Quadruple Studies

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1 The Hamiltonian

Vector potential for quadruple is:

$$\boldsymbol{a} = \left(0, 0, -\frac{k_0}{s}(x^2 + y^2)\right) \tag{1}$$

which gives the exact Hamiltonian for quadruple (h = 0):

$$\begin{aligned} H &= \frac{p_t}{\beta_0} - \sqrt{\left(p_t + \frac{1}{\beta_0}\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + \frac{k_0}{2}(x^2 - y^2) \\ &= \frac{p_t}{\beta_0} - \sqrt{p_t^2 + 2\frac{p_t}{\beta_0} + 1 - p_x^2 - p_y^2} + \frac{k_0}{2}(x^2 - y^2) \end{aligned} \tag{2}$$

Then, with the paraxial approximation by expanding the square root to second order in the dynamical variables we get:

$$\tilde{H} = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{p_t^2}{2\beta_0^2\gamma_0^2} + \frac{k_0}{2}(x^2 - y^2) \tag{3}$$

In this case we loose the effect of the variation of focusing strength with particle energy in an accelerator beam line (*chromaticity*) . To take it into account we can treat the transverse variables (x, p_x) and (y, p_y) separately from the longitudinal variables (z, p_t). This is possible, since if we neglect radiation and certain collective effects, the energy of a particle in a static magnetic field is constant. Therefore, the energy deviation p_t of a particle moving through a quadrupole will be constant. We can then expand the Hamiltonian (2) to second order in the transverse variables, while maintaining the exact dependence on p_t:

$$\tilde{H} = \frac{p_x^2}{2(1+\delta)} + \frac{p_y^2}{2(1+\delta)} + \boldsymbol{\frac{p_t}{\beta_0} - (1+\delta)} + \frac{k_0}{2}(x^2 - y^2) \tag{4}$$

where

$$(1+\delta) = \sqrt{p_t^2 + 2\frac{p_t}{\beta_0} + 1}, \tag{5}$$

and δ is the momentum deviation.

If we compare it to Ref[2], Eq.(3)

$$\tilde{H} = \frac{p_x^2}{2(1+\delta)} + \frac{p_y^2}{2(1+\delta)} + \boldsymbol{\frac{1-\beta_0^2}{2\beta_0^2} p_t^2} + \frac{k_0}{2}(x^2 - y^2) \tag{6}$$

we can see that the term $\frac{1-\beta_0^2}{2\beta_0^2} p_t^2$ in (6) is nothing else but the expansion of the term $\frac{p_t}{\beta_0} - (1+\delta)$ in (4) to the second order.

2 The Equations of Motions

For both Hamiltonians, (4) and (6), the solutions of the equations of motions in the transverse plane are the same:

$$
\begin{aligned}
\frac{dx}{ds} &= \frac{\partial \tilde{H}}{\partial p_x} = \frac{p_x}{1+\delta} \\
\frac{dp_x}{ds} &= -\frac{\partial \tilde{H}}{\partial x} = -k_0 x \\
\frac{dy}{ds} &= \frac{\partial \tilde{H}}{\partial p_y} = \frac{p_y}{1+\delta} \\
\frac{dp_y}{ds} &= -\frac{\partial \tilde{H}}{\partial y} = k_0 y
\end{aligned}
\tag{7}
$$

whereas the longitudinal ones for z are different due to second order expansion in p_t. For Hamiltonian (4) (we have dropped a constant $\frac{1}{\beta_0}$ which has no significance for the dynamics) we have:

$$
\frac{dz}{ds} = \frac{\partial \tilde{H}}{\partial p_t} = \boldsymbol{-\frac{1}{(1+\delta)}\left(p_t + \frac{1}{\beta_0}\right)} - \frac{1}{2}\left(p_t + \frac{1}{\beta_0}\right)\frac{p_x^2 + p_y^2}{(1+\delta)^3} \tag{8}
$$

while for Hamiltonian (6) we get:

$$
\frac{dz}{ds} = \frac{\partial \tilde{H}}{\partial p_t} = \boldsymbol{\frac{1-\beta_0^2}{\beta_0^2} p_t} - \frac{1}{2}\left(p_t + \frac{1}{\beta_0}\right)\frac{p_x^2 + p_y^2}{(1+\delta)^3} \tag{9}
$$

