HW 5

1

a

Let ϕ be a map between the left cosets and the right cosets of N as follows: given a left coset gN, the map is conjugation by g^{-1}.

To show that ϕ is well-defind, suppose $gN = hN$; this means $g^{-1}h \in N$. Under conjugation by g^{-1}, they are mapped to Ng and $g^{-1}hNg$ respectively, and it is required to show that $Ng = g^{-1}hNg$, which follows because $g^{-1}hN = N$.

b

Let $N \leq G$ be such a subgroup of index 2; by theorem 6, it suffices to show that $gN = Ng$ for all $g \in G$, and for this it suffices to show that the bijection ϕ defined in 1a is an identity. We have $\phi(N) = eNe = N$. Hence ϕ maps the remaining coset N^C (the complement of N in G) to itself.

c

Among S_n the even permutations are closed under composition. Furthermore, an even permutation can be written as $g = t_1 t_2 \dots t_{2n}$ where each t_i is a transposition; hence $g^{-1} = t_{2n}^{-1} \dots t_1^{-1}$ is even as well. Hence the even permutations form a subgroup.

Let $A_n \leq S_n$ be the subgroup of even permutations. Choose a transposition t; then every element of tA_n is odd. Furthermore, every odd permutation g can be written as $g = t(tg) \in tA_n$, hence $S_n = A_n \sqcup tA_n$ and A_n has index 2.

d

Consider the set $N = \{e, (1,2)(3,4), (1,3)(2,4)(1,4)(2,3)\}$ consisting of all the permutations of cycle structure $2+2$ together with the identity. By brute force computation, this is a subgroup since

$$\begin{aligned}
(1,2)(3,4)(1,3)(2,4) &= (1,4)(2,3) \\
(1,2)(3,4)(1,4)(2,3) &= (1,3)(2,4) \\
(1,3)(2,4)(1,4)(2,3) &= (1,2)(3,4)
\end{aligned}$$

Since the conjugate pnp^{-1} is n with relabelled elements, it has the same cycle structure as n; hence conjugation by $g \in S_n$ fixes N. (To be more rigorous, it suffices to verify this for all 6 transpositions in S_n since conjugation by p can be written as a sequence of conjugations by transpositions).

2

C_2 as quotient: let $N = \langle r \rangle = e, r^1, r^2 \dots r^n$. Since $|G : N| = 2, N$ is normal and the quotient Dic_n/N has order 2; hence it must be isomorphic to C_2.

D_n as quotient: consider the subgroup $E = \{1, s^2\}$ (this is a subgroup since $s^4 = 1$). To show that E is normal, it suffices to show that s^2 commutes with every element of D_n, and it suffices to show that it commutes with every generator. s^2 commutes with s is trivial, and $s^2 = r^{\frac{n}{2}}$ hence it commutes with r as well.

In terms of the matrix representation, this is $E = \{I, -I\}$, and this commutes with everything in the group since it consists of diagonal matrices.

Hence Dic_n/E is a quotient group with the same order as D_n; it remains to find an isomorphic copy of D_n in it; in fact $R = rE$ and $S = sE$ satisfies the correct relations. $\langle R \rangle = \{R^1, R^2 \dots R^{\frac{n}{2}} = E\}$ where the inner equality holds because $r^{\frac{n}{2}}\{e, r^{\frac{n}{2}}\} = \{r^{\frac{n}{2}}, e\}$ and the list has no duplicates since $r^aE = r^bE \iff \{r^a, r^{a+\frac{n}{2}}\} = \{r^b, r^{b+\frac{n}{2}}\} \iff \{[a], [a + \frac{n}{2}]\} = \{[b], [b + \frac{n}{2}]\}$ where $[\cdot]$ denotes congruence classes modulo n. Hence $r^aE = r^bE \iff$ either $a = b \pmod N$ or $a = b + \frac{n}{2} \pmod n$; in either case, $a = b \pmod{\frac{n}{2}}$. Similarly, $\langle S \rangle = \{e, s^2E\}$. Lastly, $RSRS = rsrsE = s^2E = E$.

