MAT220 - Assignment 5 - Loic Dallaire

1. a) let $c : C \to \mathbb{R}^3$
   s.t. $x^2 + y^2 = r^2 \to (x, y, r)$ is a clear bijection $\therefore [C] = [\mathbb{R}^3]$

   let $f : [0, 1) \to [0, 1) \times [0, 1) \times [0, 1)$
   s.t. $0.a_1b_1c_1a_2b_2c_2 \cdots \to (0.a_1a_2 \cdots, 0.b_1b_2 \cdots, 0.c_1c_2 \cdots)$ is also a bijection.

   We showed $[[0, 1)] = \left[[0, 1)^3\right]$
   $\therefore [\mathcal{C}] = [\mathbb{R}^3] = [\mathbb{R}]$
   $\therefore [\mathcal{C}] = c$

   b) We know that $\mathbb{Q}$ is countable. Suppose that $\mathbb{I}$ is also countable:
   then $\mathbb{Q} \cup \mathbb{I}$ is countable,
   but $\mathbb{Q} \cup \mathbb{I} = \mathbb{R}$ which is not countable so we have a contradiction
   therefore $\mathbb{I}$ is uncountable so $[\mathbb{N}] < [\mathbb{I}]$

   We can define $i : \mathbb{I} \to \mathbb{R}$ s.t. $x \to x$. This is obviously an injection so we have that $[\mathbb{N}] < [\mathbb{I}] \leq [\mathbb{R}]$. By the continuum hypothesis we have that $[\mathbb{I}] = [\mathbb{R}] = c$

   c) With induction lets show that $\mathbb{Z}^n$ is countable
   Base case is $\mathbb{Z}$ which we know to be countable.
   Assuming $\mathbb{Z}^k$ is countable for some $k \in \mathbb{N}$, show that $\mathbb{Z}^{k+1}$ is also countable:
   $\mathbb{Z}^{k+1} = \mathbb{Z}^k \times \mathbb{Z}$ and the cartesian product of two countable sets is countable therefore $\mathbb{Z}^{k+1}$ is countable. We have shown $\mathbb{Z}^n$ is countable so $[\mathbb{Z}^n] = \aleph_0$

   d) First lets look at $\mathbb{Z}_n[x]$ which is the set of polynomials with integer coefficients of degree n.

   define $f : \mathbb{Z}_n[x] \to \mathbb{Z}^{n+1}$
   s.t. $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \to (a_0, a_1, a_2, \cdots, a_n)$
   this is a bijection so $\mathbb{Z}_n[x]$ is countable.
   now consider $\cup_{i=0}^{\infty} \mathbb{Z}_i[x]$.
   this is a countable union of countable sets, and this union is also equivalent to $\mathbb{Z}[x]$ meaning that $\mathbb{Z}[x]$ is countable and therefore
   $[\mathbb{Z}[x]] = \aleph_0$

2. a) A polynomial of degree d has a countable number of roots d, since $d \in \mathbb{N}$
Define $r : \mathbb{Z}[x] \to \mathcal{P}(\mathcal{A})$ s.t. $p(n) \to \{\text{roots of } p(n)\}$
consider the image of r.
$\forall x \in \text{IM}(r)$, x is a countable set. We also previously showed that $\mathbb{Z}[x]$ is countable so $\text{IM}(r)$ is countable. Consider $\cup_{i=1}^{\infty} r(x)_i$ to be the union of all sets in $\text{IM}(r)$. This is a countable union of countable sets, this is also $\mathcal{A}$ therefore the set of algebraic numbers is countable.

b) Suppose $\mathcal{A}'$ is countable, take $\mathcal{A} \cup \mathcal{A}'$ which is the set of all numbers. The union of two countable sets is countable but the set of all numbers is not countable so we have a contradiction. therefore the set of transcendent numbers is not countable.

c) Let $\mathcal{A}_n = \{\mathcal{D} \in \mathcal{P}(\mathbb{N}) : \mathcal{D} \text{ is finite and } |\mathcal{D}| = n\}$
Let $\mathbb{N}_o^n$ be the subset of $\mathbb{N}^n$ s.t. the elements in the tuple are ordered. For example $(1, 2, 3) \in \mathbb{N}_o^n$ for $n = 3$ but $(1, 3, 2)$ is not.
$\mathbb{N}_o^n \subseteq \mathbb{N}^n \therefore [\mathbb{N}_o^n] = \aleph_0$
We can define a simple bijection:
$f : \mathcal{A}_n \to \mathbb{N}_o^n$ s.t. $\{a_1, a_2, \cdots, a_n\} \to (a_1, a_2, \cdots, a_n)$ showing that $\mathcal{A}_n$ is countable
let $\mathcal{A} = \cup_{i=1}^{\infty} \mathcal{A}_i$. This is a countable union of countable sets which is countable, it is also the set of all finite subsets of $\mathbb{N}$ therefore the set of all finite subsets of $\mathbb{N}$ is countable.

d) The set of all subsets of $\mathbb{N}$ is simply $\mathcal{P}(\mathbb{N})$ and we know $[\mathcal{P}(\mathbb{N})] = c$ which is not countable.