The equations of motion for p_t are obviously the same in both cases:

$$
\frac{dp_t}{ds} = -\frac{\partial \tilde{H}}{\partial z} = 0 \tag{10}
$$

3 The transfer map

The transfer map of the thick quadruple is obtained by the integration of the equations of motion over the length L. We define

$$k = \sqrt{\frac{k_0}{1+\delta}} \tag{11}$$

$$\begin{aligned} C &= cos(\sqrt{k}L), & \hat{C} &= cosh(\sqrt{k}L), \\ S &= sin(\sqrt{k}L), & \hat{S} &= sinh(\sqrt{k}L); \end{aligned}$$

The map for the transverse variable (in both cases) reads:

$$x(L) = C \cdot x_0 + \frac{S}{\sqrt{k}(1+\delta)} \cdot p_{x_0} \tag{12}$$

$$p_x(L) = -\sqrt{k}(1+\delta)S \cdot x_0 + C \cdot p_{x_0} \tag{13}$$

$$y(L) = \hat{C} \cdot y_0 + \frac{\hat{S}}{\sqrt{k}(1+\delta)} \cdot p_{y_0} \tag{14}$$

$$p_y(L) = \sqrt{k}(1+\delta)\hat{S} \cdot y_0 + \hat{C} \cdot p_{y_0} \tag{15}$$

For the longitudinal variables p_t stays the same in both cases, i.e.

$$p_t(L) = p_{t_0} \tag{16}$$

For z in case (8) the solution is:

$$
\begin{aligned}
z(L) = z_0 &- \boldsymbol{\frac{1}{(1+\delta)}\left(p_t + \frac{1}{\beta_0}\right)} \cdot L - \\
&\frac{1}{2}\frac{p_t + \frac{1}{\beta_0}}{(1+\delta)^2} \cdot \left\{ \frac{1}{2}k_0 \left[x^2 \left(L - \frac{C \cdot S}{\sqrt{k}} \right) - y^2 \left(L - \frac{\hat{C} \cdot \hat{S}}{\sqrt{k}} \right) \right] + \right. \\
&\frac{1}{2}\frac{1}{(1+\delta)} \left[p_x^2 \left(L + \frac{C \cdot S}{\sqrt{k}} \right) + p_y^2 \left(L + \frac{\hat{C} \cdot \hat{S}}{\sqrt{k}} \right) \right] - \\
&\left. \left[x \cdot p_x \left(1 - C^2 \right) + y \cdot p_y \left(1 - \hat{C}^2 \right) \right] \right\}
\end{aligned}
$$

and in case (9):

$$
\begin{aligned}
z(L) = z_0 &+ \boldsymbol{\frac{1-\beta_0^2}{\beta_0^2} p_t} \cdot L + \\
&\frac{1}{2}\frac{p_t + \frac{1}{\beta_0}}{(1+\delta)^2} \cdot \left\{ \frac{1}{2}k_0 \left[x^2 \left(L - \frac{C \cdot S}{\sqrt{k}} \right) - y^2 \left(L - \frac{\hat{C} \cdot \hat{S}}{\sqrt{k}} \right) \right] + \right. \\
&\frac{1}{2}\frac{1}{(1+\delta)} \left[p_x^2 \left(L + \frac{C \cdot S}{\sqrt{k}} \right) + p_y^2 \left(L + \frac{\hat{C} \cdot \hat{S}}{\sqrt{k}} \right) \right] - \\
&\left. \left[x \cdot p_x \left(1 - C^2 \right) + y \cdot p_y \left(1 - \hat{C}^2 \right) \right] \right\}
\end{aligned}
$$

As one can see, the only difference is the coefficient in front of the linear term which is coming from the choice of the "quasi"-exact solutions for p_t or the second-order approximation. It makes sense to replace second-order expansion in p_t by the exact solution for the new MAD (?). In this particular case we can easily have the "exact" solution without adding any complication to the calculation.

Reference:

1. A.Wolski, Beam Dynamics in HEP Accelerators, chapter 3, p.101-105

2. A. Latina, Implementation of a Thick Quadrupole in the MAD-X tracking module