Approximate Confidence Intervals for the Mean Direction of a von Mises Distribution
Author(s): Graham J. G. Upton
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Approximate confidence intervals for the mean direction of a von Mises distribution

BY GRAHAM J. G. UPTON
*Department of Mathematics, University of Essex, Colchester C*04 3*SQ, U.K.*

SUMMARY

Simple formulae are suggested that provide accurate confidence bounds for the mean direction of a sample from a von Mises distribution. These formulae obviate the need for calculating an estimate of the second trigonometric moment, and dispense with nomograms.

Some key words: Confidence interval; Mean direction; von Mises distribution.

Fisher & Lewis (1983), without making distributional assumptions, obtain confidence intervals for the preferred direction of a circular population. Nevertheless, the von Mises distribution is commonly assumed for cyclic data and it is therefore of interest to find simple formulae that provide confidence intervals for the mean direction of that distribution.

The major text-books on circular data (Mardia, 1972; Batschelet, 1981) present nomograms for the 95% and 99% confidence intervals for the mean direction of a von Mises distribution. These nomograms are difficult to read and refer to a restricted range of sample sizes.

A normal approximation is obtained as follows. Let $\theta_1, \ldots, \theta_n$ be a random sample of n observations from a von Mises distribution with unknown parameters κ and μ. Let

$$s = \Sigma \sin \theta_i, \quad c = \Sigma \cos \theta_i, \quad R^2 = S^2 + C^2.$$

The maximum likelihood estimate $\hat{\mu}$ of μ satisfies $\cos \hat{\mu} = C/R$ and $\sin \hat{\mu} = S/R$, and the maximum likelihood estimate $\hat{\kappa}$ of κ is the solution of

$$A(\kappa) = I_1(\kappa)/I_0(\kappa) = R/n = \bar{R}, \tag{1}$$

where the I's are modified Bessel functions. The resulting estimate of κ is biased, and an alternative procedure (Schou, 1978) is to use the estimate $\tilde{\kappa}$ which is the solution, for $R^2 \geqslant n$, of $A(\kappa) = \bar{R}A(R\kappa)$, where the function A was defined in (1) above. In cases where $R^2 < n$, the estimate $\tilde{\kappa} = 0$. Tables of $\tilde{\kappa}$ are provided by Schou (1978) and Batschelet (1981).

When the data are tightly clustered the von Mises (μ, κ) distribution is closely approximated by the normal distribution with variance $\kappa^{-\frac{1}{2}}$ (Mardia, 1972, p. 60).

Two new approximations are introduced here, one recommended for the case $\bar{R} \leqslant 0{\cdot}9$, $\kappa < 5$, approximately, and one for $\bar{R} \geqslant 0{\cdot}9$. The approximations, satisfactory for $n \geqslant 8$, are based on the likelihood ratio rests suggested by Upton (1973).

Test 3·2 of Upton (1973) was concerned with the hypotheses H_0: $\mu = \mu_0$ and H_1: $\mu \neq \mu_0$, with κ unknown and small. The likelihood ratio test specified rejection of H_0 at the α significance level if

$$R^2 > X^2 + \tfrac{1}{4}(2n^2 - X^2)Z_\alpha/n, \tag{2}$$

where $X = S \sin \mu_0 + C \cos \mu_0 = R \cos(\mu_0 - \hat{\mu})$. Substitution for X in (2), replacing the inequality by an equality leads to the penultimate interval given in Table 1.

When $\bar{R} \geqslant 0{\cdot}9$, Upton (1973, Eqn (13·7)) noted that the likelihood ratio test statistic for comparing H_0 and H_1 was approximated by

$$n \log \{(n^2 - X^2)/(n^2 - R^2)\}, \tag{3}$$

and had an asymptotic χ_1^2 distribution. Substitution for X in (3) and suitable manipulation leads to the last of the intervals given in Table 1.

