Numerical simulations of the gravity of a global monopole

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Project Number: AS48
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(Dated: April 25, 2022)

In this paper we investigate the topological defects that result from the breaking of the global O(3) symmetry to U(1) of a proposed isoscalar triplet, ϕ^a with $a = 1, 2, 3$. We find numerical solutions to the radial profile of the resulting global monopole formed in background Minkowski spacetime and when its gravity is accounted for. We demonstrate the existence of static solutions in Einstein gravity given de Sitter-like boundary conditions at the monopole centre. We find numerical solutions for the radial metric within and outside the monopole core, and use these to calculate the energy of the resultant spacetime. These solutions are found for $\Delta = (\frac{\eta}{M_{pl}})^2$ in the range $0 < \Delta \leq 1.1$, thereby demonstrating numerical solutions in the Planck energy regime. We then derive the ODE system for a global monopole in conformally coupled gravity, not seen before in literature.

I. INTRODUCTION

The discovery of the Higgs boson at the large hadron collide in Geneva, 2013 was the latest triumph of the standard model, our most successful and complete description of the fundamental forces and particles of physics. The discovery is seen as confirmation of our best theory for how particles in the standard model acquire mass, via the Higg's mechanism, and therefore also of the unification of electromagnetism and the weak interaction into the electroweak interaction at high energies. For many it also made the prospect of a grand unified theory (GUT), where all fundamental forces of the standard model were unified at some earlier epoch, even more tantalising. At the very least, it has strengthened the case that symmetries and symmetry breaking will play a crucial role in any fundamental theory.

In a GUT there is an assumed fundamental force, gauge-invariant under the symmetry action of a proposed larger symmetry group which, as the universe cools, is spontaneously broken into smaller and smaller groups. Thus, the one unified force splits into the strong, weak and electromagnetic interactions of the standard model, described by the gauge group $SU(3) \times SU(2) \times U(1)$. Notably, the breaking of these symmetries can lead to the generation of topological defects. For example, GUTs inevitably lead to the production of an abundance of magnetic monopoles [1], of which we see no experimental evidence for. This was one of the original motivations for proposing an inflationary model in 1981 [2], essentially to expand all primordial monopoles far enough away from each other that today there would be negligible densities on a horizon scale.

However, global monopoles (GM) created due to breaking of a global, not gauge, symmetry are also a possible occurrence in some species of GUTs [3]. These defects are theoretically simpler to examine and their existence could still have cosmological implications when their gravitational effects are considered. GMs are one proposal for the cause of density perturbations in the early universe which may have seeded galaxy clusters or anisotropies in the cosmic microwave background [3]. Global monopoles, themselves, have also been proposed as a possible cosmological model [4].

Analysis of these topological defects in the geometric setting of general relativity is also interesting, given the unique spacetimes they produce and the possibility of their regularising singularities when combined with black holes (BHs). They are also of great interest in their potential to violate the so-called no-hair theorem of BHs which states that all BHs can be completely described by their mass,charge and angular momentum. GMs may violate this theorem [5] and give these bodies topological charge or "scalar hair" due to the surrounding non-trivial field configuration. Thus GMs are certainly worthy of investigation.

In this paper we set up the spontaneous symmetry breaking (SSB) scalar triplet field model which allows for the formation of GMs. We then find static solutions for their radial profiles and for the surrounding spacetime. We first find numerical solutions for all values of $\sqrt{\lambda}\eta$ in Minkowski spacetime, using a shooting method to constrain a highly sensitive system. We then find numerical solutions when the self-gravity of the scalar fields is considered, by taking a minimal coupling to gravity. In this case we first find solutions for the GM radial profile in the range $0 < \Delta \leq 1$ and then also the resulting metric of spacetime within and outside the monopole core. We demonstrate their good agreement with asymptotic solutions to the system and derive an expression for the energy of the monopoles using a Komar integral. We then look at monopoles formed via a symmetry breaking parameter, η with energy greater than the Planck mass and to find a solution with horizon, prompting a discussion on the possibility of monopoles in a BH. Finally, we derive the scalar and gravitational field equations for a global monopole when using a non-minimal, specifically conformal, coupling to gravity.

II. CONVENTIONS

- ☐ Units:
  $\hbar = c = 1$, $M_{pl}^2 = \frac{\hbar c}{8\pi G}$ is the reduced Planck mass/energy.

