= \frac{\varepsilon}{r(z)} \sup_{\zeta\in C(z)} |1 - \zeta| \\
&= \frac{\varepsilon}{r(z)}(r(z) + |1 - z|) \\
&= \varepsilon\Big(1 + \frac{|1-z|}{r(z)}\Big) \\
&= \varepsilon(1 + \csc(\beta - \alpha))
\end{aligned}$$

This shows that the integral tends to 0, meaning that $\varphi'(z)$ has non-tangential limit δ at 1.

(1) $\implies$ (2) : Let us choose a convergent sequence $z_n \to \omega$. By our hypothesis,

$$\frac{1 - |\varphi(z_n)|}{1 - |z_n|} \to \delta$$

If $\eta = \varphi(\omega)$ from $\varphi(z_n) \to \eta$ non-tangentially, then from Julia's Theorem (Theorem 3.2.4),

$$\varphi(H(\omega, \lambda)) \subset H(\eta, \delta\lambda))$$

Just as done before in (2) $\implies$ (3), we will rotate $\mathbb{D}$ so that $\omega = \eta = 1$. Now, we will imagine taking the right half plane Π and mapping it to $\mathbb{D}$ such that $\infty \to 1$ using the the Cayley transform $\tau : \Pi \to \mathbb{D}$ defined by

$$z = \tau(w) = \frac{w-1}{w+1}.$$

When working in Π, our φ is transformed into $\Phi = \tau^{-1} \circ \varphi \circ \tau$. Comparing distances between $\mathbb{D}$ and Π, we see that

$$1 - z = \frac{2}{1+w}$$

and

$$1 - |z|^2 = \frac{4\mathrm{Re}(w)}{|w+1|^2}$$

Now, we have that the horocycle $H(1, \lambda) \subset \mathbb{D}$ translated into Π is the half plane

$$\Pi(\lambda) = \left\{ \mathrm{Re}(w) > \frac{1}{\lambda} \right\}.$$

Now, our subset inclusion of horocycles from Julia's Theorem becomes

$$\mathrm{Re}(\Phi(w)) \geq \frac{1}{\delta}\mathrm{Re}(w)$$

for $w \in \Pi$. Now we may write the quotient for (2) as

$$\frac{1 - \varphi(z)}{1 - z} = \frac{w+1}{\Phi(w)+1}.$$

Note that $\lim_{z \to 1} \varphi(z) = 1$, so in Π, $\lim_{w \to \infty} \Phi(w) = \infty$ for w inside the image of a Stolz region centered at 1 by τ^{-1}. This gives us the half-plane version of our non-tangential limit in (2), written as

$$\lim_{w \to \infty} \frac{\Phi(w)}{w} = \frac{1}{\delta}.$$

Now all we must prove is that

$$\text{Re}(\Phi(w)) \geq \frac{1}{\delta}\text{Re}(w) \quad \text{implies} \quad \lim_{w\to\infty} \frac{\Phi(w)}{w} = \frac{1}{\delta}.$$

First, let's define c by

$$c = \inf_{w\in\Pi} \frac{\text{Re}(\Phi(w))}{\text{Re}(w)}.$$

We then get the inequality $\text{Re}(\Phi(w)) \geq c\text{Re}(w)$ for all $w \in \Pi$. We want to show that $c \geq 1/\delta$. Let us define γ on Π by

$$\gamma(w) = \Phi(w) - cw.$$

Note that γ is holomorphic and that $\text{Re}(\gamma(w)) \geq 0$ for all $w \in \Pi$. From our definitions of c and γ, we have that

$$\inf_{w\in\Pi} \frac{\text{Re}(\gamma(w))}{\text{Re}(w)} = 0.$$

The final conclusion is that

$$\lim_{w\to\infty} \frac{\gamma(w)}{w} = 0$$

non-tangentially. The argument for why this is true is a lengthy geometric proof. It can be accomplished by letting $\varepsilon > 0$, finding a value $R = R(\varepsilon) > 0$ such that for w in some sector in Π with vertex at 0, $|w| > R$, which then implies that

$$\left|\frac{\gamma(w)}{w}\right| < \varepsilon,$$

thus giving us the non-tangential limit 0. (See Shapiro [10, Chapter 4]).

