HW 12

5.4.14

We show that $G = D \times U$ as an internal direct product. D and U are both contained in G, and every element of d commutes with every element of u. Given $g \in G$ whose diagonal elements are g_0, we have $g_0 > 0$ since otherwise, the determinant (which is the product of the diagonal elements for an upper triangular matrix) is 0, which contractis $G \subseteq GL_n(F)$. We can write $g = du$ where $d = g_0 I \in D, u \in U$.

Claim: this decomposition is unique. Suppose $g = d_1u_1 = d_2u_2$. Let $d_i = k_iI$ for some scalars k_i. Then $k_1u_1 = k_2u_2$. The upper left element of the LHS is k_1 and that of the RHS is k_2, hence $k_1 = k_2$, hence $d_1 = d_2$ and $u_1 = u_2$. taking determinants, $|d_1| = |d_2|$ hence they have equal diagonal elements, hence $d_1 = d_2$, hence $u_1 = u_2$.

The map $\phi : H \times K \to G$ given by $(h, k) \mapsto hk$ is a homomorphism. It maps (e, e) to e, it maps $(h, k)^{-1} = (h^{-1}, k^{-1})$ to to $(hk)^{-1} = h^{-1}k^{-1}$, and it maps $(h, k)(h', k') = (hh', kk')$ to $hh'kk' = (hk)(h'k')$. There is a map $G \to H \times K$ given by the unique decomposition. This is an inverse map since the composed map maps $hk \to (h, k) \to hk$, hence ϕ is an isomorphism.

5.5.5a

By definition, $G = H \rtimes K$ where $H = Z_2 \times Z_2$ and $K = \mathrm{Aut}(H)$. Let $Z_2 = \{e, x, y, xy\}$. Any automorphism of H fixes e, hence K is a subgroup of $S_{\{x,y,xy\}}$. In fact, since Z_2 is generated by x, y with $x^2 = y^2 = (xy)^2 = e$ which is symmetric in x, y, xy, any permutation of x, y, xy is an automorphism of H (e.g. the permutation $(x\ y)$ gives the homomorphism $e \mapsto e, x \mapsto y, y \mapsto x, xy \mapsto xy$). Hence $K = S_{\{x,y,xy\}}$.

$|G| = |H||K| = 4 \times 6 = 24$.

5.5.5b

Notation: given $v \in H, \sigma \in K$ let the action of K on v that defines the semidirect product be written v^σ, and let elements of K be written as $v\sigma$ where $v \in H, \sigma \in K$. Then the multiplication rule for G can be written as $v\sigma_1 w\sigma_2 = vw^{\sigma_1}\sigma_1\sigma_2$.

Consider H as a subgroup of K. There are 4 left-cosets which are $C = \{K, xK, yK, xyK\}$. G acts on C by left multiplication; denote this action by $g \cdot wK$ where $g \in G, w \in H$. This action has a permutation representation $\pi : G \to S_C$. Write $g = v\sigma$. Then $g \cdot wK = v\sigma \cdot wK = vw^\sigma\sigma K = vw^\sigma K$.

Notation: when writing elements of $\mathrm{im}\pi$ as permutations, we shall replace vK with v.

Considering the elements g for which $v = e$, we see that $g \cdot wK = w^\sigma K$. Hence $\mathrm{im}\pi$ contains $S_{\{x,y,xy\}}$.

Considering the elements of g for which σ is the identity automorphism, we see that $g \cdot wK = vwK$. Hence $\mathrm{im}\pi$ contains $(e\ x)(y\ xy)$ as the image of x, $(e\ y)(x\ xy)$ as the image of y, and $(e\ xy)(x\ y)$ as the image of xy (these are the permutations that correspond to the left regular representation of the action of H on itself by left multiplication)

We know imπ contains $(x\ y)$. Conjugating this by $(e\ x)(y\ xy)$ we see that it contains $(e\ xy)$. Conjugating this by elements of $S_{\{x,y,xy\}}$ we see that it contains $(e\ x)$ and $(e\ y)$. Hence imπ contains all the tranpositions of S_C, hence im$\pi = S_C$, which means there is a surjective homomorphism $G \to S_4$; since $|G| = |S_4|$, this is an isomorphism.

5.5.16

I will use C_k for the cyclic group of order k.

Considering C_8 as the additive group of $\mathbb{Z}/8\mathbb{Z}$, every automorphism of C_8 is multiplication by a unit of $\mathbb{Z}/8\mathbb{Z}$, i.e. the automorphisms are precisely multiplication by 1, 3, 5, or 7. Every homomorphism $C_2 \to C_8$ is determined by the image of the nonidentity element of C_2, and an automorphism of C_8 is a valid target if it is self-inverse. It is easy to check that each of those four automorphisms is self-inverse (since $-1, 3, -3$ all square to 1 modulo 8).

