HW 4

1

1.7.18

Reflexive: for all $a \in A$ we have $a \sim a$ since $a = ea$.

Symmetric: let $a, b \in A$ such that $a \sim b$, that is, $a = hb$. Then $b = h^{-1}a$. Since $h^{-1} \in H$ this means $b \sim a$.

Transitive: let $a, b, c \in A$ such that $a \sim b, b \sim c$. This means $a = h_1 b, b = h_2 c$ for some $h_1, h_2 \in H$. Then $a = h_1 \cdot (h_2 \cdot c) = (h_1 h_2) \cdot c$ hence $a \sim c$ because $h_1 h_2 \in H$.

1.7.19

Let ϕ be the map.

Injective: suppose $\phi(h_1) = \phi(h_2)$, that is $h_1 x = h_2 x$. By multiplying by x^{-1} on the right, we have $h_1 = h_2$.

Surjective: let $y \in O$ be some element in the codomain of ϕ. This means $x \sim y$, that is there exists some h with $x = hy$. Then $\phi(h^{-1}) = hh^{-1}y = y$. Here $\phi(h^{-1})$ is well-defined because $h^{-1} \in H$.

Let O_g be the orbit of g. The bijection given by ϕ tells us that $|O_g| = |H|$. Since the orbits partition G we can write $G = O_{g_1} \sqcup O_{g_2} \ldots \sqcup O_{g_k}$ for some subset $\{g_1, g_2 \ldots g_k\} \subseteq G$ which means $|G| = \sum_k |O_{g_k}| = k|H|$.

2

a

Let D_{14} act faithfully on A where $n = |A| < 7$. The group action is equivalent to a group homomorphism $D_{14} \to S_A$. Since the action is faithful, this is an injective homomorphism (since distinct elements of D_{14} are mapped to distinct permutations). By Cayley's theorem we have S_A is a subgroup of S_6; hence there is an injective homomorphism $D_{14} \to S_6$; the range of this homomorphism is a subgroup of S_6 isomorphic to D_{14}. By Lagrange's theorem the order of this subgroup divides $|S_6| = 6!$, that is $14|6!$, a contradiction.

b

We wish to construct an isomorphic copy of D_{12} in S_5, say with generating permutations r, s satisfying the usual relations. We have $ord(r) = 6$ hence r must decompose into a 3-cycle multiplied by a 2-cycle;

WLOG $r = (1,2,3)(4,5)$. If we take $s = (1,2)$ we have $rsrs = (1,2,3)(4,5)(1,2)(1,2,3)(4,5)(1,2) = (1,2,3)(1,2)(1,2,3)(1,2) = (1,2,3)(2,1,3) = e$ and $s^2 = e$.

Concretely, the action can be defined as follows: $r^i s^j \cdot x = (1,2,3)^i (1,2)^j (4,5)^i x$ for $i \in [0,6), j \in [0,1)$.

c

By an argument similar to 2a, for $n > 2$ if D_{2n} acts faithfully on a set with k elements then there is an injective homomorphism $D_{2n} \to S_k$, in particular $n|k!$. For $(n,k) = (7,6)$ this is a contradiction; the smallest factorial which is a multiple of $2 \cdot 7$ is $7!$. In general for $(n,k) = (p, p-1)$ for p prime this leads to a contradiction. However for $(n,k) = (6,5)$ there is no problem, since $6|5!$.

3

Lemma: let $H \leq G, h \in G$ and let $\phi : H \to G$ be conjugation by h. Claim: ϕ is an injective group homomorphism. Proof: for $h_1, h_2 \in H, \phi(h_1)\phi(h_2) = hh_1h^{-1}hh_2h^{-1} = hh_1h_2h^{-1} = \phi(h_1h_2)$. Furthermore $\phi(h_1) = \phi(h_2) \iff hh_1h^{-1} = hh_2h^{-1} \iff h_1 = h_2$.

a

Let G_x, G_y be two stabalizers.

For every $h \in G, a, b \in A$ such that $h \cdot a = b$ let $\phi : G_a \to G$ be conjugation by h, which is a group homomorphism.

Claim: $\text{im}\phi = G_b$. Proof: $g \in G_a \iff g \cdot a = a \iff hgh^{-1} \cdot b = b \iff hgh^{-1} \in G_b$ where the second biimplication follows because $hgh^{-1} \cdot b = hg \cdot a = h \cdot a = b$.

Diagramatically, the stable action on b corresponds to travelling to a, performing a stable action, and then traveling back to b.

Hence G_a and G_b are isomorphic as long as a and b are in the same orbit; hence if G acts transitively on A, all the stabalizers are isomorphic.

b

We can use D_8 acting on the seven binary squares (slide 5 of `https://www.math.clemson.edu/~macaule/classes/s24_math4120/slides/math4120_slides_chapter05_h.pdf`)

$\text{Stab}(0,0,0,0) = D_8$ but $r \notin \text{Stab}(0,1,1,0)$.

4

a

WLOG we can prove this for transitive actions, since the stabalizers G_o of $o \in O$ in the action of G on O are exactly the same as the stabalizers G'_o of $o \in S$ in the action of G on S.

Fix s and consider the set-function $\phi : G \to O$ defined by $\phi(g) = g \cdot s$. This function is surjective since O is transitive. For each $o \in O$ consider the preimage $\phi^{-1}(o) = \{g \in G | g \cdot s = o\}$. There exists $g_{o \to s} \in G$ such that $g_{o \to s} \cdot o = s$. Define $g_{s \to o} = g_{o \to s}^{-1}$; it is easy to check that $g_{s \to o} \cdot s = o$.

The set $g_{o \to s}\phi^{-1}(o) = \{g_{o \to s}x | x \in \phi^{-1}(o)\}$ has the same cardinality as $\phi^{-1}(o)$ (since left-multiplication in G is invertible) and each element satisfies $g_{o \to s}x \cdot s = g_{o \to s} \cdot o = s$, hence $g_{o \to s}\phi^{-1}(o) \subseteq G_s$. Also every $g \in G_s$ can be written as $g_{o \to s} g_{s \to o} g$ where $g_{s \to o} g \in \phi^{-1}(o)$.

Hence $g_{o \to s}\phi^{-1}(o) = G_s$ and ϕ is a surjection onto O where every preimage has size G_s, hence $|G| = |G_s||O|$.

b

Let $\phi : G \times G \to G$ be an action of G on itself by conjugation, that is $g \cdot a = gag^{-1}$. This is a group action because for $h, g, a \in G$ we have $h \cdot (g \cdot a) = h \cdot gag^{-1} = hgag^{-1}h^{-1} = hg \cdot a$. A conjugacy class in an orbit under this action, hence by (a) it divides $|G|$.

c

Suppose $|G| = p^n$ for some prime p and $Z(G) = \{e\}$. For all $g \in G$ let $[g]$ be the set of conjugates of g. We have $g \in Z(G) \iff \forall a \in G, aga^{-1} = g \iff [g] = \{g\}$; hence $|[g]| = 1 \iff g = e$. By 4b, $|[g]|$ divides p^n. Hence we can write G as a disjoint union of its conjugates $G = [e] \sqcup [g_1] \sqcup [g_2] \ldots [g_k]$ where for each i, $|g_i|$ divides p^n but is not equal to one, hence it is a multiple of p. Hence consider the equation $|G| = |[e]| + |[g_1]| + \ldots$ modulo p; this becomes $0 = 1 + 0 + 0 + \ldots$, a contradiction.