Translating back to $\Phi(w) = cw + \gamma(w)$, we conclude that

$$\lim_{w\to\infty} \frac{\Phi(w)}{w} = c,$$

which is what we were trying to prove, except using c instead of $1/\delta$.

Now, we can use τ to translate this claim back to $\mathbb{D}$, thus proving (1) $\implies$ (2).

Since all of the statements imply one another, we have proved the theorem. $\square$

With the Julia-Carathéodory Theorem equipped, we have the existence of a limit of a sequence of derivatives and angular derivatives of a holomorphic function as the sequence approaches a boundary point non-tangentially. Additionally, the derivative and angular derivatives correspond with one another.

This theorem is powerful in showing the existence of such derivatives and how to compute them, meaning that when studying function behavior on $\partial\mathbb{D}$, there is more information to be gathered about the derivative.

Chapter 4

EXTENSIONS

4.1 Result

After our results, the next logical question to ask is about what the contemporary research in this field is focused on. Since the Julia-Carathéodory Theorem (3.3.1) gives a description for a sequence of first derivatives of a function approaching the boundary, it is reasonable to ponder what might happen with higher order derivatives in this context.

The Julia-Carathéodory gives a meaningful description of the derivative of a function f inside some Stolz region S_τ with the formula

$$f(z) = f(\tau) + f'(z)(z - \tau) + o(|z - \tau|)$$

whenever $J_\tau(z) < \infty$.

This notion broadens the theory for further investigation such as alternative methods of proof and generalizations. The next section gives a brief overview of one of these alternative methods of proving boundary theorems.

4.2 The Hilbert Space Approach

While the methods used to prove the Julia-Carathéodory Theorem presented in this paper use techniques from Complex Analysis and Measure Theory, there exist other approaches. Notably, Donald Sarason used the theory of Hilbert spaces. He accomplishes this by replacing measures with Hilbert space operations like the inner product. One of the key changes was converting the Poisson kernel, which featured a few times in this paper, into an operator on Hilbert spaces. For further detail into how this approach was carried out, see Sarason's *Angular derivatives via Hilbert space* [9].

4.3 The Two-Variable Julia-Carathéodory Theorem

Further extensions can be found when moving into two variables. Unfortunately, some of the results that held in one variable fail to hold in two. For example, consider the function f defined by

$$f(z,w) = \frac{z+w-2zw}{2-z-w}.$$

Note that f has a singularity at $(1,1)$, but there exists a problem when looking at the non-tangential limit of a sequence of derivatives approaching $(1,1)$. For this reason, the Julia-Carathéodory Theorem doesn't fully generalize to two variables.

A key result in two variables is that for some analytic two-variable function φ,

$$J_\tau(\lambda) \text{ is bounded if and only if } \varphi \text{ has a non-tangential limit at } \tau.$$

Interestingly, the Julia quotient being bounded does not guarantee differentiability like it does in the one-variable Julia-Carathéodory Theorem. In order to get differentiability, one needs to go deeper into Hilbert space theory, which Agler, McCarthy, and Young explore in *A Carathéodory Theorem for the bidisk via Hilbert space methods* [1].

4.4 Tangential Limits

Throughout this paper and in all of the other extensions mentioned beforehand, the assumption was that of limits approaching non-tangentially. Thus the question of what would happen if we did allow sequences to converge tangentially. This concept is explored in a paper written by Pascoe, Sargent and Tully-Doyle [7].

The switch means that we analyze sequences converging in "Stolz-like" regions where the boundary of the region approaches the limit point tangentially. The result is much of our machinery breaks down. For example, Fatou's Non-Tangential Theorem (Theorem 3.1.5) and Julia's Theorem (Theorem 3.2.4) fail to hold, which means that the Julia quotient's power is limited to the non-tangential limit case.

To overcome this, a switch to something called an average Julia quotient comes into play to assess the mean behavior along horizontal strips inside the new "Stolz-like" region approaching the limit point. To understand the topic further, see *A controlled tangential Julia-Carathéodory theory via averaged Julia quotients* by Pascoe, Sargent and Tully-Doyle [7].

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