



Dicyclic group

In group theory, a **dicyclic group** (notation Dic_n or $\mathbf{Q}_{4n}, [1] \langle n, 2, 2 \rangle [2]$) is a particular kind of non-abelian group of order $4n$ ($n > 1$). It is an extension of the cyclic group of order 2 by a cyclic group of order $2n$, giving the name *di-cyclic*. In the notation of exact sequences of groups, this extension can be expressed as:

$$1 \rightarrow C_{2n} \rightarrow \text{Dic}_n \rightarrow C_2 \rightarrow 1.$$

More generally, given any finite abelian group with an order-2 element, one can define a dicyclic group.

Definition

For each integer $n > 1$, the dicyclic group Dic_n can be defined as the subgroup of the unit quaternions generated by

$$\begin{aligned} a &= e^{\frac{i\pi}{n}} = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \\ x &= j \end{aligned}$$

More abstractly, one can define the dicyclic group Dic_n as the group with the following presentation^[3]

$$\text{Dic}_n = \langle a, x \mid a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle.$$

Some things to note which follow from this definition:

- $x^4 = 1$
- $x^2 a^m = a^{m+n} = a^m x^2$
- if $l = \pm 1$, then $x^l a^m = a^{-m} x^l$
- $a^m x^{-1} = a^{m-n} a^n x^{-1} = a^{m-n} x^2 x^{-1} = a^{m-n} x$

Thus, every element of Dic_n can be uniquely written as $a^m x^l$, where $0 \leq m < 2n$ and $l = 0$ or 1. The multiplication rules are given by

- $a^k a^m = a^{k+m}$
- $a^k a^m x = a^{k+m} x$
- $a^k x a^m = a^{k-m} x$
- $a^k x a^m x = a^{k-m+n}$

It follows that Dic_n has order $4n$.^[3]

When $n = 2$, the dicyclic group is isomorphic to the quaternion group Q . More generally, when n is a power of 2, the dicyclic group is isomorphic to the generalized quaternion group.^[3]

Properties

For each $n > 1$, the dicyclic group Dic_n is a non-abelian group of order $4n$. (For the degenerate case $n = 1$, the group Dic_1 is the cyclic group C_4 , which is not considered dicyclic.)

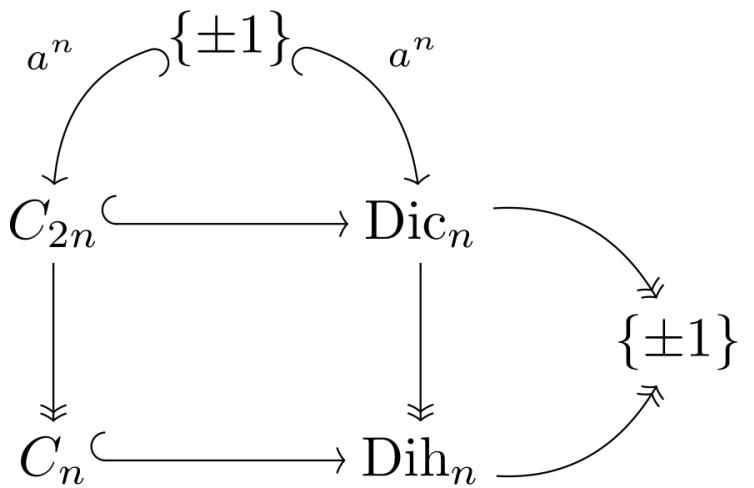
Let $A = \langle a \rangle$ be the subgroup of Dic_n generated by a . Then A is a cyclic group of order $2n$, so $[\text{Dic}_n : A] = 2$. As a subgroup of index 2 it is automatically a normal subgroup. The quotient group Dic_n / A is a cyclic group of order 2.

Dic_n is solvable; note that A is normal, and being abelian, is itself solvable.

Binary dihedral group

The dicyclic group is a binary polyhedral group — it is one of the classes of subgroups of the Pin group $\text{Pin}_-(2)$, which is a subgroup of the Spin group $\text{Spin}(3)$ — and in this context is known as the **binary dihedral group**.

