HW 3

1.3

1.4

1.6

Lemma: if $\phi : G \to H$ is a group isomoprhism, then the inverse of G (taken as a set function, which exists because ϕ is bijective) is a group isomoprhism.

Proof: it is required to show that $\phi^{-1}(xy) = \phi^{-1}(x)\phi^{-1}(y)$ for all $x, y \in H$. We have $\phi(\phi^{-1}(xy)) = \phi(\phi^{-1}(x)\phi^{-1}(y))$ because the LHS is xy and the RHS is $\phi(\phi^{-1}(x)\phi^{-1}(y)) = (\phi(\phi^{-1}(x)))(\phi(\phi^{-1}(y))) = xy$ where the first equality holds because ϕ is a group homomorphism. The result follows because ϕ is injective.

1

We prove part a by induction on n. We will use $n = 1$ as the base case, which is trivial. For the inductive step, we have $\phi(x^{n+1}) = \phi(x^n x) = \phi(x^n)\phi(x) = \phi(x)^n\phi(x) = \phi(x)^{n+1}$.

Additionally we will prove that $\phi(e) = e$. Let $\phi(e) = x$, then $x = \phi(e) = \phi(ee) = x^2$; cancelling, we have $x = e$.

For part b, it is required to prove that $\phi(x^{-1}) = \phi(x)^{-1}$, equivalently $\phi(x)\phi(x^{-1}) = e$. This follows because $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e) = e$.

The extension to $\mathbb{Z}$ then follows by applying part (a) to the element x^{-1}, since $\phi(x^{-n}) = \phi((x^{-1})^n) = (\phi(x)^{-1})^n = \phi(x)^{-n}$.

2

Let $n = ord(x)$. Then $\phi(x)^n = \phi(x^n) = \phi(e) = e$, hence $ord(\phi(x)) \leq n$.

3

Let H be abelian. Then for all $x, y \in G$ we have $xy = \phi(\phi^{-1}(x)\phi^{-1}(y)) = \phi(\phi^{-1}(y)\phi^{-1}(x)) = yx$. The proof for when G is abelian is the same proof applied to the isomoprhism ϕ^{-1}.

6

$\mathbb{Q}$ has an element of order two, namely $\frac{1}{2}$, but $\mathbb{Z}$ does not since the equation $2x = 1$ has no solution in integers.

14

17

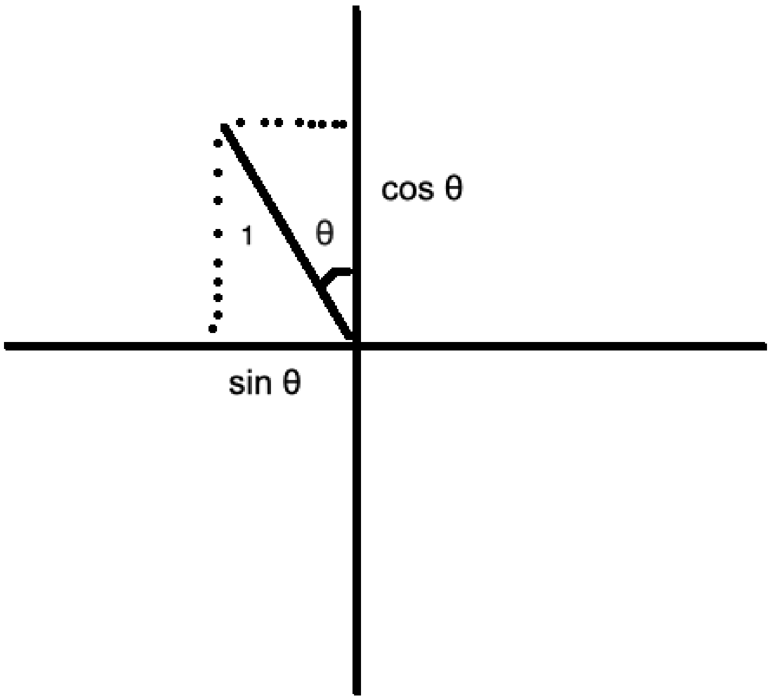
Let ϕ be the map. Suppose G is abelian. Then for all $x, y \in G$ we have $\phi(xy) = (xy)^inv = y^{-1}x^{-1} = x^{-1}y^{-1} = \phi(x)\phi(y)$ where the 3rd equation follows because G is abelian.

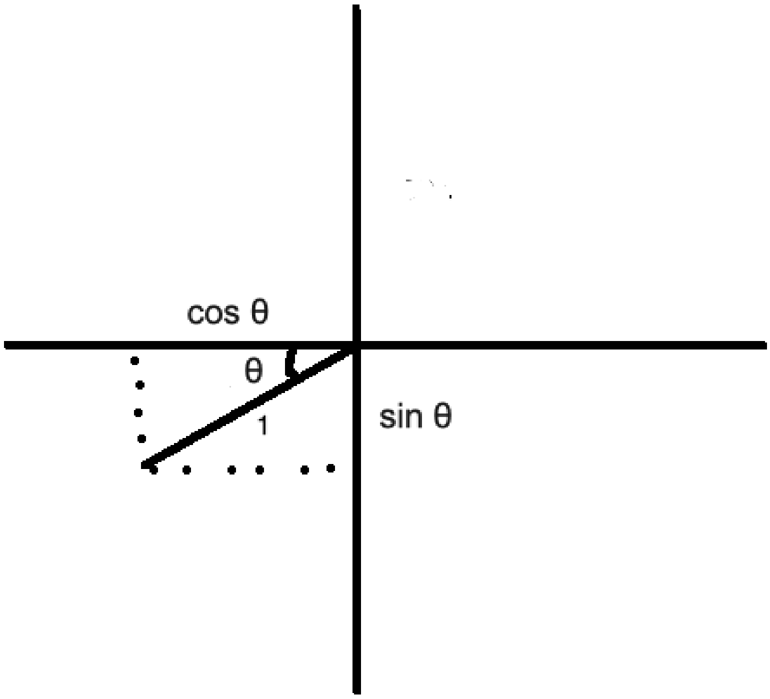
Conversely suppose ϕ is a homomorphism and let $x, y \in G$. Then $xy = (y^{-1}x^{-1})^{-1} = \phi(y^{-1}x^{-1}) = \phi(y^{-1})\phi(x^{-1}) = yx$

25a

a

Let M be the matrix in the question. Since the question presupposes that a rotation is a linear transformation, it suffices to show that M is a rotation on any basis of $\mathbb{R}^2$; we will take $\{[-1, 0], [1, 0]\}$ which M maps to $\{[-\cos\theta, -\sin\theta], [-\sin\theta, \cos\theta]\}$. A geometric proof is attached.





b

We have to show that the generator relations are satisfied.

$\phi(r)^n = I$; since $\phi(r)$ is a ccw rotation by θ, $\phi(r)^n$ is a ccw rotation by $n\theta = 2\pi$, which is the identity map.

$\phi(s)^2 = I$; this is a simple computation.

The relation $rsr = s$ can be verified by wolfram alpha `https://www.wolframalpha.com/input?i=%5B%5Bcos+x%2C+-sin+x%5D%2C+%5Bsin+x%2C+cos+x%5D%5D+%5B%5B0%2C+1%5D%2C+%5B1%2C+0%5D%5D+%5B%5Bcos+x%2C+-sin+x%5D%2C+%5Bsin+x%2C+cos+x%5D%5D`

c

It suffices to show that the $2n$ elements $\phi(r)^i \phi(s)^j$ for $i \in [0, n), j \in \{1, 2\}$ are distinct matrices. TBD