Modeling of Physical Environment

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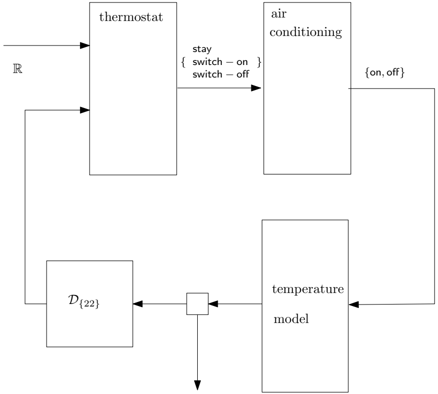
The Past

job of someone with a degree in computer science

not any more!

and if you want an interesting and well-paid job, it is not enough to know about computer systems.

Today's and Tomorrow's Computer Systems (Example)



Problem: In reality, temperature

- is in infinite (uncountable) set $\mathbb{R}$, and
- does not jump in steps.

Modeling Real-World Phenomena

The transition systems we studied up to now mainly were

- finite (e.g., , $S = \mathbb{B}^n$, $\mathbb{F}$), or, at least
- infinite, but with simple evolution (clocks).

But: this is not enough to study most real-world phenomena:

- time
- speed
- acceleration
- pressure
- temperature

Timed automata at least allowed us to model time by clocks, that

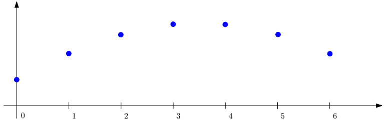
- can be set to 0, a
- run at a exactly the same speed.

What about speed, pressure, temperature etc.?

Continuous Modeling of Time

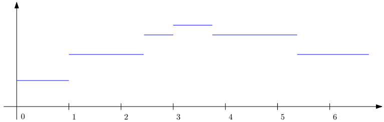
Recall:

Given a set S, a *(discrete time) signal* over S is
an infinite sequence of elements of S (i.e., $\mathbb{N}_0 \to S$).



Now:

Given a set S, a *(continuous time) signal* over S is
a function $\mathbb{R}_{\geq 0} \to S$



From now on we write

- Σ_S^C for the set of continuous time signals over S, a
- Σ_S^D for the set of discrete time signals over S.

And if clear from the context, simply Σ_S (or even Σ).

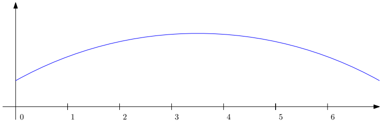
Continuous Modeling of State

State:

- continuous: $\mathbb{R}^n$
- discrete: e.g., $\mathbb{N}$, finite

All four combinations useful, but mainly:

Analog signal: continuous-time signal with continuous state.



Digital signal: discrete-time signal with discrete state.

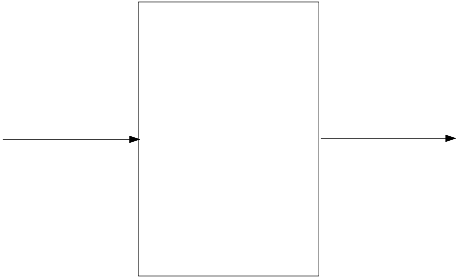
For example, classical CD:
$44100 Hz$, 2^{16} states in the interval $[-2^{15}, 2^{15} - 1]$

Discretization of time: *sampling*

Discretization of state: *quantization*

Modeling Components

In general: relation between input signals and output signals, that is:



