HW 8

4.1.1

$G_b = \{h \in G | h \cdot b = b\} = \{h \in G | h \cdot (g \cdot a) = (g \cdot a)\} = \{h \in G | hg \cdot a = (g \cdot a)\} = \{h \in G | g^{-1}hg \cdot a = a\}$

$gG_ag^{-1} = g\{h \in G | h \cdot a = a\}g^{-1} = \{ghg^{-1} \in G | h \cdot a = a\}$

let $ghg^{-1} = x$, then this is equal to $\{x \in G | g^{-1}xg \cdot a = a\}$.

$\cap_{g \in G} gG_ag^{-1} = \cap_{g \in G} G_{g \cdot a} = \cap_{b \in A} G_b$ where the last step follows because $g \cdot a$ ranges over all of A as g ranges over all of G. This last expression is the kernal of the action since it is the intersection of all the stabalizers.

4.1.2

The first part follows from 4.1.1 since $\sigma \cdot a = \sigma(a)$. Hence $\cap_{\sigma \in G} \sigma G_a \sigma^{-1}$ is the kernal of G which is the set consisting of the identity permutation.

4.1.3

Let G act on A by permutation, then this is a transitive action. Fix a. Since G is abelian, we have $a\sigma G_a \sigma^{-1} = G_a$. Hence for all $G_a = 1$. Hence if $\sigma \in G - \{1\}$ we have σ is not in G_a, hence $\sigma(a) \neq a$.

Choose $\sigma \in G - \{1\}$ and let $n = ord(\sigma)$. Then $\sigma, \sigma^2, \ldots \sigma^{n-1}$ are all non-identity. Let $a_0 \in A, a_1 = \sigma(a_0) \neq a_0, a_2 = \sigma(a_1) \neq a_1$. Since $a_2 = \sigma^2(a_0)$, we have $a_2 \neq a_0$. Continuing this way, if $a_n = \sigma^n(a_0)$ we have $a_0, a_1, \ldots, a_{n-1}$ are all distinct, hence σ is a single cycle $[a_0, a_1, \ldots, a_{n-1}]$. Claim: $n = |A|$. Suppose not, then there is some $a \in A$ not in $\{a_0, \ldots a_{n-1}\}$. Then $\sigma(a) = a$ which is a contradiction.

Claim: every permutation in G is a power of σ. This implies that $|G| = |A|$ since $ord(\sigma) = |A|$. Proof of claim: for any $\tau \in G$ we have $\sigma\tau = \tau\sigma$, hence $\sigma = \tau\sigma\tau^{-1}$. In cycle notation, $[a_0, a_1 \ldots] = [\tau(a_0), \tau(a_1), \ldots]$, hence $\tau(a_i) = a_{i+k}$ for some k where the indices are taken modulo n.

4.1.7a

The identity is clearly in G_B. Closure: if $\sigma, \tau \in G_B$ then $(\sigma\tau) \cdot B = \sigma(\tau(B)) = \sigma(B) = B$. Inverse: if $\sigma \in G_B$ then $\sigma(B) = B$ hence $B = \sigma^{-1}(B)$.

To show $\sigma \in G_a \implies \sigma \in G_B$, suppose $\sigma \in G_a$. Then $\sigma(a) = a$. Either $\sigma(B) = B$ or $\sigma(B) \cap B = 0$. But in the later case, $\sigma(a) = a \notin B$ which is a contradiction. Hence $\sigma(B) = B$.

4.1.7b

The partition covers A: let $b \in B, a \in A$. By transitivity there is a σ such that $\sigma \cdot b = a$, and $a \in \sigma(B)$. Hence a appears in a part.

The parts are disjoint: let $\sigma(B), \tau(B)$ be two parts. Suppose they intersect, then there is a $b \in B$ such that $\sigma(b) = \tau(b)$. Then $\sigma^{-1}\tau \cdot b = b$ hence $\sigma^{-1}\tau \cdot B \cap B \neq 0$ hence $\sigma^{-1}\tau \cdot B = B$, hence $\tau(B) = \sigma(B)$.

4.1.7c

Let B be a nontrivial block; then there exists $a, b, c \in A$ such that $a, c \in B, b \notin B$. Then with permutation $\sigma = [a, b], \sigma(B)$ and B have nontrivial intersection.

Let the vertices be $\{1, 2, 3, 4\}$ in clockwise order. Then $\{1, 3\}$ is a nontrivial block, since it is a diagonal and is either mapped to itself or to the other diagonal $\{2, 4\}$.

4.1.7d

$\Longleftarrow:$ We prove the contrapositivie. If G is imprimitive, it has a nontrivial block B and some $a \in B$. By part a, G_B contains G_a. G acts on the partition in part (b) (by acting on each of the elements of the parts) and hence by the orbit-stabalizer theorem, $|G_B|$ (considered as a stabalizer of this action) is strictly between $|G|$ and $|G_a|$.

$\Longrightarrow:$ We prove the contrapositive. By assumption there exists some $a \in A$ and some subgroup G' such that $G_a \subset G' \subset G$ and all the inclusions are strict. Let B be the set of elements of A fixed pointwise by G' (i.e. $B = \{b \in A | G'(b) = b\}$)... TBD

4.2.7a

This follows by Caley's theorem.

4.2.7b

Let $G = Q_8$ be isomorphic to a subgroup of S_n with n minimal, and suppose $n \leq 7$.

Then G act faithfully on X with $|X| \leq 7$ with induced homomorphism ϕ. Let $x \in X$ be arbitrary. Then $|G_x| = 1, 2, 4, 8$ since it $|G_x|$ divides $|G|$. If $|G_x| = 8, G$ acts faithfully on $X - \{x\}$, contradicting the minimality of n.

If $|G_x| = 1$ then the orbit of x has size 8, which is impossible. If $|G_x| = 4$ there are 3 choices for G_x (generated by i, j, k) all of which contain $\{-1, 1\}$ as a subgroup. If $|G_x| = 2$ then $G_x = \{-1, 1\}$ (the only subgroup of size 2).

Hence, $-1 \cdot x = x$. Since x was arbitrary, $\phi(-1)$ is the identity permutation, a contradiction.