Contents

Acknowledgements

First and foremost, I would like to thank my supervisor, Michel Van den Bergh. His expertise, incredible insight and the ease with which he navigates the mathematical landscape has made this thesis the diverse collection of techniques and ideas that it is.

I also wish to express my gratitude to Dennis Presotto, with whom I collaborated for the research in the last chapters of this thesis. Working together has not only been fruitful but also just plain fun.

Thank you also to the many people who have placed their trust in me by inviting me for a stay or to give a talk over the course of the past 5 years.

On a more personal note, I would like to thank my family: my parents, Stefan and my Capons. You have always trusted me, supported me, been patient with me and been proud of me. For this I am grateful.

Finally, thank you to my Erin. This step is done. On to the next one.

Nederlandstalige Samenvatting

De rode draad doorheen dit proefschrift is de notie van Calabi-Yau algebra, gedefinieerd door een zekere dualiteitseigenschap.
In het inleidende hoofdstuk, geven we een overzicht van hun rol in de algebra en de meetkunde. In het bijzonder beschrijven de constructie van de Ginzburg DG algebra van een quiver met potentiaal alsook de hogere preprojectieve algebra van een eindige dimensionale algebra die telkens een voorbeeld van Calabi-Yau algebra leveren. Daarnaast leggen we uit hoe de theorie van Calabi-Yau algebras toepassingen heeft in de constructie van cluster categorieën..

Het eerste hoofdstuk beschrijft werk van de auteur met Van den Bergh waarin twee resultaten opgesteld worden die toepassingen hebben in de verscheidene constructies vermeld in de inleiding. We geven een stel criteria voor Jacobi algebras waaruit we kunnen concluderen dat de onderliggende quiver met potentiaal geen lussen of 2-cycles heeft en er geen verwerft na toepassing van een mutatie. Vermits het muteren van een quiver met potentiaal overeenkomt met een afgeleide equivalentie van de bijbehorende Jacobi algebras, bekomen zo een rijke collectie afgeleid equivalente algebras. We tonen aan dat deze condities voldaan zijn voor de opgerolde algebras van een helix op een Del Pezzo oppervlak en voor getwiste groep algebras van cyclische groepsacties in dimensie 3. Het tweede resultaat handelt over de singulariteitencategorie van geïsoleerde Gorenstein singulariteiten. In het geval dat deze een niet-commutatieve crepante resolutie toelaten, kunnen we een relatieve versie van de singulariteitencategorie definiëren. Een toepassing van de theorie van minimale modellen levert een beschrijving van de singulariteitencategorie als clustercategorie. In het dimensie 3 geval alsook in het geval van cyclische quotient singulariteiten verkrijgen we hier dan expliciete formules

In het tweede hoofdstuk stellen we een deformatietheorie op voor Calabi-Yau algebras. We geven een homologe interpretatie van de definitie van Calabi-Yau algebra door behulp van een vectorruimte A, Hochschild-cocykel μ en een cykel in negatief cyclische homologie η. Deze reïnterpretatie laat ons toe op een canonieke manier een deformatietheorie te definiëren. Er is een gekend mechanisme dat een deformatietheorie associeert aan een nilpotente DG Lie algebra. We tonen aan dat er ook in het geval van deformaties van Calabi-Yau algebras zo'n nilpotente DG Lie algebra $\mathfrak{D}^\bullet(A, \mu, \eta)$ bestaat die de deformaties beschrijft. We tonen tevens aan dat de cohomologie van deze dg Lie algebra negatief cyclische homologie is en dat de gegradeerde haak samenvalt met een gekende haak die een niet-commutatieve versie van de string topologie haak van Chas en Sullivan vormt. Met wat meer werk, volgt hieruit dat de raakruimte van dit deformatieprobleem negatief cyclische homologie groep $\mathrm{HC}_{d-2}(A)$ is en dat er een canoniek gedefinieerde obstructie theorie is, die verdwijnt in het geval dat $d \leq 3$. Ten laatste beschrijven we het meetkundig geval waarin de algebra gegeven wordt door globale secties van een Calabi-Yau variteit X. In dit geval passen we een resultaat van Willwacher aan om een concretere beschrijving te geven van $\mathfrak{D}(A, \mu, \eta)$ door middel van een L_∞-quasi-isomorfisme met een DG Lie algebra van de vorm $T^{\bullet,\mathrm{poly}}(X)[[u]]$.

In het vervolg van deze thesis wordt de aandacht verschoven naar de relatie tussen Del Pezzo oppervlakken en Calabi-Yau algebras. Het is gekend dat deze klasse van oppervlakken een volle exceptionele rij heeft en dat de braid groep transitief op deze collecties werkt door mutatie. Hoofdstuk drie is een eerste stap naar een rëinterpretatie van dit resultaat in de context van niet-commutatieve meetkunde. Door de numerieke Grothendieck groep van het oppervlak te beschouwen bekomen we de notie van exceptionele basis, waarop de braid groep werkt. Dit laat ons toe om meer algemeen vrije Abelse groepen met een unimodulaire bilineaire vorm te beschouwen (zogenaamde 'lattices'). We stellen voor deze lattices een aantal axioma's op die voldaan zijn in het meetkundig geval. Deze axioma's laten ons toe om klassiek meetkundige concepten zoals de codimensie filtratie, numerieke Picard groep, intersectie vorm, rang, graad, canonische klasse,... te definiëren in deze nieuwe meer algemene context. Deze nieuwe noties laten ons tevens toe een veralgemening van Del Pezzo te beschrijven in deze context. We tonen aan dat er exact 4 lattices bestaan die aan deze condities voldoen op isomorfie na. Een eerste triviaal type, dan de numerieke Grothendieck groep horende bij het projectief vlak en het eerste Hirzebruch oppervlak en ten laatste een type dat niet bij een Del Pezzo oppervlak kan horen. De rest van deze thesis beschrijft een niet-commutatief meetkundig model voor dit type.

In hoofdstuk vier beschrijven we een nieuwe klasse algebras die nodig zijn om een lokale beschrijving van dit niet-commutatieve model te geven. Deze algebras worden geconstrueerd aan de hand van een relatief Frobenius paarS/R (een veralgemening van de notie van Frobenius algebra over een willekeurige commutatieve grondring R). In het geval R^n/R, komt deze algebra overeen met de klassieke preprojectieve algebra over de ster quiver op n vertices. Voor deze reden noemen we deze algebras veralgemeend preprojectief, en noteren deze met $\Pi_R(S)$. We tonen aan dat indien S een vrij R-moduul van rang 4 is, dat $\Pi_R(S)$ noethers en eindig over zijn centrum is. We tonen tevens aan dat deze eindige global dimensie heeft indien R en S regulier zijn.

Het laatste hoofdstuk van deze thesis is gewijd aan een constructie van het niet-commutatief oppervlak ingeleid aan het einde van hoofdstuk 3. Een heuristische redenering op basis van de expliciete vorm van de Gram matrix in type 3 duidt aan dat dit oppervlak twee canoniek gedefinieerde morfismes moet hebben naar de projectieve lijn $\mathbb{P}^1$ zodat de exceptionele rij in kwestie gevormd wordt door pullback van de standaard exceptionele rij $(\mathcal{O}_{\mathbb{P}^{\nvDash}}, \mathcal{O}_{\mathbb{P}^1}(1))$ door deze twee morfismes. Dit lijdt ons tot de theorie van niet-commutatieve $\mathbb{P}^1$ bundels over een commutatief schema te bekijken. Deze worden meer precies geconstrueerd als een symmetrische schoof $\mathbb{Z}$-algebra $\mathbb{S}(\mathcal{E})$ vertrekkende van een lokaal vrij bimoduul $\mathcal{E}$ dat in ons geval de rangen $(4, 1)$ heeft voor numerieke redenen. Indien X affien is, kunnen we deze in verband brengen met de veralgemeende preprojectieve algebra ingevoerd in het vorige hoofdstuk en dus bewijzen dat de categorie van modulen lokaal noethers is. Dit laat ons toe het niet-commutatieve schema $Z = \mathrm{Proj}(\mathbb{S}(\mathcal{E})$ te construeren. In het laatste stuk passen we gekende resultaten aan in onze context om aan te tonen dat de collectie inderdaad exceptioneel is met de gewenste Gram matrix. We tonen tevens ook aan dat Z eindige globale dimensie heeft.

Overview

This thesis is an investigation of the notion of Calabi-Yau algebras and its role in various branches of algebra and geometry. The defining property of these algebras is a certain self-duality:

Definition 1. Let $\Bbbk$ be a field. A d-Calabi-Yau algebra is an algebra A together with an isomorphism

$$\eta : \mathrm{RHom}_{A^e}(A, A^e) \xrightarrow{\simeq} \Sigma^{-d} A$$

in $\mathscr{D}(A^e)$, the derived category of the enveloping algebra $A^e = A \otimes_{\Bbbk} A^{\mathrm{op}}$.

This definition has natural variants for general commutative groundrings and for DG algebras. We shall first give an account of its use in representation theory and (noncommutative) geometry. It has been known since the work of Ginzburg [Gin] that there is an intimate relation between these algebras and quivers with potential. Given a quiver with potential $(\mathcal{Q}, \mathtt{w})$, one can construct a DG algebra $\Gamma(\mathcal{Q}, \mathtt{w})$, which now bears Ginzburg's name whose cohomology in degree 0 is the ubiquitous Jacobi algebra $\mathrm{Jac}(\mathcal{Q}, \mathtt{w})$ (or vacualgebra as it known to physicists). Combining results of Keller [Kel11] and Ginzburg, it is known that $\Gamma(\mathcal{Q}, \mathtt{w})$ is always a 3-Calabi-Yau DG algebra and that $\mathrm{Jac}(\mathcal{Q}, \mathtt{w})$ is 3-Calabi-Yau iff it is quasi-isomorphic to $\Gamma(\mathcal{Q}, \mathtt{w})$. Conversely, in [VdB10], Van den Bergh showed that if A is 3-Calabi-Yau and complete or graded, then A is always a Jacobi algebra of a certain quiver with potential. A second class of examples was described by Keller also in [Kel11]. The classical preprojective

algebra of an acyclic quiver has a natural generalization to any finite dimensional algebra of global dimension $\leq d-1$ written as $\Pi_d(A)$. This notion was introduced in [IO09] in the context of higher representation theory. In [Kel11], a DG version of these higher preprojective algebras, $\mathbf{\Pi}_d(A)$, whose 0th cohomology is $\Pi_d(A)$ was introduced. This construction always yields a d-Calabi-Yau DG algebra. Moreover in the case where $d=3$, Keller showed how to relate both constructions by exhibiting a quiver with potential $(\mathcal{Q}, \mathtt{w})$ such that $\Gamma(\mathcal{Q}, \mathtt{w}) \cong \mathbf{\Pi}_3(A)$.
Throughout our exposition, we will focus our attention on two examples. Our first example uses the above constructions to give a proof that the rolled-up algebra of a geometric helix on a Del Pezzo surface (as defined in [BS10]) is a higher preprojective DG algebra of dimension 3 in the above sense of Keller and hence also a 3-Calabi-Yau Jacobi algebra (more on that below). In the second example, we let a cyclic group act linearly on a vector space of dimension 3. it is well known that if the representation is unimodular, the twisted group ring is Calabi-Yau. Moreover it is easy to write down a quiver with potential for this algebra.
We will end our introductory chapter by explaining how (DG) Calabi-Yau algebras play a key role in the construction of cluster categories, the categorical analogue of cluster algebras.

The first chapter is an account of two results by the author and Van den Bergh concerning the properties associated to the DG algebra $\Gamma(\mathcal{Q}, \mathtt{w})$ described above (see [dTdVdB10] and [dTdVVdB13]). A major aspect of Ginzburg DG algebras (and hence of the Jacobi Algebra) is the fact that there is a combinatorial procedure, called mutation, on $(\mathcal{Q}, \mathtt{w})$ which lifts to a derived equivalence of $\Gamma(\mathcal{Q}, \mathtt{w})$ by [KY11]. This procedure has the disadvantage of only behaving well if the quiver has no loops or 2-cycles (see [IW14], for example for a discussion when this is not satisfied). Moreover, this property is not in general preserved under mutation. Our first result is an easily verified set of conditions for a 3-Calabi-Yau Jacobi algebra which allows one to conclude that the underlying quiver has no loops or 2-cycles and indeed does not acquire any under sub-

sequent mutations, allowing one to repeat the process indefinitely. We show that our conditions are satisfied in the 2 motivating examples described in the above paragraph. Moreover we also mention an example from the theory of deformed preprojective algebras from [CBH98] that shows that the conditions cannot be weakened.
Our second result pertains to the relation between Calabi-Yau algebras and cluster categories hinted at above. It is a well known fact that the singularity category of a Gorenstein isolated singularity R is Calabi-Yau since the work of Auslander in [Aus78]. In the event that R admits a noncommutative crepant resolution in the sense of [VdB02a], we consider a relative version of the singularity category and use it to obtain an explicit DG algebra whose cluster category is precisely the singularity category of R. From this one obtains in particular that the singularity category of a 3-dimensional cyclic quotient singularity satisfying some numerical conditions is the cluster category of a quiver with potential, which yields an alternative interpretation of the results obtained in [AIR]

The second chapter develops the deformation theory of Calabi-Yau algebras. We reinterpret the definition of Calabi-Yau algebra given above as a vector space A with a Hochschild 2-cochain μ which represents the associative multiplication and a negative cyclic chain η which encodes the additional Calabi-Yau duality. This reinterpretation has the advantage that one can naturally associate a deformation functor. It is this functor we wish to describe more explicitly. Following the work of Kontsevich, there is a general mechanism called Maurer-Cartan formalism which associates a deformation functor of a specific type to a nilpotent DG Lie algebra. Hence, whenever one is given a specific deformation problem, it is a natural question as to whether it is given by one of Maurer-Cartan type. If this is true then we say that the deformation problem is 'controlled' by the corresponding DG-Lie algebra. Furthermore in that case the cohomology of the DG-Lie algebra yields an obstruction theory in the classical sense of deformation theory. The archetypical example is the deformation theory of associative algebras which is controlled by the shifted

Hochschild complex $\mathfrak{C}^\bullet(A)$. It is precisely this example we will extend to the case of Calabi-Yau algebras.

The homological interplay between $\mathfrak{C}^\bullet(A)$ and the (normalized) negative cyclic complex $\overline{CC}_\bullet^-(A)$ has very rich structure which was studied by many authors (see [TT05] for our primary source of reference) and is referred to as *noncommutative calculus*. We use this interplay to construct a DG Lie algebra $\mathfrak{D}^\bullet(A, \mu, \eta)$ which as a complex is $\mathfrak{C}^\bullet(A) \oplus \overline{CC}_\bullet^-(A)[-d+1]$ and whose bracket is defined in such a way that the Maurer-Cartan elements correspond to Calabi-Yau algebras. We will show that the deformation functor associated to $\mathfrak{D}^\bullet(A, \mu, \eta)$ controls the deformation theory of Calabi-Yau algebras.
Furthermore, in [Men09], Menichi defined a bracket on the negative cyclic homology of algebras $\mathrm{HC}(A)$ of degree $d-1$, guided by the construction of Chas and Sullivan's bracket on the string homology of manifolds ([CS99]). Using noncommutative calculus once again, we will construct a quasi-isomorphism $\Psi : \mathfrak{D}^\bullet(A, \mu, \eta) \longrightarrow \Sigma^{-d+1}\overline{CC}_\bullet^-(A)$ such that the bracket on $\mathrm{HC}(A)$ induced by the bracket on $\mathrm{H}(\mathfrak{D}(A, \mu, \eta))$ through Ψ coincides with Menichi's construction. This allows us first and foremost to describe the tangent space as $\mathrm{HC}^-_{d-2}(A)$ and second, with a little more work, we also show that the obstruction theory of $\mathfrak{D}^\bullet(A, \mu, \eta)$ provided by the Maurer-Cartan formalism lies in the kernel of the canonical morphism $\mathrm{HC}^-_{d-3}(A) \longrightarrow \mathrm{HC}^{per}_{d-3}(A)$. This implies in particular that the deformation theory of a Calabi-Yau algebra of dimension ≤ 3 is unobstructed. Finally, in the case where $A = \mathcal{O}(X)$ is the ring of global sections on a smooth affine Calabi-Yau variety X, we adapt results by Willwacher (see [Wil08]) to obtain a quasi-isomorphism $(T^{\mathrm{poly},\bullet}(A)[[u]], -u\,\mathrm{div}) \longrightarrow \mathfrak{D}^\bullet(A, , \mu, \eta)$ which reduces to Kontsevich's famous formality morphism $T^{\mathrm{poly},\bullet}(A) \longrightarrow \mathfrak{C}^\bullet(A)$ when we let $u \mapsto 0$ and forget the additional η.

In the rest of this thesis, we focus our attention on the relation between Del Pezzo surfaces and Calabi-Yau algebras. It has been know since the foundational work of the 'Rudakov' seminar (see [GK04]) for example) that (full) exceptional sequences in triangulated categories have very rich structure. In particular they too are equipped with a mutation operation yielding an action of the braid group. In [KO95], this operation was studied in the case where the category in question is the bounded derived category of a Del Pezzo surface X. It is proven there that any such surface always has a full exceptional sequence and that the braid group action is transitive on the set of exceptional sequences. By applying the Serre functor indefinitely on a full exceptional sequence, one obtains the closely related notion of a helix $\mathbb{H}$ upon which this time the cylindrical braid group acts by mutation. As mentioned in the first paragraph, the associated rolled-up algebra $B(\mathbb{H})$ is a 3-Calabi-Yau Jacobi algebra if $\mathbb{H}$ is geometric. The interrelation between the mutations of $\mathbb{H}$ and mutation of the underlying quiver with potential of the algebra $B(\mathbb{H})$ is elucidated in [BS10], where it is shown in particular that mutating a quiver with potential corresponds to performing a series of braid mutations on $\mathbb{H}$. The rest of this thesis can be viewed as an effort to understand these results in the philosophy of noncommutative geometry, i.e. we take the viewpoint that $\mathscr{D}^b(X)$ satisfies some additional categorical properties making the results of [KO95] and [BS10] valid. The first step in this program is to interpret the results 'numerically'

The starting point of chapter three is the fact that by considering the numerical Grothendieck group $K(X)_{\mathrm{num}}$ of the Del Pezzo surface X, we obtain the corresponding statement that $K(X)_{\mathrm{num}}$ has an exceptional basis and that the braid group acts transitively on the set of these bases. Now, the Serre functor S on $\mathscr{D}^b(X)$ defines a unipotent linear map s for which $\mathrm{rk}(s-1) \leq 2$ on $K(X)_{\mathrm{num}}$ (this statement being new to the best of our knowledge). We consider these properties as the foundation for an axiomatic definition of a finitely generated free abelian group K with a nondegenerate bilinear form (henceforth known as a lattice) which is 'of smooth projective surface (SPS*) type'. One can define a canonical 2-step filtration for these lattices which coincides with the codimension filtration when $K = K(X)_{\mathrm{num}}$ and which in turn allows us to define the notions of rank, degree, numerical Picard group, intersection form, canonical class, etc ... all of which indeed extend the well-known geometric notions to this new more general setting. In particular, we obtain a definitions for a lattice 'of Del Pezzo surface (DPS*) type' as one with an canonical class of negative self-intersection in our new language (the reason for the opposite sign will become apparent later on). The main objective of this chapter is a classification of these these lattices of DPS* type in rank 4. In the subsequent section, we review the theory of braid mutations of exceptional sequences and cylindrical mutations of helices. By sending an exceptional sequence in $\mathscr{D}^b(X)$ to its exceptional basis given by the classes in $K(X)_{\mathrm{num}}$, we develop a similar theory of braid mutations on exceptional bases and cylindrical mutations on helices in the context of lattices. Finally, by assigning to an exceptional basis its Gram matrix we in turn develop a theory of braid mutations on so-called *exceptional matrices* and show how this action extends in two way to cylindrical braids. We show how in each of these settings, the orbits under the braid- or cylindrical action coincide. This will have the technical advantage of simplifying the computations with the braid group action.

The final part discusses a classification of lattices of DPS* type with an exceptional basis up to isomorphism. We start by considering the case of rank 3, following the ideas laid out in ([BP94]). The unipotency of the serre automorphism s on $K(X)$ mentioned in the above paragraph in fact translates into the famed Markov equation. Using Markov's classification of the solutions to this equation, it is easy to see that K is isomorphic to the Grothendieck group of $\mathbb{P}^2$. In rank 4, the same unipotency condition takes the form of a system of diophantine equations.

$$\begin{cases} a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - bad - edf - ace - bcf + abdf = 0 \\ af + bd = ce \end{cases}$$

The techniques of Markov's cannot simply be generalized to this case, as this system is notably more complicated. To tackle these equations, we do however show that one can reduce using the mutation actions described in the preceding paragraph to one of two simpler settings: either one of the vectors in the exceptional basis has rank 0 or two successive vectors in the basis satisfy

$\langle e_i, e_{i+1}\rangle = 0$. In the former case, this amounts to saying that three successive vectors of the basis generate again a lattice of SPS type, yielding an extra constraint in the form of an additional Markov equation. The latter situation translates the system of equations into a slightly generalized version of the Markov equation, whose solutions are easily described by adapting Markov's techniques. This allows us to write the Gram matrix of a lattice of DPS* type with an exceptional basis of rank 4 in one of 4 standard forms. We show that such a lattice is either of a certain trivial type, isomorphic to the Grothendieck group of $\mathbb{P} \times \mathbb{P}^1$ or $\mathbb{F}_1$ or one final so-called *exotic* type for which the Gram matrix takes the following form:

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

The rest of this thesis is dedicated to the construction of a geometric model for this type.

In the fourth chapter, we first give an account of the work done in [dTdVP14], joint with D. Presotto. We introduce a new class of rings which will be required to describe the local structure of the noncommutative geometry used in the promised geometric model. To a map of rings $R \longrightarrow S$ which satisfy a relative version of the Frobenius property, we associate a graded algebra $\Pi_R(S)$. This construction coincides with the preprojective algebra of the star quiver when we choose the morphism to be $R \longrightarrow R^n$. For this reason, we call this family of algebras generalized preprojective (not to be confused with the 'higher' preprojective algebras described in the first paragraph). We study the properties of this class of rings in the case where S is a free R-module of rank 4 (as this is the only setting of future applications). We prove that in each degree n, the R-module $\Pi_R(S)_n$ is projective and give an explicit formula for its rank. We also show that this algebra is always noetherian and finite over its center. Finally we give an upper bound for the global dimension of $\Pi_R(S)$, proving in particular that it has finite global dimension if R and S are regular rings.

The fifth chapter describes a construction of a noncommutative surface (in the sense of [AZ94]) with an exceptional sequence whose numerical Grothendieck group is of DPS* type and whose Gram matrix is given by (1) (again in joint work with D. Presotto, see [dTdVP15]). The explicit form of the Gram matrix for this lattice exhibited above suggests that this model should be equipped with two maps to $\mathbb{P}^1$ such that the exceptional sequence in question is formed by pulling back the standard one on $\mathbb{P}^1$ along those two maps in question. The theory of noncommutative $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ yields a construction of a noncommutative surface which readily comes with such maps. These surfaces were first constructed by Van den Bergh in [VdB12] as sheaf $\mathbb{Z}$-algebras. Given a locally free bimodule $\mathcal{E}$, he considers the so-called noncommutative symmetric sheaf $\mathbb{Z}$-algebra $\mathbb{S}(\mathcal{E})$ and the associated noncommutative scheme $\mathrm{Proj}(\mathbb{S}(\mathcal{E}))$. In his foundational paper, he argues that this construction yields a natural noncommutative generalization of a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ provided the bimodule has ranks $(2, 2)$. In order for our exceptional sequence to fit the required data, we have to adapt this theory to the case where the bimodule has rank $(4, 1)$.

The work of [Mor07] provides a general framework in which one can compute Euler characteristics of modules over $\mathrm{Proj}(\mathbb{S}(\mathcal{E}))$ provided one knows a number of geometric properties beforehand. We adapt the technique of 'point modules' used in [VdB12] to show that the bimodule $\mathbb{S}(\mathcal{E})_{n,m}$ is locally free in each degree and subsequently compute its left and right rank. Next, we give a local description of $\mathbb{S}(\mathcal{E})$ and show how it can be covered in the appropriate sense so that locally the category of graded modules is a direct summand of the category of graded modules over a generalized preprojective algebra $\Pi_R(S)$ as studied in the previous chapter. The properties we proved for this algebra allow us to prove that $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ is noetherian, this in turn allows us to use Mori's technique to finally show that the Grothendieck group of $\mathrm{Proj}(\mathbb{S}(\mathcal{E}))$ is of exotic type in this case as promised.

Chapter 0

All Things Calabi-Yau

The notion of Calabi-Yau traces its origin to the work of Yau on the Calabi conjecture for complex compact Kähler manifolds X with a trivial canonical bundle. Since [BK89], it is known that this is condition can be reformulated in terms of the Serre functor on the bounded derived category of X being naturally isomorphic to a shift. This purely categorical statement leads to the notion of a Calabi-Yau triangulated category over a field $\Bbbk$:

Definition 0.0.1. Let $\mathcal{C}$ be a Hom-finite triangulated category over $\Bbbk$ with a Serre functor S. We say that $\mathcal{C}$ is d-Calabi-Yau if there is a natural isomorphism

$$S \cong \Sigma^d$$

in $\mathcal{C}$ where Σ denotes the shift functor

One of Kontsevich's fundamental insights was to propose a definition for a Calabi-Yau algebra which in particular implies that the derived category of finite A-modules $\mathscr{D}^b_f(A)$ is Calabi-Yau in the above sense. This notion will be of fundamental importance throughout this thesis. We give the definition over a general commutative groundring R as this generality will be required in the sequel.

For an R-algebra A, we put $A^e = A \otimes_R A^{op}$ and use without further comment the standard equivalences between the categories of left A^e-modules, right A^e-modules and A-bimodules which are R-central.

An A^e-module is called *perfect* if it has a finite resolution by finitely generated projective A^e-modules. If A is R-flat and A is a perfect A^e-module then we say that A is *homologically smooth* over R.[1]

Definition 0.0.2. [Gin] A Calabi-Yau R-algebra of dimension d is a pair (A, η) where

1. A is an R-algebra, homologically smooth over R;
2. $\eta : \mathrm{RHom}_{A^e}(A, A^e) \xrightarrow{\simeq} \Sigma^{-d} A$ is an isomorphism in $\mathscr{D}(A^e)$.

Remark 0.0.3. *Note that the amount of freedom for the choice of η is quite limited. If (A, η) and (A, η') are Calabi-Yau algebras, then there exists a central element $z \in Z(A)$ such that $\eta' = z\eta$ (see [Dav12]).*

There is another variant of 0.0.2 of the Calabi-Yau condition that we will use throughout, namely that of a Calabi-Yau (DG)-algebra [2]

Definition 0.0.4. A Calabi-Yau DG algebra over R is a pair (Γ, η) where

- Γ is a DG algebra which is flat over R and perfect as an $\Gamma \otimes_R \Gamma^{\mathrm{op}}$ DG-module
- $\eta : \mathrm{RHom}_{\Gamma^e}(\Gamma, \Gamma^e) \longrightarrow \Sigma^{-d}\Gamma$ is an isomorphism in $\mathscr{D}(\Gamma \otimes_R \Gamma^{op})$.

Remark 0.0.5. *It is clear that this definition coincides with 0.0.2 if we consider A to be a DG algebra concentrated in degree 0*

[1] The implicit flatness hypothesis ensures that this is the correct 'derived' definition

[2] We note the slight difference in terminology from [Kel11] as Keller considers both conditions separately

0.1 the Ginzburg- and Higher Preprojective DG algebras

The work of Keller and Ginzburg ([Kel11],[Gin]) provides two important constructions which yield DG algebras that always satisfy the Calabi-Yau condition 0.0.4.

To give an account of the first construction, we briefly recall the notion of a quiver with potential: let $\mathcal{Q}$ denote a finite quiver. Let $(\Bbbk\,\mathcal{Q})^\wedge \overset{\text{def}}{=} \prod_i \Bbbk\,\mathcal{Q}_i$ denote the path algebra of $\mathcal{Q}$ over the field $\Bbbk$, completed with respect to the ideal $\Bbbk\,\mathcal{Q}_{\geq 1}$. A potential on $\mathcal{Q}$ is an element in $\prod_{i\geq 2} \Bbbk\,\mathcal{Q}_{i,cyc}$, the vector space spanned by all cycles of length ≥ 2. The data of $(\mathcal{Q}, \mathtt{w})$ is a *quiver with potential* or QP for short. We say that a potential is *reduced* if $\mathtt{w} \in \prod_{i\geq 3} \Bbbk\,\mathcal{Q}_{i,cyc}$ and *trivial* if $\mathtt{w} \in \Bbbk\,\mathcal{Q}_2$. Two potentials are *cyclically equivalent* if their difference lies in $[\Bbbk\,\mathcal{Q}^\wedge, \Bbbk\,\mathcal{Q}^\wedge]$. Two quivers with potentials $(\mathcal{Q}, \mathtt{w})$ and $(\mathcal{Q}', \mathtt{w}')$ are *right equivalent* if there is an isomorphism $\widehat{\Bbbk\,\mathcal{Q}} \longrightarrow \widehat{\Bbbk\,\mathcal{Q}'}$ such that $\phi|_{\mathcal{Q}_0} = \mathrm{Id}$ and $\phi(\mathtt{w})$ is cyclically equivalent to $\mathtt{w}'$. One can always decompose a quiver with potential in a trivial and reduced part according to the following *splitting theorem:*

Theorem 0.1.1 ([DWZ08],4.6)**.** *Any quiver with potential $(\mathcal{Q}, \mathtt{w})$ is right equivalent to a direct sum $(\mathcal{Q}^{triv}, \mathtt{w}^{triv}) \oplus (\mathcal{Q}^{red}, \mathtt{w}^{red})$ where $(\mathcal{Q}^{triv}, \mathtt{w}^{triv})$ is trivial and $(\mathcal{Q}^{red}, \mathtt{w}^{red})$ is reduced. This decomposition is unique up to right equivalence.*

We construct a DG algebra from a quiver with potential $(\mathcal{Q}, \mathtt{w})$ as follows: let $\tilde{\mathcal{Q}}$ be the graded quiver on the vertex set $\mathcal{Q}_0$ defined as follows:

- each arrow in $\mathcal{Q}_1$ defines an arrow of degree 0 in $\tilde{\mathcal{Q}}$
- for each arrow α in $\mathcal{Q}$, we add an arrow in the opposite direction α^* of degree -1 in $\tilde{\mathcal{Q}}$
- add a loop t_i in $\tilde{Q}$ of degree -2 to each vertex $i \in \mathcal{Q}_0$

We define a differential on $\Bbbk\tilde{\mathcal{Q}}^{\wedge}$ by requiring that

- $d\alpha = 0$ for $\alpha \in \mathcal{Q}_1$
- $d\alpha^* = \partial_\alpha \mathtt{w}$ where $\partial_a \mathtt{w}$ is the cyclic derivative

$$\Bbbk\,\mathcal{Q}^{\wedge} \longrightarrow \Bbbk\,\mathcal{Q}^{\wedge} : w \longrightarrow \sum_{w=u\alpha v} vu$$

- $dt_i = e_i \sum_j [\alpha_j, \alpha_j^*] e_i$

Definition 0.1.2. Let $(\mathcal{Q}, \mathtt{w})$ be a quiver with potential. The DG algebra constructed above is called the Ginzburg DG algebra, denoted $\Gamma(\mathcal{Q}, \mathtt{w})$.
The algebra

$$\mathrm{H}^0(\Gamma(\mathcal{Q}, \mathtt{w}) \cong (\Bbbk\,\mathcal{Q})^{\wedge}/\overline{\{\partial_\alpha w\,|\alpha \in \mathcal{Q}_1\}}$$

is called the Jacobi algebra of $(\mathcal{Q}, w)$, denoted $\mathrm{Jac}(\mathcal{Q}, w)$. If $\mathrm{Jac}(\mathcal{Q}, \mathtt{w})$ is finite dimensional, we say that $(\mathcal{Q}, \mathtt{w})$ is *Jacobi-finite*

Theorem 0.1.3. *[Kel11] For any quiver with potential* $(\mathcal{Q}, \mathtt{w})$ *the DG algebra* $\Gamma(\mathcal{Q}, \mathtt{w})$ *is 3-Calabi-Yau (in the sense of 0.0.4)*

It is not true in general that the Jacobi algebra is 3-Calabi-Yau. However Ginzburg proved the following criterion:

Theorem 0.1.4 ([Gin],5.3.1)**.** *The Jacobi algebra of a quiver with potential* $(\mathcal{Q}, \mathtt{w})$ *is* 3*-Calabi-Yau if and only if the cohomology of the DG algebra* $\Gamma(\mathcal{Q}, \mathtt{w})$ *is concentrated in degree 0, in which case* $\mathrm{Jac}(\mathcal{Q}, \mathtt{w}) \cong \Gamma(\mathcal{Q}, \mathtt{w})$

Conversely, it is known that not every Calabi-Yau algebra can be written as a Jacobi algebra, (see for example [Dav12] for an elegant example). There are two important settings in which this does hold however:

Theorem 0.1.5 ([Boc08],[VdB10])**.** *Let* A *be a* 3*-Calabi-Yau algebra. Assume that* A *is either graded or complete. Then there exists a quiver* $\mathcal{Q}$ *with (homogeneous) potential* $\mathtt{w}$ *such that*

$$A \cong \mathrm{Jac}(\mathcal{Q}, \mathtt{w})$$

One of the major features of quivers with potentials is that one can define an extremely rich operation called mutation on them. This rule is closely related to the rule of mutations for quivers without loops or 2-cycles, defined by Fomin and Zelevinsky in [FZ02]:

Definition 0.1.6. Let $\mathcal{Q}$ be a quiver without loops or 2-cycles and $i \in \mathcal{Q}_0$. Then the *mutation of* $\mathcal{Q}$ *at* i is the quiver $\mu_i(\mathcal{Q})$ defined as follows:

- For each path of length 2, $h \xrightarrow{\alpha} i \xrightarrow{\beta} j$ add an extra arrow $h \xrightarrow{[\alpha\beta]} j$
- replace each arrow α incident to i with an arrow α^* in the opposite direction
- remove arrows occurring in a maximal set of pairwise disjoint 2-cycles

The mutation of a QP $(\mathcal{Q}, \mathtt{w})$ is now described by the following variation of the above rule

Definition 0.1.7. Let $(\mathcal{Q}, \mathtt{w})$ be a quiver with potential and $i \in \mathcal{Q}_0$. Assume that $\mathcal{Q}$ has no loops or 2-cycles. Then the *mutation of* $(\mathcal{Q}, \mathtt{w})$ *at* i is given by $\mu_i(\mathcal{Q}, \mathtt{w})$ as the result of the following series of steps:

1. For each path of length 2, $h \xrightarrow{\alpha} i \xrightarrow{\beta} j$ add an extra arrow $h \xrightarrow{[\alpha\beta]} j$
2. replace each arrow α incident to i with an arrow α^* in the opposite direction
3. define $\mathtt{w}' = [\mathtt{w}] + \Delta$ where $[\mathtt{w}]$ is obtained by substituting $[\alpha\beta]$ for each path $h \xrightarrow{\alpha} i \xrightarrow{\beta} j$ in $\mathtt{w}$ and
$$\Delta = \sum_{\alpha,\beta \in \mathcal{Q}_1} \beta^* \alpha^* [\alpha\beta]$$
4. take the reduced part as in the splitting theorem. 0.1.1

To see how quiver mutation compares to QP mutation, we have the following easy observation:

Lemma 0.1.8. *Let $(\mathcal{Q}, \mathtt{w})$ be a quiver with potential, $i \in \mathcal{Q}_0$ and write $\mu_i(\mathcal{Q}, w) = (\mathcal{Q}', \mathtt{w}')$. Then $\mu_i(\mathcal{Q})$ is obtained from $\mathcal{Q}'$ after removing a maximal set of 2-cycles*

In general, a mutation of a quiver with potential can produce a 2-cycle (as opposed to the quiver case). In particular, mutations cannot be composed in general. We therefore make the following definition:

Definition 0.1.9. Let $\mathcal{Q}$ be a quiver without loops or 2-cycles and $\mathtt{w}$ a potential on $\mathcal{Q}$. We say that $(\mathcal{Q}, \mathtt{w})$ is nondegenerate if for any composition of mutations,

$$(\mu_{i_1} \circ \ldots \circ \mu_{i_n})(\mathcal{Q}, \mathtt{w})$$

the underlying quiver has no loops or 2-cycles

One of the major results in the next chapter is a criterion that implies that certain Jacobi algebras have an underlying quiver which is nondegenerate (we refer the curious reader to 1.2.14).

The relation between mutations of quiver with potentials and their Ginzburg DG algebras was elucidated in the following theorem:

Theorem 0.1.10. *[KY11] Let $(\mathcal{Q}, \mathtt{w})$ be a quiver with potential such that $\mathcal{Q}$ has no loops or 2-cycles and let Γ be the associated Ginzburg DG algebra. Let $i \in \mathcal{Q}_0$ and write Γ' for the Ginzburg DG algebra of $\mu_i(\mathcal{Q}, \mathtt{w})$.*

- *There is a $\Gamma \otimes \Gamma'$ DG bimodule T inducing mutually inverse equivalences*
$$(-) \overset{L}{\otimes}_{\Gamma'} T : \mathscr{D}^+(\Gamma') \longrightarrow \mathscr{D}^+(\Gamma) \text{ and } \mathrm{RHom}_\Gamma(-, T) : \mathscr{D}^+(\Gamma') \longrightarrow \mathscr{D}^+(\Gamma) \tag{1}$$
- *If $\Gamma \cong \mathrm{Jac}(\mathcal{Q}, \mathtt{w})$ has cohomology concentrated in degree 0, then so does $\Gamma' \cong \mathrm{Jac}(\mu_i(\mathcal{Q}, \mathtt{w}))$ and T is a 2-term tilting complex.*

There is a second construction, closely related to the construction Ginzburg DG algebra which yields a DG algebra which is d-Calabi-Yau for general d. The idea behind its construction takes the preprojective algebra of an acyclic quiver as its starting point.

Definition 0.1.11. Let $\mathcal{Q}$ be an acyclic quiver and R a commutative ring. Let $\overline{\mathcal{Q}}$ be the doubled quiver obtained by adding an arrow α^* in the opposite direction for every $\alpha \in \mathcal{Q}_1$. The preprojective algebra $\Pi_R(\mathcal{Q})$ is defined as

$$\Pi_R(\mathcal{Q}) \stackrel{\text{def}}{=} R\overline{\mathcal{Q}}/\left(\sum_{\alpha\in\mathcal{Q}_1}[\alpha,\alpha^*]\right)$$

In the case where the groundring is a field $R = \Bbbk$, a classical result by Ringel gives a simple homological description of this algebra. We denote by $D(-)$ the usual duality $\mathrm{Hom}_{\Bbbk}(-,\Bbbk)$.

Theorem 0.1.12 ([Rin98])**.** *The preprojective algebra of an acyclic quiver is isomorphic to the algebra $T_{\Bbbk\,\mathcal{Q}}\Theta$ where Θ is the $\Bbbk\,\mathcal{Q}$-bimodule $\mathrm{Ext}^1_{\Bbbk\,\mathcal{Q}}(D(\Bbbk\,\mathcal{Q}),\Bbbk\,\mathcal{Q})$*

Remark 0.1.13. *Note that Θ indeed is a bimodule, the left action being induced from the right action on $D(\Bbbk\,\mathcal{Q})$ and the right action coming from the right action of the second variable $\Bbbk\,\mathcal{Q}$.*

It is this characterization that is used to generalize the notion of preprojective algebra for algebras of higher global dimension:

Definition 0.1.14. [IO09] Let A be a finite dimensional $\Bbbk$-algebra of finite global dimension $\leq d-1$. The d-preprojective algebra is given by the formula:

$$\Pi_d A = T_A\,\mathrm{Ext}_A^{d-1}(D(A_A),A)$$

Higher preprojective algebras were reinterpreted in the DG context by Keller in [Kel11].[3]

Definition 0.1.15. ([Kel11, §4] Let A be a finite dimensional $\Bbbk$-algebra of global dimension $\leq (d-1)$. Let Θ be a cofibrant replacement of the inverse dualizing DG-bimodule $\mathrm{RHom}_A^\bullet(D(A),A)$ in $\mathscr{D}(A^e)$. We define the derived preprojective DG algebra as

$$\mathbf{\Pi}_d(A) \stackrel{\text{def}}{=} T_A\Theta([d-1])$$

[3]Keller in fact defines this notion in a much more general context of a homologically smooth DG category, cofibrant over a commutative groundring, but to retain clarity, we choose to remain in this setting

Theorem 0.1.16. *Let A be finite dimensional with* $\mathrm{gl.dim} \leq (d-1)$ *. Then the DG algebra $\mathbf{\Pi}_d(A)$ is d-Calabi-Yau. Moreover we have $H^0(\mathbf{\Pi}_d(A)) = \Pi_d(A)$.*

Proof. See [Kel11, theorem 4.8] □

We now have described two constructions of a 3-Calabi-Yau DG algebra using theorems 0.1.3 and 0.1.16. One starting from a quiver with potential and one starting from a finite dimensional algebra of global dimension ≤ 2. The relation between both constructions is elucidated once again in [Kel11]:

Theorem 0.1.17 ([Kel11], thm. 6.10)**.** *Let $A = k\,\mathcal{Q}\,/I$ be a finite dimensional algebra of global dimension ≤ 2. Then there exists an extension $\mathcal{Q}'$ of the quiver $\mathcal{Q}$ and a graded potential $\mathtt{w}'$ in degree 1 such that*

$$\Pi_3(A) \cong \Gamma(\mathcal{Q}', \mathtt{w}').$$

0.2 Two Examples

0.2.1 Exceptional Sequences, Helices and Their Algebras

We shall first describe these constructions in the setting of the bounded derived category of coherent sheaves on Del Pezzo surfaces. We briefly recall the required notions:

Definition 0.2.1. Let $\mathcal{T}$ be a Hom-finite triangulated category. An object E is exceptional if $\mathrm{Hom}^\bullet_{\mathcal{T}}(E, E) \cong \Bbbk$ as graded modules.
A sequence of exceptional objects $\mathbb{E} \stackrel{\text{def}}{=} (E_i)_i$ is called exceptional if it satisfies the additional condition $\mathrm{Hom}^\bullet_{\mathcal{T}}(E_i, E_j) = 0$ for $i > j$.

An exceptional sequence is *full* if it generates the whole category (i.e. the smallest triangulated category containing $\mathbb{E}$ is $\mathcal{C}$)[4] and strong if $\mathrm{Hom}^\alpha(E_i, E_j) = 0$ for $\alpha > 0$. The first example of a full, strong exceptional sequence was famously constructed on $\mathscr{D}^b(\mathbb{P}^n)$ by Beilinson

[4]note that this category is necessarily thick

Example 0.2.2 ([Bei90]). *Let $X = \mathbb{P}^n$. The sequence $(\mathcal{O}_X, \dots \mathcal{O}_X(n))$ is a full, strong exceptional sequence on $\mathscr{D}^b(X)$*

Using this example, combined with the fact that full exceptional sequences can be lifted under blowups by adding precisely one object, Kuleshov-Orlov proved their famous theorem:

Theorem 0.2.3 ([KO95]). *Let X be a Del Pezzo surface. Then $\mathscr{D}^b(X)$ has a full, strong exceptional sequence .*

A notion closely related to that of exceptional sequences is that of helix, which is constructed from a full exceptional collection by applying the Serre functor and its inverse indefinitely to it in both direction[5]

Definition 0.2.4. Let $\mathcal{T}$ be a Hom-finite triangulated category with Serre functor S. A helix of length n and period d is a $\mathbb{Z}$-indexed collection of objects $\mathbb{H} \stackrel{\text{def}}{=} (E_i)_{i \in \mathbb{Z}}$ in $\mathcal{T}$ such that

- any *thread* $(E_{i+1}, \dots E_{i+n})$ is a full exceptional sequence for each $i \in \mathbb{Z}$
- $SE_i = E_{i-n}[d]$

A helix is *geometric* if $\mathrm{Hom}^{\alpha}(E_i, E_j) = 0$ for $\alpha \neq 0$ and $i \leq j$

we say that $\mathbb{H}$ is of *type* (n, d)

Remark 0.2.5. *After a choice of d any exceptional sequence $\mathbb{E}$ of length n defines a helix $\mathfrak{H}(\mathbb{E})$ of type (n, d) though the rule*

$$E_{i-kn} = SE_{i+(k-1)n}[-d]$$

Conversely, to any helix, one can associate the initial thread $\mathfrak{E}(\mathbb{H}) \stackrel{def}{=} (E_1, \dots E_n)$ which is an exceptional sequence by the definition, yielding inverse bijections $\mathfrak{E}$ and $\mathfrak{H}$ between helices and exceptional sequences

[5]Note that any category with a full exceptional sequence is saturated and hence has a Serre functor by [BVdB03])

Now, let $\mathbb{E} = (E_1, \dots E_n)$ be an exceptional sequence. Following [BS10], we put $\mathbf{E} = \bigoplus_i E_i$ and define the endomorphism algebra of $\mathbb{E}$ to be

$$A(\mathbb{E}) = \mathrm{End}_{\mathcal{T}}(\mathbf{E}, \mathbf{E}) \tag{2}$$

If the helix $\mathbb{H}$ is geometric, we define the *rolled up algebra of* $\mathbb{H}$ to be

$$B(\mathbb{H}) = \oplus_i \mathrm{Hom}_{\mathcal{T}}(\mathbf{E}, S^{-i}\mathbf{E}[d]) \tag{3}$$

with obvious multiplication. As mentioned above, rolled up algebras form an important example in this work of a 3-Calabi-Yau algebra [6].

Lemma 0.2.6. *(see [dTdVVdB13, app. A]) Let X be a Del Pezzo surface with a full exceptional sequence $\mathbb{E}$ and associated period 2 helix $\mathbb{H} = H(\mathbb{E})$. Assume that $\mathbb{H}$ is geometric. Let $A = A(\mathbb{E})$ and $B = B(\mathbb{H})$ as above*

1. *then $B(\mathbb{H})$ is the derived tensor algebra of the A-bimodule B_1*
2. $B_1 \cong \mathrm{RHom}_{A^e}(A, A^e)[2]$
3. $\mathrm{gl.dim}(A(\mathbb{E})) \leq 2$

Proof. Let $\mathbf{E} = \bigoplus_i \mathbb{E}_i$.
To show the first point, since the Serre functor is given by $S(-) = (-) \otimes \omega_X[2]$ by Serre duality, for $k < 0$ we have

$$\mathrm{Hom}(\mathbf{E}, \omega_X^{-k} \otimes \mathbf{E}) = \mathrm{Ext}^2(\mathbf{E}, \omega_X^{k+1} \otimes \mathbf{E})^* = 0$$

Moreover, for $k \geq 0$ by exceptionality, we have

$$\mathrm{Hom}(\mathbf{E}, \omega_X^{-k} \otimes \mathbf{E}) = \mathrm{RHom}(\mathbf{E}, \omega_X^{-k} \otimes \mathbf{E})$$

hence the claim reduces to showing that for $k, l \geq 0$ the canonical map

$$\mathrm{RHom}(\mathbf{E}, \omega_X^{-k} \otimes \mathbf{E}) \overset{L}{\otimes}_{A(\mathbb{E})} \mathrm{RHom}(\mathbf{E}, \omega_X^{-l} \otimes \mathbf{E}) \longrightarrow \mathrm{RHom}(\mathbf{E}, \omega_X^{-k-l} \otimes \mathbf{E})$$

[6]Part of this result can be equally be obtained from [BS10, thm.3.6], although their definition of Calabi-Yau is our 0.0.1

is an isomorphism, which in turn is equivalent to the map

$$\mathrm{RHom}(\mathbf{E}, \omega_X^{-k} \overset{L}{\otimes} \mathbf{E}) \overset{L}{\otimes}_{A(\mathbf{E})} \mathrm{RHom}(\omega_X^l \otimes \mathbf{E}, \mathbf{E}) \longrightarrow \mathrm{RHom}(\omega_X^l \otimes \mathbf{E}, \omega_X^{-k} \otimes \mathbf{E})$$

being an isomorphism.

By the fullness of the exceptional sequence $\mathbf{E}$ is a classical generator for $\mathscr{D}^b(\mathcal{O}_X)$ and we can replace $\omega_X^{-k} \otimes \mathbf{E}$ and $\omega_X^l \otimes \mathbf{E}$ by $\mathbf{E}$ in the above composition. The statement becomes obvious in this this case. Since it is clear that $B_0 = A$, by the definition of the rolled-up algebra (3), we obtain the first statement.

The second statement requires us to compute the complex

$$\mathrm{RHom}_{\mathrm{End}(\mathbf{E})^e}(\mathrm{End}(\mathbf{E}), \mathrm{End}(\mathbf{E}) \otimes \mathrm{End}(\mathbf{E}))$$

Let $\boxtimes$ denote the *external tensor product* $\pi_1^*(-) \otimes_{X \times X} \pi_2^*(-)$ where $\pi_1, \pi_2 : X \times X \longrightarrow X$ are the canonical projections of $X \times X$. Then we have an isomorphism

$$\mathrm{End}(\mathbf{E})^e \cong \mathrm{End}(\mathbf{E} \boxtimes \mathbf{E}^*)$$

(see [Cra, 3]). $\mathbf{E}$ being a classical generator, implies that $\mathbf{E}^*$ must be so too, and hence $\mathbf{E} \boxtimes \mathbf{E}^*$ is in turn a classical generator for $\mathscr{D}^b(X \times X)$ by [BVdB03, 3.4.1]. Moreover, since exceptional objects are quasi-isomorphic to shifted locally free sheaves by [KO95], $\mathbf{E} \boxtimes \mathbf{E}^*$ is perfect and hence compact, yielding a derived equivalence

$$\mathrm{RHom}_{\mathscr{D}^b(X)}(\mathbf{E} \boxtimes \mathbf{E}^*, -) : \mathscr{D}^b(X \times X) \xrightarrow{\simeq} \mathscr{D}^b(\mathrm{End}(\mathbf{E} \boxtimes \mathbf{E}^*))$$

it is easy to check that under this equivalence

- the structure sheaf of the diagonal $\mathcal{O}_\Delta$ corresponds to the module $\mathrm{End}(\mathbf{E})$
- that the object $\mathbf{E} \boxtimes \mathbf{E}^*$ corresponds to $\mathrm{End}(\mathbb{E}) \otimes \mathrm{End}(\mathbb{E})$

implying that there is an isomorphism

$$\mathrm{RHom}_{\mathrm{End}(\mathbf{E})^e}(\mathrm{End}(\mathbf{E}), \mathrm{End}(\mathbf{E}) \otimes \mathrm{End}(\mathbf{E})) \cong \mathrm{RHom}_{X \times X}(\mathcal{O}_\Delta, \mathbf{E} \boxtimes \mathbf{E}^*)$$

where Δ is the diagonal. Using the well known formula

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_{X\times X}) = \omega_\Delta^{-1}[-2] \tag{4}$$

we obtain

$$\begin{aligned}\mathrm{RHom}_{X\times X}(\mathcal{O}_\Delta, \mathbf{E}\boxtimes\mathbf{E}^*) &= \mathrm{R}\,\Gamma(X\times X, \mathrm{R}\mathcal{H}om_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_{X\times X}) \overset{L}{\otimes}_{X\times X} \mathbf{E}\boxtimes\mathbf{E}^*)\\ &\overset{(4)}{=} R\Gamma(X\times X, \omega_\Delta^{-1} \overset{L}{\otimes}_{X\times X} \mathbf{E}\boxtimes\mathbf{E}^*)[-2]\\ &= \mathrm{RHom}_Y(\mathbf{E}, \omega_X^{-1}\otimes\mathbf{E})[-2]\end{aligned}$$

proving the second claim.
Finally, to compute the global dimension of A, first note that A is clearly finite dimensional of finite global dimension. This means that both statements above combine to show that $B(\mathbb{H})$ is a 3-preprojective algebra in the sense of 0.1.14 and as such it is a 3-Calabi-Yau algebra. More precisely, $B(\mathbb{H})$ is graded, and if we denote the shift by (-1), Keller's proof shows that that the Serre functor is given by $(-1)[3]$. This implies that the global dimension of B is exactly 3. Now, let S, T be simple $A = B_0$-modules which we view as B-modules concentrated in degree zero. The part of degree zero of a graded projective resolution of S is a projective resolution, implying that $\mathrm{gl.dim}(A) \leq 3$. Moreover, the Calabi-Yau property on B implies that

$$\mathrm{Ext}^3_{\mathrm{Gr}(B)}(S, T) = \mathrm{Hom}_{\mathrm{Gr}(B)}(T, S(-1))^* = 0$$

and hence $\mathrm{gl.dim}(A) \leq 2$ $\square$

Corollary 0.2.7. • *There is an isomorphism* $B(\mathbb{H}) \cong \mathbf{\Pi}_3(A(\mathbb{E}))$ *(see 0.1.15*

- $B(\mathbb{H})$ *is 3-Calabi-Yau*
- $B(\mathbb{H})$ *is isomorphic to the Jacobi algebra* $\mathrm{Jac}(\mathcal{Q}, \mathtt{w})$ *for some quiver with homogeneous potential of degree* 1

Proof. The first statement is a combination of 1) and 2) above. The second statement then follows from 0.1.16 (and was in fact used in its proof). To prove the final statement, since A has global dimension ≤ 2, we can apply 0.1.17, to conclude that $B \cong \Gamma(\mathcal{Q}, \mathtt{w})$ for some quiver with a homogeneous potential of degree 1 and since B is concentrated in degree 0, we obtain $B \cong \mathrm{Jac}(\mathcal{Q}, \mathtt{w})$ $\square$

0.2.2 Representations of Cyclic Groups

For the purposes of this section, we assume that $\Bbbk$ is algebraically closed of characteristic 0.

Let G be a finite group and V a finite dimensional representation of G. Recall that the McKay quiver is constructed with

- vertices corresponding to isomorphism classes of irreducible representations ρ_i of G
- n_{ij} arrows from nodes i to j if the multiplicity of the representation ρ_j in the representation $V \otimes \rho_i$ is n_{ij}

Example 0.2.8. *We shall often consider the case where $G \subset \mathrm{SL}(V)$ is a cyclic group of order n generated by an element g. The representation theory of cyclic groups implies that we can assume that $V = \bigoplus_{i=1}^{d} \Bbbk x_i$ and that g acts through the rule*

$$g * (x_1, \dots x_d) = (\xi^{w_1} x_1, \dots, \xi^{w_d} x_d) \tag{5}$$

Where ξ is an n-th root of unity. The McKay quiver $\mathcal{Q}$ of G has

- *vertices $\mathcal{Q}_0$ corresponding to the elements in $\mathbb{Z}/n\mathbb{Z}$*
- *arrows $\bar{l} \xrightarrow{\alpha_{l,i,l+w_i}} \overline{l + w_i}$ for each $\bar{l} \in Z/\mathbb{Z}$ and $1 \leq i \leq d$*

We have the following beautiful connection between the skew group ring and the McKay quiver:

Theorem 0.2.9. *Let $A = SV \# G$ and let $\mathcal{Q}$ be the McKay quiver of the representation $G \subset \mathrm{SL}(V)$, then*

- *A is d-Calabi-Yau*

- *If $d = 3$, there exists a potential such that $A \cong \mathrm{Jac}(\mathcal{Q}, \mathtt{w})$*

Proof. Both statements are well known, for the first, see for example [IR08], for the second see [Gin, thm 4.4.6] □

Returning to the example where G is a cyclic group, we can make this theorem a little more explicit

Lemma 0.2.10. *Let $G \subset \mathrm{SL}(V)$ be a cyclic group of order n and let $\dim_{\mathbb{k}}(V) = 3$.. Let $\mathtt{w}$ be the potential satisfying the conclusion of theorem 0.2.9. Then with the notations of (5) we have*

$$\mathtt{w} = \sum_{\sigma \in S_3} \epsilon_\sigma \alpha_{\sigma(1),i_1} \alpha_{\sigma(2),i_2} \alpha_{\sigma(3),i_3}$$

where $\alpha_{\sigma(1),i_1} \alpha_{\sigma(2),i_2} \alpha_{\sigma(3),i_3}$ is a 3-cycle and the sign is such that $\epsilon_\sigma = 1 \iff (\sigma(1), \sigma(2)), \sigma(3)) = (1, 2, 3)$ up to cyclic permutation

Proof. This first appeared in [CDT07] (see remark 2.9). It can also be found in [Gin], [BWS10, theorem 3.2], or [AIR, prop 5.5] □

0.3 Cluster Algebras and Categories

An important setting in which (DG)-Calabi-Yau algebras play a central role is that of cluster categories. These arose as a construction of a certain type of category in representation theory, which mimic the beautiful combinatorics of cluster algebra. Cluster algebras in turn were defined by Fomin and Zelevinsky in [FZ02], though the simple combinatorial process of seed mutation, which is again an extension of the mutation rule for quivers (0.1.6)

Definition 0.3.1. A *seed* is a pair $(\mathcal{Q}, \underline{u})$ where $\mathcal{Q}$ is a quiver without loops or 2-cycles on the vertex set $\{1, \ldots n\}$ and $\underline{u} = (u_i)_i$ a free generating set of the field $\mathbb{Q}(x_1, \ldots x_n)$

Definition 0.3.2. Let $(\mathcal{Q}, \underline{u})$ be a seed and $i \in \mathcal{Q}_0$. The mutation of $(\mathcal{Q}, \underline{u})$ at i is the seed $\mu_i(\mathcal{Q}, \underline{u}) \stackrel{\mathrm{def}}{=} (\mathcal{Q}', \underline{u}')$ where

- $\mathcal{Q}' = \mu_i(\mathcal{Q})$ as in 0.1.6
- $u_l' = u_l$ for $l \neq i$ and

$$u_i' = \frac{1}{u_i}\left(\prod_{\alpha:h\longrightarrow i} x_i + \prod_{\alpha:i\longrightarrow j} x_j\right) \tag{6}$$

the relation (6) defining u_i' is referred to as the *exchange relation*. We can collect the data of the various mutations of a seed to form the cluster algebra:

Definition 0.3.3. Let $(\mathcal{Q}, \{x_1, \dots x_n\})$ be a seed. We define *clusters* as being the sets of variables appearing in a mutation of the seed. The *cluster variables* are all the variables appearing in a cluster and finally, the *cluster algebra* is the subalgebra of $\mathbb{Q}(x_1, \dots x_n)$ generated by all cluster variables. It is denoted $\mathcal{A}_Q$

Morally, *a cluster algebra associated to a quiver has an explicit set of variables, collected in distinguished generating sets called clusters equipped with a mutation rule between these sets which is characterized by an exchange relation as in (6).*

In [BMR$^+$06], this mantra was adapted in order to associate to an acyclic quiver $\mathcal{Q}$ a category $\mathcal{C}_\mathcal{Q}$ instead, which has a similar combinatorial structure. The role of cluster variable in this context is played by rigid indecomposable objects. To exhibit its construction, recall that for a finite dimensional $\mathbb{k}$-algebra of finite global dimension A, the left derived Nakayama functor $S(-) \overset{\text{def}}{=} D(A) \overset{L}{\otimes}_A -$ is a Serre functor on the category $\mathscr{D}^b(A)$.

Definition 0.3.4. Let $\mathcal{Q}$ be an acyclic quiver. the factor category

$$\mathcal{C}_\mathcal{Q} = \mathscr{D}^b(\mathbb{k}\,\mathcal{Q})/(S^{-1}[2])$$

is the *cluster category* of $\mathcal{Q}$

The properties of this category are summarized in the following theorem which highlights the analogy between the combinatorics of the cluster algebra $\mathcal{A}_\mathcal{Q}$ and the cluster category $\mathcal{C}_\mathcal{Q}$

Theorem 0.3.5. *Let $\mathcal{Q}$ be an acyclic quiver on n vertices. Then,*

- *the category $\mathcal{C}_{\mathcal{Q}}$ is Krull-Schmidt and 2-Calabi-Yau (see 0.0.2). Moreover, the canonical functor $\mathscr{D}^b(\Bbbk\, \mathcal{Q}) \longrightarrow \mathcal{C}_{\mathcal{Q}}$ is a triangle functor.*
- *any rigid indecomposable object T_1 is part of a set $T \overset{def}{=} (T_1, \ldots, T_n)$ of pairwise nonisomorphic irreducible objects satisfying*

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{Q}}}(T_i, T_j) = 0$$

  called a cluster tilting set

- *For any cluster tilting set T and $1 \leq i \leq n$ there is a unique irreducible object T_i^* such that replacing T_i by T_i^* yields another cluster tilting set, called the mutation of T at T_i*
- *the pair (T_i, T_i^*) is characterized by $\dim_{\Bbbk} \mathrm{Ext}^1(T_i^*, T_i) = 1$. In this case there are nonsplit* exchange triangles

$$T_i \longrightarrow E \longrightarrow T_i^* \longrightarrow T_i[1] \text{ and } T_i^* \longrightarrow E' \longrightarrow T_i \longrightarrow T_i^*[1]$$

  unique up to isomorphism

Proof. This was proven in [BMR$^+$06] and we also refer to the excellent overview paper [Kel08] for additional background. □

Remark 0.3.6. *Note that in particular as part of the definition of Krull-Schmidt, $\mathcal{C}_{\mathcal{Q}}$ is* Hom*-finite. This will become a major technical issue in the sequel.*

In [IY08], Iyama and Yoshino realized that it is not necessary to involve a quiver $\mathcal{Q}$ should one want to describe categories having similar properties as 0.3.5. In fact, one can prove an analogous result for any Hom-finite 2-Calabi-Yau category. To this end we let $\mathcal{C}$ denote a Hom-finite 2-Calabi-Yau idempotent split triangulated category.

Definition 0.3.7. An object T is *cluster tilting* if

- T is a direct sum of nonisomorphic indecomposable objects.
- $\{L \in \mathcal{C} \,|\, \mathrm{Ext}^1(T, L) = 0\} = add(T)$, the category of direct sums of factors of T.

The algebra $\mathrm{End}_{\mathcal{C}}(T)$ is called the *associated cluster tilted algebra.*

Theorem 0.3.8 ([IY08]). *Let T be a cluster tilting object of $\mathcal{C}$. Let T_1 be an irreducible component of T. Then*

- *there exists a unique irreducible T_1^* such that the object $\mu_1(T)$ obtained by replacing T_1 by T_1^* in the direct sum is cluster tilting.*
- *There are non split triangles, unique up to isomorphism*

$$T_i \xrightarrow{f} E \longrightarrow T_i^* \longrightarrow T_i[1]$$

where f satisfies a universal property (i.e f is the minimal left $add((T_i)_{i \neq k})$-approximation.

The above theorem justifies the definition of cluster category:

Definition 0.3.9. A (generalized) cluster category is an (idempotent split) Hom-finite 2-Calabi-Yau category $\mathcal{C}$ together with a choice of cluster tilting object T.
The algebra $\mathrm{End}_{\mathcal{C}}(T)$ is the *cluster tilted algebra* of T.

The main technique for producing such categories is the following result due to Keller ([Kel11]

Theorem 0.3.10. *Let A be a DG algebra and assume that*

- *A is 3-Calabi-Yau*
- *$H^p(A) = 0$ for $p \geq 1$*
- $\dim_k H^0(A) \neq \infty$

Let $\mathscr{D}_{\mathrm{fd}}(A)$ denote the subcategory of $\mathscr{D}(A)$ of DG modules with finite dimensional total homology over $\Bbbk$. Then $\mathscr{D}_{\mathrm{fd}}(A) \subset \mathrm{Perf}(A)$ and the category

$$\mathcal{C}_A \overset{def}{=} \mathrm{Perf}(A)/\mathscr{D}^b_{\mathrm{fd}}(A)$$

is a cluster category where the image of A in $\mathcal{C}_A$ is a cluster tilting object with associated cluster-tilted algebra $H^0(A)$.

As an application, we obtain two ways to construct cluster categories:

Theorem 0.3.11. *Let $(\mathcal{Q}, \mathtt{w})$ be a Jacobi-finite quiver with potential (0.1.2). Then $\mathcal{C}_{(\mathcal{Q},\mathtt{w})} \overset{def}{=} \mathcal{C}_{\Gamma(\mathcal{Q},\mathtt{w})}$ is a generalized cluster category Γ with cluster tilting object $\Gamma(\mathcal{Q}, \mathtt{w})$ and associated cluster tilted algebra $\mathrm{Jac}(\mathcal{Q}, \mathtt{w})$.*

The relation between mutations of quivers with potentials and cluster tilted algebras is well-understood for quiver without loops or 2-cycles A second construction of cluster categories is given by the following: theorem:

Theorem 0.3.12. *[Ami09] Let A be a finite dimensional algebra of gldim ≤ 2 and assume that the 3-preprojective algebra $\Pi_3(A)$ (see 0.1.15) is finite dimensional. Then the category*

$$\mathcal{C}_A \overset{def}{=} \mathcal{C}_{\mathbf{\Pi}_3(A)}$$

is a cluster category in which the image of $\mathbf{\Pi}_3(A)$ is a cluster tilting object with cluster tilted algebra $\Pi_3(A)$.

Proof. This is a combination of 0.3.10 and 0.1.16 □

We end this introduction by stating that there is a higher version of the theory of cluster categories with cluster tilting object. We shall not go into details, but it is worth mentioning that Guo proved the following generalization of theorem 0.3.10 to general dimensions:

Theorem 0.3.13. *[Guo10] Assume that A is a DG algebra such that*

- *A is d-Calabi-Yau*

- $H^p(A) = 0$ *for* $p \geq 1$
- $\dim_{\Bbbk} H^0(A) \neq \infty$

Then the category $\mathcal{C}_A = \mathrm{Perf}(A)/\mathscr{D}^b_f(A)$ *is* Hom-*finite and* $d-1$ *Calabi-Yau.*

For this reason, the category is referred to as a $d-1$-cluster category. By Keller's result 0.1.16, an example of this construction is provided by

$$\mathcal{C}_{\mathbf{\Pi}_d(A)} \tag{7}$$

where A is a finite dimensional algebra of finite global dimension $\leq d-1$ such that $\Pi_d(A)$ is finite dimensional.

Chapter 1

on Nondegenerate QP's and Cluster Categories

1.1 Introduction

This chapter describes 2 results proven by the author and Van den Bergh in [dTdVdB10] and [dTdVVdB13], which yields interesting examples of the constructions described in the introductory chapter above. In the first part we exhibit an easy to check criterion which allow us to conclude as to when a Jacobi algebra has an underlying quiver with potential which is nondegenerate as defined in 0.1.9. We give a relatively simple technique to determine when a QP $(\mathcal{Q}, \mathtt{w})$ has no loops or 2-cycles assuming the existence of an additional grading on $\mathrm{Jac}(\mathcal{Q}, \mathtt{w})$ and some technical conditions on that grading (see 1.2.11). Next, we tweak these conditions a little to obtain a set of conditions which remain invariant under derived equivalence, and hence under mutation by 1.2.7 (see 1.2.14) .

We show that our set of conditions are valid in two examples:

- Our proof of theorem 0.2.7 shows that the rolled-up algebra of a geometric helix on a Del Pezzo surface is a Calabi-Yau Jacobi algebra. We show that for these algebras the conditions of our criterion are satisfied

- If $G \subset \mathrm{SL}(V)$ acts on a vector space of dimension 3, then we stated in 0.2.9 that the skew group ring is Jacobi. We shall prove that under certain extra numerical constraints, it satisfies the criterion.