A *discrete time system* with input set I and output set O is

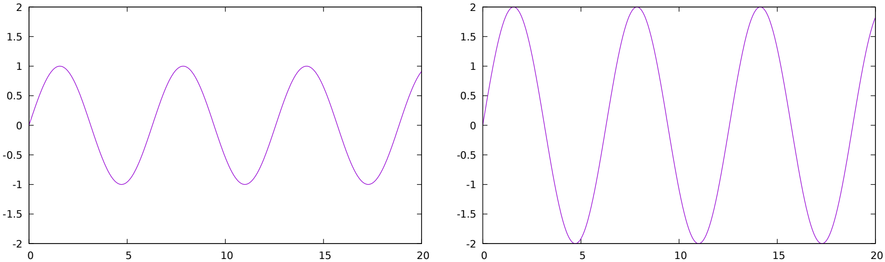
- a relation between signals over I and signals over O, that is
  - a subset of $\Sigma_I^D \times \Sigma_O^D$

A *continuous time system* with input set I and output set O is

- a relation between signals over I and signals over O, that is
  - a subset of $\Sigma_I^C \times \Sigma_O^C$

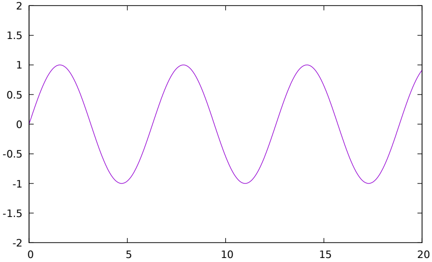
Amplifier/Gain

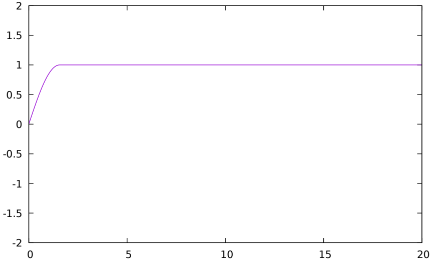
$$\{(i, o) \mid \forall t \in \mathbb{R}_{\geq 0} \,.\, o(t) = 2i(t)\}$$



Running Maximum

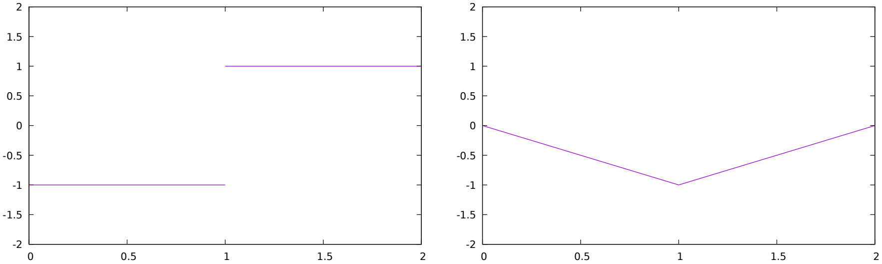
$$\{(i, o) \mid \forall t \in \mathbb{R}_{\geq 0} \,.\, o(t) = \max_{\tau \in [0,t]} i(\tau)\}$$





Integrator

$$\{(i, o) \mid \forall t \in \mathbb{R}_{\geq 0} \, . \, o(t) = s_o + \int_0^t i(\tau) d\tau\}$$



$$\{(i, o) \mid \forall t \in \mathbb{R}_{\geq 0} \, . \, \dot{o}(t) = i(t), o(0) = s_0\}$$

System Properties

A system $\mathcal{S}$ is *receptive* iff
for all $i \in \Sigma_I$ there is $o \in \Sigma_O$ s.t. $(i, o) \in \mathcal{S}$.

A system $\mathcal{S}$ $\mathcal{S}$ is *causal* iff
for all $i_1, i_2 \in \Sigma_I$, $x \in O$, t such that
for all $t' \leq t$. $i_1(t') = i_2(t')$,
there is $o \in \Sigma_O$ s.t. $(i_1, o) \in \mathcal{S}$, $o(t) = x$
iff
there is $o \in \Sigma_O$ s.t. $(i_2, o) \in \mathcal{S}$, $o(t) = x$

A system $\mathcal{S}$ is *deterministic* iff
for all $i \in \Sigma_I$ there is precisely one $o \in \Sigma_O$ s.t. $(i, o) \in \mathcal{S}$.

