HW 11

5.2.14a

The group operation is associative: it is required to prove that for $\alpha, \beta, \gamma \in \hat{G}, (\alpha\beta)\gamma = \alpha(\beta\gamma)$. This follows because $(\alpha(z)\beta(z))\gamma(z) = \alpha(z)(\beta(z)\gamma(z))$ as an equation in $\mathbb{C}$. Similarly, commutativity of multiplication in $\mathbb{C}$ implies commutativity of the group operation in $\hat{G}$.

The identity element is the constant function $\epsilon(z) = 1$ for all $z \in G$, because for any $\alpha \in \hat{G}$, $\alpha(z)\epsilon(z) = \alpha(z)$.

Given $\alpha \in \hat{G}$, the inverse α^{-1} is given by $\alpha^{-1}(z) = \alpha(z)^{-1}$ for all $z \in G$. This is well-defined because $\alpha(z) \neq 0$ for all $z \in G$ (0 is not a root of unity), and $\alpha(z)\alpha(z)^{-1} = \alpha(z)\alpha(z)^{-1} = 1$, hence $\alpha\alpha^{-1} = \epsilon$.

5.2.14b

Let $\mathbb{C}'$ be the group of roots of unity in $\mathbb{C}$. Let j be the square root of -1 in $\mathbb{C}$.

Write $G = \langle x_1 \rangle \times \langle x_2 \rangle \dots \langle x_r \rangle$ and let n_i be the order of x_i, and let $\chi_i \in \hat{G}$ be as in the hint. Let F be the function from $G \to \hat{G}$ given by $F(x_1^{e_1}, x_2^{e_2} \dots x_n^{e_n}) = \chi_1^{e_1}\chi_2^{e_2} \dots \chi_n^{e_n}$. It is easy to check that F is an injective homomorphism.

We prove that F is surjective. Let $\phi \in \hat{G}$, in other words $\phi : \langle x_1 \rangle \times \langle x_2 \rangle \dots \langle x_r \rangle \to \mathbb{C}'$ is a homomorphism. Since the x_i generate G, ϕ is uniquely determined by its action on each x_i. $\phi(x_i)$ is a root of unity of order dividing n_i, so we can write $\phi(x_i) = \chi_i^{e_i}$ for some e_i since $\chi_i(x_i) = e^{\frac{2\pi j}{n_i}}$ is a primitive n_i-th root of unity. Hence $\phi = F(x_1^{e_1}, x_2^{e_2} \dots x_n^{e_n})$.