HW 3

1a.

It suffices to show that each cycle $(a_1, a_2, \dots a_k)$ is generated by 2-cycles. This is true because $(a_1, a_2, \dots a_k) = (a_1, a_2)(a_2, a_3) \dots (a_{k-1}, a_k)$, which we can see by having it operate on an arbitrary a_j from right to left. If $j \neq k$, the first 2-cycle that does not fix its argument is (a_j, a_{j+1}) which sends it to a_{j+1}. Now the next 2-cycle is (a_{j-1}, a_j) which fixes a_{j+1}; every other 2-cycle also fixes it since the indices are decreasing. If $j = k$, each 2-cycle reduces the index of j by one, and we end up with a_1.

1b.

The identity has 0 inversions and is even.

A transposition has an odd number of inversions. Proof: count the number of inversions modulo 2. Call a pair of indices a candidate inversion if its elements are strictly increasing; the inversions are in bijection with candidate inversions which are mapped by σ to a decreasing tuple. Let the transposition be (a, b) with $a < b$. If $[i, j]$ is a candidate inversion, suppose $\{i, j\}$ is disjoint with $\{a, b\}$, then $[i, j]$ is mapped to $[\sigma(i), \sigma(j)] = [i, j]$ so $[i, j]$ does not correspond to an inversion. Hence it suffices to consider candidate inversions where at least one of the indices is a or b.

First consider the candidate inversions where exactly one of the incides is a or b. If the other index is $< a$ then the pair is $[i, a]$ which is mapped to $[i, b]$, so the candidate does not correspond to an inversion since $i < b$. Similarly if the other index is $> b$, we contribute 0 to the sum.

Consider the candidate inversions where exactly one of the indices is a or b and the other index is between a and b. For every k where $a < k < b$, we have the two distinct candidate inversions $[a, k]$ and $[k, b]$ which fall in this category, and they are mapped to $[b, k]$ and $[k, a]$. These correspond to 2 inversions and contribute 0 to the sum.

We are left with the candidate inversions $[a, b]$; this is mapped to $[b, a]$ hence it corresponds to an inversion.

1c.

The proof is similar to 1b; we want to compare the number of inversions in the list $[\sigma(1), \sigma(2), \dots \sigma(n)]$ before and after swapping two elements. WLOG those two elements are $\sigma(i)$ and $\sigma(j)$ for some $i < j$. Among the candidate inversions $[i', j']$ if $\{i', j'\}$ is disjoint from $T = \{\sigma(i), \sigma(j)\}$ then it is unaffected by the transposition; of those where exactly one of them lies in T, they are paired up because every k such that $i < k < j$ leads to two candidate inversions, and the pair either causes the number of inversions to increase by two (if $\sigma(i) < \sigma(j)$) or decrease by two (otherwise); and there is one candidate whose elements are exactly T, where the number of inversions increases or decreases by one.

1d.

Every permutation can be written as a product of transpositions (by 1a). By 1b and 1c, an even permutation can be written as a product of an even number of transpositions, and an odd permutation can be written as a product of an odd number of transpositions.

Let Q, R be odd permutations and consider $P = QR$. We can write $Q = q_1 q_2 \dots q_k$ (k odd) and $R = r_1 r_2 \dots r_l$ (l odd), means we can write $P = q_1 q_2 \dots q_k r_1 r_2 \dots r_l$ which has $k+l$ transpositions, hence P is even by 1b and 1c.

The proofs for the other 3 cases of the parity of Q, R are identical.

Section 1.4

7

It suffices to show that the number of singular matrices is $p^3 + p^2 - p$, since the total number of matrices is p^4. The singular matrices either have first row equal to 0 or not. Of the matrices with first row 0, there are p^2 choices for the second row (no restrictions). There are $p^2 - 1$ nonzero first rows, and each of them corresponds to p singular matrices (one for each multiple of the first row, since there are p distinct multiples). Hence the total number is $p^2 + p(p^2 - 1) = p^3 + p^2 - p$.

Section 1.6

Lemma: if $\phi : G \to H$ is a group isomoprhism, then the inverse of ϕ (taken as a set function, which exists because ϕ is bijective) is a group isomoprhism.

Proof: it is required to show that $\phi^{-1}(xy) = \phi^{-1}(x)\phi^{-1}(y)$ for all $x, y \in H$. We have $\phi(\phi^{-1}(xy)) = \phi(\phi^{-1}(x)\phi^{-1}(y))$ because the LHS is xy and the RHS is $\phi(\phi^{-1}(x)\phi^{-1}(y)) = (\phi(\phi^{-1}(x)))(\phi(\phi^{-1}(y))) = xy$ where the first equality holds because ϕ is a group homomorphism. The result follows because ϕ is injective.

