Automata and Grammars (BIE-AAG)

1. Basic notions

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Motivation

Motivation

- Theory of languages and automata is one of the building blocks of theoretical computer science.
- The theory offers efficient solutions to a number of basic problems.
- The right use of the theory saves time and money.
- Applications: compiler construction, model checking, verification of communication protocols, pattern matching, data compression, . . .

Course goals

Course goals

- To get to know the languages and automata theory.
- To be able to use the theory efficiently.
- To recognize classes of languages.

Course evaluation

Seminars

- homeworks: one for 5 points, the others for 0 points each (Student's responsibility.)
- 2 tests (7th week: Marast online test for 10 pts; 11th week: contact test for 25 pts)
- up to 5 points for activity at seminars
- 40 points in total
- assessment ("zápočet"): 20 points at minimum

Course evaluation

Exam

- “multiple choice”: 10 points at max.
- main test: 50 points at max.
- requirements: multiple choice min. 7 points, main test min. 25 points (successfully completed multiple choice test is considered in the next exam)

Course schedule

1. Basic notions, Chomsky hierarchy, Closure properties of CFL
2. Deterministic and nondeterministic finite automata (DFA and NFA), NFA with epsilon transitions
3. Operations on automata (removal of epsilon transitions, determinization, minimization, intersection, union)
4. Reg. expressions, RE, FA and RG conversions, Kleene theorem
5. Operations on regular grammars, conversions to FA
6. Properties of regular languages (pumping lemma, Nerode th.)
7. Context-free languages, pushdown automaton
8. Parsing of Context-free languages (nondet. vs. determinism)
9. Transducers, Mealey, Moore, conversions
10. Context-sensitive and Recursively enumerable languages, Turing machine
11. Program and circuit implementation of DFA and NFA
12. FA as a lexical analyzer, lex/flex generator

Literature

- Aho A. V, Motwani R., Ullman, J. D.: Introduction to Automata Theory, Languages, and Computation. (2nd Edition). Addison Wesley, 2001. ISBN 0-201-44124-1.
- Kozen, D. C.: Automata and Computability. Springer, 1997. ISBN 0387949070.
- Melichar B., Holub J., Mužátko P.: Languages and Translations. Praha: Publishing House of CTU, 1997. ISBN 80-01-01692-7.
- Šestáková E.: Automata and Grammars. A Collection of Excercises and Solutions. Praha, Pražská technika – Nakladatelství ČVUT, 2018.

Warning

- The slides are only auxiliary materials for lectures and cannot be considered as the only study source for tests and exam.

Basic notions

Alphabet – finite set of *symbols* (notation: Σ or T)

- binary {0,1}, ternary {Yes, No, Maybe},
  DNA {A, C, G, T}, keywords {while, do, begin, end, to, for, false, true, ...}

String over an alphabet – a finite sequence of symbols of the alphabet. E.g. “0110101”, “ACCCGT”, “while true do”

Empty sequence = *empty string* = ε.

Σ^* – set of all strings over Σ

Σ^+ – set of all nonempty strings over Σ

$\Sigma^* = \Sigma^+ \cup \{\varepsilon\}$

Basic notions

Operation *concatenation* (denoted as '.'):

- $\forall x, y \in \Sigma^*$, string $x.y$ (shortened to xy) is created by concatenating strings x and y.
- is associative, i.e., $\forall x, y, z \in \Sigma^* : (xy)z = x(yz)$,
- is not commutative, i.e., $\exists x, y \in \Sigma^* : xy \neq yx$,
- ε acts as the identity element of concatenation operation: $x\varepsilon = \varepsilon x = x$
- $a^0 = \varepsilon,\ a^1 = a,\ a^2 = aa,\ a^3 = aaa,\ \ldots$

Operation *reversal* of string (denoted as x^R):

- $x = a_1a_2a_3 \ldots a_n$, $x^R = a_na_{n-1} \ldots a_1$
- $y = abcd$, $y^R = dcba$

Basic notions

Length of a string x:

- denoted by $|x|$
- $|x| \geq 0$,
- $|x| = 0 \Leftrightarrow x = \varepsilon$

Formal language

Formal language L over Σ: $L \subseteq \Sigma^*$ (a set of strings).

operations:

- set operations: *union, intersection, difference*
- *complement* of language L_1: $\overline{L_1} = \Sigma^* \setminus L_1$ ($\overline{L_1} \cup L_1 = \Sigma^*, L_1 \cap \overline{L_1} = \emptyset$).
- *concatenation* (*product*) of languages:
  $L = L_1.L_2 = \{xy : x \in L_1, y \in L_2\}$ (L is defined over alphabet $\Sigma = \Sigma_1 \cup \Sigma_2$)
- n-th *power* of language L: $L^n = L.L^{n-1}$, $L^0 = \{\varepsilon\}$.
  Kleene star L^* of language L: $L^* = \bigcup_{n=0}^{\infty} L^n$.
  $L^* = L^+ \cup \{\varepsilon\}$,
  $L^+ = L.L^* = L^*.L = \bigcup_{n=1}^{\infty} L^n$ (Kleene plus)

Grammar

Definition

Grammar is a quadruple $G = (N, \Sigma, P, S)$, where

- N is a finite set of nonterminal symbols,
- Σ is a finite set of terminal symbols ($\Sigma \cap N = \emptyset$, denoted also by T),
- P is a set of *production rules*. It is a finite subset of $(N \cup \Sigma)^*.N.(N \cup \Sigma)^* \times (N \cup \Sigma)^*$, (element (α, β) of P is written as $\alpha \rightarrow \beta$ and called a *rule*),
- $S \in N$ is the *start symbol* of the grammar.

Grammar

Example

Grammar $G_1 = (\{A, S\}, \{0, 1\}, P, S)$, where P:

- $S \rightarrow 0A$
- $A \rightarrow 1A$
- $A \rightarrow 0.$

Note:

$\alpha \rightarrow \beta_1,\ \alpha \rightarrow \beta_2, \ldots, \alpha \rightarrow \beta_n$, can be shortened to:
$\alpha \rightarrow \beta_1 \mid \beta_2 \mid \ \ldots \ \mid \beta_n.$

Grammar

Other possibilities to describe grammar:

- Backus-Naur Form (BNF),
- Extended Backus-Naur Form (EBNF).

Example

Grammar G generates a language of unsigned integers. BNF is used to specify this grammar.

$G = (\{\langle \textit{integer} \rangle, \langle \textit{digit} \rangle\}, \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, P, \langle \textit{integer} \rangle)$

Set P contains rules:

$\langle \textit{integer} \rangle \ ::= \ \langle \textit{digit} \rangle \langle \textit{integer} \rangle \mid \langle \textit{digit} \rangle$

$\langle \textit{digit} \rangle \ ::= \ 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9.$

Grammar

Definition
$G = (N, \Sigma, P, S)$, $x, y \in (N \cup \Sigma)^*$. We say that x *derives* y *in one step* $(x \Rightarrow y)$, if there exists $(\alpha \to \beta) \in P$ and $\gamma, \delta \in (N \cup \Sigma)^*$ such that $x = \gamma\alpha\delta, y = \gamma\beta\delta$ (i.e., $\gamma\alpha\delta \Rightarrow \gamma\beta\delta$).

Example
$G_1 = (\{A, S\}, \{0, 1\}, P, S)$, $P = \{S \to 0A00, A \to 1A, A \to 0\}$.
$01A00 \Rightarrow 011A00 \quad (\gamma = 01, \alpha = A, \beta = 1A, \delta = 00)$

Definition
$\alpha \Rightarrow^k \beta$ if there exists a sequence $\alpha_0, \alpha_1, \ldots, \alpha_k$ for $k \geq 0$, of $k+1$ strings such that $\alpha = \alpha_0, \alpha_{i-1} \Rightarrow \alpha_i$ for $1 \leq i \leq k$, and $\alpha_k = \beta$. This sequence is called *derivation* of string β from string α that has length k in grammar G.

Example
G_1: $01A00 \Rightarrow^4 011111A00$

Grammar

Definition
Transitive closure of relation $\Rightarrow$: $\alpha \Rightarrow^+ \beta$ if $\alpha \Rightarrow^i \beta$ for some $i \geq 1$.
(We read '$\Rightarrow^+$' as "derives in nonzero number of steps".)

Definition
Transitive and reflexive closure of relation $\Rightarrow$: $\alpha \Rightarrow^* \beta$ if $\alpha \Rightarrow^i \beta$ for some $i \geq 0$.
(We read '$\Rightarrow^*$' as "derives in any number of steps".)

Grammar

Definition
$G = (N, \Sigma, P, S)$. String α is called a *sentential form* in grammar G, if $S \Rightarrow^* \alpha, \alpha \in (N \cup \Sigma)^*$.

Example
G_1: $S \Rightarrow^* 01A00$
G_1: $S \Rightarrow^* 011000$

Definition
A sentential form in $G = (N, \Sigma, P, S)$ that contains no nonterminal symbols is called a *sentence generated by grammar* G.