Table 1. *Alternative procedures for obtaining confidence bounds for the mean direction of a von Mises distribution*

Number	Method	Source
1	Nomograms, for limited combinations of sample size and confidence size	First published by Batschelet (1972)
2	Normal approximation for large κ: $\hat{\mu} \pm \{Z_\alpha/(R\hat{\kappa})\}^{\frac{1}{2}}$.	Mardia (1972, p. 145)
3	As above, with $\hat{\kappa}$ replaced by $\tilde{\kappa}$: $\hat{\mu} \pm \{Z_\alpha/(R\tilde{\kappa})\}^{\frac{1}{2}}$.	This paper
4	Likelihood based, $\bar{R} \leqslant 0\cdot 9$ $\hat{\mu} \pm \cos^{-1}[\{2n(2R^2 - nZ_\alpha)/(4n - Z_\alpha)\}^{\frac{1}{2}}/R]$	This paper
5	Likelihood based, $\bar{R} \geqslant 0\cdot 9$ $\hat{\mu} \pm \cos^{-1}[\{n^2 - (n^2 - R^2)\exp(Z_\alpha/n)\}^{\frac{1}{2}}/R]$	This paper

Z_α is the upper α point of a χ_1^2 distribution.

Table 2. *Half-widths, in degrees, of the confidence intervals for the mean direction of a von Mises distribution obtained using the five procedures of Table* 1.

(*a*) 95% confidence bounds

	$n=8$					$n=30$					$n=200$				
	Procedure no.					Procedure no.					Procedure no.				
$\bar{R}$	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
0·3	+	+*	+*	+	+*	58	47*	50*	57	+*	18	18*	18*	19	21*
0·5	85	52*	58*	78	67*	29	27*	28*	29	40*	11	10*	10*	11	11*
0·7	41	33*	35*	40	44*	19	17	18	19	22*	7	7	7	7	7*
0·8	32	26*	28*	33	36*	14	14	14	15	16*	5	5	5	6	5*
0·85	29	22*	24*	29	29*	12	12	12	14	13*	4	4	4	5	4*
0·9	25	18*	19*	27*	22	10	9	10	13*	10	3	4	4	5*	3
0·95	15	13*	14*	24*	15	6	7	7	12*	7	2	3	3	4*	2

(b) 99% confidence bounds

	$n=8$					$n=30$					$n=200$				
	Procedure no.					Procedure no.					Procedure no.				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
0·3	+	+*	+*	+	+*	+	62*	66*	+	+*	24	24*	24*	25	28*
0·5	+	69*	76*	+	+*	40	35*	36*	40	60*	15	14*	14	14	15*
0·7	63	44*	46*	64	+*	25	23	23	25	31*	9	9	9	9	9*
0·8	47	34*	36*	48	58*	19	18	18	21	22*	7	7	7	8	6*
0·85	41	30*	31*	43	45*	16	15	15	19	18*	6	6	6	7	5*
0·9	35	24*	25*	38*	33	13	12	13	17*	14	5	5	5	6*	4
0·95	28	17*	18*	34*	22	9	9	9	15*	9	3	3	3	6*	3

Procedure no. 1, nomogram; procedures numbers 2 and 3, forms of Mardia's test; procedures 4 and 5, likelihood ratio based. + half-width greater than 90°. * Procedure used outside recommended range.

The performance of the competing procedures is examined in Table 2 which distinguishes between cases where the method has been used in its recommended range and cases where n is too small or $\bar{R}$ takes an inappropriate value. Mardia (1972) recommended his test for $n > 30$ and κ large, using it, on his page 146, for a case with $\bar{R} = 0\cdot 74$. In the table it has been assumed that it could be used for $n \geqslant 30$ and $\bar{R} \geqslant 0\cdot 7$.

Using the nomogram values as a standard for comparison, it will be seen that the replacement of $\hat{\kappa}$ by $\tilde{\kappa}$ improves Mardia's test which appears satisfactory for $\bar{R} \geqslant 0{\cdot}4$ and $n \geqslant 30$. For smaller values of n or $\bar{R}$ the likelihood ratio tests are to be preferred, while for large n all the tests give very comparable values.

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