A system $\mathcal{S}$ is *memory-less* iff
there is $R \subseteq I \times O$ s.t. for all $(i, o) \in \Sigma_I \times \Sigma_O$,
$(i, o) \in \mathcal{S}$ iff for all t, $(i(t), o(t)) \in R$.

Apply to both continuous time and discrete time systems
($t \in \mathbb{N}_0$ vs. $t \in \mathbb{R}_{\geq 0}$)

Modeling General Real-World Phenomena

Concentrate on $\Sigma^{C}_{\mathbb{R}^n}$

How to describe such (analog) signals and systems?

In discrete-time (automata, transition systems) state is usually
a result of the previous one.

We do not have a notion of "previous state" here.

Continuous functions:

$$\forall t \,.\, x(t) = \sin t.$$

$x = \sin t$

But, not enough for describing physical laws

Further Structure of Lecture

Up to now: black-box description of continuous systems
(i.e., by their I/O behavior)

Further

- description of continuous time signals, modeling of physical systems without input/output (in analogy to transition system)
- white-box description of continuous systems
(in analogy to automaton=transition system + input/output)
- discrete vs. continuous modeling

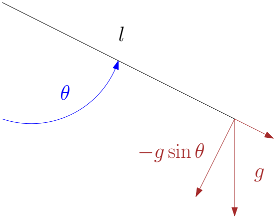
Modeling General Real-World Phenomena

Example: pendulum

Similar models are used in many areas, e.g.

- robotics motion planning
- controlling autonomous cars
- computer games (physics engines)
- processing sensor data (mobile phones) to compute position/speed etc.

Modeling General Real-World Phenomena



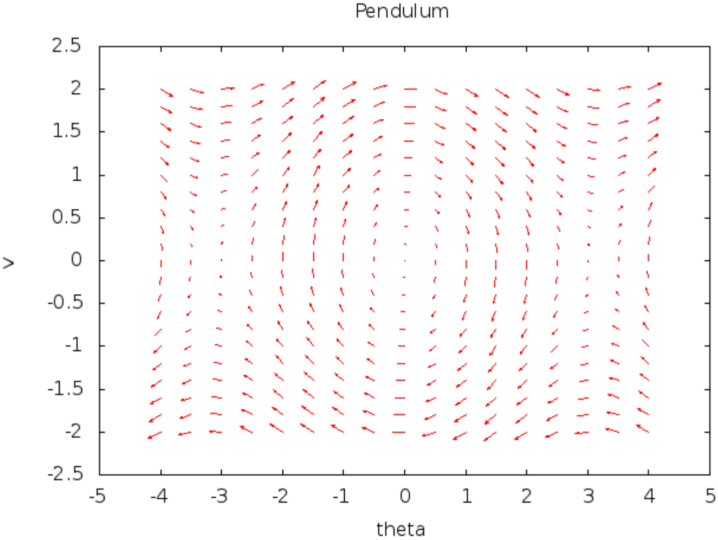
Pendulum (v = angular velocity):

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -\frac{g}{l} \sin \theta \end{aligned}$$

equilibrium: θ, v s.t. corresponding $\dot{\theta}, \dot{v}$ are zero.

may be stable ($\theta = 0, v = 0$) and unstable ($\theta = \pi, v = 0$)

Vector Fields

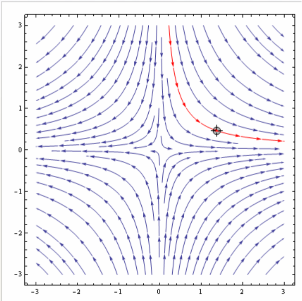
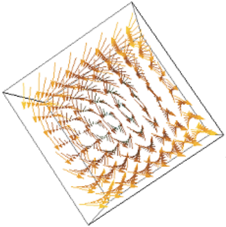
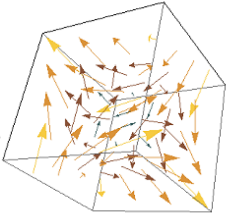