Multiplication by one: this is the semidirect product $C_2 \rtimes C_8$ where the action of C_2 on C_8 is trivial, hence it is the direct product $C_2 \times C_8$.

Multiplication by $7 = -1$: this is the semidirect product $C_2 \rtimes C_8$ where the action of C_2 on C_8 is inversion, hence it is the dihedral group.

Multiplication by 3: use exponential notation for the action. The multiplication rule is $\sigma_1\tau_1\sigma_2\tau_2 = \sigma_1\sigma_2\tau_1^{\sigma_2}\tau_2$. Let $\sigma_1 = e, \tau_2 = e$ and let τ, σ be generators of C_8 and C_2. The multiplication rule simplifies to $\tau\sigma = \sigma^\tau\tau = \sigma^3\tau$. Hence $\tau\sigma^3 = \sigma^3\tau\sigma^2 = \sigma^6\tau\sigma = \sigma^9\tau = \sigma\tau$, hence this group has the presentation $\langle \sigma, \tau \mid \sigma^8 = \tau^2 = e, \sigma\tau = \tau\sigma^3 \rangle$ which is the same presentation as 2.5.11.

Multiplication by 5: let u, v be generators of C_2 and C_8; similar to the above, we have $uv = v^5u$. Hence $uv^5 = v^5uv^4 = v^{10}uv^3 = \ldots = v^{25}u = vu$ which is the same presentation in 2.5.14.

5.5.18

Let H be a group; by Cayley's theorem it is a subgroup of S_H, namely the subgroup of permutations $P = \{\pi_h : h' \mapsto hh' \mid h \in H\}$. Take $G = N_{S_H}(H)$; then H is a normal subgroup of G by construction. Let σ be an automorphism of H; in particular σ is a permutation of H so we can treat σ as an element of S_H. The conjugation π_h^σ is the map $\sigma \circ \pi_h \circ \sigma^{-1}$ which maps h' to $\sigma(h\sigma^{-1}(h')) = \sigma(h)h'$ hence it is the map $\pi_{\sigma(h)}$. Hence conjugation by σ is an automorphism of P (this also shows $\sigma \in G$).

5.5.22a

We apply the recognition theorem (theorem 12):

$U \trianglelefteq G$: when multiplying upper-triangular matrices, the diagonal entries are multiplied with each other. Hence, $\begin{pmatrix} a & \dots & \dots \\ 0 & b & \dots \\ & & \ddots \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & \dots & \dots \\ 0 & b^{-1} & \dots \\ & & \ddots \end{pmatrix}$ and conjugation by an upper triangular matrix leaves the diagonal values unchanged.

$U \cap D = \{I\}$: let $\mathcal{M}$ be in the intersection. $\mathcal{M}$ is a diagonal matrix, and all of its diagonal elements are 1, hence it is the identity matrix.

$UD = G$: note that diagonal matrices in $GL_n(F)$ are invertible, with

$$\begin{pmatrix} a & 0 & \dots \\ 0 & b & \dots \\ & & \ddots \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & 0 & \dots \\ 0 & b^{-1} & \dots \\ & & \ddots \end{pmatrix}$$

where every diagonal element is nonzero, since otherwise the determinant would be 0. Given an arbitrary matrix $\mathcal{G} = \begin{pmatrix} a & \dots & \dots \\ 0 & b & \dots \\ & & \ddots \end{pmatrix}$ let $\mathcal{D} = \begin{pmatrix} a & 0 & \dots \\ 0 & b & \dots \\ & & \ddots \end{pmatrix}$; we have $\mathcal{G}\mathcal{D}^{-1}$ is an upper triangular matrix with diagonal elements 1; call this matrix $\mathcal{U}$. Then $\mathcal{G} = \mathcal{U}\mathcal{D}$.

5.5.22b

Elements of U are of the form $\mathcal{U} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where we can take $x \in F$ to be the image of the isomorphism. Similarly elements of D are of the form $\mathcal{D} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where $a, b \in F^{\times}$.

The action is given by conjugation of $\mathcal{U}$ by $\mathcal{D}$; we have $\mathcal{D}\mathcal{U}\mathcal{D}^{-1} = \begin{pmatrix} 1 & a^{-1}xb \\ 0 & 1 \end{pmatrix}$. Hence the action is given by $(a, b) \circ x = a^{-1}bx$.