We conclude our study of this criterion by mentioning that one can use the theory of deformed preprojective algebras as developed by Crawley-Boevey and Holland in [CBH98] to construct a non-example which show the necessity of the conditions.

Our next result uses the cluster categories defined in 0.3.9 to obtain some explicit descriptions of the singularity category. Let R be a local Gorenstein ring of Krull dimension d and let $\mathscr{D}_f(R) = \mathscr{D}(\mathrm{mod}(R))$. Recall that the singularity category is defined as the quotient $\mathrm{sing}(R) = \mathscr{D}^b_f(R)/\operatorname{Perf}(R)$, which is trivial in the case of a regular ring. If R has an isolated singularity, it is known since the work of Auslander ([Aus78]) that $\mathrm{sing}(R)$ is Hom-finite and $(d-1)$-Calabi-Yau under these conditions. It is thus a natural question to determine if $\mathrm{sing}(R)$ can in fact be described as the higher cluster category associated to a DG algebra as described in 0.3.10. The solution is provided by introducing the relative singularity category. Assume that R has a non-commutative crepant resolution $R \longrightarrow A$ (see §1.3.2). Then we consider the category $\mathrm{sing}(R, A) = \mathscr{D}^b(A)/\operatorname{Perf}(R)$. The passage of $\mathscr{D}^b(R)$ to $\mathrm{sing}(R, A)$ has an elegant description in the language of minimal DG models, which in turn yields a description of $\mathrm{sing}(R)$.
To be more precise, we consider a minimal DG model $T_l V \longrightarrow A$ (see §1.3.1) which by the finite global dimension of A must satisfy $\mathscr{D}_f(A) = \operatorname{Perf}(T_l V)$. There is a way to modify $T_l V$ to obtain a DG algebra Γ which in turn satisfies $\operatorname{Perf}(\Gamma) \cong \mathrm{sing}(R, A)$. We finally prove that modding out the category generated by the simple Γ-modules yields an equivalence $\operatorname{Perf}(\Gamma)/\mathscr{D}^b(\Gamma) \cong \mathrm{sing}(R)$. Hence in the case where the DG algebra Γ satisfies the conditions of 0.3.10, we obtain a description of the category $\mathrm{sing}(R)$ as a generalized cluster algebra.

We shall discus the cases of where A is 3-Calabi-Yau and the case of cyclic quotient singularities as applications of this construction

1.2 Nondegeneracy of Quivers with Potential

1.2.1 a Nondegeneracy Criterion

As mentioned in the introduction above, the key technical tool is the assumption of a grading on the Jacobi algebra. We therefore begin our discussion by explaining how to adapt the theory of QP's as laid out in §0.1 in the presence of a grading. To this end we fix an abelian group Λ.

Definition 1.2.1. A Λ-graded quiver with potential $(\mathcal{Q}, \mathtt{w})$ consists of a grading on $\mathcal{Q}$ such that $\mathtt{w}$ is a potential, homogeneous for the grading. The graded path algebra $\Bbbk\,\mathcal{Q}^{\wedge gr}$ is defined as the path algebra $\Bbbk\,\mathcal{Q}$ completed at sequences of paths of constant degree.
The graded Ginzburg DG algebra $\Gamma^{gr}(\mathcal{Q}, \mathtt{w})$ is given as in 0.1.2 where $\Bbbk\,\mathcal{Q}$ is replaced by $\Bbbk\,\mathcal{Q}^{gr}$. And finally, the graded Jacobi algebra of $(\mathcal{Q}, \mathtt{w})$ is defined as

$$\mathrm{Jac}^{gr}(\mathcal{Q}, \mathtt{w}) = H^0(\Gamma^{gr}(\mathcal{Q}, \mathtt{w})) = \Bbbk\,\mathcal{Q}^{\wedge gr} \,/\overline{\{\partial_\alpha \mathtt{w}\,|\alpha \in \mathcal{Q}_1\}}$$

Definition 1.2.2. Two Λ-graded QP's $(\mathcal{Q}, \mathtt{w})$ and $(\mathcal{Q}'\,\mathtt{w})$ are right graded equivalent if there is an isomorphism of graded algebras $\Bbbk\,\mathcal{Q}^{\wedge gr} \longrightarrow \Bbbk\,\mathcal{Q}'^{\wedge gr}$ such that $\phi|_{\mathcal{Q}_0} = \mathrm{Id}$ and $\phi(\mathtt{w})$ is cyclically equivalent to $\mathtt{w}'$.

The graded analogue of the splitting theorem (0.1.1) holds

Theorem 1.2.3. *Any Λ-graded quiver with potential $(\mathcal{Q}, \mathtt{w})$ is graded right equivalent to a direct sum of graded QP's $(\mathcal{Q}^{triv}, \mathtt{w}^{triv}) \oplus (\mathcal{Q}^{red}, \mathtt{w}^{red})$ where $(\mathcal{Q}^{triv}, \mathtt{w}^{triv})$ is trivial and $(\mathcal{Q}^{red}, \mathtt{w}^{red})$ is reduced. This decomposition is unique up to right graded equivalence.*

Proof. This is an immediate corollary of the fact that the right equivalence constructed in [FZ02] is in fact graded, see also [AO14, theorem 6.6] $\square$

Remark 1.2.4. *The proof of the above theorem immediately implies that the reduced and trivial components of a QP coincide with the usual ones in the non-graded splitting theorem*

The mutation rule of quivers with potential 0.1.7 has a natural graded version:

Definition 1.2.5. Let $i \in \mathcal{Q}_0$. Let $(\mathcal{Q}, \mathtt{w})$ be a Λ-graded QP where $\mathtt{w}$ is homogeneous of degree r. The graded mutation of $(\mathcal{Q}, \mathtt{w})$ is the QP $(\mathcal{Q}', \mathtt{w}') = \mu_i(\mathcal{Q}, \mathtt{w})$ is defined in 0.1.7, graded through the following rule:

- arrows both in $\mathcal{Q}$ and $\mathcal{Q}'$ have the same degree
- $|[\alpha\beta]| = |\alpha| + |\beta|$
- $|\alpha^*| = -|\alpha|$ if α starts in i
- $|\alpha^*| = -|\alpha| + r$ if α end in i

Remark 1.2.6. *Note that* $(\mathcal{Q}, \mathtt{w})$ *is graded and that* $\mathtt{w}'$ *remains homogeneous of degree* r.

With this definition we have the following analogue of theorem 1.2.7, which follows by keeping track of the grading in the proof of [KY11].

Theorem 1.2.7. *[KY11] Let* $(\mathcal{Q}, \mathtt{w})$ *be a* Λ*-graded quiver with potential such that* $\mathcal{Q}$ *has no loops or 2-cycles and let* $\Gamma = \Gamma^{gr}(\mathcal{Q}, \mathtt{w})$ *be the associated graded Ginzburg DG algebra. Let* $i \in \mathcal{Q}_0$ *and write* $\Gamma' = \Gamma^{gr}(\mu_i(\mathcal{Q}, \mathtt{w}))$

- *There is a* $\Gamma \otimes \Gamma'$ *DG bimodule* T *inducing mutually inverse equivalences*

$$(-) \overset{L}{\otimes}_{\Gamma'} T : \mathscr{D}^+(\Gamma') \longrightarrow \mathscr{D}^+(\Gamma) \text{ and } \mathrm{RHom}_\Gamma(-, T) : \mathscr{D}^+(\Gamma') \longrightarrow \mathscr{D}^+(\Gamma) \tag{1.1}$$

- *If* $\Gamma \cong \mathrm{Jac}^{gr}(\mathcal{Q}, \mathtt{w})$ *has cohomology concentrated in degree 0, then so does* $\Gamma' \cong \mathrm{Jac}^{gr}(\mu_i(\mathcal{Q}, \mathtt{w}))$ *and* T *is a 2-term tilting complex.*

Theorem 1.2.8. *Let $(\mathcal{Q}, w)$ be a Λ-graded quiver with potential and let $i \in \mathcal{Q}_0$. Then there is a derived equivalence*

$$T : \mathscr{D}^+(\Gamma^{gr}(\mathcal{Q}, \mathtt{w})) \longrightarrow \mathscr{D}^+(\Gamma^{gr}(\mu_i(\mathcal{Q}, \mathtt{w})) \tag{1.2}$$

In order to prove our main lemma 1.2.11, we need the following preparatory result, the proof of which is a straightforward computation

Lemma 1.2.9. *[deleting a vertex] Let $(\mathcal{Q}, \mathtt{w})$ be a QP. Let $0 \in \mathcal{Q}_0$ and $e_0 \in k\,\mathcal{Q}^\wedge$ be the corresponding idempotent. Let $(\mathcal{Q}^\#, \mathtt{w}^\#)$ be the quiver with potential obtained by deleting the vertex $0 \in \mathcal{Q}$.*
There is a canonical isomorphism of DG algebras

$$\Gamma(\mathcal{Q}, \mathtt{w})/\Big(\Gamma(\mathcal{Q}, \mathtt{w})e_0\Gamma(\mathcal{Q}, \mathtt{w})\Big) \xrightarrow{\simeq} \Gamma(\mathcal{Q}^\#, \mathtt{w}^\#)$$

If moreover $(\mathcal{Q}, \mathtt{w})$ is Λ-graded, then so is $(Q^\#, \mathtt{w}^\#)$ and there is an isomorphism of DG algebras

$$\Gamma^{gr}(\mathcal{Q}, \mathtt{w})/\Big(\Gamma^{gr}(\mathcal{Q}, \mathtt{w})e_0\Gamma^{gr}(\mathcal{Q}, \mathtt{w})\Big) \xrightarrow{\simeq} \Gamma^{gr}(\mathcal{Q}^\#, \mathtt{w}^\#)$$

Proof. The morphism defined by removing all instances of paths going through the vertex is the required isomorphism □

Remark 1.2.10. *Taking cohomology in degree zero for both isomorphism immediately yields analogous statements for the (graded) Jacobi algebra s of the (graded) quiver with potential.*

Lemma 1.2.11. *Let $(\mathcal{Q}, \mathtt{w})$ be a $\mathbb{Z}$-graded QP where $\mathcal{Q}$ is connected with at least 3 vertices and $\mathtt{w}$ is a reduced homogeneous potential of degree r. Let $A \overset{def}{=} \mathrm{Jac}^{gr}(\mathcal{Q}, \mathtt{w})$ and assume the following conditions hold:*

1. *A is noetherian of finite global dimension.*
2. *$\dim_{\Bbbk} A_i \neq \infty$ for all i and $\dim_{\Bbbk} A = 0$ for $i \ll 0$*
3. *For each nonzero idempotent $e \in A$, $\dim_{\Bbbk}(A/AeA) \neq \infty$.*

4. *$A/[A,A]$ contains no homogeneous elements lying in the interval* $[1, \frac{r}{2}]$

Then $\mathcal{Q}$ has no loops or 2-cycles

Proof. Assume that $\mathcal{Q}$ contains a loop $0 \xrightarrow{\alpha} 0$. Let $e = \sum_{i \neq 0} e_i \in \Bbbk\,\mathcal{Q}$ be the sum of all idempotents corresponding to vertices different from 0. Then e is a nonzero idempotent since $|\,\mathcal{Q}_0\,| \geq 3$ by hypothesis. By 1.2.9 and 1.2.10, we have $\bar{A} \stackrel{\text{def}}{=} A/AeA \cong \mathrm{Jac}^{gr}(\mathcal{Q}^0, \mathtt{w}^0)$, where $\mathcal{Q}^0$ is the quiver whose sole vertex is 0 and whose arrows are the loops around 0 and $\mathtt{w}^0$ is the potential consisting of the cycles in $\mathtt{w}$ solely passing through 0.
If $\mathtt{w}^0 = 0$ then $\mathrm{Jac}^{gr}(\mathcal{Q}^0, \mathtt{w}^0) = \Bbbk\,\mathcal{Q}^{\wedge gr}$ is an infinite dimensional algebra, which we exclude by (3). We can thus assume that $\mathtt{w}^0$ is sum of terms of degree r which are products of loops

$$l_1 \cdot \ldots \cdot l_n$$

where $n \geq 3$ as $\mathtt{w}^0$ is a reduced potential by hypothesis. Assume that the number n is minimal. It must then follow that there is a loop of l_i of degree $|l_i| < \frac{r}{2}$. Moreover, it is clear that $\overline{l_i} \neq 0$ in $\overline{A}/[\overline{A}, \overline{A}]$. Hence, in particular $l_i \neq 0$ in $A/[A,A]$. Condition (4) now states that $|l_i| \leq 0$, but then l_i is nilpotent by the second part of condition (2). Since A is noetherian and of finite global dimension by condition (1), we can apply Lenzing's theorem [Len69] to conclude that $l_i \in [A,A]$, to obtain a contradiction.

If $\mathcal{Q}$ has a 2-cycle $0 \longrightarrow 1 \longrightarrow 0$, then the argument is very similar: Let e denote the idempotent defined by $e = \sum_{i \neq 0,1} e_i$. Then e is a nonzero idempotent once again by $|\,\mathcal{Q}_0\,| \geq 3$. Using 1.2.10, $\bar{A} = A/AeA \cong \mathrm{Jac}(Q^{0,1}, \mathtt{w}^{0,1})$ is the graded Jacobi algebra of the quiver with vertices 0 and 1, whose arrows coincide with the arrows in $\mathcal{Q}$ between 0 and 1 and $\mathtt{w}^{0,1}$ is the potential whose terms are the paths consisting of 2-cycles in $\mathtt{w}$ between 0 and 1. We exclude $\mathtt{w}^{0,1} = 0$ using condition (3) to conclude that $\mathtt{w}^{0,1}$ must be a sum of 2-cycles of the form

$$\alpha_1\beta_1 \ldots \alpha_n\beta_n$$

Where $n \geq 2$ since $\mathtt{w}^{0,1}$ is reduced. Again, there must be some 2-cycle $\alpha_i\beta_i$ of degree $\leq \frac{r}{2}$ in this product. This 2-cycle is again a nonzero element in $\overline{A}/[\bar{A}, \bar{A}]$ by the minimality of $n \geq 2$. It follows that $\alpha_i\beta_i \neq 0$ in $A/[A, A]$ and $\alpha_i\beta_i$ must have degree ≤ 0 by condition (4) and in turn be nilpotent by condition (2). Applying [Len69] using condition (1) once more finally yields a contradiction. □

The above conditions can be tweaked a little to obtain a set of conditions which remain invariant under (graded) mutation of quivers with potentials, and hence provide us with a criterion for nondegeneracy. To this end, we make the following definition

Definition 1.2.12. Let R be a commutative noetherian $\mathbb{N}$-graded ring and A an $\mathbb{N}$-graded algebra, finite as a module over R. The pair A/R is *projectively Azumaya* if for all $\mathfrak{p} \in \mathrm{Proj}(R)$, $A_\mathfrak{p}$ is Azumaya over $R_\mathfrak{p}$

Lemma 1.2.13. *Let A be projectively Azumaya over a noetherian center R. Then for any nonzero idempotent e, the module A/AeA is a finite R_0-module*

Theorem 1.2.14. *Let $(\mathcal{Q}, \mathtt{w})$ be an $\mathbb{N}$-graded QP where $\mathcal{Q}$ has at least 3 vertices and $\mathtt{w}$ is reduced and let $A \overset{def}{=} \mathrm{Jac}^{gr}(\mathcal{Q}, \mathtt{w})$. Assume the following conditions hold:*

1. $\dim_{\Bbbk} A_i \neq \infty$ *for all i*
2. *A is a 3-CY and projectively Azumaya over a noetherian center*
3. *the graded module $A/[A, A]$ contains no homogeneous elements in the interval $[1, \frac{r}{2}]$*

Then $(\mathcal{Q}, \mathtt{w})$ is nondegenerate

Proof. We proceed to verify the conditions of 1.2.11 for A. The algebra is noetherian by condition (2) and has global dimension 3 again by condition (2). Condition (2) of 1.2.11 coincides with condition (1) above. Next, by the above lemma 1.2.13, for each nonzero idempotent A/AeA is a finite R_0-module, and as $\dim_{\Bbbk} R_0 \leq \dim_{\Bbbk} A_0 \neq 0$ the third condition is satisfied. Finally, the last conditions coincide in both statements. It follows that $\mathcal{Q}$ doesn't contain any loops or 2-cycles. We must thus show that $A' \stackrel{\text{def}}{=} \operatorname{Jac}^{gr}(\mu_i(\mathcal{Q}, \mathtt{w}))$ again satisfies the conditions of 1.2.11. By 1.2.8, we immediately conclude that there is a derived equivalence between A and A' given by a 2-term tilting complex T.

1. we clearly have $\dim_{\Bbbk} A'_i \neq \infty$ for $i \geq 0$ and the fact that $A'_i = 0$ for $i \ll 0$ follows from the fact T is a bounded complex hence so is $End(T)$.

2. A' is 3-Calabi-Yau by a combination of 0.1.3 and 1.2.8, It is well known that the center is invariant under derived equivalence, implying that $S \stackrel{\text{def}}{=} Z(A') \cong Z(A)$. It follows that $Z(A')$ is noetherian in particular. Finally, to prove that A' is projectively Azumaya over $Z(A')$, let $\mathfrak{p} \in \operatorname{Proj}(S)$. Since the property of being Azumaya over the center is Morita invariant, it suffices to show that there is a Morita equivalence $A_{\mathfrak{p}} \simeq A'_{\mathfrak{p}}$. This follows from [KY11, thm 6.2] with the fact that the tilting modules in question become projective after localizing at $\mathfrak{p}$.

3. It is well know that the zeroth Hochschild cohomology $\mathrm{HH}^0(A) = A/[A, A]$ is a derived invariant. Moreover, by keeping track of the additional grading in [Kel03], it follows in this case that it preserves the grading on Hochschild cohomology.

$$A'/[A', A'] = HH^0(A') \cong HH^0(A) = A/[A, A]$$

which trivially yields the result □

Remark 1.2.15. *We note that the last condition, although rather technical in nature cannot be omitted from the statement. In [CBH98], Crawley-Boevey and Holland consider a deformed version of the preprojective algebra* Π^λ *over an acyclic quiver* $\mathcal{Q}$ *where* $\lambda \in \Bbbk\,\mathcal{Q}_0$. *An explicit computation shows that* $\Pi^\lambda = Jac(\overline{\mathcal{Q}}, \overline{\mathtt{w}})$ *where* $\overline{\mathcal{Q}}$ *is the formally doubled quiver as in definition 0.1.11,* *with an extra loop t_i at each vertex and*

$$\overline{\mathtt{w}} = \left(\sum_{i \in \mathcal{Q}_0} t_i\right)\left(\sum_{\alpha \in \mathcal{Q}_1} [\alpha, \alpha^*]\right) - \sum_{i \in \mathcal{Q}_0} \frac{1}{2}\lambda_i t_i^2$$

Using methods from [CBH98] and [VB08] one proves that if $\mathcal{Q}$ is extended Dynkin, under certain numerical conditions for λ, this algebra satisfies all but the last condition in 1.2.14. Moreover, the quiver $\overline{\mathcal{Q}}$ by construction has a loop $t_i$at each vertex i and a 2-cycle for each arrow $\alpha \in \mathcal{Q}$ We refer the reader to our paper [dTdVVdB13] for proofs of these statements

1.2.2 Some Applications

In this section, we apply the theorem 1.2.14 to the two examples discussed in §0.2. We shall first consider the case of cyclic group actions on a vector space. To this end, we assume that $\Bbbk$ is an algebraically closed field of characteristic zero. We recall the following geometric fact:

Theorem 1.2.16. *Let $G \subset \mathrm{SL}(V)$ be a finite group acting on a vector space of dimension d over the field $\Bbbk$. The following are equivalent:*

- *the ring of invariants SV^G has an isolated singularity*
- *the ring $SV\#G$ is projectively Azumaya over its center SV^G*
- *the action of G on V is free outside of the origin*

If d is an odd prime, then G is a cyclic group

Proof. The three equivalences follow from [IR08, 8.4]. The fact that G must be cyclic in this case is the main result of [KN] □

We consider the case $d = 3$ and as in §0.2.2, we write $V = \Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk x_3$. We let $G = \mathbb{Z}/n\mathbb{Z}$ act on V by weights (w_1, w_2, w_3) after a choice of n-th primitive root of unity ξ. The condition that $G \subset \mathrm{SL}_3(V)$ is equivalent to

$$\sum w_i = 0 \bmod n$$

Recall from 0.2.9 that $SV\#G$ is a 3-Calabi-Yau Jacobi algebra $\mathrm{Jac}(\mathcal{Q}, \mathtt{w})$ where the quiver $\mathcal{Q}$ is the McKay quiver and the potential $\mathtt{w}$ is a signed sum of 3-cycles (and hence canonical graded of degree 3)

We can now prove:

Lemma 1.2.17. *Let $G \subset GL(V)$ be a finite group of order n and V a representation of dimension 3. The following are equivalent:*

1. *$G \subset \mathrm{SL}(V)$ and $SV\#G$ satisfies the conditions of 1.2.14*
2. *G is cyclic of order n and the weights of the action of G on V satisfy*

$$\gcd(w_i, n) = 1 \text{ and } \sum_i w_i = 0 \operatorname{mod} n\,, \forall 1 \leq i \leq n$$

Proof. By 1.2.16, the requirement that $SV\#G$ be projectively Azumaya is equivalent to the action being free outside the origin. This immediately implies that G is cyclic by 1.2.16 once again. It is easy to see that in this case the condition of being free outside the origin translates to

$$\gcd(w_i, n) = 1$$

The condition $\sum_i w_i = 0 \operatorname{mod} n\,, \forall 1 \leq i \leq n$ follows from the above discussion, proving (1) $\implies$ (2)

Conversely, assume that G is cyclic of order n with weights satisfying the above conditions. The fact that $\mathcal{Q}$ is connected and that $\mathtt{w}$ is reduced and homogeneous follows from the construction in 0.2.10. Moreover, $\mathcal{Q}$ must have at least 3 vertices, as otherwise, we have $2 = |\,\mathcal{Q}_0\,| = n$ and the condition $\sum w_i = 0$ implies that one of the weights must be zero, contradicting that the action is free outside the origin. We now use 1.2.16.

The first condition is trivially satisfied. The second condition is once again an immediate application of 1.2.16, together with the above discussion.

The second condition is 1.2.16.

The last condition is equivalent to $(A/[A, A])_1 = 0$ as the potential has degree 3. The only nonzero elements in this group are classes of loops, hence if

$(A/[A,A])_1 \neq 0$, then $\mathcal{Q}$ must have a loop. This in turn implies that one of the weights is 0 by the definition of the McKay quiver, contradicting our assumption on weights once again $\square$

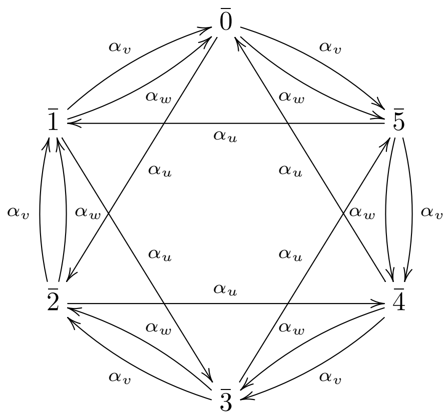
Summarizing both theorems yields 0.2.10 and 1.2.17 yields:

Theorem 1.2.18. *Assume $\mathbb{Z}/n\mathbb{Z}$ acts on V with $dim_{\Bbbk}(V) = 3$ through weights satisfying $\sum w_i = 0$ and $\gcd(w_i, n) = 1$ Let $\mathcal{Q}$ be the McKay quiver of G. Then there exists a reduced homogenous potential $\mathtt{w}$ such that*

- *$SV\#G \cong \mathrm{Jac}(\mathcal{Q}, \mathtt{w})$*
- *the QP $(\mathcal{Q}, \mathtt{w})$ is nondegenerate*

Proof. The statement in now a combination of 0.2.10, 1.2.17 and 1.2.14 $\square$

We also give a counterexample, showing that the numerical conditions on the action are in fact necessary. Let $\mathbb{Z}/6\mathbb{Z}$ act on $\Bbbk x_1 \oplus \Bbbk x_2 \oplus \Bbbk x_3$ with weights $(u, v, w) \stackrel{\text{def}}{=} (2, 5, 5)$ (note that $\gcd(2, 6) \neq 1$, and hence the second condition of 1.2.18 isn't satisfied). The McKay quiver of this group has the following pretty picture:



where we have omitted the head and tail of the arrows from the notation introduced in 0.2.8 for purposes of clarity.

This quiver is naturally $\mathbb{Z}^3$ graded, we will grade by monomials x_1, x_2, x_3 so that $|\alpha_i| = x_i$. Following the computation in 0.2.10, the potential $\mathtt{w}$ is homogeneous of degree $x_1x_2x_3$. We leave the reader to check that a mutation at $\overline{0}$ yields the 2-cycle $[x_1, x_1]x_1$ of degree x_1^3 between the vertices $\overline{2}$ and $\overline{4}$. This 2-cycle remains after taking the reduced component following 1.2.3, since by degree constraints, one can only remove 2-cycles for which the sums of the degrees of the arrows is the degree of the homogeneous potential $x_1x_2x_3$

As a second application, we turn to Del Pezzo surfaces for which we can immediately prove

Theorem 1.2.19. *Let X be a Del Pezzo surface, $\mathbb{E}$ a full exceptional sequence in $\mathscr{D}^b(X)$ and $\mathbb{H} = H(\mathbb{E})$ be the associated helix of period 2 (see 0.2.5). Assume that $\mathbb{H}$ is geometric. Then the rolled up algebra $B(\mathbb{H})$ is isomorphic to the Jacobi algebra of a nondegenerate QP*

Proof. The number of vertices of $\mathcal{Q}$ corresponds to the number of objects in $\mathbb{E}$, since it is well know that a full exceptional sequence of a Del Pezzo surface must have at least 3 objects (as a corollary of 0.2.3), we obtain that $|\,\mathcal{Q}_0\,| \geq 3$. Since it is well known that the Grothendieck group of a Del Pezzo surface has rank at least 3, it immediately follows that $\mathcal{Q}$ has 3 vertices. Moreover it is easy to see that $\mathcal{Q}$ is connected (by the description of $B(\mathbb{H})$ in [BS10, thm 3.2], for example). By 0.2.7, $B(\mathbb{H})$ is isomorphic to a 3-Calabi-Yau Jacobi algebra $\mathrm{Jac}(\mathcal{Q}, \mathtt{w})$ where $\mathcal{Q}$ is $\mathbb{N}$-graded and $\mathtt{w}$ is homogeneous of degree 1. We proceed to verify the conditions of 1.2.14.

- We have $B(\mathbb{H})_i = 0$ if $i < 0$ and $\dim_{\Bbbk} B(\mathbb{H}) \neq \infty$ since $\mathscr{D}^b(X)$ is a Hom-finite category.
- Next, $B(\mathbb{H})$ is 3-Calabi-Yau by 0.2.7 and projectively Azumaya as it is the pushforward of an Azumaya algebra on the total space of the canonical bundle ω_X (see [BS10] again)
- the last condition is vacuous as the degree of $\mathtt{w}$ is $r = 1$

$\square$

Remark 1.2.20. *We note that the work of Bridgeland-Stern ([BS10]) on the relation between mutations of quivers with potentials and mutations of exceptional sequences implies the above nondegeneracy result as well.*

1.3 Singularity Categories as Cluster Categories

In this section, we discuss a construction which allows us to describe singularity categories as cluster categories in certain cases following §1.1

1.3.1 Minimal Models

Throughout $\Bbbk$ denotes an algebraically closed field of characteristic 0 and $\Bbbk \longrightarrow l$ a separable $\Bbbk$-algebra.

Definition 1.3.1. Let A be a $\Bbbk$-algebra. A finite minimal model for A is a quasi-isomorphism $(T_l V, d) \longrightarrow A$ where $(T_l V, d)$ is a DG l-algebra such that

- V is a finitely generated graded l-bimodule
- $T_l V$ is the free graded-completed l-algebra over this bimodule
- $T_l V$ is concentrated in degree ≤ 0
- $d(V)$ lies in the two-sided ideal generated by $V \otimes_l V$

Example 1.3.2. *We have already seen an example of a finite minimal model in the context of QP's: let $(\mathcal{Q}, \mathtt{w})$ be a QP and let $l = \Bbbk\, \mathcal{Q}_0$ and be the l-bimodule $V = \Bbbk\, \mathcal{Q}_1$. Then it is easy to see that the Ginzburg DG algebra $\Gamma(\mathcal{Q}, \mathtt{w})$ defined in 0.1.2 is indeed of the required form. Moreover as the differential is* 0 *in degree* 0 *we have a morphism of complexes,*

$$\pi : \Gamma(\mathcal{Q}, \mathtt{w}) \twoheadrightarrow \operatorname{Jac}(\mathcal{Q}, \mathtt{w})$$

0.1.3 shows that π is a quasi-isomorphism if and only if $A = \mathrm{Jac}(\mathcal{Q}, \mathtt{w})$ is 3-Calabi-Yau. Hence that $\Gamma(\mathcal{Q}, \mathtt{w})$ is a finite minimal model for A if and only if A is 3-Calabi-Yau.

Theorem 1.3.3. *Assume A is complete, let $l = A/\operatorname{rad}(A)$ and assume $\dim_{\Bbbk}(l) \neq \infty$. Then A has a minimal model, unique up to isomorphism*

Proof. This is a special case of the discussion in [VdB10, appendix A] $\square$

We include this theorem solely to be complete. For our purposes, it will be natural to assume that A is a Koszul l-algebra. In this case, it is well known that the minimal model can be constructed directly (see [LV99, chapter 3] for a detailed account of this theory). We will give however provide a direct proof of this fact for the benefit of the reader. To this end let $A = T_l V/R$ where R is a finitely generated l-bimodule in $V \otimes_l V$. Define a series of l-bimodules by

$$J_n = \bigcap_{p+2+q=n} V^{\otimes p} \otimes_l R \otimes_l V^{\otimes q}$$

By definition, we have $J_1 = V$ and $J_2 = R$. Moreover, there is a canonical map

$$\delta_{i,n-i} : J_n \longrightarrow J_i \otimes_l J_{n-i}$$

given by concatenating. We shall use Sweedler style notation $\delta_{i,n-i}(a) = \delta_i(a)' \otimes \delta_i(a)''$ for $a \in J_n$ throughout.
Now, A is graded by the so-called Adams grading which gives elements in V degree 1. The maps $\delta_{i,n-i}$ become graded morphisms with this grading. We define $\tilde{V} \stackrel{\text{def}}{=} \bigoplus_{n\geq 1} J_n[n-1]$ and construct a DG algebra through the rule

- $\tilde{A} = T_l\tilde{V}$
- $da = (-1)^{i-1} \sum \delta_i(a)' \otimes \delta_i(a)''$

We leave it to the reader to check that this is indeed a DG algebra, which is a straightforward computation

Lemma 1.3.4. *Assume A is Koszul. Then the DG algebra $\tilde{A}$ is a minimal model of A where the quasi-isomorphism $\tilde{A} \longrightarrow A$ is induced from the canonical projection map $\tilde{V} \longrightarrow J_1 = V$*

Proof. The Koszul hypothesis on A implies that the following complex of graded left A-modules is exact

$$\cdots \longrightarrow A \otimes_l J_2 \longrightarrow A \otimes_l J_1 \longrightarrow A \longrightarrow l \longrightarrow 0 \tag{1.3}$$

where the differential is given by

$$d : A \otimes_l J_n \longrightarrow A \otimes_l J_{n-1} : a \otimes b \mapsto a\delta_{1,n-1}(b)' \otimes \delta_{1,n-1}(b)''$$

Put

$$M = (T_l\tilde{V})_+ \stackrel{\text{def}}{=} \bigoplus_{n\geq 1} \tilde{V}^{\otimes_l n} \subset T_l\tilde{V}$$

We consider M as a left sub DG-$\tilde{A}$-module of $\tilde{A}$. As a left graded $\tilde{A}$-module we have

$$M = \tilde{A} \otimes_l (J_1 \oplus J_2[1] \oplus J_3[2] \oplus \cdots)$$

Let $\tilde{C}$ be the cone of the inclusion map $i : M \longrightarrow \tilde{A}$. As graded $\tilde{A}$-modules we have

$$\tilde{C} = \tilde{A} \otimes_l (l \oplus J_1[1] \oplus J_2[2] \oplus \cdots)$$

As $\operatorname{coker} i = l$ the obvious map $\tilde{C} \longrightarrow l$ is a quasi-isomorphism.

Put $C = A \otimes_{\tilde{A}} \tilde{C}$. Then one checks that C is precisely the complex (1.3) without the right most l.

$$\cdots \longrightarrow A \otimes_l J_2 \longrightarrow A \otimes_l J_1 \longrightarrow A \longrightarrow 0$$

Thus $C \longrightarrow l$ is a quasi-isomorphism as well and hence so is the canonical map $\tilde{C} \longrightarrow C$.

We now equip $\tilde{C}$ with an ascending filtration of sub-DG-$\tilde{A}$-modules as follows: $F_0\tilde{C} = \tilde{A}$, $F_1\tilde{C} = \tilde{A} \otimes (\Bbbk \oplus J_1[1])$, ...etc and equip C with the similar filtration. The canonical map $\tilde{C} \longrightarrow C$ is a map of filtered DG-$\tilde{A}$-modules.

Assume we have shown that $\tilde{A} \longrightarrow A$ is a quasi-isomorphism in Adams degree $\leq n$. Then $(\tilde{C}/F_0\tilde{C})_{n+1} \longrightarrow (C/F_0C)_{n+1}$ is a quasi-isomorphism. Given that $\tilde{C}_{n+1} \longrightarrow C_{n+1}$ is a quasi-isomorphism we deduce that $F_0\tilde{C}_{n+1} = \tilde{A}_{n+1} \longrightarrow A_{n+1} = (F_0C)_{n+1}$ is also a quasi-isomorphism and the result follows by induction $\square$

It is worth making this construction more explicit in a few cases

Example 1.3.5. *let V be a vector space over $\Bbbk$ and let $A = S_{\Bbbk}V = T_{\Bbbk}V/(R)$ where $R = \{v \otimes w - w \otimes v\}$. Then $J_n \cong \bigwedge^n V$ and $\tilde{V} = \bigoplus_n \bigwedge^n V[n-1]$ and $\tilde{A} = T_{\Bbbk}\tilde{V}$. The differential is given by*

$$d(v_1 \wedge \ldots \wedge w_n) = \sum_i (-1)^{i-1}(v_1 \wedge \ldots \wedge v_i) \otimes (v_{i+1} \wedge \ldots \wedge v_n)$$

Introducing a basis on $V = \bigoplus_i \Bbbk x_i$, we obtain that $\tilde{A}$ is a DG algebra over $\Bbbk$ explicitly described as

- *as an algebra $\tilde{A} = \Bbbk\langle x_S \rangle$ where S is an subset of strictly ascending numbers $\{1, \ldots n\}$*
- *the grading on $\tilde{A}$ is given by $|x_S| = |S|$*
- *the differential is given by*

$$dx_S = \sum_{S = A \amalg B} (-1)^{|A|-1} \epsilon_{A,B} x_A \wedge x_B$$

  where the sign $\epsilon_{A,B}$ is defined as $x_S = \epsilon_{A,B} x_A \wedge x_B$.

This example can be adapted to skew group algebras as follows:

Example 1.3.6. *Assume that G is a finite group acting on a vector space W. Let $A = SW\#G$. Then $A = T_l V/R$ where $l = \Bbbk G$, V is the l-bimodule generated by W and R is the l-bimodule of symmetric relations. Repeating the same procedure leads one to a minimal model for A.*

- *$\tilde{A} = k\langle x_S \rangle \# G$ where S is an subset of strictly ascending numbers $\{1, \ldots n\}$, graded by $|x_S| = |S|$*

- *the differential is the* $\Bbbk G$*-linear morphism which acts on the variables* x_S *as*

$$dx_S = \sum_{S=A\amalg B} (-1)^{|A|-1} \epsilon_{A,B} x_A \wedge x_B$$

where the sign $\epsilon_{A,B}$ *is defined as* $x_S = \epsilon_{A,B} x_A \wedge x_B$.

1.3.2 the Relative Singularity Category

Following the introduction 1.1, R will denote a local complete Gorenstein $\Bbbk$-algebra of Krull dimension d such that $R/\mathfrak{m} \cong \Bbbk$, with an isolated singularity. It is well known that R is regular if and only if the subcategory $\operatorname{Perf}(R)$ of complexes quasi-isomorphic to bounded complexes of finitely generated projective R-modules coincides with $\mathscr{D}^b(R)$ (see [Orl46]). This leads one to study the singularities of R through the following important invariant:

$$\operatorname{sing}(R) = \mathscr{D}^b_f(R)/\operatorname{Perf}(R)$$

called the singularity category of R.
The study of this category has proven to be very fruitful. We will mention one celebrated theorem by Buchweitz in this context, which he describes as the 'raison d'être' of maximal Cohen Macaulay modules. Recall that the category of maximal Cohen Macauley modules can be defined as

$$\operatorname{MCM}(R) \overset{\text{def}}{=} \{M \in \operatorname{mod}(R) \,|\, \operatorname{Ext}^i(M,R) = 0 \,\forall i > 0\}$$

by the Gorenstein assumption on R. We denote by $\underline{\operatorname{MCM}}(R)$ its associated stable category

Theorem 1.3.7. *(Buchweitz, [Buc]) The inclusion* $\operatorname{MCM}(R) \hookrightarrow \operatorname{mod}(R)$ *induces an equivalence of categories:*

$$\underline{\operatorname{MCM}}(R) \xrightarrow{\simeq} \operatorname{sing}(R)$$

In this section, we discuss a relative version of $\operatorname{sing}(R)$ which we introduced in [dTdVdB10] and has since been studied by various authors (see [BK12] for the origin of the term 'relative singularity category').

The strongest notion of resolution of singularities in algebraic geometry is that of a *crepant* resolution. This is a morphism $\pi : X \longrightarrow \mathrm{Spec}(R)$ such that $\pi^*(\omega_R) = \omega_X$. The characteristic aspects of this notion were generalized to the setting of noncommutative rings by Van den Bergh leading to the concept of a noncommutative crepant resolution (NCCR). Under the stated conditions on R, this is an R-algebra of the form $A = \mathrm{End}_R(M)$ where M is given as a direct sum $M = \bigoplus M_i$ over a complete set of nonisomorphic indecomposable MCM modules $\{M_0, \ldots, M_k\}$ [1] such that A is of finite global dimension (we refer to the foundational paper [VdB02a] and to the overview paper [Leu11] for details). To fix notation, let $e_i : M \longrightarrow M_i$ be the projection map, and let $P_i = Ae_i$ the corresponding projective module, $S_i = P_i/\operatorname{rad} P_i$ the corresponding simple module and finally $l \stackrel{\text{def}}{=} \bigoplus_i \Bbbk e_i$ the split semisimple $\Bbbk$-algebra. Following our assumption, we have $P_0 \cong \mathrm{Hom}_R(M, R)$ and the properties of NCCR's imply that the functor $\iota := (-) \overset{L}{\otimes}_R P_0$ defines an embedding of $\mathrm{Perf}(R) \cong \langle P_0 \rangle$ into $\mathscr{D}^b(A)$. This allows us to define the singularity category of R, relative to the NCCR A as

$$\mathrm{sing}(R, A) \stackrel{\text{def}}{=} \mathscr{D}^b_f(A)/\iota(\mathrm{Perf}(R)) = \mathscr{D}^b_f(A)/\langle P_0 \rangle$$

Our result is an explicit relation between the derived category of A, the relative singularity of R and the (absolute) singularity category of R using minimal models. The explicit nature allows us to develop powerful methods for certain examples.

Theorem 1.3.8. *let $T_l V \longrightarrow A$ be a finite minimal model. Put $\Gamma = T_l V / T_l V e_0 T_l V$.*

[1] We shall always tacitly assume that $R = M_0$

1. *the modules S_i are perfect Γ-modules and there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Perf}(T_l V) \arrow[r, "(-)\overset{L}{\otimes}_{T_l V}\Gamma"] \arrow[d, "\simeq"] & \mathrm{Perf}(\Gamma) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Perf}(\Gamma)/\mathscr{D}_{\mathrm{fd}}(\Gamma) \arrow[d, "\simeq"] \\
\mathscr{D}^b_f(A) \arrow[r, "\mathrm{mod}\langle P_0\rangle"'] & \mathrm{sing}(R, A) \arrow[r, "\mathrm{mod}\langle S_{i\neq 0}\rangle"'] & \mathrm{sing}(R)
\end{tikzcd}$$

2. *The DG algebra Γ has finite dimensional cohomology in each degree. Moreover, we have*
$$H^0(\Gamma) = A/Ae_0A$$

We shall prove the commutativity of the above diagram using the following lemma

Lemma 1.3.9. *The functor*

$$\Xi : \mathscr{D}^b_f(A) \longrightarrow \mathscr{D}^b_f(R) : N \mapsto e_0N \tag{1.4}$$

induces an equivalence

$$\mathscr{D}^b_f(A)/\langle (S_i)_{i\neq 0}\rangle \cong \mathscr{D}^b_f(R)$$

Proof. Let $U \in \mathscr{D}^b_f(R)$. Since P_0 is a locally free on $\operatorname{Spec} R - \{\mathfrak{m}\}$ we know that $P_0 \otimes_R U$ has finite dimensional cohomology in each degree. Let N be such that for $n \geq N$ we have $H^{-n}(U) = 0$. We claim that for $n \geq N$ this implies that $H^{-n}(P_0 \overset{L}{\otimes}_R U)$ is an extension of $(S_i)_{i\neq 0}$, i.e. $e_0H^{-n}(P_0 \overset{L}{\otimes}_R U) = 0$. Indeed

$$e_0H^{-n}(P_0 \overset{L}{\otimes}_R U) = H^{-n}(e_0Ae_0 \overset{L}{\otimes}_R U) \cong H^{-n}(R \overset{L}{\otimes}_R U) = H^{-n}(U) = 0 \tag{1.5}$$

Define $\Phi(U) = \tau_{\geq -N}(P_0 \overset{L}{\otimes}_R U)$. Then $\overline{\Phi(U)}$ is a well defined object of the category $\mathscr{D}^b_f(A)/\langle (S_i)_{i\neq 0}\rangle$. We will prove the claim by showing that assignment

$\overline{\Phi(-)}$ yields a quasi-inverse to (1.4). If $U \in D^b_f(R)$ then the computation (1.5) shows that $\Xi\overline{\Phi}(U) = U$. Conversely assume $V \in D^b_f(A)$. Then $\overline{\Phi}\Xi(V) = \tau_{\geq -N}(P_0 \overset{L}{\otimes}_R e_0 V)$. Let C be the cone of the morphism

$$P_0 \overset{L}{\otimes}_R e_0 V = Ae_0 \overset{L}{\otimes}_R e_0 V \longrightarrow V$$

We immediately infer that $e_0 C = 0$, in other words C is zero in $D^b_f(A)/\langle (S_i)_{i\neq 0}\rangle$. Furthermore by our choice of N we have $e_0 H^{-n}(V) = H^{-n}(e_0 V) = 0$ for $n \geq N$ and hence $H^{-n}(V)$ is an extension of $(S_i)_{i\neq 0}$ for such n. Thus working modulo $\langle (S_i)_{i\neq 0}\rangle$ we have

$$\tau_{\geq -N}(P_0 \overset{L}{\otimes}_R e_0 V) = \tau_{\geq -N} V = V$$

which finishes the proof. □

The bulk of the work in the second claim lies in the following lemma

Lemma 1.3.10. *The category* $\operatorname{sing}(R, A) = \mathscr{D}^b_f(A)/\langle P_0\rangle$ *is* Hom*-finite and in addition*

$$\operatorname{Hom}_{\operatorname{sing}(R,A)}(A, A) = A/Ae_0A \tag{1.6}$$

Proof. Since A has finite global dimension, by a standard homological argument, the claim of Hom-finiteness reduces to showing that it is sufficient to prove for any $M \in \mathscr{D}^b_f(A)$:

$$\dim \operatorname{Hom}_{\operatorname{sing}(R,A)}(A, M) \neq \infty \tag{1.7}$$

We make one further reduction as follows: if $N \in \operatorname{mod}(A)$ then there there is a map

$$\phi : P_0^n \longrightarrow N$$

such that $\operatorname{coker}\phi \in \langle (S_i)_{i\neq 0}\rangle$. Using ϕ we can resolve any object $M \in \mathscr{D}^b(A)$ by a complex $P \in \langle P_0\rangle$ such that $\operatorname{cone}(P \longrightarrow M)$ modulo $\langle (S_i)_{i\neq 0}\rangle$ is an object M_1 in $\operatorname{mod}(A)[n]$ for $n \gg 0$. This further reduces the claim (1.7) to one of two cases where either $M \in \mathrm{mod}(A)$ or $M = S_i[p]$ for $i \neq 0$ and $p \in \mathbb{Z}$.

To deal with the latter, let $M = S_i[p]$ and let N be an extension of $(S_i)_{i\neq 0}$ in $\mathrm{mod}(A)$. Then $\mathrm{Hom}^i_{D^b_f(A)}(P_0, N) = 0$ from which we easily deduce

$$\mathrm{Hom}^i_{\mathrm{sing}(R,A)}(A, N) = \begin{cases} N & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

and the claim follows in this case

Now assume $M \in \mathrm{mod}(A)$. We first show that in this case

$$\mathrm{Hom}^i_{\mathrm{sing}(R,A)}(A, M) = 0 \tag{1.9}$$

for $i > 0$. Let p be a map $A \longrightarrow M[i]$ in $\mathrm{sing}(R, A) = \mathscr{D}^b_f(A)/\langle P_0\rangle$. Then by Verdier localization, p is represented in $\mathscr{D}^b_f(A)$ by a diagram of the following kind

$$\begin{tikzcd} & C \arrow[dl, "q"'] \arrow[dr, "p'"] & \\ A & & M[i] \end{tikzcd}$$

such that $P = \mathrm{cone}\, q \in \langle P_0\rangle$. We then obtain a morphism of distinguished triangles

$$\begin{tikzcd} P \arrow[r] \arrow[d, equal] & C \arrow[r, "q"] \arrow[d, "p'"] & A \arrow[d, dotted] \\ Q \arrow[r] & M[i] \arrow[r] & Z \end{tikzcd}$$

where $Z = (Q \longrightarrow M[i])$ is a complex

$$0 \longrightarrow Q_t \longrightarrow \cdots \longrightarrow Q_{i+2} \longrightarrow Q_{i+1} \longrightarrow Q_i \longrightarrow M \longrightarrow 0$$

for some t such that $Q_j \cong P_0^{a_j}$.

We deduce $\mathrm{Hom}_{D^b_f(A)}(A, Z) = 0$ and hence p' factors through Q. Thus p (which is the image of p' in $D^b_f(A)/\langle P_0\rangle$) is the zero map. This finishes the proof of (1.9) for $i > 0$.

To understand $\mathrm{Hom}_{\mathrm{sing}(R,A)}(A, M)$ let $\bar{M} = \operatorname{coker}\phi$. Applying (1.9) together with (1.8) to the exact sequence

$$0 \longrightarrow \operatorname{im}\phi \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0$$

we obtain

$$\mathrm{Hom}_{\mathrm{sing}(R,A)}(A, M) = \overline{M}$$

Thus this is finite dimensional as well. Applying this identity with $M = A$ we immediately obtain (1.6). We leave it to the reader ti check that this isomorphism respects the multiplications and is hence a ring-isomorphism

Proof of theorem 1.3.8. The simple modules are indeed perfect by [KY11, 2.19]. We proceed to show the properties of the diagram 1. We begin by showing that the downward functors are equivalences

- $\mathrm{Perf}(T_l V) \longrightarrow \mathscr{D}_f^b(A)$ as $\mathrm{Perf}(T_l V) \cong \mathrm{Perf}(A)$ since $T_l V \longrightarrow A$ is a quasi-isomorphism and $\mathrm{Perf}(A) \cong \mathscr{D}_f^b(A)$ since A has finite global dimension
- For the second claim, we must show that $\mathrm{Perf}(\Gamma) \cong \mathscr{D}_f^b(A)/\langle P_0\rangle$. We have $\mathrm{Perf}(T_l V) \cong \mathrm{Perf}(A) \cong \mathscr{D}_f^b(A)$ and we now invoke [Kel11, lemma 7.2] and its proof.
- to see that the downward right functor is in fact an equivalence, we first note that $\mathscr{D}_{\mathrm{fd}}^b(\Gamma)$ coincides with the triangulated category generated by the simple modules (as in [KY11, thm. 2.19a]) and hence

  $$\mathrm{Perf}(\Gamma)/\mathscr{D}_{\mathrm{fd}}(\Gamma) \cong \mathrm{sing}(R, A)/\langle (S_i)_{i\neq 0}\rangle \cong \mathscr{D}_f^b(A)/\langle P_0, (S_i)_{i\neq 0}\rangle$$

  We now compute:

  $$\begin{aligned}\mathscr{D}_f^b(A)/\langle P_0, (S_i)_{i\neq 0}\rangle &= \Big(D_f^b(A)/\langle (S_i)_{i\neq 0}\rangle\Big)/\langle P_0\rangle \\ &= \mathscr{D}_f^b(R)/\langle P_0\rangle \\ &= \mathscr{D}_f^b(R)/\mathrm{Perf}(R) \\ &= \mathrm{sing}(R)\end{aligned}$$

where we used lemma 1.3.9

For the second claim, note that

$$H^n(\Gamma) = \mathrm{Hom}_{\mathrm{Perf}(\Gamma)}(\Gamma, \Gamma[n])$$

hence since $\mathrm{Perf}(\Gamma) \cong \mathrm{Perf}(T_l V) \cong \mathscr{D}^b_f(A)$ once again, the second statement is an immediate consequence of 1.3.10 $\square$

We can apply this theorem in the case of 3-Calabi-Yau algebras, using 1.3.2

Theorem 1.3.11. *Assume that R has Krull dimension 3 and let $R \longrightarrow A$ be an NCCR for R, which is complete. Then*

- $A \cong \mathrm{Jac}(\mathcal{Q}, \mathtt{w})$
- *there is an equivalence of categories*

$$\mathrm{sing}(R) \cong \mathcal{C}_{(\mathcal{Q}^0, \mathtt{w}^0)}$$

where $(\mathcal{Q}^0, \mathtt{w}^0)$ is obtained by removing a vertex from $(\mathcal{Q}, \mathtt{w})$

Proof. It is well known that an NCCR is always Calabi-Yau. Since A is complete, it follows from 0.1.5 that A is the Jacobi algebra of a QP $(Q, \mathtt{w})$. Applying example 1.3.2 yields that the Ginzburg DG algebra $\Gamma(\mathcal{Q}, \mathtt{w})$ is a minimal model. The construction laid out in 1.3.8, together with 1.2.9 now yields the result. $\square$

Example 1.3.12. *Let $R = \Bbbk[[t, x, y, z]]/(xy - zt)$ be the ordinary double point singularity. Then R has a noncommutative crepant resolution $\mathrm{End}_R(R \oplus I)$ with $I = (x, z)$. The corresponding quiver is*

$$\begin{tikzcd} 0 \arrow[r, bend left=40, "p"] \arrow[r, bend left=15, "q"] & 1 \arrow[l, bend left=15, "s"] \arrow[l, bend left=40, "r"] \end{tikzcd}$$

with super potential

$$w = psqr - prqs$$

We see that $\mathcal{Q}^0$ *consists of the single vertex* 1 *and* $\mathtt{w}^0 = 0$. *Hence*

$$\Gamma(\mathcal{Q}^0, \mathtt{w}^0) = \Bbbk[t]$$

with $\deg t = -2$. *The associated cluster category is simply the category of* $\mathbb{Z}_2$*-graded vector spaces* SupVec($\Bbbk$) *and we recover the well-known fact*

$$\mathrm{sing}(R) \cong \mathrm{SupVec}(\Bbbk)$$

1.3.3 an Application to Cyclic Quotient Singularities

Assume that $G \stackrel{\text{def}}{=} \mathbb{Z}/n\mathbb{Z} \subset GL(V)$ acts on $V = \bigoplus_i^d \Bbbk x_i$ with weights $(\xi^{w_1}, \ldots, \xi^{w_d})$. (following the notation in §0.2.2 as always). We assume the additional conditions

$$\gcd(w_i, n) = 1 \text{ and } \sum_i w_i = 0 \bmod n$$

so that $G \subset \mathrm{SL}(V)$ by 1.2.17, $R = SV^G$ is a Gorenstein isolated singularity by 1.2.16 (R is in fact Gorenstein for any finite group by Watanabe's theorem). Moreover, the following is well known:

Lemma 1.3.13. *The canonical morphism* $SV^G \longrightarrow SV\#G$ *is an NCCR*

Proof. This is a consequence of 'Auslander's McKay correspondence '[Aus78] (see also [Leu11, thm J.2] for a discussion) or [IT10] for a detailed proof □

Hence to apply our construction, we need to explicitly compute a minimal model for $SV\#G$, This was done in example 1.3.6. Recall that for a finite group G and an irreducible character $\chi : G \longrightarrow \mathbb{C}$, there is an associated primitive idempotent

$$e_i = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g$$

Specifying to the case of a cyclic group of order n, Each irreducible character is given as

$$\chi_i : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}^* : \overline{a} \mapsto \xi^{ia}$$

so that in this case we obtain

$$e_i = \frac{1}{n}\sum_{\bar{a}=0}^{n-1} \xi^{ai}\bar{a}$$

We have $\Bbbk G = \bigoplus \Bbbk e_i$. By the construction of $SV\#G$ we have the relation $\bar{a}x_i = \xi^{aw_i}\bar{a}$ which translates into

$$e_j x_i = x_i e_{j+w_i}$$

This, together with example 1.3.6 shows the following:

Lemma 1.3.14. *The algebra $A \overset{def}{=} SV\#G$ has a minimal model $\tilde{A}$ over $l = \Bbbk G$ given by the free l-algebra $\tilde{A} = l\langle x_S\rangle_S$ where $S \subset [n]$ is a choice of numbers in ascending order subject to the relations*

$$e_l \cdot x_S = x_S \cdot e_{l+d(S)}$$

where $d(S) = \sum_{i\in S} w_i$

We will reformulate this example in terms of quivers. We described in §0.2.2 how to compute the McKay quiver $\mathcal{Q}$ of G as indexed by edges $\mathbb{Z}/n\mathbb{Z}$ and arrows $x_{l,i,l+w_i}$. [AIR, prop 5.5] now shows that $SV\#G$ is the path algebra of the McKay quiver modulo relations:

$$x_{l,i,l+w_i} \cdot x_{l+w_i,k,l+w_i+w_k} = x_{l,k,l+w_k} \cdot x_{l+w_k,i,l+w_k+w_i}$$

(see §0.2.10, for a superpotential inducing these relations in the case $d = 3$) This allows us to give a second description of the finite minimal model of $SV\#G$ as follows: let $\tilde{\mathcal{Q}}$ denote the graded quiver with same vertices as $\mathcal{Q}$ and arrows $x_{j,S,j+d(S)}$ of homological degree $-|S|+1$ going from j to $j+d(S)$. Then $\tilde{A} = \Bbbk\tilde{\mathcal{Q}}, d)$ with differential given by

$$dx_{j,S,j+d(S)} = \sum_{S=A\coprod B, A\neq\emptyset, B\neq\emptyset} (-1)^{|A|-1}\epsilon_{A,B} x_{j,A,j+d(A)} \cdot x_{j+d(A),B,j+d(S)} \tag{1.10}$$

Let $\tilde{Q}^0$ be the quiver obtained from $\tilde{Q}$ by dropping all arrows adjacent to 0. Thus the arrows in $\tilde{Q}^0$ are of the form $x_{j,S,j+d(S)}$ with $j \neq 0$, $j + d(S) \neq 0$. We can now specialize theorem 1.3.8 to the current situation we obtain the following result:

Proposition 1.3.15. *Let $A = \Bbbk[x_1, \ldots, x_n]$ and assume that the cyclic group $G = \mathbb{Z}/m\mathbb{Z}$ acts linearly on A with weights $(\xi^{a_1}, \ldots, \xi^{a_n})$ satisfying the additional properties $\sum_i a_i = 0$ mod 0 and $\gcd(a_i, m) = 1$. Then*

$$\operatorname{sing}(\hat{A}^G) \cong \operatorname{Perf}(\Bbbk\tilde{\mathcal{Q}}^0, d)/\langle (S_i)_{i=1,\ldots,m-1}\rangle$$

where the differential is given by (1.10) (taking into account that arrows adjacent to the vertex 0 should be suppressed on the righthand side). The DG algebra $(\Bbbk\tilde{\mathcal{Q}}^0, d)$ has finite dimensional cohomology and

$$H^0(k\tilde{Q}^0, d) \cong \hat{A}/\hat{A}e_0\hat{A}$$

Proof. The morphism $\hat{A}^G \longrightarrow \hat{A}\#G$ is an NCCR by 1.3.13. We described the minimal model $(\Bbbk\tilde{\mathcal{Q}}, d) \longrightarrow \hat{A}\#G$ above. The result now follows from theorem 1.3.8, by noting that the associated DG algebra Γ is precisely given by removing the vertex 0

□

We conclude our example by mentioning that this example yields an interpretation of $\operatorname{sing}(R)$ as a higher cluster category as described in 0.3.13. This in particular gives an alternate explanation of Auslander's result that the singularity category of $\hat{A}^G$ is $d-1$-Calabi-Yau. To this end, we consider the following construction of a quiver algebra: Let P be the quiver with the same vertices as $\tilde{Q}^0$ but only with ascending arrows subject once again to the relations

$$x_{l,i,l+w_i} \cdot x_{l+w_i,k,l+w_i+w_k} = x_{l,k,l+w_k} \cdot x_{l+w_k,i,l+w_k+w_i}$$

Let C be the resulting path algebra. As in [10, 15] we define the inverse dualizing complex of C as

Proposition 1.3.16. *There is a quasi-isomorphism of DG algebras*

$$(k\tilde{Q}^0, d) \xrightarrow{\simeq} \mathbf{\Pi}_n C$$

In particular $\operatorname{sing}(\hat{A}^G)$ *is a generalized* $d-1$*-cluster category as in (7) of theorem 0.3.13.*

Proof. We refer the reader to our paper [dTdVdB10, prop 6.6.1] for a detailed proof. □

Chapter 2

the Deformation Theory of Calabi-Yau Algebras

2.1 Introduction and Statement of Results

In this chapter we describe the deformation theory of Calabi-Yau algebras. We will tweak the definition given in 0.0.2 a little in order to better suit our purposes. More precisely, Calabi-Yau algebra will consist of an ordinary associative algebra A together with a cocycle η_0 in negative cyclic homology $\mathrm{HC}_d^-(A)$ which encodes the duality $\mathrm{RHom}_{A^e}(A, A^e) \xrightarrow{\sim} A^e[d]$ (see 2.3.7 below). This interpretation allows us to adapt the classical deformation theory of associative algebras to the setting of Calabi-Yau algebras. In fact, we consider the category $\mathrm{Test}_{\Bbbk}$ of so-called test algebras over the field $\Bbbk$, that is, commutative, local finite-dimensional $\Bbbk$-algebras $(R, \mathfrak{m})$, whose maximal ideal is nilpotent and satisfies $R/\mathfrak{m} = \Bbbk$. A deformation functor[1] is a pseudo-functor $\mathrm{Test}_{\Bbbk} \longrightarrow \mathrm{Gd}$ where Gd is the 2-category of groupoids. In the classical deformation theory of associative algebras one assigns to a test algebra $(R, \mathfrak{m})$ the

[1] We shall also use refer to this as a deformation theory

groupoid of associative R-algebras which reduce to A after tensoring with $\Bbbk$ to obtain a deformation functor

$$\mathrm{Def}_A : \mathrm{Test}_{\Bbbk} \longrightarrow \mathrm{Gd}\,.$$

We extend this by associating to the Calabi-Yau algebra (A, η_0) the groupoid of algebra deformations of A equipped with a negative cyclic chain lifting η_0 in the appropriate sense (see §2.4). It will immediately follow from the definition that these algebra deformations are themselves d-Calabi-Yau algebras (2.4.2) and we can thus obtain the deformation functor of a Calabi-Yau algebra (A, η_0):

$$\mathrm{Def}_{A,\eta_0} : \mathrm{Test}_{\Bbbk} \longrightarrow \mathrm{Gd}$$

An important way to generate deformation functors is through the Maurer-Cartan equation associated to a nilpotent DG Lie algebra $\mathfrak{g}^\bullet$ (see §2.5). One considers the set $\mathrm{MC}(\mathfrak{g}^\bullet)$ of Maurer-Cartan elements $x \in \mathfrak{g}^1$ satisfying the equation

$$dx + \frac{1}{2}[x, x] = 0$$

and shows that the group $\exp(\mathfrak{g}^\bullet) \overset{\text{def}}{=} \{\exp(x)\,|x \in \mathfrak{g}^0\}$, with product given by the Baker-Campbell-Hausdorff formula, acts on $\mathrm{MC}(\mathfrak{g}^\bullet)$, endowing it with the structure of a groupoid. This allows one to define a pseudo-functor

$$\mathcal{MC}(\mathfrak{g}^\bullet) : \mathrm{Test}_{\Bbbk} \longrightarrow \mathrm{Gd} : (R, \mathfrak{m}) \mapsto \mathrm{MC}(\mathfrak{g}^\bullet \otimes \mathfrak{m}).$$

The interesting invariants associated to a deformation theory have a simple description for this type of deformation functor: the tangent space to the deformation theory -defined as the groupoid associated to the algebra of dual numbers- is given by $H^1(\mathfrak{g}^\bullet)$. Moreover, although obstruction theories for deformation functors need not be unique, in this case there is a canonical choice of obstruction space $\mathcal{O}(\mathfrak{g}^\bullet)$ which can be computed as an explicit subspace of $H^2(\mathfrak{g}^\bullet)$.

Returning to the classical case of deformations of associative algebras, it is well know that shifted Hochschild cochain complex $\mathfrak{C}^\bullet(A)$ can be endowed with a bracket giving it the structure of a DG Lie algebra. Moreover, the associated deformation functor $\mathcal{MC}(\mathfrak{C}^\bullet(A))$ *controls* the deformation theory of A in the sense that there is a natural transformation

$$\pi_A : \mathcal{MC}(\mathfrak{C}^\bullet(A)) \longrightarrow \mathrm{Def}_A$$

from the Maurer-cartan functor of $\mathfrak{C}^\bullet(A)$ to the deformation functor of the associative algebra A such that for any test algebra $(R, \mathfrak{m})$, the morphism of groupoids $\pi(R, \mathfrak{m})$ is essentially surjective on objects and surjective on morphisms. We can again adapt this result to our setting of Calabi-Yau deformations as follows: The higher operations on $\mathfrak{C}^\bullet(A)$ interact with the (normalized) negative cyclic complex $\overline{CC}_\bullet^-(A)$ to produce the structure of a *noncommutative calculus* (see [TT05]). In particular there is an operation L, the *Lie derivative* which gives $\overline{\mathrm{CC}}_\bullet^-(A)$ the structure of a DG Lie module over $\mathfrak{C}^\bullet(A)$. This allows us to construct the semi-direct product DG Lie algebra $\mathfrak{C}^\bullet(A) \ltimes \overline{\mathrm{CC}}_\bullet^-(A)[d-1]$. Moreover, the properties of η_0 show that we can deform the differential by adding $[\eta_0, -]$ to produce a DG Lie algebra $\mathfrak{D}^\bullet(A, \mu, \eta_0)$. [23]. Our first result in this chapter is that this DG Lie algebra controls the deformation theory of the Calabi-Yau algebra $\mathfrak{D}^\bullet(A, \eta_0)$ in the sense explained above.

Theorem A. *(see 2.6.3) There is a morphism of pseudo-functors*

$$\pi_{A,\eta_0} : \mathcal{MC}(\mathfrak{D}^\bullet(A, \eta_0)) \longrightarrow \mathrm{Def}_{(A,\eta_0)}$$

such for any $(R, \mathfrak{m})$ *in* $\mathrm{Test}_{\Bbbk}$*, the morphism of groupoids* $\pi(R, \mathfrak{m})$ *is essentially surjective on objects and surjective on morphisms.*

[2] In fact this is slightly imprecise as $\mathfrak{D}^\bullet(A, \eta_0)$ is only determined up to a non-unique isomorphism. The actual definition of $\mathfrak{D}^\bullet(A, \eta_0)$ depends on the lift of η_0 to an explicit cycle in a suitable complex but we will ignore this subtlety in the introduction.

[3] This in similar in spirit to [Ter06] which treats the deformation of finite dimensional A_∞-algebras with a non-degenerate inner product. We do note however that this analogy is merely conceptual as we do not necessarily require A to be finite dimensional or to carry an inner product

As a formal consequence we obtain a bijection between equivalence classes of deformations

$$\mathcal{MC}(\mathfrak{D}^\bullet(A,\eta_0)\otimes\mathfrak{m})/\simeq\;\longleftrightarrow\;\mathrm{Def}_{A,\eta_0}(R)/\simeq.$$

It is also easy to see that the morphism π_{A,η_0} of the deformation theory of Calabi-Yau deformations agrees with the morphism π_A of the usual theory of associative algebra deformations of A in an obvious sense (see §2.6.2).

In the next section, we give an explicit description of the cohomology of $\mathfrak{D}^\bullet(A,\eta_0)$ and its induced graded Lie bracket. The negative cyclic homology of A is endowed with a 'string topology' bracket of degree $d-1$ constructed from the cup product on Hochschild homology by using Poincaré duality [4]. This bracket coincides with the one on $\mathfrak{D}^\bullet(A,\eta_0)$ in the following sense:

Theorem B. *(see 2.7.1) There is a quasi-isomorphism of complexes*

$$\Psi:\mathfrak{D}^\bullet(A,\eta_0)\xrightarrow{\mathrm{qis}}\Sigma^{\bullet+d-1}\,\mathrm{CC}^-(A)$$

such that the bracket of degree $d-1$ on $\mathrm{HC}^\bullet(A)$ induced from $\mathrm{H}(\mathfrak{D}^\bullet(A,\eta_0))$ coincides with the string topology bracket.

The fact that the deformations of Calabi-Yau algebras are controlled by the Maurer-Cartan functor $\mathfrak{D}^\bullet(A,\eta_0)$ together with the above description of its homology allows us to describe the tangent space and, with a little more work, also give a constraint on the obstruction space $\mathcal{O}(\mathfrak{D}(A,\eta_0))$ we mentioned above.

Theorem C.

- *the tangent space of $\mathrm{Def}_{(A,\eta_0)}$ is precisely $\mathrm{HC}^-_{d-2}(A)$*
- *there is a natural obstruction theory for $\mathrm{Def}_{(A,\eta_0)}$ which lies in in the kernel of the map $\mathrm{HC}^-_{d-3}(A)\longrightarrow HC^{\mathrm{per}}_{d-3}(A)$*

[4] The name comes from an alternative description of this bracket wich was given by Menichi in [Men09] based on intuition coming from Chas-Sullivan's string topology ([CS99]

The first statement is a formal consequence of theorem B. The second statement is theorem 2.8.1 below. It follows in particular that if $\mathrm{HC}^-_{d-3}(A) \longrightarrow \mathrm{HC}^{\mathrm{per}}_{d-3}(A)$ is injective then the deformation theory of A as Calabi-Yau algebra is unobstructed. This happens for example if $d \leq 3$ (see Corollary 2.8.8 and lemma 2.8.9 below).