$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -\frac{g}{l}\sin\theta \end{aligned}$$

http://en.wikipedia.org/wiki/Pendulum_(mathematics)

Vector Fields

Intuition: assigns to each allowed value of real variables,
a direction into which the values will evolve

Formally: for $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}^n$

Equilibrium: $x \in S$ s.t. $f(x) = 0$

Discrete-time analogon: deterministic transition systems, state diagram

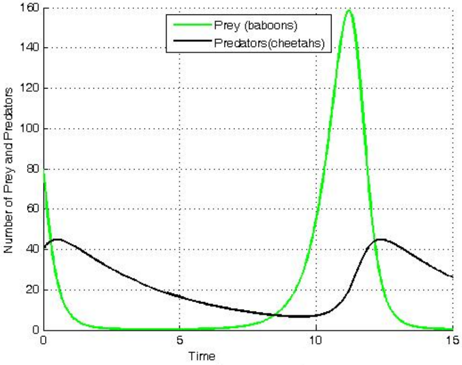
Analogy to path of transition system?

Ordinary Differential Equations and Their Solution

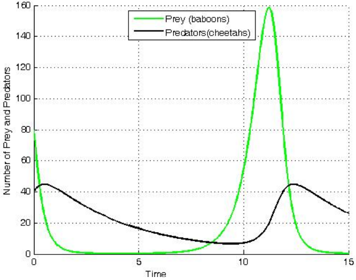
Lotka-Volterra model, x: prey, y: predator (continuous abstraction)

$$
\begin{aligned}
\dot{x} &= \alpha x - \beta y x \\
\dot{y} &= -\gamma y + \delta x y
\end{aligned}
$$

Scilab Demo: Simul./ODEs



Ordinary Differential Equations and Their Solution



Ordinary differential equations:

$\dot{x} = f(x)$, where f is a vector field $f : S \to \mathbb{R}^n$

We look for a function x: $\mathbb{R}_{\geq 0} \to \mathbb{R}^n$, that follows the vector field, i.e.

In the representation with time axis:

for all time $t \in \mathbb{R}_{\geq 0}$,

every curve has a slope that corresponds to $f(x(t))$

A *solution* of the equation $\dot{x} = f(x)$ is $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, s.t.

for all $t \in \mathbb{R}_{\geq 0}$, $\dot{x}(t) = f(x(t))$

Solution of Differential Equations

Such a solution is also called a *trajectory* of the differential equation

See also: path of transition system.

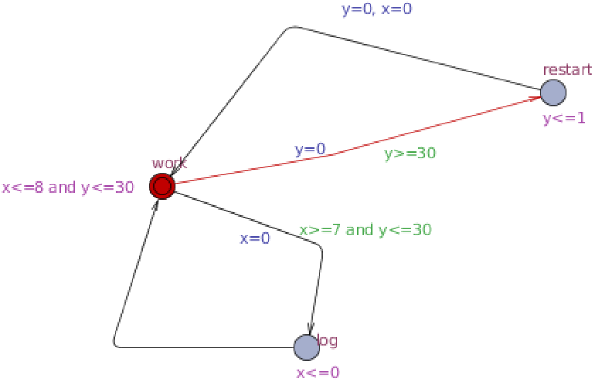
Usually, as for transition systems,
we state initial conditions
initial value problem (IVP)

discrete	continuous
transition function $f : S \to S$	vector field $f : S \to \mathbb{R}^n$
	differential equation $\dot{x} = f(x)$
$\forall t \in \mathbb{N}_0 \,.\, s(t+1) = f(s(t))$	$\forall t \in \mathbb{R}_{\geq 0} \,.\, \dot{x}(t) = f(x(t))$
state diagram	vector field visualization
path	solution

up to now: everything deterministic

Timed Automata?