1

We prove part a by induction on n. We will use $n = 1$ as the base case, which is trivial. For the inductive step, we have $\phi(x^{n+1}) = \phi(x^n x) = \phi(x^n)\phi(x) = \phi(x)^n \phi(x) = \phi(x)^{n+1}$.

Additionally we will prove that $\phi(e) = e$. Let $\phi(e) = x$, then $x = \phi(e) = \phi(ee) = x^2$; cancelling, we have $x = e$.

For part b, it is required to prove that $\phi(x^{-1}) = \phi(x)^{-1}$, equivalently $\phi(x)\phi(x^{-1}) = e$. This follows because $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e) = e$.

The extension to $\mathbb{Z}$ then follows by applying part (a) to the element x^{-1}, since $\phi(x^{-n}) = \phi((x^{-1})^n) = (\phi(x)^{-1})^n = \phi(x)^{-n}$.

2

Let $n = ord(x)$. Then $\phi(x)^n = \phi(x^n) = \phi(e) = e$, hence $ord(\phi(x)) \leq n$. Similarly, by the lemma, $ord(\phi(x)) \geq n$, hence $ord(\phi(x)) = n$.

The trivial homomorphism $\phi : G \to H$ that sends every element to the identity shows that this is not true for just homomorphisms, since there exists groups with different number of elements of order 4 (e.g. Z_4 has two such element and the quaternion group has 6)

3

Let H be abelian. Then for all $x, y \in G$ we have $xy = \phi(\phi^{-1}(x)\phi^{-1}(y)) = \phi(\phi^{-1}(y)\phi^{-1}(x)) = yx$. The proof for when G is abelian is the same proof applied to the isomoprhism ϕ^{-1}.

For a sujective homomorphism $\phi : G \to H$, if G is abelian then H is abelian. Let $h_1, h_2 \in H$. We know that there are elements g_1, g_2 satisfying $g_1 = h_1, g_2 = h_2$, and applying ϕ to the identity $g_1 g_2 = g_2 g_1$ leads to $h_1 h_2 = h_2 h_1$.

6

$\mathbb{Z}$ is generated by 1. If $\mathbb{Q}$ is isomorphic to $\mathbb{Z}$ then $\phi(1)$ also generates $\mathbb{Q}$ since any element $n \in \mathbb{Z}$ can be mapped to $n\phi(i)$. Let $q = \phi(1)$.

It is easy to see that $q \neq 0$. Then the element $\frac{q}{2}$ is not in $\mathbb{Q}$ because the equation $nq = \frac{q}{2}$ has no integer solutions in n.

14

We will first prove exercise 26. First, $e \in H$; since H is nonempty, take some $x \in H$, we have $e = xx^{-1}$ which is a product of two elements in H. The fact that the group operation in H is associative and that e is an identity under that operation follow by considering those elements as elements of G.

It remains to show that $\ker \phi$ is closed under the group operation and under inverse. First, suppose $x, y \in \ker G$, which means $\phi(x) = \phi(y) = e$. Then $\phi(xy) = \phi(x)\phi(y) = e^2 = e$ hence $\phi(xy) \in \ker G$. Secondly, $\phi(x^{-1}) = \phi(x)^{-1} = e^{-1} = e$, hence $x^{-1} \in \ker G$.

ϕ is injective $\iff$ $\ker G = \{e\}$. $\implies$: let $x \in \ker G$, that is $\phi(x) = e = \phi(e)$; applying the injectivity of ϕ to the equation $\phi(x) = \phi(e)$ yields $x = e$. $\impliedby$: suppose $\phi(x) = \phi(y)$ for some $x, y \in G$. We can rewrite this as $e = \phi(x)^{-1}\phi(y) = \phi(x^{-1}y)$, hence $x^{-1}y \in \ker G$, hence $x^{-1}y = e$, hence $x = y$.