In the final part of this chapter, we focus on the 'commutative case'. I.e. we let $A = \mathcal{O}(X)$ be the algebra of global section on a smooth affine Calabi-Yau variety X of dimension d. In this setting case the element η_0 can be interpreted as a volume form through the Hochschild-Kostant-Rosenberg theorem. Let $T^{\mathrm{poly},\bullet}(A)$ be the Lie algebra of poly-vector fields on A (see §2.9 below). Kontsevich's famous result from [Kon03] yields an L_∞-quasi-isomorphism

$$T^{\mathrm{poly},\bullet}(A) \xrightarrow{\mathrm{qis}} \mathfrak{C}^\bullet(A)$$

This was later extended to an L_∞-quasi-isomorphism between L_∞-modules over $T^{\mathrm{poly},\bullet}(A)$

$$(\overline{\mathrm{CC}}^-_\bullet(A), \mathsf{b} + u\mathsf{B}) \longrightarrow (\Omega^\bullet(A)[[u]], ud)$$

by Willwacher in [Wil08] [5]. In our final result we will use this morphism to obtain an analogous statement for the DG Lie algebra $\mathfrak{D}^\bullet(A, \eta_0)$: the volume form η_0 defines a divergence operator div on $T^{\mathrm{poly},\bullet}(A)$, and using this differential, we prove:

Theorem D. *Let u have degree 2. There is an commutative diagram*

$$\begin{tikzcd}
(T^{\mathrm{poly},\bullet}(A)[[u]], -u\,\mathrm{div}) \arrow[r, "\sim"] \arrow[d, "u\mapsto 0"'] & \mathfrak{D}^\bullet(A, \eta_0) \arrow[d, "\phi"] \\
T^{\mathrm{poly},\bullet}(A) \arrow[r, "\sim"'] & \mathfrak{C}^\bullet(A)
\end{tikzcd} \tag{2.1}$$

where the horizontal maps are isomorphisms in the homotopy category of DG Lie algebras.

[5] He proves this for cyclic chains, but the result follows by extending u-linearly

2.2 Preliminaries on the Hochschild- and Cyclic Complex

In this section we recall the basic operations on the Hochschild and cyclic complexes (introducing notation and conventions for the rest of the chapter along the way).

Convention 2.2.1. *We mix homological and cohomological indices, using the classical convention $X_i = X^{-i}$.*

Let R be a commutative ring and let B be an R-algebra[6]. The complexes $\mathcal{C}_\bullet(B)$ and $\mathcal{C}^\bullet(B)$ will denote the usual relative Hochschild (co)chain of B over R:

$$\mathcal{C}^\bullet(B) = \bigoplus_n \mathrm{Hom}_R(\Sigma B^{\otimes n}, B)$$
$$\mathcal{C}_\bullet(B) = \bigoplus_n B \otimes (\Sigma B)^{\otimes n}$$

(here and below, *all unadorned tensor products are over R*) We also use the following notation for the shifted Hochschild cochain complex

$$\mathfrak{C}^\bullet(B) = \Sigma\mathcal{C}^\bullet(B) = \bigoplus_n \mathrm{Hom}_R(\Sigma B^{\otimes n}, \Sigma B)$$

Convention 2.2.2. *If $x \in \mathfrak{C}^n(B)$ then we write $|x| = n - 1$. Thus $|x|$ refers to the cohomological degree of x*

Finally it will be convenient to pass to the *normalized versions* of these complexes:

$$\bar{\mathcal{C}}^\bullet(B) = \bigoplus_n \mathrm{Hom}_R(\Sigma(B/R)^{\otimes n}, B)$$
$$\bar{\mathcal{C}}_\bullet(B) = \bigoplus_n B \otimes \Sigma(B/R)^{\otimes n}$$

with a similar definition for $\bar{\mathfrak{C}}^\bullet(B)$. The following is well known:

[6]We reserve the notation of a $\Bbbk$-algebra A to denote a specific Calabi-Yau algebra we wish to deform

Lemma 2.2.3. *the canonical morphisms* $\bar{\mathcal{C}}^\bullet(B) \longrightarrow \mathcal{C}^\bullet(B)$ *and* $\mathcal{C}_\bullet(B) \longrightarrow \bar{\mathcal{C}}_\bullet(B)$ *are quasi-isomorphisms (and hence so are its shifted versions)*

Proof. see for example [Wei94, Thm 8.3.8, lemma 8.3.7]. □

Recall that if M is a complex of B^e-modules then its *Hochschild homology and cohomology* are respectively defined as

$$\mathrm{HH}_i(B, M) = \mathrm{H}^{-i}(M \overset{L}{\otimes}_{B^e} B)$$
$$\mathrm{HH}^i(B, M) = \mathrm{H}^i(\mathrm{RHom}_{B^e}(B, M))$$

As usual we write $\mathrm{HH}_i(B) = \mathrm{HH}_i(B, B)$ and similarly $\mathrm{HH}^i(B) = \mathrm{HH}^i(B, B)$. One has

$$\mathrm{HH}_i(B) = H^{-i}(\mathcal{C}_\bullet(B))$$

and if B is a projective R-module then

$$\mathrm{HH}^i(B) = H^i(\mathcal{C}^\bullet(B))$$

Convention 2.2.4. *We shall assume that* B *is a projective* R*-module so that the above equality is always satisfied.*

2.2.1 the Hochschild Cochain Complex

The standard algebraic structures on the Hochschild cochain complex can all be deduced from its structure as a *brace algebra* (for an excellent account see [GV95] as we shall merely summarize its results). Recall that braces are maps

$$\mathfrak{C}^\bullet(B) \otimes \ldots \otimes \mathfrak{C}^\bullet(B) \longrightarrow \mathfrak{C}^\bullet(B) : x \otimes x_1 \otimes \cdots \otimes x_m \mapsto x\{x_1, \ldots, x_m\}$$

defined explicitly by

$$x\{x_1, \ldots, x_m\}(b_1, \ldots b_n) = \sum_{0 \le i_1 \ldots \le i_m \le n} (-1)^\epsilon x(b_1, \ldots b_{i_1}, x_1(b_{i_1+1}, \ldots b_{i_1+|x_1|+1}), \ldots, b_{i_m}, x_m(b_{i_m+1}, \ldots, b_{i_m+|x_m|+1}), \ldots b_n)$$

where the sign is given by $\epsilon = \sum_1^m |x_k| i_k$ The corresponding *Lie bracket* on $\mathfrak{C}^\bullet(B)$ is

$$[x, y] = x\{y\} - (-1)^{|x||y|} y\{x\}$$

Let $\mu \in \mathfrak{C}^1(B) = \mathrm{Hom}(\Sigma B \otimes \Sigma B, \Sigma B)$ denote the *"inverse" multiplication* $\mu(b_1, b_2) = -b_1 b_2$. Then $[\mu, \mu] = 0$ and hence

$$dx = [\mu, x] \tag{2.2}$$

defines a differential of degree one on $\mathfrak{C}^\bullet(B)$. The *cupproduct* on $\mathfrak{C}^\bullet(B)$ is defined by

$$x \cup y = (-1)^{|x|} \mu\{x, y\}$$

This is an associative product of degree one on $\mathfrak{C}^\bullet(B)$, or equivalently an associative product of degree zero in the unshifted $\mathcal{C}^\bullet(B)$. The main properties of these operations can be summarized as follows

Theorem 2.2.5. *The operations on the Hochschild cochain complex satisfy the following properties:*

- $(\mathcal{C}^\bullet(B), d, \cup)$ *is a DG algebra.*
- $(\mathfrak{C}^\bullet(B), d, [\ ,\ ])$ *is a DG-Lie algebra.*
- *for cohomology classes,* $\overline{x}, \overline{y}, \overline{z} \in \mathrm{HH}^\bullet(B)$, *we have the graded Leibniz rule:*
$$[\overline{x}, \overline{y} \cup \overline{z}] = [\overline{x}, \overline{y}] \cup \overline{z} + \overline{y}(-1)^{|x|(|y|+1)}[\overline{x}, \overline{z}].$$
- *for* x *and* $y \in \mathrm{HH}^\bullet(B)$ *the cup product is graded commutative:*
$$\bar{x} \cup \bar{y} = (-1)^{|x||y|} \bar{y} \cup \bar{x}.$$

Proof. see [GV95] □

As a result $\mathrm{HH}^\bullet(A)$ has the structure of a so-called *Gerstenhaber algebra.*

Remark 2.2.6. *Up to suitable -and for us irrelevant- signs the cup product* $\cup$ *coincides with the Yoneda product given by the isomorphism* $\mathrm{HH}^\bullet(B) \cong \mathrm{Ext}^\bullet_{B^e}(B, B)$

2.2.2 the Hochschild Chain Complex

The combination of the Hochschild cochain- and chain complexes yields a much more complex structure. We refer to ([TT05, CR11]) for more details. The first basic operation is the *contraction.*

$$i : \mathfrak{C}^\bullet(B) \otimes \mathcal{C}_\bullet(B) \longrightarrow \mathcal{C}_\bullet(B)$$

defined through the formula

$$i_x(b_0 \otimes \ldots \otimes b_n) \stackrel{\text{def}}{=} b_0 x(b_1, \ldots b_d) \otimes b_{d+1} \otimes \ldots \otimes b_n$$

for $x \in \mathfrak{C}^\bullet(B)$ and $b_0 \otimes \ldots \otimes b_n \in \mathcal{C}_\bullet(B)$. Note that $|i_x| = |x| + 1$. We have

$$i_x i_y = (-1)^{(|x|+1)(|y|+1)} i_{y \cup x} \tag{2.3}$$

Convention 2.2.7. *The contraction is often written as a cap product:*

$$i_x(-) = x \cap -.$$

Remark 2.2.8. *The cap product $\cap$ on $\mathrm{HH}_\bullet(B)$ coincides with the action of $HH^\bullet(B)$ on $\mathrm{HH}_\bullet(B) = H^{-\bullet}(B \otimes^L_{B^e} B)$ through its action on the second factor (see e.g. [CRVdB10, Prop 11.1, 12.1])*

The second basic operation is the *Lie derivative*

$$L : \mathfrak{C}^\bullet(B) \otimes \mathcal{C}_\bullet(B) \longrightarrow \mathcal{C}_\bullet(B)$$

given explicitly as

$$\begin{aligned} L_x(b_0 \otimes \ldots \otimes b_n) := & \sum_{i=0}^{n-|x|-1} (-1)^{|x|i} b_0 \otimes \ldots \otimes b_i \otimes x(b_{i+1}, \ldots, b_{i+|x|+1}) \otimes \ldots \otimes b_n \\ & + \sum_{i=n-|x|}^{n} (-1)^{n(i+1)+|x|} x(b_{i+1}, \ldots b_n, b_0, \ldots, b_{|x|-n+i}) \otimes \ldots \otimes b_i \end{aligned}$$

The Lie derivative defines an action of $\mathfrak{C}^\bullet(B)$ on $\mathcal{C}_\bullet(B)$ for which $|L_x| = |x|$ and

$$[L_x, L_y] = L_{[x,y]} \tag{2.4}$$

The *Hochschild differential* on $\mathcal{C}_\bullet(B)$ is defined as

$$\mathsf{b} = L_\mu \tag{2.5}$$

Convention 2.2.9. *If we simply wish to consider* b *as a differential without referring to its properties, we shall write* d *as always.*

Some basic properties of these two actions are summarized in the following theorem:

Theorem 2.2.10. *the space of Hochschild chains* $(C_\bullet(B), \mathsf{b})$ *is equipped with*

- *an action* i *of the graded algebra* $(C^\bullet(B), \cup)$ *on the graded vector space* $(C_\bullet(B), \mathsf{b})$.
- *a DG Lie action* L *of the DG Lie algebra* $(\mathfrak{C}^\bullet(B), d, [,])$ *on the complex* $(\mathcal{C}_\bullet(B), \mathsf{b})$.

satisfying the following compatibilities:

1. *for* $x, y \in \mathfrak{C}^\bullet(B)$, $[L_x, i_y] = i_{[x,y]}$
2. *for* $x, y \in \mathfrak{C}^\bullet(B)$, $L_{a\cup b} = L_x i_y + (-1)^{|x|} i_x L_y$
3. *for* $x \in \mathfrak{C}^\bullet(B)$ $[\mathsf{b}, i_x] = L_x$
4. *for* $x \in \mathfrak{C}^\bullet(B)$: $[\mathsf{b}, i_x] + i_{dx} = 0$

Proof. See for example [TT05], although some parts of the statement are quite trivial. As an example, from (2.2) and (2.4) one obtains

$$[\mathsf{b}, L_x] = L_{dx} \tag{2.6}$$

showing that $(\mathcal{C}_\bullet(B), \mathsf{b})$ is not only a graded - but a DG Lie module over $\mathfrak{C}^\bullet(B)$. □

This theorem implies in particular that both the action of i and L are well defined on homology and that $(\mathrm{HH}^\bullet(B), \mathrm{HH}_\bullet(B))$ is an example of what is referred to as a *calculus* in [TT05, 3.2]

The last basic operation we need is the *Connes differential*.

$$\mathsf{B} : \mathcal{C}_\bullet(B) \longrightarrow \mathcal{C}_\bullet(B)$$

with formula

$$\mathsf{B}(b_0 \otimes \ldots \otimes b_n) = \sum_{i=0}^{n} (-1)^{ni} 1 \otimes b_i \otimes \ldots \otimes b_n \otimes b_0 \otimes \ldots \otimes b_{i-1}$$

$$+ \sum_{0}^{n} (-1)^{n(i+1)} b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_n \otimes b_0 \otimes \ldots \otimes b_{i-2}$$

It is well-known that $|\mathsf{B}| = -1$, $\mathsf{bB} + \mathsf{Bb} = 0$, $\mathsf{B}^2 = 0$. One summarizes these properties as follows:

Theorem 2.2.11. $(\mathcal{C}_\bullet(B), \mathsf{b}, \mathsf{B})$ *is a mixed complex*

Proof. see for example [Lod98, section 2.1] □

We will also need an important equality that holds for *normalized* cyclic chains: if $x \in \bar{\mathfrak{C}}^\bullet(B)$ is a normalized chain then i_x, L_x remain well-defined operations on $\bar{\mathcal{C}}_\bullet(B)$. We have the following

Lemma 2.2.12. *Assume $x \in \bar{\mathfrak{C}}^\bullet(B)$. Then on $\bar{\mathcal{C}}_\bullet(B)$ we have*

$$[\mathsf{B}, L_x] = 0 \tag{2.7}$$

Remark 2.2.13. *The formula 2.7 does not hold for unnormalized cyclic chains.*

2.2.3 the Negative Cyclic Complex

Let u be a variable of degree 2 and put

$$\mathrm{CC}_\bullet^-(B) = \mathcal{C}_\bullet(B)[[u]].$$

Equipped with the *cyclic differential* $\mathsf{b}+u\mathsf{B}$, this is the *negative cyclic complex.* Performing the same construction with the normalized Hochschild chain complex $\overline{\mathcal{C}}_\bullet(B)$, we obtain the *normalized negative cyclic complex* $\overline{\mathrm{CC}}_\bullet^-(B)$. We can extend the operations on $\bar{\mathcal{C}}_\bullet(B)$ discussed in 2.2.1 to $\mathrm{CC}_\bullet^-(B)$ by making them u-linear. This applies in particular to i_x and L_x. Combining (2.6) and (2.7) we obtain

$$[\mathsf{b} + u\mathsf{B}, L_x] = L_{dx}, \tag{2.8}$$

which immediately shows the following:

Lemma 2.2.14. *With the action of the Lie derivative L, the complex $\overline{\mathrm{CC}}_\bullet^-(B)$ becomes a DG Lie module over the DG Lie algebra $\bar{\mathfrak{C}}(A)$.*

The relation between the contraction i_x with and the cyclic differential is more subtle however. In [TT05] (see also [Get92]) Tamarkin and Tsygan define for $x \in \bar{\mathfrak{C}}^\bullet(B)$ a graded endomorphism S_x of $\bar{\mathcal{C}}_\bullet(B)$ (depending linearly on x) of degree $|S_x| = |x| - 1$ and such that the following identity holds

$$[\mathsf{b} + u\mathsf{B}, (i + uS)_x] + (i + uS)_{dx} = uL_x \tag{2.9}$$

on $\overline{\mathrm{CC}}_\bullet^-(B)$. This identity will be important for us in the sequel. Note that it implies (2.8). Finally, we mention the following special case of [TT05, Prop. 3.3.4] which will be crucial later on.

Lemma 2.2.15. *Let $x, y \in \bar{\mathfrak{C}}^\bullet(B)$ be such that $dx = dy = 0$. Then $[L_x, (i + uS)_y]$ is homotopic to $(-1)^{|x|}(i_{[x,y]} + uS_{[x,y]})$.*

Convention 2.2.16. *(A comment on base change) IF $\Bbbk$ is a field, A a $\Bbbk$-algebra and $\Bbbk \longrightarrow R$ is a morphism of commutative rings then for $B = A \otimes_\Bbbk R$ it is clear that $\mathcal{C}_{R,\bullet}(B) \cong \mathcal{C}_\bullet(A) \otimes_\Bbbk R$. Since the negative cyclic complex involves a product this is not true for $\mathrm{CC}_{R,\bullet}^-(B)$. It is true if R is finite dimensional over $\Bbbk$ however. Similarly in that case we have $\mathcal{C}_R^\bullet(B) \cong \mathcal{C}^\bullet(A) \otimes_\Bbbk R$. In the sequel we will not mention these base change isomorphisms explicitly.*

Convention 2.2.17. *(a word on signs) The operations $i_x, L_x, S_x, \mathsf{b}, \mathsf{B}$ of degree $|x| + 1$, $|x|$, $|x| - 1$, 1, -1 were defined as acting on $\bar{\mathcal{C}}_\bullet(B)$. We define* *corresponding operations on shifts* $\Sigma^r\overline{\mathcal{C}}_\bullet(B)$ *using the usual Koszul convention:*

$$\begin{aligned} i_x(s^r b) &= (-1)^{r(|x|+1)} s^r i_x(b) \\ L_x(s^r b) &= (-1)^{r|x|} s^r L_x(b) \\ S_x(s^r b) &= (-1)^{r(|x|-1)} s^r S_x(b) \\ \mathsf{b}(s^r b) &= (-1)^r s^r \mathsf{b}(b) \\ \mathsf{B}(s^r b) &= (-1)^r s^r \mathsf{b}(B) \end{aligned}$$

where s *is the* degree change operator $|sb| = |b| - 1$. *The relations between* $i_x, L_x, S_x, \mathsf{b}, \mathsf{B}$ *stated in §2.2.2 and §2.2.2 carry over to all shifts* $\Sigma^r\overline{\mathcal{C}}_\bullet(B)$ *without any sign changes, since all terms in the identities (necessarily) have the same degree*

2.3 Reinterpreting the Calabi-Yau Condition

We now show how one can encode the extra Calabi-Yau duality from 0.0.2 using negative cyclic classes, allowing us to give an alternate description of Calabi-Yau algebras:

Lemma 2.3.1. *Assume* B *is homologically smooth (see 0.0.2). Let* M *be a perfect* B^e*-module. Then there is a canonical isomorphism*

$$\mathrm{HH}_i(B, M) \cong \mathrm{Hom}_{B^e}(\Sigma^i\, \mathrm{RHom}_{B^e}(M, B^e), B) \tag{2.10}$$

in $\mathscr{D}^+(R)$.

Proof. There is an obvious morphism

$$B \otimes_{B^e} M \longrightarrow \mathrm{Hom}_{B^e}(\mathrm{Hom}_{B^e}(M, B^e), B) \tag{2.11}$$

in $\mathrm{Mod}(R)$, giving rise to a morphism

$$B \overset{L}{\otimes}_{B^e} M \longrightarrow \mathrm{Hom}_{B^e}(\mathrm{RHom}_{B^e}(M, B^e), B) \tag{2.12}$$

in $\mathscr{D}^+(R)$. We must show that this is a quasi-isomorphism. Since M is perfect we may replace it with a complex of finitely generated projective B^e-modules, which in turn reduces the claim to the case $M = B^e$. It is clear that the first morphism (2.11) is an isomorphism in this case. □

The above lemma justifies the following definition:

Definition 2.3.2. Let B be a homologically smooth algebra R and $\eta \in \mathrm{HH}_d(B)$. We say that η is *nondegenerate* if its image under (2.10) is an isomorphism.

This allows us to make a first step towards restating the definition (0.0.2) of a d-Calabi-Yau algebra over R. We temporarily redefine a Calabi-Yau algebra as a couple (B, η) where

- B is a homologically smooth R-algebra
- η is a non-degenerate element of $\mathrm{HH}_d(B)$

In order to fully take advantage of the rich structure relating the negative cyclic complex and the Hochschild complex described in 2.2.3, we will massage this temporary definition further below.
The following theorem -known as Poincaré duality- was first stated in [VdB02b]. We give a sketch of a proof and elucidate the exact nature of the duality morphism as it wasn't stated explicitly in[VdB02b]

Proposition 2.3.3. *("Poincaré duality") Let $\eta \in \mathrm{HH}_d(B)$ be such that (B, η) is d-Calabi-Yau in the above sense. Then for each i, the map*

$$\mathrm{HH}^i(B) \longrightarrow \mathrm{HH}_{d-i}(B) : \mu \mapsto \mu \cap \eta \tag{2.13}$$

is an isomorphism

Proof. The isomorphism (2.12) in $\mathscr{D}^+(R)$

$$\mathrm{RHom}_{B^e}(\mathrm{RHom}_{B^e}(B, B^e), \overset{\downarrow}{B}) \cong B \otimes^L_{B^e} \overset{\downarrow}{B} \tag{2.14}$$

is in fact compatible with the $\mathrm{RHom}_{B^e}(B, B)$-actions on the marked copies of B.

By our definition of Calabi-Yau, the class $\eta \in H^{-d}(B \otimes^L_{B^e} B)$ must correspond to an isomorphism $\eta^+ : \mathrm{RHom}_{B^e}(B, B^e) \to \Sigma^{-d} B$ under (2.14). This yields an isomorphism

$$\mathrm{RHom}_{B^e}(\Sigma^{-d}B, \overset{\downarrow}{B}) \longrightarrow \mathrm{RHom}_{B^e}(\mathrm{RHom}_{B^e}(B, B^e), \overset{\downarrow}{B}) : \theta \mapsto \theta \circ \eta^+ \quad (2.15)$$

which is again compatible with the marked $\mathrm{RHom}_{B^e}(B, B)$-actions. Composing (2.14) and (2.15) yields an isomorphism

$$\xi : \mathrm{RHom}_{B^e}(\Sigma^{-d}B, \overset{\downarrow}{B}) \longrightarrow B \otimes^L_{B^e} \overset{\downarrow}{B}$$

which sends Id_B to η. Taking into account the remark 2.2.8, the compatibility with the $\mathrm{RHom}_{B^e}(B, B)$-actions implies that ξ transforms $\cup$ into $\cap$ on the level of cohomology. More precisely

$$\xi(\mu \cup \sigma) = \pm \mu \cap \xi(\sigma)$$

The lemma now follows by taking $\sigma = \mathrm{Id}_B$. □

This immediately implies the following observation:

Corollary 2.3.4. *Let $\eta \in \mathrm{HH}_d(B)$ be such that (B, η) is d-Calabi-Yau. Then*

$$\begin{aligned} \mathrm{HH}^i(B) &= 0 \qquad \textit{for } i \notin [0, d] \\ \mathrm{HH}_i(B) &= 0 \qquad \textit{for } i \notin [0, d] \quad \square \end{aligned}$$

We remind the reader that $\mathrm{CC}^-_\bullet(B) = (\mathcal{C}_\bullet(B)[[u]], \mathsf{b} + u\mathsf{B})$ is the negative cyclic complex with corresponding homology by $\mathrm{HC}^-_\bullet(B)$ as in §2.2.3.

Proposition 2.3.5. *Let $\eta \in \mathrm{HH}_d(B)$ be such that (B, η) is d-Calabi-Yau. Then $\mathrm{HC}^-_i(B) = 0$ for $i > d$ and furthermore the map*

$$\pi : \mathrm{CC}^-_\bullet(B) \longrightarrow \mathcal{C}_\bullet(B) : \sum b_i u^i \mapsto b_0$$

induces an isomorphism $\mathrm{HC}^-_d(B) \cong \mathrm{HH}_d(B)$.

Proof. We prove this through a spectral sequence argument. We view $\mathrm{CC}^-_\bullet(B)$ as a double complex with b pointing vertically upwards and $u\mathsf{B}$ pointing horizontally to the right. By Corollary 2.3.4 we have $\mathrm{HH}_i(B) = 0$ for $i > d$. Hence if we filter $\mathrm{CC}^-_\bullet(B)$ by column degree then the E^1 term of the resulting spectral sequence looks like

$$\begin{tikzcd}
0 & \mathrm{HH}_{d-2}(B) \arrow[r, "u\mathsf{B}"] & u\,\mathrm{HH}_{d-1}(B) \arrow[r, "u\mathsf{B}"] & u^2\,\mathrm{HH}_d(B) \\
0 & \mathrm{HH}_{d-1}(B) \arrow[r, "u\mathsf{B}"] & u\,\mathrm{HH}_d(B) & 0 \\
0 & \mathrm{HH}_d(B) & 0 & 0 \\
0 & 0 & 0 & 0
\end{tikzcd}$$

This implies the result □

This allows us extend the notion of nondegeneracy to the negative cyclic complex:

Definition 2.3.6. Let B be a homologically smooth R-algebra. We say that an element $\eta \in \mathrm{HC}^-_d(B)$ is nondegenerate if $\pi(\eta)$ is non-degenerate as in 2.3.2.

This brings us to our ultimate definition of Calabi-Yau algebra which we will use for the rest of this chapter :

Definition 2.3.7. (Restatement of definition 0.0.2.) A Calabi-Yau algebra of dimension d over R is a couple (B, η) where B is a homologically smooth R-algebra and η is a non-degenerate element of $\mathrm{HC}^-_d(B)$.

The preceding discussion shows:

Theorem 2.3.8. *Let B be a homologically smooth R-algebra. Then the following are equivalent:*

- *B is d-Calabi-Yau*
- *there exists an $\eta \in HC_d^-(B)$ such that (B, η) is Calabi-Yau*

Remark 2.3.9. *In the more general setting of DG algebras this is no longer the case (the main hurdle being the existence of cohomology in negative degrees, which breaks the spectral sequence argument we made). It is generally believed that definition 2.3.7 is the "correct" definition for a d-Calabi-Yau algebra in the DG case. We refer to the works [KS09] and Keller [Kel11] for this point of view*

2.4 Deformations of Calabi-Yau Algebras

In this section we fix a d-Calabi-Yau algebra (A, η_0) over a field $\mathbb{k}$ as in definition 2.3.7 and we explain how one can associate a deformation theory to (A, η_0), illustrating some of its basic properties along the way.

Let $\mathrm{Test}_{\mathbb{k}}$ be the category of commutative, finite dimensional, local $\mathbb{k}$-algebras $(R, \mathfrak{m})$ such that $R/\mathfrak{m} = \mathbb{k}$, henceforth known as test algebras. For $(R, \mathfrak{m}) \in \mathrm{Test}_{\mathbb{k}}$ we define a groupoid $\mathrm{Def}_{A,\eta_0}(R)$ as follows

- The objects in $\mathrm{Def}_{A,\eta_0}(R)$ are triples (B, s, η) such that B is a flat R-algebra, $s : B \to A$ is an R-algebra morphism inducing an isomorphism $B \otimes_R \mathbb{k} \to A$ and η is an element of $\mathrm{HC}_d^-(B)$ such that $s(\eta) = \eta_0$.
- A morphism $(B_1, s_1, \eta_1) \to (B_2, s_2, \eta_2)$ is a commutative diagram

$$\begin{tikzcd} B_1 \arrow[rr, "\phi"] \arrow[dr, "s_1"'] & & B_2 \arrow[dl, "s_2"] \\ & A & \end{tikzcd}$$

  such that $\eta_2 = \phi(\eta_1)$.

This indeed defines a groupoid as the following lemma shows:

Lemma 2.4.1. *A morphism* $(B, s_1, \eta_1) \longrightarrow (B_2, s_2, \eta_2)$ *in the category* Def_{A,η_0} *is necessarily invertible*

Proof. One easily deduces that B_1 and B_2 are isomorphic to $A \otimes_{\Bbbk} R$ as R-modules, we can thus assume $B_1 = B_2 = A \otimes_{\Bbbk} R$. Any R-linear map $\phi : A \otimes_{\Bbbk} R \longrightarrow A \otimes_{\Bbbk} R$ is determined by the $\Bbbk$-linear map $\phi|_A : A \longrightarrow A \otimes R$. Using the exact sequences

$$0 \longrightarrow \mathfrak{m}^{i+1} \longrightarrow \mathfrak{m}^i \longrightarrow \mathfrak{m}^i/\mathfrak{m}^{i+1} \longrightarrow 0$$

and the fact that $\mathfrak{m}$ is nilpotent, we obtain a direct sum decomposition

$$R = \bigoplus_0^n \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

It follows that $\phi|_A$ is given as a sum of maps $\phi_i : A \longrightarrow A \otimes_{\Bbbk} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ with $\phi_0 = \mathrm{Id}_A \otimes 1$. As this map is invertible, one can use the classical inverse formula to obtain an inverse for $\sum \phi_i$, and hence for ϕ. $\square$

To be able to rightfully claim that Def_{A,η_0} describes the Calabi-Yau deformations of (A, η_0) we need the following elementary lemma:

Lemma 2.4.2. *Assume that* $(B, s, \eta) \in \mathrm{Def}_{A,\eta_0}(R)$. *Then* (B, η) *is d-Calabi-Yau.*

Proof. We have to show that B is a perfect B^e-module and that η induces an isomorphism $\eta^+ : \mathrm{RHom}_{B^e}(B, B^e) \to \Sigma^{-d} B$. Since R is finite dimensional every flat R-module is R-projective. This applies in particular to B and B^e. Let

$$0 \to P_u \to \cdots \to P_0 \to A \to 0$$

be a finite resolution of A by finitely generated projective A^e-modules. It is easy to see that this resolution can be lifted step by step to a resolution

$$0 \to Q_u \to \cdots \to Q_0 \to B \to 0$$

where the Q_i are finitely generated projective B^e-modules satisfying $Q_i \otimes_R \Bbbk \cong P_i$. In particular B is perfect.

Since η^+ is now a map between perfect modules, $H = \operatorname{cone} \eta^+$ is itself perfect. Moreover, as $s(\eta) = \eta_0$, using 2.12, it it easy to see that $\eta^+ \overset{L}{\otimes} \Bbbk \cong \eta_0^+$ and hence $(\operatorname{cone} \eta^+) \overset{L}{\otimes} \Bbbk \cong \operatorname{cone}(\eta^+ \overset{L}{\otimes} \Bbbk) \cong \operatorname{cone} \eta_0^+ = 0$. It now suffices to note that if H is perfect and $H \overset{L}{\otimes} \Bbbk = 0$ then $H = 0$. □

Varying the choice of test algebra $(R, \mathfrak{m})$ results in a pseudo-functor

$$\mathrm{Def}_{A,\eta_0} : \mathrm{Test}_{\Bbbk} \to \mathrm{Gd} : (R, \mathfrak{m}) \mapsto Def_{A,\eta_0}(R)$$

It will be more convenient to work with a variant of the groupoid $\mathrm{Def}_{A,\eta_0}(R)$ which is easier to describe cohomologically. We remind the reader of the base change convention exhibited in §2.2.16 which we will use throughout. As in §2.2.1 let $-\mu_0 \in \mathfrak{C}^1(A)$ be the multiplication map on A and let $\hat{\eta}_0$ be a lift of η_0 to $\overline{\mathrm{CC}}_d^-(A)$. We define an associated groupoid $\mathrm{Def}^\flat_{A,\mu_0,\hat{\eta}_0}(R)$ as follows: The objects are couples (μ, η) where

- $\mu \in \mathfrak{C}^1(A) \otimes_k R$ is such that $-\mu$ defines a unital associative multiplication on $A \otimes_k R$;
- $\mu \bmod \mathfrak{m} = \mu_0$;
- $\eta \in \overline{\mathrm{CC}}_d^-(A) \otimes_k R$;
- $(L_\mu + u\mathsf{B})(\eta) = 0$;
- $\eta \bmod \mathfrak{m} = \hat{\eta}_0$.

Concerning the 4th condition, recall that by (2.5), L_μ is the Hochschild differential of the algebra $(A \otimes_{\Bbbk} R, \mu)$. Hence $L_\mu + u\mathsf{B} = \mathsf{b}_\mu + u\mathsf{B}$ is the cyclic differential for the algebra $(A \otimes_k R, \mu)$.
A morphism $(\mu_1, \eta_1) \to (\mu_2, \eta_2)$ in $\mathrm{Def}^\flat_{A,\mu_0,\hat{\eta}_0}(R)$ is a couple (f, ξ) where

1. f is a unital map of R-algebras $f : (A \otimes_k R, -\mu_1) \longrightarrow (A \otimes_k R, -\mu_2)$;

2. $f \otimes \mathbb{k} = \mathrm{Id}$;
3. ξ is an element of $\overline{\mathrm{CC}}^{-}_{d+1}(A) \otimes_k \mathfrak{m}$;
4. $(L_{\mu_2} + u\mathsf{B})(\xi) = \phi(\eta_1) - \eta_2$.

The composition of morphisms

$$(\mu_1, \eta_1) \xrightarrow{(f',\xi')} (\mu_2, \eta_2) \xrightarrow{(f,\xi)} (\mu_2, \eta_2)$$

is defined by

$$(f, \xi) \circ (f', \xi') = (\phi \circ f', \phi(\xi') + \xi) \tag{2.16}$$

The following observation is necessary yet trivial:

Lemma 2.4.3. *the above construction defines a groupoid structure on* $\mathrm{Def}^{\flat}_{A,\hat{\eta_0}}$

Proof. The reader will check that composition is associative. Let (f, ξ) be a morphism in$\mathrm{Def}^{\flat}_{A,\mu_0,\hat{\eta_0}}$. Then the first two conditions of morphisms imply that f is bijective. It follows that $(f^{-1}, f^{-1}(\xi))$ is the inverse morphism. □

We again obtain a pseudo-functor

$$\mathrm{Def}^{\flat}_{A,\eta_0} : \mathrm{Test}_{\mathbb{k}} \to \mathrm{Gd} : (R, \mathfrak{m}) \mapsto \mathrm{Def}^{\flat}_{A,\eta_0}(R)$$

This pseudo-functor is not exactly equivalent to the one defined for Calabi-Yau deformations, but satisfies the following weakened property of 'controlling', which is very useful in the context of deformation functors

Definition 2.4.4. Let $F, G : \mathrm{Test}_k \longrightarrow \mathrm{Gd}$ be two deformation functors. We say that F controls G if there is a natural transformation $F \longrightarrow G$ of pseudo-functors such that for any $(R, \mathfrak{m}) \in \mathrm{Test}_{\mathbb{k}}$, the morphism of groupoids $F(R, \mathfrak{m}) \longrightarrow G(R, \mathfrak{m})$ is essentially surjective on objects and surjective on morphisms.

Below we will use the notation $\overline{\eta}$ for the cohomology class of a cocycle.

Proposition 2.4.5. $\mathrm{Def}^{b}_{A,\hat{\eta_0}}$ *controls* Def_{A,η_0}. *More precisely, the assignment*

$$\begin{cases} \mathrm{Ob}(\mathrm{Def}^{\flat}_{A,\hat{\eta}_0}(R)) \longrightarrow \mathrm{Ob}(\mathrm{Def}_{A,\eta_0}(R)) & : (\mu,\eta) \mapsto ((A\otimes_k R, -\mu), \text{“}\operatorname{mod}\mathfrak{m}\text{”}, \bar{\eta}) \\ \mathrm{Mor}(\mathrm{Def}^{\flat}_{A,\hat{\eta}_0}(R)) \longrightarrow \mathrm{Mor}(\mathrm{Def}_{A,\eta_0}(R)) & : (f,\xi) \mapsto f \end{cases}$$

is a morphism of groupoids, essentially surjective on objects and surjective on morphisms.

Proof. We first prove essential surjectivity. Let $(B, s, \psi) \in \mathrm{Def}_{A,\eta_0}(R)$. Then since R is a finite dimensional local $\mathbb{k}$-algebra and B is R-flat we have an isomorphism $\phi : B \cong A \otimes_{\mathbb{k}} R$ as R-modules and it is easy to see that this isomorphism may be chosen to make the following diagram commutative

$$\begin{tikzcd} B \arrow[rr, "\phi"] \arrow[dr, "s"'] & & A\otimes_{\mathbb{k}} R \arrow[dl, "\operatorname{mod}\mathfrak{m}"] \\ & A & \end{tikzcd}$$

We now transfer the multiplication on B to $A \otimes_{\mathbb{k}} R$ where it becomes an element of $-\mu \in \mathfrak{C}^1(A) \otimes_{\mathbb{k}} R$ which modulo $\mathfrak{m}$ is equal to $-\mu_0$. We do the same with $\psi \in \mathrm{HC}^-_d(B)$ and we choose an element $\eta \in \overline{\mathrm{CC}}^-_d(A) \otimes_{\mathbb{k}} R$ such that $(L_\mu + u\mathsf{B})(\eta) = 0$, $\bar{\eta} = \phi(\psi)$. Thus in $\mathrm{Def}_{A,\eta_0}(R)$ we have

$$(B, s, \psi) \cong ((A \otimes_k R, -\mu), -\operatorname{mod}\mathfrak{m}, \bar{\eta})$$

This proves essential surjectivity. Now we prove surjectivity on morphisms. Let (μ_1, η_1), $(\mu_2, \eta_2) \in \mathrm{Ob}(\mathrm{Def}^{\flat}_{A,\hat{\eta}_0}(R))$ and let f be a unital algebra morphism

$$(A \otimes_k R, -\mu_1) \longrightarrow (A \otimes_k R, -\mu_2)$$

inducing the identity modulo $\mathfrak{m}$ and satisfying $f(\bar{\eta}_1) = \bar{\eta}_2$.

It follows that $f(\eta_1) - \eta_2$ is a boundary in the negative cyclic complex of $(A \otimes_k R, -\mu_2)$. In other words there exists $\xi \in \overline{\mathrm{CC}}^-_{d+1}(A) \otimes_k R$ such that

$$\phi(\eta_1) - \eta_2 = (L_{\mu_2} + u\mathsf{B})(\xi)$$

We have to show that we may in fact choose $\xi \in \overline{\mathrm{CC}}^-_{d+1}(A) \otimes_k \mathfrak{m}$. Since f is the identity modulo $\mathfrak{m}$ we have

$$\begin{aligned}\phi(\eta_1) - \eta_2 \bmod \mathfrak{m} &= \eta_1 - \eta_2 \bmod \mathfrak{m} \\ &= \hat{\eta}_0 - \hat{\eta}_0 \\ &= 0\end{aligned}$$

It follows that $\phi(\eta_1) - \eta_2 \in \overline{\mathrm{CC}}_d^-(A) \otimes_k \mathfrak{m}$ and hence $d\xi \bmod \mathfrak{m} = 0$. Since $\mathrm{HC}^-_{d+1}(A) = 0$ by Proposition 2.3.5 we see that there exists $\gamma \in \overline{\mathrm{CC}}^-_{d+2}(A) \otimes_k R$ such that $(L_{\mu_2} + u\mathsf{B})(\gamma) \cong \xi \bmod \mathfrak{m}$. In other words

$$\xi' = \xi - (L_{\mu_2} + u\mathsf{B})(\gamma) \in \overline{\mathrm{CC}}_d^-(A) \otimes_{\Bbbk} \mathfrak{m}$$

Then the couple (f, ξ') is a pre-image for f. $\square$

For the sake of completeness we state the following:

Proposition 2.4.6. *Let $\hat{\eta}_0' \in \overline{\mathrm{CC}}_0^-(A)$ be a different lift of η_0. Then $\mathrm{Def}^\flat_{A,\hat{\eta}_0'}(R)$ and $\mathrm{Def}^\flat_{A,\hat{\eta}_0}(R)$ are isomorphic.*

We could easily prove this here directly, however we will postpone the proof until §2.6 where we reinterpret $\mathrm{Def}^\flat_{A,\hat{\eta}_0}(R)$ in terms of the Maurer-Cartan equation.

2.5 the Maurer-Cartan Formalism

In this section we briefly recall the construction of the deformation functor associated to a DG-Lie algebra.

Let $\mathfrak{g}^\bullet$ be a DG-Lie algebra over $\Bbbk$. The set

$$\mathrm{MC}(\mathfrak{g}^\bullet) \stackrel{\mathrm{def}}{=} \left\{ y \in \mathfrak{g}^1 \;\middle|\; dy + \frac{1}{2}[y, y] = 0 \right\}$$

is the set of solutions to the *Maurer-Cartan equation* in $\mathfrak{g}^\bullet$. If $\mathfrak{g}^\bullet$ is a nilpotent DG Lie algebra, this set has a natural structure of a groupoid which we now describe. Let $\widehat{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, completed at the augmentation ideal. Then the group $\exp(\mathfrak{g})$ is by definition the set of group like elements in $\widehat{U}(\mathfrak{g})$. It is well-known and easy to see that there is a bijection

$$\exp : \mathfrak{g} \longrightarrow \exp(\mathfrak{g}) : x \mapsto e^x$$

between the primitive and the group like elements in $\widehat{U}(\mathfrak{g})$. $\widehat{U}(\mathfrak{g}^0)$ acts on the graded Lie algebra $\mathfrak{g}^\bullet$ using the adjoint action and hence so does the *gauge group* $G(\mathfrak{g}^\bullet) \stackrel{\text{def}}{=} \exp(\mathfrak{g}^0)$. This action does not commute with the differential and in particular it does not preserve $\mathrm{MC}(\mathfrak{g}^\bullet)$. However the following modified *gauge action* does:

$$\begin{aligned} \exp(x) * y &\stackrel{\text{def}}{=} e^{\operatorname{ad} x}(y) - \frac{e^{\operatorname{ad} x} - 1}{\operatorname{ad} x}(dx) \\ &= e^{\operatorname{ad} x}(y) - \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\operatorname{ad} x)^n(dx) \end{aligned} \tag{2.17}$$

where $x \in \mathfrak{g}^0$, $y \in \mathfrak{g}^1$ and $(\operatorname{ad} x)(u) = [x, u]$. An elegant derivation of this action is given by Manetti in [Man04, §V.4]. One first formally adjoins an element δ of degree one to $\mathfrak{g}^\bullet$ and considers the DG Lie algebra $\mathfrak{g}^\bullet \oplus \Bbbk\delta$ where δ satisfies the rules:

$$dx = [\delta, x],\ d\delta = 0 \text{ and } [\delta, \delta] = 0$$

We use this to rewrite (2.17):

Lemma 2.5.1.

$$\exp(x) * y = e^{\operatorname{ad} x}(y + \delta) - \delta \tag{2.18}$$

Proof. We have

$$\begin{aligned}
e^{\operatorname{ad} x}(\delta + y) &= \delta + y + \sum_{1}^{\infty} \frac{1}{n!} \operatorname{ad}^n(x)(\delta + y) \\
&= \delta + y + \sum_{1}^{\infty} \frac{1}{n!} \big(\operatorname{ad}^n(x)(\delta) + ad^n(x)(y) \big) \\
&= \delta + \sum_{1}^{\infty} \frac{1}{n!} \big(\operatorname{ad}^n(x)(\delta) + e^{\operatorname{ad} x}(y) \big) \\
&= \delta + \sum_{0}^{\infty} \frac{1}{n+1!} \big(\operatorname{ad}^n(x)(-dx)) + e^{\operatorname{ad} x}(y) \big) \\
&= \delta + e^{\operatorname{ad} x}(y) - \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\operatorname{ad} x)^n (dx)
\end{aligned}$$

□

This shows that the action preserves the set $\mathrm{MC}(\mathfrak{g}^\bullet)$ since for $y \in \mathfrak{g}^1$ it is easy to check that

$$y \in \mathrm{MC}(\mathfrak{g}^\bullet) \iff [y + \delta, y + \delta] = 0.$$

and the latter equation is preserved under the action $e^{\operatorname{ad} x}$. Hence, in the sequel we view $\mathrm{MC}(\mathfrak{g}^\bullet)$ as a groupoid through the $G(\mathfrak{g}^\bullet)$-action given by (2.17). Now assume that $(R, \mathfrak{m}) \in \mathrm{Test}_{\Bbbk}$ is a test algebra over $\Bbbk$. Given an arbitrary DG-Lie algebra $\mathfrak{g}^\bullet$ over $\Bbbk$, the vector space $\mathfrak{g}^\bullet \otimes_k \mathfrak{m}$ becomes a nilpotent DG-Lie algebra by extending the differential and the bracket in the obvious way. We thus obtain a pseudo-functor

$$\mathcal{MC}(\mathfrak{g}^\bullet) : \mathrm{Test}_{\Bbbk} \longrightarrow \mathrm{Gd} : (R, \mathfrak{m}) \mapsto \mathrm{MC}(\mathfrak{g}^\bullet \otimes_{\Bbbk} \mathfrak{m})$$

This is the "deformation functor" associated to $\mathfrak{g}^\bullet$. For this type of functor we can give an explicit description of some important notions from deformation theory. Recall that the *tangent space* of a deformation functor is the groupoid associated to the $\Bbbk$-algebra of dual numbers $\Bbbk[\epsilon]/(\epsilon^2)$. We can compute that

$$T^1(\mathcal{MC}\,(\mathfrak{g}^\bullet)) \stackrel{\text{def}}{=} \mathcal{MC}\,\mathfrak{g}^\bullet(\Bbbk[X]/(X^2)) = \mathrm{H}^1(\mathfrak{g}^\bullet) \tag{2.19}$$

Any deformation functor also comes with a family of obstruction theories which roughly determine when a deformation can be lifted along small extensions of rings in Test_k in a functorial way (see [Man09] for a detailed account of the general situation). Although there are a priori many different such theories for a given deformation functor, the particular setting of a Maurer-Cartan functor allows us to define a canonical one, denoted $\mathcal{O}(\mathfrak{g}^\bullet)$. Let $(S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a surjective morphism in $\mathrm{Test}_{\Bbbk}$ with one-dimensional kernel $\Bbbk s \subset \mathfrak{n}$. Let $x \in \mathfrak{g}^1 \otimes \mathfrak{m}$ be a solution to the Maurer-Cartan equation. Lift x to an arbitrary element $\hat{x}$ of $\mathfrak{g}^1 \otimes \mathfrak{n}$ and let $p(\hat{x}) \in \mathfrak{g}^2$ be such that $p(\hat{x})s = d\hat{x} + \frac{1}{2}[\hat{x}, \hat{x}]$. Then clearly $dp(\hat{x}) = 0$ and furthermore the cohomology class $o(x) \stackrel{\text{def}}{=} \overline{p(\hat{x})} \in \mathrm{H}^2(\mathfrak{g}^\bullet)$ does not depend on the chosen lift $\hat{x}$ of x. It is easy to see that $o(x) = 0$ if and only if x can be lifted to an element of $\mathrm{MC}(\mathfrak{g}^\bullet \otimes n)$. The class of $o(x)$ is called the *obstruction class* of x and the *obstruction space* $O(\mathfrak{g}^\bullet)$ is the linear span in $\mathfrak{g}^2$ of all such $o(x)$ as we vary the morphisms $(S, n) \to (R, m)$ with one-dimensional kernel and all $x \in \mathrm{MC}(\mathfrak{g}^\bullet \otimes m)$ as above (we refer once again to [Man09, §4] for a more detailed discussion in this case).

Finally, we shall need to perform the following twist operation in the sequel: for $y \in \mathrm{MC}(\mathfrak{g}^\bullet)$, by definition $\mathfrak{g}^\bullet_y$ is the DG-Lie algebra which is $\mathfrak{g}^\bullet$ as graded Lie algebra but which has the deformed differential

$$d_y = d + [y, -] \tag{2.20}$$

It follows from the Maurer-Cartan equation that ${\mathfrak{g}^\bullet}_y$ is indeed a DG Lie algebra. We have the following

Lemma 2.5.2. *for $x \in \mathfrak{g}^0$, the morphism*

$$e^{\operatorname{ad} x} : \mathfrak{g}^\bullet_y \longrightarrow \mathfrak{g}^\bullet_{\exp(x)*y} \tag{2.21}$$

is an isomorphism of DG-Lie algebras

Proof. This is an immediate application of (2.18) $\square$

2.6 the DG-Lie Algebra $\mathfrak{D}^\bullet(A,\eta)$

2.6.1 Controlling Calabi-Yau Deformations

Following 2.3.7, we will consider a d-Calabi-Yau algebra $(A,\bar{\eta}_0)$ where $\eta_0 \in \overline{\mathrm{CC}}_d^-(A)$ satisfies $(L_{\mu_0} + u\mathsf{B})(\eta_0) = 0$, with $-\mu_0 \in \mathfrak{C}^1(A)$ being the multiplication on A. In this section we associate a DG-Lie algebra $\mathfrak{D}^\bullet(A,\eta_0)$ to A and prove that its deformation functor (see §2.5) is isomorphic to the functor $\mathrm{Def}^\flat_{A,\eta_0}$ introduced in §2.4. As a corollary to 2.6.3, we obtain that this DG Lie algebra $\mathfrak{D}(A,\eta_0)$ controls the deformation theory of the Calabi-Yau algebra (A,η_0)

If $\mathfrak{g}^\bullet$ is a DG-Lie algebra and $M^\bullet$ a DG Lie module over $\mathfrak{g}^\bullet$, then the direct sum complex $\mathfrak{g}^\bullet \oplus M^\bullet$ becomes a DG-Lie algebra when endowed with the following bracket:

$$[(g,m),(g',m')] \overset{\text{def}}{=} ([g,g'], gm' - (-1)^{|g'||m|}g'm) \tag{2.22}$$

The resulting DG-Lie algebra is called the *semi-direct product* of $\mathfrak{g}^\bullet$ and $M^\bullet$ and is denoted by $\mathfrak{g}^\bullet \ltimes M^\bullet$

By (2.2.10) we have a DG-Lie action L of $\mathfrak{C}^b(A)$ on the negative cyclic chains. Using 2.2.17, this yields an action on the shifted negative cyclic chains also:

$$\bar{\mathfrak{C}}^\bullet(A) \times \Sigma^{-d-1}\overline{\mathrm{CC}}_\bullet^-(A) \longrightarrow \Sigma^{-d-1}\overline{\mathrm{CC}}_\bullet^-(A) : (x,\eta) \mapsto L_x\eta$$

Consequently, we can form the corresponding semi-direct product $\mathfrak{D}^\bullet(A)^\sharp = \bar{\mathfrak{C}}^\bullet(A) \ltimes \Sigma^{-d-1}\overline{\mathrm{CC}}_\bullet^-(A)$.

The element $x = (0, s^{-d-1}\eta_0) \in \mathfrak{D}^\bullet(A,\eta_0)^\sharp$ satisfies $dx = 0$ and $[x,x] = 0$. Hence it satisfies the Maurer-Cartan equation. Put $\mathfrak{D}^\bullet(A,\eta_0) = \mathfrak{D}^\bullet(A)^\sharp_x$, with notation as in §2.5.

Theorem 2.6.1. *Let* $(R,\mathfrak{m}) \in \mathrm{Test}_{\Bbbk}$. *There is an isomorphism of groupoids*

$$\Phi(R) : \mathrm{MC}(\mathfrak{D}^\bullet(A,\eta_0) \otimes_k \mathfrak{m}) \longrightarrow \mathrm{Def}^\flat_{A,\eta_0}(R)$$

which on objects is given by

$$(\mu, s^{-d-1}\eta) \mapsto (\mu_0 + \mu, \eta_0 + \eta) \tag{2.23}$$

Corollary 2.6.2. *There is a natural transformation of pseudo-functors*

$$\Phi : \mathcal{MC}(\mathfrak{D}^\bullet(A, \eta_0)) \longrightarrow \mathrm{Def}^\flat_{A,\eta_0}$$

which, when evaluated on $(R, \mathfrak{m}) \in \mathrm{Test}_k$*, is an isomorphism of groupoids.*

As an immediate corollary of 2.6.1 and 2.6.3, we obtain:

Corollary 2.6.3. *The deformation theory* $\mathcal{MC}(\mathfrak{D}^\bullet(A, \eta_0))$ *controls the Calabi-Yau deformations of* (A, η_0) *as in 2.4.4*

We shall prove theorem 2.6.1 by a subsequent series of lemmas. Throughout we fix $(R, \mathfrak{m}) \in \mathrm{Test}_\Bbbk$. The following lemma says that $\Phi(R)$ behaves correctly on objects.

Lemma 2.6.4. *Let* $\mu \in \bar{\mathfrak{C}}^\bullet_1(A) \otimes_\Bbbk \mathfrak{m}$ *and* $\eta \in \overline{\mathrm{CC}}^-_d(A) \otimes_\Bbbk \mathfrak{m}$*. The following are equivalent:*

1. $(\mu, s^{-d-1}\eta) \in \mathrm{MC}(\mathfrak{D}^\bullet(A, \eta_0) \otimes_\Bbbk \mathfrak{m})$*;*
2. $(\mu_0 + \mu, \eta_0 + \eta) \in \mathrm{Def}^\flat_{A,\eta_0}(R)$*.*

Proof. We will work out what it means for $(\mu, s^{-d-1}\eta) \in \mathfrak{D}^1(A, \eta_0) \otimes_\Bbbk \mathfrak{m}$ to satisfy the Maurer-Cartan equation. To simplify the notations we write $\eta_0' = s^{-d-1}\eta_0$, $\eta' = s^{-d-1}\eta$. We compute

$$\begin{aligned}\frac{1}{2}[(\mu, \eta'), (\mu, \eta')] + d_{\mathfrak{D}}(\mu, \eta') &= \frac{1}{2}([\mu, \mu], 2L_\mu(\eta')) + ([\mu_0, \mu], (L_{\mu_0} + u\mathsf{B})(\eta')) + [(0, \eta_0'), (\mu, \eta')] \\ &= \frac{1}{2}([\mu, \mu], 2L_\mu(\eta')) + ([\mu_0, \mu], (L_{\mu_0} + u\mathsf{B})(\eta')) + [0, L_\mu(\eta_0')] \\ &= (\frac{1}{2}[\mu, \mu] + [\mu_0, \mu], L_\mu(\eta') + (L_{\mu_0} + u\mathsf{B})(\eta') + L_\mu(\eta_0')) \\ &= ([\mu_0 + \mu, \mu_0 + \mu], (L_{\mu+\mu_0} + u\mathsf{B})(\eta' + \eta_0'))\end{aligned}$$

where in the last line we have used $[\mu_0, \mu_0] = 0$, $(L_{\mu_0} + u\mathsf{B})(\eta_0') = 0$. Thus if $(\mu_0 + \mu, \eta_0 + \eta) \in \mathrm{Def}^{\flat}_{A,\eta_0}(R)$ then $(\mu, s^{-d-1}\eta) \in \mathrm{MC}(\mathfrak{D}^\bullet(A, \eta_0) \otimes_{\mathbb{k}} \mathfrak{m})$. To prove the converse the only thing we still need to check is that $-(\mu_0 + \mu)$ defines a *unital* multiplication on $A \otimes_k R$. This follows immediately from the fact that $-\mu_0$ is unital and μ is normalized. $\square$

The next two lemmas will help us describe the gauge group action of the group $G(\mathfrak{D}^\bullet(A, \eta)) = \exp(\mathfrak{D}^0(A, \eta_0))$.

Lemma 2.6.5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{k}$ and let M a Lie module over $\mathfrak{g}$. Then there is an isomorphism of groups*

$$\exp(\mathfrak{g}) \ltimes M \longrightarrow \exp(\mathfrak{g} \ltimes M) : (\exp(n), m) \mapsto \exp(n, 0)\exp(0, m)$$

Proof. This can bee seen as a consequence of the fact that there is an isomorphism of groups $U(\mathfrak{g} \ltimes M) \cong U(\mathfrak{g}) \ltimes \mathrm{Sym}(M)$ More explicitly, consider the map

$$\Phi : \exp(\mathfrak{g} \rtimes M) \longrightarrow \exp(\mathfrak{g} \rtimes M) : \exp(g, m) \longrightarrow \exp(0, m)\exp(g, 0)$$

Since $[(0, m), (0, m)] = 0$, according to the Baker-Campbell-Hausdorff formula we have

$$\exp(g, \frac{ge^g}{e^g - 1}m) = \exp(g, 0)\exp(0, m)$$

Moreover, we have the following exchange relation:

$$\exp(g, 0)\exp(0, m) = \exp(0, e^g m)\exp(g, 0) \tag{2.24}$$

Indeed:

$$\begin{aligned}
\exp(0,m)exp(g,0) =& (\exp(-g,0)exp(0,-m))^{-1} \\
&= \exp(-g, \frac{-ge^{-g}}{e^{-g}-1}(-v)) \\
&= \exp(g, -\frac{ge^{-g}}{e^{-g}-1}v) \\
&= \exp(g, \frac{g}{1-e^{-g}}(e^{-g}v)) \\
&= \exp(g, \frac{ge^{g}}{e^{g}-1}(e^{-g}(v)) \\
&= \exp(g,0)\exp(0,e^{-g}v).
\end{aligned}$$

Hence the above map is given by the composition:

$$\Phi = \exp(Id, \frac{ge^{g}}{e^{g}-1}(-)) \circ \exp(Id, e^{-g}(-))$$

Now, the first map is clearly bijective and the second one is so by a final application of the Baker-Campbell-Hausdorff formula.

Lemma 2.6.6. *Let $\mathfrak{g}^{\bullet\sharp}$ be a nilpotent DG-Lie algebra over $\Bbbk$ and $M^\bullet$ a nilpotent DG Lie module over $\mathfrak{g}^{\bullet\sharp}$. Consider the DG algebra $\mathfrak{g}^\bullet$ which is $\mathfrak{g}^{\bullet\sharp} \ltimes M^\bullet$ as graded Lie algebras and which is equipped with a deformed differential $(d_{\mathfrak{g}}, d_M) + d_0$ where $d_0 : \mathfrak{g}^\bullet \longrightarrow M$ is of the form $g \mapsto (-1)^{|g|} g m_0$ for suitable $m_0 \in M^1$. Then for $g \in \mathfrak{g}^0$, $m \in M^0$ and $(g_1, m_1) \in \mathfrak{g}^1$ we have*

$$\begin{aligned}
\exp(g,0) * (g_1,m_1) &= (\exp(g) * g_1, e^{g}(m_1 - m_0) + m_0) \\
\exp(0,m) * (g_1,m_1) &= (g_1, m_1 - (g_1 + d_M)m)
\end{aligned}$$

Proof. We compute

$$\begin{aligned}
\exp(g,0)*(g_1,m_1) &= e^{\mathrm{ad}(g,0)}(g_1,m_1) - \sum_n \frac{1}{(n+1)!}\,\mathrm{ad}^n(g,0)(d_{\mathfrak{g}}(g,0)) \\
&= (e^{\mathrm{ad}\,g}g_1, e^g m_1) - \sum_n \frac{1}{(n+1)!}\,\mathrm{ad}^n(g,0)(d_{\mathfrak{g}}g, d_0 g) \\
&= (e^{\mathrm{ad}\,g}g_1, e^g m_1) - \sum_n \frac{1}{(n+1)!}(\mathrm{ad}^n(g)(d_{\mathfrak{g}}g), g^{n+1}m_0) \\
&= (e^g * g_1, e^g(m_1-m_0)+m_0)
\end{aligned}$$

Similarly:

$$\begin{aligned}
\exp(0,m)*(g_1,m_1) &= e^{\mathrm{ad}(0,m)}(g_1,m_1) - \sum \frac{1}{(n+1)!}\,\mathrm{ad}^n(0,m)(d_{\mathfrak{g}}(0,m)) \\
&= (g_1,m_1) - (0, g_1 m) - (0, d_M m) \\
&= (g_1, m_1 - (g_1+d_M)m)
\end{aligned}$$

□

We will also use the following variant of (2.18):

Lemma 2.6.7. *Let $\mathfrak{g}^\bullet$ be a nilpotent DG-Lie algebra with inner differential $d=[\mu_0,-]$. Then for $x\in\mathfrak{g}^0$, $y\in\mathfrak{g}^1$ one has*

$$\exp(x)*y = e^{\mathrm{ad}\,x}(y+\mu_0)-\mu_0.$$

Proof. Direct evaluation of the righthand side yields the formula (2.17) for $\exp(x)*y$. □

Proof of theorem 2.6.1. We start by verifying that (2.23) indeed defines a map of groupoids. To this end we have to define $\Phi(R)$ on maps. Note that by lemma 2.6.5 each element of $\exp(\mathfrak{D}^0(A,\mu_0,\eta_0)\otimes_{\Bbbk}\mathfrak{m})$ can be uniquely written as

$$\exp(0,s^{-d-1}\xi)\exp(f,0)$$

for $f\in\bar{\mathfrak{C}}^0(A)\otimes_{\Bbbk}\mathfrak{m} = \mathrm{Hom}(A/\Bbbk,A)\otimes_{\Bbbk}\mathfrak{m}\subset\mathrm{Hom}(A,A)\otimes_{\Bbbk}\mathfrak{m}$ and $\xi\in\overline{\mathrm{CC}}^-_{d+1}(A)\otimes_{\Bbbk}\mathfrak{m}$. We put $\phi=e^f$. Then $\phi\in\mathrm{Hom}(A,A)\otimes_k R$ is such that $\phi \bmod \mathfrak{m} = \mathrm{Id}_A$.

Assume that

$$\big(\exp(0, s^{-d-1}\xi) * \exp(f,0)\big) * (\mu_1, s^{-d-1}\eta_1) = (\mu_2, s^{-d-1}\eta_2) \tag{2.25}$$

We define $\Phi(R)$ on maps as follows

$$\Phi(R)(\exp(0, s^{-d-1}\xi)\exp(f,0)) = (e^f, (-1)^d\xi) \tag{2.26}$$

For this to be well defined we should have a morphism

$$(\phi, (-1)^d\xi) : (\mu_0 + \mu_1, \eta_0 + \eta_1) \longrightarrow (\mu_0 + \mu_2, \eta_0 + \eta_2)$$

in $\mathrm{Def}^\flat_{A,\eta_0}(R)$. In other words:

(a) $\phi : (A \otimes_{\Bbbk} R, -(\mu_0 + \mu_1)) \longrightarrow (A \otimes_{\Bbbk} R, -(\mu_0 + \mu_2))$ is an R-algebra morphism;

(b) $\phi(\eta_0 + \eta_1) = \eta_0 + \eta_2 + (-1)^d(L_{\mu_0+\mu_2} + u\mathsf{B})(\xi)$.

Put $\eta_i' = s^{-d-1}\eta_i$ for $i = 0,1,2$, $\xi' = s^{-d-1}\xi$. We invoke lemma 2.6.6 with $m_0 = -\eta_0'$. Then (2.25) yields

$$(\mu_2, \eta_2') = \Big(\exp(f) * \mu_1, e^f(\eta_0' + \eta_1') - \eta_0' - L_{\exp(f)*\mu_1}(\xi') - (L_{\mu_0} + u\mathsf{B})(\xi')\Big) \tag{2.27}$$

We may compute $\exp(f) * \mu_1$ inside unnormalized cochains $\mathcal{C}^\bullet(A)$ and then we may invoke lemma 2.6.7. We find

$$\exp(f) * \mu_1 = e^{\mathrm{ad}\, f}(\mu_0 + \mu_1) - \mu_0$$

Furthermore a direct computation shows that

$$\begin{aligned} e^{\mathrm{ad}\, f}(\mu_0 + \mu_1) &= e^f \circ (\mu_0 + \mu_1) \circ (e^{-f}, e^{-f}) \\ &= \phi \circ (\mu_0 + \mu_1) \circ (\phi^{-1}, \phi^{-1}) \end{aligned}$$

Hence (2.27) translates into

$$\begin{aligned} \mu_0 + \mu_2 &= \phi \circ (\mu_0 + \mu_1) \circ (\phi^{-1}, \phi^{-1}) \\ \eta_0' + \eta_2' &= \phi(\eta_0' + \eta_1') - (L_{\mu_0+\mu_2} + u\mathsf{B})(\xi') \end{aligned}$$

The first of these equations yields (a). The second yields (b) taking into account that $L_{\mu_0+\mu_2} + u\mathsf{B}$ has degree one, which induces a sign change.

It remains to show that our assignment respects compositions. By lemma 2.6.5 we have for $f, g, h \in \bar{\mathfrak{C}}^0(A) \otimes_{\Bbbk} \mathfrak{m}$ such that $\exp(h) = \exp(g)\exp(f)$, $\nu, \xi \in \overline{\mathrm{CC}}^-_{-d-1}(A) \otimes_{\Bbbk} \mathfrak{m}$:

$$\begin{aligned}\Phi(R)\big(&\exp(0, s^{-d-1}\nu)\exp(g,0) \circ \exp(0, s^{-d-1}\xi)\exp(f,0)\big)\\ &= \Phi(R)\big(\exp(0, s^{-d-1}\nu)\exp(0, s^{-d-1}e^g\xi)\exp(g,0)\exp(f,0)\big)\\ &= \Phi(R)\big(\exp(0, s^{-d-1}(\nu + e^g\xi))\exp(h,0)\big)\\ &= (e^h, (-1)^d(\nu + e^g\xi))\\ &= (e^g e^f, (-1)^d(\nu + e^g\xi))\end{aligned}$$

and

$$\begin{aligned}\Phi(R)(\exp(0, s^{-d-1}\nu)\exp(g,0)) &\circ \Phi(R)(\exp(0, s^{-d-1}\xi)\exp(f,0))\\ &= (e^g, (-1)^d\nu)(e^f, (-1)^d\xi)\\ &= (e^g e^f, (-1)^d(\nu + e^g\xi))\end{aligned}$$

by (2.16). We conclude that $\Phi(R)$ is indeed a map of groupoids. By lemma 2.6.4 it is bijective on objects, and running the above computation backwards, starting from (2.26), we see that it also bijective on maps. Thus $\Phi(R)$ is an isomorphism of groupoids. □

The following result implies proposition 2.4.6.

Proposition 2.6.8. *Assume that $\eta_0, \eta_0' \in \overline{\mathrm{CC}}^-_d(A)$ induce the same element in $\mathrm{HC}^-_d(A)$. Then $\mathfrak{D}^\bullet(A, \eta_0) \cong \mathfrak{D}^\bullet(A, \eta_0')$.*

Proof. From (2.21) one sees that it is sufficient to show that $(0, s^{-s-1}\eta_0), (0, s^{-d-1}\eta_0')$ are in the same $G(\mathfrak{D}^\bullet(A)^\sharp)$ orbit. Pick $\xi \in \overline{\mathrm{CC}}^-_{d+1}(A)$ such that $\eta_0' = \eta_0 + (-1)^d(L_{\mu_0} + u\mathsf{B})\xi$. We compute using (2.17)

$$\begin{aligned}\exp(0, s^{-d-1}\xi) * (0, s^{-d-1}\eta_0) &= (0, s^{-d-1}\eta_0) - (0, (L_{\mu_0} + u\mathsf{B})(s^{-d-1}\xi))\\ &= (0, s^{-d-1}\eta_0') \quad \square\end{aligned}$$

2.6.2 Relation with Hochschild Cohomology

Let (A, η_0) be a d-Calabi-Yau $\mathbb{k}$-algebra and let $-\mu_0$ be the multiplication of A. Let $(R, \mathfrak{m}) \in \mathrm{Test}_{\mathbb{k}}$. We may define pseudo-functors Def_A, $\mathrm{Def}^{\flat}_A : \mathrm{Test} \longrightarrow \mathrm{Gd}$ in the same way as Def_{A,η_0} and $\mathrm{Def}^{\flat}_{A,\eta_0}$, simply by ignoring the datum of η_0. The induced morphism

$$\mathrm{Def}^{\flat}_A(R) \longrightarrow \mathrm{Def}_A(R)$$

is essentially surjective on objects and surjective on morphisms. Furthermore there is an isomorphism of groupoids

$$\Phi(R) : \mathrm{MC}(\overline{\mathfrak{C}^{\bullet}}(A) \otimes_{\mathbb{k}} \mathfrak{m}) \longrightarrow \mathrm{Def}^{\flat}_A(R) : \mu \mapsto \mu_0 + \mu$$

The obvious morphism of DG-Lie algebras

$$\phi : \mathfrak{D}^{\bullet}(A, \eta_0) \longrightarrow \overline{\mathfrak{C}^{\bullet}}(A) : (\mu, \eta) \mapsto \mu$$

makes the following diagram commutative:

$$\begin{tikzcd}
\mathcal{MC}(\mathfrak{D}^{\bullet}(A, \eta_0)) \arrow[r, "\phi"] \arrow[d, "\Phi"'] & \mathcal{MC}(\overline{\mathfrak{C}^{\bullet}}(A)) \arrow[d, "\Phi"] \\
\mathrm{Def}_{A,\eta} \arrow[r, "\text{forget } \eta"'] & \mathrm{Def}_A
\end{tikzcd}$$

2.7 Homology of $\mathfrak{D}^{\bullet}(A, \eta_0)$

We remain in the setting where $(A, \bar{\eta}_0)$ is a d-Calabi-Yau algebra with multiplication $-\mu_0$. In this section we prove that the homology of the DG Lie algebra $\mathfrak{D}^{\bullet}(A, \eta_0)$ is isomorphic to $\mathrm{HC}^{-}_{-\bullet+d-1}(A)$. Furthermore we show that the induced Lie bracket on $\mathrm{HC}^{-}_{-\bullet+d-1}(A)$ is given Menichi's string topology bracket [Men09]. In our statements and computations we will use the following conventions:

- Taking homology classes is indicated by overlining.