Example
G_1: $S \Rightarrow^* 011000$

Grammar

Definition
$L(G) = \{w : w \in \Sigma^*, \exists S \Rightarrow^* w\}$ is the *language generated by grammar* $G = (N, \Sigma, P, S)$.

(Language generated by grammar G is the set of all sentences generated by grammar G.)

Remark
Note that ε is a sentence generated by grammar G, if $\varepsilon \in L(G)$.

Example
$G_1 = (\{A, S\}, \{0, 1\}, \{S \to 0A, A \to 1A, A \to 0\}, S)$ generates language $L(G_1) = \{01^n0 : n \geq 0\}$.

The following derivations exist in grammar G_1:
$S \Rightarrow 0A \Rightarrow 00$
$S \Rightarrow 0A \Rightarrow 01A \Rightarrow 010$
$S \Rightarrow 0A \Rightarrow 01A \Rightarrow 011A \Rightarrow 0110$

Grammar

Definition
Grammars G_1 and G_2 are *equivalent* if they generate the same language.
That means $L(G_1) = L(G_2)$.

Classification of grammars

Noam Chomsky (*7.12.1928 Philadelphia)

- work in the field of grammars of both formal and natural languages

Definition

$G = (N, \Sigma, P, S)$. We say that G is:

0. *Unrestricted* (type 0), if it satisfies the general grammar definition.
1. *Context-sensitive* (type 1), if every rule from P is of the form $\gamma A\,\delta \to \gamma\alpha\delta$, where $\gamma, \delta \in (N \cup \Sigma)^*, \alpha \in (N \cup \Sigma)^+, A \in N$, or the form $S \to \varepsilon$ in case that S is not present in the right-hand side of any rule.
2. *Context-free* (type 2), if every rule is of the form $A \ \to \alpha$, where $A \in N, \alpha \in (N \cup \Sigma)^*$.
3. *Regular* (type 3), if every rule is of the form $A \to aB$ or $A \to a$, where $A, B \in N, a \in \Sigma$, or the form $S \to \varepsilon$ in case that S is not present in the right-hand side of any rule.