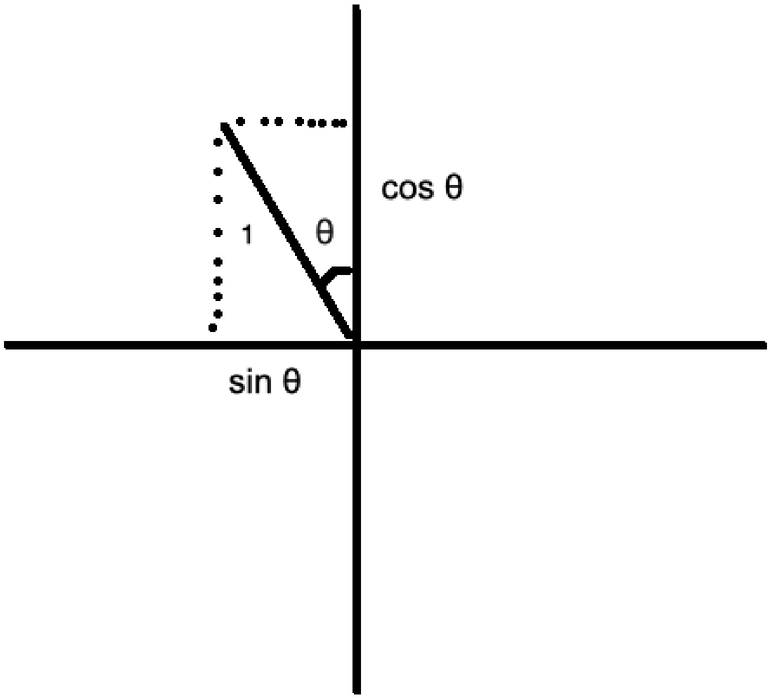
17

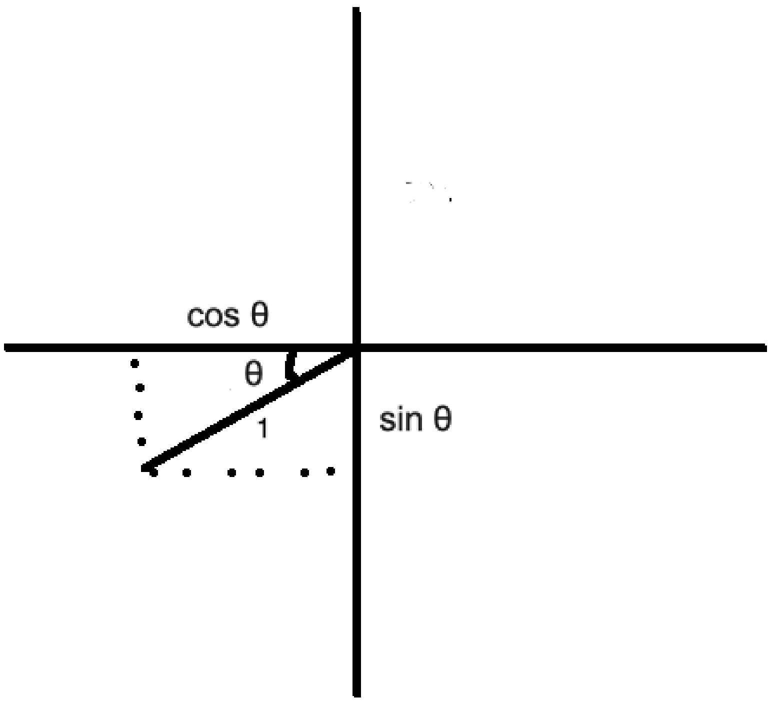
Let ϕ be the map. Suppose G is abelian. Then for all $x, y \in G$ we have $\phi(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = \phi(x)\phi(y)$ where the 3rd equation follows because G is abelian.

Conversely suppose ϕ is a homomorphism and let $x, y \in G$. Then $xy = (y^{-1}x^{-1})^{-1} = \phi(y^{-1}x^{-1}) = \phi(y^{-1})\phi(x^{-1}) = yx$

25a

Let M be the matrix in the question. Since the question presupposes that a rotation is a linear transformation, it suffices to show that M is a rotation on any basis of $\mathbb{R}^2$; we will take $\{[-1,0],[1,0]\}$ which M maps to $\{[-\cos\theta, -\sin\theta], [-\sin\theta, \cos\theta]\}$. A geometric proof is attached.





25b

We have to show that the generator relations are satisfied.

$\phi(r)^n = I$; since $\phi(r)$ is a ccw rotation by θ, $\phi(r)^n$ is a ccw rotation by $n\theta = 2\pi$, which is the identity map.

$\phi(s)^2 = I$; this is a simple computation.

The relation $\phi(r)\phi(s)\phi(r) = \phi(s)$ can be verified by wolfram alpha `https://www.wolframalpha.com/input?i=%5B%5Bcos+x%2C+-sin+x%5D%2C+%5Bsin+x%2C+cos+x%5D%5D+%5B%5B0%2C+1%5D%2C+%5B1%2C+0%5D%5D+%5B%5Bcos+x%2C+-sin+x%5D%2C+%5Bsin+x%2C+cos+x%5D%5D`

25c

It suffices to show that the $2n$ elements $\phi(r)^i\phi(s)^j$ for $i \in [0, n), j \in \{0, 1\}$ are distinct matrices. First, if two such elements have different j, they have different determinants, since $\det \phi(r) = 1, \det \phi(s) = -1$. Next, we will prove that all the $\{\phi(r)^i\}_i$ are distinct, which is sufficient since $\phi(s)$ is injective when considered as a linear map. If we consider the first column of each of those matrices and map the column $[a, b]$ to the complex number $a + bi$ the $\{\phi(r)^i\}_i$ are mapped to $[0, z, z^2 \dots z^{n-1}]$ where $z = e^{i\theta}$. These are the n (distinct) roots of unity since $\theta = 2\pi/n$, hence the first columns (and hence matrices) are distinct.