- Depending on the context $\cong$ will mean either "up to homotopy" (when discussing maps) or "up to addition of a coboundary" (when discussing elements).

Theorem 2.7.1. *The map*

$$\Psi : \mathfrak{D}^\bullet(A,\eta_0) \longrightarrow \Sigma^{-d+1}\overline{\mathrm{CC}}^-_\bullet(A) : (\mu, s^{-d-1}\eta) \mapsto (-1)^{|\mu|-1}(i_\mu+uS_\mu)(s^{-d+1}\eta_0)+us^{-d+1}\eta$$

is a quasi-isomorphism of complexes.

Proof. To simplify the notation we put

$$I_\mu = i_\mu + uS_\mu \tag{2.28}$$

We first check that Ψ does indeed commute with differentials. Write $\eta_0' = s^{-d-1}\eta_0$ and $\eta' = s^{-d-1}\eta$. Then

$$\Psi(\mu,\eta') = s^2((-1)^{|\mu|-1}I_\mu\eta_0' + u\eta') \tag{2.29}$$

and hence

$$\begin{aligned}
(d\circ\Psi)(\mu,\eta') &= (\mathsf{b}+u\mathsf{B})s^2((-1)^{|\mu|-1}I_\mu\eta_0' + u\eta') \\
&= s^2((-1)^{|\mu|-1}(\mathsf{b}+u\mathsf{B})I_\mu\eta_0' + u(\mathsf{b}+u\mathsf{B})\eta') \\
&= s^2((-1)^{|\mu|-1}[\mathsf{b}+u\mathsf{B},I_\mu](\eta_0') + u(\mathsf{b}+u\mathsf{B})\eta') && (\text{since } (\mathsf{b}+u\mathsf{B})\eta_0'=0) \\
&= s^2((-1)^{|\mu|-1}(uL_\mu - I_{d\mu})\eta_0' + u(\mathsf{b}+u\mathsf{B})\eta') && (\text{by } (2.9)) \\
&= s^2((-1)^{|\mu|}I_{d\mu}\eta_0' + u((\mathsf{b}+u\mathsf{B})\eta' - (-1)^{|\mu|}L_\mu\eta_0')) \\
&= \Psi(d\mu, (\mathsf{b}+u\mathsf{B})\eta' - (-1)^{|\mu|}L_\mu\eta_0') && (\text{by } (2.29)) \\
&= (\Psi\circ d)(\mu,\eta')
\end{aligned}$$

To see that Ψ is indeed a quasi-isomorphism, consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{-d-1}\overline{\mathrm{CC}}^-_\bullet(A) \arrow[r] \arrow[d, "\Psi"'] & \mathfrak{D}^\bullet(A,\eta_0) \arrow[r] \arrow[d, "\Psi"'] & \bar{\mathfrak{C}}^\bullet(A) \arrow[r] \arrow[d, "\overline{\Psi}"] & 0 \\
0 \arrow[r] & u\Sigma^{-d+1}\overline{\mathrm{CC}}^-_\bullet(A) \arrow[r] & \Sigma^{-d+1}\overline{\mathrm{CC}}^-_\bullet(A) \arrow[r] & \Sigma^{-d+1}\bar{\mathcal{C}}_\bullet(A) \arrow[r] & 0
\end{tikzcd}$$

The map on the left is multiplication by u which is an isomorphism. The map $\overline{\Psi}$ is given on cohomology by

$$\overline{\mu} \mapsto \pm\overline{I_\mu \eta_0 \bmod u} = \pm\overline{i_\mu \pi(\eta_0)}$$

where π is as in Proposition 2.3.5. Hence $\overline{\Psi}$ is an isomorphism by Proposition 2.3.3. From the five lemma we conclude that the middle arrow is an isomorphism on cohomology as well. □

We now describe the Lie bracket on $\mathrm{HC}^-_\bullet(A)$ induced by the quasi-isomorphism Ψ. As already used in the above proof the map

$$- \cap \pi(\bar{\eta}_0) : \mathrm{HH}^i(A) \longrightarrow \mathrm{HH}_{d-i}(A)$$

is invertible by Poincaré duality 2.3.3. Let us denote its inverse by j. Using j, one can transport the cup product on Hochschild cohomology $\mathrm{HH}^\bullet(A)$ to a product on Hochschild homology $\mathrm{HH}_\bullet(A)$

$$\cdot : \mathrm{HH}_i(A) \times \mathrm{HH}_j(A) \longrightarrow \mathrm{HH}_{i+j-d}(A)$$

with explicit formula

$$a \cdot b = (j(a) \cup j(b)) \cap \pi(\bar{\eta}_0)$$

or in a form more suitable for us below

$$i_{\mu_1}\pi(\bar{\eta}_0) \cdot i_{\mu_2}\pi(\bar{\eta}_0) = i_{\mu_1 \cup \mu_2}\pi(\bar{\eta}_0) \tag{2.30}$$

Theorem 2.7.2. *The Lie bracket induced on*

$$H^\bullet(\Sigma^{-d+1}\overline{\mathrm{CC}}^-_\bullet(A)) = \mathrm{HC}_{\bullet+d-1}(A)$$

by the quasi-isomorphism Ψ is given by

$$[-,-] : \mathrm{HC}^-_n(A) \times \mathrm{HC}^-_m(A) \longrightarrow \mathrm{HC}^-_{n+m-d+1}(A) : (\eta_1, \eta_2) \mapsto (-1)^{|\eta_1|+d}\mathsf{B}(\pi(\eta_1) \cdot \pi(\eta_2))$$

where B is given by

$$\mathsf{B} : \mathrm{HH}_q(A) \longrightarrow \mathrm{HC}^-_{q+1}(A) : \bar{\nu} \mapsto \overline{\mathsf{B}\nu}$$

We first need the following technical lemma.

Lemma 2.7.3. *Let $\mu \in \bar{\mathfrak{C}}^\bullet(A)$ and $\eta \in \overline{\mathrm{CC}}_\bullet^-(A)$ be cocycles. Then $L_\mu\eta$ and $\mathsf{B}i_\mu\pi(\eta)$ are both cocycles in $\overline{\mathrm{CC}}_\bullet^-(A)$ and $\overline{\mathsf{B}i_\mu\pi(\eta)} = \overline{L_\mu\eta}$ in $\mathrm{HC}_\bullet^-(A)$.*

Proof. $L_\mu\eta$ is a cocycle by 2.8. $\mathsf{B}i_\mu\pi(\eta)$ is a cocycle since $\pi(\eta)$ is a cocycle in $\bar{\mathcal{C}}_\bullet(A)$ and

$$(\mathsf{b}+u\mathsf{B})(\mathsf{B}i_\mu\pi(\eta)) = \mathsf{bB}i_\mu\pi(\eta) = -\mathsf{Bb}i_\mu\pi(\eta) = 0$$

where the last equality follows from 2.2.10(4).

For the second claim, we first multiply by u:

$$\begin{aligned} u(L_\mu\eta - \mathsf{B}i_\mu\pi(\eta)) &= [\mathsf{b}+u\mathsf{B}, I_\mu]\eta - u\mathsf{B}i_\mu\pi(\eta) && \text{(by (2.9))}\\ &= (\mathsf{b}+u\mathsf{B})I_\mu\eta - u\mathsf{B}i_\mu\pi(\eta) && \text{(since } (\mathsf{b}+u\mathsf{B})\eta = 0)\\ &= (\mathsf{b}+u\mathsf{B})(I_\mu\eta - i_\mu\pi(\eta)) && \text{(since } \mathsf{b}i_\mu\pi(\eta) = 0) \end{aligned}$$

Now, $\pi(I_\mu\eta - i_\mu(\pi(\eta)) = i_\mu\pi(\eta) - i_\mu\pi(\eta) = 0$, which means that $I_\mu\eta - i_\mu\pi(\eta)$ is divisible by u. Thus it follows that

$$L_\mu\eta - \mathsf{B}i_\mu\pi(\eta) = (\mathsf{b}+u\mathsf{B})(u^{-1}(I_\mu\eta - i_\mu\pi(\eta)))$$

hence the claim. □

Proof of theorem 2.7.2. Let $(\mu_1, s^{-d-1}\eta_1)$ and $(\mu_2, s^{-d-1}\eta_2)$ be two cocycles in $\mathfrak{D}^\bullet(A, \eta_0)$. We must prove for $\eta_i' \overset{\text{def}}{=} s^{-d-1}\eta_i$

$$\overline{s^{d-1}\Psi([(\mu_1,\eta_1'),(\mu_2,\eta_2')])} = [\overline{s^{d-1}\Psi(\mu_1,\eta_1')}, \overline{s^{d-1}\Psi(\mu_2,\eta_2')}] \tag{2.31}$$

We will first compute the left-hand side of (2.31). Writing out the differential in $\mathfrak{D}^\bullet(A, \eta_0')$ explicitly, the fact that (μ_1, η_1'), (μ_2, η_2') are cocycles implies

$$\begin{gathered} d\mu_1 = d\mu_2 = 0 \\ (\mathsf{b}+u\mathsf{B})\eta_1' - (-1)^{|\mu_1|}L_{\mu_1}\eta_0' = (\mathsf{b}+u\mathsf{B})\eta_2' - (-1)^{|\mu_2|}L_{\mu_2}\eta_0' = 0 \end{gathered} \tag{2.32}$$

where $\eta_0' = s^{-d-1}\eta_0'$. We compute

$$\begin{aligned} x &\stackrel{\text{def}}{=} s^{d-1}\Psi([(\mu_1,\eta_1'),(\mu_2\eta_2')]) \\ &= s^{d-1}\Psi([\mu_1,\mu_2], L_{\mu_1}\eta_2' - (-1)^{|\mu_1||\eta_2'|}L_{\mu_2}\eta_1') \\ &= s^{d+1}((-1)^{|\mu_1|+|\mu_2|-1}I_{[\mu_1,\mu_2]}\eta_0' + u(L_{\mu_1}\eta_2' - (-1)^{|\mu_1||\mu_2|}L_{\mu_2}\eta_1')) \end{aligned} \tag{2.33}$$

where we have used (2.29) and the fact that $|\eta_2'| = |\mu_2|$.

We now consider the boundary element $(\mathsf{b}+u\mathsf{B})I_{\mu_1}\eta_2'$. By (2.9), we have

$$(\mathsf{b}+u\mathsf{B})I_{\mu_1}\eta_2' - (-1)^{|\mu_1|+1}I_{\mu_1}(\mathsf{b}+u\mathsf{B})\eta_2' + I_{d\mu_1}\eta_2' = uL_{\mu_1}\eta_2'$$

Taking into account (2.32) this becomes

$$\begin{aligned} (\mathsf{b}+u\mathsf{B})I_{\mu_1}\eta_2' &= (-1)^{|\mu_1|-1}I_{\mu_1}(\mathsf{b}+u\mathsf{B})\eta_2' + uL_{\mu_1}\eta_2' \\ &= (-1)^{|\mu_1|-1+|\mu_2|}I_{\mu_1}L_{\mu_2}\eta_0' + uL_{\mu_1}\eta_2' \end{aligned}$$

and similarly

$$(\mathsf{b}+u\mathsf{B})I_{\mu_2}\eta_1' = (-1)^{|\mu_2|-1+|\mu_1|}I_{\mu_2}L_{\mu_1}\eta_0' + uL_{\mu_2}\eta_1'$$

We now subtract both boundaries with appropriate sign from (2.33) to obtain the following homologous cocycle

$$\begin{aligned} x &\cong (-1)^{|\mu_1|+|\mu_2|-1}s^{d+1}(I_{[\mu_1,\mu_2]}\eta_0' - I_{\mu_1}L_{\mu_2}\eta_0' + (-1)^{|\mu_1||\mu_2|}I_{\mu_2}L_{\mu_1}\eta_0') \\ &= (-1)^{|\mu_1|+|\mu_2|-1}s^{d+1}(I_{[\mu_1,\mu_2]} - I_{\mu_1}L_{\mu_2} + (-1)^{|\mu_1||\mu_2|}I_{\mu_2}L_{\mu_1})\eta_0' \end{aligned} \tag{2.34}$$

By lemma 2.2.15 and (2.32):

$$[L_{\mu_1}, I_{\mu_2}] - (-1)^{|\mu_1|}I_{[\mu_1,\mu_2]} \cong 0$$

Thus

$$\begin{aligned} I_{[\mu_1,\mu_2]} &\cong (-1)^{|\mu_1|}(L_{\mu_1}I_{\mu_2} - (-1)^{|L_{\mu_1}||I_{\mu_2}|}I_{\mu_2}L_{\mu_1}) \\ &= (-1)^{|\mu_1|}(L_{\mu_1}I_{\mu_2} - (-1)^{|\mu_1|(|\mu_2|+1)}I_{\mu_2}L_{\mu_1}) \\ &= (-1)^{|\mu_1|}L_{\mu_1}I_{\mu_2} - (-1)^{|\mu_1||\mu_2|}I_{\mu_2}L_{\mu_1} \end{aligned}$$

Substituting this in (2.34) we obtain

$$x \cong (-1)^{|\mu_1|+|\mu_2|-1} s^{d+1}((-1)^{|\mu_1|} L_{\mu_1} I_{\mu_2} \eta'_0 - I_{\mu_1} L_{\mu_2} \eta'_0)$$

Next we observe, using (2.9)

$$\begin{aligned}
[\mathsf{b} + u\mathsf{B}, I_{\mu_1} I_{\mu_2} - (-1)^{(|\mu_1|+1)(|\mu_2|+1)} I_{\mu_2 \cup \mu_1}] \\
&= [\mathsf{b} + u\mathsf{B}, I_{\mu_1}] I_{\mu_2} + (-1)^{|\mu_1|+1} I_{\mu_1} [\mathsf{b} + u\mathsf{B}, I_{\mu_2}] - (-1)^{(|\mu_1|+1)(|\mu_2|+1)} [\mathsf{b} + u\mathsf{B}, I_{\mu_2 \cup} \\
&= u(L_{\mu_1} I_{\mu_2} + (-1)^{|\mu_1|+1} I_{\mu_1} L_{\mu_2} - (-1)^{(|\mu_1|+1)(|\mu_2|+1)} L_{\mu_2 \cup \mu_1})
\end{aligned} \tag{2.35}$$

and also using (2.3):

$$\begin{aligned}
I_{\mu_1} I_{\mu_2} - (-1)^{(|\mu_1|+1)(|\mu_2|+1)} I_{\mu_2 \cup \mu_1} \bmod u &= i_{\mu_1} i_{\mu_2} - (-1)^{(|\mu_1|+1)(|\mu_2|+1)} i_{\mu_2 \cup \mu_1} \\
&= 0
\end{aligned}$$

In other words $I_{\mu_1} I_{\mu_2} - (-1)^{(|\mu_1|+1)(|\mu_2|+1)} I_{\mu_2 \cup \mu_1}$ is divisible by u and we obtain from (2.35)

$$L_{\mu_1} I_{\mu_2} + (-1)^{|\mu_1|+1} I_{\mu_1} L_{\mu_2} \cong (-1)^{(|\mu_1|+1)(|\mu_2|+1)} L_{\mu_2 \cup \mu_1}$$

Substituting this back in (2.35) we find

$$\begin{aligned}
x &\cong (-1)^{|\mu_1|(|\mu_2|+1)} s^{d+1} L_{\mu_2 \cup \mu_1} \eta'_0 \\
&= (-1)^{|\mu_1|(|\mu_2|+1)} s^{d+1} \mathsf{B} i_{\mu_2 \cup \mu_1} \pi(\eta'_0) && \text{(by lemma 2.7.3)} \\
&\cong (-1)^{|\mu_2|+1} s^{d+1} \mathsf{B} i_{\mu_1 \cup \mu_2} \pi(\eta'_0) && \text{(by §2.2.1)} \\
&\cong (-1)^{|\mu_2|+1} (-1)^{(|\mu_1|+|\mu_2|+1)(d+1)} \mathsf{B} i_{\mu_1 \cup \mu_2} \pi(\eta_0)
\end{aligned}$$

and hence by (2.30)

$$\bar{x} = (-1)^{|\mu_2|+1} (-1)^{(|\mu_1|+|\mu_2|+1)(d+1)} \mathsf{B}(i_{\mu_1} \pi(\bar{\eta}_0) \cdot i_{\mu_2} \pi(\bar{\eta}_0))$$

To compute the righthand side of (2.31) we note

$$\begin{aligned}
\pi(s^{d-1} \Psi(\mu_i, \eta'_i)) &= \pi(s^{d+1}((-1)^{|\mu_i|-1} I_{\mu_i} \eta'_0 + u\eta'_i)) && \text{(by (2.29))} \\
&= (-1)^{|\mu_i|-1} (-1)^{(|\mu_i|+1)(d+1)} i_{\mu_i} \pi(\eta_0)
\end{aligned}$$

so that

$$\begin{aligned}[\overline{s^{d-1}\Psi(\mu_1,\eta_1')}, \overline{s^{d-1}\Psi(\mu_2,\eta_2')}] &= (-1)^{|\mu_1|+d}\mathsf{B}(\overline{s^{d-1}\Psi(\mu_1,\eta_1')} \cdot \overline{s^{d-1}\Psi(\mu_1,\eta_1')}) \\ &= (-1)^{|\mu_1|+d+|\mu_1|+|\mu_2|}(-1)^{(|\mu_1|+|\mu_2|)(d+1)}\mathsf{B}(\overline{i_{\mu_1}\pi(\eta_0)} \cdot \overline{i_{\mu_2}\pi(\eta_0)}) \\ &= (-1)^{|\mu_2|+1}(-1)^{(|\mu_1|+|\mu_2|+1)(d+1)}\mathsf{B}(i_{\mu_1}\pi(\bar{\eta}_0) \cdot i_{\mu_2}\pi(\bar{\eta}_0))\end{aligned}$$

finishing the proof. □

2.8 Obstructions

Recall from §2.5 that there is a natural obstruction theory $O(\mathfrak{g}^\bullet)$ for deformation functors $\mathcal{MC}(\mathfrak{g}^\bullet)$ of Maurer-Cartan type. The obstruction space is given as the linear span in $H^2(\mathfrak{g}^\bullet)$ of all cohomology classes $o(x) = d\hat{x} + \frac{1}{2}[\hat{x}, \hat{x}]$ of lifts $\hat{x} \in \mathfrak{g}^1 \otimes \mathfrak{n}$ of $x \in \mathrm{MC}(\mathfrak{g}^\bullet \otimes \mathfrak{m})$ for all morphisms $(S, \mathfrak{n}) \to (R, \mathfrak{m})$ with one-dimensional kernel. Clearly $o(x)$ and hence $O(\mathfrak{g}^\bullet)$ is functorial under DG-Lie algebra morphisms.

The *periodic cyclic complex* $\mathrm{CC}^{\mathrm{per}}_\bullet(A)$ of a $\Bbbk$-algebra A is obtained by inverting the element u in $\mathrm{CC}^-_\bullet(A)$. Its homology will be denoted by $\mathrm{HC}^{\mathrm{per}}_\bullet(A)$. The following is the main result of this section.

Theorem 2.8.1. *Let $(A, \bar{\eta})$ be a d-Calabi-Yau algebra. Then the composition*

$$\mathcal{O}(\mathfrak{D}^\bullet(A,\eta)) \hookrightarrow H^2(\mathfrak{D}^\bullet(A,\eta)) \overset{\text{Thm 2.7.1}}{\cong} \mathrm{HC}^-_{d-3}(A) \to \mathrm{HC}^{per}_{d-3}(A)$$

is zero.

The main ingredient in the proof is a result by Tsygan and Daletskii (see [DT99]) which extends the Lie derivative action of $\mathfrak{C}^\bullet(A)$ on $\overline{CC}^-_\bullet(A)$ to an L_∞ action of a complex $(\mathfrak{C}^\bullet(A)[u,\epsilon])$ on $\overline{CC}^-_\bullet(A)$. For the benefit of the reader, we collect all the required notions on the language of L_∞-algebras, -modules and -morphisms in the section below:

2.8.1 a Reminder on L_∞-Algebras

Let $\mathfrak{g}^\bullet$ be a graded $\mathbb{k}$-vector space. An L_∞-structure on $\mathfrak{g}^\bullet$ is most elegantly defined as a square zero, degree one coderivation Q on the symmetric coalgebra $S^c(\Sigma\mathfrak{g}^\bullet)$. Such an L_∞-structure is determined by its Taylor coefficients $\partial^n Q$ which are maps

$$S^n(\Sigma\mathfrak{g}^\bullet) \longrightarrow \Sigma\mathfrak{g}^\bullet$$

Convention 2.8.2. *Here and in related situations below we always assume that zeroth order Taylor coefficient are zero.*

A DG-Lie algebra can be made into an L_∞-algebra by putting

$$\partial^1 Q(sg) = -sdg,\ \partial^2 Q(sg, sh) = (-1)^{|g|} s[g,h],\ \text{and } \partial^n Q = 0 \text{ for } n \geq 3$$

A morphism of L_∞-algebras $\psi : (\mathfrak{g}^\bullet, Q) \to (\mathfrak{h}^\bullet, Q)$ is a coalgebra morphism $\psi : S^c(\Sigma\mathfrak{g}^\bullet) \longrightarrow S^c(\Sigma\mathfrak{h}^\bullet)$ commuting with Q. It is in turn also determined by its Taylor coefficients

$$\partial^n\psi : S^n(\Sigma\mathfrak{g}^\bullet) \longrightarrow \Sigma\mathfrak{h}^\bullet$$

If $V^\bullet$ is a graded $\mathbb{k}$-vector space then an L_∞-$\mathfrak{g}^\bullet$-module structure on $V^\bullet$ is a square zero, degree one differential $R : S^c(\Sigma\mathfrak{g}^\bullet)\otimes V^\bullet \to S^c(\Sigma\mathfrak{g}^\bullet)\otimes V^\bullet$ satisfying

$$(Q \otimes \mathrm{Id}_{S^c\mathfrak{g}} \otimes \mathrm{Id}_V + \mathrm{Id}_{S^c\mathfrak{g}} \otimes R) \circ (\Delta \otimes \mathrm{Id}_V) = (\Delta \otimes \mathrm{Id}_V) \circ R$$

as morphisms $S^c(\Sigma\mathfrak{g}^\bullet) \otimes V^\bullet \longrightarrow S^c(\Sigma\mathfrak{g}^\bullet) \otimes S^c(\Sigma\mathfrak{g}^\bullet) \otimes V^\bullet$ An L_∞-$\mathfrak{g}^\bullet$-module structure R on $V^\bullet$ is entirely determined by the maps

$$\partial^{n+1} R : S^n(\Sigma\mathfrak{g}^\bullet) \otimes V^\bullet \longrightarrow V^\bullet$$

If $\mathfrak{g}^\bullet$ is a DG-Lie algebra and $V^\bullet$ is a DG-module over it then $V^\bullet$ can be made into an L_∞-module over $\mathfrak{g}$ by putting

$$\partial^1 R(v) = dv,\ \partial^2 R(sg, v) = g \cdot v \text{ and } \partial^n R = 0 \text{ for } n \geq 3$$

Assume that $(V^\bullet, R)$, $(W^\bullet, R)$ are L_∞-$\mathfrak{h}^\bullet$-modules. An L_∞ morphism $\mu : V^\bullet \longrightarrow W^\bullet$ is a comodule map $\mu : S^c(\Sigma\mathfrak{g}) \otimes V^\bullet \longrightarrow S^c(\Sigma\mathfrak{g}) \otimes W^\bullet$. commuting with R. It is determined by its Taylor coefficients

$$\partial^n \mu : S^n(\Sigma \mathfrak{g}^\bullet) \otimes V^\bullet \longrightarrow W^\bullet$$

The DG Lie algebra $\mathfrak{D}^\bullet(A, \eta)$ is constructed by subsequently applying a sequence of operations: shifting a DG-Lie module (2.2.17), forming the semi-direct product between a DG-Lie algebra and module (2.22), twisting a DG Lie algebra using a Maurer-Cartan element (2.20). Moreover, to extend Kontsevich's L_∞-morphism of DG Lie algebras to an L_∞-morphism of DG-Lie modules we shall also need to pull such modules back. We shall give a short account of these operations in the L_∞-setting

- if $V^\bullet$ is an L_∞-$\mathfrak{g}^\bullet$-module then so are $\Sigma^m V^\bullet$ for all m using the obvious sign convention

  $$\partial^{n+1} R(sg_1, \ldots, sg_n, s^m v) = (-1)^{m(n+|g_1|+\cdots+|g_n|)} \partial^{n+1} R(sg_1, \ldots, sg_n, v)$$

- The L_∞-structures on $\mathfrak{g}^\bullet$ and $\Sigma V^\bullet$can be combined to make the direct sum $\mathfrak{g}^\bullet \oplus V^\bullet$ into an L_∞-algebra. We will denote the resulting L_∞-algebra by $\mathfrak{g}^\bullet \ltimes V^\bullet$ and call it the *semi-direct product* of $\mathfrak{g}^\bullet$. This is an obvious generalization of the semi-direct product of a DG-Lie algebra with a DG-module which was used in §2.6.

- Given an L_∞-morphism $\psi : \mathfrak{g}^\bullet \longrightarrow \mathfrak{h}^\bullet$ and an $L\infty$-module $V^\bullet$ over $\mathfrak{h}$, the pullback $V^\bullet_\psi$ of $V^\bullet$ is defined as follows:

  $$\begin{aligned}\partial^{n+1} R_\psi(sg_1, \ldots, sg_n, v) = \\ \sum_{t, 1 \leq m_1 < \cdots < m_{t-1} < n} & \pm \partial^{t+1} R(\partial^{m_1}\psi(sg_{i_1}, \ldots, sg_{i_{m_1}}), \partial^{m_2 - m_1}\psi(sg_{i_{m_1+1}}, \ldots, sg_{i_{m_2}}), \\ & \ldots, \partial^{n-m_{t-1}}\psi(sg_{i_{m_{t-1}+1}}, \ldots, sg_n), v)\end{aligned}$$

  where for all j: $i_{m_j+1} < \cdots < i_{m_{j+1}}$ and the sign is the Koszul sign of the corresponding shuffle of the $(sg_i)_i$. By construction we have a canonical L_∞-morphism

  $$\psi_V : \mathfrak{g}^\bullet \ltimes V^\bullet_\psi \longrightarrow \mathfrak{h}^\bullet \ltimes V^\bullet$$

which restricted to $S^n(\Sigma\mathfrak{g})$ coincides with $\partial^n\psi$.

- If $\mathfrak{g}^\bullet$ is equipped with some type of topology, we can consider the set of Maurer-Cartan element $\omega \in \mathfrak{g}^1$ satisfying the L_∞-Maurer-Cartan equation

$$\sum_{i\geq 1} \frac{1}{i!}(\partial^i Q)(\underbrace{\omega \cdots \omega}_{i}) = 0$$

Remark 2.8.3. *One has to assume that one is in a situation where all occurring series are convergent and standard series manipulations are allowed. In our application below the series are in fact finite.*

We can twist the L_∞-structure on $\mathfrak{g}^\bullet$ by defining ${\mathfrak{g}^\bullet}_\omega$ as the same graded vector space with

$$(\partial^i Q_\omega)(\gamma) = \sum_{j\geq 0} \frac{1}{j!}(\partial^{i+j} Q)(\underbrace{\omega \cdots \omega}_{j}\gamma) \qquad (\text{for } i > 0) by [\text{Yek06}] \quad (2.36)$$

- We can transport the element ω along ψ by defining a morphism ψ_ω and the resulting $\omega' \in \mathfrak{h}^1$ as

$$(\partial^i \psi_\omega)(\gamma) = \sum_{j\geq 0} \frac{1}{j!}(\partial^{i+j}\psi)(\underbrace{\omega \cdots \omega}_{j}\gamma) \qquad (\text{for } i > 0) \qquad (2.37)$$

$$\omega' = \sum_{j\geq 1} \frac{1}{j!}(\partial^j \psi)(\underbrace{\omega \cdots \omega}_{j}) \qquad (2.38)$$

Then [Yek06, 3.19-3.20] shows that ω' again a solution of the Maurer-Cartan equation on $\mathfrak{h}^\bullet$ and that ψ_ω is an L_∞-map $\mathfrak{g}^\bullet_\omega \longrightarrow \mathfrak{h}^\bullet_{\omega'}$.

2.8.2 the Proof of the Obstruction Theorem

The proof is an application of the following beautiful result by Tsygan and Daletskii [Tsy99, Thm 1] (see also [DT99]).

Theorem 2.8.4. *The DG Lie action of $\mathfrak{C}^\bullet(A)$ on $\mathrm{CC}^-_\bullet(A)$ can be extended to a u-linear L_∞-action of the DG-Lie algebra $(\mathfrak{C}^\bullet(A)[u,\epsilon], d + u\partial/\partial\epsilon)$, with*

$|\epsilon| = 1$, $\epsilon^2 = 0$ *and such that*

$$\begin{aligned} \partial^1 R(\gamma) &= d\gamma \\ \partial^2 R(s\sigma, \gamma) &= L_\sigma \gamma \\ \partial^2 R(s(\epsilon\sigma), \gamma) &= I_\sigma \gamma \end{aligned}$$

for $\sigma \in \mathfrak{C}^\bullet(A)$, $\gamma \in \mathrm{CC}^-_\bullet(A)$ *using the notations of* §*2.2 and the definition of* I_σ *as in in 2.28*

Remark 2.8.5. *The claim about* $\partial^2 R(s(\epsilon\sigma), \gamma)$ *isn't explicitly mentioned in the statement of[Tsy99, Thm 1]. It does however easily follow from the proof.*

In the rest of this section (A, η) is a d-Calabi-Yau algebra.

Lemma 2.8.6. *There is a commutative diagram of complexes*

$$\begin{tikzcd}[column sep=small] (\mathfrak{C}^\bullet(A) \ltimes \Sigma^{-d-1}\mathrm{CC}^-_\bullet(A))_{(0,\eta')} \arrow[rr] \arrow[dr, "\Psi"'] & & (\mathfrak{C}^\bullet(A)[u,\epsilon] \ltimes \Sigma^{-d-1}\mathrm{CC}^-_\bullet(A))_{(0,\eta')} \arrow[dl, "\Psi'"] \\ & \Sigma^{-d+1}\mathrm{CC}^-_\bullet(A)) & \end{tikzcd} \tag{2.39}$$

where

- Ψ *was introduced in theorem 2.7.1;*
- $\eta = s^{-d-1}\eta'$*;*
- *the horizontal map is a twist (see* §*2.8.1) of the map obtained from the obvious inclusion of DG-Lie algebras* $(\mathfrak{C}^\bullet(A), d) \hookrightarrow (\mathfrak{C}^\bullet(A)[u,\epsilon], d + \partial/\partial\epsilon)$.
- Ψ' *restricted to* $\mathfrak{C}^\bullet(A)[u,\epsilon]$ *is* u*-linear and satisfies*

$$\begin{aligned} \Psi'(\sigma) &= (-1)^{|\sigma|+1} I_\sigma \eta' \\ \Psi'(\epsilon\sigma) &= 0 \end{aligned} \tag{2.40}$$

for $\sigma \in \mathfrak{C}^\bullet(A)$.

- Ψ' *restricted to* $\Sigma^{-d-1}\,\mathrm{CC}^{-}_{\bullet}(A)$ *is multiplication by* u*.*

Proof. The commutativity of the diagram is clear. We only have to show that Ψ' commutes with the differential. For Ψ' restricted to $\Sigma^{-d-1}\,\mathrm{CC}^{-}_{\bullet}(A)$ this is obvious. As far as the restriction of Ψ' to $\mathfrak{C}^{\bullet}(A)[u,\epsilon]$ is concerned: the only non-trivial case (given that Ψ already commutes with the differential) is the evaluation on an element of $\epsilon\mathfrak{g}$.

Using (2.36) we find for $\sigma \in \mathfrak{C}^{\bullet}(A)$

$$d_{(0,\eta')}(\epsilon\sigma) = (d(\epsilon\sigma), (-1)^{|g|} I_{\sigma}\eta')$$

Given (2.40) we have to show

$$\Psi'(d_{(0,\eta')}(\epsilon\sigma) = 0$$

We compute

$$\begin{aligned}
\Psi'(d_{(0,\eta')}(\epsilon\sigma)) &= \Psi'(d(\epsilon\sigma), (-1)^{|\sigma|} I_{\sigma}\eta') \\
&= \Psi'(-\epsilon d\sigma + u\sigma, (-1)^{|\sigma|} I_{\sigma}\eta') \\
&= (-1)^{|\sigma|+1} u I_{\sigma}\eta' + (-1)^{|\sigma|} u I_{\sigma}\eta' \\
&= 0 \quad \square
\end{aligned}$$

Lemma 2.8.7. *Consider* $\Sigma^{-d+1}\,\mathrm{CC}^{per}_{\bullet}(A)$ *as an abelian DG-Lie algebra. Then there exists an* L_{∞}*-morphism*

$$\Delta : \mathfrak{D}^{\bullet}(A,\eta) \to \Sigma^{-d+1}\,\mathrm{CC}^{per}_{\bullet}(A)$$

such that the following diagram is commutative

$$\begin{tikzcd}
H^{\bullet}(\mathfrak{D}^{\bullet}(A,\eta)) \arrow[r, "H^{\bullet}(\Psi)"] \arrow[dr, "H^{\bullet}(\Delta)"'] & H^{\bullet}(\Sigma^{-d+1}\,\mathrm{CC}^{-}_{\bullet}(A)) \arrow[d, "canonical"] \\
& H^{\bullet}(\Sigma^{-d+1}\,\mathrm{CC}^{per}_{\bullet}(A))
\end{tikzcd}$$

Proof. To simplify the notations put $\mathfrak{g}^\bullet = \mathfrak{C}^\bullet(A)$, $V^- = \Sigma^{-d-1}\operatorname{CC}^-_\bullet(A)$, $V^{\mathrm{per}} = \Sigma^{-d-1}\operatorname{CC}^-_\bullet(A)$. Thus we get L_∞-morphisms (see §2.8.1)

$$(\mathfrak{g}^\bullet \ltimes V^-)_{(0,\eta')} \to (\mathfrak{g}^\bullet[u,\epsilon] \ltimes V^-)_{(0,\eta')} \to (\mathfrak{g}^\bullet[u,u^{-1},\epsilon] \ltimes V^{\mathrm{per}})_{(0,\eta')} \xleftarrow{c} (0 \ltimes V^{\mathrm{per}})_{(0,\eta')} \cong V^{\mathrm{per}} \overset{\times u}{\cong} \Sigma^2 V^{\mathrm{per}} \tag{2.41}$$

Here c goes in the wrong direction but it is easy to see that $(\mathfrak{g}^\bullet[u,u^{-1},\epsilon], d + u\partial/\partial\epsilon)$ is acyclic. Hence c is an quasi-isomorphism. This means that there is an L_∞-quasi-isomorphism c' which goes in the opposite direction and which inverts c on the level of cohomology. Taking the composition of everything we obtain an L_∞-morphism

$$(\mathfrak{g}^\bullet \ltimes V^-)_{(0,\eta')} \longrightarrow \Sigma^2 V^{\mathrm{per}}$$

which is the desired Δ.

It remains to show that Δ and Ψ are compatible on the level of cohomology. This follows from the following commutative diagram whose upper row is a compressed version of (2.41) and whose lower row we obtain from (2.39).

$$\begin{tikzcd}
(\mathfrak{g}^\bullet \ltimes V^-)_{(0,\eta')} \arrow[r] \arrow[d, "\Psi"'] \arrow[rrr, bend left, "\Delta"] & (\mathfrak{g}^\bullet[u,u^{-1},\epsilon] \ltimes V^{\mathrm{per}})_{(0,\eta')} \arrow[r, shift left, "c'"] \arrow[r, phantom, "\cong"] \arrow[d, "\Psi'"] & V^{\mathrm{per}} \arrow[l, shift left, "c"] \arrow[r, "\times u"'] \arrow[d, equal] & \Sigma^2 V^{\mathrm{per}} \arrow[d, equal] \\
\Sigma^2 V^- \arrow[r] \arrow[rrr, bend right, "\text{canonical}"'] & \Sigma^2 V^{\mathrm{per}} & V^{\mathrm{per}} \arrow[l, "\times u"] \arrow[r, "\times u"] & \Sigma^2 V^{\mathrm{per}}
\end{tikzcd} \quad \square$$

Proof of theorem 2.8.1. The theorem follows from lemma 2.8.7 together with the functoriality of obstruction spaces under L_∞-morphisms and the fact that the obstruction space of an abelian Lie algebra is trivial. $\square$

Corollary 2.8.8. *If the map* $\operatorname{HC}^-_{d-3}(A) \longrightarrow \operatorname{HC}^{per}_{d-3}(A)$ *is injective then the deformation theory of A is unobstructed.*

This corollary applies for example in the case $d \leq 3$ by the following well-known lemma.

Lemma 2.8.9. $\mathrm{HC}_n^-(A) \longrightarrow \mathrm{HC}_n^{per}(A)$ *is an isomorphism for* $n \leq 0$.

Proof. There is an exact sequence

$$\mathrm{HC}_{n-1}(A) \longrightarrow \mathrm{HC}_n^-(A) \longrightarrow \mathrm{HC}_n^{\mathrm{per}}(A) \longrightarrow \mathrm{HC}_{n-2}(A)$$

(e.g. [Lod98, Prop. 5.1.5]) where $\mathrm{HC}_\bullet(A)$ denotes ordinary cyclic homology. The complex computing ordinary cyclic homology is concentrated in homological degrees ≥ 0. Hence $\mathrm{HC}_n(A) = 0$ for $n < 0$. This finishes the proof. □

2.9 the Case of Calabi-Yau Varieties

In this section we will use formality results from [Dol06, Kon03, Sho03, Tsy99, Wil08] so we will assume that the ground field $\Bbbk$ contains $\mathbb{R}$. Let $A = \mathcal{O}(X)$ where X is a smooth affine d-dimensional Calabi-Yau variety over $\Bbbk$. Let $T^{\mathrm{poly},\bullet}(A) = \Gamma(X, \bigwedge^\bullet TX)$ denote the poly-vector fields on X. The Schouten-Nijenhuis bracket defines a Lie algebra structure on $T^{\mathrm{poly},\bullet}(A)$. We will implicitly assume that $T^{\mathrm{poly},\bullet}(A)$ is shifted so that the bracket has degree 0, in which case $T^{\mathrm{poly},\bullet}(A)$ becomes a graded Lie algebra. In 1997 Kontsevich proved the following famous theorem

Theorem 2.9.1 ([Kon03], see also [CFT02, Yek05])**.** *(Kontsevich Formality) There is an L_∞-isomorphism*[7]

$$\mathfrak{U} : (T^{\mathrm{poly},\bullet}(A), 0) \longrightarrow (\mathfrak{C}^\bullet(A), d)$$

We let $\Omega^\bullet(A)$ be the differential forms on X (not shifted) and fix a volume form $\eta \in \Omega^d(A)$. The Hochschild-Kostant-Rosenberg theorem furnishes an isomorphism

$$\mathrm{HKR} : \Omega_d(A) \xrightarrow{\simeq} \mathrm{HH}_d(A)$$

It follows that η defines an element in $\mathrm{HH}_d(A)$ and hence by Proposition 2.3.5, we obtain a cycle in $\mathrm{CC}_d^-(A)$ which we will still write as η by abuse of notation.

[7] The reader will find a brief overview of the language of L_∞ algebras in the next section

By [Gin, ex. 3.2.1], the fact that η is a volume form implies that (A, η) is a Calabi-Yau algebra in the sense of 2.3.7. Let

$$\operatorname{div} : T^{\mathrm{poly},\bullet}(A) \longrightarrow T^{\mathrm{poly},\bullet-1}(A)$$

be the divergence operator corresponding to η characterized via the following identity

$$d_{dR}(\gamma \cap \eta) = \operatorname{div} \gamma \cap \eta$$

where $\cap$ denotes the classical contraction and d_{dR} is the de Rham differential. It immediately follows that $\operatorname{div}^2 = 0$. Moreover div acts as a derivation with respect to the Schouten-Nijenhuis bracket, defining a dg Lie algebra $(T^{\mathrm{poly},\bullet}(A), -\operatorname{div})$. Finally, the Tian-Todorov lemma states that

$$(1)^{|\gamma_1|}[\gamma_1, \gamma_2] = \operatorname{div}(\gamma_1\gamma_2) - \operatorname{div}(\gamma_1)\gamma_2 - (1)^{|\gamma_1|+1}\gamma_1 \operatorname{div} \gamma_2$$

implying that $(T^{\mathrm{poly},\bullet}, -\operatorname{div}, \wedge)$ is a BV algebra (see [Sch06, theorem 2.3] for a complete proof).

Theorem 2.9.2. *There exists an L_∞-quasi-isomorphism*

$$(T^{\mathrm{poly},\bullet}(A)[[u]], -u \operatorname{div}) \cong \mathfrak{D}^\bullet(A, \eta)$$

yielding a commutative square in the homotopy category of DG-Lie algebras which extends Kontsevich's formality L_∞-quasi-isomorphism2.9.1 in the following way:

$$\begin{tikzcd} (T^{\mathrm{poly},\bullet}(A)[[u]], -u \operatorname{div}) \arrow[r, "\sim"] \arrow[d, "u\mapsto 0"'] & \mathfrak{D}^\bullet(A, \eta) \arrow[d, "\phi"] \\ T^{\mathrm{poly},\bullet}(A) \arrow[r, "\mathfrak{U}"'] & \mathfrak{C}^\bullet(A) \end{tikzcd} \tag{2.42}$$

2.9.1 Applying Formality to $\mathfrak{D}(A, \eta)$

Recall that the operation L endows the normalized negative cyclic complex $\overline{CC}_\bullet^-(A)$ with the structure of a DG Lie-module over $\mathfrak{C}^\bullet(A)$. Using Kontsevich's L_∞-morphism $\mathfrak{U}$ (2.9.1) we can pull this structure back (see §2.8.1) and

consider $\overline{CC}^{-}_{\bullet}(A)$ as a DG Lie module -module over $T^{\text{poly},\bullet}(A)$.
In turn it is well known that the complex of differential forms $\Omega^{\bullet}(X)$ in turns becomes an L_∞-module over $T^{\text{poly},\bullet}$ through the classical Lie derivative. The main result of [Wil08] (see also [Sho03, Tsy99]) after adding the formal variable u in degree 2 reads:

Theorem 2.9.3. *There is a quasi-isomorphism of L_∞-modules over $T^{\text{poly},\bullet}(A)$*

$$(\overline{\mathrm{CC}}^{-}_{\bullet}(A), \mathsf{b} + u\mathsf{B}) \longrightarrow (\Omega^{\bullet}(A)[[u]], ud)$$

This yields a roof of L_∞-quasi-morphisms of graded DG-Lie algebras

$$\begin{tikzcd}[column sep=small] & T^{\text{poly},\bullet}(A) \ltimes \Sigma^{-d-1}\overline{\mathrm{CC}}^{-}_{\bullet}(A) \arrow[dl, "\mathfrak{S}"'] \arrow[dr, "\mathfrak{U}"] & \\ T^{\text{poly},\bullet}(A) \ltimes \Sigma^{-d-1}\Omega^{\bullet}(A)[[u]] & & \bar{\mathfrak{C}}^{\bullet}(A) \ltimes \Sigma^{-d-1}\overline{\mathrm{CC}}^{-}_{\bullet}(A) \end{tikzcd} \tag{2.43}$$

We in turn obtain a new roof by twisting with $(0, \eta')$ where $\eta' = s^{-d-1}\eta$ (following §2.8.1).

$$\begin{tikzcd}[column sep=small] & (T^{\text{poly},\bullet}(A) \ltimes \Sigma^{-d-1}\overline{\mathrm{CC}}^{-}_{\bullet}(A))_{(0,\eta')} \arrow[dl, "\mathfrak{S}_{(0,\eta')}"'] \arrow[dr, "\mathfrak{U}_{(0,\eta')}"] & \\ \mathfrak{T}^{\bullet}(A, \eta) & & \mathfrak{D}^{\bullet}(A, \eta) \end{tikzcd} \tag{2.44}$$

where we denote

$$\mathfrak{T}^{\bullet}(A, \eta) = (T^{\text{poly},\bullet}(A) \ltimes \Sigma^{-d-1}\Omega^{\bullet}(A)[[u]])_{(0,\eta')}$$

The complexes here are 2-step filtered. The arrows are quasi-isomorphisms since if we take the associated graded complexes for the 2-step filtrations we find the same arrows as in (2.43). To simplify $\mathfrak{T}^{\bullet}(A, \eta)$, we make use the of the divergence operator as mentioned in the introduction:

$$\text{div} : T^{\bullet,\text{poly}}(A) \longrightarrow T^{\bullet-1,\text{poly}}(A)$$

which satisfies the following identity

$$d(\gamma \cap \eta) = \operatorname{div} \gamma \cap \eta$$

We conclude immediately

$$\operatorname{div}^2 = 0$$

theorem 2.9.2 reduces to showing

Proposition 2.9.4. *There is an L_∞-isomorphism of DG-Lie algebras*

$$\delta : (T^{\mathrm{poly},\bullet}(A)[[u]], -u \operatorname{div}) \longrightarrow \mathfrak{T}^\bullet(A, \eta)$$

The proof of this statement implicitly combines two L_∞-quasi-isomorphism which are rather technical technical in nature:

- there exists an L_∞-isomorphism

$$(T^{\mathrm{poly},\bullet}(A)[[u]], -u \operatorname{div}) \longrightarrow (T^{\mathrm{poly},\bullet}(A) \ltimes \mathfrak{a}, -u \operatorname{div}) \tag{2.45}$$

  where $\mathfrak{a}$ is the abelian graded Lie algebra on the vector space $uT^{\mathrm{poly},\bullet}(A)[[u]]$ with action of $T^{\mathrm{poly},\bullet}(A)$ on $\mathfrak{a}$ given by

$$\gamma \star a = [\gamma, a] + (-1)^{|\gamma|} \operatorname{div} \gamma \cup a$$

- there is an isomorphism of DG Lie algebras

$$\begin{aligned} \delta' : (T^{\mathrm{poly},\bullet}(A) \ltimes \mathfrak{a}, -u \operatorname{div}) &\longrightarrow \mathfrak{T}^\bullet(A, \eta) \\ (\gamma, a) &\mapsto (\gamma, (-1)^{|a|} u^{-1} a \cap \eta') \end{aligned} \tag{2.46}$$

BV-Algebras and the Proof of the L_∞-isomorphism (2.45)

Recall that a DG-BV-algebra is a quadruple $(\mathfrak{g}^\bullet, d, \Delta, \cup)$ where $(\mathfrak{g}^\bullet, d)$ is a complex, $\cup$ is a commutative, associative product of degree[8] 1 on $\mathfrak{g}^\bullet$ compatible with d and Δ is a differential of degree -1.

[8]As always our grading conventions are such that Lie brackets have degree zero.

$(\mathfrak{g}^\bullet, d, [-,-])$ is a DG-Lie algebra when endowed with the bracket $[-,-]$ defined by:

$$[g,h] = (-1)^{|g|+1}(\Delta(g \cup h) - \Delta g \cup h - (-1)^{|g|+1} g \cup \Delta h)$$

In this case $\cup$ and $[-,-]$ are related by the Leibniz rule:

$$[g, h_1 \cup h_2] = [g, h_1] \cup h_2 + (-1)^{|g|(|h_1|+1)} h_1 \cup [g, h_2]$$

It is shown in [KKP08, Ter08] that if $\mathfrak{g}^\bullet$ is a DG-BV-algebra then $(\mathfrak{g}^\bullet((u)), d + u\Delta)$ is homotopy abelian. The same proof goes over without change to the case where for $u\mathfrak{g}^\bullet[[u]], d + u\Delta)$ but not for $(\mathfrak{g}^\bullet[[u]], d + u\Delta)$.
Our aim in this section is to make $(\mathfrak{g}^\bullet[[u]], d + u\Delta)$ as "commutative as possible" (see proposition 2.9.8 below) by making at least its sub-DG-Lie algebra $(u\mathfrak{g}^\bullet[[u]], d+u\Delta)$ abelian. This is not completely straightforward since in order to do this we have to twist the action of $\mathfrak{g}^\bullet$ on $u\mathfrak{g}^\bullet[[u]]$.

The fact that $(\mathfrak{g}^\bullet((u)), d+u\Delta)$ and $(u\mathfrak{g}^\bullet[[u]], d+u\Delta)$ are homotopy abelian is in fact a special case of a general result in [ST08]. For the benefit of the reader we will give a complete proof of this result below. Afterwards we will reuse the proof to treat the DG-BV-algebra $(\mathfrak{g}^\bullet[[u]], d + u\Delta)$. It is convenient to use the following (ad hoc) definition.

Definition 2.9.5. A BV_- algebra is a DG-Lie algebra $\mathfrak{g}^\bullet$ equipped with a commutative, associative product $\cup$ of degree -1, compatible with d, such that

$$[g,h] = (-1)^{|g|+1}(d(g \cup h) - dg \cup h - (-1)^{|g|+1}(g \cup dh) \tag{2.47}$$

and

$$[g, h_1 \cup h_2] = [g, h_1] \cup h_2 + (-1)^{|g|(|h_1|+1)} h_1 \cup [g, h_2] \tag{2.48}$$

Lemma 2.9.6. *[ST08] Let $\mathfrak{g}^\bullet$ be a BV_--algebra and let $\mathfrak{a}^\bullet$ be the same as $\mathfrak{g}^\bullet$ but with the Lie bracket set to zero. Then there is a L_∞-morphism $\psi : \mathfrak{g}^\bullet \longrightarrow \mathfrak{a}^\bullet$ such $\partial^1\psi$ is the identity. In other words $\mathfrak{g}^\bullet$ is homotopy abelian.*

Example 2.9.7. *Let* $(\mathfrak{g}^\bullet, d, \Delta, \cup)$ *be a DG-BV-algebra. Then* $(u\mathfrak{g}^\bullet[[u]], d + u\Delta, [-,-], u^{-1}(- \cup -))$ *is a* BV_-*-algebra and hence by the previous lemma* $(u\mathfrak{g}^\bullet[[u]], d+u\Delta)$ *is homotopy abelian. The same reasoning applies to* $(\mathfrak{g}^\bullet((u)), d+u\Delta)$*.*

Proof of lemma 2.9.6. Put $V^\bullet = \Sigma\mathfrak{g}^\bullet$. The coderivation Q on $S^cV^\bullet$ corresponding to the DG-Lie structure is given by (see §2.8.1

$$\partial^1 Q : V^\bullet \longrightarrow V^\bullet : sg \mapsto -s\,dg$$

$$\partial^2 Q : S^2V^\bullet \longrightarrow V^\bullet : (sg, sh) \mapsto (-1)^{|g|} s[g,h]$$

and all other $\partial^n Q$ are zero.

For simplicity of notation we put

$$sg_1 \cdot sg_2 \cdots sg_n = s(g_1 \cup \cdots \cup g_n)$$

From (2.47)(2.48) we obtain:

$$\begin{aligned}\partial^1 Q(v_1 \cdot v_2 \cdots v_n) &= \sum_i \epsilon_i \partial^1 Q(v_i) v_1 \cdots \hat{v}_i \cdots v_n \\ &+ \sum_{i<j} \epsilon_{i,j} \partial^2 Q(v_i, v_j) v_1 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n\end{aligned} \tag{2.49}$$

where the signs are determined by

$$\begin{aligned}v_1 \cdot v_2 \cdots v_n &= \epsilon_i v_i \cdot v_1 \cdots \hat{v}_i \cdots v_n \\ &= \epsilon_{i,j} v_i \cdot v_j \cdot v_1 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n\end{aligned}$$

Consider $\partial^1 Q$ as a coderivation of $S^cV^\bullet$ and let $\psi : S^cV^\bullet \longrightarrow S^cV^\bullet$ be the coalgebra automorphism determined by

$$\partial^n \psi(v_1, \ldots, v_n) = v_1 \cdot v_2 \cdots v_n$$

Then (2.49) becomes

$$\partial^1 Q \circ \psi = \psi \circ Q$$

which finishes the proof. □

Proposition 2.9.8. *Let $(\mathfrak{g}^\bullet, d, \Delta, \cup)$ be a DG-BV-algebra. Let $\mathfrak{a}^\bullet$ be the graded vector space $u\mathfrak{g}^\bullet[[u]]$. The following operation*

$$h \star a = [h, a] + (-1)^{|h|+1}\Delta(h) \cup a \tag{2.50}$$

for $h \in \mathfrak{g}^\bullet$, $a \in \mathfrak{a}^\bullet$ makes $\mathfrak{a}^\bullet$ into a graded $\mathfrak{g}^\bullet$-representation. Furthermore $d + u\Delta$ defines a derivation on the Lie algebra $\mathfrak{g}^\bullet \ltimes \mathfrak{a}^\bullet$ and finally there is an L_∞-isomorphism

$$\phi : \mathfrak{g}^\bullet[[u]] \longrightarrow (\mathfrak{g}^\bullet \ltimes \mathfrak{a}^\bullet, d + u\Delta)$$

such that $\partial^1\phi$ is the identity.

Proof. In the proof below we identify the underlying vector spaces of $\mathfrak{g}^\bullet[[u]]$ and $\mathfrak{g}^\bullet \ltimes \mathfrak{a}$ in the obvious way. The fact that (2.50) defines indeed a representation as well as compatibility with differentials is an easy direct verification: Now put $V^\bullet = \Sigma\mathfrak{a}^\bullet$, $W^\bullet = \Sigma\mathfrak{g}^\bullet$. Let Q be the coderivation on $S^c(W^\bullet \oplus V^\bullet)$ corresponding to $\mathfrak{g}^\bullet[[u]]$. We observe that $\partial^1 Q|W^\bullet = \partial^1 Q_1 + \partial^2 Q_2$ where $\partial^1 Q_1 = -d$ and $\partial^1 Q_2 = -u\Delta$. Let Q' be the coderivation on $S^c(W^\bullet \oplus V^\bullet)$ corresponding to $(\mathfrak{g}^\bullet \ltimes \mathfrak{a}^\bullet, d + u\Delta)$. We have $\partial^1 Q' = \partial^1 Q$. Furthermore

$$\begin{aligned} \partial^2 Q'(w_1, w_2) &= \partial^2 Q(w_1, w_2) && \text{for } w_1, w_2 \in W^\bullet \\ \partial^2 Q'(v_1, v_2) &= 0 && \text{for } v_1, v_2 \in V^\bullet \end{aligned}$$

and for $h \in \mathfrak{g}^\bullet$, $a \in \mathfrak{a}^\bullet$

$$\begin{aligned} \partial^2 Q'(sh, sa) &= (-1)^{|h|} s(h \star a) \\ &= (-1)^{|h|} s[h, a] - s(\Delta h \cup a) \\ &= \partial^2 Q(sh, sa) + \partial^1 Q_2(sh) \cdot sa \end{aligned}$$

where as above $x \cdot y = u^{-1}(x \cup y)$. In other words

$$\partial^2 Q'(w, v) = \partial^2 Q(w, v) + \partial^1 Q_2(w) \cdot v \qquad \text{for } w \in W^\bullet,\ v \in V^\bullet \tag{2.51}$$

We now construct the desired L_∞-morphism. By definition $\partial^n \psi = \mathrm{Id}$ for $n = 1$. For $n > 1$, $i \geq 1$, $w_1, \ldots, w_i \in W^\bullet$, $v_1, \ldots, v_j \in V^\bullet$ we put

$$\partial^n \psi(w_1, \ldots, w_i, v_1, \ldots, v_j) = 0$$

and

$$\partial^n \psi(v_1, \ldots, v_j) = v_1 \cdot v_2 \cdots v_n$$

We now verify

$$\psi \circ Q = Q' \circ \psi$$

We must evaluate both sides on $S^i W^\bullet \otimes S^j V^\bullet$. If $i = 0$ then the desired equality follows from the proof of lemma 2.9.6. If $i > 2$ then both sides are zero so this case is trivial as well. If $i = 2$ then both sides are zero unless $j = 0$ in which case we reduce to $\partial^2 Q | S^2 W^\bullet = \partial^2 Q' | S^2 W^\bullet$.

We concentrate on the case $i = 1$. We find

$$(Q' \circ \psi)(w_1, v_1, \ldots, v_j) = \partial^2 Q'(w_1, v_1 \cdot v_2 \cdots v_n)$$

and

$$\begin{aligned}(\psi \circ Q)(w_1, v_1, \ldots, v_j) &= \partial^1 Q_2(w_1) \cdot v_1 \cdots v_j + \sum_l \pm \partial^2 Q(w_1, v_l) \cdot v_1 \cdots \hat{v}_l \cdots v_j \\ &= \partial^1 Q_2(w_1) \cdot v_1 \cdots v_j + \partial^2 Q(w_1, v_1 \cdot v_2 \cdots v_j)\end{aligned}$$

We conclude by (2.51). □

The above lemma allows us to yield a proof of 2.45.

Lemma 2.9.9. *there exists an L_∞-isomorphism*

$$(T^{\mathrm{poly},\bullet}(A)[[u]], -u \operatorname{div}) \longrightarrow (T^{\mathrm{poly},\bullet}(A) \ltimes \mathfrak{a}, -u \operatorname{div})$$

where $\mathfrak{a}$ is the abelian graded Lie algebra on the vector space $uT^{\mathrm{poly},\bullet}(A)[[u]]$. The action of $T^{\mathrm{poly},\bullet}(A)$ on $\mathfrak{a}$ is given by

$$\gamma \star a = [\gamma, a] + (-1)^{|\gamma|} \operatorname{div} \gamma \cup a$$

Proof. By [Sch06], we have the following equality

$$(-1)^{|\gamma_1|}[\gamma_1, \gamma_2] = \operatorname{div}(\gamma_1 \gamma_2) - \operatorname{div}(\gamma_1)\gamma_2 - (-1)^{|\gamma_1|+1} \gamma_1 \operatorname{div} \gamma_2$$

implying that $(T^{\mathrm{poly},\bullet}(A), -\operatorname{div}, \cup)$ is a BV-algebra. The result is now an immediate application of 2.9.8 □

The second required isomorphism can be computed directly:

Lemma 2.9.10. *The morphism from 2.46*

$$\delta' : (T^{\mathrm{poly},\bullet}(A) \ltimes \mathfrak{a}, -u\,\mathrm{div}) \longrightarrow \mathfrak{T}^\bullet(A,\eta)$$

is an isomorphism of DG-Lie algebras.

Proof. First we show that

$$\delta' : \mathfrak{a} \longrightarrow \Sigma^{-d-1}\Omega^\bullet(A)[[u]] : a \mapsto (-1)^{|a|}u^{-1}(a \cap \eta')$$

is compatible with the action of $T^{\mathrm{poly},\bullet}(A)$. We compute for $\gamma \in T^{\mathrm{poly},\bullet}(A)$ and $a \in \mathfrak{a}$.

$$\begin{aligned}
\delta'(\gamma \star a) &= \delta'([\gamma, a] + (-1)^{|\gamma|}\,\mathrm{div}\,\gamma \cup a)\\
&= (-1)^{|\gamma|+|a|}u^{-1}([\gamma, a] + (-1)^{|\gamma|}\,\mathrm{div}\,\gamma \cup a) \cap \eta'\\
&= (-1)^{|\gamma|+|a|}u^{-1}((-1)^{|\gamma|}\,\mathrm{div}(\gamma \cup a)\\
&- (-1)^{|\gamma|}\,\mathrm{div}(\gamma) \cup a + \gamma \cup \mathrm{div}(a) + (-1)^{|\gamma|}\,\mathrm{div}(\gamma) \cup a) \cap \eta'\\
&= (-1)^{|\gamma|+|a|}u^{-1}((-1)^{|\gamma|}\,\mathrm{div}(\gamma \cup a) + \gamma \cup \mathrm{div}\,a) \cap \eta'\\
&= (-1)^{|\gamma|+|a|}u^{-1}((-1)^{|\gamma|}d(\gamma \cap (a \cap \eta')) + \gamma \cap d(a \cap \eta'))\\
&= L_\gamma(\delta'(a))
\end{aligned}$$

Now we check compatibility with the differential of δ' on an element $a \in \mathfrak{a}$.

$$\begin{aligned}
\delta'(-u\,\mathrm{div}\,a) &= -(-1)^{|a|+1}\,\mathrm{div}\,a \cap \eta'\\
&= (-1)^{|a|}d(a \cap \eta')\\
&= d(\delta'(a))
\end{aligned}$$

Finally we check compatibility with the differential of δ' on $\gamma \in T^{\mathrm{poly},\bullet}(A)$.

$$\begin{aligned}
\delta'(-u\,\mathrm{div}\,\gamma) &= -(-1)^{|\gamma|+1}\,\mathrm{div}\,\gamma \cap \eta'\\
&= (-1)^{|\gamma|}d(\gamma \cap \eta')\\
&= (-1)^{|\gamma|}L_a\eta'\\
&= [(0,\eta'),(\gamma,0)] \quad \square
\end{aligned}$$

We finally have all the required tools to prove theorem 2.9.2:

Proof of theorem 2.9.2. It suffices to combine diagram (2.44) with proposition 2.9.4, taking into account that an L_∞-quasi-isomorphism yields an isomorphism in the homotopy category of DG-Lie algebras. □

Chapter 3

Numerical Classification of Exceptional Sequences in Rank 4

3.1 Introduction and Statement of Results

We discussed in §0.2 how one can construct a 3-Calabi-Yau algebra starting from an exceptional sequence on a Del Pezzo surface X. From the point of view of noncommutative geometry, it is natural to try and impose conditions on a triangulated category which reflect the geometry of a Del Pezzo surface in order to produce new Calabi-Yau algebras using exceptional sequences in the same way. This chapter can be seen as a first step towards this goal in which we investigate the Grothendieck groups of such categories in the presence of exceptional sequences.

The set of exceptional sequences in a given triangulated category has very rich structure. It was proven in [GK04] that the braid group B_n acts on these sequences through an operation called mutation. In [KO95], Kuleshov and

Orlov investigated the mutation action in the case where $\mathcal{T} = \mathscr{D}^b(X)$. They showed in particular that this category indeed has a full, strong exceptional sequence and that the mutation action is transitive on $\mathscr{D}^b(X)$.

The Grothendieck group K of $\mathcal{T}$ is equipped with an Euler form defined through the formula $\langle X, Y\rangle \stackrel{\text{def}}{=} \sum_i (-1)^i \dim_{\Bbbk} \operatorname{Hom}(X, Y[i])$. The classes in a full exceptional sequence E in $\mathcal{T}$ define an ordered basis $\overline{E}$ of K for which the Gram matrix M is upper triangular with ones on the diagonal. We call such an ordered basis again exceptional and a matrix of this form exceptional as well. There is an obvious way to define a braid group action on exceptional bases of free abelian groups with a unimodular form (which we call a 'lattice') as in §3.4.3 and on exceptional matrices (see §3.4.4) in general in such a way that the maps

$$\mathcal{T} \longrightarrow K(\mathcal{T}) \longrightarrow \mathrm{SL}_n(\mathbb{Z}) : E \mapsto \overline{E} \mapsto M$$

are B_n-equivariant.

We will also consider the closely related notion of helix mutation. In this case the cylindrical braid group CB_n acts on the set of helices. This action extends the action of B_n on exceptional sequences described above in the appropriate sense (see 3.4.9). We can similarly define helices in the general context of lattices, define a cylindrical braid group action on them and show that the action on helices is compatible with the action on exceptional bases in a similar way (see 3.4.14). In the setting of exceptional matrices, one can define two obvious cylindrical braid group actions however. We shall show that the orbits under both actions actually coincide (3.4.23). This will prove to be a useful technical tool when making explicit computations.

In this chapter we propose that the results of Kuleshov-Orlov on the beautiful properties of exceptional sequences in $\mathscr{D}^b(X)$ described above should be a consequence of some extra structure present in this triangulated category. This extra structure comes from the Serre functor S which translates into a unique automorphism s on the Grothendieck group $K(X)$ for which the formula $\langle v, sw\rangle = \langle w, v\rangle$ holds. If we let s act on the numerical Grothendieck group $K(X)_{\mathrm{num}} = K(X)/\operatorname{rad}\langle -, -\rangle$, the fact that X is smooth and projective implies two fundamental properties, the second of which is new to the best of our knowledge:

Theorem E (see 3.3.7)**.** *Let X be a smooth projective surface over $\mathbb{C}$. Then the Serre automorphism s on $K(X)_{num}$ satisfies the following conditions:*

- $(s-1)^3 = 0$
- $\mathrm{rk}(s-1) \leq 2$

We consider more generally the setting of a lattice K (as mentioned, to us this means a finitely generated free abelian group with a unimodular bilinear form) and impose axioms on K based on the above theorem [1]. We refer to this as a lattice of smooth projective surface* ('SPS*') type. There is a canonical 2-step filtration F^iK on K which generalizes the codimension filtration. We call F^2K and F^1K/F^2 respectively the rank group and the numerical Picard group. The form $\langle -,-\rangle$ restricts to a symmetric non-degenerate form $(-,-)$ on the numerical Picard group (after tensoring with $\mathbb{Q}$), which we call the negative intersection form. Using this filtration, it is possible to define many other notions such as rank, degree, canonical class, in this more general context of an SPS* lattice. All these notions coincide with the usual geometric ones in the case where $K = K(X)_{\mathrm{num}}$ is the numerical Grothendieck group of X (see 3.3.26,3.3.11). Moreover in §3.3.1, we exhibit a formula reminiscent of the Riemann-Roch theorem. In view of our wish to reinterpret the Kuleshov-Orlov theory valid for Del Pezzo surfaces in the more general setting of lattices, our focus will lie on lattices of SPS type for which the self-intersection of the canonical class is negative, which we aptly call of Del Pezzo surface ('DPS*') type [2]..