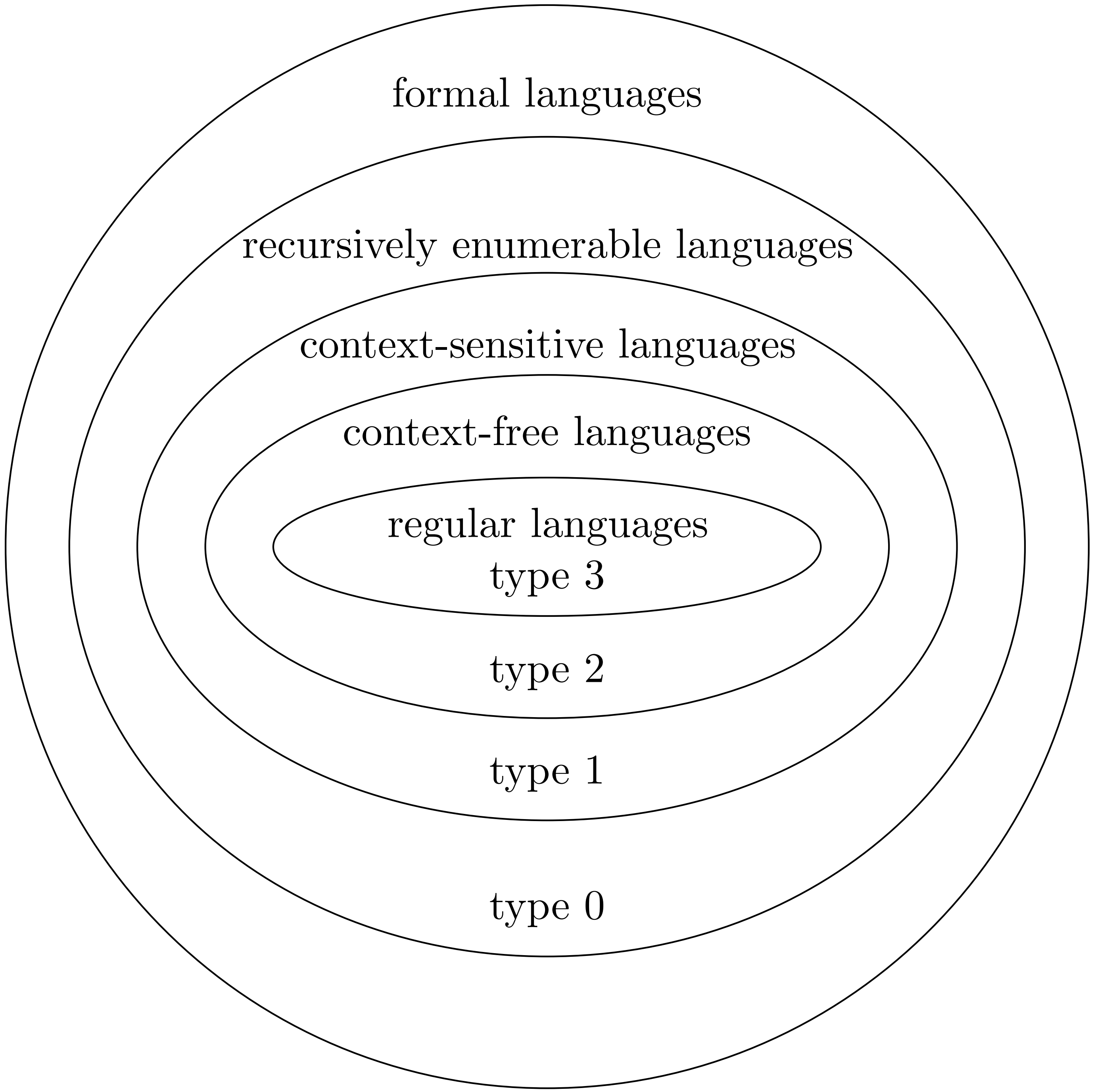
Classification of languages

Definition

We say that language

0. is *recursively enumerable* (type 0), if $\exists$ unrestricted grammar which generates it.
   - recognized by Turing machine.
1. is *context-sensitive* (type 1), if $\exists$ context-sensitive grammar which generates it.
   - recognized by linear bounded Turing machine (also called linear bounded automaton).
2. is *context-free* (type 2), if $\exists$ context-free grammar which generates it.
   - recognized by nondeterministic pushdown automaton.
3. is *regular* (type 3), if $\exists$ regular grammar which generates it.
   - recognized by finite automaton.

Classification of languages



Classification of languages

Example

$G_1 = (\{S\}, \{0,1\}, \{S \to 0S, S \to 1S, S \to 1, S \to 0\}, S)$ is a regular grammar and generates language $L(G_1) = \{0,1\}^+$.

$G_2 = (\{S\}, \{0,1\}, \{S \to 0S1, S \to 01\}, S)$ is context-free and generates language $L(G_2) = \{0^n1^n : n \geq 1\}$.

$G_3 = (\{S, A\}, \{0,1\}, \{S \to 0A1, S \ \to 01, 0A \to 00A1, A \ \to 01\}, S)$ is context-sensitive and generates $L(G_3) = \{0^n1^n : n \geq 1\}$.

$G_4 = (\{S\}, \{0,1\}, \{S \to 0S1, S \to 01, 0S1 \to S\}, S)$ is unrestricted and generates language $L(G_4) = \{0^n1^n : n \geq 1\}$.

Grammars G_2, G_3 and G_4 are equivalent, because $L(G_2) = L(G_3) = L(G_4) = \{0^n1^n : n \geq 1\}$.

$L(G_1)$ is regular language, $L(G_2)$ is not, but $L(G_2) \subseteq L(G_1)$.

Closure Properties of CFL

Closure property:

If certain languages are context-free, and a language L is formed from them by certain operations, then L is also context-free.

Theorem
The class of context-free languages is closed under operations *union*, *product*, and *iteration*.

Grammars and ops. over languages

Algorithm Grammar for a *union* of languages.

Input: Context-free grammars $G_1 = (N_1, \Sigma, P_1, S_1)$ and
$G_2 = (N_2, \Sigma, P_2, S_2)$ generating languages L_1 and L_2, $N_1 \cap N_2 = \emptyset$.

Output: Context-free grammar G, $L(G) = L_1 \cup L_2$.

```
1: $G \leftarrow (N_1 \cup N_2 \cup \{S\}, \Sigma, P_1 \cup P_2 \cup \{S \to S_1 \mid S_2\}, S)$, where
   $S \notin N_1 \cup N_2$.
```

W.L.O.G., the sets of terminal symbols are expected to be equal.

Grammars and ops. over languages

Algorithm Grammar for a *product* of languages.

Input: Context-free grammars $G_1 = (N_1, \Sigma, P_1, S_1)$ and
$G_2 = (N_2, \Sigma, P_2, S_2)$ generating languages L_1 and L_2, $N_1 \cap N_2 = \emptyset$.