discrete	continuous
transition function $f : S \to S$	vector field $f : S \to \mathbb{R}^n$
	differential equation $\dot{x} = f(x)$
$\forall t \in \mathbb{N}_0 \,.\, s(t+1) = f(s(t))$	$\forall t \in \mathbb{R}_{\geq 0} \,.\, \dot{x}(t) = f(x(t))$
state diagram	vector field visualization



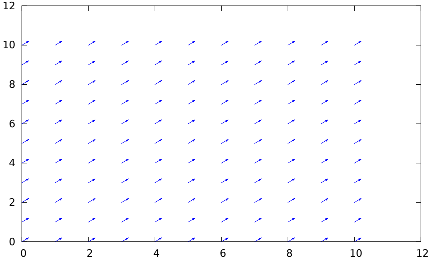
Timed Automata?

discrete	continuous
transition function $f : S \to S$ $\forall t \in \mathbb{N}_0 \ . \ s(t+1) = f(s(t))$	vector field $f : S \to \mathbb{R}^n$ $\forall t \in \mathbb{R}_{\geq 0} \ . \ \dot{x}(t) = f(x(t))$
state diagram	vector field visualization
location action transition	clocks/clock assignments delay transition

Example of delay transition:

$$(work, \{x \mapsto 0, y \mapsto 4\}) \xrightarrow{7} (work, \{x \mapsto 7, y \mapsto 11\})$$

Corresponding vector field?



Corresponds to differential equations $\dot{x} = 1$ for every clock $x \in X$

Timed automata represent only endpoints of solutions

Non-determinism

Usually we have non-determinism coming from

- system environment (e.g., user, weather)
- unknown details
- unmodeled details

How does this look like for differential equations?

Example: $\dot{x} = 2x + 0.4$, where we do not know the constant 0.4 precisely.

Common notation: $\dot{x} = 2x + 0.4 \pm 0.1$, $\dot{x} = 2x + [0.3, 0.5]$ for
$\dot{x} \in \{2x + \delta \mid \delta \in [0.3, 0.5]\}$, or $2x + 0.3 \leq \dot{x} \leq 2x + 0.5$

No unique direction ($f : S \to \mathbb{R}^n$), but

- a set of possibilities $F : S \to 2^{\mathbb{R}^n}$, or
- a relation $r : S \times \mathbb{R}^n$.

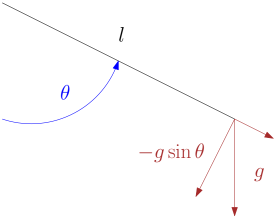
Result:

- differential inclusion $\dot{u} \in F(u)$, or
- differential relation $r(u, \dot{u})$ (e.g., differential inequalities)

discrete	continuous
transition function $f : S \to S$ $\forall t \in \mathbb{N}_0 \,.\, s(t+1) = f(s(t))$	vector field $f : S \to \mathbb{R}^n$ differential equation $\dot{x} = f(x)$ $\forall t \in \mathbb{R}_{\geq 0} \,.\, \dot{x}(t) = f(x(t))$
transition function $F : S \to 2^S$ $\forall t \in \mathbb{N}_0 \,.\, s(t+1) \in F(s(t))$	$F : S \to 2^{\mathbb{R}^n}$ differential inclusion $\dot{x} \in F(x)$ $\forall t \in \mathbb{R}_{\geq 0} \,.\, \dot{x}(t) \in F(x(t))$
transition relation $T \subseteq S \times S$ $\forall t \in \mathbb{N}_0 \,.\, (s(t), s(t+1)) \in T$	$r \subseteq S \times \mathbb{R}^n$ differential relation/inequality $\forall t \in \mathbb{R}_{\geq 0} \,.\, r(x(t), \dot{x}(t))$
state diagram path	vector field visualization solution

Empty space can be filled
(delay system in analogy to differentiation operator)

