

FINITE GROUPS WITH MANY INVOLUTIONS

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ABSTRACT. It is shown that a finite group in which more than $3/4$ of the elements are involutions must be an elementary abelian 2-group. A group in which exactly $3/4$ of the elements are involutions is characterized as the direct product of the dihedral group of order 8 with an elementary abelian 2-group.

1. INTRODUCTION

It is a standard exercise in an introductory algebra class to show that if G is a group such that $x^2 = 1$ for all $x \in G$, then G is abelian. It follows that if G is also finite, then it is an elementary abelian 2-group.

We will show that in fact any finite group in which more than $3/4$ of its elements are involutions must satisfy the same conclusion, that is, it must be an elementary abelian 2-group. Further, we will also characterize a group for which the proportion of involutions is exactly $3/4$ as the direct product of a dihedral group of order 8 with an elementary abelian 2-group.

Although none of the results in this paper depend on computer calculation, explorations using the GAP program [3] were important in formulating the results.

2. DEFINITIONS AND EXAMPLES

A good reference for basic group theory is Aschbacher's text [1]. In particular, Section 45 contains results on the number of involutions in a finite group and additional references to the literature.

Let G denote a finite group, written multiplicatively with identity element 1. We refer to any element $x \in G$ such that $x^2 = 1$, including the identity element, as an *involution*. We write $J(X)$ for the set of involutions in a subset X of a group, and write $j(X)$ for its cardinality $|J(X)|$. We also find it convenient to define the invariant $\alpha(G) = j(G)/|G|$ representing the proportion of involutions in the group G . Note that $\alpha(G) \in (0, 1]$.

Proposition 2.1. *If G is a finite abelian group, then $J(G)$ is an elementary abelian subgroup of G , and $j(G)$ is a power of 2, dividing $|G|$.*

Proof. One easily verifies that $J(G)$ is closed under inversion in general and under the group operation when G is abelian. It is then a subgroup. Since every element has order 2, it is an elementary abelian 2-group, and its order, $j(G)$ is therefore a power of 2 dividing $|G|$. \square

Corollary 2.2. *If G is a finite abelian group and $\alpha(G) > 1/2$, then G is an elementary abelian 2-group.* \square

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In general, of course, $J(G)$ is not a subgroup of G . Dihedral groups form the most important class of non-abelian groups with many involutions.

Example 2.3. If $G = D_{2n}$, the dihedral group of order $2n$, then

$$j(G) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n+2 & \text{if } n \text{ is even} \end{cases}.$$

In particular

$$\alpha(D_{2n}) = \begin{cases} \frac{1}{2} + \frac{1}{2n} & \text{if } n \text{ is odd} \\ \frac{1}{2} + \frac{1}{n} & \text{if } n \text{ is even} \end{cases}$$

and $\alpha(D_{2n}) \leq 3/4$, unless $n = 2$ and the dihedral group is actually elementary abelian.

3. PRELIMINARY RESULTS

We record some basic, useful, facts about counting involutions in finite groups.

Lemma 3.1. *If $G = H \times K$, then $J(G) = J(H) \times J(K)$, $j(G) = j(H) \times j(K)$, and $\alpha(G) = \alpha(H) \times \alpha(K)$.*

Proof. One simply observes that a pair (h, k) in G is an involution if and only if both h and k are involutions. \square

Lemma 3.2. *If G is a finite group with a normal subgroup H , then $j(G) \leq |H| \times j(G/H)$, and $\alpha(G) \leq \alpha(G/H)$.*

Proof. Clearly each involution of G maps to an involution (perhaps trivial) in G/H , and over any involution of G/H there are at most $|H|$ involutions in G . The result follows. \square

Lemma 3.3. *If G is a finite group with a central subgroup H , then $j(G) \leq j(G/H)j(H)$, and $\alpha(G) \leq \alpha(G/H)\alpha(H)$.*

Proof. As before, each involution $x \in G$ maps to an involution $\bar{x} \in G/H$. Over the involution \bar{x} of G/H are the elements of the form xh , $h \in H$. Since H is central, an element of the form xh is an involution if and only if h is an involution. The result follows. \square

Lemma 3.4. *If G is a group expressed as a semidirect product $NQ \cong N \rtimes Q$, then*

$$J(G) = \{nq : n \in N, q \in Q, q^2 = 1, qnq = n^{-1}\}$$

Proof. The proof is an easy calculation: If $q^2 = 1$ and q inverts n , then $(nq)^2 = nqnq = nqnq^{-1} = nn^{-1} = 1$. Conversely, if nq is an involution, then so is its image q in the quotient group Q . Therefore $1 = (nq)^2 = nqnq^{-1}$, implying that $qnq^{-1} = n^{-1}$. \square

Note that $G = N$ is the disjoint union of the cosets Nq , $q \in Q$, and a coset Nq contains involutions if and only if q is an involution, and the involutions in Nq are in one-to-one correspondence with the elements of N that are inverted by the action of q on N by conjugation.