- ☐ Signs:
  Metric signature $(+---)$
  Einstein's field equations $G_{\mu\nu} = -\kappa T_{\mu\nu}$

- ☐ Indices:
  $a = 1, 2, 3$
  $\mu, \nu, \sigma = 0, 1, 2, 3$

III. THE MODEL

We propose the existence of a Higgs-like triplet of scalar fields, $\mathbf{\Phi} = (\Phi^1, \Phi^2, \Phi^3)$, posessing an internal O(3) symmetry explicit in the form of the Lagrangian density given by Eqn. 1,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^a \partial^\mu \Phi^a - \frac{\lambda}{4}(\Phi^a\Phi^a - \eta^2)^2 \,. \tag{1}$$

The potential postulated is a "Mexican hat", where λ and η are constant parameters of the model. This potential admits spontaneous symmetry breaking to a choice of vacuua state at the degenerate minima $|\mathbf{\Phi}| = \eta$.

We follow the prescription set by most models of a GUT, and assume the hot big bang cosmological model. Thus, in the early epoch of the universe it was so hot that the equilibrium value of our scalar fields had zero magnitude, residing at the local maximum of the potential. In this unbroken phase the field solution remains invariant under the O(3) symmetry of the Lagrangian. We then assume the expansion of the universe cooled the universe below some critical temperature and a phase transition occurred, wherein the fields spontaneously took one of the degenerate minima and a corresponding internal orientation of the triplet, breaking the O(3) symmetry.

Due to local fluctuations in the field, this choice of orientation would be different in causally disconnected regions of the universe, and therefore in the broken phase we see the topological defects, global monopoles, arise. This process, and the inevitability of these defects forming was proven by Kibble [6].

We investigate static global monopoles in a hedgehog configuration, given by the ansatz in Eqn. 2,

$$\Phi^a = \eta f(r)\frac{x^a}{r} \,, \tag{2}$$

with radial coordinate, r, and standard standard Cartesian coordinates, x^a. The radial profile function, $f(r)$, is required to satisfy the boundary conditions,

$$\begin{aligned} f(0) &= 0, \\ f(r \to \infty) &= 0. \end{aligned} \tag{3}$$

These are set by the requirement that the field be single-valued at its centre and thus be lifted off its vacuum expectation value (VEV), but take on the non-zero VEV expected of the broken symmetry phase far from its core.

IV. GLOBAL MONOPOLE IN MINKOWSKI SPACETIME

In the first case we approximate the energy of the monopole as having no effect on a background Minkowski spacetime, i.e any gravitational effects of the monopole were ignored. We then vary the classical field theory action and, using the principle of least action, obtain Eqn. 4, the Euler-Lagrange equations of motion,

$$\partial_\mu \partial^\mu \Phi^a + \lambda(\Phi^a \Phi^a - \eta^2)\Phi^a = 0\,. \qquad (4)$$

We then apply our hedgehog configuration ansatz, Eqn. 2 for a static monopole. This gives a second order non-linear ordinary differential equation (ODE) which we then reparameterise by changing to a dimensionless variable $x = \sqrt{\lambda}\eta r$ and thus convert into a parameterless Eqn,

$$f''(x) + \frac{2}{x}f'(x) - \frac{2}{x^2}f(x) - f(f^2 - 1) = 0\,. \quad (5)$$

Without loss of generality we can then set $\sqrt{\lambda}\eta = 1$ and simply find other $\sqrt{\lambda}\eta$ solutions by rescaling x.

It is then necessary to implement a shoot match to meet the boundary conditions, Eqns. 3 at large x. This involves varying the initial conditions at the origin to minimise the error in the resultant plot at the boundary. However, the ODE is very sensitive to the value of $f'(0)$ and so requires an extremely precisely specified initial gradient to behave correctly out to infinity. With machine precision we can only push the correct asymptotic behaviour out so far as with too large an initial gradient the solution eventually diverges. With too small an initial gradient it shows oscillations which decay to zero.

In Fig. 1 we show this behaviour but, importantly, that our numerical solution with correctly specified initial conditions well matches the asymptotic solution to Eqn. 5, given by a power series expansion. Thus our finite precision is satisfactory as, clearly, $f(x)$ and $f'(x)$ agree with the asymptotic solution to a high degree. We can therefore piece together the numerical and asymptotic solution and thus find a solution for the entire domain. By rescaling x we also have a solution for all symmetry breaking energy scales.

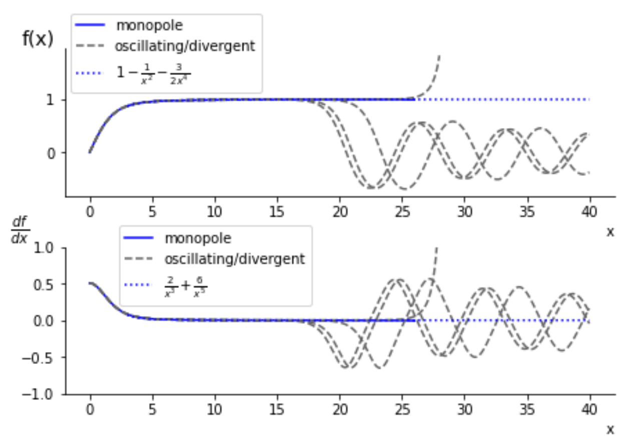

FIG. 1. The radial profile $f(x)$ is very sensitive to the gradient at its centre. However we find a monopole solution for $\sqrt{\lambda}\eta = 1$.