The connection with the binary cyclic group C_{2n} , the cyclic group C_n , and the dihedral group Dih_n of order $2n$ is illustrated in the diagram at right, and parallels the corresponding diagram for the Pin group. Coxeter writes the *binary dihedral group* as $\langle 2,2,n \rangle$ and *binary cyclic group* with angle-brackets, $\langle n \rangle$.



There is a superficial resemblance between the dicyclic groups and dihedral groups; both are a sort of "mirroring" of an underlying cyclic group. But the presentation of a dihedral group would have $x^2 = 1$, instead of $x^2 = a^n$; and this yields a different structure. In particular, Dic_n is not a semidirect product of A and $\langle x \rangle$, since $A \cap \langle x \rangle$ is not trivial.

The dicyclic group has a unique involution (i.e. an element of order 2), namely $x^2 = a^n$. Note that this element lies in the center of Dic_n . Indeed, the center consists solely of the identity element and x^2 . If we add the relation $x^2 = 1$ to the presentation of Dic_n one obtains a presentation of the dihedral group Dih_n , so the quotient group $\text{Dic}_n / \langle x^2 \rangle$ is isomorphic to Dih_n .

There is a natural 2-to-1 homomorphism from the group of unit quaternions to the 3-dimensional rotation group described at quaternions and spatial rotations. Since the dicyclic group can be embedded inside the unit quaternions one can ask what the image of it is under this homomorphism.

The answer is just the dihedral symmetry group Dih_n . For this reason the dicyclic group is also known as the **binary dihedral group**. Note that the dicyclic group does not contain any subgroup isomorphic to Dih_n .

The analogous pre-image construction, using $\text{Pin}_+(2)$ instead of $\text{Pin}_-(2)$, yields another dihedral group, Dih_{2n} , rather than a dicyclic group.

Generalizations

Let A be an abelian group, having a specific element y in A with order 2. A group G is called a **generalized dicyclic group**, written as **Dic(A, y)**, if it is generated by A and an additional element x , and in addition we have that $[G:A] = 2$, $x^2 = y$, and for all a in A , $x^{-1}ax = a^{-1}$.

Since for a cyclic group of even order, there is always a unique element of order 2, we can see that dicyclic groups are just a specific type of generalized dicyclic group.

The dicyclic group is the case $(p, q, r) = (2, 2, n)$ of the family of binary triangle groups $\Gamma(p, q, r)$ defined by the presentation:[1] (https://groupprops.subwiki.org/wiki/Binary_von_Dyck_group)

$$\langle a, b, c \mid a^p = b^q = c^r = abc \rangle.$$

Taking the quotient by the additional relation $abc = e$ produces an ordinary triangle group, which in this case is the dihedral quotient $\text{Dic}_n \rightarrow \text{Dih}_n$.

See also

- binary polyhedral group
 - binary cyclic group, $\langle n \rangle$, order $2n$
 - binary tetrahedral group, $2T = \langle 2, 3, 3 \rangle$, [2] order 24
 - binary octahedral group, $2O = \langle 2, 3, 4 \rangle$, [2] order 48
 - binary icosahedral group, $2I = \langle 2, 3, 5 \rangle$, [2] order 120

References

1. Nicholson, W. Keith (1999). *Introduction to Abstract Algebra* (2nd ed.). New York: John Wiley & Sons, Inc. p. 449. ISBN 0-471-33109-0.
 2. Coxeter&Moser: Generators and Relations for discrete groups: $\langle l, m, n \rangle : R^l = S^m = T^n = RST$
 3. Roman, Steven (2011). *Fundamentals of Group Theory: An Advanced Approach*. Springer. pp. 347–348. ISBN 9780817683016.
- Coxeter, H. S. M. (1974), "7.1 The Cyclic and Dicyclic groups", *Regular Complex Polytopes*, Cambridge University Press, pp. 74–75 (<https://books.google.com/books?id=9BY9AAAAIAAJ&pg=PA74>).

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External links

- Dicyclic groups on GroupNames (<http://groupnames.org/#?dicyclic>)

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