HW 4

1

1.7.18

Reflexive: for all $a \in A$ we have $a \sim a$ since $a = ea$.

Symmetric: let $a, b \in A$ such that $a \sim b$, that is, $a = hb$. Then $b = h^{-1}a$. Since $h^{-1} \in H$ this means $b \sim a$.

Transitive: let $a, b, c \in A$ such that $a \sim b, b \sim c$. This means $a = h_1 b, b = h_2 c$ for some $h_1, h_2 \in H$. Then $a = h_1 \cdot (h_2 \cdot c) = (h_1 h_2) \cdot c$ hence $a \sim c$ because $h_1 h_2 \in H$.

1.7.19

Let ϕ be the map.

Injective: suppose $\phi(h_1) = \phi_(h_2)$, that is $h_1 x = h_2 x$. By multiplying by x^{-1} on the right, we have $h_1 = h_2$.

Surjective: let $y \in O$ be some element in the codomain of ϕ. This means $x \sim y$, that is there exists some h with $x = hy$. Then $\phi(h^{-1}) = hh^{-1}y = y$. Here $\phi(h^{-1})$ is well-defined because $h^{-1} \in H$.

Let O_x be the orbit of x. For all $x, y \in G$ we have $|O_x| = |H| = |O_y|$. Since the orbits partition G we can write $G = H_{i_1} \sqcup H_{i_2} \ldots \sqcup H_{i_k}$ which means $|G| = k|H|$.

2

a

Let D_{14} act faithfully on A where $n = |A| < 7$. The group action is equivalent to a group homomorphism $D_{14} \to S_A$. Since the action is faithful, this is an injective homomorphism (since distinct elements of D_{14} are mapped to distinct permutations). By Cayley's theorem we have S_A is a subgroup of S_6; hence there is an injective homomorphism $D_{14} \to S_6$; the range of this homomorphism is a subgroup of S_6 isomorphic to D_{14}. By Lagrange's theorem the order of this subgroup divides $|S_6| = 6!$, that is $14|6!$, a contradiction.

b

We wish to construct an isomorphic copy of D_{12} in S_5, say with generating permutations r, s satisfying the usual relations. We have $ord(r) = 6$ hence r must decompose into a 3-cycle multiplied by a 2-cycle;

WLOG $r = (1, 2, 3)(4, 5)$. If we take $s = (4, 5)$ we have $rsrs = (1, 2, 3)(4, 5)(4, 5)(1, 2, 3)(4, 5)(4, 5) = e$ and $s^2 = e$.

Concretely, the action can be defined as follows: $r^i s^j \cdot x = (1, 2, 3)^i (4, 5)^{i+j} x$ for $i \in [0, 6), j \in [0, 1)$.

c

By the argument above, if D_{2n} acts faithfully on a set with k elements then there is an injective homomorphism $D_{2n} \to S_k$, in particular $n|k!$. For $(n, k) = (7, 7)$ this is a contradiction; the smallest factorial which is a multiple of $2 \cdot 7$ is $7!$. In general for $(n, k) = (p, p)$ for p prime this leads to a contradiction. However for $(n, k) = (6, 5)$ this is fine, since $6|5!$.