Output: Context-free grammar G such that $L(G) = L_1.L_2$.

```
1: $G \leftarrow (N_1 \cup N_2 \cup \{S\}, \Sigma, P_1 \cup P_2 \cup \{S \rightarrow S_1S_2\}, S)$, where $S \notin N_1 \cup N_2$.
```

Grammars and ops. over languages

Algorithm Grammar for a *Kleene star* of a language.

Input: Context-free grammar $G = (N, \Sigma, P, S)$ generating language L.
Output: Context-free grammar G' such that $L(G') = L^*$.

```
1: $G' \leftarrow (N \cup \{S'\}, \Sigma, P \cup \{S' \rightarrow SS', S' \rightarrow \varepsilon\}, S')$, where $S' \notin N$.
```

Finite languages

Definition
Language $L \subset \Sigma^*$ is called *finite* if $\exists n \in \mathbb{N}$: $|L| < n$.

Theorem
The smallest class of languages that contains all finite languages and languages created from finite languages by a finite number of operations

a) union,
b) product,
c) Kleene star,
d) complement,

is the set (class) of regular languages.

Parse tree for CF languages

A parse tree is a graphical representation of a syntactical structure of a sentential form.

Parse tree for CF languages

Definition
Given grammar $G = (N, \Sigma, P, S)$. *Parse tree* is an tree having the following properties:

1. The nodes of the parse tree are labeled with terminal symbol, nonterminal symbols, and symbol ε (provided it is the only child node of his parent node).
2. The root of the tree is labeled with the start symbol S.
3. If a node has at least one descendant, it is labeled by a nonterminal symbol.
4. If $n_1, n_2, \ldots, n_k$ are the direct descendants of node n that is labeled with symbol A and these symbols are (from left to right) labeled with symbols $A_1, A_2, \ldots, A_k$, then $A \to A_1 A_2 \ldots A_k$ is a rule in P.
5. The leaves of the parse tree form, from left to right, a sentential form or a sentence in grammar G, which is the *result* of the parse tree.

Parse tree for CF languages

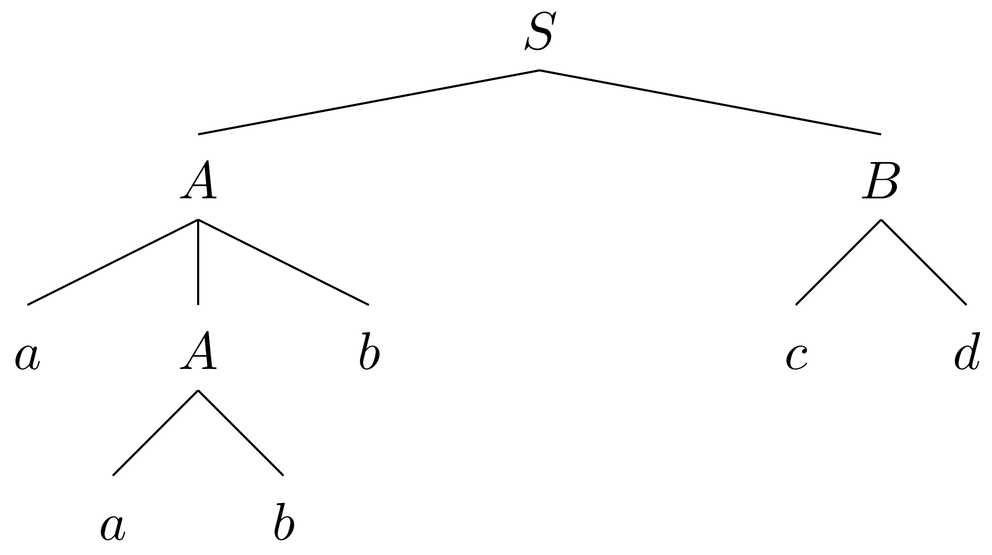
Example

$G = (\{S, A, B\}, \{a, b, c, d\}, P, S)$, where P:

(1) $S \to AB$ (2) $A \to aAb$ (3) $A \to ab$

(4) $B \to cBd$ (5) $B \to cd$

$S \Rightarrow AB \Rightarrow aAbB \Rightarrow aabbB \Rightarrow aabbcd.$



$$S \Rightarrow AB \Rightarrow Acd \Rightarrow aAbcd \Rightarrow aabbcd$$
$$S \Rightarrow AB \Rightarrow aAbB \Rightarrow aAbcd \Rightarrow aabbcd$$