Proposition 3.5. *If G is a finite group, $S < G$ a Sylow 2-subgroup with normalizer $N = N_G(S)$, then $\alpha(G) \leq |S|/|N|$.*

Proof. Every involution lies in some Sylow 2-subgroup and all Sylow 2-subgroups are conjugate. Therefore $J(G) \subset \bigcup_{g \in G} gSg^{-1}$. We need only take the union over a set of coset representatives of N in G . It follows that

$$j(G) \leq |G/N|(|S| - 1) + 1 \leq |G/N| \times |S| = |G||S|/|N|$$

Dividing through by $|G|$, the result follows. \square

Corollary 3.6 ([2], Corollary 4.4). *If G is a finite group with $\alpha(G) > 1/2$ and S is a Sylow 2-subgroup, then $N_G(S) = S$.* \square

Proposition 3.7. *If G is a finite group such that $j(G) > |G|/2$, then the center $Z(G)$ is an elementary abelian 2-group.*

The example of a dihedral group D_{2n} , n odd, shows that the center $Z(G)$ of such a group with $\alpha > 1/2$ could be trivial.

Proof. Let $Z = Z(G)$ and let S be a Sylow 2-subgroup of G . By Corollary 3.6, $N_G(S) = S$, so $Z \leq S$. Therefore Z is a 2-group; let $|Z| = 2^a$ and $j(Z) = 2^b$ for some $0 \leq b \leq a$. Now we have

$$\begin{aligned} |G|/2 &< j(G) \\ &\leq j(G/Z)j(Z) \quad (\text{by Lemma 3.3}) \\ &= j(G/Z)2^b \\ &\leq (|G|/|Z|)2^b. \end{aligned}$$

It follows that $|Z| < 2^{b+1}$. But Z is a 2-group of order at least $j(Z) = 2^b$, so $|Z| = j(Z)$ and $Z = J(Z)$. Since every element of Z is an involution, the result follows. \square

Note that the inequality in the hypothesis of the proposition must be strict: Consider $G = C_4 \times (C_2)^{n-2}$. The order of G is 2^n , and $j(G) = 2^{n-1} = |G|/2$, but $Z(G) = G$, which is not an elementary abelian 2-group.

We recall a crucial result for handling non-2-groups, which provided the starting point for the present investigation.

Theorem 3.8 (Edmonds [2], Theorem 4.1). *If $|G| = 2^n m$, m odd and $n \geq 1$, then $j(G) \leq 2^{n-1}(m+1)$.* \square

Corollary 3.9. *If $|G| = 2^n m$, m odd and $n \geq 1$, then $\alpha(G) \leq \frac{m+1}{2m} = \frac{1}{2} + \frac{1}{2m}$. If $m > 1$, then $\alpha(G) \leq \frac{2}{3}$.* \square

Edmonds [2] also proved that a finite group with $j(G) = 2^{n-1}(m+1)$ is the direct product of C_2^{n-1} and a group of order $2m$ of dihedral type, i.e., a split extension of an abelian group of order m by the cyclic group of order 2 acting by inversion.

4. GROUPS WITH $\alpha > 3/4$

Here we present our characterization of groups in which more than $3/4$ of the elements are involutions.

Theorem 4.1. *If G is a finite group and $\alpha(G) > 3/4$, then G is an elementary abelian 2-group.*

Proof. By Corollary 3.9, G must be of order 2^n for some n . We proceed by induction on n . We may suppose $n > 3$, as the case when $n \leq 3$ follows easily by inspection of the lists of groups of small order, at most 8. Because G is a 2-group, its center is nontrivial, and so contains an involution. Let a be a central involution in G . By Lemma 3.3, we have $j(G) \leq j(G/\langle a \rangle)j(\langle a \rangle)$; hence

$$j(G/\langle a \rangle) \geq j(G)/2 > 3|G|/8 = 3|G/\langle a \rangle|/4$$

Therefore, by the inductive hypothesis, $G/\langle a \rangle$ is an elementary abelian 2-group of order 2^{n-1} . So we have a central extension

$$C_2 \twoheadrightarrow G \twoheadrightarrow (C_2)^{n-1}$$

which we must show to be a direct product. It is a direct product if and only if G is elementary abelian.

