Manfrino, Ortega, Delgato Inequalities A Mathematical Olympiad Approach

1.25 (Difference of AM and GM)

Let $p = \sqrt{a}, q = \sqrt{b}$. We have

$$\begin{aligned} p \leq \frac{1}{2}(p+q) \leq q \\ \frac{1}{2}\frac{p+q}{p} \leq 1 \leq \frac{1}{2}\frac{p+q}{q} \\ \frac{1}{2}\frac{p^2-q^2}{p} \leq p-q \leq \frac{1}{2}\frac{p^2-q^2}{q} \\ \frac{1}{4}\frac{p^2-q^2}{p^2} \leq (p-q)^2 \leq \frac{1}{4}\frac{p^2-q^2}{q^2} \end{aligned}$$

(Can also be proven by direct computation)

Lesson: AM-GM can be factorized

1.33 $x^4 + y^4 + 8 \geq 8xy$

Looks like a special case of 4-term AMGM, $x^4 + y^4 + p^4 + q^4 \geq 4xypq$. Comparing coefficients, we have $p^4 + q^4 = 8, pq = 2$, hence $p = q = \sqrt{2}$

1.38 $a > 1 \implies a^n - 1 > n(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}})$

$$\begin{aligned} a^n - 1 &= (a-1)(1 + a + a^2 + \dots a^{n-1}) \\ &\geq (a-1)(a^{1+2+\dots n-1})^{1/n} \cdot n \\ &= n(a-1)a^{\frac{n-1}{2}} \\ &= RHS \end{aligned}$$

1.39 $(1+a)(1+b)(1+c) \implies 8 \implies abc \leq 1$

$1 = \frac{1+a}{2}\frac{1+b}{2}\frac{1+c}{2} \geq \sqrt{abc}$. Now square.

1.40 $\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab + bc + ca$

By muirhead, since $(3, -1, 0) > (1, 1, 0)$. Can also expand to get rid of the -1.

1.41 $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$

By muirhead, since $(2, 2, 0) > (2, 1, 1)$.

1.42

1.58 $x^4 + y^4 + z^2 \geq \sqrt{8}xyz$

LHS $= x^4 + y^4 + \frac{z^2}{2} + \frac{z^2}{2}$. Now apply 4-term AMGM.

Good example of weighted AMGM