HW 5

1

a

Let ϕ be a map between the left cosets and the right cosets of N as follows: given a left coset gN, the map is conjugation by g^{-1}.

To show that ϕ is well-defind, suppose $gN = hN$; this means $g^{-1}h \in N$. Under conjugation by g^{-1}, they are mapped to Ng and $g^{-1}hNg$ respectively, and it is required to show that $Ng = g^{-1}hNg$, which follows because $g^{-1}hN = N$.

b

Let $N \leq G$ be such a subgroup of index 2; by theorem 6, it suffices to show that $gN = Ng$ for all $g \in G$, and for this it suffices to show that the bijection ϕ defined in 1a is an identity. We have $\phi(N) = eNe = N$. Hence ϕ maps the remaining coset N^C (the complement of N in G) to itself.

c

Among S_n the even permutations are closed under composition. Furthermore, an even permutation can be written as $g = t_1 t_2 \dots t_{2n}$ where each t_i is a transposition; hence $g^{-1} = t_{2n}^{-1} \dots t_1^{-1}$ is even as well. Hence the even permutations form a subgroup.

Let $A_n \leq S_n$ be the subgroup of even permutations. Choose a transposition t; then every element of tA_n is odd. Furthermore, every odd permutation g can be written as $g = t(tg) \in tA_n$, hence $S_n = A_n \sqcup tA_n$ and A_n has index 2.

d

Consider the set $N = \{e, (1,2)(3,4), (1,3)(2,4)(1,4)(2,3)\}$ consisting of all the permutations of cycle structure $2 + 2$ together with the identity. By brute force computation, this is a subgroup since

$$
\begin{aligned}
(1,2)(3,4)(1,3)(2,4) &= (1,4)(2,3) \\
(1,2)(3,4)(1,4)(2,3) &= (1,3)(2,4) \\
(1,3)(2,4)(1,4)(2,3) &= (1,2)(3,4)
\end{aligned}
$$

Since the conjugate pnp^{-1} is n with relabelled elements, it has the same cycle structure as n; hence conjugation by $g \in S_n$ fixes N. (To be more rigorous, it suffices to verify this for all 6 transpositions in S_n since conjugation by p can be written as a sequence of conjugations by transpositions).

2

C_2 as quotient: let $N = \langle r \rangle = e, r^1, r^2 \dots r^n$. Since $|G : N| = 2, N$ is normal and the quotient Dic_n/N has order 2; hence it must be isomorphic to C_2.

D_n as quotient: consider the subgroup $E = \{1, s^2\}$ (this is a subgroup since $s^4 = 1$). To show that E is normal, it suffices to show that s^2 commutes with every element of D_n, and it suffices to show that it commutes with every generator. s^2 commutes with s is trivial, and $s^2 = r^{\frac{n}{2}}$ hence it commutes with r as well.

In terms of the matrix representation, this is $E = \{I, -I\}$, and this commutes with everything in the group since it consists of diagonal matrices.

Hence Dic_n/E is a quotient group with the same order as D_n; it remains to find an isomorphic copy of D_n in it; in fact $R = rE$ and $S = sE$ satisfies the correct relations. $\langle R \rangle = \{R^1, R^2 \dots R^{\frac{n}{2}} = E\}$ where the inner equality holds because $r^{\frac{n}{2}}\{e, r^{\frac{n}{2}}\} = \{r^{\frac{n}{2}}, e\}$ and the list has no duplicates since $r^a E = r^b E \iff \{r^a, r^{a+\frac{n}{2}}\} = \{r^b, r^{b+\frac{n}{2}}\} \iff \{[a], [a + \frac{n}{2}]\} = \{[b], [b + \frac{n}{2}]\}$ where $[\cdot]$ denotes congruence classes modulo n. Hence $r^a E = r^b E \iff$ either $a = b \pmod{N}$ or $a = b + \frac{n}{2} \pmod{n}$; in either case, $a = b \pmod{\frac{n}{2}}$. Similarly, $\langle S \rangle = \{e, s^2 E\}$. Lastly, $RSRS = rsrsE = s^2 E = E$.