HW 9

4.1.3

Let G act on A by permutation, which by assumption is a transitive and faithful action. Fix a. Since G is abelian, for all $\sigma \in G$ we have $a\sigma G_a \sigma^{-1} = G_a$. By 4.1.2, $\cap_{\sigma \in G} G_a = \{e\}$, hence $G_a = e$, which means $\{\sigma \in G | \sigma(a) = a\} = \{e\}$, which means for all $\sigma \in G$ with $\sigma \neq e, \sigma \notin G_a$, which means $\sigma(a) \neq a$.

Choose some $g \in G$ and define the map $\phi : A \to G$ by $\phi(g) = g \cdot a$. ϕ is surjective since G acts transitively on A. ϕ is injective. Proof: suppose $g \cdot a = h \cdot a$. Then $(g^{-1}h) \cdot a = a$, hence $g^{-1}h = e$, hence $g = h$. Hence ϕ is a bijection, so $|A| = |G|$.

4.2.9

Let p be a prime, G have prime power order, $H \leq G$ with $[G : H] = p$. By corollary 4.2.5, since p is the smallest prime dividing $|G|, H$ is normal.

Let G be a group of order p^2. By Cauchy's theorem, there is an element of order p, say r. Then the index of $\langle r \rangle$ is $p^2/p = p$. By the above theorem, $\langle r \rangle$ is normal in G.

4.2.10

Let G be a non-abelian group of order 6. By Cauchy's theorem, G has an element of order 2, say s. Let $H = \langle s \rangle$ and consider the action of G on the left cosets of H, which we can label H, H', H'', with associated homomorphism $\phi_H \leq S_{\{H,H',H''\}} \cong S_3$. The stababilzer of H is $G_H = H$, and all the stabalizers are conjugate to G_H.

By theorem 4.2.3.3, $\ker \pi_H$ is some subgroup of H, so it is etiher either $\{e\}$ or H. If it is $\{e\}$ then ϕ_H is injective and there is an injective homomorphism $G \to S_3$; since both sides have size 6 we have $G \cong S_3$.

Otherwise, since $\ker \phi_H = H$ is the intersection of every stabalizer, every stabalizer must be H, so H is a normal subgroup. It remains to be shown that in this case, $G \cong C_6$.

Proof 1: Let $g \in G$ be arbitrary. Then $gH = Hg$, which means $\{g, gs\} = \{g, sg\}$, which means $gs = sg$, hence H commutes with every element of G. By Cauchy's theorem, G has an element of order 3, say r, with $rs = sr$. The subgroup $\langle r, s \rangle$ has order 6 (since it contains subgroups of order 2 and 3), with presentation $\langle r, s | rs = sr, r^3 = s^2 = e, \ldots \rangle$ where $\ldots$ indicates some additional relations. Hence there is a surjective homomorphism from $C_2 \times C_3 \to \langle r, s \rangle$; since both sides have size 6 we have $G \cong C_2 \times C_3$.

Proof 2: By Cauchy's theorem, there exists an element of order 3, say r, and $\langle r \rangle$ is a normal subgroup of G (since it has index 2). Hence G has two normal subgroups N_1, N_2 of size 2 and 3. By the diamond isomorphism theorem, $G' = N_1 N_2$ is a subgroup of G of size 6 (since its size must be divisible by 2 and 3 but be less than 6). $N_1 \cap N_2 = \{e\}$ by order considerations. Hence by exercise 3.3.7, $G = G' \cong N_1 \times N_2$.

4.2.11

In the cycle representation of $\pi(x)$, consider the cycle containing e. The cycle must be $(e, x, x^2, \dots x^{n-1})$ by definition of n.

Similarly, consider an arbitrary $g \in G$ with $g \notin \{e, x, x^2 \dots x^{n-1}\}$, and consider the cycle containing g. The elements $\{g, xg, x^2 g, \dots x^{n-1} g\}$ are all distinct since $x^a g = x^b \iff x^a = x^b$. Hence the cycle in $\pi(x)$ containing g must be $(g, xg, x^2 g, \dots x^{n-1} g)$.

From proposition 3.5.25, $\pi(x)$ is odd $\iff$ the number of cycles of even length is odd $\iff$ n is even and m is odd. For the last biimplication, either n is odd (in which case there are no cycles of even length) or n is even (in which case there are m cycles of even length).

4.2.12

Let $H = \{g \in G | \pi(g) \text{ is even}\}$. It is easy to check that H is a subgroup of G. Let $k \in G$ such that $\pi(k)$ is odd. Then the bijective map $\phi : G \to G$ given by $\phi(g) = kg$ sends H to $G - H$ and $G - H$ to H, hence $2|H| = |G|$.

4.2.13

Let G be as in the question and π be the left regular representation. By Cauchy's theorem, there is an element of order 2, say s. Since $|s|$ is even and $|G|/|s| = k$ is odd, by 4.2.11 $\pi(s)$ is an odd permutation. By 4.2.12, G has a subgroup of index 2.

4.3.13

Let G be a finite group with $|G| = n$. If $n = 2$ then G has 2 conjugacy classes. Now suppose G has 2 conjugacy classes, and let G act on itself by conjugation; the nonidentity elements form an orbit of size $n - 1$. Since the size of the orbit divides the size of the group, $n - 1$ divides n, hence $n = 2$.

4.3.23

The normalizer $N_G(M)$ is a subgroup of G containing M, that is $M \leq N_G(M) \leq G$. By maximality of M, either $N_G(M) = M$ or $N_G(M) = G$.

If M is a maximal subgroup of G and M is not normal, then $N_G(M) \neq G$ hence $N_G(M) = M$. Consider the action of G on $P(G)$ by conjugation. The stabilizer of M is $N_G(M) = M$, hence the orbit of M has size $[G : M]$. Each orb (i.e. element in the image of the action, contained in the orbit) is a subgroup conjugate to M and hence the number of nonidentity elements is $|M| - 1$. Hence the number of nonidentity elements of G contained in conjugates of M is at most $[G : M](|M| - 1)$.

4.3.24

Since the subgroup lattice is a finite partial order, G is contained in some maximal subgroup $M \neq G$. It suffices to show that $G \neq \cup_{g \in G} gMg{-}1$, since $gHg{-}1 \subseteq gMg{-}1$.

The number of nonidentity elements of G contained in conjugates of M is $|G| - 1 \le [G : M](|M| - 1) = \frac{|G|}{|M|}(|M| - 1) = |G| - \frac{|G|}{|M|} \le |G| - 2$, a contradiction. Here the first inequality follows from 4.3.23 and the second from $|G|/|M| \ge 2$ since M is a proper subgroup of G.

4.3.26

?