Dipole Studies

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1 The Hamiltonian

Vector potential for quadruple is:

$$\boldsymbol{A} = \left(0, 0, -B_0 x + \frac{B_0 h x^2}{2(1+hx)}\right) \tag{1}$$

$$\boldsymbol{a} = \frac{q}{P_0}\boldsymbol{A} = \left(0, 0, -k_0 x + \frac{k_0 h x^2}{2(1+hx)}\right) \tag{2}$$

where q is the particle charge, P_0 is the reference momentum, and $k_0 = \frac{q}{P_0 B_0}$ the normalised field strength.

The exact Hamiltonian for dipole in this case is:

$$\begin{aligned} H &= \frac{p_t}{\beta_0} - (1+hx)\sqrt{\left(p_t + \frac{1}{\beta_0}\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + (1+hx)k_0\left(x - \frac{hx^2}{2(1+hx)}\right) \\ &= \frac{p_t}{\beta_0} - (1+hx)\sqrt{p_t^2 + 2\frac{p_t}{\beta_0} + 1 - p_x^2 - p_y^2} + (1+hx)k_0\left(x - \frac{hx^2}{2(1+hx)}\right) \end{aligned}$$

The expanded to the second order in the transverse variables, while maintaining the exact dependence on p_t, Hamiltonian:

$$\tilde{H} = \frac{p_x^2}{2(1+\delta)} + \frac{p_y^2}{2(1+\delta)} + \boldsymbol{\frac{p_t}{\beta_0} - (1+\delta)} - h(1+\delta)x + k_0 x\left(1 + \frac{1}{2}hx\right) \quad (3)$$

where

$$(1+\delta) = \sqrt{p_t^2 + 2\frac{p_t}{\beta_0} + 1}, \quad (4)$$

and δ is the momentum deviation.

If we compare it to Ref[2], Eq.(4)

$$\tilde{H} = \frac{p_x^2}{2(1+\delta)} + \frac{p_y^2}{2(1+\delta)} + \boldsymbol{\frac{1-\beta_0^2}{2\beta_0^2}p_t^2} - h(1+\delta)x + k_0 x\left(1 + \frac{1}{2}hx\right) \quad (5)$$

we can see that the term $\frac{1-\beta_0^2}{2\beta_0^2}p_t^2$ in (5) is nothing else but the expansion of the term $\frac{p_t}{\beta_0} - (1+\delta)$ in (3) to the second order.

2 The Equations of Motions

For both Hamiltonians, (3) and (5), the solutions of the equations of motions in the transverse plane are the same:

$$\begin{aligned}
\frac{dx}{ds} &= \frac{\partial\tilde{H}}{\partial p_x} = \frac{p_x}{1+\delta} \\
\frac{dp_x}{ds} &= -\frac{\partial\tilde{H}}{\partial x} = -k_0(hx+1) + h(1+\delta) \\
\frac{dy}{ds} &= \frac{\partial\tilde{H}}{\partial p_y} = \frac{p_y}{1+\delta} \\
\frac{dp_y}{ds} &= 0
\end{aligned} \quad (6)$$

whereas the longitudinal ones for z are different due to second order expansion in p_t. For Hamiltonian (3) (the constant $\frac{1}{\beta_0}$ which has no significance for the dynamics was dropped) we have:

$$
\begin{aligned}
\frac{dz}{ds} &= \frac{\partial \tilde{H}}{\partial p_t} \\
&= -\boldsymbol{\frac{1}{(1+\delta)}\left(p_t + \frac{1}{\beta_0}\right)} - \frac{p_t + \frac{1}{\beta_0}}{1+\delta} hx - \frac{1}{2}\left(p_t + \frac{1}{\beta_0}\right)\frac{p_x^2 + p_y^2}{(1+\delta)^3} \\
&= -\boldsymbol{\frac{1}{(1+\delta)}\left(p_t + \frac{1}{\beta_0}\right)} - \frac{p_t + \frac{1}{\beta_0}}{1+\delta}\left(hx + \frac{1}{2}\frac{p_x^2 + p_y^2}{(1+\delta)^2}\right)
\end{aligned}
\tag{7}
$$

while for Hamiltonian (5) we get:

$$
\frac{dz}{ds} = \frac{\partial \tilde{H}}{\partial p_t} = \boldsymbol{\frac{1-\beta_0^2}{\beta_0^2} p_t} - \frac{p_t + \frac{1}{\beta_0}}{1+\delta}\left(hx + \frac{1}{2}\frac{p_x^2 + p_y^2}{(1+\delta)^2}\right) \tag{8}
$$

The equations of motion for p_t are obviously the same in both cases:

$$
\frac{dp_t}{ds} = -\frac{\partial \tilde{H}}{\partial z} = 0 \tag{9}
$$

3 The transfer map

The transfer map of the thick dipole is obtained by the integration of the equations of motion over the length L. We define

$$
k = \sqrt{\frac{k_0}{1+\delta}} \tag{10}
$$

$$C = cos(\sqrt{hk}L),$$
$$S = sin(\sqrt{hk}L);$$

The map for the transverse variable (in both cases) reads:

$$x(L) = C \cdot x_0 + \frac{S}{\sqrt{hk}(1+\delta)} \cdot p_{x_0} + \left(\frac{1}{k} - \frac{1}{h}\right)(1-C) \tag{11}$$

$$p_x(L) = -\sqrt{hk}(1+\delta) \cdot S \cdot x_0 + C \cdot p_{x_0} + (1+\delta) \cdot \left(\sqrt{\frac{h}{k}} - \sqrt{\frac{k}{h}}\right) \cdot S \tag{12}$$

$$y(L) = y_0 + \frac{p_{y_0}}{(1+\delta)} \cdot L \tag{13}$$

$$p_y(L) = p_{y_0} \tag{14}$$

For the longitudinal variables p_t stays the same in both cases, i.e.

