HW 7

3.3.1

Let $G = GL_n(F)$, $S = SL_n(F)$, $\phi : G \to S$ be defined by $f(m) = m/\det(m)$. This is a surjective mapping because f(S) = S. Furthermore $f^{-1}(m) = \{m, 2m, \dots (q-1)m\}$ hence f is a q-1-to-1 mapping.

3.3.3

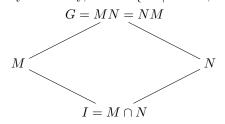
Let G be a group, H normal in G, K a subgroup of G, [G:H]=p. From the second isomorphism theorem we know that KH is a subgroup of G, and we have the lattice



let [G:KH]=a,[KH:H]=b. Since p=ab we have (a,b)=(1,p) or (a,b)=(p,1). In the first case, KH=HK=G. By the second isomorphism theorem, $H\cap K$ is a normal subgroup of K and $K/H\cap K\sim KH/H$ hence $[K:H\cap K]=[KH:H]=p$. In the second case, we have [KH:H]=1 hence KH=H hence $K\subseteq H$.

3.3.7

By normality, $MN = \{mn | m \in M, n \in N\} = \{mm^{-1}n'm | m \in M, n' \in N\} = NM$.



By the diamond isomorphism theorem, we have $G/M \sim N/I$ and $G/N \sim M/I$. Hence it suffices to show that $G/I \sim N/I \times M/I$.

We have $M/I \times N/I = \{(mI, nI) | m \in M, n \in N\}$. Let $\phi: M/I \times N/I \to G/I$ be given by $\phi(mI, nI) = mnI$. This is well-defined as follows: suppose (mI, nI) = (m'I, nI), that is mI = m'I, hence mIn = m'In, hence mII = m'nI. A similar argument shows that the choice of representative for N/I does not matter.

This is a surjective group homomorphism (the homomorphism laws are easily verified) since any $g \in G$ can be written as g = mn for some $m \in M, n \in N$, and $\phi(mI, nI) = gI$. Hence it suffices to show that this is an injective homomorphism. Suppose mnI = m'n'I, then $nn'^{-1}I = m^{-1}m'I$, and this coset must be I (since toherwise, it lies in G - I, which is composed of a disjoint union of a subset of M and a subset of N), hence $m^{-1}m'I = I$, hence mI = m'I. A similar argument shows that nI = n'I.