Using these geometric notions in our new more general context allows us to

[1] More precisely, we impose the additional conditions that $\langle,\rangle$ is nondegenerate and that $(s-1)^2 \neq 0$, which are natural in the presence of an exceptional basis

[2] The reader may note the sign difference from the usual definition coming from the fact that our notion of intersection form differs from the classical one by a sign

classify lattices of SPS type with an exceptional basis in rank 3. More precisely, for such a lattice K, following [BP94], we note that the unipotency of the Serre automorphism gives a restraint on the nontrivial coefficients of the Gram matrix in the form of the Markov equation

$$a^2 + b^2 + c^2 = abc$$

Markov's theorem [Mar79] concerning this equation now proves that one can mutate the exceptional basis so that the resulting coefficients are given by the standard solution $(3, 6, 3)$. This yields the result that K is isomorphic to the Grothendieck group of $\mathbb{P}^2$.

We use this result and the technique of its proof to exhibit a classification of lattices of DPS* type with a rank 4 exceptional basis (e_1, e_2, e_3, e_4). The unipotency of the Serre automorphism in this case yields a system of 2 Diophantine equations

$$\begin{cases} a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - bad - edf - ace - bcf + abdf = 0 \\ af + bd = ce \end{cases}$$

whose solutions contrarily to the rank 3 case cannot be described in a straightforward manner analogous to Markov's approach. The key to analyzing this situation is to consider two simpler cases.

- if the element e_1 has an element of rank 0, it is not to difficult to show that (e_2, e_3, e_4) is in turn a sublattice of rank 3 which is again of SPS type. By the previous result, we can perform a mutation so that $(d, e, f) = (3, 6, 3)$
- If the exceptional basis satisfies $\langle e_2, e_3 \rangle = 0$, then the Markov equation in rank 4, together with this extra condition is equivalent to the following generalization of the Markov equation

$$ka^2 + kb^2 + c^2 = kabc$$

Our main reduction result using the tools provided by the different (cylindrical) braid action developed above is that one can always mutate so that either of these additional constraints are satisfied. This allows us to prove the following classification result:

Theorem F. *Let K be a lattice of DPS* type with an exceptional basis of length 4. Then K is isomorphic to one of four lattices for which the Gram matrix has a basis of one of 4 respective nonisomorphic forms*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first type is a trivial extension the Grothendieck group of $\mathbb{P}^2$ to a rank 4 lattice. The next two matrices correspond to the Grothendieck groups of the surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ respectively. The last type however doesn't correspond to the Grothendieck group of a Del Pezzo surface (see 3.5.18), and we shall dedicate the last chapter of this thesis to the construction of a noncommutative geometric model for this lattice

3.2 Geometric Conditions on Grothendieck Groups

Let X be a smooth projective variety of dimension d and let $S \stackrel{\text{def}}{=} (-) \otimes_X \omega_X[d]$ denote the Serre functor on $\mathscr{D}^b(X)$. Then s induces an automorphism s on the Grothendieck group $K(X)$. The following is well known:

Theorem 3.2.1. *s satisfies $((-s)^d - 1)^{d+1} = 0$ on $K(X)$.*

Proof. see [Dol08, 3.3.8] □

It is a trivial observation that s remains a well-defined automorphism on the numerical Grothendieck group $K(X)_{\text{num}} = K(X)/\operatorname{rad}(\langle -, - \rangle$ If X is a surface, the rank of $(s-1)$ on $K(X)_{\text{num}}$ can be bounded by 2 by using Chern classes. Although the proof of this result is quite elementary, this result seems to be new to the best of our knowledge.

Lemma 3.2.2. $\mathrm{rk}(s-1) \leq 2$ *on* $K(X)_{num}$

Proof. As mentioned above, we shall use the Chern character (we refer for example to [Kar77] for the basic properties we use below):

$$\mathrm{ch} : K(X) \longrightarrow H^{\mathrm{even}}(X, \mathbb{Q})$$

Define the homological Grothendieck group by the quotient

$$K(X)_{\mathrm{hom}} \stackrel{\mathrm{def}}{=} K(X)/\ker(\mathrm{ch}).$$

We first show that the identity induces a well defined morphism $\pi : K(X)_{\mathrm{hom}} \twoheadrightarrow K(X)_{\mathrm{num}}$. to this end, let $\mathcal{E}$ be a vector bundle on X. Since $c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$, the splitting principle implies that $ch(\mathcal{E}^\vee) = \tau(ch(\mathcal{E}))$ where $\tau \in \mathrm{Aut}(H^{\mathrm{even}}(X, \mathbb{Q}))$ is defined through the rule

$$\tau(v_{2i}) \stackrel{\mathrm{def}}{=} (-1)^i (v_{2i})$$

This, together with the Hirzebruch-Riemann-Roch theorem, yields:

$$\begin{aligned}
\langle \mathcal{E}, \mathcal{F} \rangle &= \chi(\mathcal{E}^\vee \otimes \mathcal{F}) \\
&= (ch(\mathcal{E}^\vee \otimes \mathcal{F})\mathrm{Todd}(X))_{\mathrm{top}} \\
&= ((\tau(ch(\mathcal{E}))ch(\mathcal{F})\mathrm{Todd}(X))_{\mathrm{top}}
\end{aligned}$$

where Todd(X) denotes the Todd genus of X. This in particular implies that $\ker(\mathrm{ch}) \subset \mathrm{rad}\,\langle,\rangle$, proving the well-definedness of the morphism π. The surjectivity of π which shows that we may prove that $\mathrm{rk}(s-1) \leq 2$ on $K(X)_{\mathrm{hom}}$ instead. Moreover, since ch is an injective morphism on $K(X))_{\mathrm{hom}}$, the claim finally to reduces to $\mathrm{rk}(\mathrm{ch}(s-1)) \leq 2$ on $\mathrm{H}^{\mathrm{even}}(X, \mathbb{Q})$. We compute:

$$\begin{aligned}
\mathrm{ch}(s-1)\mathcal{E} &= \mathrm{ch}(\mathcal{E} \otimes \omega_X[2]) - \mathrm{ch}(\mathcal{E}) \\
&= \mathrm{ch}(\mathcal{E})ch(\omega_X) - \mathrm{ch}(\mathcal{E}) \\
&= \mathrm{ch}(\mathcal{E})(c_1(\omega_X) + \frac{c_1(\omega)^2}{2}) \\
:&= \mathrm{ch}(s-1)\,\mathrm{ch}(\mathcal{E})
\end{aligned}$$

This action is given by multiplication by an element of degree ≥ 2 and

$$\begin{aligned}\mathrm{ch}(s-1)(H^{\mathrm{even}}(X,\mathbb{Q})) &= \mathrm{ch}(s-1)(\mathrm{H}^0(X,\mathbb{Q})) \oplus \mathrm{ch}(s-1)(\mathrm{H}^2(X,\mathbb{Q}) \oplus \mathrm{H}^4(X,\mathbb{Q}))\\ &\subset \mathrm{ch}(s-1)\big(\mathrm{H}^0(X,\mathbb{Q}) \oplus \mathrm{H}^4(X,\mathbb{Q})\big)\end{aligned}$$

Now $\mathrm{H}^0(X,\mathbb{Q}) = \Gamma(X,\mathbb{Q}) = \mathbb{Q}$ and $\mathrm{H}^4(X,\mathbb{Q}) = \mathbb{Q}$ since X is an orientable complex surface. Hence $\mathrm{rk}\,\mathrm{ch}(s-1) \leq 2$ and the claim is proven. $\square$

Throughout, we shall make use of the following 'folklore' theorem which relates the intersection form on the Picard group to the Euler form on the Grothendieck group

Lemma 3.2.3. *Let X be a smooth complex projective surface*

1. *For smooth curves C and C' intersecting transversely and line bundles $\mathcal{U}$ and $\mathcal{V}$ on C and C' respectively, we have*

$$C \cdot C' = -\langle \mathcal{U}, \mathcal{V}\rangle$$

2. *For line bundles $\mathcal{L}$ and $\mathcal{L}'$ on X, we have*

$$\mathcal{L} \cdot \mathcal{L}' = -\langle \mathcal{L} - \mathcal{O}_X, \mathcal{L}' - \mathcal{O}_X\rangle$$

   where the first product is the standard intersection pairing on the Picard group of a smooth surface.

Proof. We proceed to prove the first claim:

$$\langle \mathcal{U}, \mathcal{V}\rangle = \dim_{\mathbb{C}}\left(\mathrm{Hom}_X(\mathcal{U},\mathcal{V})\right) - \dim_{\mathbb{C}}\left(\mathrm{Ext}^1_X(\mathcal{U},\mathcal{V})\right) + \dim_{\mathbb{C}}\left(\mathrm{Ext}^2_X(\mathcal{U},\mathcal{V})\right)$$

We analyze each term separately. We have

$$\mathrm{Hom}_X(\mathcal{U},\mathcal{V}) = \Gamma\left(X, \mathcal{H}om_X(\mathcal{U},\mathcal{V})\right).$$

Now, at a point $P \in X$, we have $\mathcal{H}om_X(\mathcal{U},\mathcal{V})_P = \mathrm{Hom}_{\mathcal{O}_{X,P}}(\mathcal{U}_P,\mathcal{V}_P)$ since both sheaves are coherent. If $P \notin C \cap C'$ this module is clearly 0. In the other situation, we can argue locally: let $A \overset{\mathrm{def}}{=} \mathcal{O}_{X,P}$ be a regular local ring with maximal ideal $\mathfrak{m}$. Then $\mathcal{U}_P = A/(f)$ and $\mathcal{V}_P = A/(g)$ for some distinct prime ideals (f) and (g) such that $(f, g) = \mathfrak{m}$. It is now an easy fact left to the reader that there are no nonzero A-linear morphisms $A/(f) \longrightarrow A/(g)$. It follows that in this case we also obtain $\mathrm{Hom}_{\mathcal{O}_{X,P}}(\mathcal{U}_P, \mathcal{V}_P) = 0$, implying that $\mathcal{H}om_X(\mathcal{U}, \mathcal{V}) = 0$ and in particular $\mathrm{Hom}_X(\mathcal{U}, \mathcal{V}) = 0$

Next, by Serre duality $\mathrm{Ext}^2_X(\mathcal{U}, \mathcal{V}) = \mathrm{Hom}_X(\mathcal{V}, \mathcal{U} \otimes \omega_X) = 0$ by an application of the previous argument as $\mathcal{U} \otimes \omega_X$ is again a line bundle.

Finally, to compute $\mathrm{Ext}^1_X(\mathcal{U}, \mathcal{V})$, we use the local to global spectral sequence

$$\mathrm{H}^p\left(X, \mathcal{E}xt^q_X(\mathcal{U}, \mathcal{V})\right) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{U}, \mathcal{V}) \tag{3.1}$$

We have already shown that $\mathrm{H}^p\left(X, \mathcal{E}xt^q(\mathcal{U}, \mathcal{V})\right)$ for $q = 0$. Moreover, the $\mathcal{E}xt$-sheaves have no higher cohomology: indeed, we can write $\mathcal{E}xt^q_X(\mathcal{U}, \mathcal{V}) = i_*\mathcal{F}$ where $i : C \cap C' \longrightarrow X$ is the canonical embedding and $\mathcal{F}$ is some quasi-coherent sheaf on $C \cap C'$. The Leray spectral sequence

$$H^p(X, R^q i_*\mathcal{F}) \implies H^{p+q}(C \cap C', \mathcal{F})$$

collapses since i is an affine morphism to yield

$$\mathrm{H}^p(X, \mathcal{E}xt^q_X(\mathcal{U}, \mathcal{V})) = \mathrm{H}^p(X, i_*\mathcal{F}) = \mathrm{H}^p(C \cap C', \mathcal{F})$$

This latter group is clearly zero for $p > 0$ as $C \cap C'$ is a finite set of points. Thus, $\mathrm{H}^p(X, \mathcal{E}xt^q(\mathcal{U}, \mathcal{V}))$ is nonzero only for $q = (1, 2)$ and $p = 0$. This implies that the local-to-global spectral sequence (3.1) degenerates to the equation

$$H^0(X, \mathcal{E}xt^1_X(\mathcal{U}, \mathcal{V})) = \mathrm{Ext}^1_X(\mathcal{U}, \mathcal{V})$$

again we obtain

$$\begin{aligned}\Gamma(X, \mathcal{E}xt^1_X(\mathcal{U}, \mathcal{V})) &= \oplus_{P \in C \cap C'}(\mathcal{E}xt^1_X(\mathcal{U}, \mathcal{V}))_P \\ &= \oplus_{P \in C \cap C'} \mathrm{Ext}^1_{\mathcal{O}_{X,P}}(\mathcal{U}_P, \mathcal{V}_P)\end{aligned}$$

Which is supported on solely on point of intersection. For each point, as above

$$\dim_{\mathcal{O}_{X,P}}(\mathrm{Ext}^1_X(\mathcal{U}_P, \mathcal{V}_P)) = \dim_{\mathbb{C}} \mathrm{Ext}^1_A(A/(f), A/(g)) = 1$$

Hence we obtain that $\dim_{\mathbb{C}}(\mathrm{Ext}^1_X(\mathcal{U}, \mathcal{V}_X)) = C \cdot C'$ and $-\langle \mathcal{U}, \mathcal{V} \rangle = C \cdot C'$ as required.

To show the second claim, we let $\mathcal{L}$ and $\mathcal{L}'$ be line bundles on X this time. By a standard Bertini type argument (see e.g. the proof of [Har97, lemma V.1.3]) we may assume that

$$\mathcal{L} = \mathcal{O}_X(D - E)$$
$$\mathcal{L}' = \mathcal{O}_X(D' - E')$$

where D, E, D', E' are smooth curves intersecting each other pairwise transversally. In $K(X)$, we have

$$\begin{aligned} \mathcal{L} - \mathcal{O}_X &= \mathcal{O}_X(D - E) - \mathcal{O}_X \\ &= \mathcal{O}_X(D) \otimes_X (\mathcal{O}_X(-E) - \mathcal{O}_X(-D)) \\ &= \mathcal{O}_X(D) \otimes_X ((\mathcal{O}_X - \mathcal{O}_X(-D)) - (\mathcal{O}_X - \mathcal{O}_X(-E))) \\ &= \mathcal{O}_D(D) - \mathcal{O}_E(D). \end{aligned}$$

and similarly

$$\mathcal{L}' - \mathcal{O}_X = \mathcal{O}_{D'}(D') - \mathcal{O}_{E'}(D').$$

Hence

$$\begin{aligned} \langle \mathcal{L} - \mathcal{O}_X, \mathcal{L}' - \mathcal{O}_X \rangle = \langle \mathcal{O}_D(D), \mathcal{O}_{D'}(D') \rangle - \langle \mathcal{O}_D(D), \mathcal{O}_{E'}(D') \rangle \\ - \langle \mathcal{O}_E(D), \mathcal{O}_{D'}(D') \rangle + \langle \mathcal{O}_E(D), \mathcal{O}_{E'}(D') \rangle. \end{aligned} \tag{3.2}$$

On the other hand, by the definition of the intersection pairing on $\mathrm{Pic}(X)$, we have

$$\mathcal{L} \cdot \mathcal{L}' = \mathcal{O}_X(D - E) \cdot \mathcal{O}_X(D' - E') = D \cdot D' - D \cdot E' - E \cdot D' + E \cdot E' \tag{3.3}$$

The result now follows by comparing 3.2 and 3.3 and applying the first claim.

□

3.3 Lattices of SPS* Type

In this section, we use the conditions inferred in the previous section as inspiration for a set of axioms on a finitely generated free abelian group K with a nondegenerate bilinear form $\langle -, - \rangle$ (henceforth known as a 'lattice'). These axioms allow us to define a *codimension* filtration, which in turn gives rise to *rank* and *degree* functions as well as a *numerical Picard group* endowed with a *negative intersection* form. We conclude the section by showing that all notions coincide with the known ones in the case where K is the numerical Grothendieck group of a smooth projective surface satisfying a certain extra condition [3].

Convention 3.3.1. *We shall assume additionally that the form $\langle -, - \rangle$ is unimodular. This is always trivially satisfied for the numerical Grothendieck groups of surfaces with an exceptional sequence, which is precisely what we aim to generalize.*

The map s used in the previous section can be abstractly described as follows:

Definition 3.3.2. A right Serre automorphism is a map $s \in \mathrm{Aut}(K)$ such that $\langle v, sw \rangle = \langle w, v \rangle, \forall v, w \in K$. A left Serre automorphism is a map $t \in \mathrm{Aut}(K)$ such that $\langle tv, w \rangle = \langle w, v \rangle$

We summarize the basic properties of Serre automorphisms in the following lemma:

Lemma 3.3.3. *We have the following facts:*

- *K has a unique right Serre automorphism s*
- *s is an orthogonal linear map*

[3] We ask the reader to be careful with this statement, as some notions only coincide after tensoring with $\mathbb{Q}$ or up to a sign (see 3.3.18)

- *s is the right Serre automorphism if and only if s^{-1} is the left Serre automorphism*

Proof. Let M be the Gram matrix of the form with respect to some chosen basis with associated coordinate map $co : K \longrightarrow \mathbb{Z}^n$ such that

$$\langle v, w\rangle =^t co(v).M.co(w)$$

Then $S \stackrel{\text{def}}{=} M^{-1} \cdot M^t$ is a matrix with integral coefficients by 3.3.1 which defines an automorphism s on K which clearly satisfies the required property. This s is unique by the nondegeneracy of the form
It is orthogonal as $\langle sv, sw\rangle = \langle w, sv\rangle = \langle v, w\rangle$. Finally $\left\langle s^{-1}v, w\right\rangle = \langle v, sw\rangle = \langle w, v\rangle$, which shows that s^{-1} is a left Serre automorphism, the other implication follows by symmetry

□

The previous section motivates the following definition:

Definition 3.3.4. Let K be a finitely generated free abelian group with unimodular form $\langle -, -\rangle$ and Serre automorphism s. We say that K is of smooth projective surface (SPS) type if

- s is unipotent.
- $\mathrm{rk}(s - 1) \leq 2$.

Throughout, the following condition will be crucial to our various constructions

$$(*) \qquad (s-1)^2 \neq 0$$

We will say that K is of SPS* type if it is of SPS type and satisfies the above condition $(*)$

Remark 3.3.5. *The easiest example of a situation where (*) is not satisfied is a Calabi-Yau surface X. Indeed, In this case the Serre functor is given by shifting by 2, and hence $(s-1) = 0$ on $K(X)$*

Remark 3.3.6. *Note that if K is of SPS* type, we have* $\mathrm{rk}(s-1) = 2$*, as otherwise* $1 = \mathrm{rk}(s-1) = \mathrm{rk}(s-1)^2$*, contradicting the fact that* $(s-1)$ *is nilpotent*

The results of the previous section can be summarized as:

Theorem 3.3.7. *Let X be a smooth projective surface over $\mathbb{C}$, then $K(X)_{num}$ is of SPS type.*

Proof. It is well known that $K(X)_{\mathrm{num}}$ is torsion free and hence free. Moreover, the chern character shows that $K(X)_{\mathrm{hom}} = K(X)/\ker(\mathrm{ch})$ injects into $\mathrm{H}^{\mathrm{even}}(X,\mathbb{C})$ and hence is finitely generated and finally the proof of 3.2.2 shows that $K(X)_{\mathrm{num}}$ is also finitely generated.
Lemma 3.2.1 shows that $(s-1)$ is unipotent on $K(X)_{\mathrm{num}}$ showing the first condition. The second condition is 3.2.2. □

Convention 3.3.8. *We shall sometimes use dimension arguments. To this end, we let $V = K \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote the induced bilinear form and the associated Serre automorphism $s \otimes 1$ again by $\langle -,- \rangle$ and s respectively.*

As we shall be concerned mostly with lattices of rank at most 4, the following lemma, which tells us that the second condition in 3.3.4 is void in those cases, will be very handy.

Lemma 3.3.9. *Assume K has rank ≤ 4. If s is unipotent satisfying (*) then K is of SPS* type.*

Proof. We need to show that $\mathrm{rk}(s-1) \leq 2$. We argue with V instead following 3.3.8.
Since we have

$$\{v,w\} \overset{\mathrm{def}}{=} \langle w,v \rangle - \langle v,w \rangle = \langle v,(s-1)w \rangle$$

it follows immediately that the radical of the antisymmetrisation $\{-,-\}$ of the form $\langle -,- \rangle$ is the space $\ker(s-1)$.
Hence $V/\ker(s-1) \cong \mathrm{im}(s-1)$ is endowed with a nondegenerate antisymmetric

form and must be even dimensional in particular. As $(s-1)$ is nilpotent, it cannot be surjective and hence $\mathrm{rk}(s-1) \neq 4$. Furthermore, since $(s-1)^2 \neq 0$, we have $\mathrm{rk}(s-1) \neq 0$. It follows that $\mathrm{rk}(s-1) = 2$ as required. $\square$

Lemma 3.3.10. *Assume K is of SPS* type. There exists a filtration on K*

$$F^3(K) \overset{def}{=} 0 \subset F^2K \subset F^1K \subset F^0(K) \overset{def}{=} K$$

such that

- $(s-1)F^iK \subset F^{i+1}K$ *if* $i \neq 1$ *and* $(s-1)F^1K \otimes \mathbb{Q} = F^2K \otimes \mathbb{Q}$
- $F^0(K)/F^1K \cong F^2K \cong \mathbb{Z}$

Proof. Let $F^1K \overset{\text{def}}{=} \ker(s-1)^2$ and $F^2K \overset{\text{def}}{=} \mathrm{im}(s-1)^2$.
As $(s-1)$ is nilpotent, $\mathrm{rk}(s-1)^i > \mathrm{rk}(s-1)^j$ whenever $i < j$. Hence the hypothesis $\mathrm{rk}(s-1) = 2$ immediately implies $(s-1)^3 = 0$ and that $\mathrm{rk}\, F^2K = \mathrm{rk}(s-1)^2 = 1$, proving simultaneously that this indeed defines a filtration and that the second condition is satisfied.
We go on to show that $(s-1)$ decreases the filtration.
The three inclusions $(s-1)(K) \subset F^1K$, $(s-1)F^2K \subset 0$ and $F^2K \subset (s-1)F^1K$ are all trivial consequences of $(s-1)^3 = 0$, hence the only thing to prove is that the last inclusion is in fact an equality after tensoring with $\mathbb{Q}$. To this end, following convention 3.3.8, write $K \otimes \mathbb{Q} = V$ and let $n = \dim_{\mathbb{Q}}(V)$. We must show that

$$\dim_{\mathbb{Q}}((s-1)F^1(V)) = \dim_{\mathbb{Q}}(F^2(V)) = 1$$

The properties deduced above imply that have a well defined $\mathbb{Q}$-linear surjective morphism

$$(s-1)|_{F^1(V)} : F^1(V)/\ker(s-1) \twoheadrightarrow (s-1)F^1(V)$$

This immediately implies

$$\begin{aligned}\dim_{\mathbb{Q}}(F^1(V)/(s-1)) =& \dim_{\mathbb{Q}} \ker(s-1)^2 - \dim_{\mathbb{Q}} \big(\ker(s-1)\big)\\ =& n - \dim_{\mathbb{Q}} \mathrm{rk}(s-1)^2 - (n - \mathrm{rk}(s-1))\\ =& n-1-(n-2) = 1\\ \geq& \dim_{\mathbb{Q}}((s-1)F^1(V))\end{aligned}$$

thus, since $(s-1)F^1(V) \neq 0$ -as this would imply in particular that $(s-1)^2 = 0$- we obtain the required equality.

□

This filtration has the following, more intrinsic characterization, after tensoring over the rationals

Lemma 3.3.11. *Let V be a $\mathbb{Q}$-vector space of dimension n with bilinear form $\langle -, - \rangle$ and Serre automorphism s. Let F'^iV be a filtration*

$$0 \stackrel{def}{=} F'^3V \subset F'^2V \subset F'1V \subset F'^0V = V$$

such that

- $(s-1)F'^iV \subset F'^{i+1}V$
- $\dim_{\mathbb{Q}} F'^2V = 1 = n - \dim_{\mathbb{Q}} F'^1V$

then F coincides with the filtration on V defined above

Proof. Let F denote the filtration constructed above. As the dimensions of $F^i \otimes \mathbb{Q}$ and F'^i coincide, it suffices to show the appropriate inclusions. Since $(s-1)^2F'^1(V) = 0$, we have $F'^1(V) \subset F^1(V)$ and we immediately have $(s-1)^2(V) = F^2(V) \subset F'^2(V)$ in a similar vein. □

Corollary 3.3.12. *Let X be a smooth projective surface such that $K(X)_{\mathrm{num}}$ satisfies (*). The codimension filtration coincides with the filtration on the vector space $K(X)_{\mathrm{num}} \otimes_{\mathbb{Z}} \mathbb{Q}$ defined above*

Proof. It is well known that the codimension filtration of a smooth projective surface satisfies the requirements of 3.3.11. □

The above result justifies the following definition:

Definition 3.3.13. Let K be of SPS* type. The filtration on K constructed in 3.3.10 is called the *codimension filtration*

Convention 3.3.14. *From now on, we shall assume additionally for the rest of this section that K is a lattice of SPS* type for the rest of this section, so that the existence of the codimension filtration is guaranteed*

Definition 3.3.15. We define the numerical Picard group of K as

$$\mathrm{Num}(K) \stackrel{\mathrm{def}}{=} F^1K/F^2K$$

Lemma 3.3.16. *Let K be a lattice of SPS* type. Restriction induces a non-degenerate symmetric form on $\mathrm{Num}_{\mathbb{Q}}(K) \stackrel{def}{=} \mathrm{Num}(K) \otimes \mathbb{Q}$.*

Proof. Since $\mathbb{Q}$ is divisible over $\mathbb{Z}$, it is $\mathbb{Z}$-flat and if we write $V = K \otimes \mathbb{Q}$ following 3.3.8, we may prove the claim for the group $\mathrm{Num}(V)$ instead.
We first show that $F^2(V)$ lies in the (right) radical of the restriction of $\langle -, - \rangle$ on $F^1(V)$. Let $v \in F^1(V)$ and $(s-1)^2(w) \in F^2(V)$. Then

$$\left\langle v, (s-1)^2 w \right\rangle = \left\langle s^{-2}(s-1)^2 v, w \right\rangle = 0$$

since $(s-1)^2 v = 0$. The proof is similar for the left radical, showing that the form is indeed well defined on $\mathrm{Num}(V)$.
Conversely, let $v \in F^1(V)$ such that $\langle v, - \rangle = 0$ on $F^1(V)$, then in particular we have $\langle v, (s-1)w \rangle = 0$ for all $w \in K$. Since

$$\left\langle (s^{-1}-1)v, w \right\rangle = \langle v, (s-1)w \rangle$$

we have $(s^{-1}-1)v = 0$ and hence $(s-1)(v) = 0$. It follows that $v \in \ker(s-1) = (s-1)\ker(s-1)^2 = (s-1)F^1(V)$ as both have the same dimension by 3.3.10. Since $(s-1)F^1V = F^2V$ by the first condition of 3.3.10, the form is indeed nondegenerate and symmetric on $\mathrm{Num}(V)$. □

Definition 3.3.17. The restriction of $\langle -,- \rangle$ to F^1K is called the negative intersection form and denoted by $(-,-)$. By the above lemma, it induces a well defined nondegenerate symmetric form on $\mathrm{Num}_{\mathbb{Q}}(K)$ which we also denote by $(-,-)$

The name negative intersection form is justified by the following:

Lemma 3.3.18. *Let X be a smooth projective surface over $\mathbb{C}$. There is an isomorphism*

$$\Phi : \mathrm{Num}_{\mathbb{Q}}(X) \longrightarrow \mathrm{Num}\left(K(X)_{\mathrm{num}} \otimes \mathbb{Q}\right)$$

such that $\Phi(V.W) = -\Phi(V).\Phi(W)$

Proof. Let $F^i_{\mathrm{cod}}K(X)$ denote the classical codimension filtration on the Grothendieck group of X.
The morphism

$$\Phi : \mathrm{CH}^1(X) \xrightarrow{\simeq} F^1_{\mathrm{cod}}K(X)/F^2_{\mathrm{cod}}K(X) : C \mapsto \mathcal{O}_C$$

which sends the class of a curve C in the 1st Chow group to the class of its structure sheaf is well known to be an isomorphism. We consider the induced map after tensoring with $\mathbb{Q}$. Since algebraic equivalence is finer than numerical equivalence, we obtain an isomorphism

$$\mathrm{Num}_{\mathbb{Q}}(X) \xrightarrow{\simeq} \left(F^1_{\mathrm{cod}}K(X)/F^2_{\mathrm{cod}}K(X))/\Phi\big(\,\mathrm{rad}(- \cdot -)\big)\right) \otimes \mathbb{Q}$$

By 3.3.12, we can drop the subscript cod and moreover, by 3.2.3, the radical of the Euler form $\langle -,- \rangle$ on $K(X)$ coincides with $\Phi(\mathrm{rad}(- \cdot -))$, yielding an isomorphism

$$\Phi : \mathrm{Num}_{\mathbb{Q}}(X) \longrightarrow \mathrm{Num}\left(K(X)_{\mathrm{num}} \otimes \mathbb{Q}\right)$$

The fact that $\Phi(V.W) = -\Phi(V).\Phi(W)$ is again an application of 3.2.3 $\square$

Convention 3.3.19. *We recall that our running assumption is that K denotes a lattice of SPS* type following 3.3.14. Using the fact that $K/F^1K \simeq \mathbb{Z}$ by 3.3.10, we -once and for all- fix an element $e \in K$ such that $\overline{e}$ generates K/F^1K.*

Definition 3.3.20. The element $\omega \overset{\text{def}}{=} (s-1)e$ is the *canonical class* of K. The *degree* of K is $\delta(K) \overset{\text{def}}{=} (\omega, \omega)$

Lemma 3.3.21. *$\delta(K)$ is an integer which is independent of the choice of e.*

Proof. $\delta(K)$ is an integer by construction.
Any other element generating K/F^1K must be of the form $e' \overset{\text{def}}{=} \pm(e+\gamma)$ for some $\eta \in F^1K = \ker(s-1)^2$, if we let $\omega' \overset{\text{def}}{=} (s-1)e'$, then

$$(\omega', \omega') = (\omega, \omega) + (\omega, (s-1)\gamma) + ((s-1)\gamma, \omega) + ((s-1)\gamma, (s-1)\gamma)$$

only the first term in this sum is nonzero since

$$(s-1)\gamma \in (s-1)F^1(K) \subset F^2(K) \subset \operatorname{rad}(-,-)$$

by 3.3.16, proving the claim. $\square$

We can use e to express the bilinear form on $V \overset{\text{def}}{=} K \otimes_{\mathbb{Z}} \mathbb{Q}$ in a new way:

Lemma 3.3.22. *Let $v \in V \setminus F^1V$. There exists unique $r_v \in \mathbb{Q}$ and $\eta_v \in F^1K$ such that $v = r_v(e + \eta_v)$. Moreover, $r_v \in \mathbb{Z}$ if $v \in K$*

Proof. Let $v \in K$. Since $\overline{e}$ generates K/F^1K, we have $v = ne + \gamma$ for unique $n \in \mathbb{Z}$ and $\gamma \in F^1K$. Hence $v = n(e + \frac{1}{n}\gamma)$, and we let $n = r_v$, $\frac{1}{n}\gamma = \eta_v$.
The result follows for $K_{\mathbb{Q}}$ by extension to the rationals. $\square$

Lemma 3.3.23. *The maps $d, r : K \longrightarrow \mathbb{Z}$ defined by*

1. $r(v) = \begin{cases} 0 & \textit{if } v \in F^1K \\ r_v & \textit{otherwise.} \end{cases}$

2. $d(v) = \begin{cases} (v, \omega), & \textit{if } v \in F^1K \\ r_v(\eta_v, \omega) & \textit{otherwise.} \end{cases}$

are linear and independent of the choice of e up to a sign

Proof. Let $v, w \in K$.
$r(-)$ indeed takes values in $\mathbb{Z}$ by (3.3.22). To see that $d(-)$ also does, note that

$$d(v) = \langle v, \omega \rangle - r_v \langle e, \omega \rangle \tag{3.4}$$

which is indeed an integer.
The morphism r is linear as it coincides with the morphism

$$K \longrightarrow K/F^1K \xrightarrow{\simeq} \mathbb{Z}$$

the last map being provided by the choice of generator e after convention 3.3.19.
The morphism d is linear as the equation 3.4 is an expression which is linear in v.

To show that d and r only depend on the sign of e, let e' be another element such that $\overline{e}'$ generates K/F^1K. Then $e' = \pm(e + \gamma)$ with $\gamma \in F^1K$. It follows that $r_v^e = \pm r_v^{e'}$ and $\eta_v^{e'} = \pm(\eta_v^e - \gamma)$. This shows that r is independent up to a sign and d is so since $(\gamma, -) = 0$ on F^1K $\square$

Definition 3.3.24. The above maps r and d are called *rank* and *degree* respectively.

Convention 3.3.25. *Throughout this chapter, we shall freely make use of the notations η_v, $r(v)$ and $d(v)$.*

The definitions of rank and degree agree with the usual ones in the case of Grothendieck groups of smooth projective surfaces satisfying the condition (*):

Lemma 3.3.26. *Let X be a complex smooth projective surface and $K = K(X)_{num}$. Then $\mathcal{O}_X$ is a generator for K/F^1K. Moreover, if we let $e = \mathcal{O}_X$, then for a coherent sheaf $\mathcal{F}$ and a curve C*

1. $r(\mathcal{F}) = \mathrm{rk}(\mathcal{F})$
2. $\eta_{\mathcal{O}_X(C)} = \mathcal{O}_C(C)$

3. $\delta(K) = -(\omega \cdot \omega)$

Proof. It is well known that $\mathcal{O}_X$ is a generator for K/F^1K. We thus write $e = \mathcal{O}_X$ The functions rk and r are both zero on F^1K and satisfy $1 = \mathrm{rk}(\mathcal{O}_X) = r(\mathcal{O}_X)$. It follows that they must coincide.
Next, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0$$

This implies that $\mathcal{O}_C(C) = \mathcal{O}_X(C) - \mathcal{O}_X$ Since $\mathcal{O}_C(C) \in F^1K$ and $r(\mathcal{O}_X(C)) = 1$, we obtain $\eta_{\mathcal{O}(C)} = \mathcal{O}_C(C)$
Finally, in $K(X)$, we have

$$(s-1)e = (s-1)\mathcal{O}_X = \omega_X - \mathcal{O}_X$$

The result now follows immediately from the formula

$$-\langle \omega_X - \mathcal{O}_X, \omega_X - \mathcal{O}_X \rangle = \omega_X \cdot \omega_X$$

which was proven in 3.2.3, (2) □

Convention 3.3.27. *If $K = K(X)_{\mathrm{num}}$ is the numerical Grothendieck group of a smooth projective surface X satisfying (*), we shall tacitly assume that $e = \mathcal{O}_X$ so that ω indeed coincides with the class of the canonical bundle.*

3.3.1 a Riemann-Roch-Type Formula

We conclude this section by using the decomposition 3.3.22 to exhibit a Riemann-Roch-type formula for the bilinear form in terms of the negative intersection form 3.3.17 and the rank and degree function 3.3.24.

Theorem 3.3.28. *Assume v and w have nonzero ranks.*
If v and w satisfy $\langle v, v\rangle = \langle w, w\rangle = 1$, then

$$\langle v, w\rangle = \frac{r(v)r(w)}{2}\left(\frac{1}{r(v)^2} + \frac{1}{r(w)^2} + (\eta_v - \eta_w, \eta_w - \eta_v - \omega)\right)$$

Moreover, if $\langle w, v\rangle = 0$ *then*

$$\langle v, w\rangle = r(v)r(w)\big(\eta_w - \eta_v, \omega\big) = r(v)d(w) - r(w)d(v)$$

Proof. This is a simple computation. First we have $\frac{1}{r(v)^2} = \langle \eta_v + e, \eta_v + e\rangle$ and $\frac{1}{r(w)^2} = \langle \eta_w + e, \eta_w + e\rangle$.
hence

$$\begin{aligned}\langle v, w\rangle =& \langle r(v)(\eta_v + e), r(w)(\eta_w + e)\rangle \\ =& r(v)r(w)\left\langle \eta_v + \eta_w - \eta_w + e, \eta_w + e\right\rangle \\ =& r(v)r(w)\left(\frac{1}{r(w)^2} + \langle \eta_v - \eta_w, \eta_w + e\rangle\right)\end{aligned}$$

and similarly

$$\langle v, w\rangle = r(v)r(w)\left(\frac{1}{r(v)^2} + \langle \eta_v + e, \eta_w - \eta_v\rangle\right)$$

taking the average of both identities, we obtain

$$\begin{aligned}\langle v, w\rangle =& \frac{r(v)r(w)}{2}\left(\frac{1}{r(v)^2} + \frac{1}{r(w)^2} + \langle \eta_v - \eta_w, \eta_w + e\rangle + \langle \eta_v + e, \eta_w - \eta_v\rangle\right) \\ =& \frac{r(v)r(w)}{2}\left(\frac{1}{r(v)^2} + \frac{1}{r(w)^2} + \langle \eta_v - \eta_w, \eta_w + e - s\eta_v - se\rangle\right)\end{aligned}$$

Hence

$$\begin{aligned}\langle v, w\rangle =& \frac{r(v)r(w)}{2}\left(\frac{1}{r(v)^2} + \frac{1}{r(w)^2} + (\eta_v - \eta_w, \eta_w - s\eta_v - \omega)\right) \\ =& \frac{r(v)r(w)}{2}\left(\frac{1}{r(v)^2} + \frac{1}{r(w)^2} + (\eta_v - \eta_w, \eta_w - \eta_v - \omega)\right)\end{aligned}$$

where the last line follows from 3.3.16.
Now assume moreover that $\langle w, v\rangle = 0$. Then by the previous formula

$$\frac{1}{r(v)^2} + \frac{1}{r(w)^2} + (\eta_w - \eta_v, \eta_v - \eta_w - \omega) = 0$$

Hence

$$(\eta_v - \eta_w, \eta_v - \eta_w - \omega) = \frac{1}{r(v)^2} + \frac{1}{r(w)^2}$$

Thus

$$\begin{aligned}
\langle v, w\rangle =& \frac{r(v)r(w)}{2}\Big((\eta_v - \eta_w, \eta_v - \eta_w - \omega) + (\eta_v - \eta_w, \eta_w - \eta_v - \omega)\Big) \\
=& \frac{r(v)r(w)}{2}\left(\eta_v - \eta_w, -2\omega\right) \\
=& r(v)r(w)(\eta_w - \eta_v, \omega) \\
&= r(v)d(w) - r(w)d(v)
\end{aligned}$$

□

3.4 Mutating Exceptional Sequences, - Bases and - Matrices

This section contains very little new material, but as references seem scarce we include this exposition for the purposes of clarity.

3.4.1 Braids, Cylindrical and Signed

We recall the following definitions of (signed) (cylindrical) braids: for $n > 1$, the braid group B_n is the free group on $\{(\sigma_i)_{1\ldots n-1}\}$ subject to the relations

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ whenever $|i - j| > 1$
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-1$

The cylindrical braid group is the free group CB_n on symbols $\{(\sigma_{i\in\mathbb{Z}_n}, \rho\}$ subject to the relations

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ whenever $\overline{i-j}$ and $\overline{j-i} \neq 1$
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $\forall i \in \mathbb{Z}_n$

- $\rho\sigma_i\rho^{-1} = \sigma_{i+1}\ \forall i \in \mathbb{Z}_n$

Remark 3.4.1. *The relations imply that CB_n is isomorphic to $B_n \rtimes \langle\rho\rangle$, the braid group by the infinite cyclic group*

There is a signed version of both groups: the signed braid group ΣB_n is generated by $((\epsilon_i)_{1\ldots n}, (\sigma_i)_{1\ldots n-1})$ and subject to the usual braid relations as described above and the additional relations

- $\epsilon_i^2 = 1$ and $\epsilon_i\epsilon_j = \epsilon_j\epsilon_i$
- $\epsilon_i\sigma_j = \sigma_j\epsilon_i$ if $|i-j| \neq 1$
- $\epsilon_i\sigma_i\epsilon_{i+1} = \sigma_i$ for $1 \leq i \leq n-1$

Similarly the group of signed cylindrical braids is the group ΣCB_n with generators $(\epsilon_{k\in\mathbb{Z}_n}, \sigma_{i\in\mathbb{Z}_n}, \rho)$ subject to the usual cylindrical braid relations described above and

- $\epsilon_i^2 = 1$ and $\epsilon_i\epsilon_j = \epsilon_j\epsilon_i$
- $\epsilon_i\sigma_j = \sigma_j\epsilon_i$ if $i-j, j-i \neq 1$
- $\epsilon_i\sigma_i\epsilon_{i+1} = \sigma_i$ otherwise.

Remark 3.4.2. *There is a more geometric interpretation of these groups: the braid group is group of paths between two sets of n collinear points in $\mathbb{R}^3$ up to homotopy and the cylindrical braid group is the same group where the n points are placed on a circle. To obtain the signed version, we place a sign at the start of each strand of a braid and compose by multiplying signs along each strand. In this interpretation, the generators ϵ_k are realized by the trivial braid with a $(-)$-sign above the k-th strand and a (+) sign elsewhere.*

These 4 different types of braids come with an automorphism given by 'flipping the braid over' as follows:

Lemma 3.4.3. *There are involutions*

$$r : B_n \longrightarrow B_n : \sigma_i \mapsto \sigma_{n-i}$$
$$r : CB_n \longrightarrow CB_n : (\sigma_i, \rho) \mapsto (\sigma_{n-i}, \rho^{-1})$$
$$r : \Sigma B_n \longrightarrow \Sigma B_n : (\epsilon_k, \sigma_i) \mapsto (\epsilon_{n-k+1}, \sigma_{n-i})$$
$$r : \Sigma CB_n \longrightarrow \Sigma CB_n : (\epsilon_k, \sigma_i, \rho) \mapsto (\epsilon_{n-k+1}, \sigma_{n-i}, \rho^{-1})$$

Proof. Clearly all maps have order 2 and hence the only nontrivial part of the claim is the welldefinedness. It is readily verified that the relations are compatible in all cases.

□

There are two canonical projections from cylindrical braids to usual braids which are related through conjugation by r. Visually, one can interpret them as 'opening up the circle' and placing them on the line

Lemma 3.4.4. *The maps $\pi_F, \pi_B : CB_n \longrightarrow B_n$ given as*

$$\begin{cases} \pi_F(\sigma_i) = \sigma_i \text{ if } 0 < i < n \\ \pi_F(\sigma_n) = \sigma_1 \dots \sigma_{n-1}\sigma_{n-2}^{-1} \dots \sigma_1^{-1} \\ \pi_F(\rho) = \sigma_1 \dots . \sigma_{n-1} \end{cases}$$

and

$$\begin{cases} \pi_B(\sigma_i) = \sigma_i \text{ if } 0 < i < n \\ \pi_B(\sigma_n) = \sigma_1^{-1} \dots \sigma_{n-2}^{-1}\sigma_{n-1} \dots \sigma_1 \\ \pi_B(\rho) = \sigma_1^{-1} \dots \sigma_{n-1}^{-1} \end{cases}$$

are surjective group morphisms satisfying

$$\pi_B = r \circ \pi_F \circ r^{-1}$$

Proof. The relation $\pi_F = r \circ \pi_B \circ r^{-1}$ is trivially verified. We only need to show that π_F is a group morphism, which reduces to showing that the image of the relations in CB_n are trivial. Only the following three are nontrivial:

- $\pi_F(\sigma_{n-1})\pi_F(\sigma_n)\pi_F(\sigma_{n-1}) = \pi_F(\sigma_{n-1})\pi_F(\sigma_n)\pi_F(\sigma_{n-1})$
- $\pi_F(\sigma_1)\pi_F(\sigma_n)\pi_F(\sigma_1) = \pi_F(\sigma_n)\pi_F(\sigma_1)\pi_F(\sigma_n)$
- $\pi_F(\rho)\pi_F(\sigma_i)(\pi_F(\rho^{-1}) = \pi_F(\sigma_i)$

We leave this as an exercise for the reader. □

There is the obvious signed extension of the theorem:

Lemma 3.4.5. *The maps $\pi_F, \pi_B : \Sigma CB_n \longrightarrow \Sigma B_n$ defined as*

$$\pi_F(\epsilon_k, \sigma_i, \rho) = (\epsilon_k, \pi_F(\sigma_i), \pi_F(\rho))$$

and

$$\pi_F(\epsilon_k, \sigma_i, \rho) = (\epsilon_k, \pi_B(\sigma_i), \pi_B(\rho))$$

are surjective group morphisms satisfying

$$\pi_B = r \circ \pi_F \circ r^{-1}$$

Proof. This is once again a straightforward computation. □

Definition 3.4.6. We call π_F the front and π_B the back projection of $(\Sigma)CB_n$ onto $(\Sigma)B_n$

Remark 3.4.7. *These projections have an intuitive interpretation as follows: if we consider a cylindrical braid between two sets of points lying on a circle, we can make a cut in the circles at the back and unbend them to the front to obtain two sets of n collinear points resulting in a standard braid. This is precisely the back projection. The front projection is the result of a similar process where the cut is made at the front*

3.4.2 Mutating Exceptional Sequences and Helices

We start by recalling the theory of mutations of exceptional sequences as developed in [GK04]. Throughout, $\mathcal{T}$ will denote an Ext-finite triangulated category over $\Bbbk$ with Serre functor S. Recall that exceptional sequences and helices were introduced and discussed in 0.2.1 and 0.2.4 in the introductory chapter

Definition 3.4.8. For an exceptional pair (E,F) of objects in $\mathcal{T}$, we define $L_E F$ as the cone of the morphism[4]

$$\mathrm{Hom}^\bullet_{\mathcal{T}}(E,F)\otimes E \longrightarrow F \longrightarrow L_E F \tag{3.5}$$

and we denote $\sigma(E,F) = (L_E F, E)$.
For an exceptional sequence $\mathbb{E} \overset{\mathrm{def}}{=} (E_1, \dots E_n)$ we define

- for $i \in 1, \dots n-1$, the mutation at i by

$$\sigma_i(\mathbb{E}) \overset{\mathrm{def}}{=} (E_1, \dots, \sigma(E_i, E_{i+1}), \dots, E_n)$$

- for $1 \le k \le n$ the shift at k, by

$$\epsilon_k(E_1, \dots, E_k, \dots E_n) \overset{\mathrm{def}}{=} (E_1, \dots, E_k[1], \dots E_n)$$

Recall from 0.2.5 that after a choice of order d, an exceptional sequence $\mathbb{E}$ gives rise to a helix of order d, $\mathbb{H} = \mathfrak{H}(\mathbb{E})$ and that conversely, each helix $\mathbb{H}$ is defined by its initial thread $\mathbb{E} = \mathfrak{E}(\mathbb{H})$.
We define for $i \in \mathbb{Z}_n$,

- the translation by

$$\rho(\mathbb{H}) = (E_{i-1})_{i\in\mathbb{Z}}$$

- for $i \in \mathbb{Z}_n$, the left mutation at i by

$$\sigma_i(\mathbb{H})) = \rho^{-i+1}\mathfrak{H}(\sigma_1(\mathfrak{E}(\rho^{i-1}(\mathbb{H})))$$

- for $k \in \mathbb{Z}_n$, the shift at k by

$$\epsilon_k(\mathbb{H}) = \mathfrak{H}(\epsilon_k(\mathfrak{E}(\mathbb{H})))$$

[4] Since the pair is exceptional, the cone is unique up to a *unique* isomorphism in this case, see [GK04]

We denote the set of full exceptional sequences of length n by $E_n(\mathcal{T})$ and the set of helices of type (n,d) by $H_{n,d}(\mathcal{T})$. If $n = \operatorname{rk} K(X)$, we drop the index altogether. To state the main results of mutating exceptional sequences and helices in triangulated categories, we invoke the notation $\operatorname{Sym}(X)$ for the set of bijections on a set X. By 0.2.5, there is a bijection

$$\Omega : \operatorname{Sym}(H_{(n,d)}(\mathcal{T})) \longrightarrow \operatorname{Sym}(E_n(\mathcal{T})) : f \mapsto \mathfrak{E} \circ f \circ \mathfrak{H}$$

Theorem 3.4.9. • *The following assignments define group actions on the set of exceptional sequences and helices respectively*

$$\begin{cases} \Sigma B_n \longrightarrow \operatorname{Sym}(E_n(\mathcal{T})) : (\sigma_i, \epsilon_k) \mapsto (\sigma_i(-), \epsilon_k(-)) \\ \Sigma CB_n \longrightarrow \operatorname{Sym}(H_{(n,d)}(\mathcal{T})) : (\sigma_i, \rho, \epsilon_k) \mapsto (\sigma_i(-), \rho(-), \epsilon_k(-)) \end{cases}$$

- *the above rules induces a commutative diagram of groups*

$$\begin{tikzcd} \Sigma CB_n \arrow[r, "\pi_F"] \arrow[d] & \Sigma B_n \arrow[d] \\ S(H_{(n,d)}(\mathcal{T})) \arrow[r, "\Omega"'] & S(E_n(\mathcal{T})) \end{tikzcd}$$

Proof. see [GK04, 2.4.2 and 2.8.6] □

As mentioned in the introduction, mutations and exceptional sequences are particularly well-behaved when $\mathcal{T} = \mathscr{D}^b(X)$ is the bounded derived category of coherent sheaves on a Del Pezzo surface.

theorem (Kuleshov-Orlov)**.** *Let X be a Del Pezzo surface and $n = \operatorname{rk} K(X)$ Then*

- *$\mathscr{D}^b(X)$ has a full strong exceptional sequence*
- *any exceptional sequence of length n is full.*
- *the braid group acts transitively on the set of exceptional sequences in $(\mathscr{D}^b(X))$*

3.4.3 Mutating Exceptional Bases

By considering the Grothendieck groups of triangulated categories, we obtain exceptional bases and induced mutation actions. In the present setting, we consider a free abelian group K with a unimodular bilinear form $\langle -, - \rangle$ and corresponding Serre automorphism s.

Definition 3.4.10. A vector $e \in K$ is exceptional if $\langle e, e \rangle = 1$.
A sequence $(e_1, \dots e_n)$ of vectors in K is exceptional if $\langle e_i, e_j \rangle = 0$ for $j < i$.

Definition 3.4.11. A *helix* in K is a sequence $(e_i)_{i \in \mathbb{Z}}$ such that any thread $(e_i, \dots e_{i+n})$ is an exceptional basis and $e_{k-n} = se_k$.

Remark 3.4.12. *It is easy to see that there is a one-to-one correspondence between helices and exceptional bases by analogy with 0.2.5. To an exceptional basis, we assign a helix $\mathfrak{H}(E)$ through the rule $e_{i-n} = se_i$. Taking the first n vectors $\mathfrak{E}(H)$ of a helix H in turn results in an exceptional basis. We call E the initial thread of H and write $E = \mathfrak{E}(H)$. We say that the helix H is* generated *by E and write $H = \mathfrak{H}(E)$*

.

The definition 3.4.8 can be adapted to the setting of lattices

Definition 3.4.13. If (v, w) is an exceptional couple, the left mutation of (v, w) is defined as

$$\sigma(v, w) \stackrel{\text{def}}{=} (w - \langle v, w \rangle v, v).$$

For an exceptional basis $E \stackrel{\text{def}}{=} (e_1, \dots, e_n)$, we define

- and for $1 \leq i \leq n-1$, the left mutation at i as

$$\sigma_i(E) \stackrel{\text{def}}{=} (e_1, \dots, \sigma(e_i, e_{i+1}), \dots, e_n)$$

- for $1 \leq k \leq n$, the sign change

$$\epsilon_k(e_1, \dots, e_k, \dots e_n) \stackrel{\text{def}}{=} (e_1, \dots, -e_k, \dots e_n)$$

For a helix $H = (e_i)_{i\in\mathbb{Z}}$ generated by E and $i \in \mathbb{Z}$ we define:

- the translation as
$$\rho(H) = (e_{i-1})_{i\in\mathbb{Z}}$$
- the left mutation at i as
$$\sigma_i(H(E)) = \rho^{-i+1}\mathfrak{H}(\sigma_1(\mathfrak{E}(\rho^{i-1}(H)))$$
- the sign change as
$$\epsilon_k(H)(E) = H(\epsilon_k(E))$$

We denote the set of exceptional bases on K by $E(K)$ and the set of helices by $H(K)$. Once again, we have a bijection

$$\Omega : \mathrm{Sym}(H(K)) \longrightarrow \mathrm{Sym}(E(K)) : f \longrightarrow \mathfrak{E} \circ f \circ \mathfrak{H}.$$

We obtain a version of 3.4.9 for exceptional bases and helices on lattices:

Theorem 3.4.14. • *The following assignments define group actions on the sets of exceptional bases and helices*

$$\begin{cases} \Sigma B_n \longrightarrow S(E(K)) : (\sigma_i, \epsilon_j) \mapsto (\sigma_i(-), \epsilon_j(-)) \\ \Sigma CB_n \longrightarrow S(H(K))(\sigma_i, \rho, \epsilon_j) \mapsto ((\sigma_i(-), \rho(-), \epsilon_j(-)) \end{cases}$$

- *We have the following commutative diagram of morphisms of groups*

$$\begin{tikzcd} \Sigma CB_n \arrow[r, "\pi_F"] \arrow[d] & \Sigma B_n \arrow[d] \\ \mathrm{Sym}(H(K)) \arrow[r, "\Omega"'] & \mathrm{Sym}(E(K)) \end{tikzcd}$$

- *the maps* $\begin{cases} E(\mathcal{T}) \longrightarrow E(K(\mathcal{T})) : E \mapsto \overline{E} \\ H(\mathcal{T}) \longrightarrow H(K(\mathcal{T})) : H \mapsto \overline{H} \end{cases}$ *are* ΣB_n*, resp.* ΣCB_n*-equivariant*

Proof. We only discuss the commutativity of the diagram and leave the rest of the statement as an easy exercise for the reader. The claim is clearly trivial for the sign changes ϵ_k and for the elements $\sigma_1, \ldots, \sigma_{n-1}$. To show the claim for ρ, we have to prove that for an exceptional basis E with associated helix $H = \mathfrak{H}(E)$,

$$\pi_F(\rho)(E) = (\rho \circ \Omega^{-1})(E) = \mathfrak{E}(\rho(\mathfrak{H}(E)))$$

Now, for any helix H, generated by a thread $(e_1, \ldots, e_n)$, we have

$$E(\rho.H) = (se_n, \ldots e_{n-1})$$

and $\pi_F(\rho)(e_1, \ldots e_n) = (\sigma_1 \ldots \sigma_{n-1})(e_1, \ldots e_n) = (v, e_1 \ldots e_{n-1})$, for some v. We must show that $v = se_n$.
Now since both lie in $\langle e_1, \ldots e_{n-1}\rangle^{\perp}$, they lie in the same 1-dimensional space. Moreover as both are exceptional, we conclude $v = \pm se_n$. Thus with respect to the exceptional basis E, both maps $\mathfrak{E}(\rho(\mathfrak{H})$ and $\pi_F(\rho)$ have same the matrix perhaps up to a sign in the first column. To show that the signs coincides, it is sufficient to show that they the determinant of both matrices coincides. Let M_{σ_i} be the matrix of the mutation σ_i with respect to the basis $(e_1, \ldots, e_n)$ we see that M_{σ_i} is the identity matrix everywhere except on the 2×2 subspace corresponding to (e_i, e_{i+1}). On that subspace the matrix is given by

$$\begin{pmatrix} -\langle e_i, e_{i+1}\rangle & 1 \\ 1 & 0 \end{pmatrix}$$

which has determinant (-1). Hence $\det(M_{\sigma_i})) = -1$ and $\det(\pi_F(\rho) = (-1)^{n-1}$. Now, we consider the matrix associated to ρ. Since $\rho(e_i) = e_{i-1}$ for $i > 1$, after expanding the determinant using the first column, we see that the only nonzero minor is the last one and $\det(M_\rho) = (-1)^{n+1} a_n$ where $se_n = \sum_1^n a_i e_n$. But we have

$$a_n = \left\langle e_n, \sum a_i e_i \right\rangle = \langle e_n, se_n\rangle = 1$$

hence $\det(\rho) = (-1)^{n+1} = (-1)^{n-1} = \det(\pi_F)$, which finishes the proof of the claim in the case of the element ρ. The final nontrivial case is that of the element σ_n. This is a straightforward computation: we have

$$\sigma_n.\mathfrak{H}(e_1, \ldots e_n) = \mathfrak{H}(se_n, e_2, \ldots e_{n-1}, s^{-1}e_1 - \left\langle e_n, s^{-1}e_1 \right\rangle e_n)$$

and

$$\begin{aligned}
\pi_F(\sigma_n)(e_1, \ldots, e_n) &= \pi_F(\rho).\sigma_{n-1}\pi_F(\rho)^{-1}(e_1, \ldots, e_n) \\
&= \pi_F(\rho).\sigma_{n-1}(e_2, \ldots e_n, s^{-1}e_1) \\
&= \pi_F(\rho)(e_2, \ldots s^{-1}e_1 - \left\langle e_n, s^{-1}e_1 \right\rangle e_n, e_n) \\
&= (se_n, e_2, \ldots e_{n-1}, s^{-1}e_1 - \left\langle e_n, s^{-1}e_1 \right\rangle e_n)
\end{aligned}$$

□

Corollary 3.4.15. *The assignments $\mathfrak{E}$ and $\mathfrak{H}$ defined in 3.4.12 induce bijections on the orbit spaces*

$$\overline{\mathfrak{E}}, \overline{\mathfrak{H}} : H(K)/\Sigma CB_n \longleftrightarrow E(K)/\Sigma B_n$$

Proof. Let $H, H' \in H(K)$ and $\sigma \in CB_n$. If $H = \sigma.H'$, then $\mathfrak{E}(H) = \mathfrak{E}(\sigma(H)) = \pi_F(\sigma)\mathfrak{E}(H')$ by the above theorem, which shows that $\overline{\mathfrak{E}}$ is well defined. It is clearly surjective and moreover if $\mathfrak{E}(H) = \tau.\mathfrak{E}(H')$, then by the surjectivity of π, $\tau = \pi_F(\gamma)$ and $\mathfrak{E}(H) = \pi_F(\gamma)(\mathfrak{E}(H) = \mathfrak{E}(\tau(H))$ and $H = \tau' H'$ by the injectivity of $\mathfrak{E}$. It is clear that $\overline{\mathfrak{H}}$ is the inverse bijections.

□

3.4.4 Mutating Exceptional Matrices

There is a third and final setting in which we shall make use of braid group actions (and its variations described in §3.4), this time on a subgroup of $\mathrm{SL}_n(\mathbb{Z})$. The braid group action on exceptional bases described in the previous section induces an action on the corresponding Gram matrices which we will describe explicitly in this section. The analogue of theorem 3.4.9 will be a little more subtle however as this action extends to a cylindrical braid group action in *two* canonical ways (using the morphisms described in 3.4.4), both of which will be relevant in the sequel. We will prove however that the orbit space of the braid group action and two cylindrical group actions all coincide (3.5.16)

Definition 3.4.16. A matrix M in $\mathrm{SL}_n(\mathbb{Z})$ is exceptional if it is upper triangular with $M_{i,i} = 1$. We denote the set of exceptional matrices by $E(\mathrm{SL}_n(\mathbb{Z}))$

It will be convenient to use the notation $M_{\{i,j\}}$ to denote $M_{i,j}$ if $i \leq j$

Definition 3.4.17. Let M be an exceptional matrix. For $1 \leq n-1$, we define the mutation at i as the exceptional matrix $\sigma_i(M)$ by

$$\sigma_i(M)_{\{k,l\}} = \begin{cases} \quad M_{\{k,l\}} & \text{if } i, i+1 \notin \{k,l\} \\ \quad M_{\{k,i\}} & \text{if } l = i+1 \\ \quad M_{\{k,i+1\}} - M_{\{k,i\}}M_{\{i,i+1\}} & \text{if } l = i \\ - \quad M_{\{i,i+1\}} & \text{if } \{k,l\} = \{i,i+1\} \end{cases}$$

for $1 \leq k \leq n$, similarly, $\epsilon_k(M)$ is defined by

$$\epsilon_k(M)\{i,j\} = \begin{cases} \quad M_{\{i,j\}} & \text{if } k \neq i,j \\ - \quad M_{\{i,j\}} & \text{otherwise} \end{cases}$$

Lemma 3.4.18. *The assignment*

$$\Sigma B_n \longrightarrow \mathrm{Sym}(E(\mathrm{SL}_n(\mathbb{Z}))); (\sigma_i, \epsilon_k) \mapsto (\sigma_i(-), \epsilon_k(-))$$

defines a group action in such a way that the canonical map

$$E(K) \longrightarrow E(\mathrm{SL}_n(\mathbb{Z}))$$

which sends an exceptional basis to its Gram matrix is ΣB_n-equivariant

Proof. Let E be an exceptional basis with Gram matrix M. We leave it to the reader to check that the operation $\sigma_i(M)$ indeed coincides with the Gram matrix of the mutated basis $\sigma_i(E)$. The sign actions coincide for trivial reasons □

As mentioned above, there are two ways to extend this action to a cylindrical braid group action. The first is a consequence of the following trivial observation:

Lemma 3.4.19. *Let G denote the image of the cylindrical group action on helices in K*

$$\Sigma CB_n \longrightarrow \mathrm{Sym}(H(K))$$

defined in 3.4.13. Then assigning to a helix the Gram matrix of its initial thread as in 3.4.12 yields a morphism of groups $G \longrightarrow S\,(E(\mathrm{SL}_n(\mathbb{Z}))$

Proof. It suffices to note that the formulas in 3.4.13 are invariant under orthogonal isomorphism □

The combination of 3.4.18 and 3.4.19 allows us to define an group action on the set of exceptional matrices

$$*_1 : \Sigma CB_n \longrightarrow G \longrightarrow \mathrm{Sym}(E(SL_n(\mathbb{Z})) \tag{3.6}$$

Moreover, as an immediate corollary of 3.4.14, we obtain the following

Theorem 3.4.20. *The group action $*_1$ fits inside the following commutative diagram:*

$$\begin{tikzcd}
\Sigma CB_n \arrow[rr, "\pi_F"] \arrow[dr, "*_1"'] & & \Sigma B_n \arrow[dl] \\
& \mathrm{Sym}(E(\mathrm{SL}_n(\mathbb{Z})) &
\end{tikzcd}$$

Proof. This is a trivial consequence of the construction of $*_1$ □

We construct a second action through the following operations:

Definition 3.4.21. For $M \in E(\mathrm{SL}_n(\mathbb{Z}))$, we define the shift ρ as follows:

$$\begin{cases} \rho(M)_{\{i,j\}} = M_{\{i-1,j-1\}} & i \neq 1 \\ \rho(M)_{i,j} = -M_{j,n} & i = 1 \end{cases}$$

Theorem 3.4.22. *The assignment*

$$*_2 : \Sigma CB_n \longrightarrow \mathrm{Sym}(E(\mathrm{SL}_n(\mathbb{Z}))) : (\sigma_i, \rho, \epsilon_k) \mapsto (\sigma_i(-), \rho(-), \epsilon_k(-))$$

defines a group action. Moreover, there is a commutative diagrams

$$\begin{tikzcd} \Sigma CB_n \arrow[rr, "\pi_B"] \arrow[dr, "*_2"'] & & \Sigma B_n \arrow[dl] \\ & \mathrm{Sym}(E(\mathrm{SL}_n(\mathbb{Z}))) & \end{tikzcd}$$

Proof. We leave it to the reader to show that the map is in fact a group action. The above diagram trivially commutes for the elements σ_i and ϵ_k. For ρ, assume that M is the Gram matrix of an exceptional basis $(e_1, \ldots, e_n)$. We compute

$$\begin{aligned} &\pi_B(\rho)(e_1, \ldots e_n) \\ =&(\sigma_1^{-1} \ldots \sigma_{n-1}^{-1})(e_1, \ldots e_n) \\ =&(e_n, e_1 - \langle e_1, e_n \rangle e_n, \ldots, e_i - \langle e_i, e_n \rangle e_n, \ldots, e_{n-1} - \langle e_{n-1}, e_n \rangle e_n) \\ \stackrel{\text{def}}{=}&(e'_1, \ldots, e'_n) \end{aligned}$$

and this sequence clearly satisfies the rule

$$\begin{cases} \langle e'_i, e'_j \rangle = \langle e_{i-1}, e_{j-1} \rangle & i \neq 1 \\ \langle e_1, , e_j \rangle = - \langle e_j, e_n \rangle & \text{otherwise} \end{cases}$$

It follows that the Gram matrix of the resulting exceptional basis indeed coincides with $\rho(M)$.
Finally for the element σ_n, the claim follows from the fact that σ_n indeed acts as the mutation 'on the couple $(n, 1)$' which coincides with $\pi_B(\rho)^{-1}\sigma_1\pi_B(\rho)$. □

Corollary 3.4.23. *The following three orbit spaces coincide:*

$$\mathrm{SL}_n(\mathbb{Z})/\Sigma B_n = \mathrm{SL}_n(\mathbb{Z})/(\Sigma CB_n, *_1) = \mathrm{SL}_n(\mathbb{Z})/(\Sigma CB_n, *_2).$$

3.5 Classifying Exceptional Bases in Rank ≤ 4

3.5.1 the Rank 3 Case

We now direct our attention to classifying lattices of rank 3 of SPS* type defined in 3.3.4 up to isomorphism. Note that by the rank 3 hypothesis, lemma 3.3.9 shows that this is equivalent to s being unipotent, the second condition of definition 3.3.4 being automatically fulfilled. Most of the results in this section are well known and follow from the 19th-century work of Markov in ([Mar79]). This section serves more as a preparation for the more involved rank 4 case. The archetypical example of a lattice of SPS* type of rank 3 is the Grothendieck group of $X = \mathbb{P}^2$, which has a full exceptional sequence of the form

$$\left(\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\right)$$

The associated exceptional basis for $K(X)$ has Gram matrix

$$\Pi \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.7}$$

Lemma 3.5.1. *Let K be a lattice with a bilinear form $\langle -, - \rangle$ with Gram matrix*

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

in a certain basis. Then K has a Serre automorphism which is unipotent if and only if the Markov equation

$$a^2 + b^2 + c^2 = abc. \tag{3.8}$$

is satisfied.

Proof. the proof of 3.3.3 shows that

$$S = M^{-1}.{}^tM = \begin{bmatrix} 1 & -a & ac-b \\ a & -a^2+1 & a^2c-ab-c \\ b & -ab+c & abc-b^2-c^2+1 \end{bmatrix}$$

From this, it follows that the characteristic polynomial of $s-1$ is

$$P_{s-1}(X) = X^3 + (a^2+b^2+c^2-abc)X^2 + (a^2+b^2+c^2-abc)X$$

Hence $(s-1)$ is nilpotent if and only if all lower terms of $P_{s-1}(X)$ vanish which is clearly equivalent to the Markov equation.

□

Theorem 3.5.2 (Markov)**.** *Let M' be an exceptional matrix with coefficients satisfying the Markov equation (3.8), then M lies in the same orbit as* Π *defined above (5.38) under ΣB_n with the action 3.4.17.*

Proof. This was proven in [Mar79] and is discussed in §3.5.5. Indeed, a combination of lemmas3.5.21 and 3.5.20 in the case where $k=1$ shows that any such M lies in the same orbit under $(CB_n, *_2)$ as the exceptional matrix M' with coefficients $a=\pm 3, b=\pm 6, c=\pm 3$, with an even number of minus signs. The resulting 4 matrices lie in the same orbit under as Π under the sign action. □

Theorem 3.5.3. *Let K be of SPS* type of rank* 3 *with an exceptional basis. Then there is an orthogonal isomorphism $K \cong K(\mathbb{P}^2)$.*

Proof. Let K and K' be two such lattices. Then 3.5.2 shows that there is an orthogonal isomorphism between them. □

3.5.2 Additional Geometric Conditions

In order to prove a similar classification result as theorem3.5.3 in the rank 4 case, we need to impose two additional conditions. The first relates to the signature of the negative intersection form:

Definition 3.5.4. Let K be a lattice of SPS* type. Then we say that K satisfies the "Hodge index condition' (HI for short) if the negative intersection form on $\mathrm{Num}_{\mathbb{Q}}(K)$ described in 3.3.17 is indefinite.