Description of Components with Input and Output



Pendulum (v = angular velocity):

$$\begin{aligned}\dot{\theta} &= v \\ \dot{v} &= -\frac{g}{l}\sin\theta - iv\end{aligned}$$

input i: braking force

Description of Components with Input and Output

Example:

$$\dot{s} = i$$

Shortcut for

$$\forall t \in \mathbb{R}_{\geq 0} \,.\, \dot{s}(t) = i(t)$$

$$\forall t \in \mathbb{R}_{\geq 0} \,.\, s(t) = \int_0^t i(\tau) d\tau$$

Example:

$$\dot{s} = i, o = 2s$$

Shortcut for

$$\forall t \in \mathbb{R}_{\geq 0} \,.\, \dot{s}(t) = i(t), o(t) = 2s(t)$$

In General

$$\dot{s} = f(s, i), o = g(i, s)$$

Shortcut for

$$\forall t \in \mathbb{R}_{\geq 0} \; . \; \dot{s}(t) = f(s(t), i(t)), o(t) = g(i(t), s(t))$$

In other words

$(i(t), s(t), \dot{s}(t), o(t)) \in R$ where

$$R = \{(i, s, s', o) \mid s' = f(s, i), o = g(i, s)\}$$

For defining a system with inputs and outputs we need such a relation

Description of Components with Input and Output

A *continuous automaton* is a quintuple (n, p, q, S_0, R), where

- $n, p, q \in \mathbb{N}$ (then we call $\mathbb{R}^n$ state space, $\mathbb{R}^p$ input space, $\mathbb{R}^q$ output space)
- $S_0 \subseteq \mathbb{R}^n$ (set of *initial states*)
- $R \subseteq \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q$ (*transition relation*) s.t.
  for all $i \in \mathbb{R}^p$, $s \in \mathbb{R}^n$,
  there is $s' \in \mathbb{R}^n$, $o \in \mathbb{R}^q$ s.t. $(i, s, s', o) \in R$

The pair of signals $(i, o) \in \Sigma_I \times \Sigma_O$ is a *behavior* of the automaton iff there is an $s \in \Sigma_S$ s.t.

- $s(0) \in S_0$,
- for all $t \in \mathbb{R}_{\geq 0}$, $(i(t), s(t), \dot{s}(t), o(t)) \in R$

An automaton T *represents* the system

$$[\![T]\!] := \{(i, o) \in \Sigma_I \times \Sigma_O \mid (i, o) \text{ is a behavior of } T\}$$

Examples

(note $\mathbb{R}^0 = \{()\}$, we do not distinguish $\mathbb{R}^1$ and $\mathbb{R}$)

$$(0, 0, 1, \{()\}, \{((), (), (), 1)\})$$
$$(0, 0, 1, \{()\}, \{(i, s, s', o) \in \mathbb{R}^0 \times \mathbb{R}^0 \times \mathbb{R}^0 \times \mathbb{R}^1 \mid o = 1\})$$

source with constant output

$$(0, 1, 1, \{()\}, \{(i, (), (), 2i) \mid i \in \mathbb{R}\})$$
$$(0, 1, 1, \{()\}, \{(i, s, s', o) \in \mathbb{R}^1 \times \mathbb{R}^0 \times \mathbb{R}^0 \times \mathbb{R}^1 \mid o = 2i\})$$

gain/amplifier

$$(0, 2, 1, \{()\}, \{((i_1, i_2), (), (), i_1 + i_2) \mid i_1 \in \mathbb{R}, i_2 \in \mathbb{R}\})$$
$$(0, 2, 1, \{()\}, \{(i, s, s', o) \in \mathbb{R}^2 \times \mathbb{R}^0 \times \mathbb{R}^0 \times \mathbb{R}^1 \mid i = (i_1, i_2), o = i_1 + i_2\})$$

input adder (see also table lookup)

$$(1, 0, 1, \mathbb{R}, \{((), s, s^2 + 1, s) \mid s \in \mathbb{R}\})$$
$$(1, 0, 1, \mathbb{R}, \{(i, s, s', o) \in \mathbb{R}^0 \times \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \mid s' = s^2 + 1, o = s\})$$