It is worth noting that the value of initial gradient I found through this numerical integration, $f'(0) = 0.5060432978484631$ agrees well with the flat space estimates of (Mazharimousavi,2017) and was consistent with my value obtained when taking the classical limit of my analysis with gravity. Fig. 1 also gives an approximate size for the monopole core within which the value of $f(x)$ changes between 0 and 1. This is of order $x = 1$ and so we assume a core size of $r \sim \frac{1}{\sqrt{\lambda}\eta}$. This is needed to give a scale of the core energy and as a consistency check when we consider the monopole's gravitational effects.

To determine the core energy we take the Noether current generated by infinitesimal spacetime translations of the Lagrangian density in Eqn. 1 and calculate the conserved charge associated with its time translation symmetry, equivalent to an integral over the Hamiltonian density, $\mathcal{H}$. Hence,

$$E = \int d^3x\mathcal{H}$$
$$= \int d^3x(\frac{1}{2}\partial_\mu \Phi^a \partial^\mu \Phi^a + \frac{\lambda}{4}(\Phi^a\Phi^a - \eta^2)^2). \tag{6}$$

After again imposing our hedehog ansatz we arrive at the total energy of our system,

$$E = 4\pi\eta^2 \int_0^\infty (\frac{1}{2}f'^2 r^2 + \frac{1}{2}f^2 + \frac{\lambda}{4}\eta^2(f^2-1)^2 r^2)dr. \tag{7}$$

This integral is clearly divergent and reproduces the known result that finite energy static global monopoles are impossible in flat spacetime [7]. While neglecting the gravity of monopoles is crude and leads to unphysical results, it is still useful. The flat space energy of the field within a sphere of radius of the characteristic monopole core size gives an estimate for the monopole mass which can be used as a dimensional consistency check. Carrying out this integral gives an order of magnitude estimate of $E_{core} \sim \lambda^{-\frac{1}{2}}\eta$

V. GLOBAL MONOPOLE MINIMALLY COUPLED TO GRAVITY

In order to solve this divergent energy, we next considered the gravitational effects of the monopole. Thus, the derivatives are upgraded to covariant ones in the scalar field Lagrangian density, as shown in Eqn. 8. Working in reduced Planck units, the Einstein-Hilbert action takes the form of Eqn. 9.

$$\mathcal{L} = \frac{1}{2}\nabla_\mu \Phi^a \nabla^\mu \Phi^a - \frac{\lambda}{4}(\Phi^a\Phi^a - \eta^2)^2), \tag{8}$$

$$S = \int d^4x\sqrt{-g}(\frac{M_{pl}^2}{2}R + \mathcal{L}). \tag{9}$$

We then vary Eqn. 9 to obtain the equations of motion of the fields, Eqn. 10 and the energy-momentum (EM) tensor, Eqn. 12, respectively.

$$\nabla_\mu \nabla^\mu \Phi^a + \lambda(\Phi^a\Phi^a - \eta^2)\Phi^a = 0, \tag{10}$$

$$T_{\mu\nu} = \nabla_\mu \Phi^a \nabla^\mu \Phi^a \tag{11}$$
$$- g_{\mu\nu}(\frac{1}{2}\nabla_\sigma \Phi^a \nabla^\sigma \Phi^a - \frac{\lambda}{4}(\Phi^a\Phi^a - \eta^2)^2). \tag{12}$$

we then assume a general, spherically symmetric and static metric with spacetime interval given by

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{13}$$

and with this metric calculate the non-zero components of the Ricci tensor:

$$\begin{aligned} R^t_t &= \frac{-B''}{2AB} + \frac{B'}{4AB}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B'}{rAB} \\ R^r_r &= \frac{-B''}{2AB} + \frac{B'}{4AB}(\frac{A'}{A} + \frac{B'}{B}) - \frac{A'}{rA^2} \\ R^\theta_\theta = R^\varphi_\varphi &= \frac{1}{r^2} + \frac{1}{2rA}(\frac{A'}{A} - \frac{B'}{B}) - \frac{1}{r^2 A}. \end{aligned} \tag{14}$$