If G is not an elementary abelian 2-group, there must be an element of order 4. Suppose G contains an element of order 4; call it x . Then, under the natural projection onto the quotient group $G/\langle a \rangle$, $\langle x \rangle$ is the inverse image of a cyclic subgroup of order 2. Because that cyclic subgroup must be normal in the abelian quotient, $\langle x \rangle \triangleleft G$. Consequently, conjugation by an element of G must send x to either x or x^{-1} , since those are the only two candidates inside $\langle x \rangle$. In particular, the length of the orbit of x under conjugation is at most 2. In fact, since by Proposition 3.7 there are no elements of order 4 in $Z(G)$,

$$|\text{cl}(x)| = [G : C(x)] = 2,$$

where $\text{cl}(x)$ denotes the conjugacy class of x , and $C(x)$ denotes its centralizer. Now note that

$$j(C(x)) = j(G) - j(G - C(x)) > 3|G|/4 - |G|/2 = |G|/4 = |C(x)|/2.$$

We can therefore apply Proposition 3.7 to $C(x)$. It follows that $Z(C(x))$ is an elementary abelian 2-group, contradicting the fact that x , an element of order 4, is necessarily in the center of its own centralizer. So G contains no element of order 4 and is thus an elementary abelian 2-group, as required. This completes the inductive step and hence the proof of the theorem. \square

The inequality in the hypothesis of the theorem is the best possible, as the results in the following section demonstrate.

5. GROUPS WITH $\alpha = 3/4$

The simplest group with $\alpha = 3/4$ is the dihedral group D_8 of order 8. We will show here that a finite group G with $\alpha(G) = 3/4$ is of the form $D_8 \times C_2^k$.

Lemma 5.1. *Suppose G is a finite group of order 2^n such that $\alpha(G) = 3/4$, and suppose there is a surjection $\pi : G \rightarrow D_8$. Then $K = \ker \pi \cong C_2^{n-3}$, the surjection π splits, so that G is a semidirect product $K \rtimes D_8$, and the semidirect product is in fact a direct product.*

Proof. Each involution of G maps to one of the six involutions of D_8 . Over each one of these involutions there are at most $|K| = 2^{n-3}$ involutions. In order to reach $\alpha(G) = 3/4$ it is necessary that all elements in the preimage of any of the involutions of D_8 must be involutions. We conclude that K consists entirely of involutions, so that $K \cong C_2^{n-3}$. Choose two involutions $x, y \in G$, mapping to two non-commuting

involutions in D_8 , which necessarily generate D_8 . Then in G the subgroup $\langle x, y \rangle$ is a dihedral group, which π maps onto D_8 . Now $(xy)^2$ maps to the central involution of D_8 . Therefore $(xy)^2$ is also an involution in G . We conclude that $\langle x, y \rangle$ defines a copy of D_8 , expressing G as a semidirect product $K \rtimes D_8$. Now an element (a, b) of such a semidirect product is an involution if and only if b is an involution and b conjugates a to its inverse. Since K is an elementary abelian 2-group, $a^{-1} = a$, so (a, b) is an involution if and only if b commutes with a . And for every involution $b \in D_8$ we must have (a, b) an involution for all $a \in K$. We conclude that all involutions in D_8 act trivially on K . Since D_8 is generated by involutions, the entire group D_8 acts trivially on K and the semidirect product is a direct product, as required. \square

Theorem 5.2. *Let G be a finite group such that $\alpha(G) = 3/4$. Then $|G| = 2^n$, $n \geq 3$, and $G \cong D_8 \times C_2^{n-3}$.*

Proof. By Corollary 3.9 it follows that G is a nonabelian 2-group, of order 2^n , say, where $n \geq 3$. We therefore proceed by induction on n . When $n = 3$ the result follows from the elementary classification of finite groups of order 8.

Let $Z = Z(G)$ be the nontrivial center of G , which by Proposition 3.7 is known to be an elementary abelian 2-group C_2^r . Consider the quotient group $Q = G/Z$. By Lemma 3.3 we know that $3|G|/4 = j(G) \leq j(Q)|Z|$. It follows that $j(Q) \geq 3|Q|/4$. If $j(Q) = 3|Q|/4$, then by induction on order, we may assume that $Q \cong D_8 \times C_2^t$, for some t . In this case G clearly admits a surjection onto D_8 , so Lemma 5.1 implies that G has the required properties.

If $j(Q) \neq 3|Q|/4$, then $j(Q) > 3|Q|/4$, and $Q \cong C_2^s$ where $r + s = n$, by Theorem 4.1. We have a central extension

$$C_2^r \twoheadrightarrow G \xrightarrow{\pi} C_2^s$$

where C_2^r is the full center of G , and $\pi: G \rightarrow C_2^s$ is the projection map.