$$p_t(L) = p_{t_0} \tag{15}$$

For z in case (7) the solution is:

$$
\begin{aligned}
z(L) = & -\boldsymbol{\frac{1}{(1+\delta)}\left(p_t+\frac{1}{\beta_0}\right)} \cdot L - \\
& \frac{p_{t_0}+\frac{1}{\beta_0}}{1+\delta}\left(\frac{(h\,k\,x+k-h)\,\sqrt{h}\sqrt{k}S}{h\,k^2} - \frac{p_{x_0}\,(C-1)}{(1+\delta)\,k} + \frac{(h-k)\ L}{k}\right) + \\
& -\frac{1}{2}\left(p_{t_0}+\frac{1}{\beta_0}\right)\frac{1}{(1+\delta)^3} \cdot \Big\{x_0^2 \cdot h \cdot k_0 \cdot (1+\delta) \cdot \left(\frac{L}{2} - \frac{1}{\sqrt{h}\sqrt{k}}\frac{C \cdot S}{2}\right) + \\
& \quad p_{x_0}^2\left(\frac{L}{2} + \frac{1}{\sqrt{h}\sqrt{k}}\frac{C \cdot S}{2}\right) + \\
& \quad x_0 \cdot (1+\delta)\left(-(1+\delta)\sqrt{k}\frac{C \cdot S}{\sqrt{h}} + (1+\delta)\sqrt{h}\frac{C \cdot S}{\sqrt{k}} + (k_0 - (1+\delta)\,h) \cdot L\right) + \\
& \quad p_{x_0} \cdot \left(-\frac{(1+\delta)}{k} \cdot C^2 + \frac{(1+\delta)}{h} \cdot C^2 + \frac{(1+\delta)}{k} - \frac{(1+\delta)}{h}\right) + \\
& \quad x_0 \cdot p_{x_0} \cdot (1+\delta)\left(C^2-1\right) + p_{y_0}^2 L + \\
& \quad - \sqrt[3]{\frac{1+\delta}{h}}\sqrt{k_0} \cdot \frac{C \cdot S}{2} + (1+\delta)^2\,\frac{C \cdot S}{\sqrt{h}\sqrt{k}} - (1+\delta)^2\,\frac{\sqrt{h}}{\sqrt[3]{k}}\frac{C \cdot S}{2} + \\
& + (1+\delta)^2 \cdot \frac{k}{2h}L + (1+\delta)^2\,\frac{h}{2k}L - (1+\delta)^2\,L\Big\}. \qquad (16)
\end{aligned}
$$

and in case (8):

$$z = \frac{\mathbf{1-\beta_0^2}}{\boldsymbol{\beta_0^2}}\boldsymbol{L\cdot p_{t_0}} - \frac{p_{t_0}+\frac{1}{\beta_0}}{1+\delta}\left(\frac{(h\,k\,x+k-h)\sqrt{h}\sqrt{k}S}{h\,k^2} - \frac{p_{x_0}\,(C-1)}{(1+\delta)\,k} + \frac{(h-k)\;L}{k}\right)+$$

$$-\frac{1}{2}\left(p_{t_0}+\frac{1}{\beta_0}\right)\frac{1}{(1+\delta)^3}\cdot\left\{x_0^2\cdot h\cdot k_0\cdot(1+\delta)\cdot\left(\frac{L}{2}-\frac{1}{\sqrt{h}\sqrt{k}}\frac{C\cdot S}{2}\right)+\right.$$

$$p_{x_0}^2\left(\frac{L}{2}+\frac{1}{\sqrt{h}\sqrt{k}}\frac{C\cdot S}{2}\right)+$$

$$x_0\cdot(1+\delta)\left(-(1+\delta)\sqrt{k}\frac{C\cdot S}{\sqrt{h}}+(1+\delta)\sqrt{h}\frac{C\cdot S}{\sqrt{k}}+(k_0-(1+\delta)\,h)\cdot L\right)+$$

$$p_{x_0}\cdot\left(-\frac{(1+\delta)}{k}\cdot C^2+\frac{(1+\delta)}{h}\cdot C^2+\frac{(1+\delta)}{k}-\frac{(1+\delta)}{h}\right)+$$

$$x_0\cdot p_{x_0}\cdot(1+\delta)\left(C^2-1\right)+p_{y_0}^2L+$$

$$-\sqrt[3]{\frac{1+\delta}{h}}\sqrt{k_0}\cdot\frac{C\cdot S}{2}+(1+\delta)^2\frac{C\cdot S}{\sqrt{h}\sqrt{k}}-(1+\delta)^2\frac{\sqrt{h}}{\sqrt[3]{k}}\frac{C\cdot S}{2}+$$

$$\left.+(1+\delta)^2\cdot\frac{k}{2h}L+(1+\delta)^2\frac{h}{2k}L-(1+\delta)^2L\right\}. \tag{17}$$

As one can see, the difference is the coefficient in front of the linear term which is coming from the choice of the "quasi"-exact solutions for p_t or the second-order approximation. It makes sense to replace second-order expansion in p_t by the exact solution for the new MAD (?). In this particular case we can easily have the "exact" solution without adding any complication to the calculation.

Reference:

1. A.Wolski, Beam Dynamics in HEP Accelerators, chapter 3, p.101-105

2. A. Latina, Implementation of a Thick Dipole in the MAD-X tracking module