After imposing the usual hedgehog ansatz in this metric, the resulting EM tensor is diagonal with non-zero components given by :

$$\begin{aligned} T^t_t &= \frac{\eta^2 f'^2}{2A} + \frac{\eta^2 f^2}{r^2} + \frac{\lambda}{4}\eta^2(f^2-1)^2 \\ T^r_r &= -\frac{\eta^2 f'^2}{2A} + \frac{\eta^2 f^2}{r^2} + \frac{\lambda}{4}\eta^2(f^2-1)^2 \\ T^\theta_\theta = T^\varphi_\varphi &= \frac{\eta^2 f'^2}{2A} + \frac{\lambda}{4}\eta^2(f^2-1)^2. \end{aligned} \tag{15}$$

We now have all the machinery to calculate Einstein's field equations (EFEs). I follow the procedure outlined by Harari et al [8], taking judicious combinations of the EFEs to arrive at more conveniently formulated ones. These are:

$$(\frac{r}{A})' = 1 - (\frac{\eta}{M_{pl}})^2 T^t_t, \tag{16}$$

$$\frac{A'}{A} + \frac{B'}{B} = (\frac{\eta}{M_{pl}}^2)\left[T^r_r - T^t_t\right]. \tag{17}$$

We again rescale the radial coordinate by the core size and introduce $\Delta = (\frac{\eta}{M_{pl}})^2$, a dimensionless parameter. This sets the energy scale of our symmetry breaking parameter relative to the Planck mass and so illustrates $\Delta = 1$ as the regime where we imagine Einsteinian gravity needs to be modified by quantum gravity. We then reformulate these equations into a system of first order ODEs, suitable for numerical integration:

$$\dot{f} = u \tag{18}$$

$$\dot{u} = -\frac{A}{x}\left[u\left(1 - \Delta f^2 - \frac{\Delta}{4}x^2(f^2-1)\right) - \frac{2f}{x} - xf(f^2-1)\right] - \frac{u}{x} \tag{19}$$

$$(\frac{\dot{x}}{A}) = 1 - \Delta\left[\frac{xu^2}{2}(\frac{x}{A}) + f^2 + \frac{x^2}{4}(f^2-1)^2\right] \tag{20}$$

$$\dot{B} = -B(\frac{1}{x^2} - \frac{(\frac{\dot{x}}{A})}{\frac{x}{A}} + \Delta x f'^2) \tag{21}$$

We first investigate the numerical solution for the radial profile of the monopole as this system behaved similarly to the Minkowski case. Again, the solution either diverged or oscillated if the initial condition on the gradient was not specified with infinite precision. So our method is again to shoot match far enough away from the core that we may piece together the numerical and asymptotic solutions. In Fig. 2 this behaviour is revealed for the classical limit $\Delta << 1$ where we see that once the initial gradient is specified correctly, the solutions well match our asymptote.

With this method it was possible to find GM solutions formed at symmetry breaking energies ranging from the classical, $\Delta << 1$, to the Planckian $\Delta \approx 1$ regime. As shown in Fig. 3, these solutions are remarkably similar, with differences in the field profile or gradient only becoming noticeable for the extreme case of $\Delta = 1$.

Solving the ODE system of Eqns. 18 - 21 also gives a numerical solution for the radial metric function. However, finding the asymptotic radial metric solution which any numerical solution ought to match is more involved in this case. We begin by substituting our asymptotic conditions $f(x) \approx 1$ and $\dot{f}(x) \approx 0$ into the EM

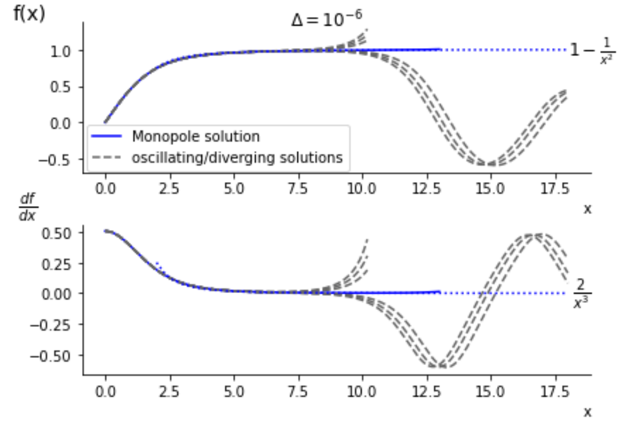

FIG. 2. Radial profile of the monopole minimally coupled to gravity in the classical limit $\Delta = 10^{-6}$.