Theorem 3.5.5. *Let X be a smooth projective surface. Then $K(X)_{\mathrm{num}}$ satisfies condition HI.*

Proof. The intersection pairing on $\mathrm{Num}(X)$ has signature $(+1, -1 \ldots, -1)$ by [Har97, rem 1.9.1]. Theorem 3.2.3 now shows that $\mathrm{Num}(K(X)_{\mathrm{num}})$ has signature $(-1, +1, \ldots, +1)$. □

The second condition we shall require is a translation of the Del Pezzo condition to our new more general setting:

Definition 3.5.6. We say that K satisfies condition DP if

$$\langle (s-1)v, (s-1)v \rangle < 0$$

for some v.

Lemma 3.5.7. *The following are equivalent:*

1. *K satisfies the DP condition*
2. *$\langle (s-1)v, (s-1)v \rangle < 0$ whenever $r_v \neq 0$*
3. *$\delta(K) < 0$.*

Proof. Let $r(w) \neq 0$ and write $w = r_w(\eta_w + e)$ with $r_w \neq 0$. Then

$$\begin{aligned} \langle (s-1)w, (s-1)w \rangle &= r_w^2((s-1)(\eta_v + e), (s-1)(\eta_v + e)) \\ &= r_w^2((s-1)e, (s-1)e) \end{aligned}$$

where the last equality comes from the fact that

$$(s-1)\eta_v \in (s-1)F^1K \subset F^2K \subset \mathrm{rad}(-,-)$$

It follows that the sign of $\langle (s-1)w, (s-1)w \rangle$ is independent of the choice of w if $r_w \neq 0$. The claim follows. □

Definition 3.5.8. We say that K is of DPS* type if

- K is of SPS* type and
- K satisfies the conditions HI and DP.

Corollary 3.5.9. *Let X be a Del Pezzo surface. Then $K(X)_{\mathrm{num}}$ is of DPS* type.*

Proof. This is a combination of 3.3.7, 3.5.5, 3.3.26(3) and 3.5.7(3) together with the fact that the degree of a del Pezzo surface is positive. □

3.5.3 a Reduction Argument

The rest of this chapter is devoted proving a similar classification result as 3.5.3 when K has an exceptional basis of rank 4, provided we assume the additional DPS* condition from definition3.5.8. To this end, we fix a lattice of SPS* type K, and we will specialize to the DPS* case later on. Our first step is to reduce to two simpler types of lattices. To this end, we prove two lemmas. The first provides a reduction step in the case where the exceptional basis contains a vector of rank 0, the second shows that we can reduce to the case where one number $\langle e_i, e_{i+1}\rangle$ must be 0 if all basis vectors have nonzero rank.

Lemma 3.5.10 (dimension-reduction)**.** *Let $(e_1, \ldots, e_n)$ be an exceptional basis on K. If $r(e_1) = 0$ then the Serre automorphism on $(e_2, \ldots, e_n)$ is unipotent.*

Proof. If $r(e_1) = 0$, then $(s-1)^2(e_1) = 0$. It is easy to see that $(e_2, \ldots, e_n) = {}^{\perp}(e_1)$. Let

$$\pi : K \longrightarrow (e_2, \ldots, e_n) : v \mapsto v - \langle v, e_1\rangle\, e_1$$

denote the orthogonal projection.
Then $\pi \circ s$ is the Serre automorphism on $(e_2, \ldots, e_n)$, since for $v, w \in {}^{\perp}e_1$, we have

$$\langle v, sw - \langle sw, e_1\rangle\, e_1\rangle = \langle w, sv\rangle = \langle v, w\rangle\,.$$

We must thus show that $(\pi \circ s)$ is unipotent. Indeed, we have

$$(\pi\circ s-1)^2(v) = (s-1)^2(v)-\langle sv, e_1\rangle (s-1)(e_1)-\left\langle (s^2-s)v - \langle sv, e_1\rangle se_1, e_1\right\rangle e_1$$

and the last term in this expression is 0 as

$$\left\langle (s^2-s)v, e_1\right\rangle = \left\langle v, (s^{-2}-s^{-1})e_1\right\rangle = \left\langle v, (s^{-1}-1)e_1\right\rangle = \langle sv, e_1\rangle,$$

where we used $(s-1)^2(e_1) = 0$ as $r(e_1) = 0$ (see 3.3.24) and $\langle se_1, e_1\rangle = \left\langle s^2 e_1, se_1\right\rangle = \langle (2s-1)e_1, e_1\rangle = 1$, hence

$$\left\langle (s^2-s)v - \langle sv, e_1\rangle se_1, e_1\right\rangle e_1 = \langle sv, e_1\rangle e_1 - \langle sv, e_1\rangle \langle se_1, e_1\rangle e_1 = 0$$

Using this, we obtain

$$\begin{aligned}(\pi\circ s-1)^3(v) &= (s-1)^3(v) - \langle sv, e_1\rangle (s-1)^2(e_1)\\ &\quad - \left\langle s(s-1)^2 v - \langle sv, e_1\rangle (s-1)e_1, e_1\right\rangle e_1\\ &= -\left\langle s(s-1)^2 v - \langle sv, e_1\rangle (s-1))e_1, e_1\right\rangle e_1\\ &= -\left\langle (s-1)^2 v - \langle sv, e_1\rangle (s^2-s)e_1, e_1\right\rangle e_1\\ &= -(\langle sv, e_1\rangle - \langle sv, e_1\rangle)e_1\\ &= 0.\end{aligned}$$

□

If all the vectors in an exceptional basis have nonzero rank, we shall need to simplify the situation in a different way. To this end for a collection of vectors $(v_1, \ldots, v_k)$ in K, we define the rank of the sequence by

$$\mathcal{M}(v_1, \ldots, v_n) \stackrel{\text{def}}{=} \sum_1^n |r(v_i)|. \tag{3.9}$$

Lemma 3.5.11. *Let (v, w) be an exceptional couple with strictly positive ranks. Assume the following two conditions are satisfied:*

- $(\eta_w - \eta_v, \eta_w - \eta_v) > 0$

- $h \stackrel{def}{=} \langle v, w\rangle > 0$

Then $\mathcal{M}(\tau(v,w)) < \mathcal{M}(v,w)$ where τ is one of the two braids σ or $\sigma^{-1} \in B_2$.

Proof. We compute

$$\begin{aligned}
0 < (\eta_w - \eta_v, \eta_w - \eta_v) &= ((\eta_w + e) - (\eta_v + e), (\eta_w + e) - (\eta_v + e)) \\
&= \frac{1}{r(v)^2} - (\eta_v + e, \eta_w + e) + \frac{1}{r(w)^2} \\
&\stackrel{\text{def}}{=} \frac{1}{r(v)^2} - \frac{h}{r(v)r(w)} + \frac{1}{r(w)^2} \\
&= \frac{1}{r(v)r(w)}\left(r(v)^2 - hr(v)r(w) + r(w)^2\right)
\end{aligned} \tag{3.10}$$

where by the Riemann-Roch type theorem 3.3.28,

$$h \stackrel{\text{def}}{=} \langle v, w\rangle = r(v)r(w)(\eta_w - \eta_v, \omega) > 0.$$

Consider the quadratic form

$$Q : \mathbb{Q}^2 \longrightarrow \mathbb{Q} : (x, y) \mapsto x^2 - hxy + y^2$$

Let $[-,-]$ denote the associated symmetric bilinear form. That is,

$$[--] : \mathbb{Q}^2 \times \mathbb{Q}^2 : \mathbb{Q} : ((x, y), (a, b)) \longrightarrow ax + yb - \frac{h}{2}(ay + xb)$$

Then the above expression becomes

$$\begin{aligned}
0 < Q(r(v), r(w)) &= [(r(v), r(w)), (r(v), r(w))] \\
&= r(v)[(1, 0), (r(v), r(w))) + r(w)[(0, 1), (r(v), r(w))]
\end{aligned}$$

It follows that one of the two terms on the left hand side is positive. We assume the latter and leave the other case to the reader[5].
Then

$$0 < [(0, 1), (r_v, r_w)] = r(w) - \frac{h}{2}r(v)$$

[5] In the other case, the element $\tau \stackrel{\text{def}}{=} \sigma^{-1}$ is necessary to conclude the theorem

which immediately implies $2r(w) - hr(v) < 0$ or $r(w) - hr(v) < r(w)$. Since we trivially also have $hr(v) - r(w) < r(w)$, we conclude $|hr(v) - r(w)| < |r(w)|$. And thus

$$\mathcal{M}(\sigma(v,w)) = |hr(w) - r(v)| + |r(v)| < |r(w)| + |r(v)| = \mathcal{M}((v,w)).$$

□

The following lemma is crucial to showing that the conditions of one of the two reduction lemmas 3.5.10 or 3.5.11 are always satisfied in the rank 4 case. To simplify notation, we fix an exceptional basis (e_1, e_2, e_3, e_4). We let $e_5 \stackrel{\text{def}}{=} s^{-1}e_1$ and write $r(e_i) = r_i$, $\eta_{e_i} = \eta_i$, etc.

Lemma 3.5.12. *If $r_i > 0, \forall\, 0 \leq i \leq 4$ and $\prod_i \langle e_i, e_{i+1} \rangle \neq 0$, then there exists an index $0 \leq i \leq 4$ such that the conditions enumerated in lemma 3.5.11 are satisfied for the exceptional couple (e_i, e_{i+1}).*

Proof. We write $T_i = \eta_{i+1} - \eta_i$. Note that since $T_i \in F^1(K)$ the bilinear form is symmetric on these T_i's by lemma3.3.16. We start by inferring some conditions:

1. by the decomposition 3.3.22, we have

$$\begin{aligned}(T_i, T_{i+2}) = (T_{i+2}, T_i) = (\eta_{i+3} - \eta_{i+2}, \eta_{i+1} - \eta_i)& \\ =& ((\eta_{i+3} + e) - (\eta_{i+2} + e), (\eta_{i+1} + e) - (\eta_i + e)) \\ =& \frac{\langle e_{i+3}, e_{i+1} \rangle}{r_{i+3} r_{i+1}} - \frac{\langle e_{i+2}, e_{i+1} \rangle}{r_{i+2} r_{i+1}} + \frac{\langle e_{i+3}, e_i \rangle}{r_{i+3} r_i} - \frac{\langle e_{i+2}, e_i \rangle}{r_{i+2} r_i} \\ =& 0.\end{aligned}$$

2. A similar argument shows

$$(T_i, T_{i+1}) = -\frac{1}{r_{i+1}^2} < 0.$$

3. We deduce that if $(T_i, T_i) < 0$, by the computation in (3.10) we have $\langle e_i, e_{i+1} \rangle > 0$ and hence also $(T_i, \omega) > 0$ by the Riemann-Roch-type theorem 3.3.28.

4. by the DP condition 3.5.6, we have

$$(T_1 + T_2 + T_3 + T_4, \omega) = (-\omega, \omega) = -\delta(K) > 0.$$

We prove the result by showing that any sequence of elements (T_1, T_2, T_3, T_4) satisfying the above three inferred conditions must have an element T_i simultaneously satisfying $(T_i, T_i) > 0$ and $(T_i, \omega) > 0$.

For this purpose, we first simplify the bilinear form using coordinate changes: consider the real plane $\mathrm{Num}(K) \otimes_{\mathbb{Z}} \mathbb{R}$, obtained by base change from the numerical Picard group 3.3.15. The form $(-,-)$ is indefinite by condition HI of 3.5.8. It follows that if we pick an element L orthogonal to ω and normalize both ω and L we obtain a basis, which we again denote as (ω, L) in which the associated quadratic form is given by

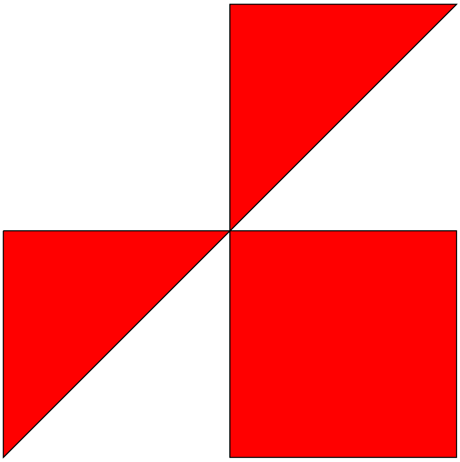
$$Q((x, y)) = y^2 - x^2$$

We make one more coordinate transformation: one easily computes that with respect to the basis $(\frac{L+\omega}{2}, \frac{L-\omega}{2})$ the quadratic form is simply $Q((x, y)) = x.y$. Note that with this choice of ω, the inequalities in (3) and (4) remain valid. It finally follows that the negative intersection form on $\mathrm{Num} \otimes_{\mathbb{Z}} \mathbb{R}$ is given by

$$((x, y), (a, b)) = \frac{1}{2}(xb + ya) \tag{3.11}$$

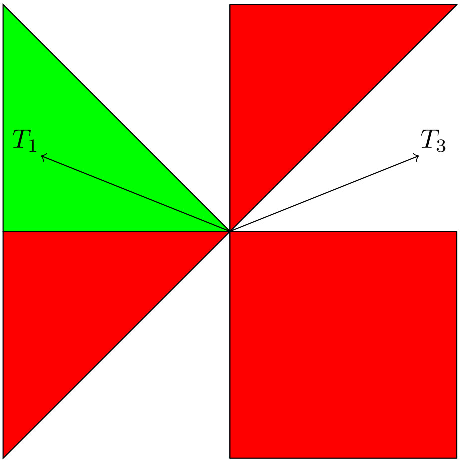
with respect to this basis. Note that for this form we have $(x, y)^{\perp} = \mathbb{R}(-x, y)$. We denote the coordinates of T_i with respect to this basis by (x_i, y_i).

Since the coordinates of ω are given by $(1, -1)$, we see that $((x, y), \omega) > 0 \iff y > x$. Also $((x, y), (x, y)) > 0$ precisely when x and y have the same sign. The third condition tells us that if $x.y < 0$ then $y > x$. Hence the regions where both the required conditions are simultaneously satisfied are $0 < x < y$, $x < y < 0$ and the fourth quadrant Q_4. Finally, since $(T_i, \omega) = \frac{\langle e_i, e_{i+1} \rangle}{r_i r_{i+1}} = 0$ is excluded by hypothesis, we can exclude the line $x = y$. Visually, we exclude the red zone in the following picture

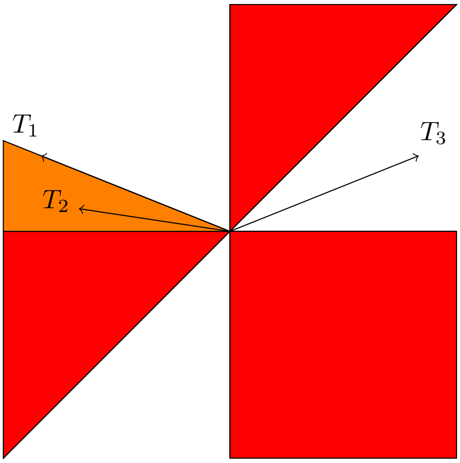


Next, Since $(T_1 + T_2 + T_3 + T_4, \omega) > 0$, by (4) there must be an index for which $(T_i, \omega) > 0$. Applying the shift $\pi_F(\rho)$, (see 3.4.3) we assume that $i = 1$ so that $(T_1, \omega) > 0$. The above conditions then imply that the coordinates of T_1 satisfy $x_1 y_1 > 0$ and by the picture above, T_1 lies somewhere in the 2nd quadrant and since $(T_1, T_3) = 0$, T_3 lies on the reflection of the line $\mathbb{R}T_1$ about the y-axis.

There are two options now: T_3 lies either in the first quadrant or in the third. We assume it lies in the first, in which case $(T_3, T_3) > 0$and let the reader treat the other case. By the above figure, we must have $y_3 < x_3$, hence $-x_1 > y_1$ and T_1 lies strictly under the line $\mathbb{R}(1, -1)$ as in the following figure where T_1 lies in the green zone and T_3 is its reflection



We finally look at T_2. Since $(T_2, T_1) < 0$ and $(T_1, T_1) < 0$ by $(T_1, \omega) > 0$, T_2 lies on the same side of as T_1 of the line defined by the condition $(-, T_1) = 0$. This condition defines the line $T_1^\perp = \mathbb{R}T_3$ as T_1. Similarly, since $(T_3, T_3) > 0$ and $(T_3, T_2) < 0$, T_2 lies on the opposite side of T_3 of the line $\mathbb{R}T_1$. Hence, T_2 must lie somewhere in the intersection of these two half planes. Taking into account the red area, we have colored this intersection orange



But since $(T_2, T_4) = 0$, T_4 must lie in the reflection of the orange zone about the y-axis (the other option, reflection along the x-axis is colored red already) since $(T_3, T_3) > 0$. This area lies in the half plane of vectors v such that $(v, T_3) > 0$, contradicting the fact that $(T_3, T_4) < 0$. □

Corollary 3.5.13. *Let K be of DPS* type with an exceptional basis $(e_1, \ldots e_4)$. Then E is equivalent under ΣB_n to a basis $(e'_1, \ldots, e'_4)$ satisfying either of the following properties*

- $r(e'_1) = 0$
- $\langle e'_2, e'_3 \rangle = 0$.

Proof. If a vector $e_i \in E$ has rank 0, then using 3.4.15, we can shift with the element $\pi_F(\rho)$ until we obtain an exceptional basis E' where $e'_1 = e_i$, so that the first condition is satisfied.

If all vectors in E have nonzero rank, after applying the sign action, we may assume that all ranks are positive. If $\langle e_i, e_{i+1} \rangle = 0$ for some i, repeated application of 3.4.15 again yields the required collection. If not, we can apply 3.5.11 and the above lemma 3.5.12 to obtain a mutation such that the resulting exceptional bases E' satisfies

$$0 < \mathcal{M}(e'_1 \ldots e'_4) < \mathcal{M}(e_1, \ldots, e_4)$$

(see 3.9). This process must stop, resulting in an exceptional basis satisfying the second condition. □

3.5.4 Classifying Exceptional Bases in Rank 4

The reduction lemma 3.5.13 was the chief technical step required to obtain a classification for lattices of DPS* type with an exceptional sequence up to mutation. We begin by describing the analogue of the Markov equation 3.8 in the rank 4 case:

Lemma 3.5.14. *Assume K is a free abelian group with a bilinear form $\langle -,-\rangle$ which has Gram matrix*

$$\begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.12}$$

in a certain basis. Then K has a Serre automorphism s wich is unipotent if and only if the system of equations

$$\begin{cases} a^2+b^2+c^2+d^2+e^2+f^2-bad-edf-ace-bcf+abdf=0 \\ af+bd=ce \end{cases} \tag{3.13}$$

is satisfied.

Proof. The proof is similar to 3.5.1. The Serre automorphism is given by $s \stackrel{\text{def}}{=} M^{-1}.{}^tM$ as in 3.3.3. The unipotency is equivalent to the characteristic polynomial of $s-1$ having no terms of lower degree. Explicitly writing out this polynomial, reveals that this condition is equivalent to the two above equations □

We shall refer to the above system of diophantine equations as the Markov equation in rank 4. Before describing our classification result, we list four archetypical examples of lattices wich satisfy these equations:

- there is a obvious trivial type (type 0), given by extending the rank 3 case by zeroes: let K be the lattice $\mathbb{Z}^4$ and consider the bilinear form whose Gram matrix with respect to the standard basis is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathscr{D}^b(X)$ has a full exceptional sequence given by $(\mathcal{O}_X, \mathcal{O}_X(1,0), \mathcal{O}_X(0,1), \mathcal{O}_X(1,1))$. This yields an exceptional basis for $K(X)$ with Gram matrix

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  we refer to this lattice as type 1.

- Let $X = \mathbb{F}_1$, the blow up of $\mathbb{P}^2$ at the origin. In this case $\mathscr{D}^b(X)$ has a full exceptional sequence of the form $(\mathcal{O}_X, \mathcal{O}_X(1,0), \mathcal{O}_X(1,1), \mathcal{O}_X(2,2))$ by [Orl92]. The Gram matrix of the bilinear form on $K(X)$ with respect to this basis is given by

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  we refer to this lattice as type 2.

- The last example is a little more subtle. We consider the lattice $K = \mathbb{Z}^4$ and endow it with a bilinear form whose Gram matrix with respect to the standard basis is given by

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  we refer to this lattice as type 3.

Lemma 3.5.15. *The above four lattices are mutually nonisomorphic and of DPS* type*

Proof. We leave the reader to verify that all four lattices satisfy the conditions of 3.3.4 and the additional condition (*) (recall that the Serre automorphism has matrix $M^{-1} \cdot M^t$) To show that these lattices satisfy the additional DP condition, we compute their degrees. For type 0, the vector $e_2 = (0, 1, 0, 0)$ is a generator for $K/F^1(K)$ satisfying the required condition of 3.3.19. We then consider the associated $\omega = (s - 1)e$ and compute $\delta = bl\omega\omega = -9$. For the other three cases we make the choice of $e_1 = (1, 0, 0, 0)$. and compute that the degrees are -8,-8 and -20 for types 1,2 and 3 respectively. This computation also immediately shows that all four types satisfy HI (although this also follows from 3.5.9 for types 0,1 and 2).

We immediately also infer that all types are nonisomorphic except possibly 1 and 2 since δ is an invariant of K by 3.3.21.
To show that types 1 and 2 are nonisomorphic, we note that the Serre automorphism on type 1 is the identity modulo 2, which is not the case in type 2. □

Lemma 3.5.16. *Let K be of DPS* type with an exceptional basis E. Then there exists a series of mutations of E such that the Gram matrix of the resulting basis is of one of four forms listed above.*

Proof. First assume that E contains an element of rank 0. Using the orbits of the cylindrical braid actions 3.4.15, we may shift the basis so that this element is e_1. It follows that $K = \text{span}(e_2, e_3, e_4)$ is a lattice of DPS* type. Using the action of ΣB_3 on this subspace, we can mutate until the Gram matrix has the standard form

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

We obtain that the Gram matrix for the bilinear form on K with respect to

(e_1, e_2, e_3, e_4) must be of the form

$$\begin{bmatrix} 1 & a & b & c \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Plugging these coefficients in the Markov equation (3.8), we obtain that $a = n$, $b = 2n$ and $c = n$ for some $n \in \mathbb{Z}$. Now, we use that K satisfies the additional DP condition of definition 3.5.6. Picking $e = (0, 1, 0, 0)$ as a generator for K/F^1K, we obtain that $\delta = \langle (s-1)e, (s-1)e \rangle = n^2 - 9$. Hence $n = 0, 1, 2$.

- If $n = 0$, we obtain the matrix

  $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

  of trivial type.

- If $n = 1$, then we have

  $$\epsilon_2 \pi_B(\rho) \sigma_3 \sigma_1 \sigma_2 \left(\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  which is the Gram matrix of $K(\mathbb{F}_1)$, our second type

- If $n = 2$, the sequence of mutations

  $$\epsilon_4 \pi_B(\rho^2) \sigma_2^2 \pi_B(\rho) \epsilon_3 \left(\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  showing that this case is equivalent to the third type.

We thus assume that no vector in E has zero rank. Then 3.5.13 tells us that $\langle e_i, e_{i+1}\rangle$ for a certain i, which we may assume to be 2 using 3.4.15. Plugging this extra condition into the second equation of (3.13), we obtain

$$\begin{bmatrix} 1 & rx & sx & z \\ 0 & 1 & 0 & ry \\ 0 & 0 & 1 & sy \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for $(r, s, x, y, z) \in \mathbb{Z}$.
The first equation of 3.13 now translates into the generalized Markov equation

$$kx^2 + ky^2 + z^2 = kxyz \tag{3.14}$$

where $k = r^2 + s^2$. We show in the appendix 3.5.21 that this equation has solutions exactly when $k = 1, 2$ or 5.

- if $k = 1$, then (x, y, z) are solutions to the classical Markov equation in degree 3. Moreover, since we may assume that $r = 0$ (for example), we end up with the matrix
$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
Applying $\epsilon_4 \pi_B(\rho)$ results in the Gram matrix of trivial type.

- if $k = 2$, then $r = s = 1$. By a combination of 3.5.21 and 3.5.20 we can mutate so that
$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
which is equal to the Gram matrix of the first type, $K(\mathbb{P}^1 \times \mathbb{P}^1)$.

- Finally if $k = 5$, there are four cases. We treat one and leave the reader to check that the other three are all equivalent using the appropriate shifts and signs ($\pi_B(\rho)$ and ϵ_k). We assume that $r = 1$ and $s = 2$. The classification of solutions 3.5.21 and 3.5.20, shows that we can mutate to obtain a solution of one of 2 types. We assume $(x, y, z) = (2, 1, 5)$, the other type being completely similar. Then the resulting matrix is precisely

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

which is the matrix corresponding to the third type □

Restating the above lemma a little, we obtain the main result of this chapter:

Theorem 3.5.17. *Let K be of DPS* type. Then K is isomorphic to one of the 4 standard types listed above.*

Proof. This is a reformulation of 3.5.16. □

Remark 3.5.18. *One can wonder as to what the geometry behind the type 3 model can look like. Since the degree of the surface is -20 following the proof of 3.5.16, it is clear that this type does not correspond to a Del Pezzo surface. Instead, we will construct a model using noncommutative geometry.*

3.5.5 Solutions to the Generalized Markov Equation

In this section, we adapt the results of [KN98] to our setting: for $k \in \mathbb{N}$ and $x, y, z \in \mathbb{Z}$, we consider the generalized Markov equation

$$M : kx^2 + ky^2 + z^2 = kxyz$$

There is an obvious type of transformation on the set of Markov solutions:

Lemma 3.5.19. *The following 3 bijections*

1. $\mu_1 : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^3 : (x, y, z) \longrightarrow (x, z, xz - y)$
2. $\mu_2 : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^3 : (x, y, z) \longrightarrow (kxy - z, y, x)$
3. $\mu_3 : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^3 : (x, y, z) \longrightarrow (y, yz - x, z)$

are preserve the subset of solutions to M

The above proposition thus yields a group action of the group G generated by $\langle \mu_1, \mu_2, \mu_3, \rangle$ on the set of solutions of M. An important aspect of this action is how it relates to mutation of exceptional matrices (§3.4.4):

Lemma 3.5.20. *Let r, s be fixed. Let $k = r^2 + s^2$ and consider the map*

$$\alpha : \mathbb{Z}^3 \longrightarrow E(\mathrm{SL}_4(\mathbb{Z})) : (x, y, z) \longrightarrow \begin{bmatrix} 1 & rx & sx & z \\ 0 & 1 & 0 & ry \\ 0 & 0 & 1 & sy \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we have using the notation of the action $*_2$ of ΣCB_n on $E(\mathrm{SL}_4(\mathbb{Z}))$ (see 3.4.4)

$$\alpha \circ \mu_3 = \sigma_1 \circ \alpha,\ \alpha \circ \mu_2 = (\rho^3 \sigma_1 \sigma_2) \circ \alpha \text{ and } \alpha \circ \mu_3 = (\rho^3 \sigma_1^{-1} \sigma_2^{-1}) \circ \alpha$$

Proof. This is a dreary check left to the reader □

Theorem 3.5.21. *The Markov equation has solutions exactly when $k = 1, 2$ or 5. In which case every solution is equivalent under the action of G in 3.5.19 to*

1. $(\pm 3, \pm 6, \pm 3)$ *for* $k = 1$
2. $(\pm 1, \pm 2, \pm 5)$ *and* $(\pm 2, \pm 1, \pm 5)$ *for* $k = 5$
3. $(\pm 2, \pm 2, \pm 4)$ *for* $k = 2$

where either no or exactly 2 signs are negative

Proof. See [KN98, prop 3.5] □

Chapter 4

Generalized Preprojective Algebras

4.1 Introduction and Statement of the Results

In this chapter, we describe joint work with Dennis Presotto ([dTdVP14]) in which we investigated a new class of algebras which are a relative version of the classical preprojective algebra on the star quiver. Our main motivation for studying these algebras lies in the fact that they provide a local description of the noncommutative geometry required to construct a model for a 'lattice of DPS* type 3' from §3.5.4. More precisely, we shall consider a noncommutative notion of a $\mathbb{P}^1$-bundle in chapter 5 and we will prove there that over an affine scheme, the category of sheaves over this noncommutative space form a summand of category of modules over the type of ring consider here [1].

We start by considering a relative version of the notion of a Frobenius pair of commutative noetherian rings $R \longrightarrow S$ (see definition 4.2.2). To such a pair, we associate an $\mathbb{N}$-graded R-algebra $\Pi_R(S)$ which is the quotient of the $R \oplus S$-

[1] we refer the impatient reader to 5.3.18 for the precise version of this statement

tensor algebra over a certain $R \oplus S$-bimodule modulo some elementary relations (0.1.11). The fact that this definition coincides with the preprojective algebra of a star quiver in the split case $R \longrightarrow R^n$ follows easily (see lemma 4.2.10). We shall investigate the ringtheoretic properties of these algebras when S is of rank 4 over R. More precisely, we prove 3 results of interest:

Theorem G. *[see 4.3.9] for each degree d, the R-module $\Pi_R(S)_d$ is projective and we have*

$$\mathrm{rk}(\Pi_R(S)_d) = \begin{cases} 5(d+1) & \textit{if } d \textit{ is even} \\ 4(d+1) & \textit{if } d \textit{ is odd} \end{cases}$$

Subsequently, we show how one can construct a morphism

$$\sigma_{R,S} : R[(Z(\Pi_R(S))_4]^{\oplus n} \longrightarrow \Pi_R(S) \tag{4.1}$$

and prove

Theorem H (see 4.5.2 and 4.5.1)**.** *$\sigma_{R,S}$ is surjective, in particular $\Pi_R(S)$ is Noetherian and finite over its center.*

In the final section, we bound the global dimension of generalized preprojective algebras as follows

Theorem I. *[see 4.6.1] If R and S have finite global dimension, then so does $\Pi_R(S)$. We have the following explicit upper bound:*

$$\mathrm{gl.dim}(\Pi_R(S)) \leq \max\Big(\mathrm{gl.dim}(R), \mathrm{gl.dim}(S)\Big) + 2$$

Theorems G and I are proven using a similar technique by increasing the order of generality of R. In the first step, we assume $R = \Bbbk$ to be an algebraically closed field. We introduce a notion of deformation (4.2.5) for Frobenius algebras over R and show that over an algebraically closed field all Frobenius pairs fit inside a simple directed diagram of such deformations (see figure 4.3). Since the dimension of the vector space $\Pi_R(S)_d$ increases and the morphism $\sigma_{R,S}$ from (4.1) preserves its surjectivity after applying a deformation, both theorems in this setting reduce to some explicit computations for the Frobenius algebra $\Bbbk[s^2, t^2]$, to which the extra section §4.7 is dedicated. The result in the case where $R = \Bbbk$ is any field then follows from dimension reasons resp. faithfully flatness. Extending the result to the general setting where R can be any ring turns out to require a convoluted series of implications. Simplifying a little, we could say that we prove that given any morphism of rings $f : R \longrightarrow R'$

- the construction of $\Pi_R(S)$ commutes with base change under f
- Forming the center $Z(\Pi_R(S))_4$ in degree 4 and the morphism $\sigma_{R,S}$ also commutes with base change under f

The first claim is a formal consequence of the construction, whereas the second is not trivial at all and specific to $\Pi_R(S)$. It follows from the fact that the center of $\Pi_R(S)$ in degree 4 is a split submodule of $\Pi_R(S)_4$, projective of rank 2 (see 4.4.2, 4.4.3 and 4.4.4): Diagrammatically, we summarize our method of proof as follows:

$$\begin{tikzcd}
 & \text{alg. closed field} \arrow[r] & \text{field} \arrow[r, "{[\text{Gro71, 1.4.4}]}"] & \text{local domain} \arrow[dr] & \\
\text{2 specific cases} \arrow[ur, "4.2.7"] & & & & \text{domain} \arrow[d, "4.3.8"] \\
 & \text{ring} & \text{local ring} \arrow[l] & & \text{local ring with } \overline{\Bbbk} = \Bbbk \arrow[ll, "4.3.7"']
\end{tikzcd} \tag{4.2}$$

4.2 Frobenius Pairs and Generalized Preprojective Algebras

4.2.1 Frobenius Pairs

Convention 4.2.1. *Throughout S/R will denote a pair of commutative rings equipped with a ring morphism $R \longrightarrow S$. Moreover we will always assume R is Noetherian, although some of the results also hold in higher generality.*

Definition 4.2.2. We say that S/R is *relative Frobenius* of rank n if:

- S is a free R-module of rank n.
- $\mathrm{Hom}_R(S, R)$ is isomorphic to S as S-module.

Remark 4.2.3.

- *It is clear that if R is a field, then a relative Frobenius pair coincides with a finite dimensional Frobenius algebra in the classical sense.*
- *Let $e_1, \ldots, e_n$ be any basis for S as an R-module. Then it is easy to see that the second condition is equivalent to the existence of an element λ in $\mathrm{Hom}_R(S, R)$ such that the R-matrix $(\lambda(e_i e_j))_{i,j}$ with respect to this basis is invertible.*
- *We may equally well assume that S/R is projective of rank n. However all results we prove may be reduced to the free case by suitably localizing the ring R.*

If R is an algebraically closed field, it is an easy exercise to describe all such algebras:

Lemma 4.2.4. *Let $\Bbbk$ be an algebraically closed field and F a commutative Frobenius algebra of dimension 4 over $\Bbbk$. Then F is isomorphic to one of the* *following algebras:*

$$\begin{cases} \Bbbk \oplus \Bbbk \oplus \Bbbk \oplus \Bbbk \\ \Bbbk[t]/(t^2) \oplus \Bbbk \oplus \Bbbk \\ \Bbbk[s]/(s^2) \oplus \Bbbk[t]/(t^2) \\ \Bbbk[t]/(t^3) \oplus \Bbbk \\ \Bbbk[t]/(t^4) \\ \Bbbk[s,t]/(s^2,t^2) \end{cases}$$

Proof. Recall the following classical facts

1. a direct sum of Frobenius algebras is Frobenius.
2. a finite dimensional commutative local $\Bbbk$-algebra is Frobenius if and only if it has a unique minimal ideal.

It follows immediately that $\Bbbk[t]/(t^n)$ is Frobenius (of dimension n) over $\Bbbk$ as it has a unique minimal ideal (t^{n-1}) and that $\Bbbk[x_1,\dots,x_n]/(x_1^2,\dots,x_n^2)$ is also Frobenius (of dimension 2^n) with unique minimal ideal $(x_1 \cdot \ldots \cdot x_n)$. By the first observation, the algebras in the above list certainly are Frobenius.
Now let F be Frobenius of dimension 4. Since F is Artinian, the structure theorem for Artinian rings (see for example [AM69, theorem 8.7]) states that F must (uniquely) decompose as a direct sum of local, Artinian $\Bbbk$-algebras:

$$F \cong F_1 \oplus \ldots \oplus F_n$$

We can now use the classification of local $\Bbbk$-algebras of small rank in [Poo08, Table 1].
If $n=4$, then clearly $F = \Bbbk \oplus \Bbbk \oplus \Bbbk \oplus \Bbbk$.
If $n=3$, then $F \cong A_1 \oplus \Bbbk \oplus \Bbbk$ where $\dim_{\Bbbk}(A_1) = 2$, hence $A_1 \cong \Bbbk[t]/(t^2)$ which is Frobenius.
If $n=2$, then either F splits as a sum of 2-dimensional local $\Bbbk$-algebras, in which case we again obtain $F \cong k[s]/(s^2) \oplus \Bbbk[t]/(t^2)$ or $F = A_1 \oplus \Bbbk$ where $\dim_{\Bbbk}(A_1) = 3$. This again yields 2 possibilities: either $A_1 \cong \Bbbk[t]/(t^3)$, which is Frobenius, or $A_1 \cong \Bbbk[s,t]/(s,t)^2$. The latter is however not Frobenius, because

it is not self-injective (the morphism $A_1 t \longrightarrow A_1 : t \mapsto s$ cannot be lifted to $A_1 \longrightarrow A_1$).
Finally, assume $n = 1$. In this case F is a local $\Bbbk$-algebra of dimension 4 and by [Poo08] takes one of the five following forms:

$$\begin{cases} \Bbbk[t]/(t^4) \\ \Bbbk[s,t]/(s^2,t^2) \\ \Bbbk[s,t]/(s^2,st,t^3) \\ \Bbbk[s,t,u]/(s,t,u)^2 \\ \Bbbk[s,t]/(s^2+t^2,st) \quad (\text{if } \mathrm{ch}(\Bbbk)=2) \end{cases}$$

The first two algebras are Frobenius whereas the other three are not as they are not self-injective by a similar argument as above.

□

The 6 Frobenius algebras listed in the above lemma are related to each other by an appropriate notion of deformation:

Definition 4.2.5. Let F and G be Frobenius algebras over $\Bbbk$.
A *Frobenius deformation* of F to G is a $\Bbbk[[u]]$-algebra D such that $D/\Bbbk[[u]]$ is relatively Frobenius and

1. $D/uD \cong F$ as a $\Bbbk$-algebra
2. $D_{(u)} \cong G \otimes_{\Bbbk} \Bbbk((u))$ as a $\Bbbk((u))$-algebra

we write $F \overset{def}{\dashrightarrow} G$

Remark 4.2.6. *Instead of requiring that $D/\Bbbk[[u]]$ be relative Frobenius we may equivalently require that D be free over $\Bbbk[[u]]$ with rank equal to the dimension of F. The condition that $\mathrm{Hom}_{\Bbbk[[u]]}\left(D, \Bbbk[[u]]\right)$ should be isomorphic to D as a D-module is immediate by the corresponding condition on $F/\Bbbk$.*

Lemma 4.2.7. *There is a diagram of Frobenius deformations*

$$\begin{tikzcd}
 & & \mathbb{k}[t]/(t^2) \oplus \mathbb{k}[s]/(s^2) \arrow[dr, dashed, "def", "4"'] & & \\
\mathbb{k}[s,t]/(s^2,t^2) \arrow[r, dashed, "def", "1"'] & \mathbb{k}[t]/(t^4) \arrow[ur, dashed, "def", "2"'] \arrow[dr, dashed, "def", "3"'] & & k[t]/(t^2) \oplus \mathbb{k} \oplus \mathbb{k} \arrow[r, dashed, "def", "6"'] & \mathbb{k}^{\oplus 4} \\
 & & \mathbb{k}[t]/(t^3) \oplus \mathbb{k} \arrow[ur, dashed, "def", "5"'] & &
\end{tikzcd} \tag{4.3}$$

Proof. We first let $F \overset{\text{def}}{=} \mathbb{k}[s,t]/(s^2,t^2)$ and $G \overset{\text{def}}{=} \mathbb{k}[t]/(t^4)$ and describe $F \overset{def}{\dashrightarrow} G$ Let $R \overset{\text{def}}{=} \mathbb{k}[[u]]$, $\mathbb{K} \overset{\text{def}}{=} \mathbb{k}((u))$ and define

$$D \overset{\text{def}}{=} R[s,t]/(us - t^2, s^2, t^4)$$

We claim that D defines a deformation from F to G. It is clear that $D/uD \cong F$ as a $\mathbb{k}$-algebra and the map

$$D \longrightarrow \mathbb{K}[t]/(t^4) : u \mapsto u, s \mapsto t^2/u, t \mapsto t$$

factors through an isomorphism

$$D_{(u)} \longrightarrow \mathbb{K}[t]/(t^4) = G \otimes_{\mathbb{k}} \mathbb{K}$$

Hence by the above remark it suffices to check that D is a free R-module of rank 4. This is obviously the case with $e_1 = 1, e_2 = s, e_3 = t, e_4 = st$ providing an R-basis for D.

The other cases are similar. We first use the Chinese remainder theorem to find an alternate presentation for F of the forms $\mathbb{k}[t]/(f(t))$. Then for each deformation $F \overset{def}{\dashrightarrow} G$, we try to find an alternate presentation for $G \otimes_{\mathbb{k}} \mathbb{K}$ (again using the Chinese remainder theorem) of the form $\mathbb{K}[t]/(g(t))$ in such a way that $g(t)|_{u=0} = f(t)$. We then exhibit an R-algebra $D \overset{\text{def}}{=} R[t]/(g(t))$. We finally leave the reader to check that in each of our choices, $(1, t, t^2, t^3)$ defines an R-basis

.

number	2	3	4	5	6* ($char(\Bbbk) \neq 2$)
$g(t)$	$t^2(t-u)^2$	$t^3(t-u)$	$(t-1)^2t(t-u)$	$t^2(t-1)(t-u)$	$(t^2-u^2)(t^2-1)$

[2]

□

4.2.2 Generalized Preprojective Algebras

We shall invoke the following notation: for a relative Frobenius pair S/R, let $M \stackrel{\text{def}}{=} {}_RS_S$. This $R-S$-bimodule can be considered as an $R \oplus S$ bimodule by letting the R-component act on the left and the S-component on the right, the other actions being trivial. Similarly, we let $N \stackrel{\text{def}}{=} {}_SS_R$ and consider it an $R \oplus S$-bimodule by only letting the S-component act on the left and the R-component act on the right, the other actions again begin trivial. We now define

$$T(R,S) \stackrel{\text{def}}{=} T_{R\oplus S}(M \oplus N)$$

Note that by construction, we have $M \otimes_{R\oplus S} M = N \otimes_{R\oplus S} N = 0$, hence

$$T(R,S)_2 = (M_{R\oplus S}N) \oplus (N \otimes_{R\oplus S} M) = ({}_RS \otimes_S S_R) \oplus ({}_SS \otimes_R S_R)$$

The algebra we are interested in, will be a quotient of $T(R,S)$ as follows: by the second condition of definition 4.2.2, there exists a generator λ of $\mathrm{Hom}_R(S,R)$ as an S-module. The R-bilinear form $\langle a,b\rangle \stackrel{\text{def}}{=} \lambda(ab)$ is then clearly nondegenerate and we can find dual R-bases $(e_i)_i$, $(f_j)_j$ satisfying

$$\lambda(e_i f_j) = \delta_{ij}$$

[2]* In the case where $\Bbbk$ has characteristic 2, one has to choose $D = R[t]/(t(t-u)) \oplus R^{\oplus 2}$ for the 6th deformation. Then $(1,0,0),(t,0,0),(0,1,0),(0,0,1)$ provides an R-basis for D.

Definition 4.2.8. For a relative Frobenius pair, the *generalized preprojective algebra* $\Pi_R(S)$ is given by

$$T(R,S)/(\text{rels})$$

where the relations (rels) are in degree 2 given by

$$1 \otimes 1 \in {}_R S \otimes_S S_R$$

$$\sum_i e_i \otimes f_i \in {}_S S \otimes_R S_S$$

Remark 4.2.9. *Up to isomorphism, the above construction is independent of choice of generator and dual basis.*

The name generalized preprojective algebra is motivated by the following:

Lemma 4.2.10. *Let S be the ring $R^{\oplus n}$.*
Then $\Pi_R(S)$ is isomorphic to the preprojective algebra over R associated to the quiver with one central vertex and n outgoing arrows as defined in 0.1.11.

Proof. Let $e_1, \dots, e_n$ be the set of complete orthogonal idempotents in S and write $x_1, \dots, x_n$ (resp. $y_1, \dots, y_n$) $\in \Pi_R(S)_1$ for the corresponding elements in the bimodules N (respectively M) discussed at the beginning of §4.2.2. We can describe the tensor algebra $T(R,S)$ as the free algebra

$$F \stackrel{\text{def}}{=} R\langle e_1, \dots, e_n, x_1, \dots, x_n, y_1, \dots, y_n \rangle$$

subject to the relations

1. $e_i e_j = \delta_{ij} e_i$.
2. $e_i x_j = \delta_{ij} x_i$ and $y_i e_j = \delta_{ij} y_i$
3. $x_i e_j = e_i y_j = 0$
4. $x_i x_j = y_i y_j = 0$

The first relation defining $\Pi_R(S)$ is given by $1 \otimes 1 \in M \otimes_S N$. The first unit, 1 is given by $1 = \sum x_i$ whereas the second unit satisfies $1 = \sum y_i$, we obtain

5. $y_1x_1 + \ldots + y_nx_n = 0$

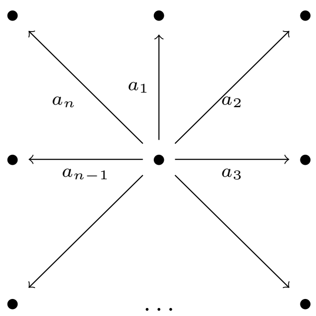
To compute the second relation, we note that

$$\lambda : S \to R : \sum_{i=1}^{n} r_i e_i \mapsto \sum_i r_i$$

is a generator of $\mathrm{Hom}_R(S, R)$ as an S-module and hence $(e_i)_i$ is a basis, self-dual for the associated form $\langle -, - \rangle$. The relation inside ${}_S S \otimes_R S_S$ now becomes

6. $x_1y_1 + \ldots + x_ny_n = 0$

It now remains to show that F subject to the above 6 relations is isomorphic to the preprojective algebra of the quiver $\mathcal{Q}$:



We let $\overline{\mathcal{Q}}$ denote the formally doubled quiver of $\mathcal{Q}$ and consider the map $F \longrightarrow R\overline{\mathcal{Q}}$ defined by

- sending e_i to the outer node n_i
- sending y_i to the arrow a_i and x_i to the formal inverse a_i^*

The first 4 relations now precisely describe the multiplication in the path algebra of $\overline{Q}$ and the relations 5 an 6 precisely map to the two relations defining a preprojective algebra $\sum a_i a_i^* = 0 = \sum a_i^* a_i$ $\square$

4.3 Computing $\mathrm{rk}(\Pi_R(S)_d)$

The construction of $\Pi_R(S)$ is compatible with base change in the following way:

Lemma 4.3.1 (Base Change for $\Pi_R(S)$)**.** *Let S/R be relative Frobenius of finite rank and $R \longrightarrow R'$ a morphism of rings. Then*

1. *$(R' \otimes_R S)/R'$ is relative Frobenius of rank n*
2. *there is a canonical isomorphism*

$$R' \otimes_R \Pi_R(S) \cong \Pi_{R'}(R' \otimes_R S)$$

Proof. Assume that S/R is relative Frobenius with generator λ and basis $\{e_1, \ldots, e_n\}$. Then it is a straightforward verification that $(R' \otimes_R S)/R'$ is relative Frobenius with generator $1 \otimes \lambda$ and basis $\{1 \otimes e_1,\ \ldots,\ 1 \otimes e_n\}$. With this data we can thus construct the algebra $\Pi_{R'}(R' \otimes_R S)$. Moreover we have a series of isomorphisms

$$R' \otimes_R ({}_R S_S \oplus {}_S S_R) \cong {}_{R'}(R' \otimes_R S)_{R' \otimes_R S} \oplus {}_{R' \otimes_R S}(R' \otimes_R S)_{R'}$$

as an $(R', R' \otimes_R S)$-bimodule. This implies that we obtain a canonical isomorphism

$$R' \otimes_R T(R, S) \cong T(R', R' \otimes_R S)$$

which by our choice of basis preserves the relations, in turn inducing an isomorphism

$$R' \otimes_R \Pi_R(S) \cong \Pi_{R'}(R' \otimes_R S)$$

□

To prove that the R-modules $\Pi_R(S)_d$ are projective and to compute their ranks following the technique of diagram (4.2), we first treat the case where R is an algebraically closed field. We have the following lemma relating the dimension of these vector spaces under deformation:

Lemma 4.3.2. *Let F and G be Frobenius algebras over $\Bbbk$ and let $F \overset{def}{\dashrightarrow} G$ be a Frobenius deformation. Then for all d, we have*

$$\dim_{\Bbbk}(\Pi_{\Bbbk}(F)_d) \geq \dim_{\Bbbk}(\Pi_{\Bbbk}(G)_d)$$

Proof. Let $R = \Bbbk[[u]]$ and $\mathbb{K} = \Bbbk((u))$.
Let $m = \dim_{\Bbbk}(\Pi_{\Bbbk}(F)_d)$. Assume that D is the R-algebra deforming F to G provided by the definition 4.2.5. Then since

$$\Pi_{\Bbbk}(F) = \Pi_{\Bbbk}(\Bbbk \otimes_R D) = \Bbbk \otimes_R \Pi_R(D)$$

by lemma 4.3.1, Nakayama's lemma implies that a $\Bbbk$-basis of cardinality m for $\Pi_{\Bbbk}(F)_d$ lifts to a set of generators for $\Pi_R(D)_d$. Moreover, as

$$\mathbb{K} \otimes_{\Bbbk} \Pi_{\Bbbk}(G) = \Pi_{\mathbb{K}}(\mathbb{K} \otimes_{\Bbbk} G) = \Pi_{\mathbb{K}}(\mathbb{K} \otimes_R D) = \mathbb{K} \otimes_R (\Pi_R(D)),$$

this set of generators contains a $\mathbb{K}$-basis for $\mathbb{K} \otimes \Pi_{\Bbbk}(G)$. It follows that

$$\dim_{\mathbb{K}}(\mathbb{K} \otimes_{\Bbbk} (\Pi_{\Bbbk}(G)_d) = \dim_{\Bbbk}(\Pi_{\Bbbk}(G)_d) \leq m$$

□

Convention 4.3.3. *From now on we will only focus on the rank 4 case for the rest of the paper. I.e. when using the notation S/R, we will always assume this is a relative Frobenius pair of rank 4. Similarly all upcoming Frobenius algebras F or G will have dimension 4 over $\Bbbk$.*

We will now prove that in the case of Frobenius algebras of rank 4 the above inequality is actually an equality. We first compute the ranks in two explicit cases:

Lemma 4.3.4. *We have*

$$\dim_{\Bbbk}\left(\Pi_{\Bbbk}\left(\frac{\Bbbk[s,t]}{(s^2,t^2)}\right)_d\right) \leq \begin{cases} 5(d+1) & \text{if } d \text{ is even} \\ 4(d+1) & \text{if } d \text{ is odd} \end{cases}$$

Proof. This is the content of 4.7.1 □

Lemma 4.3.5. *Let $\Bbbk$ be an algebraically closed field, then*

$$\dim_{\Bbbk}\left(\Pi_{\Bbbk}(\Bbbk^{\oplus 4})_d\right) = \begin{cases} 5(d+1) & \text{if } d \text{ is even} \\ 4(d+1) & \text{if } d \text{ is odd} \end{cases}$$

Proof. By lemma 4.2.10, $\Pi_{\Bbbk}(S)$ is the preprojective algebra over $\Bbbk$ associated to the extended Dynkin quiver of $\mathcal{Q} = \widetilde{D_4}$.

Let $\overline{\mathcal{Q}}$ be the formally doubled quiver. Let 0 denote the central vertex and 1, 2, 3, 4 the outer vertices. Then for each $d \in \mathbb{N}$ we consider the matrix $W_d \in \mathbb{N}^{5\times 5}$ where $(W_d)_{ij}$ gives the number of paths of length d in $\overline{\mathcal{Q}}$ starting at vertex i and ending at vertex j, modulo relations. Finally write $W(t) = \sum_{d=0}^{\infty} W_d t^d \in \mathbb{N}^{5\times 5}[[t]]$. Then by [EE07, Proposition 3.2.1] we have

$$W(t) = \frac{1}{1 - t\cdot C + t^2}$$

Where C is the adjacency matrix of $\overline{\mathcal{Q}}$, i.e.

$$\begin{aligned} W(t) &= \left(1 - t\cdot\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + t^2\right)^{-1} \\ &= \frac{1}{(1-t^2)^2(1+t^2)}\cdot\begin{pmatrix} (1+t^2)^2 & t(1+t^2) & t(1+t^2) & t(1+t^2) & t(1+t^2) \\ t(1+t^2) & 1-t^2+t^4 & t^2 & t^2 & t^2 \\ t(1+t^2) & t^2 & 1-t^2+t^4 & t^2 & t^2 \\ t(1+t^2) & t^2 & t^2 & 1-t^2+t^4 & t^2 \\ t(1+t^2) & t^2 & t^2 & t^2 & 1-t^2+t^4 \end{pmatrix} \end{aligned}$$

This gives the desired result as the Hilbert series of $\Pi_{\Bbbk}(S)$ now becomes

$$\begin{aligned}
h_{\Pi_{\Bbbk}(S)}(t) &= \sum_{d=0}^{\infty}\left(\sum_{i,j=0}^{4}(W_d)_{i,j}\right)t^d \\
&= \sum_{i,j=0}^{4}\sum_{d=0}^{\infty}(W_d)_{i,j}t^d \\
&= \frac{(1+t^2)^2+8t(1+t^2)+4(1-t^2+t^4)+12t^2}{(1-t^2)^2(1+t^2)} \\
&= \frac{5+8t+5t^2}{(1-t^2)^2} \\
&= (5+8t+5t^2)\sum_{l=0}^{\infty}(l+1)t^{2l} \\
&= \sum_{l=0}^{\infty}(5l+5(l+1))t^{2l}+8(l+1)t^{2l+1} \\
&= \sum_{l=0}^{\infty}(5(2l+1))t^{2l}+4((2l+1)+1)t^{2l+1}
\end{aligned}$$

□

These two computations immediately apply the required result over a field:

Corollary 4.3.6. *Let $\Bbbk$ be a field and F a Frobenius algebra (of rank 4 by 4.2.1) over $\Bbbk$ then:*

$$\dim_{\Bbbk}(\Pi_{\Bbbk}(F)_d) = \begin{cases} 5(d+1) & \text{if } d \text{ is even} \\ 4(d+1) & \text{if } d \text{ is odd} \end{cases}$$

Proof. By lemma 4.3.1 we can reduce to the case where $\Bbbk$ is algebraically closed. The statement then follows as a combination of lemmas 4.2.4, 4.2.7, 4.3.2, 4.3.4 and 4.3.5 □

To extend the result from fields to general rings we will need the following two lemmas. Combined, they essentially show that locally a relative Frobenius pair is constructed through base change of a relative Frobenius pair where the ground ring is a polynomial ring over the integers.

Lemma 4.3.7. *Let R be a local ring with residue field $\Bbbk$. Then there is a faithfully flat morphism $R \longrightarrow \overline{R}$ where $\overline{R}$ is a local ring whose residue field is the algebraic closure $\overline{\Bbbk}$ of $\Bbbk$.*

Proof. This is an immediate application of [GD71, 10.3.1] □

Lemma 4.3.8. *Let R be a local ring with an algebraically closed residue field $\Bbbk$. Let S/R be relative Frobenius of rank 4 (following convention 4.2.1). Then there exists a domain $\tilde{R}$, together with a morphism $\tilde{R} \longrightarrow R$ and a ring $\tilde{S}$ with $\tilde{S}/\tilde{R}$ relative Frobenius of rank 4 such that $\tilde{S} \otimes_{\tilde{R}} R \cong S$.*
Moreover $\tilde{R}$ can be chosen to be of the form

$$\tilde{R} = \Big(\mathbb{Z}[x_1, \ldots, x_m]\Big)_f$$

the localization of a polynomial ring over $\mathbb{Z}$ at some non-zero element f.

Proof. We prove the theorem in a specific case and sketch the other cases, leaving some details to the reader for the sake of brevity. By lemmas 4.3.1 and 4.2.7, $S \otimes_R \Bbbk$ is one of 6 Frobenius algebras. The case we consider is $S \otimes_R \Bbbk = \Bbbk[s,t]/(s^2,t^2)$. Let $\tilde{s}, \tilde{t} \in S$ be lifts of s and t. Since $\{1, s, t, st)\}$ is a basis for $S_\Bbbk$, By Nakayama's lemma $\{1, \tilde{s}, \tilde{t}, \tilde{s}\tilde{t}\}$ forms a set of R-generators for S. In particular we can write:

$$\begin{aligned} \tilde{s}^2 &= a_1 + b_1\tilde{s} + c_1\tilde{t} + d_1\tilde{s}\tilde{t} \\ \tilde{t}^2 &= a_2 + b_2\tilde{s} + c_2\tilde{t} + d_2\tilde{s}\tilde{t} \end{aligned}$$

where the coefficients $a_1, \ldots, d_2$ all lie in the maximal ideal $\mathfrak{m}$ of R (because $s^2 = t^2 = 0$ in $S \otimes_R k$). We subsequently obtain a canonical morphism

$$\pi : R[\tilde{s}, \tilde{t}]/(a_1 + b_1\tilde{s} + c_1\tilde{t} + d_1\tilde{s}\tilde{t} - \tilde{s}^2, a_2 + b_2\tilde{s} + c_2\tilde{t} + d_2\tilde{s}\tilde{t} - \tilde{t}^2) \longrightarrow S$$

such that $\pi \otimes_R \Bbbk$ is the identity morphism on $\Bbbk$. This immediately implies that π is surjective. Moreover since S is free over R, we have $0 = \ker(\pi \otimes_R \Bbbk) = \ker(\pi) \otimes_R \Bbbk$ and $\ker(\pi) = 0$ by Nakayama's lemma. π is thus an isomorphism. There is a canonical morphism

$$A \overset{\text{def}}{=} \mathbb{Z}[a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2] \longrightarrow R$$

Let $f \overset{\text{def}}{=} 1 - d_1 d_2$ and denote $\tilde{R} = A_f$. As the image of f in R is invertible (because d_1, d_2 lie in the maximal ideal of $\mathfrak{m}$ of R), the above morphism factors through a morphism $\tilde{R} \longrightarrow R$. Finally set $\tilde{S} = \tilde{R}[\tilde{s}, \tilde{t}]/(a_1 + b_1\tilde{s} + c_1\tilde{t} + d_1\tilde{s}\tilde{t} - \tilde{s}^2, a_2 + b_2\tilde{s} + c_2\tilde{t} + d_2\tilde{s}\tilde{t} - \tilde{t}^2)$. By construction we have

$$\tilde{S} \otimes_{\tilde{R}} R \cong R[\tilde{s}, \tilde{t}]/(a_1 + b_1\tilde{s} + c_1\tilde{t} + d_1\tilde{s}\tilde{t} - \tilde{s}^2, a_2 + b_2\tilde{s} + c_2\tilde{t} + d_2\tilde{s}\tilde{t} - \tilde{t}^2) \overset{\pi}{\cong} S.$$

It hence suffice to prove $\tilde{S}/\tilde{R}$ is relative Frobenius of rank 4. For this note that $(e_i)_1^4 \overset{\text{def}}{=} \{1, \tilde{s}, \tilde{t}, \tilde{s}\tilde{t})\}$ is an $\tilde{R}$-basis for $\tilde{S}$ and if we let $\lambda \in \mathrm{Hom}_{\tilde{R}}(\tilde{S}, \tilde{R})$ denote the projection onto the component $\tilde{R}\tilde{s}\tilde{t}$, the matrix of $\lambda(e_i d \cdot e_j)$ is of the form

$$\Theta \overset{\text{def}}{=} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & d_1 & 1 & * \\ 0 & 1 & d_2 & * \\ 1 & * & * & * \end{bmatrix}$$

Hence Θ has determinant $1 - d_1 d_2$, which by construction is invertible in $\tilde{R}$, proving that $\tilde{S}$ is indeed Frobenius of rank 4 over $\tilde{R}$ by remark 4.2.3.
In the 5 other cases from lemma 4.2.4 we have $S \otimes_R \Bbbk = \Bbbk[t]/(t^4 + at^3 + bt^2 + ct + d)$ for some coefficients $a, b, c, d \in \Bbbk$ and we can choose $\tilde{R}, \tilde{S}$ to be of the form $\tilde{R} \overset{\text{def}}{=} \mathbb{Z}[\alpha, \beta, \gamma, \delta]$ and $\tilde{S} \overset{\text{def}}{=} \tilde{R}[t]/(t^4 + \alpha t^3 + \beta t^2 + \gamma t + \delta)$. For each choice of $\alpha, \beta, \gamma, \delta$ we have that $\tilde{S}/\tilde{R}$ is relative Frobenius of rank 4, because the corresponding matrix Θ will have determinant exactly 1. We leave the details to the reader. $\square$

We can now prove the main theorem of this section:

Theorem 4.3.9. $\Pi_R(S)_d$ *is projective. Moreover, we have*

$$\mathrm{rk}(\Pi_R(S)_d) = \begin{cases} 5(d+1) & \text{if } d \text{ is even} \\ 4(d+1) & \text{if } d \text{ is odd} \end{cases}$$

Proof. First let R be a local domain with residue field $\Bbbk$ and field of fractions $\mathbb{K}$. By Corollary 4.3.6 and lemma 4.3.1 we have for each degree d:

$$\begin{aligned} \dim_{\mathbb{K}}(\mathbb{K} \otimes_R \Pi_R(S)_d) &= \dim_{\mathbb{K}}(\Pi_{\mathbb{K}}(\mathbb{K} \otimes_R S)_d) \\ &= \dim_{\Bbbk}(\Pi_{\Bbbk}(\Bbbk \otimes_R S)_d) = \dim_{\Bbbk}(\Bbbk \otimes_R \Pi_R(S)_d) \end{aligned}$$

Theorem [Gro71, 1.4.4] implies that $\Pi_R(S)$ is free of the stated ranks.

Next, let R be any domain. Then for each $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_\mathfrak{p} \otimes_R \Pi_R(S) \cong \Pi_{R_\mathfrak{p}}(R_\mathfrak{p} \otimes S)$ is a generalized preprojective algebra over the local domain $R_\mathfrak{p}$ and hence in each degree a free module of the stated rank (recall that R is always assumed noetherian by 4.2.3). As these ranks do not depend on the choice of $\mathfrak{p}$, Serre's theorem (see for example [Ser55]) now implies that $\Pi_R(S)_d$ is projective of the stated rank.

For the next step, we let R be a local ring with algebraically closed residue field. Then by lemma 4.3.8 there is a domain $\tilde{R}$, a morphism $\tilde{R} \longrightarrow R$ and an $\overline{R}$-algebra $\tilde{S}$ such that $\tilde{S}/\tilde{R}$ is relative Frobenius of rank 4 and $S \cong \tilde{S} \otimes_{\tilde{R}} R$. By the above $\Pi_{\tilde{R}}(\tilde{S})_d$ is a projective $\tilde{R}$-module of the given ranks and hence $\Pi_R(S)_d = \Pi_{\tilde{R}}(\tilde{S})_d \otimes R$ is a projective R-module of the above rank.

To extend the result to general local rings, we invoke lemma 4.3.7 to find a faithfully flat morphism $R \longrightarrow \overline{R}$. By the above $\Pi_{\overline{R}}(\overline{R} \otimes S)_d \cong \overline{R} \otimes \Pi_R(S)_d$ is a free $\overline{R}$-module of the desired rank. By the faithfully flatness of $R \longrightarrow \overline{R}$, $\Pi_R(S)_d$ is itself a free R-module of the desired rank.

Finally we extend the statement from local rings to general commutative rings by again applying Serre's theorem. $\square$

We conclude this section by mentioning the following lemma which is a slight improvement of theorem 4.3.9. It will be needed for technical reasons in §4.6. Since the proof closely follows that of 4.3.9, we will limit ourselves with a mere sketch of the proof

Lemma 4.3.10. *$(1_R \cdot \Pi_R(S))_d$ and $(1_S \cdot \Pi_R(S))_d$ are projective R-modules of ranks respectively*

$$\begin{cases} d+1 & \text{if } d \text{ is even} \\ 2(d+1) & \text{if } d \text{ is odd} \end{cases}$$

and

$$\begin{cases} 4(d+1) & \text{if } d \text{ is even} \\ 2(d+1) & \text{if } d \text{ is odd} \end{cases}$$

Proof. We have

$$\Pi_R(S) = 1_R \cdot \Pi_R(S) \oplus 1_S \cdot \Pi_R(S)$$

which immediately shows that both modules are indeed projective. This decomposition is compatible with base change and Frobenius deformations in the obvious ways. An argument similar to the proof of theorem 4.3.9 shows that it suffices compute ranks in the two specific cases $S = \Bbbk^{\oplus 4}$ and $S = \Bbbk[s,t]/(s^2,t^2)$. For the first case we notice that the Hilbert function $h_{1_\Bbbk \cdot \Pi_R(S)}(t)$ can be deduced from the proof of lemma 4.3.5 by adding the entries in the first column of $W(t)$, resulting in

$$\begin{aligned} h_{1_\Bbbk \cdot \Pi_\Bbbk(S)}(t) &= \frac{(1+t^2)^2 + 4 \cdot t(1+t^2)}{(1-t^2)^2(1+t^2)} \\ &= \frac{1+6t+t^2}{(1-t^2)^2} \\ &= (1+6t+t^2)\sum_{l=0}^{\infty}(l+1)t^{2l} \\ &= \sum_{l=0}^{\infty}(2l+1)t^{2l} + \sum_{l=0}^{\infty}2((2l+1)+1)t^{2l+1} \end{aligned}$$

Similarly we find

$$h_{1_S \cdot \Pi_\Bbbk(S)}(t) = \sum_{l=0}^{\infty}4(2l+1)t^{2l} + \sum_{l=0}^{\infty}2((2l+1)+1)t^{2l+1}$$

For the second explicit case $S = \Bbbk[s,t]/(s^2,t^2)$ the result is an immediate corollary of the "Type I-II"-classification of the generators for $\Pi_\Bbbk(S)$ described in §4.7. □

4.4 Base Change for $Z_4(R,S)$ and $\mathrm{rk}(Z_4(R,S))$

Convention 4.4.1. *To ease notation, we shall denote the center of* $\Pi_R(S)$ *in degree* d *by*

$$Z_d(R,S) \stackrel{def}{=} Z(\Pi_R(S)_d)$$

Throughout this section we prove the following results for the center in degree 4:

Theorem 4.4.2. *$Z_4(R, S)$ is a split R-submodule of $\Pi_R(S)_4$.*

Theorem 4.4.3. *Let S/R be relative Frobenius of rank 4 and $R \longrightarrow R'$ a morphism of rings. Then the canonical base change map*

$$Z_4(R, S) \otimes_R R' \longrightarrow Z_4(R', S \otimes_R R')$$

is an isomorphism.