source with output from ODE $\dot{s} = s^2 + 1$

Further Example

$$(1, 1, 1, S_0, \{(i, o, i, o) \mid i \in \mathbb{R}, o \in \mathbb{R}\})$$

$$(1, 1, 1, S_0, \{(i, s, s', o) \mid s' = i, o = s\})$$

A pair $(i, o) \in \Sigma_{\mathbb{R}}^{C} \times \Sigma_{\mathbb{R}}^{C}$ is a behavior of this system iff there is $s \in \Sigma_{S}^{C}$ s.t.

- $s(0) \in S_0$,
- for all $t \in \mathbb{R}_{\geq 0}$. $\dot{s}(t) = i(t), o(t) = s(t)$

The latter condition can be simplified to

$$\dot{o}(t) = i(t)$$

Hence, the automaton represents the integrator

$$\{(i, o) \mid \forall t \in \mathbb{R}^{\geq 0} \,.\, o(t) = s_0 + \int_0^t i(\tau) d\tau, s_0 \in S_0\}$$

Terminology

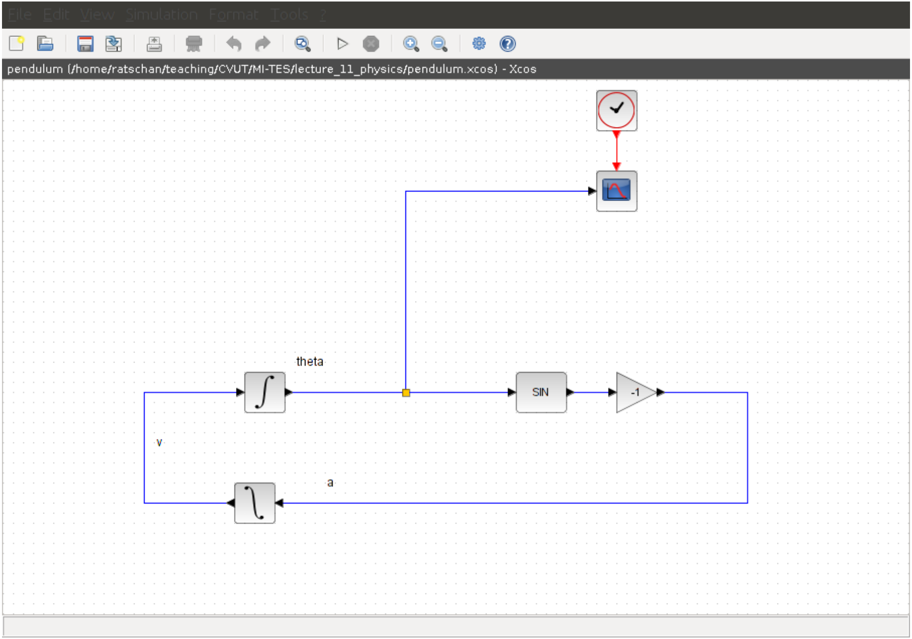
Control theory/engineering uses the terms:

- SISO (single input, single output system): $p = 1, q = 1$
- MIMO (multiple input, multiple output system): $p > 1, q > 1$
- LTI (linear, time-invariant system): Transition relation is given in the form

  $$\dot{x} = Ax + Bu, y = Cx + Du,$$

  where x denotes state, u input, y output.