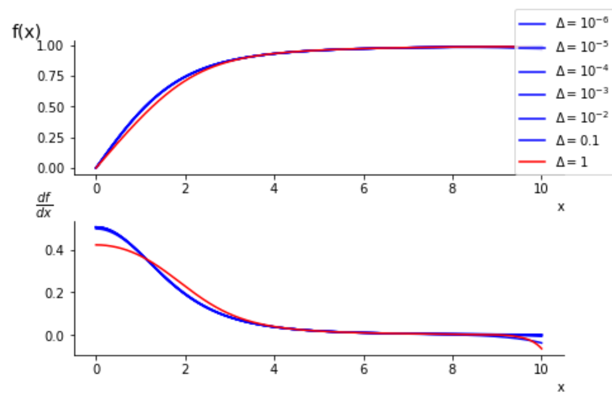

FIG. 3. The shape of the radial profile, and the gradient of the field at the centre of the monopole is quite insensitive to Δ.

tensor time-time component of Eqns. 15. In doing so we may then directly integrate Eqn. 16 to obtain the asymptotic solution,

$$A(x)^{-1} = 1 - \Delta - \frac{C}{x}, \tag{22}$$

where C is, for now, an arbitrary integration constant. Rewriting Eqn. 17 as

$$\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = \Delta \dot{f}^2, \tag{23}$$

we see in the asymptotic limit $d(\ln AB) = const. \implies AB = const.$ We then use our freedom to rescale our time coordinate, to specify that constant as $AB = 1$ at spatial infinity. This then also gives the asymptotic time component of our metric,

$$B(x) = 1 - \Delta - \frac{C}{x}. \tag{24}$$

The asymptotic spacetime around a global monopole is thus manifestly not flat, and as discussed in Barriola and Vilenkin [9] has a solid angle deficit given by Δ.

To aid the numerical integration of the radial metric function we have formulated the ODE in terms of $\frac{x}{A}$ as it allows us to impose suitable boundary conditions at the centre of the monopole, $x = 0$. These can be justified by considering the EM components, Eqns. 15 at the monopole centre in the limit that the potential energy of the scalar field dominates the kinetic energy. In this limit the EM tensor is proportional to the metric and we can treat our EM tensor as instead acting as a constant, positive curvature, cosmological constant. We thus expect the metric to approach a de Sitter spacetime near the centre of the monopole. With this reasoning, we impose $\frac{x}{A} \to 0$ as our initial condition and can then numerically integrate to obtain the metric solution, plotted in Fig. 4.

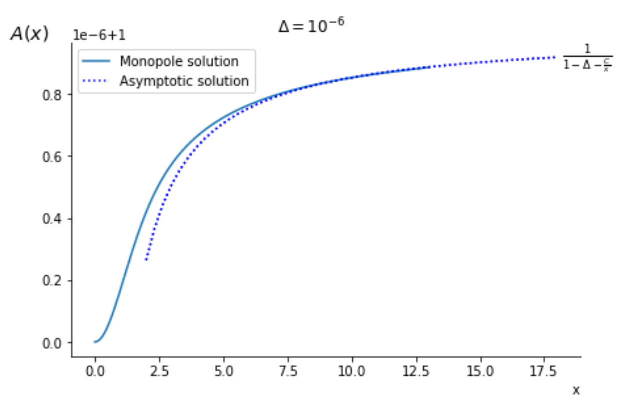

FIG. 4. $A(r)$ showing de Sitter character near the origin and asymptoting to spacetime with solid deficit angle. Here $C \approx -1.47$.

The numerical solution reveals the de Sitter character in the concave shape near the centre and is, again, well matched to the asymptotic solution. The asymptote is specified by an integration constant, C, which is obtained in the proceeding analysis of the monopole energy in the next section.

VI. ENERGY OF THE MONOPOLE

Whereas in Minkowski space it was sufficient to define the energy of the monopole via a Noether charge that would be conserved in any given hypervolume, it is more difficult in general relativity. In flat spacetime global conservation of energy relied upon using the zero divergence of the current associated with time translations when applying Stoke's theorem. However in curved spacetime we only have a vanishing *covariant* divergence of the stress-energy *tensor*, which has no corresponding Stoke's law - and as such covariant conservation is more subtle.

Instead we must use the Komar integral, relying on the fact that our spacetime is static and thus posesses a killing vector $K^\mu = (1, 0, 0, 0)$. With this and the Ricci tensor, we can construct a covariantly conserved current, $J^\mu_R = K_v R^{\mu v}$. On applying stokes theorem to a given hypervolume, as outlined in Carroll [10], with convenient normalisation, we derive the total energy of the spacetime enclosed.

$$E_R = \frac{1}{4\pi} \int_{\partial\Sigma} d^2 z \sqrt{\gamma(\partial\Sigma)} n_\mu \sigma_\mu \nabla^\mu K^v. \tag{25}$$

Here, γ is the metric induced on our hypersurface, $\partial\Sigma$. It has a timelik and future-pointing unit vector, n_μ, and a spacelike and outward-pointing unit vector, σ_μ.

