# HW 11

## 5.2.14a

The group operation is associative: it is required to prove that for $\alpha, \beta, \gamma \in \hat{G}, (\alpha\beta)\gamma = \alpha(\beta\gamma)$. This follows because $(\alpha(z)\beta(z))\gamma(z) = \alpha(z)(\beta(z)\gamma(z))$ as an equation in $\mathbb{C}$. Similarly, commutativity of multiplication in $\mathbb{C}$ implies commutativity of the group operation in $\hat{G}$.

The identity element is the constant function $\epsilon(z) = 1$ for all $z \in G$, because for any $\alpha \in \hat{G}$, $\alpha(z)\epsilon(z) = \alpha(z)$.

Given $\alpha \in \hat{G}$, the inverse $\alpha^{-1}$ is given by $\alpha^{-1}(z) = \alpha(z)^{-1}$ for all $z \in G$. This is well-defined because $\alpha(z) \neq 0$ for all $z \in G$ (0 is not a root of unity), and $\alpha(z)\alpha(z)^{-1} = \alpha(z)\alpha(z)^{-1} = 1$, hence $\alpha\alpha^{-1} = \epsilon$.

## 5.2.14b

Let $\mathbb{C}'$ be the group of roots of unity in $\mathbb{C}$. Let $j$ be the square root of $-1$ in $\mathbb{C}$.

Write $G = \langle x_1 \rangle \times \langle x_2 \rangle \dots \langle x_r \rangle$ and let $n_i$ be the order of $x_i$, and let $\chi_i \in \hat{G}$ be as in the hint. Let $F$ be the function from $G \to \hat{G}$ given by $F(x_1^{e_1}, x_2^{e_2} \dots x_n^{e_n}) = \chi_1^{e_1}\chi_2^{e_2} \dots \chi_n^{e_n}$. It is easy to check that $F$ is an injective homomorphism.

We prove that $F$ is surjective. Let $\phi \in \hat{G}$, in other words $\phi : \langle x_1 \rangle \times \langle x_2 \rangle \dots \langle x_r \rangle \to \mathbb{C}'$ is a homomorphism. Since the $x_i$ generate $G, \phi$ is uniquely determined by its action on each $x_i$. $\phi(x_i)$ is a root of unity of order dividing $n_i$, so we can write $\phi(x_i) = \chi_i^{e_i}$ for some $e_i$ since $\chi_i(x_i) = e^{\frac{2\pi j}{n_i}}$ is a primitive $n_i$-th root of unity. Hence $\phi = F(x_1^{e_1}, x_2^{e_2} \dots x_n^{e_n})$.

## 5.2.15

Let us use the convention that if $G = \langle a \rangle \times \langle b \rangle$ then $a$ has order 8 and $b$ has order 4 (this is always possible since the direct product gives $G \cong C_8 \times C_4$). If $G = \langle a \rangle \times \langle b \rangle$ we have $G = \langle a^3 \rangle \times \langle b \rangle$ as well since $\langle a \rangle = \langle a^3 \rangle$; similarly we can replace $a$ with $a^3, a^5, a^7$ and we can replace $b$ with $b^3$.

Since $ord(a) = 8$ we have $a = x^i y^j$ for some $i \in (0, 8)$ with $i$ coprime with 8 since otherwise the 8 exponents of $x$ in $x^i y^j, x^{2i} y^{2j} \dots$ are not distinct. We can replace $a$ with some power of a $a'$ such that $a' = xy^k, G \cong \langle a' \rangle \times \langle b \rangle$. Hence we first classify all such $a', b$.

If $b$ is an element of $\langle y \rangle$, we have $b \neq y^2$ (otherwise $b$ has order 2), and some power of $b$ is $y^{-1}$, hence some power is $y^-k$. Hence $\langle a' \rangle \times \langle b \rangle$ contains $x$ as well as $y^{-1}$, and since $a'$ has order 8 and $b$ has order 4, $G \cong \langle a' \rangle \times \langle b \rangle$. Hence we have $G \cong \langle xy^j \rangle \times \langle y \rangle$ and $G \cong \langle xy^j \rangle \times \langle y^3 \rangle$ for $j \in [0, 3]$. Taking odd powers of $xy^j$ we have $G \cong \langle x^i y^j \rangle \times \langle y^k \rangle$ where $i, k$ odd.

If $b$ is not an element of $\langle y \rangle$ we have $a = xy^i, b = x^j y^k$ with $j$ even. Similarly to the above, we replace $b$ with powers of $b$ such that $j = 2$ so $a = xy^i, b = x^2 y^k$. Let $l$ be arbitrary; since $y^l$ can be written a product of powers of $a$ and $b$ in a unique way, $y^l$ is some power of $a^{-2}b = y^{k-2i}$. Working modulo 4,

$k-2i$ must be congruent to 1 or $-1$, or $k = 2i + -1$. Since $2i$ is either 0 or 2 modulo 4, we have $k = 1$ or 3. Hence we have $a = xy^i, b = x^2y^j$ where $i$ is arbitrary and j is even. Taking odd powers of $a$ gives $a = x^h y^i$, $h$ odd. Taking odd powers of $b$ we have $b = x^2y^j$ or $b = x^{-2}y^{-j}$, j even, which is equivalent to $x^6y^j$, $j$ even.

Hence the full list is: either

$G \cong \langle x^i y^j \rangle \times \langle y^k \rangle$ where $i, k$ odd.

$G \cong \langle x^h y^i \rangle \times \langle x^i y^j \rangle$ where $h$ odd, $i = 2, 6, j$ even.

## 5.4.4

First, every commutator in $S_n$ is even since $\epsilon(x^{-1}y^{-1}xy) = \epsilon(xx^{-1})\epsilon(yy^{-1})$ where $\epsilon$ is the sign-counting homomorphism.

Using the identity $[(12), (23)] = (12)(23)(12)(23) = (132)$ we see that any 3-cycles lies in the commutator. Since the 3-cycles in $S_4$ generate $A_4$, $S_4' = A_4$.

Alternatively, from the identity $[(12)(34), (23)] = (14)(23)$ we have $S_4'$ contains the group generated by 3-cycles (a subgroup of order 3) as well as $V_4$ (of order 4); hence it has size at least 12, hence it is $A_4$.

For $A_4'$: looking at the subgroup lattice of $S_4$, the only candidates are the trivial group, $V_4$ and $A_4$. It cannot be the trivial group since $A_4/A_4'$ is abelian. Also since $A_4/V_4 = C_3$ is abelian, by 7.4 $A_4' \leq V_4$. Hence $A_4' = V_4$.

## 5.4.5

Similar to the above, the 3-cycles generate $A_n$ and are contained in $S_n'$ hence $S_n' = A_n$.

## 5.4.14

We show that $G = D \times U$ as an internal direct product. $D$ and $U$ are both contained in $G$.