HW 7

3.2.11

For each coset of H in K, choose a representative element, so that the set of representatives is $\{h_\alpha | \alpha \in A\}$, and the set of cosets is $\{h_\alpha H | \alpha \in A\}$. Similarly, for each coset of K in G, choose a representative element, so that the set of representatives is $\{k_\beta | \beta \in B\}$, and the set of cosets is $\{k_\beta K | \beta \in B\}$.

The set $\{k_\beta h_\alpha | \alpha \in A, \beta \in B\}$ is a set of representatives for the cosets of H in G, and none of the representatives are for the same coset. To see that every coset is represented, let $g \in G$, then $gK = k_\beta K$ for some $\beta \in B$. Hence $k_\beta^{-1} g \in K$, hence $k_\beta^{-1} gH = h_\alpha H$ for some $\alpha \in A$, hence $gH = k_\beta h_\alpha H$.

To see that none of the representatives are for the same coset, suppose $k_\alpha h_\beta H = k_{\alpha'} h_{\beta'} H$ for some $\alpha, \alpha' \in A, \beta, \beta' \in B$. This means $k_\alpha K = k_{\alpha'} K$ since $K = h_\beta HK = h_{\beta'} HK$. Hence $\alpha = \alpha'$ and hence $h_\beta H = h_{\beta'} H$. Hence $\beta = \beta'$.

We have $|G : H| = \{k_\beta h_\alpha | \alpha \in A, \beta \in B\}, |G : K| = \{k_\beta | \beta \in B\}, |K : H| = \{h_\alpha | \alpha \in A\}$, hence $|G : H| = |G : K||K : H|$.

3.3.1

Let $G = GL_n(F), S = SL_n(F), \phi : G \to S$ be defined by $f(m) = m/\det(m)$. This is a surjective mapping because $f(S) = S$. Furthermore $f^{-1}(m) = \{m, 2m, \dots (q-1)m\}$ hence f is a $q-1$-to-1 mapping.

3.3.3

Let G be a group, H normal in G, K a subgroup of G, $[G : H] = p$. From the second isomophism theorem we know that KH is a subgroup of G, and we have the lattice

$$\begin{tikzcd} G \arrow[d, dash, "a"] \\ KH \arrow[d, dash, "b"] \\ H \end{tikzcd}$$

let $[G : KH] = a, [KH : H] = b$. Since $p = ab$ we have $(a, b) = (1, p)$ or $(a, b) = (p, 1)$. In the first case, $KH = HK = G$. By the second isomophism theorem, $H \cap K$ is a normal subgroup of K and $K/H \cap K \sim KH/H$ hence $[K : H \cap K] = [KH : H] = p$. In the second case, we have $[KH : H] = 1$ hence $KH = H$ hence $K \subseteq H$.

3.3.7

By normality, $MN = \{mn | m \in M, n \in N\} = \{mm^{-1}n'm | m \in M, n' \in N\} = NM$.

$$\begin{tikzcd}[column sep=small]
 & G = MN = NM \arrow[dl, dash] \arrow[dr, dash] & \\
M \arrow[dr, dash] & & N \arrow[dl, dash] \\
 & I = M \cap N &
\end{tikzcd}$$

By the diamond isomophism theorem, we have $G/M \sim N/I$ and $G/N \sim M/I$. Hence it suffices to show that $G/I \sim N/I \times M/I$.

We have $M/I \times N/I = \{(mI, nI) | m \in M, n \in N\}$. Let $\phi : M/I \times N/I \to G/I$ be given by $\phi(mI, nI) = mnI$. This is well-defined as follows: suppose $(mI, nI) = (m'I, nI)$, that is $mI = m'I$, hence $mIn = m'In$, hence $mnI = m'nI$. A similar argument shows that the choice of representative for N/I does not matter.

This is a surjective group homomorphism (the homomorphism laws are easily verified) since any $g \in G$ can be written as $g = mn$ for some $m \in M, n \in N$, and $\phi(mI, nI) = gI$. Hence it suffices to show that this is an injective homomorphism. Suppose $mnI = m'n'I$, then $nn'^{-1}I = m^{-1}m'I$, and this coset must be I (since toherwise, it lies in $G - I$, which is composed of a disjoint union of a subset of M and a subset of N), hence $m^{-1}m'I = I$, hence $mI = m'I$. A similar argument shows that $nI = n'I$.