discrete	continuous
transition function $f : S \to S$ $\forall t \in \mathbb{N}_0 \,.\, s(t+1) = f(s(t))$	vector field $f : S \to \mathbb{R}^n$ differential equation $\dot{x} = f(x)$ $\forall t \in \mathbb{R}_{\geq 0} \,.\, \dot{x}(t) = f(x(t))$
transition function $F : S \to 2^S$ $\forall t \in \mathbb{N}_0 \,.\, s(t+1) \in F(s(t))$	$F : S \to 2^{\mathbb{R}^n}$ differential inclusion $\dot{x} \in F(x)$ $\forall t \,.\, \dot{x}(t) \in F(x(t))$
transition relation $T \subseteq S \times S$ $\forall t \in \mathbb{N}_0 \,.\, (s(t), s(t+1)) \in T$	$r \subseteq S \times \mathbb{R}^n$ differential relation/inequality $\forall t \,.\, r(x(t), \dot{x}(t))$
state diagram path	vector field visualization solution
(discrete time) automaton $\forall t \in \mathbb{N}_0 \,.\, (i(t), s(t), s(t+1), o(t)) \in R$	(continuous time) automaton $\forall t \in \mathbb{R}_{\geq 0} \,,\, (i(t), s(t), \dot{s}(t), o(t)) \in R$
(discrete time) system	(continuous time) system


File Edit View Simulation Format Tools ?
pendulum (/home/ratschan/teaching/CVUT/MI-TES/lecture_11_physics/pendulum.xcos) - Xcos
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Examples of Software Packages

https://en.wikipedia.org/wiki/TORCS

(racing car simulation)

http://www.solidthinking.com/embed_land.html

(simulation of embedded systems including physical environment)

http://gazebosim.org/

(robot simulation)

Choice of Model

Digital electronics: always discrete model?
Physical surroundings: always continuous?

Sometimes, already the physical system contains discrete aspects.

For example:

- physical contact: bouncing ball
- technical device has discrete aspects: switches, car gears
- discrete modeling artifact: linearization

Sometimes, continuity already in computer systems:

- real-time requirements: protocols (after 10 seconds, do this)
- computation of continuous output:
  music, simulation of continuous phenomena
- continuous abstraction of computer systems: data streams

And, of course, there is analogue circuits

Hierarchy of Abstractions

In general: Type of model (continuous, discrete, probabilistic)
is not an inherent property of the reality we are modeling,
but dependent on the application and modeling level:

Electronics:

- Programming languages
- Assembly language
- Hardware desription languages
- Boolean Logic
- Transistor level description
- Electromagnetic field (partial differential equations: maxwell equations)
- Particle (atomic)
- Quantum mechanics

Physical systems:

- Item database
- Newtonian mechanics
- Statistical thermodynamics
- Particle (atomic)

Conclusion

discrete	continuous
transition function $f : S \to S$ $\forall t \in \mathbb{N}_0 \,.\, s(t+1) = f(s(t))$	vector field $f : S \to \mathbb{R}^n$ dif. rovnice $\dot{x} = f(x)$ $\forall t \in \mathbb{R}_{\geq 0} \,.\, \dot{x}(t) = f(x(t))$
transition function $F : S \to 2^S$ $\forall t \in \mathbb{N}_0 \,.\, s(t+1) \in F(s(t))$	$F : S \to 2^{\mathbb{R}^n}$ dif. inkluze $\dot{x} \in F(x)$ $\forall t \in \mathbb{R}_{\geq 0} \,.\, \dot{x}(t) \in F(x(t))$
transition relation $T \subseteq S \times S$ $\forall t \in \mathbb{N}_0 \,.\, (s(t), s(t+1)) \in T$	$r \subseteq S \times \mathbb{R}^n$ dif. relace/nerovnice $\forall t \in \mathbb{R}_{\geq 0} \,.\, r(x(t), \dot{x}(t))$
state diagram path	vector field visualization solution
(discrete time) automaton $\forall t \in \mathbb{N}_0 \,.\, (i(t), s(t), s(t+1), o(t)) \in R$	(continuous time) automaton $\forall t \in \mathbb{R}_{\geq 0} \,,\, (i(t), s(t), \dot{s}(t), o(t)) \in R$
(discrete time) system	(continuous time) system
LTL BMC SAT unbounded model checking ...	

Conclusion

reality	physical world	computation
usual models	continuous	discrete

For computer scientists it is more and more important to
feel at home in both worlds.