

HW 9

4.1.3

Let G act on A by permutation, which by assumption is a transitive and faithful action. Fix a . Since G is abelian, for all $\sigma \in G$ we have $a\sigma G_a\sigma^{-1} = G_a$. By 4.1.2, $\cap_{\sigma \in G} G_a = \{e\}$, hence $G_a = e$, which means $\{\sigma \in G \mid \sigma(a) = a\} = \{e\}$, which means for all $\sigma \in G$ with $\sigma \neq e$, $\sigma \notin G_a$, which means $\sigma(a) \neq a$.

Choose some $g \in G$ and define the map $\phi : A \rightarrow G$ by $\phi(g) = g \cdot a$. ϕ is surjective since G acts transitively on A . ϕ is injective. Proof: suppose $g \cdot a = h \cdot a$. Then $(g^{-1}h) \cdot a = a$, hence $g^{-1}h = e$, hence $g = h$. Hence ϕ is a bijection, so $|A| = |G|$.

4.2.9

Let p be a prime, G have prime power order, $H \leq G$ with $[G : H] = p$. By corollary 4.2.5, since p is the smallest prime dividing $|G|$, H is normal.

Let G be a group of order p^2 . By Cauchy's theorem, there is an element of order p , say r . Then the index of $\langle r \rangle$ is $p^2/p = p$. By the above theorem, $\langle r \rangle$ is normal in G .

4.2.10

Let G be a non-abelian group of order 6. By Cauchy's theorem, G has an element of order 2, say s . Let $H = \langle s \rangle$ and consider the action of G on the left cosets of H , which we can label H, H', H'' , with associated homomorphism $\phi_H \leq S_{\{H, H', H''\}} \cong S_3$. The stabilizer of H is $G_H = H$, and all the stabilizers are conjugate to G_H .

By theorem 4.2.3.3, $\ker \pi_H$ is some subgroup of H , so it is either $\{e\}$ or H . If it is $\{e\}$ then ϕ_H is injective and there is an injective homomorphism $G \rightarrow S_3$; since both sides have size 6 we have $G \cong S_3$.

Otherwise, since $\ker \phi_H = H$ is the intersection of every stabilizer, every stabilizer must be H , so H is a normal subgroup. It remains to be shown that in this case, $G \cong C_6$.

Proof 1: Let $g \in G$ be arbitrary. Then $gH = Hg$, which means $\{g, gs\} = \{g, sg\}$, which means $gs = sg$, hence H commutes with every element of G . By Cauchy's theorem, G has an element of order 3, say r , with $rs = sr$. The subgroup $\langle r, s \rangle$ has order 6 (since it contains subgroups of order 2 and 3), with presentation $\langle r, s \mid rs = sr, r^3 = s^2 = e, \dots \rangle$ where \dots indicates some additional relations. Hence there is a surjective homomorphism from $C_2 \times C_3 \rightarrow \langle r, s \rangle$; since both sides have size 6 we have $G \cong C_2 \times C_3$.

Proof 2: By Cauchy's theorem, there exists an element of order 3, say r , and $\langle r \rangle$ is a normal subgroup of G (since it has index 2). Hence G has two normal subgroups N_1, N_2 of size 2 and 3. By the diamond isomorphism theorem, $G' = N_1N_2$ is a subgroup of G of size 6 (since its size must be divisible by 2 and 3 but be less than 6). $N_1 \cap N_2 = \{e\}$ by order considerations. Hence by exercise 3.3.7, $G = G' \cong N_1 \times N_2$.

4.2.11

In the cycle representation of $\pi(x)$, consider the cycle containing e . The cycle must be $(e, x, x^2, \dots, x^{n-1})$ by definition of n .

Similarly, consider an arbitrary $g \in G$ with $g \notin \{e, x, x^2, \dots, x^{n-1}\}$, and consider the cycle containing g . The elements $\{g, xg, x^2g, \dots, x^{n-1}g\}$ are all distinct since $x^a g = x^b g \iff x^a = x^b$. Hence the cycle in $\pi(x)$ containing g must be $(g, xg, x^2g, \dots, x^{n-1}g)$.

From proposition 3.5.25, $\pi(x)$ is odd \iff the number of cycles of even length is odd $\iff n$ is even and m is odd. For the last biimplication, either n is odd (in which case there are no cycles of even length) or n is even (in which case there are m cycles of even length).

4.2.12

Let $H = \{g \in G \mid \pi(g) \text{ is even}\}$. It is easy to check that H is a subgroup of G . Let $k \in G$ such that $\pi(k)$ is odd. Then the bijective map $\phi : G \rightarrow G$ given by $\phi(g) = kg$ sends H to $G - H$ and $G - H$ to H , hence $2|H| = |G|$.

4.2.13

Let G be as in the question and π be the left regular representation. By Cauchy's theorem, there is an element of order 2, say s . Since $|s|$ is even and $|G|/|s| = k$ is odd, by 4.2.11 $\pi(s)$ is an odd permutation. By 4.2.12, G has a subgroup of index 2.

4.3.13

Let G be a finite group with $|G| = n$. If $n = 2$ then G has 2 conjugacy classes. Now suppose G has 2 conjugacy classes, and let G act on itself by conjugation; the nonidentity elements form an orbit of size $n - 1$. Since the size of the orbit divides the size of the group, $n - 1$ divides n , hence $n = 2$.

4.3.23

The normalizer $N_G(M)$ is a subgroup of G containing M , that is $M \leq N_G(M) \leq G$. By maximality of M , either $N_G(M) = M$ or $N_G(M) = G$.

If M is a maximal subgroup of G and M is not normal, then $N_G(M) \neq G$ hence $N_G(M) = M$. Consider the action of G on $P(G)$ by conjugation. The stabilizer of M is $N_G(M) = M$, hence the orbit of M has size $[G : M]$. Each orb (i.e. element in the image of the action, contained in the orbit) is a subgroup conjugate to M and hence the number of nonidentity elements is $|M| - 1$. Hence the number of nonidentity elements of G contained in conjugates of M is at most $[G : M](|M| - 1)$.

4.3.24

Since the subgroup lattice is a finite partial order, G is contained in some maximal subgroup $M \neq G$. It suffices to show that $G \neq \cup_{g \in G} gMg^{-1}$, since $gHg^{-1} \subseteq gMg^{-1}$.

The number of nonidentity elements of G contained in conjugates of M is $|G| - 1 \leq [G : M](|M| - 1) = \frac{|G|}{|M|}(|M| - 1) = |G| - \frac{|G|}{|M|} \leq |G| - 2$, a contradiction. Here the first inequality follows from 4.3.23 and the second from $|G|/|M| \geq 2$ since M is a proper subgroup of G .

4.3.26

By assumption, G acts transitively by permutation on $A = \{a_1, \dots, a_n\}$. Then the orbit of a_1 is all of A , hence $\text{Stab}(a_i) \sim \text{Stab}(a_j)$ where \sim denotes conjugate subgroups. Furthermore, $\text{Stab}(a_1) \neq G$ since otherwise G cannot act transitively on A .

Suppose for a contradiction that every $\sigma \in G$ has some fixed point a_i , that is $\sigma \in \text{Stab}(a_i)$. Then $G = \cup_{\sigma \in G} \text{Stab}(a_i)$, which expresses G as a union of conjugates of $\text{Stab}(a_1)$, contradicting 4.3.24.