Over any involution in Q there is either a full set of 2^r involutions (all obtained by multiplying one such involution by an element of Z) or there are no involutions. Thus $j(G) = |\pi(J(G))| \times 2^r$ and $|\pi(J(G))| = 3 \times 2^{s-2}$.

Since G is non-abelian and generated by $J(G)$, there must be two non-commuting involutions $x, y \in G$, which map nontrivially to $\bar{x}, \bar{y} \in Q$, such that $(xy)^2 \neq 1$. In particular, $\bar{x}\bar{y} \neq 1$. Then $a = (xy)^2$ is an element of Z . In particular $\bar{x}\bar{y}$ is an involution of Q that is not the image of an involution of G .

Then $\langle x, y \rangle \cong D_8$. and $D_8 \cap Z = \langle a \rangle$ for the central involution $a = (xy)^2$. Note that a is also a commutator $[x, y]$. Extend $\{a\}$ to a basis a, f_2, \dots, f_r of Z , as a vector space over the field of two elements, and let $Z' = \langle f_2, \dots, f_r \rangle$.

Suppose Z' is nontrivial. As a subgroup of the center, Z' is normal in G . Consider the quotient group $R = G/Z'$. Note that R is not abelian since the commutator a is not in Z' . Moreover, as before, $\alpha(R) \geq 3/4$. Since R is not abelian, we cannot have $\alpha(R) > 3/4$, by Theorem 4.1. Therefore $\alpha(R) = 3/4$. Since Z' is nontrivial, $|R| < |G|$, so that by induction on order the group R can be expressed as a direct product of D_8 and an elementary abelian 2-group. In particular, R , and hence G , maps onto D_8 . By Lemma 5.1 G itself can be expressed as a direct product of D_8 and an elementary abelian 2-group.

It remains to consider the case where $Z = \langle a \rangle$, and $R = G/Z \cong C_2^{n-1}$. We will show that under this assumption we necessarily have $n = 3$ and $G = D_8$. As above

we have the two involutions $x, y \in G$ that generate a D_8 and whose images $\bar{x}, \bar{y} \in R$ generate a summand C_2^2 of R . We aim to show that this D_8 is the whole group.

Now D_8 is normal in G with quotient S isomorphic to C_2^{n-3} . Let $\pi : G \rightarrow S$ be the quotient map. For any involution $t \in G$ such that $\pi(t) = \bar{t}$ is nontrivial in S , the subgroup $\langle x, y, t \rangle$ in G is a semidirect product $D_8 \rtimes C_2$, where $t \in C_2$ acts on D_8 by conjugation.

The automorphism group of D_8 is “well-known.” Compare Hall [4], Exercise 1, page 90. It is abstractly isomorphic to D_8 , although, of course, not every automorphism is an inner automorphism. Among the 8 automorphisms there are 6 involutions. These involutions each invert at most 6 elements of D_8 , since D_8 is nonabelian. Three of them, including the identity, invert 6 elements.

It follows that an element of S is hit by up to six involutions or by none. Now $\frac{3}{4}|G| = j(G) \leq |\pi(J(G))| \times 6 \leq 6 \times 2^{n-3}$. Therefore the inequality is an equality and $\pi(J(G)) = S$. That is, each element of S is hit by exactly 6 involutions of G .

We conclude that each such coset $\langle x, y \rangle t$ contains exactly 6 involutions and that for each element of S there is an involution t that maps to it. A coset $\langle x, y \rangle t$ contains involutions if and only if t is an involution. Therefore the set-theoretic difference $G - \langle x, y \rangle$ consists entirely of involutions, each of which acts on D_8 as one of the involution automorphisms that inverts 6 elements.

In particular $G - D_8$ consists entirely of involutions. But then

$$J(G) = J(D_8) \cup (G - D_8)$$

and

$$(3/4) \times 2^n = 6 + (2^n - 8)$$

It follows that $n = 3$ and in this case we already know the result. This completes the proof. \square

REFERENCES

- [1] Michael Aschbacher, *Finite Group Theory*, Second edition, Cambridge University Press, 2000.
- [2] Allan L. Edmonds, *The partition problem for equifacetal simplices*, Contributions to Algebra and Geometry, **50** (2009), 195-213.
- [3] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2005, www.gap-system.org.
- [4] Marshall Hall, Jr., *The Theory of Groups*, The Macmillan Co., New York, N.Y. 1959. Reprinted by the Chelsea Publishing Co., New York, 1976.

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