Theorem 4.4.4. *$Z_4(R, S)$ is a projective R-module of rank 2.*

The proofs of these theorems are heavily intertwined, we shall prove them according to the following diagram of implications:

theorem 4.4.4 when R is a field

$\Downarrow$

theorems 4.4.4 and 4.4.2 when R is a local domain

$\Downarrow$

theorem 4.4.2 for general R

$\Downarrow$

theorem 4.4.3 for general R

$\Downarrow$

theorem 4.4.4 for general R

In several of these steps we use the fact that in each degree d, the center $Z_d(R, S)$ can be obtained as kernel of a morphism between (projective) R-modules. For this, note that by the above section §4.3, there exists R-bases

$a_1^0 \ldots, a_5^0$ for $\Pi_R(S)_0$ and $a_1^1 = e_1, \ldots, a_8^1$ for $\Pi_R(S)_1$. Moreover, since $\Pi_R(S)$ is generated in degrees 0 and 1, for each d there is a map

$$\phi_{R,S} : \Pi_R(S)_d \longrightarrow \Pi_R(S)_d^{\oplus 5} \oplus \Pi_R(S)_{d+1}^{\oplus 8} : x \mapsto \Big(\left([x, a_i^0]\right)_i , \left([x, a_j^1]\right)_j \Big) \tag{4.4}$$

whose kernel is precisely $Z_d(R,S)$. I.e. there is a left-exact sequence

$$0 \longrightarrow Z_d(R,S) \longrightarrow \Pi_R(S)_d \xrightarrow{\phi_{R,S}} \Pi_R(S)_d^{\oplus 5} \oplus \Pi_R(S)_{d+1}^{\oplus 8} \tag{4.5}$$

This discussion allows us to immediately prove that the center is compatible with change of ground ring for flat morphisms:

Lemma 4.4.5 (flat base change)**.** *Let $R \longrightarrow R'$ be a flat morphism of rings. Then the canonical map*

$$R' \otimes_R Z_d(R,S) \longrightarrow Z_d(R' \otimes_R S)$$

is an isomorphism

Proof. The construction of $\phi_{R,S}$ is compatible with base change and the tensor product with flat modules preserves kernels. Hence

$$\begin{aligned} R' \otimes_R Z_d(R,S) = R' \otimes_R (\ker(\phi_{R,S})) &= (R' \otimes_R \ker(\phi_{R,S})) \\ &= \ker(\phi_{R',R'\otimes S}) = Z_d(R', R' \otimes S) \end{aligned}$$

$\square$

Following our philosophy established in the introduction (see 4.2) we shall first compute the dimension of $Z_4(k,S)$ in two specific cases. The first is $S = \mathbb{k}^4$. To simplify the computation, we first give an alternate description of $\Pi_{\mathbb{k}}(S)$ using the McKay correspondence.
Recall that the binary dihedral group is given by

$$BD_{2n} = < a, b \mid a^4 = 1, a^2 = b^2, ab = ba^3 >$$

Lemma 4.4.6. *Let $\mathbb{k}$ be an algebraically closed field with* $\mathrm{ch}(\mathbb{k}) \neq 2$ *and* $F = \mathbb{k}^{\oplus 4}$, *then* $\Pi_{\mathbb{k}}(F)$ *is Morita equivalent to the skew group ring* $\mathbb{k}[x,y]\#BD_8$ *where the binary dihedral group of order 8 acts on* $\mathbb{k}[x,y]$ *through its natural action on the complex* $x-y$*-plane:*

$$a \cdot x = ix, \; a \cdot y = -iy, \; b \cdot x = y, \; b \cdot y = x$$

Proof. Let $\mathcal{Q} \stackrel{\text{def}}{=} \widetilde{D_4}$ be the star quiver on 4 vertices (or equally, the Dynkin quiver of type D_4) and denote by $\overline{\mathcal{Q}}$ the associated formally doubled quiver. Then $\overline{\mathcal{Q}}$ is the McKay-quiver of BD_8 and by [CBH98] (which was already announced in [RVdB89]) the preprojective algebra on $\mathcal{Q}$ is Morita equivalent to $k[x,y]\#BD_8$, the result now follows from lemma 4.2.10. □

Lemma 4.4.7. *Let $R = k$ be an algebraically closed field with* $\mathrm{ch}(\mathbb{k}) \neq 2$ *then* $\dim_{\mathbb{k}}(Z_4(\mathbb{k}, \mathbb{k}^{\oplus 4})) = 2$.

Proof. By lemma 4.4.6 and the fact that the center of a ring is invariant under Morita equivalence, we only need to show that the degree 4 polynomials in $k[x,y]$ invariant under the action of BD_8 span a 2-dimensional vector space. One easily checks that these invariants are given by $kx^2y^2 \oplus k(x^4+y^4)$. □

In order to include the case of characteristic 2, we need to compute the center explicitly through brute force.

Lemma 4.4.8. *Let $\mathbb{k}$ be an algebraically closed field of characteristic 2, then* $\dim_{\mathbb{k}}(Z_4(\mathbb{k}, \mathbb{k}^{\oplus 4})) = 2$.

Proof. Let a,b,c,d denote the 4 idempotent elements. We exhibit two linearly independent central elements of degree 4 in $\Pi_{\mathbb{k}}(F)$. $\Pi_{\mathbb{k}}(F)$ is generated by elements of two types:

Type I) Elements of the form $f * ef * ef \ldots * ef * e(f*)$ where each $*$ is either a,b,c or d

Type II) Elements of the form $*ef*ef\ldots*ef*e(f*)$ where each $*$ is either a,b,c or d

The relations on $\Pi_{\Bbbk}(F)$ imply the following relations for these generators:

- $aefa = befb = cefc = defd = 0$
- $fae + fbe + fce + fde = 0$

We first find elements in degree 2 which are normalizing with respect to some automorphism σ satisfying $\sigma^2 = \mathrm{Id}$. One such element is the following:

$$u_{ab/cd} \stackrel{\text{def}}{=} aefb+befa+cefd+defc+fae+fbe = aefb+befa+cefd+defc+fce+fde$$

The equality follows from the above relations combined with the fact that $\mathrm{ch}(\Bbbk) = 2$ (we use the subscript to denote that this element only depends on the partition of $\{a, b, c, d\}$ in the subsets $\{a, b\}$ and $\{c, d\}$. By symmetry there are 2 other such elements, namely $u_{ac/bd}$ and $u_{ad/bc}$.)
This element is normalizing with respect to the automorphism

$$\sigma_{ab/cd} : a \mapsto b,\ b \mapsto a,\ c \mapsto d,\ d \mapsto c$$

since for example:

$$(ce)u_{ab/cd} = cefde = u_{ab/cd}(de) = u_{ab/cd} \cdot \sigma_{ab/cd}(ce)$$

and similarly for a, b and d. By symmetry we obtain the same for the elements $u_{ac/bd}$ and $u_{ad/bc}$. Since the automorphisms have order 2, it follows that the squares of these three elements are in fact central.

$$\begin{aligned}
x_{ab/cd} \stackrel{\text{def}}{=} u^2_{ab/cd} &= aefbefa + befaefb + cefdefc + defcefd + faefbe + fbefae \\
x_{ac/bd} \stackrel{\text{def}}{=} u^2_{ac/bd} &= aefcefa + befdefb + cefaefc + defbefd + faefce + fcefae \\
x_{ad/bc} \stackrel{\text{def}}{=} u^2_{ad/bc} &= aefdefa + befcefb + cefbefc + defaefd + faefde + fdefae
\end{aligned}$$

We claim that these 3 elements are pairwise linearly independent. Indeed, suppose for example that $x_{ab/cd}$ and $x_{ac/bd}$ were linearly dependent. Then $faefbe + fbefae$ and $faefce + fcefae$ should be linearly dependent. By the nature of the relations and the fact that we are working in characteristic 2 there are only 3 possibilities:

- $faefbe + fbefae = 0$
- $faefce + fcefae = 0$
- $faefbe + fbefae + faefce + fcefae = 0$

The first two options are obviously impossible and third option gives $faefde + fdefae = 0$, leading to a contradiction. Hence $x_{ab/cd}$ and $x_{ac/bd}$ must be linearly independent. On the other hand one checks that these elements satisfy the relation $x_{ab/cd} + x_{ac/bd} + x_{ad/bc} = 0$. This shows that there are 3 central elements satisfying one linear relation, hence $\dim_{\Bbbk}(Z_4(\Bbbk, \Bbbk^{\oplus 4})) = 2$. □

For the second specific case we have:

Lemma 4.4.9. *Let* $\Bbbk$ *be a field, then* $dim_{\Bbbk}(Z_4(\Bbbk, \Bbbk[s,t]/(s^2,t^2))) = 2$.

Proof. The proof is similar to the above argument and based on an explicit description of generators. We refer the reader to §4.7.2 □

In order to compute the dimension of $(Z_4(\Bbbk, F))$ for any Frobenius algebra F, we use the following lemma, which relates these dimensions under Frobenius deformation:

Lemma 4.4.10. *Let* F *and* G *be two Frobenius algebras over a field* $\Bbbk$ *such that* $F \overset{def}{\dashrightarrow} G$. *Then for each* $d \in \mathbb{N}$,

$$\dim_{\Bbbk}(Z_d(\Bbbk, F)) \geq \dim_{\Bbbk}(Z_d(\Bbbk, G))$$

Proof. Let D be the algebra deforming F to G as in definition 4.2.5 and denote $R = \Bbbk[[u]]$, $\mathbb{K} = \Bbbk((u))$. Using the morphism (4.4), we write $Z_d(R, D) = \ker(\phi)$ and let Φ be the matrix corresponding to ϕ.
Let $\Phi_{\mathbb{K}}$ denote the same matrix with coefficients viewed in the fraction field $\mathbb{K}$ and $\Phi_{\Bbbk}$ denote the matrix with coefficients viewed in the residue field $\Bbbk$. Then by construction,

$$\ker(\Phi_{\mathbb{K}}) = \ker(\mathbb{K} \otimes_R \phi) = Z_d(\mathbb{K}, \mathbb{K} \otimes_R D)$$
$$\ker(\Phi_{\Bbbk}) = \ker(\Bbbk \otimes_R \phi) = Z_d(\Bbbk, \Bbbk \otimes_R D)$$

Now,

$$\begin{aligned}\dim_{\Bbbk}(Z_d(\Bbbk, G)) &= \dim_{\mathbb{K}}(\mathbb{K} \otimes_{\Bbbk} (Z_d(\Bbbk, G)) \\ &= \dim_{\mathbb{K}}(Z_d(\mathbb{K}, \mathbb{K} \otimes_{\Bbbk} G)) \\ &= \dim_{\mathbb{K}}(Z_d(\mathbb{K}, \mathbb{K} \otimes_R D)) \\ &= \dim_{\mathbb{K}} \ker(\Phi_{\mathbb{K}})\end{aligned}$$

Since clearly $\dim_{\Bbbk}(\ker(\Phi_{\Bbbk}) \geq \dim_{\Bbbk}(\ker(\Phi_{\mathbb{K}}))$, the claim follows. □

Lemma 4.4.11. *For any field $\Bbbk$ and Frobenius algebra F of dimension 4, we have $dim_{\Bbbk}(Z_4(\Bbbk, F)) = 2$*

Proof. If $\Bbbk$ is algebraically closed, this follows from lemmas 4.2.4, 4.2.7, 4.4.10, 4.4.7, 4.4.8 and 4.4.9.
For the general case we use lemma 4.4.5. □

Lemma 4.4.12. *theorems 4.4.4 and 4.4.2 hold in the case where $(R, \mathfrak{m})$ is a local domain.*

Proof. Let $\phi_{R,S}$ be the morphism in (4.4), then $\phi_{R,S}$ is a morphism between free R-modules of finite rank and hence can be represented by a matrix Φ with respect to some chosen basis for $V \stackrel{\text{def}}{=} \Pi_R(S)_4$ and $W \stackrel{\text{def}}{=} \Pi_R(S)_4^{\oplus 5} \oplus \Pi_R(S)_5^{\oplus 8}$. Let $\Phi_{\Bbbk}$ be the matrix obtained by replacing each entry of Φ by its corresponding class in the residue field $\Bbbk \stackrel{\text{def}}{=} R/\mathfrak{m}$, then $\Phi_{\Bbbk}$ is a matrix-representation for $\Bbbk \otimes_R \phi_{R,S}$ using the induced $\Bbbk$-basis for $\Bbbk \otimes_R V$ and $\Bbbk \otimes_R W$. Let $a = \operatorname{rk}(\Phi_{\Bbbk})$. Then there is an invertible $a \times a$ submatrix $\Psi_{\Bbbk}$ in $\Phi_{\Bbbk}$. The corresponding submatrix Ψ of Φ has a determinant which does not lie in $\mathfrak{m}$ and is thus itself invertible. By a suitable base change on V and W we can now rewrite Φ in the form:

$$\Phi = \left[\begin{array}{c|c} \mathrm{Id}_{a\times a} & 0 \\ \hline 0 & \Psi' \end{array}\right]$$

where all entries of Ψ' lie in $\mathfrak{m}$ (any entry not in $\mathfrak{m}$ would give rise to an invertible submatrix of rank $a+1$ by elementary row and column operations).

It follows that we can decompose V and W into a direct sum of free submodules $V = V_1 \oplus V_2$ and $W = W_1 \oplus W_2$ such that $\phi_{R,S} = \phi_1 \oplus \phi_2$ where $\phi_1 : V_1 \xrightarrow{\sim} W_1$ and $\phi_2 : V_2 \longrightarrow W_2$ satisfies $\Bbbk \otimes_R \phi_2' = 0$. This implies that $Z_4(\Bbbk, \Bbbk \otimes_R S) = (\Bbbk \otimes_R \phi_{R,S}) = \Bbbk \otimes_R V_2$ and hence V_2 is free of rank 2 by lemma 4.4.11.
Now, by construction $\ker(\phi_{R,S}) \subset V_2$ and hence $\mathbb{K} \otimes \ker(\phi_{R,S}) \subset K \otimes V_2$. But then, since $\mathbb{K}$ is flat over R, lemma 4.3.1 gives:

$$\begin{aligned}\dim_{\mathbb{K}}(\mathbb{K} \otimes_R \ker(\phi_{R,S})) &= \dim_{\mathbb{K}}(\mathbb{K} \otimes Z_4(R,S)) \\ &= \dim(Z_4(\mathbb{K}, \mathbb{K} \otimes_R S) = 2 = \dim(\mathbb{K} \otimes V_2)\end{aligned}$$

It follows that $\ker(\phi_{R,S}) = V_2$ from which $\phi_2 = 0$ and hence $Z_4(R,S) \hookrightarrow \Pi_R(S)_4$ splits. It also follows that $Z_4(R,S)$ is projective of finite rank since $\Pi_R(S)_4$ is so by 4.3.9 and this rank must equal 2 by lemma 4.4.11. □

We can now finish the proofs of the main results of this section. This is done in a way similar to the proof of theorem 4.3.9:

Proof of theorem 4.4.2. By lemma 4.4.12 we already know that the result holds if R is a local domain and by the local nature of splitting (see for example [Lam07, ex. 4.13, p.105]) hence also if R is any domain.

Now let R be a local ring with algebraically closed residue field. Then by lemma 4.3.8 S/R is a base change of $\tilde{S}/\tilde{R}$ by a morphism $\tilde{R} \longrightarrow R$ for some domain $\tilde{R}$ and the result follows in this case as the base change of a split embedding is a split embedding.

If R is any local ring, we can consider the faithfully flat morphism $R \longrightarrow \overline{R}$ provided by lemma 4.3.7. As the residue field of $\overline{R}$ is algebraically closed the monomorphism $\phi_{\overline{R}, S \otimes \overline{R}} = \phi_{R,S} \otimes \overline{R}$ from (4.4) is split. This implies that $\phi_{R,S}$ must be split itself by the lemma 4.4.13 below.

Finally, again using the local nature of splitting, we obtain the result for any ring R. □

Lemma 4.4.13. *Let R be a local ring and let $R \longrightarrow \overline{R}$ be as in lemma 4.3.7. Let $\phi : A \hookrightarrow B$ be an embedding of finitely generated R-modules such that B is projective. Moreover assume $\phi \otimes \overline{R} : A \otimes \overline{R} \hookrightarrow B \otimes \overline{R}$ is split. Then ϕ is a split injection*

Proof. Let $\Bbbk$ be the residue field of R and $\overline{\Bbbk}$ its algebraic closure, then there is a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & \Bbbk \\ \downarrow & & \downarrow \\ \overline{R} & \longrightarrow & \overline{\Bbbk} \end{array}$$

As $\phi \otimes \overline{R}$ is split, $\phi \otimes \overline{\Bbbk}$ is injective. The above commutative diagram combined with the faithfully flatness of $\Bbbk \longrightarrow \overline{\Bbbk}$ implies $\phi \otimes \Bbbk$ is also injective. Let $C = \mathrm{coker}(\phi)$, then we have a long exact sequence

$$\ldots \longrightarrow \mathrm{Tor}_1^R(B, \Bbbk) \longrightarrow \mathrm{Tor}_1^R(C, \Bbbk) \longrightarrow A \otimes \Bbbk \xrightarrow{\phi \otimes \Bbbk} B \otimes \Bbbk \longrightarrow C \otimes \Bbbk \longrightarrow 0$$

Since B is a projective R-module, it is flat, implying $\mathrm{Tor}_1^R(B, \Bbbk) = 0$. From this it follows that $\mathrm{Tor}_1^R(C, \Bbbk) = 0$ and because R is a local noetherian ring, this implies C is a projective R-module and the exact sequence $0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$ is indeed split. $\square$

Proof of theorem 4.4.3. This is an immediate consequence of theorem 4.4.2 and the fact that the construction of $\phi_{R,S}$ in (4.5) is compatible with base change. $\square$

Proof of theorem 4.4.4. First let R be a local domain with residue field $\Bbbk$ and field of fractions $\Bbbk$. Then by lemma 4.4.11,

$$\begin{aligned} \dim_{\mathbb{K}}(\mathbb{K} \otimes_R (Z_4(R, S)) &= \dim_{\mathbb{K}}(Z_4(\mathbb{K}, \mathbb{K} \otimes_R S)) = 2 \\ &= \dim_{\Bbbk}(Z_4(\Bbbk, \Bbbk \otimes_R S)) = \dim_{\Bbbk}(\Bbbk \otimes_R Z_4(R, S)) \end{aligned}$$

Hence by [Gro71, ch. 1, cor. 4.4], $Z_4(R, S)$ is free of rank 2.

If R is a domain, then for any $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_\mathfrak{p}$ is a local domain such that $R_\mathfrak{p} \otimes_R Z_4(R,S) = Z_4(R_\mathfrak{p}, R_\mathfrak{p} \otimes_R S)$ is a free module of rank 2. Serre's theorem then proves that $Z_4(R,S)$ is projective of rank 2.

Now let R be a local ring with algebraically closed residue field and let $\tilde{S}/\tilde{R}$ be as in lemma 4.3.8. Then we know that $Z_4(\tilde{R}, \tilde{S})$ is projective over $\tilde{R}$ of rank 2. Hence $Z_4(R,S) = Z_4(\tilde{R}, \tilde{S}) \otimes R$ is free of rank 2 over R.

To extend the statement to general local rings we just use lemma 4.3.7.

Finally Serre's theorem extends the statement to non-local rings as well. $\square$

4.5 $\Pi_R(S)$ is Noetherian and Finite Over Its Center

We recall the reader that we follow convention 4.2.1 and that S/R relative Frobenius of rank 4 over a noetherian commutative ring R. In this section we give a proof of the following fact:

Theorem 4.5.1. $\Pi_R(S)$ *is noetherian.*

To this end, we define a map

$$\sigma_{R,S} : R[Z_4(R,S)]^{\oplus N} \to \Pi_R(S)$$

as follows: by the first condition of the definition of a relative Frobenius pair, 4.2.2 we may choose an R-basis (x,y,z,w) for S. Let e be the element corresponding to $1_S \in N$ and f be the element corresponding to $1_S \in M$ (recall the definition for N and M from §4.2.2). There is a map

$$\pi : R < x,y,z,w,e,f > \longrightarrow T_{R \oplus S}(M \oplus N)$$

Where x,y,z,w have degree 0 and e,f have degree 1 in $R < x,y,z,w,e,f >$. The R-module $T(R,S)_0$ is generated by $(1_R, x, y, z, w)$ and these 5 elements are the images under π of the corresponding elements in $R < x,y,z,w,e,f >$,

hence π is surjective in degree 0.
Moreover, $T(R,S)_1 = {}_RS_S \oplus {}_SS_R$ is generated by $(xe, ye, ze, we, fx, fy, fz, fw)$ as an $R-R$-bimodule and hence π is also surjective in degree 1.
Finally since $T(R,S)$ is a tensor algebra, it is generated in degree 0 and 1 and π is surjective. Composing with the canonical quotient map $T(R,S) \twoheadrightarrow \Pi_R(S)$ yields a surjection

$$\chi : R < x, y, z, w, e, f > \twoheadrightarrow \Pi_R(S)$$

Now, the R-module $\Pi_R(S)_{\leq 6}$ is generated the image of the words of length at most 6 in $\{e, f\}$. We can reduce this set by making the following remarks:

1. since $\{1_R, x, y, z, w\}$ forms an R-basis for $\Pi_R(S)_0$, we can assume that any subword of degree zero is precisely a letter in this set
2. by the definition of the multiplication of $\Pi_R(S)$, we have $e^2 = f^2 = 0$

Hence if we let H be the finite set set of words in $\{x, y, z, w, e, f\}$ of length at most 6 in $\{e, f\}$ such any two instances of x, y, z, w are separated by at least one e or f, we obtain $\chi(R \cdot H) = \Pi_R(S)_{\leq 6}$. If we list this set as

$$H = \{a_1, \ldots, a_n\}$$

we can define $\sigma_{R,S}$ as

$$\sigma_{R,S} : R[Z_4(R,S)]^{\oplus n} \to \Pi_R(S) : (z_i)_{i=1}^n \mapsto \sum_{i=1}^n z_i \chi(a_i)$$

We shall prove the following theorem

Theorem 4.5.2. *$\sigma_{R,S}$ is a surjective map. In particular $\Pi_R(S)$ is finite over its center.*

From this theorem 4.5.1 will readily follow as $Z_4(R,S)$ is finitely generated over R by 4.4.4. In turn we prove theorem 4.5.2 first for fields following the diagram 4.2. The general result will the follow quickly.

First we give some base change arguments: Let $R \longrightarrow R'$ be any morphism of rings, then by theorem 4.4.3 we have a diagram

$$\begin{tikzcd} R'[Z_4(R', R' \otimes_R S)]^{\oplus n} \arrow[r, "\sigma_{R', R' \otimes_R S}"] & \Pi'_R(R' \otimes_R S) \\ R' \otimes_R R[Z_4(R,S)]^{\oplus n} \arrow[u, "\zeta"] \arrow[r, "R' \otimes_R (\sigma_{R,S})"'] & R' \otimes_R \Pi_R(S) \arrow[u, "\eta"'] \end{tikzcd} \tag{4.6}$$

Lemma 4.5.3. *For any morphism $\varphi : R \longrightarrow R'$, the diagram in (4.6) is commutative.*

Proof. We recall that if S/R be relative Frobenius with generator λ and basis $\{e_1, \ldots, e_n\}$. Then $(R' \otimes S)/R'$ is relative Frobenius with generator $1_{R'} \otimes \lambda$ and basis $\{1_{R'} \otimes e_1, \ldots, 1_{R'} \otimes e_n\}$ (see lemma 4.3.1). Following the successive steps in the construction of $\sigma_{R', R' \otimes_R S}$ we see that

$$\begin{cases} \Pi_{R'}(R' \otimes_R S) & = 1_{R'} \otimes \Pi_R(S) \\ \chi_{R'} & = 1_{R'} \otimes \chi_R \\ H_{R'} & = 1_{R'} \otimes H_R \end{cases}$$

Let z_i be an element in $R[Z_4(R,S)]$ considered as the ith component of $R[Z_4(R,S)]^{\oplus n}$, then

$$\begin{aligned} \eta \circ (1_{R'} \otimes_R (\sigma_{R,S}))\,(r' \otimes z_i) &= \eta\,(r' \otimes z_i \chi_R(a_i)) \\ &= r'(1 \otimes z_i \chi_R(a_i)) \\ &= r'(1 \otimes z_i)(1 \otimes \chi_R(a_i)) \\ &= r'(1 \otimes z_i)(\chi_{R'}(1 \otimes a_i)) \\ &= \sigma_{R', R' \otimes S}(r'(1 \otimes z_i)) \\ &= \sigma_{R', R' \otimes S} \circ \zeta(r' \otimes z_i) \quad \square \end{aligned}$$

As in the proof of theorems 4.3.9 and 4.5.1, to prove the claim in the case of fields, we relate the surjectivity of σ under Frobenius deformation:

Lemma 4.5.4. *Let F and G be Frobenius algebras over $\Bbbk$ such that $F \overset{def}{\dashrightarrow} G$. If $\sigma_{\Bbbk,F}$ is surjective, then so is $\sigma_{\Bbbk,G}$*

Proof. Let D be the algebra deforming F to G provided by the definition 4.2.5 and write $R \overset{\text{def}}{=} \Bbbk[[u]]$ and $\mathbb{K} \overset{\text{def}}{=} \Bbbk((u))$.
Then lemma 4.4.12 and lemma 4.3.1 imply that the vertical maps in (4.6) are isomorphisms, hence $\Bbbk \otimes_R \sigma_{R,D} = \sigma_{\Bbbk,F}$. Thus Nakayama's lemma implies that $\sigma_{R,D}$ is surjective whenever $\sigma_{\Bbbk,F}$ is. A second application of (4.6) together with lemma 4.3.1 and theorem 4.4.3 shows that $\mathbb{K} \otimes_{\Bbbk} \sigma_{\Bbbk,G} = \mathbb{K} \otimes_R \sigma_{R,D}$, showing that $\mathbb{K} \otimes_{\Bbbk} \sigma_{\Bbbk,G}$ is surjective in this case and hence also $\sigma_{\Bbbk,G}$ since $\mathbb{K}$ is faithfully flat over $\Bbbk$. □

Lemma 4.5.5. *Let $F \overset{def}{=} \Bbbk[s,t]/(s^2,t^2)$. Then the map $\sigma_{\Bbbk,F}$ is surjective*

Proof. This is proven in $A.3$. □

Corollary 4.5.6. *Let F be Frobenius over a field $\Bbbk$. Then $\sigma_{\Bbbk,F}$ is surjective*

Proof. If $\Bbbk$ is algebraically closed, then any Frobenius algebra F over $\Bbbk$ can be obtained from $\Bbbk[s,t]/(s^2,t^2)$ by a finite number of Frobenius deformations following the diagram 4.3. Hence this case follows immediately from lemma 4.5.5 and lemma 4.5.4.
For a general field we use that $\overline{\Bbbk}$ is faithfully flat over $\Bbbk$. □

Proof of theorem 4.5.2. If R is a local ring, then $\Bbbk \otimes_R \sigma_{R,S} \cong \sigma_{\Bbbk,\Bbbk\otimes_R S}$ and the result follows by the above and an application of Nakayama's lemma.
If R is any ring, for any $\mathfrak{p} \in \operatorname{Spec}(R)$, we have $R_\mathfrak{p} \otimes_R \sigma_{R,S} = \sigma_{R_\mathfrak{p},R_\mathfrak{p}\otimes_R S}$, which is a surjective morphism. As this holds for all $\mathfrak{p} \in \operatorname{Spec}(R)$, $\sigma_{R,S}$ is itself surjective.

□

4.6 The Global Dimension of $\Pi_R(S)$

In this section we prove the following:

Theorem 4.6.1. *The global dimension of* $\Pi_R(S)$ *is bounded by the number*

$$\max\Big(\mathrm{gl.dim}(R), \mathrm{gl.dim}(S)\Big) + 2$$

We first bound the projective dimension of R and S as $\Pi_R(S)$-modules.

Lemma 4.6.2. *There is a projective resolution of* $R \oplus S$ *of the following form:*

$$0 \longrightarrow \Pi_R(S)(-2) \xrightarrow{\alpha_2} ({}_S S_R \oplus {}_R S_S) \otimes \Pi_R(S)(-1) \xrightarrow{\alpha_1} \Pi_R(S) \xrightarrow{\alpha_0} R \oplus S \longrightarrow 0 \tag{4.7}$$

Proof. α_0 is the canonical projection with kernel $\Pi_R(S)_{\geq 1}$. This module is generated by $\Pi_R(S)_1 = {}_S S_R \oplus {}_R S_S$, hence $\mathrm{im}(\alpha_1) = \ker(\alpha_0)$. Since the relations of $\Pi_R(S)$ are generated in degree 2, we also have $\mathrm{im}(\alpha_2) = \ker(\alpha_1)$. Only the injectivity of α_2 remains to be checked.
The sequence splits into the following two subsequences:

$$0 \longrightarrow 1_R \cdot \Pi_R(S)(-2) \longrightarrow 1_S \cdot \Pi_R(S)(-1) \longrightarrow 1_R \cdot \Pi_R(S) \longrightarrow R \longrightarrow 0 \tag{4.8}$$

$$0 \longrightarrow 1_S \cdot \Pi_R(S)(-2) \longrightarrow (1_R \cdot \Pi_R(S)(-1))^{\oplus 4} \longrightarrow 1_S \cdot \Pi_R(S) \longrightarrow S \longrightarrow 0 \tag{4.9}$$

By lemma 4.3.1 exactness can be checked after localization at each prime ideal of R, hence we may assume all terms in (4.8) and (4.9) are free R-modules of finite rank in each degree by lemma 4.3.10. The claim reduces to the following relation on the Hilbert series: for each $d \in \mathbb{N}$ we must have

$$\begin{aligned} h_{d-2}(1_R \cdot \Pi_R(S)(-2)) - h_{d-1}(1_S \cdot \Pi_R(S)(-1)) + h_d(1_R \cdot \Pi_R(S)) - \delta_{d0} &= 0 \\ h_{d-2}(1_S \cdot \Pi_R(S)(-2)) - 4h_{d-1}(1_R \cdot \Pi_R(S)(-1)) + h_d(1_R \cdot \Pi_R(S)) - 4\delta_{d0} &= 0 \end{aligned}$$

(where $h_d(-)$ denotes the rank of the degree d-part as an R-module)
Using lemma 4.3.10 we see that this is indeed the case. ☐

Lemma 4.6.3. *Each simple* $\Pi_R(S)$*-module is either a simple* R*-module or a simple* S*-module.*

Proof. Each simple R or S-module is clearly simple when considered as a $\Pi_R(S)$-module. Conversely if M is a simple $\Pi_R(S)$-module, then $M = 1_R M$ or $M = 1_S M$ since $M = 1_R M \oplus 1_S M$. Moreover we claim that $\Pi_R(S)_{\geq 1} M = 0$ or equivalently $\Pi_R(S)_1 M = 0$. For this assume for example that $M = 1_R M$. If $x \in {}_S S_R$ then

$$xM = (1_S x)M = 1_S(xM) = 0$$

and if $x \in {}_R S_S$ then

$$xM = (x 1_S)M = x(1_S M) = 0$$

Hence only the R-component in degree 0 acts non-trivially on M, it follows in particular that M is also a simple R-module. The case $M = 1_S M$ is completely similar. $\square$

Proof of theorem 4.6.1. It suffices to check that if M is a simple $\Pi_R(S)$-module then:

$$\mathrm{pd}_{\Pi_R(S)}(M) \leq \max\Big(\mathrm{gl.dim}(R), \mathrm{gl.dim}(S)\Big) + 2$$

By lemma 4.6.3, M is a simple R-module or a simple S-module. We assume the former, the other case being completely similar. Let $P_\bullet \longrightarrow M$ be a resolution of M by projective R-modules of length $\mathrm{pd}_R(M) \leq \mathrm{gl.dim}(R)$. Then for each i, by lemma 4.6.2 we have

$$\mathrm{pd}_{\Pi_R(S)}(P_i) \leq \mathrm{pd}_{\Pi_R(S)}(R) \leq \mathrm{pd}_{\Pi_R(S)}(R \oplus S) \leq 2$$

A standard long exact sequence-argument now gives the desired result. $\square$

4.7 Explicit Computations for $S = \dfrac{\Bbbk[s,t]}{(s^2,t^2)}$

We describe $\Pi_\Bbbk(S)$ through generators and relations:

- $\Pi_\Bbbk(S)_0 = k \oplus S$. Let a denote $(1_\Bbbk, 0)$ and $b = (0, 1_S)$ then since $a + b = 1$, $a, 1, s, t, st$ is a $\Bbbk$-basis for $\Pi_\Bbbk(S)_0$. It is clear that this set satisfies the relations

$$a^2 = a, as = sa = at = ta = 0$$

- $\Pi_{\Bbbk}(S)_1 = {}_{\Bbbk}S_S \oplus {}_SS_{\Bbbk}$. Let f be $(1_S, 0)$ and $e = (0, 1_S)$, then we can write $\Pi_{\Bbbk}(S)_1 = fS \oplus Se$. Hence $f, fs, ft, fst, e, se, te, ste$ is a $\Bbbk$-basis for $\Pi_{\Bbbk}(S)_1$. By construction, each generator $\neq 1$ of $\Pi_{\Bbbk}(S)_0$ acts nontrivially on exactly one side of each component. Hence we have the relations

  $$ea = e, af = f, ae = fa = 0, es = et = sf = tf = 0$$

  Note that this implies $e^2 = f^2 = 0$ since for example

  $$e^2 = (ea)e = e(ae) = 0$$

- It is clear that the relation $1 \otimes 1 \in {}_{\Bbbk}S_S \otimes {}_SS_{\Bbbk}$ takes the form $fe = 0$. To compute the second relation, note that projection onto $\Bbbk st$ provides the duality isomorphism $\mathrm{Hom}_R(S, R) \cong S$ (see lemma 4.2.7). It immediately follows that (e, se, te, ste) is dual to (fst, ft, fs, f) in the sense of definition 0.1.15. The relation now takes the form

  $$efst + seft + tefs + stef = 0 \tag{4.10}$$

To summarize $\Pi_{\Bbbk}(S)$ is a quotient of the free algebra $\Bbbk < a, s, t, e, f >$ by the relations

$$\begin{cases} s^2 = t^2 = st - ts = 0 \\ a^2 = a, as = sa = at = ta = 0 \\ ea = e, af = f, ae = fa = 0, es = et = sf = tf = 0 \\ fe = efst + seft + tefs + stef = 0 \end{cases}$$

Note that $\Pi_{\Bbbk}(S)$ is a graded algebra via $\deg(a) = \deg(s) = \deg(t) = 0$ and $\deg(e) = \deg(f) = 1$.

4.7.1 an Explicit Set of Generators for $\Pi_{\Bbbk}(S)$

In this subsection we give sets of generators in each degree, hence giving an upper bound for $\dim_{\Bbbk}(\Pi_{\Bbbk}(S)_d)$. More explicitly we prove that

$$\dim_{\Bbbk}\left(\Pi_{\Bbbk}\left(\frac{k[s,t]}{(s^2,t^2)}\right)_d\right) \leq \begin{cases} 5(d+1) & \text{if } d \text{ is even} \\ 4(d+1) & \text{if } d \text{ is odd} \end{cases}$$

which proves 4.3.4 For this we make the following remarks:

- In each degree there are generators of two types:

Type I) Elements of the form $f * ef * ef \ldots * ef * e(f(*))$ where each $*$ is either s, t or st

Type II) Elements of the form $(*)ef * ef \ldots * ef * e(f(*))$ where each $*$ is either s, t or st

- Let $\mathcal{R}$ denote the relation (4.10), then $f\mathcal{R}e, t\mathcal{R}, s\mathcal{R}, st\mathcal{R}$ take the form

$$fsefte = -ftefse \tag{4.11}$$

$$steft = -tefst \tag{4.12}$$

$$stefs = -sefst \tag{4.13}$$

$$stefst = 0 \tag{4.14}$$

- As a consequence of the above equalities, we know that for any non-zero element there is at most one appearance of st. For example:

$$f\underline{st}efsef\underline{st} = fstef(sefst) = -fstef(stefs) = -f(stefst)efs = 0$$

We say any of the above elements is of bidegree (m, n) if there are m appearances of s and n appearances of t. It is easy to see that the above relations do not violate this bidegree and that it turns $\Pi_{\Bbbk}(S)$ into a $\mathbb{Z} \times \mathbb{Z}$-graded ring. Using the above remarks we create (minimal) sets of generators by a case-by-case study:

- Case 1: d even and Type I

  All words in this case take the form $(f*e)\ldots(f*e)$. We can use relations (4.11), (4.12), (4.13) to write the element in the form $\pm(fse)^i(fste)^\varepsilon(fte)^j$ where $\varepsilon = 0, 1$. For $\varepsilon = 0$ we have $\frac{d}{2}+1$ choices for i and j and for $\varepsilon = 1$ we have $\frac{d}{2}$ choices, giving a total of $d+1$ generators.

- Case 2: d even and Type II

  These are elements of the form $(*)(ef*)\dots(ef*)ef(*)$ and since there is at most one occurrence of st the bidegree satisfies $\frac{d}{2}-1 \leq m+n \leq \frac{d}{2}+2$. If $m+n = \frac{d}{2}-1$ the element can be written in the form $\pm(efs)^m(eft)^n ef$, giving $\frac{d}{2}$ choices. Similarly if $m+n = \frac{d}{2}+2$ the element can be written in the form $\pm(sef)^{m-1}st(eft)^{n-1}$. Giving $\frac{d}{2}+1$ choices.

  Assume $m+n = \frac{d}{2}$. If $(n,m) = (\frac{d}{2},0)$ (or $(n,m) = (0,\frac{d}{2})$) we have 2 generators: $(sef)^{\frac{d}{2}}$ and $(efs)^{\frac{d}{2}}$ (or $(tef)^{\frac{d}{2}}$ and $(eft)^{\frac{d}{2}}$).

  In all other cases we need 3 generators: $(sef)^m(tef)^n$, $(efs)^m(eft)^n$ and $(efs)^{m-1}efstef(tef)^{n-1}$. This gives a total of $\frac{3d}{2}+1$ generators for this sub-case.

  Finally assume $m+n = \frac{d}{2}+1$. If $(m,n) = (\frac{d}{2}+1,0)$ (or $(m,n) = (0,\frac{d}{2}+1)$) we have 1 generator: $(sef)^{\frac{d}{2}}s$ (or $(tef)^{\frac{d}{2}}t$).

  In all other cases we need 3 generators: $(sef)^m(tef)^{n-1}t$, $(efs)^{m-1}efst(eft)^{n-1}$ and $(sef)^{m-1}stef(tef)^{n-1}$. This gives a total of $\frac{3d}{2}+1$ generators for this sub-case.

  For case 2 this results in $\dfrac{d}{2}+\left(\dfrac{3d}{2}+1\right)+\left(\dfrac{3d}{2}+2\right)+\left(\dfrac{d}{2}+1\right) = 4(d+1)$ generators.

  Finally adding up the number of generators from Case 1 and Case 2 yields $5(d+1)$ generators.

- Case 3: d odd and Type I

  All elements in this case take the form $(f*e)(f*e)\dots(f*e)f(*)$. By a completely similar argument as above, we conclude that generators can be chosen of the following forms:

  $(fse)^m(fte)^n f$, $(fse)^m(fte)^{n-1}ft$, $(fse)^{\frac{d-1}{2}}fs$, $(fse)^{n-1}(fte)^{m-1}fst$ and $fste(fse)^{n-1}(fte)^{m-1}f$. This gives a total of

  $$\frac{d+1}{2}+\frac{d+1}{2}+1+\frac{d+1}{2}+\frac{d-1}{2} = 2(d+1)$$

  generators

- Case 4: d odd and Type II

Elements in this case are of the form $(*)e(f*e)(f*e)\dots(f*e)$. Note that any such word can be obtained by taking a word from Case 3, reading it from right to left and interchanging e and f. Applying this "procedure" to the generators of Case 3 yields a set of generators for the current case by symmetry. Hence in the current case we have $2(d+1)$ generators, adding up to $4(d+1)$ generators in case d is odd.

4.7.2 the Dimension of $Z_4(\Bbbk, S)$

Consider the elements

$$u \overset{\text{def}}{=} sef + efs + fse \text{ and } v \overset{\text{def}}{=} tef + eft + fte \tag{4.15}$$

It is easy to see that u is normalizing with respect to the automorphism σ on $\Pi_{\Bbbk}(S)$ which sends t to $-t$ and is the identity on the other generators. As $\sigma^2 = \mathrm{Id}$ we have as an immediate consequence that u^2 is central. A completely similar discussion yields that v^2 is central. Using the relations defining $\Pi_{\Bbbk}(S)$ we can write u^2 and v^2 as

$$sefsef + efsefs + fsefse \text{ and } teftef + efteft + ftefte$$

In what follows we explain why there are no other central elements. This is done by constructing a basis for $\Pi_{\Bbbk}(S)_4$ which reduces the search for central elements to some standard linear algebraic computations. By the arguments in 4.7.1, the following 25 elements generate $\Pi_{\Bbbk}(S)_4$:

Type I)
- 1 element of bidegree $(1,1)$: $fsefte$
- 1 element of bidegree $(2,0)$: $fsefse$
- 1 element of bidegree $(0,2)$: $ftefte$
- 1 element of bidegree $(2,1)$: $fsefste$
- 1 element of bidegree $(1,2)$: $fstefte$

Type II)
- 1 element of bidegree $(1,0)$: $efsef$

– 1 element of bidegree $(0, 1)$: $eftef$
– 2 elements of bidegree $(2, 0)$: $sefsef, efsefs$
– 2 elements of bidegree $(0, 2)$: $teftef, efteft$
– 1 element of bidegree $(3, 0)$: $sefsefs$
– 1 element of bidegree $(0, 3)$: $tefteft$
– 3 elements of bidegree $(1, 1)$: $seftef, efseft, efstef$
– 3 elements of bidegree $(2, 1)$: $sefseft, efsefst, sefstef$
– 3 elements of bidegree $(1, 2)$: $sefteft, efsteft, steftef$
– 1 element of bidegree $(2, 2)$: $sefsteft$
– 1 element of bidegree $(3, 1)$: $sefsefst$
– 1 element of bidegree $(1, 3)$: $stefteft$

Corollary 4.3.6 implies that they form a $\mathbb{k}$-basis for $\Pi_{\mathbb{k}}(S)_4$.

Since the center of a graded ring is a homogeneous subring, we can write $Z_d(\mathbb{k}, S)$ as

$$Z_d(\mathbb{k}, S) = \bigoplus_{(m,n)} Z_d(\mathbb{k}, S)_{m,n}$$

Where $Z_d(k, S)_{m,n}$ consists of the central elements in $\Pi_{\mathbb{k}}(S)$ of degree d and bidegree (m, n). It follows that generators for $Z_4(\mathbb{k}, S)$ can be chosen as linear combinations of elements of fixed bidegree. This reduces the computations to 12 linear combinations of at most 4 elements. A brute force computation shows that $sefsef + efsefs + fsefse$ and $teftef + efteft + ftefte$ are the only linear combinations that are central.

4.7.3 the Surjectivity of $\sigma_{\mathbb{k},S}$

Let u and v be the normalizing elements defined above in 4.15. Let $V \subset \Pi_{\mathbb{k}}(S)_2$ be the $\mathbb{k}$-vector space spanned by u and v. Let μ_3 be the multiplication morphism given by the composition

$$\mu_3 : V \otimes \Pi_{\mathbb{k}}(S)_1 \longrightarrow \Pi_{\mathbb{k}}(S)_2 \otimes \Pi_{\mathbb{k}}(S)_1 \longrightarrow \Pi_{\mathbb{k}}(S)_3$$

Then we use a brute force computation to show that μ_3 must be surjective. I.e. we show that any element of $\Pi_{\Bbbk}(S)_3$ can be written as a linear combination of elements of the form $u \cdot x$ or $v \cdot x$ with $x \in \Pi_{\Bbbk}(S)_1$. It suffices to check this for the generators of $\Pi_{\Bbbk}(S)_3$:

Type I) : elements of the form $f * ef(*)$.
These can all be put into the form $fsef(*)$ or $ftef(*)$ where $*$ is either s, t or st. Now use $fsef(*) = u \cdot f(*)$ and similarly $ftef(*) = v \cdot f(*)$.

Type II) : elements of the form $(*)ef * e$

- $efse = u \cdot e$ and $efte = v \cdot e$
- $sefse = u \cdot se$ and $tefte = v \cdot te$
- $sefste = u \cdot ste$ and $tefste = v \cdot ste$
- $sefte = -tefse - eftse = v \cdot (-se)$
- $tefse = -sefte - efste = u \cdot (-te)$

Which shows that μ_3 is indeed surjective.

Now for each degree d we have a commutative diagram

$$\begin{tikzcd}
V \otimes \Pi_{\Bbbk}(S)_{d+1} \arrow[r] & \Pi_{\Bbbk}(S)_{d+3} \\
V \otimes \Pi_{\Bbbk}(S)_1 \otimes \Pi_{\Bbbk}(S)_d \arrow[u] \arrow[r, "\mu_3 \otimes \Pi_{\Bbbk}(S)_d"'] & \Pi_{\Bbbk}(S)_3 \otimes \Pi_{\Bbbk}(S)_d \arrow[u]
\end{tikzcd}$$

where the top horizontal arrow must be a surjection as the other three are surjective. Hence by induction (and the fact that $V \otimes -$ is right exact) we have for each $n \in \mathbb{N}$ a surjection

$$\mu_{2n+\epsilon} : V^{\otimes n} \otimes \Pi_{\Bbbk}(S)_\epsilon \longrightarrow V^{\otimes n-1} \otimes \Pi_{\Bbbk}(S)_{2+\epsilon} \longrightarrow \ldots \longrightarrow \Pi_{\Bbbk}(S)_{2n+\epsilon}$$

Next let W be the vector space spanned by u^2 and v^2, then for each n and $\omega = 1, 2$ there is a surjection

$$W^{\otimes n} \otimes V^{\otimes \omega} \twoheadrightarrow V^{\otimes 2n+\omega}$$

and we have a commutative diagram

$$\begin{tikzcd}
W^{\otimes n} \otimes V^{\otimes \omega} \otimes \Pi_{\Bbbk}(S)_\epsilon \arrow[r] \arrow[d] & W^{\otimes n} \otimes \Pi_{\Bbbk}(S)_{2\omega+\epsilon} \arrow[d, "\rho_{4n+2\omega+\epsilon}"] \\
V^{\otimes 2n+\omega} \otimes \Pi_{\Bbbk}(S)_\epsilon \otimes \Pi_{\Bbbk}(S)_d \arrow[r, "\mu_{4n+2\omega+\varepsilon}"'] & \Pi_{\Bbbk}(S)_{4n+2\omega+\varepsilon}
\end{tikzcd}$$

where $\rho_{4n+2\omega+\epsilon}$ must be surjective because the other three morphisms are. Then using the commutative triangle

$$\begin{tikzcd}
W^{\otimes n} \otimes \Pi_{\Bbbk}(S)_{2\omega+\epsilon} \arrow[rr, "\rho_{4n+2\omega+\epsilon}"] \arrow[dr] & & \Pi_{\Bbbk}(S)_{4n+2\omega+\epsilon} \\
& \Bbbk[Z_4(\Bbbk, S)]_n \otimes \Pi_{\Bbbk}(S)_{2\omega+\epsilon} \arrow[ur, "\overline{\rho_{4n+2\omega+\epsilon}}"'] &
\end{tikzcd}$$

we must have that $\overline{\rho_{4n+2\omega+\epsilon}} : \Bbbk[Z_4(\Bbbk, S)]_n \otimes \Pi_{\Bbbk}(S)_{2\omega+\epsilon} \longrightarrow \Pi_{\Bbbk}(S)_{4n+2\omega+\epsilon}$ must be surjective. As $2\omega + \epsilon$ takes the values 3,4,5,6 we have an induced surjection:

$$\overline{\rho} : \Bbbk[Z_4(k, S)] \otimes \Pi_{\Bbbk}(S)_{\leq 6} \twoheadrightarrow \Pi_{\Bbbk}(S)$$

(where we included $\Pi_{\Bbbk}(S)_d$ for $d = 0, 1, 2$ on the left hand side to guarantee surjectivity in these three lowest degrees).

Now $\sigma_{\Bbbk,S}$ factors as $\overline{\rho} \circ \varsigma$ where ς is the morphism:

$$\varsigma : \Bbbk[Z_4(\Bbbk, S)]^{\oplus N} \longrightarrow \Bbbk[Z_4(\Bbbk, S)] \otimes \Pi_{\Bbbk}(S)_{\leq 6} : (z_i)_{i=1}^N \mapsto \sum_{i=1}^N z_i \otimes \chi(a_i)$$

By the choice of the a_i in H, ς is surjective and hence also $\sigma_{\Bbbk,S}$ proving the lemma.

Chapter 5

the Exotic Rank 4 Surface

5.1 Introduction and Statement of Results

One of the main results of chapter 3 (3.5.17) states that any lattice with an exceptional basis of rank 4 of DPS* type must either be trivial, isomorphic to the Grothendieck group of a Del Pezzo surface or to a lattice of so called 'type 3' given as $\mathbb{Z}^4$, equipped with a bilinear from whose Gram matrix is given by

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This chapter is devoted to a construction made by the author in collaboration with D. Presotto that can be viewed as a noncommutative surface Z together with a full exceptional sequence E of four Z-modules whose classes in the Grothendieck group $K(Z)$ form a basis in which the Euler form has the above Gram matrix. The top-left and bottom-right 2×2 submatrices, show that the couples (E_1, E_2) and (E_3, E_4) are isomorphic to the standard sequence $(\mathcal{O}_X, \mathcal{O}_X(1))$ on $K(\mathbb{P}^1)$. We heuristically conclude that Z should be equipped with 2 'maps' (in the noncommutative sense) $\Pi_0, \Pi_1 : Z \longrightarrow \mathbb{P}^1$ such that E

is obtained by pulling back $(\mathcal{O}_X, \mathcal{O}_X(1))$ along both:

$$E = \left(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1))\right) \tag{5.1}$$

The construction of this *exotic rank 4 surface* is an adaptation of Van den Bergh's theory of noncommutative $\mathbb{P}^1$-bundles over a smooth finite type $\Bbbk$-scheme X as developed in [VdB12]. In that paper, Van den Bergh proposes a new construction which results in a sheafified notion of a $\mathbb{Z}$-algebra as follows: let $\mathcal{E}$ be a *coherent* X-bimodule (see 5.2.2) which is locally free on both sides. Then there is an appropriate notion of left- and right dual ${}^*\mathcal{E}$ resp. $\mathcal{E}^*$ (5.2.9). Applying the construction indefinitely yields higher duals ${}^{*m}\mathcal{E}$ resp. $\mathcal{E}^{*m}$, which by naturality come with a counit morphism

$$\gamma_m : \mathcal{O}_\Delta \longrightarrow \mathcal{E}^{*m} \otimes_X \mathcal{E}^{*m+1}$$

Van den Bergh defines the *symmetric sheaf* $\mathbb{Z}$*-algebra* $\mathbb{S}(\mathcal{E})$ as

- $\mathbb{S}(\mathcal{E})_{m,m} = \mathcal{O}_X$
- $\mathbb{S}(\mathcal{E})_{m,m+1} = \mathcal{E}^{*m}$
- $\mathbb{S}(\mathcal{E})$ is freely generated by $\mathbb{S}(\mathcal{E})$ subject to the relations given by the images of the morphism γ_m

This is recalled in 5.2.4. There is an associated category of graded modules $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ which is Grothendieck (5.2.8). The intuition behind the definition of $\mathbb{S}(\mathcal{E})$ comes from the fact that in the case where $\mathcal{E}$ is a central bimodule of rank $(2,2)$, the definition coincides with the notion of a $\mathbb{P}^1$-bundle over X in the sense that there is an equivalence between their categories of graded modules. For the convenience of the reader, we provide an explicit proof of this in 5.2.20.

Pushing our heuristic intuition further, if we assume that the exceptional sequence E is strong (as in definition 0.2.1), we have

$$\mathrm{Hom}_Z(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1)) = \langle \Pi_1^*\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1)\rangle = 4$$

which seems to suggest that the bimodule $\mathcal{E}$ should have rank 4 on the left and a similar argument shows that rE should have rank 1 on the right. This leads one to adapt Van den Bergh's work under the assumption that the bimodule $\mathcal{E}$ is locally free of rank $(4, 1)$ instead. To construct the noncommutative scheme $\mathrm{Proj}(\mathbb{S}(\mathcal{E}))$ and establish its properties, we shall first prove two facts in the setting: The first is a description of $\mathbb{S}(\mathcal{E})$ in the case where the base scheme X is affine. More precisely, we relate $\mathbb{S}(\mathcal{E})$ to the generalized preprojective algebras introduced in the previous chapter as follows:

Theorem J. *(see 5.3.10 and 5.3.15 together with 5.3.18) Let $\mathcal{E}$ be a bimodule of rank $(4, 1)$ over X. Then there is a finite affine open cover $U_i \subset X$ such that the category $\mathrm{Gr}(\mathbb{S}(\mathcal{E})|_{U_i})$ identifies with a direct summand of $\mathrm{Gr}(\Pi_{R_i}(S_i))$ where $R_i \longrightarrow S_i$ is relatively Frobenius of rank 4*

Second, we adapt the technique of point modules which was developed in [VdB12] for the rank $(2, 2)$ case to the rank $(4, 1)$ case. This proves to be a substantial modification, requiring an adaptation of the very definition of point module. We use this theory to prove that $\mathbb{S}(\mathcal{E})_{n,m}$ is a locally free bimodule in each degree. This, together with the previous result allows us to adapt the ideas of [Mor07] and [Nym04a] to obtain a proof of the fact that $\mathrm{Gr}(\mathbb{S}(\mathcal{E})$ is a locally noetherian category. We summarize:

Theorem K. *(see 5.3.1 and 5.4.5) Let $\mathcal{E}$ be a bimodule of rank $(4, 1)$ over X. Then*

- $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ *is a noetherian category*
- *Each bimodule $\mathbb{S}(\mathcal{E})_{n,m}$ is locally free and the ranks can be explicitly computed by (5.4.5).*

This allows us to consider the noncommutative scheme $Z = \mathrm{Proj}(\mathbb{S}(\mathcal{E})$ in the language of [AZ94]. Z comes with a sequence of maps $\Pi_n : Z \longrightarrow X$ (again in the sense of [AZ94]) given by taking the n-th degree of the graded module. We describe these and show that Π_0 and Π_1 contain all the information on these maps in a certain sense. With these definitions, we finally prove

Theorem L. *(See 5.4.22) Let $\mathcal{E}$ be a $\mathbb{P}^1$-bimodule of rank $(4,1)$. Let $\mathbb{S}(\mathcal{E}))$ be the associated symmetric sheaf $\mathbb{Z}$-algebra and put $Z = \mathrm{Proj}(\mathbb{S}(\mathcal{E})$. Then*

$$\left(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1))\right)$$

is a full strong exceptional sequence on Z for which the Gram matrix of the Euler form is given by

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.2 Symmetric Sheaf $\mathbb{Z}$-Algebras

5.2.1 Definitions and Construction

We begin by giving a summary of the material needed to construct symmetric sheaf $\mathbb{Z}$-algebras following the article [VdB12].

Convention 5.2.1. *In this chapter $\Bbbk$ denotes an algebraically closed field. W, X and Y will denote smooth varieties (that is smooth, integral*[1]*, separated and of finite type over $\Bbbk$).*

Definition 5.2.2. A coherent $X-Y$ bimodule $\mathcal{E}$ is a coherent $\mathcal{O}_{X\times Y}$-module such that the support of $\mathcal{E}$ is finite over X and Y. We denote the corresponding category by $\mathrm{bimod}(X-Y)$. More generally an $X-Y$-bimodule is a quasi-coherent $\mathcal{O}_{X\times Y}$-module which is a filtered direct limit of objects in $\mathrm{bimod}(X-Y)$. The category of $X-Y$-bimodules is denoted $\mathrm{BiMod}(X-Y)$. A bimodule $\mathcal{E}$ is called locally free if $\pi_{X*}(\mathcal{E})$ and $\pi_{Y*}(\mathcal{E})$ are locally free, where π_X denotes the standard projection $X\times Y \longrightarrow X$ and π_Y is defined similarly. If $\mathrm{rk}(\pi_{X*}(\mathcal{E})) = m$ and $\mathrm{rk}(\pi_{Y*}(\mathcal{E})) = n$, we write $\mathrm{rk}\,\mathcal{E} = (m,n)$.

[1]One could leave out this condition, which leads to the more general setting of disjoint unions of varieties, we choose not to do this for purposes of clarity

For W, X and Y the tensor product of $\mathcal{O}_{W\times X\times Y}$-modules induces a tensor product

$$\mathrm{BiMod}(W-X)\otimes\mathrm{BiMod}(X-Y)\longrightarrow\mathrm{BiMod}(W-Y):(\mathcal{E},\mathcal{F})\mapsto\mathcal{E}\otimes_X\mathcal{F}$$

through the formula

$$\mathcal{E}\otimes\mathcal{F}\stackrel{\text{def}}{=}\pi_{W\times Y*}\big(\pi^*_{W\times X}(\mathcal{E})\otimes_{W\times X\times Y}\pi^*_{X\times Y}(\mathcal{F})\big)$$

For each $\mathcal{E}\in\mathrm{BiMod}(W-X)$ this defines a functor :

$$-\otimes_X\mathcal{E}:\mathrm{Qcoh}(W)\longrightarrow\mathrm{Qcoh}(X):\mathcal{M}\mapsto\mathcal{M}\otimes_X\mathcal{E}\stackrel{\text{def}}{=}\pi_{X*}\big(\pi^*_W(\mathcal{M})\otimes_{W\times X}\mathcal{E}\big) \tag{5.2}$$

which is right exact in general and exact if $\mathcal{E}$ is locally free on the left. We mention that [VdB12, lemma 3.1.1.] shows that this functor determines the bimodule $\mathcal{E}$ uniquely.

Definition 5.2.3. Let W be a variety with morphisms $u:W\longrightarrow X$, $v:W\longrightarrow Y$. If $\mathcal{U}\in\mathrm{Qcoh}(W)$, then we denote $(u,v)_*\mathcal{U}\in\mathrm{BiMod}(X-Y)$ as ${}_u\mathcal{U}_v$. One easily checks:

$$-\otimes{}_u\mathcal{U}_v=v_*(u^*(-)\otimes_W\mathcal{U}) \tag{5.3}$$

A bimodule isomorphic to one of the form ${}_u\mathcal{U}_u\cong{}_{\mathrm{Id}}(u_*\mathcal{U})_{\mathrm{Id}}$ is called *central.*

This chapter will use the language of sheaf $\mathbb{Z}$-algebras, a 'sheafified version' of a classical $\mathbb{Z}$-algebra.

Definition 5.2.4. Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of smooth varieties.
A sheaf $\mathbb{Z}$-algebra $\mathcal{A}$ is a collection of X_i-X_j-bimodules $\mathcal{A}_{ij}$ equipped with multiplication maps $\mu_{i,j,k}$ and identity maps u_i:

$$\mu_{i,j,k}:\mathcal{A}_{i,j}\otimes\mathcal{A}_{j,k}\longrightarrow\mathcal{A}_{ik}\text{ and }u_i:\mathcal{O}_{X_i}\longrightarrow\mathcal{A}_{i,i}$$

such that the usual associativity

$$\begin{tikzcd}
\mathcal{A}_{i,j}\otimes\mathcal{A}_{j,k}\otimes A_{k,l} \arrow[r, "\mu_{i,j,k}\otimes 1"] \arrow[d, "1\otimes\mu_{j,k,l}"'] & \mathcal{A}_{i,k}\otimes\mathcal{A}_{k,l} \arrow[d, "\mu_{i,k,l}"] \\
\mathcal{A}_{i,j}\otimes\mathcal{A}_{j,l} \arrow[r, "\mu_{i,j,l}"'] & \mathcal{A}_{i,l}
\end{tikzcd}$$

and unit diagram

$$\begin{tikzcd} \mathcal{O}_{X_i} \otimes \mathcal{A}_{i,j} \arrow[rr, "u_i \otimes 1"] \arrow[dr] & & \mathcal{A}_{i,i} \otimes \mathcal{A}_{i,j} \arrow[dl, "\mu_{i,i,j}"] \\ & \mathcal{A}_{i,j} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_{i,j} \otimes \mathcal{O}_{X_i} \arrow[rr, "1 \otimes u_i"] \arrow[dr] & & \mathcal{A}_{i,j} \otimes \mathcal{A}_{j,j} \arrow[dl, "\mu_{i,j,j}"] \\ & \mathcal{A}_{i,j} & \end{tikzcd}$$

commute

In a similar vain, we introduce the notion of graded module over a sheaf $\mathbb{Z}$-algebra:

Definition 5.2.5. Let $\mathcal{A}$ be a sheaf $\mathbb{Z}$-algebra.
A graded $\mathcal{A}$-module is a sequence of X_i-modules $\mathcal{M}_i$ together with maps

$$\mu_{i,j} : \mathcal{M}_i \otimes \mathcal{A}_{ij} \longrightarrow \mathcal{M}_j$$

such that the associativity,

$$\begin{tikzcd} \mathcal{M}_i \otimes \mathcal{A}_{i,j} \otimes A_{j,k} \arrow[r, "\mu_{i,j,k} \otimes 1"] \arrow[d, "1 \otimes \mu_{j,k,l}"'] & \mathcal{M}_j \otimes \mathcal{A}_{j,k} \arrow[d, "\mu_{i,k,l}"] \\ \mathcal{M}_j \otimes \mathcal{A}_{j,k} \arrow[r, "\mu_{i,j,l}"'] & \mathcal{M}_k \end{tikzcd}$$

and unit diagram

$$\begin{tikzcd} \mathcal{O}_{X_i} \otimes \mathcal{M}_i \arrow[rr, "u_i \otimes 1"] \arrow[dr] & & \mathcal{A}_{i,i} \otimes \mathcal{M}_i \arrow[dl, "\mu_{i,i,j}"] \\ & \mathcal{A}_i & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}_i \otimes \mathcal{O}_{X_i} \arrow[rr, "1 \otimes u_i"] \arrow[dr] & & \mathcal{M}_i \otimes \mathcal{A}_{i,i} \arrow[dl, "\mu_{i,j,j}"] \\ & \mathcal{M}_i & \end{tikzcd}$$

commute. A morphism of graded $\mathcal{A}$-modules $f : \mathcal{M} \longrightarrow \mathcal{N}$ is a collection of X_i-module morphisms $f_i : \mathcal{M}_i \longrightarrow \mathcal{N}_i$ such that the diagram

$$\begin{tikzcd} \mathcal{M} \otimes \mathcal{A}_{i,j} \arrow[r, "f_i"] \arrow[d] & \mathcal{N} \otimes \mathcal{A}_{i,j} \arrow[d] \\ \mathcal{M}_j \arrow[r, "f_j"'] & \mathcal{N}_j \end{tikzcd}$$

commutes. The associated category is denoted $\mathrm{Gr}(\mathcal{A})$

Definition 5.2.6. An $\mathcal{A}$-module is right bounded if $\mathcal{M}_i = 0$ for $i \gg 0$. An $\mathcal{A}$-module is called torsion if it is a filtered colimit of right bounded modules. Let $\mathrm{Tors}(\mathcal{A})$ be the subcategory of $\mathrm{Gr}(\mathcal{A})$ consisting of torsion modules. Then if $\mathrm{Gr}(\mathcal{A})$ is a locally noetherian category [2], $\mathrm{Tors}(\mathcal{A})$ is a localizing subcategory and the corresponding quotient category is denoted by $\mathrm{Proj}(\mathcal{A})$.
This construction yields a projection functor $p : \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Proj}(\mathcal{A})$ with right adjoint ω (see [Smi99]).

Remark 5.2.7. *It an easy observation that* $\mathrm{Gr}(A)$ *is abelian and that all universal constructions are defined 'degreewise'*

The fundamental example of a graded right $\mathcal{A}$-module is given by the collection $e_n\mathcal{A}$ satisfying

$$\big(e_n\mathcal{A}\big)_i = \mathcal{A}_{n,i} \tag{5.4}$$

Theorem 5.2.8. *Let* $\mathcal{A}$ *be a sheaf* $\mathbb{Z}$*-algebra. Then* $\mathrm{Gr}(\mathcal{A})$ *is Grothendieck.*

Proof. Let $(\mathcal{M}_i, f_{ij})$ be a direct system of graded $\mathcal{A}$-modules. In each degree d, we obtain a direct system of quasi-coherent X_d-modules $(\mathcal{M}^d, f^d_{ij})$. Since $\mathrm{Qcoh}(X_n)$ is Grothendieck, we can form the direct limit in each degree to obtain a sequence of X_n-modules $\mathcal{L}_n \stackrel{\text{def}}{=} \varinjlim(\mathcal{M}^n_i, f^n_{ij})$. If we fix a couple (n, m), the universality of the direct limit naturally defines a map

$$\mathcal{A}_{n,m} \otimes \mathcal{L}_n = \mathcal{A}_{n,m} \otimes \varinjlim(X^n_i, f^n_{ij}) \longrightarrow \varinjlim(X^m_i, f^m_{ij}) = \mathcal{L}_m$$

showing that $\mathcal{L}$ is in fact a graded $\mathcal{A}$-module. The fact that $\mathcal{L}$ is a direct limit and that the formation of $\mathcal{L}$ is exact is an easy consequence of the construction. Next, for each, i let $\mathcal{G}^j_i$ be a collection of generators for $\mathrm{Qcoh}(X_i)$. Then the collection

$$\{\mathcal{G}^j_i \otimes e_i\mathcal{A} \mid n \in \mathbb{Z}, \mathcal{N} \in \mathcal{N}^n\} \tag{5.5}$$

forms a set of generators for $\mathrm{Gr}(\mathcal{A})$. $\square$

[2] As mentioned in the introduction, this property is nontrivial and in fact one of the main results of this chapter

The first crucial ingredient in our construction is a certain duality between locally free bimodules. The this end, we recall that we assume that5.2.1 that X and Y are smooth varieties over $\Bbbk$

Lemma 5.2.9. *(see [VdB12, §4]) Let $\mathcal{E} \in \mathrm{bimod}(X-Y)$ be a locally free coherent bimodule. Then there is a unique object -the right dual- $\mathcal{E}^* \in \mathrm{bimod}(Y-X)$ such that the functor*

$$- \otimes_Y \mathcal{E}^* : \mathrm{Qcoh}(Y) \longrightarrow \mathrm{Qcoh}(X)$$

defined in 5.2 is right adjoint to the functor $- \otimes_X \mathcal{E}$, i.e for $\mathcal{M} \in \mathrm{Qcoh}(X)$ and $\mathcal{N} \in \mathrm{Qcoh}(Y)$, there is a natural isomorphism:

$$\mathrm{Hom}_Y(\mathcal{M} \otimes \mathcal{E}, \mathcal{N}) \cong \mathrm{Hom}_X(\mathcal{M}, \mathcal{N} \otimes \mathcal{E}^*).$$

Remark 5.2.10. *Van den Bergh also gives an explicit formula (see the discussion following prop. 4.1.6 in [VdB12]): if $\mathcal{E} = {}_u\mathcal{U}_v$ then $\mathcal{E}^*$ is given by ${}_v\,\mathcal{H}om_W(\mathcal{U}, v^!\mathcal{O}_Y)_u$*

The opposite notion leads to the left dual: an object ${}^*\mathcal{E}$ such that

$$\mathrm{Hom}_X(\mathcal{N} \otimes {}^*\mathcal{E}, \mathcal{M}) \cong \mathrm{Hom}_Y(\mathcal{N}, \mathcal{M} \otimes \mathcal{E})$$

By Yoneda's lemma we have $\mathcal{E} = {}^*(\mathcal{E}^*) = ({}^*\mathcal{E})^*$. Repeated application of duals leads to the following notation as well:

$$\mathcal{E}^{*n} = \begin{cases} \mathcal{E}\overbrace{^{* \cdots *}}^{n} & n \geq 0 \\ \overbrace{^{* \cdots *}}^{-n}\mathcal{E} & n < 0 \end{cases}$$

In the sequel it will be convenient to invoke the following notation:

Convention 5.2.11. *For the smooth varieties X and Y, we shall often implicitly consider the sequence $(X_i)_{i\in\mathbb{Z}}$ defined as*

$$X_n = X \textit{ if } n \textit{ is even and } Y \textit{ if } n \textit{ is odd}$$

From the adjointness properties of the duals defined above, there are unit and counit morphisms:

$$\begin{aligned} i_n : \mathcal{O}_{X_n} &\longrightarrow \mathcal{E}^{*n} \otimes \mathcal{E}^{*n+1} \\ j_n : \mathcal{E}^{*n} \otimes \mathcal{E}^{*n-1} &\longrightarrow \mathcal{O}_{X_n} \end{aligned} \tag{5.6}$$

Our next ingredient is that of a nondegenerate bimodule.

