Manfrino, Ortega, Delgato Inequalities A Mathematical Olympiad Approach

1.25 (Difference of AM and GM)

Let $p = \sqrt{a}, q = \sqrt{b}$. We have

$$\begin{aligned} p \leq \frac{1}{2}(p+q) &\leq q \\ \frac{1}{2}\frac{p+q}{p} \leq 1 &\leq \frac{1}{2}\frac{p+q}{q} \\ \frac{1}{2}\frac{p^2-q^2}{p} \leq p-q &\leq \frac{1}{2}\frac{p^2-q^2}{q} \\ \frac{1}{4}\frac{p^2-q^2}{p^2} \leq (p-q)^2 &\leq \frac{1}{4}\frac{p^2-q^2}{q^2} \end{aligned}$$

(Can also be proven by direct computation)

Lesson: AM-GM can be factorized

1.33 $x^4 + y^4 + 8 \geq 8xy$

Looks like a special case of 4-term AMGM, $x^4 + y^4 + p^4 + q^4 \geq 4xypq$. Comparing coefficients, we have $p^4 + q^4 = 8, pq = 2$, hence $p = q = \sqrt{2}$

1.38 $a > 1 \implies a^n - 1 > n(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}})$

$$\begin{aligned} a^n - 1 &= (a-1)(1 + a + a^2 + \ldots a^{n-1}) \\ &\geq (a-1)(a^{1+2+\ldots n-1})^{1/n} \cdot n \\ &= n(a-1)a^{\frac{n-1}{2}} \\ &= RHS \end{aligned}$$

1.39 $(1+a)(1+b)(1+c) \implies 8 \implies abc \leq 1$

$1 = \frac{1+a}{2}\frac{1+b}{2}\frac{1+c}{2} \geq \sqrt{abc}$. Now square.

1.40 $\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab + bc + ca$

By cyclic (see section), since $(3, -1, 0) > (1, 1, 0)$. Can also expand to get rid of the -1.

1.41 $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$

By muirhead, since $(2, 2, 0) > (2, 1, 1)$.

1.51 $a + b + c = 1 \implies \left(\frac{1}{a} + 1\right)\left(\frac{1}{b} + 1\right)\left(\frac{1}{c} + 1\right) \geq 64$

This is equivalent to $(1+a)(1+b)(1+c) \geq 64abc$. Using $1 = a+b+c$, $(2a+b+c)(a+2b+c)(a+b+2c) \geq 64abc$. Apply 4-term AMGM, e.g. the first factor is $\geq (a^2bc)^{\frac{1}{4}}$.

Note the starting inequality is sharp when $a = b = c$. 3-term doesn't work since it is not sharp in that case.

1.52 $a + b + c = 1 \implies \left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8$

Equivalent to $(b + c)(a + c)(a + b) \geq 8abc$ which holds by AMGM on each factor.

1.53

Unsolved...

1.54

Let $p = \frac{1}{1+a}$. Then the inequality is equivalent to 1.52.

1.55

Equivalent to $HM(a, b) + HM(b, c) + HM(a, c) \leq 3 * AM(a, b, c)$. Rewrite the RHS as $AM(a, b) + AM(b, c) + AM(a, c)$.

Same strategy used to show that $3 * AM(x, y, z) \geq GM(x, y) + GM(y, z) + GM(x, z)$.

1.56

1.57

1.58 $x^4 + y^4 + z^2 \geq \sqrt{8}xyz$

LHS $= x^4 + y^4 + \frac{z^2}{2} + \frac{z^2}{2}$. Now apply 4-term AMGM.

1.59

Using the substitution $x = 1 + p$ we have $(1+p)^2 p + (1+q)^2 q \geq 8pq$. We have $(1+p)^2 \geq 4p$ by AMGM. Then $4(p^2 + q^2) \geq 8pq$ which holds by AMGM.

We know we first have to apply AMGM to the $1 + p$ term by dimensional analysis.

Good example of weighted AMGM

IMO 2000 Q2

$$A + B + C \leq AB + BC + CA$$

Cyclic $(1, 0, 0) > (p, q, 0)$

Let $a, b, c > 0, p + q = 1$. Define $(r, s, t) = \frac{1}{3} \sum_{cyc} a^r b^s c^t$. Then $(1, 0, 0) > (p, q, 0)$.

Proof: we have $pa + qb \geq a^p b^q, pb + qc \geq b^p + c^q, pc + qa \geq c^p a^q$ by weighted AMGM. Summing them gives $(p + q)(a + b + c) \geq (p, q, 0)$.