

HW 9

4.1.3

4.2.9

Let p be a prime, G have prime power order, $H \leq G$ with $[G : H] = p$. By corollary 4.2.5, since p is the smallest prime dividing $|G|$, H is normal.

Let G be a group of order p^2 . By Cauchy's theorem, there is an element of order p , say r . Then the index of $\langle r \rangle$ is $p^2/p = p$. By the above theorem, $\langle r \rangle$ is normal in G .

4.2.10

Let G be a non-abelian group of order 6. By Cauchy's theorem, G has an element of order 2, say s . Let $H = \langle s \rangle$ and consider the action of G on the left cosets of H , which we can label H, H', H'' , with associated homomorphism $\phi_H \leq S_{\{H, H', H''\}} \cong S_3$. The stabilizer of H is $G_H = H$, and all the stabilizers are conjugate to G_H .

By theorem 4.2.3.3, $\ker \pi_H$ is some subgroup of H , so it is either $\{e\}$ or H . If it is $\{e\}$ then ϕ_H is injective and there is an injective homomorphism $G \rightarrow S_3$; since both sides have size 6 we have $G \cong S_3$.

Otherwise, since $\ker \phi_H = H$ is the intersection of every stabilizer, every stabilizer must be H , so H is a normal subgroup. It remains to be shown that in this case, $G \cong C_6$.

Proof 1: Let $g \in G$ be arbitrary. Then $gH = Hg$, which means $\{g, gs\} = \{g, sg\}$, which means $gs = sg$, hence H commutes with every element of G . By Cauchy's theorem, G has an element of order 3, say r , with $rs = sr$. The subgroup $\langle r, s \rangle$ has order 6 (since it contains subgroups of order 2 and 3), with presentation $\langle r, s | rs = sr, r^3 = s^2 = e, \dots \rangle$ where \dots indicates some additional relations. Hence there is a surjective homomorphism from $C_2 \times C_3 \rightarrow \langle r, s \rangle$; since both sides have size 6 we have $G \cong C_2 \times C_3$.

Proof 2: By Cauchy's theorem, there exists an element of order 3, say r , and $\langle r \rangle$ is a normal subgroup of G (since it has index 2). Hence G has two normal subgroups N_1, N_2 of size 2 and 3. By the diamond isomorphism theorem, $G' = N_1 N_2$ is a subgroup of G of size 6 (since its size must be divisible by 2 and 3 but be less than 6). $N_1 \cap N_2 = \{e\}$ by order considerations. Hence by exercise 3.3.7, $G = G' \cong N_1 \times N_2$.

4.2.11

In the cycle representation of $\pi(x)$, consider the cycle containing e . The cycle must be $(e, x, x^2, \dots, x^{n-1})$ by definition of n .

Similarly, consider an arbitrary $g \in G$ with $g \notin \{e, x, x^2, \dots, x^{n-1}\}$, and consider the cycle containing g . The elements $\{g, xg, x^2g, \dots, x^{n-1}g\}$ are all distinct since $x^a g = x^b \iff x^a = x^b$. Hence the cycle in $\pi(x)$ containing g must be $(g, xg, x^2g, \dots, x^{n-1}g)$.

From proposition 3.5.25, $\pi(x)$ is odd \iff the number of cycles of even length is odd $\iff n$ is even and m is odd. For the last biimplication, either n is odd (in which case there are no cycles of even length) or n is even (in which case there are m cycles of even length).

4.2.12

Let $H = \{g \in G \mid \pi(g) \text{ is even}\}$. It is easy to check that H is a subgroup of G . Let $k \in G$ such that $\pi(k)$ is odd. Then the bijective map $\phi : G \rightarrow G$ given by $\phi(g) = kg$ sends H to $G - H$ and $G - H$ to H , hence $2|H| = |G|$.

4.2.13

Let G be as in the question and π be the left regular representation. By Cauchy's theorem, there is an element of order 2, say s . Since $|s|$ is even and $|G|/|s| = k$ is odd, by 4.2.11 $\pi(s)$ is an odd permutation. By 4.2.12, G has a subgroup of index 2.

4.3.13

Let G be a finite group with $|G| = n$. If $n = 2$ then G has 2 conjugacy classes. Now suppose G has 2 conjugacy classes, and let G act on itself by conjugation; the nonidentity elements form an orbit of size $n - 1$. Since the size of the orbit divides the size of the group, $n - 1$ divides n , hence $n = 2$.

4.3.23

4.3.24

4.3.26