For computing the total spacetime energy, we take our hypersurface to be a 2-sphere at spatial infinity with obvious metric. Taking the unit vectors $n_\mu = (-\sqrt{B}, 0, 0, 0)$ and $\sigma_\mu = (0, \sqrt{A}, 0, 0)$ and using the asymptotic solutions, Eqns. 22 and 24, we can then calculate the total energy, and hence mass, of our monopole>

$$\begin{aligned} E_R &= \frac{1}{4\pi}\int_{S^2} d\theta d\phi x^2 \sin\theta \frac{1}{2}\frac{\dot{B}}{\sqrt{AB}} \\ &= \frac{1}{4\pi}\int_{S^2} d\theta d\phi \sin\theta \frac{C}{2} \equiv M \\ &\implies C = 2M. \end{aligned} \tag{26}$$

We have therefore shown that we can associate the integration constant, C, of our asymptotic solution with the total mass of our system and so we rewrite this solution as,

$$A(x)^{-1} = 1 - \Delta - \frac{2M(x)}{x}. \tag{27}$$

Furthermore, given the fast convergence of our numerical solutions to their asymptotic forms, we expect that on explicitly integrating Eqn. 20 we will find a monopole mass which also converges. This integral, which gives the mass as a function of distance from its centre is given in Eqn. 28.

Fig. 5 shows the result of carrying out this integration, given our numerical solutions, for $0 < \Delta \leq 1$. This reveals the bizarre fact that a global monopole has a negative mass, confirming the analysis of Harari et al [8].

$$2M(x) = \Delta \int_0^x \frac{x^2 \dot{f}^2}{2A} + (f^2 - 1) + \frac{1}{4}x^2(f^2-1)^2) dx \tag{28}$$

We note in the asymptote of Fig. 5 the approximately linear relationship $M(x) \approx -\frac{3}{2}\frac{\Delta}{2}$ in reduced Planck units. On restoring ordinary mass and length units to Eqn. 22, and adjusting the constant accordingly, we can see $M(r) \approx -6\pi\frac{\eta}{\sqrt{\lambda}}$. This concurs with the order-of-magnitude calculation in our flat space analysis.

VII. MONOPOLES WITH HORIZONS

The integration constant, C, of our large x radial metric solution Eqn. 22 having a negative value presents the opportunity for a horizon within a monopole spacetime. This arises in the $\Delta > 1$ regime as we see the radial metric becomes singular for $x \to \frac{C}{1-\Delta}$, now a positive number in this regime.

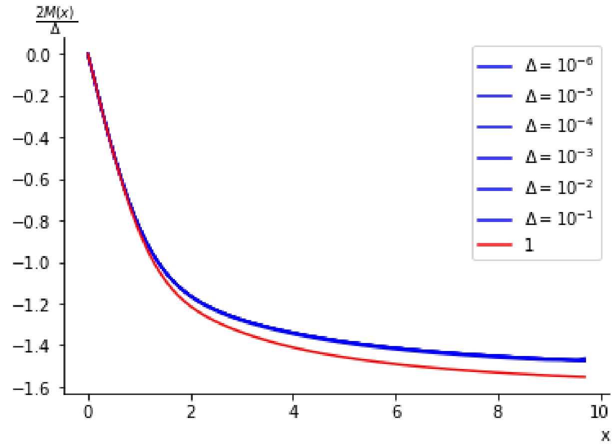

FIG. 5. $\frac{2M(x)}{\Delta}$ is negative but no longer divergent when its self gravity has been considered but approaches $\approx -\frac{3}{2}$ in regime $\Delta \leq 1$.

We therefore repeat the analysis for $\Delta > 1$, first finding a suitably matched numerical solution up to large x and then piecing it together with an asymptotic solution with integration constant found through Eqn. 28. The solution for $\Delta = 1.1$, is plotted in Fig. 6

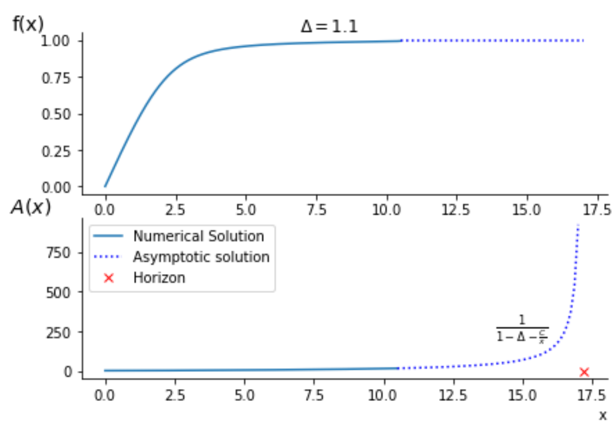

FIG. 6. Regular monopole solutions appear possible in $\Delta > 1$ regime and possess a horizon. Here this occurs at $x = -10C \approx 17.18$