3.1.36

Suppose $G/Z(G)$ is cyclic with generator $xZ(G)$. This means that any coset can be written as $(xZ(G))^a = x^aZ(G)$ for some $a \in \mathbb{Z}$. Since the cosets partition G, every element of G is of the form x^az for some $a \in \mathbb{Z}, z \in Z(G)$.

Let $g_1, g_2 \in G$; we can write $g_1 = x^az_1, g_2 = x^bz_2$ for some $a, b \in \mathbb{Z}, z_1, z_2 \in Z(G)$. Then $g_1g_2 = x^az_1x^bz_2 = x^ax^bz_1z_2 = x^{a+b}z_1z_2 = x^bx^az_2z_1 = x^bz_2x^az_1 = g_2g_1$.

3.1.41

Let $M = \{x^{-1}y^{-1}xy | x, y \in G\}$, and $N = \bar{M}$.

Notation: let $[a, b] = a^{-1}b^{-1}ab$ be the comutator of a and b. We have $[b^{-1}, a^{-1}]ab = bab^{-1}a^{-1}ab = ba$.

For $n \in N, g \in G$ we show that $gng^{-1} \in N$.

By the definition of N, we can write $n = c_1c_2 \dots c_k$ where each $c_i = [x_i, y_i]$.

$$\begin{aligned}
gn &= gc_1c_2 \dots c_k \\
&= [g^{-1}, c_1^{-1}]c_1gc_2 \dots c_k \\
&= [g^{-1}, c_1^{-1}]c_1[g^{-1}, c_2^{-1}]c_2g \dots c_k \\
&\dots \\
&= [g^{-1}, c_1^{-1}]c_1[g^{-1}, c_2^{-1}]c_2 \dots [g^{-1}, c_k^{-1}]g
\end{aligned}$$

Hence $gng^{-1} = [g^{-1}, c_1^{-1}]c_1[g^{-1}, c_2^{-1}]c_2 \dots [g^{-1}, c_k^{-1}]$. This belongs to N since every factor is a comutator.

3.1.43

For each $g \in G$ let $P(g)$ be the partition containing g. Let $N = P(e)$ where e is the identity in G.

Notation: let Q be a part. Q^{-1} is the unique part satisfying $Q^{-1}Q = N$ where the operation is quotient multiplication (this exists and is unique because the partition is a group under quotient multiplication).

Claim: N is closed under the group operation. Suppose $a, b \in N$. By the well-definedness of the quotient operation, $P(ab) = P(ee) = N$. Hence $ab \in N$.

Claim: N is closed under inverse. For $a \in N$ we have $P(a^{-1})$ must be an identity element under quotient multiplication; since identities are unique in groups, $P(a^{-1}) = N$ hence $a^{-1} \in N$.

By the above two claims, N is a subgroup of G. We will prove that every part is a left coset of N, and every left coset is a part. This suffices to prove that N is normal by proposition 5 (page 81).

Claim: $P(g) = P(g') \iff g^{-1}g' \in N$. Proof: $\implies$: suppose $P(g) = P(g')$. Consider the equation $P(g)^{-1}P(g') = N$, which is true because $P(g)^{-1} = P(g')^{-1}$. By the well-definedness of quotient multiplication the LHS is $P(g^{-1}g')$ hence $P(g^{-1}g') \in N$ hence $g^{-1}g' = N$. $\impliedby$: suppose $g^{-1}g' \in N$, then $N = P(g^{-1}g') = P(g)^{-1}P(g')$. By multiplying the outer equation on the left with $P(g)$, we have $P(g) = P(g')$.

Now $g' \in P(g) \iff P(g') = P(g) \iff g^{-1}g' \in N \iff g' \in gN$ where the first biimplication holds because the P's form a partition. This shows that $P(g) = gN$, hence each part is a left coset. The fact that every left coset is a part follows because the parts partition (i.e. cover) G.