Increasing Δ further results in the horizon being pulled closer to the monopole core but otherwise shows the same behaviour. The existence of GMs with a regular centre and a horizon makes them an attractive prospect for regularising, i.e. removing the singularity of, BHs. They look particularly appealing when given a large positive mass, as their asymptotic spacetime looks Schwarzschild [9]. This would be the case if a BH swallowed a monopole. However, in that case the monopole core is no longer dominated by the scalar potential energy, and we are unable to justify the use of de Sitter core boundary conditions. Unfortunately we found no way to regularise the singularity. This is unsurprising, given the no-go BH Theorem [11] which states that there are no regular BH solutions in gravity with minimal coupling to scalar fields, for any potential or for any asymptotic spacetime.

VIII. NON-MINIMAL COUPLING

The logical next step was to consider a non minimal coupling to gravity. In Einstein Gravity the strength of the gravitational interaction is set by the coefficient before the Ricci scalar in the action. Thus in SI units we talk about its strength through the gravitational constant, G. Or in Planck units it is set by the Planck Mass, M_{pl}.

One class of alternate theories of gravity involves allowing the strength of gravity to be modified by a scalar field, so called scalar-tensor theories of gravity. In our case, this is implemented by a non-minimal coupling of the scalar field to the Ricci scalar, and a new action for the theory, given by

$$S = \int d^4x\sqrt{-g}\big(f(\mathbf{\Phi}))R + \frac{1}{2}\nabla_\mu\Phi^a\nabla^\mu\Phi^a - \frac{\lambda}{4}(\Phi^a\Phi^a - \eta^2)^2\big), \quad (29)$$

where we see the value of our triplet of scalar fields modulates the effective strength of gravity through

$$f(\mathbf{\Phi}) = \frac{M_{pl}^2}{2} - \frac{\xi}{2}\mathbf{\Phi}^2. \quad (30)$$

We again vary the action with respect to the scalar fields and the metric to give the equations of motion for the scalar field, Eqn. 31, and the modified EFEs, Eqn. 32:

$$\nabla_\mu\nabla^\mu\Phi^a + \lambda(\Phi^a\Phi^a - \eta^2)\Phi^a + \xi R\Phi^a = 0\,, \quad (31)$$

$$f(\mathbf{\Phi})G_{\mu\nu} = \frac{1}{2}T_{\mu\nu} + \nabla_\mu\nabla_\nu f(\mathbf{\Phi}) - g_{\mu\nu}\nabla_\mu\nabla^\mu f(\mathbf{\Phi})\,. \quad (32)$$

A conformal transformation of the metric, $g_{\mu\nu}(x) \to \Omega(x)g_{\mu\nu}(x)$ is an active transformation of the metric which preserves the causal relations of the spacetime, but changes the physical length scales.

Choosing $\xi = \frac{1}{6}$ admits a simple conformal transformation that leaves the action, and therefore the physical theory invariant [12]. Thus, monopole solutions with this coupling may be of theoretical interest when considering topological defects in more sophisticated conformal field theories (CFTs), such as in string theory [13]. A description of this, however, would be vastly above the scope of this paper.

With this choice of ξ, we once again impose our ansatz for a spherically symmetric metric and a hedgehog configuration for the fields and rescale by the monopole core size. For notational convenience we introduce the parameter $\gamma = \frac{M_{pl}}{\eta} = \Delta^{-\frac{1}{2}}$. Taking the trace of Eqn. 32 we can thus calculate the Ricci scalar for our spacetime as

$$R = \frac{-\lambda\eta^2}{(\gamma^2 + \frac{1}{6}f^2)}\left[\frac{2f^2}{x^2} + \frac{2\dot{f}^2}{A} + (f^2-1)^2 + (\frac{1}{2B}\frac{\dot{B}}{A} + \frac{2}{xA})f\dot{f} + f\ddot{f}\right]. \quad (33)$$

We substitute this into our equation of motion, Eqn. 31 and again take judicious combinations of the components of Eqn. 32. Doing so, we then arrive at the system of ODEs for our minimally coupled monopole,

$$\ddot{f} = \frac{1}{\gamma^2 + (1-A)\xi f^2} A(\gamma^2 + \xi + \xi f^2) f(f^2 - 1) + 2\xi f \dot{f}^2 + 2A\gamma^2 \frac{f}{x^2} + 4A\xi \frac{f^3}{x^2} + \gamma^2 \Big(\frac{2}{x} + (\frac{\dot{B}}{A})\Big) \dot{f} \tag{34}$$

$$\frac{\dot{A}}{A} = \frac{1}{\gamma^2 + \xi f^2} \Big[(2x\xi - \gamma^2 - \xi f^2) \frac{\dot{B}}{B} + 4\xi f \dot{f} - x\dot{f}^2\Big] \tag{35}$$

$$\ddot{B} = \frac{-\dot{B}}{2}(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}) - \frac{\dot{B}}{x} + \frac{B\dot{A}}{xA} + \frac{4}{x^2}(B - AB) \frac{B}{\gamma^2 + \xi f^2}\left(\dot{f}^2 + \frac{1}{2}A(f^2-1)^2 - \frac{4\xi f \dot{f}}{x}\right) \tag{36}$$

Unfortunately for this unwieldy ODE system we were unable to make progress with a shooting method towards matching the asymptotic conditions of a monopole. The system is again extremely sensitive to initial conditions, of which a greater number now need to be specified, as compared to the minimal coupling case. And so, we did not find a regular monopole solution. However, this is not to suggest that a monopole solution does not exist - but rather that a more sophisticated numerical analysis may be needed to produce one. Perhaps reformulation in terms of the conformally transformed metric may have produced a simpler system to be solved.

It would, of course, be of great interest to see if these solutions exist, and whether there is a global monopole solution in non-minimal coupling that regularises a BH. The ability of $f(\mathbf{\Phi})$ to go to zero, effectively switching off gravity for large enough values of $\mathbf{\Phi}$ does seem to be an intriguing mechanism for achieving this.

IX. CONCLUSIONS

Topological defects will remain a feature of many cosmological and theoretical theories and so it is useful to test their existence and define the physical quantities of these otherwise intangible entities. These tests will invariably be a mixture of pen and paper thinking and numerical analysis, often having to meet tightly constraining boundary conditions. We have demonstrated the limited effectiveness of using shooting methods to plot solutions for one class of these defects, static global monopoles. In doing so, we have found well behaved GM solutions in background Minkowski spacetime and when their own self-gravity is considered. This included solutions in the regime where the energy of the symmetry breaking parameter of the theory approached the Planck mass, and also monopole solutions with a horizon when this energy was exceeded.

We have also combined the machinery of the Komar integral with these numerical solutions to plot the masses of the monopoles in the sub-Planckian range, recovering their negative value when a de Sitter core is assumed. In further investigation we would focus on the de Sitter nature of the GM cores, and focus on the possibility of inflationary monopoles, apposite given the Higgs-like model we have studied. The cosmological implications of, for example, the fractal inflating monopoles described by Linde [4] would be particularly engaging.

Unfortunately, we were unable to use a global monopole to regularise BHs, particularly as the shooting method failed at producing a numerical solution when employing conformal coupling of the scalar fields to gravity. But the novel derivation of the equations of motion for the metric and field in this coupling presciption does deserve a more sophisticated analysis using more powerful numerical techniques. Relaxation methods would be the next avenue we would employ to attempt to solve this system, and then use the numerical solutions to calculate the mass of a conformally coupled GM. This would then hopefully lead to investigation of its ability to regularise a BH in a large mass limit.

[1] J. P. Preskill, Cosmological production of superheavy magnetic monopoles, Phys. Rev. Lett. **43**, 1365 (1979).
[2] A. H. Guth, Inflationary universe: A possible solution to the horizon and flatness problems, Phys. Rev. D **23**, 347 (1981).
[3] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, 2000).
[4] A. Linde, Monopoles as big as a universe, Physics Letters B **327**, 208 (1994).
[5] N. Dadhich, K. Narayan, and U. A. Yajnik, Schwarzschild black hole with global monopole charge, Pramana **50**, 307 (1998).
[6] T. W. B. Kibble, Topology of Cosmic Domains and Strings, J. Phys. A **9**, 1387 (1976).
[7] K. A. Bronnikov, B. E. Meierovich, and E. R. Podolyak, Global monopole in general relativity, Journal of Experimental and Theoretical Physics **95**, 392 (2002).
[8] D. Harari and C. Loustó, Repulsive gravitational effects of global monopoles, Phys. Rev. D **42**, 2626 (1990).
[9] M. Barriola and A. Vilenkin, Gravitational Field of a Global Monopole, Phys. Rev. Lett. **63**, 341 (1989).
[10] S. M. Carroll, *Spacetime and Geometry* (Cambridge University Press, 2019).
[11] K. A. Bronnikov, Spherically symmetric false vacuum: No-go theorems and global structure, Phys. Rev. D **64**, 064013 (2001).
[12] E. Winstanley, On the existence of conformally coupled scalar field hair for black holes in (anti-)de sitter space, Foundations of Physics **33**, 111 (2003).
[13] C. A. Keller, *Conformal aspects of string theory*, Ph.D. thesis, Zurich, ETH (2008).