Definition 5.2.12. We say that $\mathcal{Q} \in \text{bimod}(X - W)$ is *invertible* if there exists a bimodule $\mathcal{Q}^{-1} \in \text{bimod}(W - X)$ such that

$$\mathcal{Q} \otimes_W \mathcal{Q}^{-1} \cong \mathcal{O}_X \text{ and } \mathcal{Q}^{-1} \otimes_X \mathcal{Q} \cong \mathcal{O}_W.$$

If there are bimodules $\mathcal{E} \in \text{bimod}(X - Y)$ and $\mathcal{F} \in \text{bimod}(Y - W)$ such that $\mathcal{Q} \subset \mathcal{E} \otimes_Y \mathcal{F}$, the we say the inclusion is *nondegenerate* if the following composition

$$\mathcal{E}^* \otimes_X \mathcal{Q} \longrightarrow \mathcal{E}^* \otimes_X \otimes (\mathcal{E} \otimes_Y \mathcal{F}) \longrightarrow \mathcal{F}$$

is an isomorphism.

Definition 5.2.13. Let $(X_i)_{i \in \mathbb{Z}}$ be a sequence of smooth varieties over $\Bbbk$ and let $\mathcal{E}_i$ be locally free $X_i - X_{i+1}$-bimodules. Then the *tensor sheaf $\mathbb{Z}$-algebra* $\mathbb{T}(\{\mathcal{E}_i\})$ is the sheaf $\mathbb{Z}$-algebra generated by the $\{\mathcal{E}_i\}$. More precisely:

$$\mathbb{T}(\{\mathcal{E}_i\})_{m,n} = \begin{cases} 0 & n < m \\ {}_{\text{Id}}(\mathcal{O}_{X_m})_{\text{Id}} & n = m \\ \mathcal{E}_m \otimes \ldots \otimes \mathcal{E}_{n-1} & n > m \end{cases}$$

If X and Y are smooth varieties and $\mathcal{E}$ a locally free $X - Y$-bimodule, the standard tensor algebra is the sheaf $\mathbb{Z}$-algebra $\mathbb{T}(\mathcal{E})$ constructed by applying the convention 5.2.11 and considering

$$\mathcal{E}_n = \mathcal{E}^{*n}$$

in the above definition

We can now state the definition of the main object of study in this chapter: the *symmetric sheaf $\mathbb{Z}$-algebra*.

Definition 5.2.14. Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of smooth varieties over $\Bbbk$ and let $\mathcal{E}_i$ be locally free $X_i - X_{i+1}$-bimodules. Suppose that for each i we are given a nondegenerate $X_i - X_{i+2}$-bimodule $\mathcal{Q}_i \subset \mathcal{E}_i \otimes \mathcal{E}_{i+1}$. Then the *symmetric sheaf $\mathbb{Z}$-algebra* $\mathbb{S}(\{\mathcal{E}_i\}, \{\mathcal{Q}_i\})$ is the quotient of $\mathbb{T}(\{\mathcal{E}_i\})$ by the relations $(\mathcal{Q}_i)_i$. I.e. $\mathbb{S}(\{\mathcal{E}_i\}, \{\mathcal{Q}_i\})_{m,n}$ is defined as

$$\begin{cases} \mathbb{T}(\{\mathcal{E}_i\})_{m,n} & n \leq m+1 \\ \mathbb{T}(\{\mathcal{E}_i\})_{m,n}/[(\mathcal{Q}_m \otimes \ldots) + (\mathcal{E}_m \otimes Q_{m+1} \otimes \ldots) + \ldots + (\ldots \otimes Q_{n-2})] & n \geq m+2 \end{cases}$$

If X and Y are smooth varieties and $\mathcal{E}$ an $X - Y$-bimodule, the standard symmetric sheaf $\mathbb{Z}$-algebra $\mathbb{S}(\mathcal{E})$ is constructed form the standard tensor sheaf $\mathbb{Z}$-algebra $\mathbb{T}(\mathcal{E})$ by considering the following sequence of nondegenerate invertible bimodules:

$$\mathcal{Q}_n = i_n\left(\mathcal{O}_{X_n}\right) \subset \mathcal{E}^{*n} \otimes \mathcal{E}^{*n+1} \tag{5.7}$$

A fundamental operation in the context of sheaf $\mathbb{Z}$-algebras is that of twisting by a sequence of invertible bimodules:

Theorem 5.2.15. *Let $(X_i)_i$ and $(Y_i)_i$ be sequences of smooth varieties over $\Bbbk$ and $\mathcal{A}$ a sheaf $\mathbb{Z}$-algebra on $(X_i)_i$.*
Given a collection of invertible $X_i - Y_i$-bimodules $(\mathcal{T}_i)_i$, one can construct a sheaf $\mathbb{Z}$-algebra $\mathcal{B}$ by

$$\mathcal{B}_{ij} \stackrel{def}{=} \mathcal{T}_i^{-1} \otimes \mathcal{A}_{ij} \otimes \mathcal{T}_j$$

called the twist of $\mathcal{A}$ by $(\mathcal{T}_i)_i$.
There is an equivalence of categories given by the functor

$$\mathcal{T} : \mathrm{Gr}(\mathcal{A}) \cong \mathrm{Gr}(\mathcal{B}) : \mathcal{M}_i \longrightarrow \mathcal{M}_i \otimes \mathcal{T}_i$$

Moreover, every symmetric sheaf $\mathbb{Z}$-algebra can be obtained from a standard symmetric one by performing such a twist operation.

Proof. This is proven [VdB12, §5.1] □

5.2.2 The Rank $(2, 2)$ Case

In this section, we give a proof of the result that $\mathrm{Proj}(\mathbb{S}(\mathcal{E}))$ is commutative in the case where $\mathcal{E}$ has rank $(2, 2)$. As mentioned in the introduction however, our primary concern is the rank $(4, 1)$ case. This section's sole purpose is to acquire a little geometric intuition for the object $\mathbb{S}(\mathcal{E})$. As such it may be skipped by the reader without any trouble.
We first begin by introducing notation for the $\mathbb{Z}$-graded-to-$\mathbb{Z}$-algebra construction in our setting

Convention 5.2.16. *Let $\mathcal{G}$ be a graded algebra in the monoidal category* $\mathrm{bimod}(X)$ *Then we denote by $\widehat{\mathcal{G}}$ the sheaf $\mathbb{Z}$-algebra over X whose (i, j)-component is the X-bimodule $\mathcal{G}_{j-i}$*

Remark 5.2.17. *It is clear that in the above situation, taking the direct sum yields an equivalence:*

$$\mathrm{Gr}(\mathcal{G}) \xrightarrow{\simeq} \mathrm{Gr}(\widehat{\mathcal{G}}) : (\mathcal{M})_i \mapsto \bigoplus_{i \in \mathbb{Z}} \mathcal{M}_i$$

If $\mathcal{E}$ is an X-bimodule, then it is easily seen that the graded sheaf of algebras $\mathrm{Sym}(\mathcal{E})$ satisfies these conditions. The following lemma (which was already announced but not proven in [VdB12]) shows that symmetric sheaf $\mathbb{Z}$-algebras over central bimodules rank (2,2) indeed essentially coincide with sheaves of commutative graded algebras:

Lemma 5.2.18. *Let $\mathcal{V}$ be a locally free X-module of rank 2. There is an equivalence of the form*

$$\mathrm{Gr}(\mathbb{S}({}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}})) \xrightarrow{\mathcal{T}} \mathrm{Gr}\left(\widehat{\mathrm{Sym}_{X \times X}({}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}})}\right) \xrightarrow{\cong} \mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V}))$$

where $\mathcal{T}$ is given by twisting through $\left(\left(\bigwedge^2 \mathcal{V}\right)^{\lfloor \frac{i}{2} \rfloor}\right)_{i \in \mathbb{Z}}$ as in theorem 5.2.15.

Proof. We first describe the second equivalence: by the remark 5.2.17, we may remove the hat and simply consider the sheaf of graded algebras $\mathrm{Sym}_{X\times X}\,\mathrm{Id}({}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}})$. The second equivalence now follows tautologically from the definitions, since in each degree d, d',

$$\mathcal{M}_d \otimes \mathrm{Sym}_{X\times X}({}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}})_{d'} = \mathcal{M}_d \otimes {}_{\mathrm{Id}}\left(\mathrm{Sym}_X(\mathcal{V})\right)_{\mathrm{Id}} \overset{5.2.1}{=} \mathcal{M}_d \otimes_X \mathrm{Sym}_X(\mathcal{V})_{d'}$$

implying that both multiplications coincide. We now explain the first equivalence:
Let $\mathcal{E} = {}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}}$. Using the explicit expression for the dual given in 5.2.10, we obtain

$$\mathcal{E}^* = {}_{\mathrm{Id}}\mathcal{H}om(\mathcal{V}, \mathrm{Id}^! \, \mathcal{O}_X)_{\mathrm{Id}} = {}_{\mathrm{Id}}(\mathcal{V}^*)_{\mathrm{Id}}$$

In particular the equalities $\mathcal{E}^{*2n} = \mathcal{E} = {}_{\mathrm{Id}}(\mathcal{V})_{\mathrm{Id}}$ and $\mathcal{E}^{*2n+1} = \mathcal{E}^* = {}_{\mathrm{Id}}(\mathcal{V}^*)_{\mathrm{Id}}$ hold for all n. Since the pairing $\mathcal{V} \otimes \mathcal{V} \longrightarrow \Lambda^2\mathcal{V}$ is perfect, there is an isomorphism

$$\mathcal{V}^* \otimes (\Lambda^2\mathcal{V}) \xrightarrow{\cong} \mathcal{V} \tag{5.8}$$

Let $(\mathcal{T}_i)_i = (\bigwedge^2 \mathcal{V})^{\lfloor \frac{i}{2} \rfloor}$. It follows from the definition of $\mathbb{T}(\mathcal{E})$, that as sheaf $\mathbb{Z}$-algebras, we have

$$\mathbb{T}(\mathcal{E}) = \widehat{T_X(\mathcal{V})}$$

By theorem 5.2.15 applying the twist by the sequence $(\mathcal{T}_i)$ yields an equivalence

$$\mathrm{Gr}(\mathbb{T}(\mathcal{E})) \to \mathrm{Gr}(\widehat{T_X(\mathcal{V})}) : \left(\mathcal{M}_i\right)_i \mapsto \left(\mathcal{M}_i \otimes (\Lambda^2\mathcal{V})^{\lfloor \frac{i}{2} \rfloor}\right)_i$$

specifically in each component:

$$\mathbb{T}(\mathcal{E})_{m,n} \cong {}_{\mathrm{Id}}\left((\Lambda^2\mathcal{V})^{\lfloor \frac{m}{2} \rfloor} \otimes T_X(\mathcal{V})_{n-m} \otimes (\Lambda^2\mathcal{V})^{-\lfloor \frac{n}{2} \rfloor}\right)_{\mathrm{Id}} \tag{5.9}$$

We now claim that the twisting in (5.9) induces a twisting

$$\mathbb{S}(\mathcal{E})_{m,n} \cong {}_{\mathrm{Id}}\left((\Lambda^2\mathcal{V})^{\lfloor \frac{m}{2} \rfloor} \otimes \mathrm{Sym}_X(\mathcal{V})_{n-m} \otimes (\Lambda^2\mathcal{V})^{-\lfloor \frac{n}{2} \rfloor}\right)_{\mathrm{Id}}$$

and hence an equivalence of categories:

$$\mathrm{Gr}(\mathbb{S}(\mathcal{E})) \to \mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V})) : \left(\mathcal{M}_i\right)_i \mapsto \bigoplus_i \mathcal{M}_i \otimes (\Lambda^2\mathcal{V})^{\lfloor \frac{i}{2} \rfloor} \tag{5.10}$$

So we are left with proving the claim. For this we must understand what happens under (5.9) to the relations that define $\mathbb{S}(\mathcal{E})$ as a quotient of $\mathbb{T}(\mathcal{E})$. As the relations are generated in degree 2, it suffices to consider $\mathbb{S}(\mathcal{E})_{m,m+2} \otimes {}_{\mathrm{Id}}(\Lambda^2\mathcal{V})_{\mathrm{Id}}$. This is the quotient of

$$\mathbb{T}(\mathcal{E})_{m,m+2} \otimes {}_{\mathrm{Id}}(\Lambda^2\mathcal{V})_{\mathrm{Id}} \cong {}_{\mathrm{Id}}\left(T_X(\mathcal{V})_2\right)_{\mathrm{Id}} = {}_{\mathrm{Id}}\left(\mathcal{V} \otimes \mathcal{V}\right)_{\mathrm{Id}}$$

by the relation

$$i\big({}_{\mathrm{Id}}(\mathcal{O}_X)_{\mathrm{Id}}\big) \otimes {}_{\mathrm{Id}}(\Lambda^2\mathcal{V})_{\mathrm{Id}} \subset {}_{\mathrm{Id}}\left(\mathcal{V} \otimes \mathcal{V}^* \otimes \Lambda^2\mathcal{V}\right)_{\mathrm{Id}} \cong {}_{\mathrm{Id}}\left(\mathcal{V} \otimes \mathcal{V}\right)_{\mathrm{Id}}.$$

We have to check that this relation is exactly the one that defines $\mathrm{Sym}_X(\mathcal{V})$ as a quotient of $\mathrm{T}_X(\mathcal{V})$. The latter relation is defined locally, so it suffices to check on a trivializing open subset U for $\mathcal{V}$. If $\mathcal{V}|_U \cong \mathcal{O}_X|_U\, u \oplus \mathcal{O}_X|_U\, v$ then $i\big({}_{\mathrm{Id}}(\mathcal{O}_X)_{\mathrm{Id}}\big)$ is locally given by $u \otimes u^* + v \otimes v^*$. One checks that the isomorphism (5.8) maps $u^* \otimes (u \wedge v)$ to v and $v^* \otimes (u \wedge v)$ to $-u$, the induced relation in $\mathcal{V} \otimes \mathcal{V}$ is locally given by $u \otimes v - v \otimes u$, the defining relation of $\mathrm{Sym}_X(\mathcal{V})$. □

We have the following result:

Proposition 5.2.19. *Let $\mathcal{E}$ be any $X - Y$-bimodule of rank $(2,2)$. Then $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ is noetherian*

This proposition ensures that we can perform the Proj construction on $\mathbb{S}(\mathcal{E})$ if the rank of $\mathcal{E}$ is (2,2) by 5.2.6. The resulting noncommutative scheme is equivalent to a projective bundle over X as follows:

Corollary 5.2.20. *With the assumptions of the previous theorem we have an induced equivalence:*

$$\Phi : \mathrm{Proj}(\mathbb{S}({}_{\mathrm{Id}}(\mathcal{V})_{\mathrm{Id}})) \xrightarrow{\simeq} \mathrm{Proj}(\mathrm{Sym}_X(\mathcal{V})) \xrightarrow{\simeq} \mathrm{Qcoh}(\mathbb{P}_X(\mathcal{V}))$$

Proof. The equivalence given in (5.10) obviously maps torsion modules onto torsion modules, hence it factors to yield an equivalence $\mathrm{Proj}(\mathbb{S}({}_{\mathrm{Id}}(\mathcal{V})_{\mathrm{Id}}) \xrightarrow{\simeq} \mathrm{Proj}(\mathrm{Sym}_X(\mathcal{V}))$.

The second equivalence is a well known result from classical algebraic geometry and is given by the following pair of functors

$$\begin{tikzcd} \mathrm{Proj}(\mathrm{Sym}_X(\mathcal{V})) \arrow[rr, bend left, "\widetilde{(-)}"] & & \mathrm{Qcoh}(\mathbb{P}_X(\mathcal{V})) \arrow[ll, bend left, "p \circ \Gamma_* \overset{\text{def}}{=} p\left[\oplus_i \pi_*\big((-)(i)\big)\right]"] \end{tikzcd}$$

Where π is the canonical projection $\pi : \mathbb{P}_X(\mathcal{V}) \longrightarrow X$. ☐

5.2.3 Truncation Functors and Periodicity

Let $\mathcal{A}$ be a sheaf $\mathbb{Z}$-algebra over a sequence of varieties $(X_i)_{i\in\mathbb{Z}}$. Then we can define a sequence of *truncation* functors as follows: for each $m \in \mathbb{Z}$, we can consider the functor

$$\mathrm{Gr}(\mathcal{A}) \xrightarrow{(-)_m} \mathrm{Qcoh}(X_m)$$

We shall need the following easy result on these functors:

Lemma 5.2.21. *Let $e_m\mathcal{A}$ be the right $\mathcal{A}$-module defined in 5.4. There is an adjoint pair*

$$- \otimes e_m\mathcal{A} \;\dashv\; (-)_m$$

Proof. The proof of this is standard and left to the reader ☐

Our next result shows that there is a certain 2-periodic behavior among these functors. To this end, for $n \in \mathbb{Z}$, we denote by $\mathcal{A}(n)$ the sheaf $\mathbb{Z}$-algebra

$$\mathcal{A}(n)_{i,j} = \mathcal{A}_{n+i,n+j} \tag{5.11}$$

Proposition 5.2.22. *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of smooth varieties and $\mathcal{A}$ be a symmetric sheaf $\mathbb{Z}$-algebra. There is an autoequivalence α on* $\mathrm{Gr}(\mathcal{A})$ *inducing* *a commutative diagram for each m:*

$$\begin{tikzcd} \mathrm{Gr}(\mathcal{A}) \arrow[r, "(-)_m"] \arrow[d, "\alpha"'] & \mathrm{Qcoh}(X_m) \arrow[d, "\otimes \omega_{X_m}"] \\ \mathrm{Gr}(\mathcal{A}) \arrow[r, "(-)_{m+2}"'] & \mathrm{Qcoh}(X_m) \end{tikzcd}$$

Proof. By theorem 5.2.15 $\mathcal{A}$ is Morita equivalent to a symmetric sheaf $\mathbb{Z}$-algebra $\mathbb{S}(\mathcal{E})$ in standard form with $\mathcal{E} \in \mathrm{bimod}(X - Y)$ (using the notation 5.2.11). Moreover by [VdB12, 4.1.7], we have

$$\mathcal{E}^{*2} \cong \omega_X^{-1} \otimes \mathcal{E} \otimes \omega_Y$$

Hence the twist by $(\omega_{X_i})_{i\in\mathbb{Z}}$ yields an equivalence

$$\mathcal{T} : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \xrightarrow{\cong} \mathrm{Gr}(\omega^{-1} \otimes \mathbb{S}(\mathcal{E}) \otimes \omega) \xrightarrow{\cong} \mathrm{Gr}(\mathbb{S}(\mathcal{E}^{*2}))$$

where we used the short-hand notation

$$\left(\omega^{-1} \otimes \mathbb{S}(\mathcal{E}) \otimes \omega\right)_{m,n} = \omega_{X_m}^{-1} \otimes \mathbb{S}(\mathcal{E})_{m,n} \otimes \omega_{X_n}.$$

Next, the construction of a standard symmetric sheaf $\mathbb{Z}$-algebra implies that there is an equivalence $\Psi : \mathrm{Gr}(\mathbb{S}(\mathcal{E})(2)) \longrightarrow \mathrm{Gr}(\mathbb{S}(\mathcal{E}^{*2}))$ (where we used the notation (5.11). We now simply define

$$\alpha \overset{\mathrm{def}}{=} (-2) \circ \Psi^{-1} \circ \mathcal{T} : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \longrightarrow \mathrm{Gr}(\mathbb{S}(\mathcal{E}^{*2})) \longrightarrow \mathrm{Gr}(\mathbb{S}(\mathcal{E})(2)) \longrightarrow \mathrm{Gr}(\mathbb{S}(\mathcal{E}))$$

$\square$

In the commutative case of a symmetric sheaf $\mathbb{Z}$-algebra constructed with a central bimodule of rank (2,2) (as discussed in 5.2.2) the 0^{th} truncation functor coincides with the pushforward functor in the following sense:

Theorem 5.2.23. *Let $\mathcal{V}$ be a vector bundle on X of rank 2 and consider the associated symmetric sheaf $\mathbb{Z}$-algebra $\mathbb{S}({}_{\mathrm{Id}}(\mathcal{V})_{\mathrm{Id}})$.*
Let

$$\Phi : \mathrm{Proj}(\mathbb{S}({}_{\mathrm{Id}}(\mathcal{V})_{\mathrm{Id}})) \longrightarrow \mathrm{Qcoh}(\mathbb{P}_X(\mathcal{V}))$$

be the equivalence provided by Corollary 5.2.20. Then the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{Gr}(\mathbb{S}({}_{\mathrm{Id}}(\mathcal{V})_{\mathrm{Id}})) \arrow[dr, "(-)_0"] & \\
\mathrm{Proj}(\mathbb{S}({}_{\mathrm{Id}}(\mathcal{V})_{\mathrm{Id}})) \arrow[ur, "\omega"] \arrow[dr, "\Phi"'] & & \mathrm{Qcoh}(X) \\
 & \mathrm{Qcoh}(\mathbb{P}_X(\mathcal{V})) \arrow[ur, "\pi_*"'] &
\end{tikzcd}$$

Proof. Let $Z \overset{\text{def}}{=} \mathbb{P}_X(\mathcal{V})$ and $\mathcal{A} \overset{\text{def}}{=} \mathbb{S}({}_{\mathrm{Id}}(\mathcal{V})_{\mathrm{Id}})$. The formula we need to prove explicitly is

$$\pi_*\Big(\oplus_i \widetilde{(-) \otimes \mathcal{T}_i}\Big) \cong \big(\omega(-)\big)_0$$

where $\mathcal{T}_i = \big(\big(\bigwedge^2 \mathcal{V}\big)^{\lfloor \frac{i}{2} \rfloor}\big)_{i\in\mathbb{Z}}$ is given as in the statement of 5.2.18.
Now by lemma 5.2.21 and the definition of ω, the functor $\big(\omega(-)\big)_0$ is right adjoint to $p\big((-) \otimes e_0\mathcal{A}\big)$. Another formal computation using corollary 5.2.20 shows that $\pi_*\Big(\oplus_i \widetilde{(-) \otimes \mathcal{T}_i}\Big)$ is right adjoint to the functor $\mathcal{T}^{-1}\left[(p \circ \Gamma_*)\big(\pi^*(-)\big)\right]$. This functor in turn being equal to $p\left[\big(\pi_*\big(\pi^*(-)(i)\big) \otimes \mathcal{T}_i^{-1}\big)_i\right]$, which by the projection formula, simplifies to $p\left(\big((-) \otimes \pi_*\mathcal{O}_Z(i) \otimes \mathcal{T}_i^{-1}\big)_i\right)$. The unicity of adjoint functors thus reduces the claim to proving the isomorphism

$$\big((-) \otimes \pi_*\mathcal{O}_Z(i) \otimes \mathcal{T}_i\big)_i \cong (-) \otimes e_0\mathcal{A} \tag{5.12}$$

Since $\mathrm{rk}(\mathcal{E}) \geq 2$, [Har97, Proposition II.7.11.a] implies that there is an isomorphism $\pi_*\big(\mathcal{O}_Z(i)\big) = \mathrm{Sym}_X(\mathcal{V})_i$. Now, by the choice of $\mathcal{T}_i$, we have $\mathrm{Sym}_X(\mathcal{V})_i = \mathcal{A}_{0i} \otimes \mathcal{T}_i$. The 2-periodicity (5.12) thus becomes

$$\big((-)\otimes\pi_*\mathcal{O}_Z(i)\otimes\mathcal{T}_i^{-1}\big)_i = \big((-)\otimes\mathcal{A}_{0i}\otimes\mathcal{T}_i\otimes\mathcal{T}_i^{-1}\big)_i = \big((-)\otimes\mathcal{A}_{0i}\big)_i = (-)\otimes e_0\mathcal{A}$$

proving the claim. $\square$

We also have a different 1-periodic behaviour for the truncation functors in this case:

Proposition 5.2.24. *Let $\mathcal{V}$ be a locally free sheaf of rank 2 on X and $\mathbb{S}({}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}})$ the associated symmetric sheaf $\mathbb{Z}$-algebra. Then there is an equivalence β and for each n, a line bundle $\mathcal{L}_n$ on X making the diagram*

$$\begin{tikzcd} \mathrm{Gr}(\mathbb{S}({}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}})) \arrow[r, "(-)_n"] \arrow[d, "\beta"'] & \mathrm{Qcoh}(X) \arrow[d, "-\otimes\mathcal{L}_n"] \\ \mathrm{Gr}(\mathbb{S}({}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}})) \arrow[r, "(-)_{n+1}"'] & \mathrm{Qcoh}(X) \end{tikzcd}$$

commute.

Proof. By lemma 5.2.18 there is a sequence of $X-X$-bimodules $\mathcal{T}_i$ such that the following is an equivalence of categories

$$\mathrm{Gr}(\mathbb{S}({}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}})) \longrightarrow \mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V})) : (\mathcal{M}_i)_i \mapsto \bigoplus_i \mathcal{M}_i \otimes \mathcal{T}_i$$

Let (-1) denote the inverse shift functor on $\mathrm{Gr}(\mathrm{Sym}_X(\mathcal{V}))$, i.e. $(\mathcal{M}(-1))_i = \mathcal{M}_{i-1}$ and define β as the autoequivalence making the diagram

$$\begin{tikzcd} \mathrm{Gr}(\mathbb{S}({}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}})) \arrow[r, "\mathcal{T}"] \arrow[d, "\beta"'] & \mathrm{Sym}_X(\mathcal{V}) \arrow[d, "(-1)"] \\ \mathrm{Gr}(\mathbb{S}({}_{\mathrm{Id}}\mathcal{V}_{\mathrm{Id}})) \arrow[r, "\mathcal{T}"'] & \mathrm{Sym}_X(\mathcal{V}) \end{tikzcd}$$

commute. Since we clearly have $(-)_{n+1} \circ (-1) = (-)_n$, we get the required result by choosing the line bundle $\mathcal{L}_n \overset{\mathrm{def}}{=} \mathcal{T}_n \otimes \mathcal{T}_{n+1}^{-1}$ with $\mathcal{T}_n$ as in the proof of lemma 5.2.18. $\square$

Remark 5.2.25. *the previous result of 1-periodicity clearly implies 2-periodicity after repeated application in the sense that*

$$(-)_{n+2} \circ \beta^2 = \big(\mathcal{L}_{n+1} \otimes \mathcal{L}_n\big) \otimes (-)_n$$

hence one can wonder whether this periodicity coincides with proposition 5.2.22. This is not the case in general. Indeed, from the explicit form of $\mathcal{T}$ in proposition 5.2.22 and β in Proposition 5.2.24*, we obtain*

$$\mathcal{L}_n = (\bigwedge^2 \mathcal{V})^{\lfloor \frac{n}{2} \rfloor} \otimes (\bigwedge^2 \mathcal{V})^{-\lfloor \frac{n+1}{2} \rfloor}$$

and $\mathcal{L}_{n+1} \otimes \mathcal{L}_n = \left(\bigwedge^2(\mathcal{V})\right)^{-1}$*, which obviously does not coincide with* $\omega_{X/S}$ *in general.*

5.3 Noetherianity of $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$

As explained in the introduction, it is the case of a bimodule $\mathcal{E}$ of rank $(4,1)$ that we are particularly interested in. This section is dedicated to proving one of the important geometric properties of $\mathbb{S}(\mathcal{E})$ in this setting:

Theorem 5.3.1. *Let X and Y be smooth varieties over $\Bbbk$ and $\mathcal{E} \in \mathrm{bimod}(X-Y)$ be locally free of rank (4,1). Then the category $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ is locally noetherian.*

As an immediate consequence, the category $\mathrm{Tors}(\mathbb{S}(\mathcal{E}))$ is localizing and we can form the noncommutative scheme $Z \stackrel{\text{def}}{=} \mathrm{Proj}(\mathbb{S}(\mathcal{E})$. Lemma [Smi99, 14.19] now also shows:

Theorem 5.3.2. *Under the conditions of 5.3.1, Z is a noetherian noncommutative scheme*

The next lemma shows that under these assumptions, the bimodule $\mathcal{E}$ can written in a convenient form using a line bundle on Y and a finite map f of degree 4.

Lemma 5.3.3. *Assume that X, Y are smooth varieties of finite type over $\Bbbk$ and $\mathcal{E}$ is a locally free $X-Y$-bimodule of rank $(n,1)$. Then there is a line bundle $\mathcal{L}$ on Y and a finite surjective morphism $f : Y \longrightarrow X$ of degree n such that $\mathcal{E} \cong {}_f\mathcal{L}_{\mathrm{Id}}$ (see 5.2.3).*[3]

Proof. Let $W \subset X \times Y$ be the scheme theoretic support of $\mathcal{E}$ and denote the projections $W \longrightarrow X$, $W \longrightarrow Y$ by g, h respectively:

[3]note that f is automatically flat here

$$\begin{tikzcd} & \operatorname{Supp}(\mathcal{E}) = W \arrow[d, hook, "\iota"] \arrow[ddl, bend right, "g"'] \arrow[ddr, bend left, "h"] & \\ & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

By definition g, h are finite morphisms. Furthermore $\mathcal{E} \cong {}_g\mathcal{F}_h$ for $\mathcal{F} \in \operatorname{coh}(W)$ such that $\operatorname{Supp}\mathcal{F} = W$. By Lemma 5.3.4 below we conclude that h is an isomorphism and that $\mathcal{F}$ is a line bundle on Z. Put $\mathcal{L} = h_*\mathcal{F}$, $f = gh^{-1}$. Then $\mathcal{E} \cong {}_f\mathcal{L}_{\mathrm{Id}}$. Since $\mathcal{L}$ is a line bundle, $f_*\mathcal{L}$ and $f_*\mathcal{O}_Y$ are locally isomorphic (e.g. by Lemma 5.3.8 below). So $f_*\mathcal{O}_Y$ is locally free of rank n as well and therefore f is flat of degree n. $\square$

Lemma 5.3.4. *Assume that $h : W \longrightarrow Y$ is a finite morphism between smooth varieties, $\mathcal{F}$ is a coherent sheaf on W whose scheme theoretic support is W and $h_*\mathcal{F}$ is locally free of rank one. Then h is an isomorphism and $\mathcal{F}$ is a line bundle on Z.*

Proof. Since h is finite it is affine, we may assume that $Y = \operatorname{Spec} R$, $W = \operatorname{Spec} S$ and $\mathcal{F} = \tilde{F}$ for F an S-module which is invertible as R-module. The composition of

$$R \xrightarrow{h} S \xrightarrow{s \mapsto (f \mapsto sf)} \operatorname{End}_R(F) \cong R$$

is the identity and the middle map is injective since W is the scheme-theoretic support of $\mathcal{F}$. It follows that all maps are isomorphisms. The claim follows. $\square$

Convention 5.3.5. *Inspired by 5.3.3, we will often consider the setting where X, Y are smooth varieties over $\Bbbk$ with a finite surjective morphism $f : Y \longrightarrow X$ of degree 4 and and $\mathcal{E}$ is an $X - Y$ bimodule given as $\mathcal{E} \cong {}_f\mathcal{L}_{\mathrm{Id}}$ for a line bundle $\mathcal{L}$ on Y.*

5.3.1 Restricting to an Open Subset

The first step in the proof of theorem 5.3.1 is showing that there is an appropriate notion of restricting a sheaf $\mathbb{Z}$-algebra to an open subset and that the statement of theorem 5.3.1 can be reduced to an open cover in this sense.
To this end, we let $\mathcal{A}$ denote a sheaf $\mathbb{Z}$-algebra over a sequence of smooth varieties X_i and $\mathcal{U} = (U^m)_{m\in\mathbb{Z}}$ be a sequence of affine open subsets $U^m \subset X_m$. For a bimodule $\mathcal{F} \in \mathrm{bimod}(X_m - X_{m+1})$, and a graded $\mathcal{A}$-module $\mathcal{M}$ we will use the notation $(-)|_{\mathcal{U}}$ to denote the restriction to the corresponding open subset. I.e.

$$\begin{aligned}
\mathcal{F}|_{\mathcal{U}} &\stackrel{\mathrm{def}}{=} \mathcal{F}|_{U^m\times U^{m+1}} \\
(\mathcal{A}|_{\mathcal{U}})_{m,n} &\stackrel{\mathrm{def}}{=} (\mathcal{A}_{m,n})|_{\mathcal{U}} = (\mathcal{A}_{m,n})|_{U^m\times U^n} \\
(\mathcal{M}|_{\mathcal{U}})_m &\stackrel{\mathrm{def}}{=} (\mathcal{M}_m)|_{U^m}
\end{aligned}$$

To ensure that the restrictions of $\mathcal{A}$ to an open subset remains a sheaf $\mathbb{Z}$-algebra, we need the following technical condition:

Lemma 5.3.6. *Let $\mathcal{A}$ be a sheaf $\mathbb{Z}$-algebra and $\mathcal{U}$ as above such that m, n:*

$$\mathrm{Supp}(\,(\mathcal{A}_{m,n})|_{U^m\times X_n}) \subset U^m \times U^n \text{ and } \mathrm{Supp}(\,(\mathcal{A}_{m,n})|_{X_m\times U^n}) \subset U^m \times U^n$$

then

i) $\mathcal{A}|_U$ has an induced algebra structure.

ii) Restriction of modules to U defines a functor $|_U : \mathrm{Gr}(\mathcal{A}) \to \mathrm{Gr}(\mathcal{A}|_U)$

Proof. i) We must show that for all $l, m, n \in \mathbb{Z}$ there are multiplication morphisms $\mathcal{A}_{l,m}|_U \otimes \mathcal{A}_{m,n}|_U \to \mathcal{A}_{l,n}|_U$ induced by the morphisms $\mathcal{A}_{l,m} \otimes \mathcal{A}_{m,n} \to \mathcal{A}_{l,n}$. It is evident that the latter induces a morphism of $U^l - U^n$-bimodules as follows:

$$(\mathcal{A}_{l,m} \otimes \mathcal{A}_{m,n})|_U \to \mathcal{A}_{l,n}|_U$$

Now the claim follows from the following chain of isomorphisms:

$$\begin{aligned}
(\mathcal{A}_{l,m}\otimes\mathcal{A}_{m,n})|_U &= \left(\pi_{X_l,X_n*}\left(\pi^*_{X_l,X_m}(\mathcal{A}_{l,m})\otimes_{X_l\times X_m\times X_n}\pi^*_{X_m,X_n}(\mathcal{A}_{m,n})\right)\right)\Big|_{U^l\times U^n}\\
&= \pi_{U^l,U^n*}\left(\left(\pi^*_{X_l,X_m}(\mathcal{A}_{l,m})\otimes_{X_l\times X_m\times X_n}\pi^*_{X_m,X_n}(\mathcal{A}_{m,n})\right)\Big|_{U^l\times X_m\times U^n}\right)\\
&= \pi_{U^l,U^n*}\left(\pi^*_{X_l,X_m}(\mathcal{A}_{l,m})\big|_{U^l\times X_m\times U^n}\otimes\pi^*_{X_m,X_n}(\mathcal{A}_{m,n})\big|_{U^l\times X_m\times U^n}\right)\\
&= \pi_{U^l,U^n*}\left(\pi^*_{U^l,X_m}(\mathcal{A}_{l,m}|_{U^l\times X_m})\otimes_{U^l\times X_m\times U^n}\pi^*_{X_m,U^n}(\mathcal{A}_{m,n}|_{X_m\times U^n})\right)\\
&= \pi_{U^l,U^n*}\left(\pi^*_{U^l,U^m}(\mathcal{A}_{l,m}|_{U^l\times U^m})\otimes_{U^l\times U^m\times U^n}\pi^*_{U^m,U^n}(\mathcal{A}_{m,n}|_{U^m\times U^n})\right)\\
&= \mathcal{A}_{l,m}|_U\otimes\mathcal{A}_{m,n}|_U
\end{aligned}$$

where π_{U^l,X_m} and π_{U^l,U^m} are the projections $\pi_{U^l,X_m}: U^l\times X_m\times U^n\to U^l\times X_m$ and $\pi_{U^l,U^m}: U^l\times U^m\times U^n\to U^l\times U^m$, with similar definitions for π_{X_m,U^n} and π_{U^m,U^n}.
The first equality is the definition of tensor product of bimodules

$$\mathrm{bimod}(X_l-X_m)\times\mathrm{bimod}(X_m-X_n)\to\mathrm{bimod}(X_l-X_n)$$

The second equality follows from the commutation of pushforward and restriction of sheaves. The third equality follows from the commutation of tensor product of sheaves and restriction. The fourth equality follows from the commutation of pullback and restriction of sheaves. The fifth equality follows the assumption of the lemma. The last equality is the definition of multiplication

$$\mathrm{bimod}(U^l-U^m)\times\mathrm{bimod}(U^m-U^n)\to\mathrm{bimod}(U^l-U^n)$$

ii) This essentially reduces to showing $(\mathcal{M}_i\otimes\mathcal{A}_{i,j})|_{U_j}=(\mathcal{M}|_U)_i\otimes(\mathcal{A}|_U)_{i,j}$ which is completely similar to i).

□

Our main motivation to study restriction of sheaf $\mathbb{Z}$-algebra lies in the following result whose proof is straightforward:

Lemma 5.3.7. *Let $\mathcal{U}_i$ be a finite set of sequences such that for each $m \in \mathbb{Z}$, $\bigcup_i (U^m)_i = X_m$. Assume that $\mathcal{A}$ is a sheaf $\mathbb{Z}$-algebra such that the conditions in lemma 5.3.6 are satisfied for all U_l, then*

$$\forall l : \mathcal{M}|_{\mathcal{U}_l} \in \mathrm{Gr}(\mathcal{A}|_{U_l}) \text{ is noetherian } \Rightarrow \mathcal{M} \in \mathrm{Gr}(\mathcal{A}) \text{ is noetherian}$$

Proof. Suppose we are given an ascending chain of sub-objects of $\mathcal{M}^n \subset \mathcal{M}$ in $\mathrm{Gr}(\mathcal{A})$ such that the restriction of this chain to any of the sequence $\mathcal{U}_l$ stabilizes. As there are only finitely many $\mathcal{U}_l$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all l: $(\mathcal{M}^n)|_{\mathcal{U}_l} = (\mathcal{M}^{n+1})\big|_{\mathcal{U}_l}$. The graded modules $\mathcal{M}^n$ and $\mathcal{M}^{n+1}$ must coincide. □

Following the convention 5.3.5, we now consider the case where $\mathcal{A} = \mathbb{S}(\mathcal{E})$ for $\mathcal{E} =_f \mathcal{L}_{\mathrm{Id}}$. Then, for an affine open subset $U \subset X$ we define the associated sequence $\mathcal{U}$ by $U^m \subset X_m$ as follows:

$$U^i = \begin{cases} U & \text{if } i \text{ is even} \\ f^{-1}(U) & \text{if } i \text{ is odd} \end{cases}$$

Note that U^i is indeed an affine open subset because f is a finite morphism. The results of lemma 5.3.6 in this context can be stated as follows:

Corollary 5.3.8. *For any $U \subset X$,*

i) $\mathbb{S}(\mathcal{E})|_U$ *has an algebra structure induced by* $\mathbb{S}(\mathcal{E})$

ii) There is a functor $|_U : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \to \mathrm{Gr}(\mathbb{S}(\mathcal{E})|_U)$

iii) There is an isomorphism of symmetric sheaf $\mathbb{Z}$-algebras: $\mathbb{S}(\mathcal{E})|_U \cong \mathbb{S}(\mathcal{E}|_U)$

Proof. i+ii) As $\mathcal{E}$ is given as ${}_f(\mathcal{L})_{\mathrm{Id}}$ following convention 5.3.5, the conditions in lemma 5.3.6 are trivially satisfied for $\mathcal{A} = \mathbb{S}(\mathcal{E})$. For iii) We first show that for all $m \in \mathbb{N}$ there is a natural isomorphism

$$\theta_{\mathcal{E}} : (\mathcal{E}^{*m})|_U = (\mathcal{E}|_U)^{*m} \tag{5.13}$$

Using remark 5.2.10 we see by induction that for each $m \geq 0$ there is a line bundle $\mathcal{L}_m$ such that

$$\begin{aligned}
\mathcal{E}^{*2m} &= {}_f(\mathcal{L}_m)_{\mathrm{Id}} \\
\mathcal{E}^{*2m+1} &= {}_{\mathrm{Id}}(\mathcal{L}_m)_f
\end{aligned}$$

where $\mathcal{L}_0 = \mathcal{L}$. The explicit form of the dual in 5.2.10 shows that it suffices to exhibit isomorphisms

$${}_f \left(\mathcal{H}om_Y(\mathcal{L}_m, f^!\mathcal{O}_X)\right)_{\mathrm{Id}}\big|_U \cong {}_{f|_U} \left(\mathcal{H}om_{f^{-1}(U)}((\mathcal{L}_m)|_{f^{-1}(U)}, (f|_U)^!\mathcal{O}_U)\right)_{Id_U}$$

However as restriction to open affine subsets commutes with f_*, $\mathcal{H}om_Y$ and $f^!$, this isomorphism is immediate.
This implies that (5.13) is valid for $m < 0$ as well: indeed, it suffices to show this for $m = -1$. In this case. there is at least a morphism $\nu_{\mathcal{E}} :^* (\mathcal{E}|_U) \longrightarrow (^*\mathcal{E})\,|_U$. To show that $\eta_{\mathcal{E}}$ is an isomorphism, we may apply $(-)^*$, since this is a fully faithful functor. This yields a commutative diagram

$$\begin{tikzcd}
\left(^*(\mathcal{E}|_U)\right)^* \arrow[r, "\simeq"] \arrow[d, "\mu^*_{\mathcal{E}}"'] & \mathcal{E}|_U \arrow[dd, "1"] \\
\left((^*\mathcal{E})^*|_U\right) \arrow[d, "\theta_{\mathcal{E}}"'] & \\
\left((^*\mathcal{E})^*\right)|_U \arrow[r, "\simeq"'] & \mathcal{E}
\end{tikzcd}$$

proving the claim.

Finally, the naturality of $\theta_{\mathcal{E}}$ immediately implies that the restricted unit morphisms $i_m|_U$ coincides with

$${}_{\mathrm{Id}}\left(\mathcal{O}_{U^m}\right)_{\mathrm{Id}} \longrightarrow \left(\mathcal{E}|_U\right)^{*m} \otimes \left(\mathcal{E}|_U\right)^{*m+1}$$

Implying in particular that $\theta_{\mathcal{E}}$ induces an isomorphism

$$i_m({}_{\mathrm{Id}}\left(\mathcal{O}_{U^m}\right)_{\mathrm{Id}}) \cong i_m({}_{\mathrm{Id}}\left(\mathcal{O}_{X^m}\right)_{\mathrm{Id}})|_{U^m}$$

and we can extend $\theta_{\mathcal{E}}$ to an isomorphism

$$\theta : \mathbb{S}(\mathcal{E})|_U \cong \mathbb{S}(\mathcal{E}|_U) \qquad \square$$

5.3.2 Covering by Relative Frobenius Pairs

The above lemma 5.3.7 shows that verifying that an $\mathcal{A}$-module is noetherian can be done locally. In this subsection we construct an open cover $X = \bigcup_l U_l$ for which the categories $\mathrm{Gr}(\mathbb{S}(\mathcal{E})|_{U_l})$ can explicitly described (see 5.3.10). The first step in this description is a finite cover U_i for which the maps on rings of sections are relatively Frobenius as in 4.2.2. In the next section, we shall subsequently use this to relate the sections of the symmetric sheaf $\mathbb{Z}$-algebra on such affine cover to the generalized preprojective algebras as defined in the previous chapter (4.2.8). Throughout, we shall make use of the following lemma, well-known to experts:

Lemma 5.3.9. *Let $f : Y \longrightarrow X$ be a finite morphism of smooth varieties. Let $\mathcal{L}$ be a line bundle on Y and $p \in X$. Then there is an open subset $U \subset X$ containing p, such that $\mathcal{L}|_{f^{-1}(U)} \cong \mathcal{O}_{f^{-1}(U)}$.*

Proof. Since affine open subsets form a base and f is affine (as it is finite), we can reduce to the case where $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$ are affine varieties over $\Bbbk$ and S is finitely generated over R and $\mathcal{L} = \tilde{L}$ for some invertible S-module L. Let $\mathfrak{p}$ be the prime ideal in $\mathrm{Spec}(R)$ corresponding to $f(p) \in X$, then $S_{\mathfrak{p}} \overset{\text{def}}{=} S \otimes_R R_{\mathfrak{p}}$ is a semi-local ring, hence every finitely generated projective $S_{\mathfrak{p}}$-module of constant rank is free and in particular the Picard group of $S_{\mathfrak{p}}$ is trivial. Consequently, there exists an $l \in L$ such that

$$S_{\mathfrak{p}} \xrightarrow{\cdot l} L_{\mathfrak{p}}$$

is an isomorphism.
Now consider the morphism $S \xrightarrow{\cdot l} L$ with kernel K and cokernel C. Then there is an exact sequence

$$0 \longrightarrow K \longrightarrow S \xrightarrow{\cdot l} L \longrightarrow C \longrightarrow 0 \tag{5.14}$$

K is a finitely generated R-submodule of S by the noetherianity of R. L is finitely generated over R, being an invertible S-module. It follows that C is finitely generated over R as a quotient of L. Now let $\alpha_1, \dots, \alpha_n$ be a set of generators for $\Bbbk$, then as $K \otimes R_{\mathfrak{p}} = 0$ there exist elements $x_1, \dots, x_n \in R \backslash \mathfrak{p}$ such that $\alpha_1 x_1 = \dots = \alpha_n x_n = 0$. Set $x \overset{\text{def}}{=} x_1 \cdot \ldots \cdot x_n \in R \backslash \mathfrak{p}$, then $\alpha \cdot x = 0$ for all $\alpha \in K$. Similarly there is a $x' \in R \backslash \mathfrak{p}$ such that $\beta \cdot x' = 0$ for all $\beta \in C$. Now define $z = x \cdot x'$, then $K \otimes R_z = C \otimes R_z = 0$ implying that $\cdot l$ defines an isomorphism

$$S \otimes R_z \xrightarrow{\cong} L \otimes R_z$$

$U = \mathrm{Spec}(R_z)$ then is the desired open subset. $\square$

We can now prove the main lemma of this subsection, which yields a cover on which many desirable geometric properties are satisfied:

Lemma 5.3.10. *Write $\mathcal{E} = {}_f(\mathcal{L})_{\mathrm{Id}}$ as in lemma 5.3.3. There is a finite cover $X = \bigcup_l U_l$ by affine open subsets $U_l = \mathrm{Spec}(R_l)$ such that:*

i) $\mathcal{L}|_{f^{-1}(U_l)}$ *is a trivial $\mathcal{O}_{f^{-1}(U_l)}$-module*

ii) $\omega_Y|_{f^{-1}(U_l)}$ *is a trivial $\mathcal{O}_{f^{-1}(U_l)}$-module*

iii) $\omega_X|_{U_l}$ *is a trivial $\mathcal{O}_{U_l}$-module*

iv) $f^{-1}(U_l) = \mathrm{Spec}(S_l)$ *where S_l/R_l is relative Frobenius of rank 4.*

Proof. We first note the following two facts:

- Let $\mathrm{Spec}(R)$ be an affine open subset on which i), ii), iii) or iv) holds. Then the same statement holds for any standard open $\mathrm{Spec}(R_f) \subset \mathrm{Spec}(R)$. This is obvious for i), ii) and iii). For iv) it follows from the first statement of 4.3.1.
- Let $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R')$ be affine open subsets of X, then their intersection is covered by open subsets which are simultaneously distinguished in each space, in other words subsets of the form $\mathrm{Spec}(R_f) = \mathrm{Spec}(R'_g)$

By these two facts it suffices to find affine open covers for i), ii), iii) and iv) separately. For i) and ii) such a cover exists by lemma 5.3.9 and the fact that ω_Y is a line bundle on the smooth variety Y. The existence of a cover satisfying iii) is immediate from the fact that ω_X is a line bundle. We have reduced the claim to exhibiting a cover satisfying iv).
Now by lemma 5.3.9: $f^!\omega_X$ is completely determined by $f_*\left(f^!\omega_X\right)$ and we have an isomorphism of $f_*\mathcal{O}_Y$-modules

$$f_*\left(f^!\omega_X\right) \stackrel{\text{def}}{=} \mathcal{H}om_X(f_*\mathcal{O}_Y, \omega_X) \cong f_*\omega_Y \tag{5.15}$$

As moreover f is also surjective and flat, there is a cover $X = \bigcup_l U_l$ with $U_l = \operatorname{Spec}(R_l)$ and $f^{-1}(U_l) = \operatorname{Spec}(S_l)$ where S_l is a free R_l-module of rank 4 for each l. By the previous arguments we can assume that ii) and iii) are also satisfied on this cover. In this case, replacing f by its restriction $f^{-1}(U_l) \longrightarrow U_l$, (5.15) reads

$$f_*\left(f^!\mathcal{O}_{U_l}\right) \stackrel{\text{def}}{=} \mathcal{H}om_{U_l}(f\mathcal{O}_{f^{-1}(U_l)}, \mathcal{O}_{U_l}) \cong f_*\mathcal{O}_{f^{-1}(U_l)}$$

and taking sections yields the required isomorphism of S_l-modules:

$$\operatorname{Hom}_{R_l}(S_l, R_l) \cong S_l \qquad \square$$

5.3.3 from Periodic $\mathbb{Z}$-Algebras to Graded Algebras

The previous section showed how we can reduce the statement of 5.3.1 to the case where X and Y are affine, and satisfy some convenient geometric properties (see 5.3.10). In this section, we provide a second technical tool which allows us to reduce to the case where the $\mathbb{Z}$-algebra comes from a graded algebra. The $\widehat{(-)}$-construction (see 5.2.16) assigns a $\mathbb{Z}$-algebra to a graded algebra. In this section, we shall conversely show that a periodic $\mathbb{Z}$-algebras A gives rise to a graded algebra $\overline{A}$ such that $\operatorname{Gr}(A)$ is a direct summand of the category $\operatorname{Gr}(\overline{A})$. We start by describing the following slight generalization of $\mathbb{Z}$-algebras in order to be able to easily apply the result in our required setting:

Definition 5.3.11. Let $\Bbbk$ be a commutative groundring and let $(R_i)_{i\in\mathbb{Z}}$ be a sequence of commutative $\Bbbk$-algebras. A *bimodule $\mathbb{Z}$-algebra* over $(R_i)_{i\in\mathbb{Z}}$ is a collection of $R_i - R_j$-bimodules $A_{i,j}$ together with multiplication maps

$$A_{i,j} \otimes_{R_j} A_{j,l} \longrightarrow A_{i,l}$$

and R_i-linear unit maps $R_i \longrightarrow A_{ii}$ satisfying the usual $\mathbb{Z}$-algebra axioms. If $R_i = R\,\forall i$, then A is a bimodule $\mathbb{Z}$-algebra over R

Definition 5.3.12. Let A be a $\mathbb{Z}$-algebra over $(R_i)_{i\in\mathbb{Z}}$ and $d > 0$ an integer. Assume that for each i, we have $R_{i+d} = R_i$. We say A is d-periodic if there is an isomorphism of $\mathbb{Z}$-algebras $\varphi : A \xrightarrow{\sim} A(d)$. I.e. there is a collection of $R_i - R_j$-bimodule isomorphisms $\{\varphi_{ij} : A_{i,j} \xrightarrow{\sim} A_{i+d,j+d}\}_{i,j}$ compatible with the multiplication and unit maps.

Let A be d-periodic and let $R \overset{\text{def}}{=} \bigoplus_{i=0}^{d-1} R_i$. We construct a graded R-algebra $\overline{A}$ as follows: let $\overline{A}_n$ be a $d \times d$-matrix with entries

$$\left(\overline{A}_n\right)_{i,j} = \begin{cases} A_{i,i+n} & \text{if } j - i \equiv n \pmod{d} \\ 0 & \text{else} \end{cases} \tag{5.16}$$

(Where we use the convention that the numbering of rows and columns of the matrix starts at 0 instead of 1.)
By way of example,

$$\overline{A}_1 = \begin{pmatrix} 0 & A_{0,1} & 0 & \dots & 0 \\ 0 & 0 & A_{1,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{d-2,d-1} \\ A_{d-1,d} & 0 & 0 & \dots & 0 \end{pmatrix}$$

Each $\overline{A}_n$ is naturally a left (resp. right) R-module by letting a d-tuple $(r_0, \dots r_{d-1})$ act as a diagonal matrix D with entries $D_{ii} \overset{\text{def}}{=} r_i$ on the left (resp. right). Moreover, there is a canonical multiplication map

$$\overline{A}_n \otimes_R \overline{A}_m \longrightarrow \overline{A}_{n+m}$$

given by the ordinary matrix multiplication and applying the periodicity isomorphisms ϕ_{ij} whenever necessary. The $(R_i)_{i\in\mathbb{Z}}$-linearity of the $\mathbb{Z}$-algebra multiplication implies that the above maps are indeed R-bilinear.

Lemma 5.3.13. *Suppose A is d-periodic, then the above maps define a graded (unital) R-algebra structure on the R-module $\overline{A} \overset{def}{=} \oplus_{i\in\mathbb{Z}}\overline{A}_i$*

Proof. The reader checks that the compatibility of the periodicity isomorphisms with the $\mathbb{Z}$-algebra multiplication maps implies that the multiplication is associative. The algebra has a unit given by

$$1 = \begin{pmatrix} e_0 & 0 & \dots & 0 \\ 0 & e_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_{d-1} \end{pmatrix} \in \overline{A}_0$$

where e_i is the unit in A_{ii}. □

There is a convenient description of the category of graded right $\overline{A}$-modules as follows: let $M \in \mathrm{Gr}(\overline{A})$. Then by definition we have a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$. Moreover, each R-module M_i in turn has a direct sum decomposition given by $M_i = \bigoplus_{j=0}^{d-1} e_j M_i$. We define $M_i^j \overset{\text{def}}{=} e_j M_i$.
This decomposition allows us to give a description of the $\overline{A}$-module structure of M. For a matrix $\overline{a} \in \overline{A}_m$, $e_j.\overline{a}$ only has one nonzero entry at position $(j, j+m)$. It follows from the right R-structure on A_m that $e_j\overline{a} = \overline{a}.e_{j+m}$ (where we consider $j+m$ mod d following 5.16). Thus the right action of $\overline{A}_m$ on M_i^j becomes a map of the form $M_i^j \otimes A_{j,j+m} \longrightarrow M_{i+m}^{j+m}$ or equivalently for $l = j+m$,

$$M_i^j \otimes A_{j,l} \longrightarrow M_{i+l-j}^l$$

we now have:

Lemma 5.3.14. *Suppose A is d-periodic and let $\mathcal{C}$ be the category defined as follows:*

1. *an object is a collection of R-modules $(M_i^j)_{i\in\mathbb{Z},0\leq j\leq d-1}$, such that M_i^j is an R_j-module together with multiplication maps*

$$\mu_{i,j,l}^M : M_i^j \otimes A_{j,l} \longrightarrow M_{i+l-j}^l$$

for each i,j,l (where l and $i+l-j$ should be interpreted modulo d) satisfying the obvious compatibility condition for multiplication and unit.

2. *a morphism is a collection $f_{i,j}$ of R_j- linear maps $M_i^j \longrightarrow N_i^j$ such that*

$$f_{i+l-j,l} \circ \mu_{i,j,l}^M = \mu_{i,j,l}^N \circ (f_{i,j} \otimes A_{j,l})$$

Then there is a canonical isomorphism of categories $\mathcal{C} \cong \mathrm{Gr}(\overline{A})$

Proof. The above discussion shows that the assignment $M \longrightarrow (M_l e_i)_{l\in\mathbb{Z},0\leq i\leq n-1}$ is well defined and essentially surjective. A morphism of graded modules $f : M \longrightarrow N$ will satisfy $f(M_i e_j) \subset N_i e_j$ and we can define $f_{i,j}$ as the restriction to these submodules. The A-linearity guarantees that $(f_{i,j})_{i,j}$ indeed defines a morphism in $\mathcal{C}$ and since $\oplus M_i e_j = M$ it is clear that this assignment is faithful. Since any collection of maps $f_{i,j}$ satisfying the above compatibility with the multiplication will sum up to an $\overline{A}$-linear map, the assignment is also full. □

Lemma 5.3.15. *There exists a decomposition*

$$\mathcal{C} = \mathcal{C}_0 \oplus \ldots \oplus \mathcal{C}_{d-1}$$

where $\mathcal{C}_n$ is the full subcategory of $\mathcal{C}$ whose objects are collections of R-modules $(M_i^j)_{i\in\mathbb{Z},0\leq j\leq d-1}$ where $M_i^j = 0$ unless $j - i \equiv n$ (mod d).

Proof. This follows immediately from the construction of $\mathcal{C}$ and the fact that $j - i = l - (l + i - j)$. Hence, if $(\mathcal{M}_i^j)_{ij}$ is a non-zero object in $\mathcal{C}_n$, then so is $(\mathcal{M}_{l+i-j}^l)_{ij}$ for all l. □

Proposition 5.3.16. *There is an exact embedding of categories*

$$\overline{(-)} : \mathrm{Gr}(A) \hookrightarrow \mathrm{Gr}(\overline{A})$$

moreover the essential image is a direct summand of $\mathrm{Gr}(\overline{A})$.

Proof. Let M be an A-module with multiplication maps $\mu_{i,m} : M_i \otimes_R A_m \longrightarrow M_{i+m}$ and let $\mathcal{C}$ be as above. We define an object $\overline{M}$ in $\mathcal{C}$ by

$$\overline{M}_i^j = \begin{cases} M_i & \text{if } j \equiv i \bmod d \\ 0 & \text{else} \end{cases}$$

where the multiplication is given by

$$\overline{\mu}_{i,j,l} = \begin{cases} \mu_{i,l-j} & \text{if } j \equiv i \bmod d \\ 0 & \text{else} \end{cases}$$

This assignment clearly defines an exact embedding $\mathrm{Gr}(A) \xrightarrow{\simeq} \mathcal{C}_0 \hookrightarrow \mathcal{C}$, finishing the proof by lemmas 5.3.14 and 5.3.15. $\square$

5.3.4 a Local Description of $\mathbb{S}(\mathcal{E})$

In this final step in the preparation of the proof of theorem 5.3.1, we complete the local description of $\mathbb{S}(\mathcal{E})$. By 5.3.10, we have reduced the claim to the case where X and Y are affine. By our hypothesis on X and $\mathcal{E}$ (see 5.2.1,5.3.5), we assume that $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ are affine varieties over $\Bbbk$ such that S/R is relative Frobenius of rank 4 and $\omega_X \cong \mathcal{O}_X$, $\omega_Y \cong \mathcal{L} \cong \mathcal{O}_Y$. After applying the global section functor, we obtain a bimodule $\mathbb{Z}$-algebra in the sense of 5.3.11 which is 2-periodic. The graded algebra associated to this $\mathbb{Z}$-algebra by the construction in §5.3.3 is precisely the generalized preprojective algebra defined in 4.2.8 and studied in chapter 2.

We start by introducing some auxiliary notations. Recall the convention 5.2.11 and let $\mathcal{A}$ be a sheaf $\mathbb{Z}$-algebra over X_i. There is a $\mathbb{Z}$-algebra over $\Bbbk$, $\Gamma(\mathcal{A})$ defined in each component by

$$\Gamma(\mathcal{A})_{i,j} \stackrel{\text{def}}{=} \Gamma(X_i \times X_j, \mathcal{A}_{i,j})$$

since each component $\Gamma(\mathcal{A})_{i,j}$ is an $R-S$ or $S-R$ bimodule depending on the parity of the indices, $\Gamma(\mathcal{A})$ is in fact a $\mathbb{Z}$-algebra over commutative groundring $R \oplus S$ as in the discussion in the beginning of §4.2.2. The equivalence between quasi-coherent sheaves over an affine scheme and modules over the ring of global sections can easily be adapted to our setting to yield an equivalence:

$$\Gamma : \mathrm{Gr}(\mathcal{A}) \xrightarrow{\simeq} \mathrm{Gr}(\Gamma(\mathcal{A})) : \{\mathcal{M}_n\}_{n\in\mathbb{Z}} \mapsto \{\Gamma(X_n, \mathcal{M}_n)\}_{n\in\mathbb{Z}}$$

The following is an immediate consequence of the assumptions of this section:

Lemma 5.3.17. *The $\mathbb{Z}$-algebra $\Gamma(\mathbb{S}(\mathcal{E}))$ is 2-periodic in the sense that*

$$\Gamma(\mathbb{S}(\mathcal{E}))_{i,j} = \Gamma(\mathbb{S}(\mathcal{E}))_{i+2,j+2}$$

Proof. By 5.2.22, there are isomorphisms $\mathbb{S}(\mathcal{E}))_{i+2,j+2} \cong \omega_i^{-1} \otimes \mathbb{S}(\mathcal{E})) \otimes \omega_j$. By the assumptions in the beginning of this section, both canonical bundles are trivial, implying that $\mathbb{S}(\mathcal{E})_{i,j} = \mathbb{S}(\mathcal{E})_{i+2,j+2}$. The result follows after applying $\Gamma(-)$. $\square$

Using lemma 5.3.13, the 2-periodic $\mathbb{Z}$-algebra $\Gamma(\mathbb{S}(\mathcal{E}))$ gives rise to a graded algebra $\overline{\Gamma(\mathbb{S}(\mathcal{E}))}$. We now have:

Lemma 5.3.18. *Let $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ be smooth affine varieties such that S/R is relative Frobenius of rank 4. Let $f : Y \to X$ be the induced morphism and $\mathcal{E} = {}_f(\mathcal{O}_Y)_{\mathrm{Id}}$. Then $\overline{\Gamma(\mathbb{S}(\mathcal{E}))} \cong \Pi_R(S)$.*

Proof. Consider the quotient map

$$\mathbb{T}(\mathcal{E}) \twoheadrightarrow \mathbb{S}(\mathcal{E})$$

Taking global sections in each component $\Gamma(X_i \times X_j, (-)_{i,j})$ yields a surjection

$$\Gamma(\mathbb{T}(\mathcal{E})) \twoheadrightarrow \Gamma(\mathbb{S}(\mathcal{E})).$$

as $X_i \times X_j$ is affine.
Since the functor $\overline{(-)}$ preserves surjectivity (see Proposition 5.3.16), we obtain a map

$$\pi : \overline{\Gamma(\mathbb{T}(\mathcal{E}))} \twoheadrightarrow \overline{\Gamma(\mathbb{S}(\mathcal{E}))}.$$

We first show that there is a canonical isomorphism of $R \oplus S$-modules

$$\overline{\Gamma(\mathbb{T}(\mathcal{E}))} \cong T(R, S) \tag{5.17}$$

For this (as $\Gamma\left(\mathbb{S}(\mathcal{E})\right)$ is clearly generated in degrees 0 and 1) it suffices to show the following three facts

- $\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_0 \cong T(R,S)_0 = R \oplus S$ as rings
- $\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_1 \cong T(R,S)_1 \cong {}_R S_S \oplus {}_S S_R$ as $R \oplus S$-modules
- the multiplication map yields isomorphisms

$$\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_1 \otimes \overline{\Gamma(\mathbb{T}(\mathcal{E}))}_n \xrightarrow{\cong} \overline{\Gamma(\mathbb{T}(\mathcal{E}))}_{n+1}$$

For the first statement, we compute:

$$\begin{aligned}
\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_0 &= \begin{pmatrix} \Gamma(\mathbb{T}(\mathcal{E}))_{0,0} & 0 \\ 0 & \Gamma(\mathbb{T}(\mathcal{E}))_{1,1} \end{pmatrix} \\
&= \begin{pmatrix} \Gamma\left(X \times X, {}_{\mathrm{Id}}\left(\mathcal{O}_X\right)_{\mathrm{Id}}\right) & 0 \\ 0 & \Gamma\left(Y \times Y, {}_{\mathrm{Id}}\left(\mathcal{O}_Y\right)_{\mathrm{Id}}\right) \end{pmatrix}
\end{aligned}$$

moreover, we have

$$\begin{aligned}
\Gamma\left(X \times X, {}_{\mathrm{Id}}\left(\mathcal{O}_X\right)_{\mathrm{Id}}\right) &= \mathrm{Hom}\left(\mathcal{O}_{X\times X}, \Delta_*\left(\mathcal{O}_X\right)\right) \\
&= \mathrm{Hom}\left(\Delta^*\left(\mathcal{O}_{X\times X}\right), \mathcal{O}_X\right) \\
&= \mathrm{Hom}\left(\mathcal{O}_X, \mathcal{O}_X\right) \\
&\cong R
\end{aligned}$$

And similarly $\Gamma\left(Y \times Y, {}_{\mathrm{Id}}\left(\mathcal{O}_Y\right)_{\mathrm{Id}}\right) \cong S$. combining these calculations yields

$$\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_0 \cong \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \cong R \oplus S$$

In a completely similar fashion, we check the second condition:

$$\begin{aligned}
\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_1 &= \begin{pmatrix} 0 & \Gamma(\mathbb{T}(\mathcal{E}))_{0,1} \\ \Gamma(\mathbb{T}(\mathcal{E}))_{1,2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Gamma\left(X \times Y, \mathcal{E}\right) \\ \Gamma\left(Y \times X, \mathcal{E}^*\right) & \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Gamma\left(X \times Y, {}_f(\mathcal{O}_Y)_{\mathrm{Id}}\right) \\ \Gamma\left(Y \times X, {}_{\mathrm{Id}}(\mathcal{O}_Y)_f\right) & 0 \end{pmatrix} \\
&\cong \begin{pmatrix} 0 & {}_RS_S \\ {}_SS_R & 0 \end{pmatrix} \cong {}_RS_S \oplus {}_SS_R
\end{aligned}$$

To check the final condition, we have the isomorphisms

$$\mathbb{T}(\mathcal{E})_{i,i+1} \otimes \mathbb{T}(\mathcal{E})_{i+1,i+n+1} \longrightarrow \mathbb{T}(\mathcal{E})_{i,i+n+1}$$

We now apply the functor $\Gamma(X_i \times X_{i+n+1}, -)$ and note that since all varieties are affine, the tensor product and $\Gamma(-)$ commute, resulting in an isomorphism

$$\Gamma(\mathbb{T}(\mathcal{E}))_{i,i+1} \otimes \Gamma(\mathbb{T}(\mathcal{E}))_{i+1,i+n+1} \longrightarrow \Gamma(\mathbb{T}(\mathcal{E}))_{i,i+n+1}$$

application of the functor $\overline{(-)}$ yields

$$\overline{\Gamma(\mathbb{T}(\mathcal{E}))}_1 \otimes \overline{\Gamma(\mathbb{T}(\mathcal{E}))}_n \xrightarrow{\simeq} \overline{\Gamma(\mathbb{T}(\mathcal{E}))}_{n+1}$$

we have thus constructed the required isomorphism (5.17). Finally, we prove that the relations defining $\Pi_R S$ coincide with the kernel of π, i.e. there is a commutative diagram:

$$\begin{tikzcd}
\overline{\Gamma(\mathbb{T}(\mathcal{E}))} \arrow[r, "\pi"] \arrow[d, "\cong"'] & \overline{\Gamma(\mathbb{S}(\mathcal{E}))} \arrow[d, "\cong"] \\
T(R,S) \arrow[r, "\overline{\pi}"'] & \Pi_R(S)
\end{tikzcd}$$

The isomorphisms in the previous step yield isomorphisms:

$$\zeta_0 : \mathrm{Hom}_{X \times X}({}_{\mathrm{Id}}\left(\mathcal{O}_X\right)_{\mathrm{Id}}, \mathcal{E} \otimes \mathcal{E}^*) \xrightarrow{\simeq} \mathrm{Hom}_R(R, {}_RS_S \otimes_S S_R)$$

$$\zeta_1 : \mathrm{Hom}_{Y\times Y}({}_{\mathrm{Id}}(\mathcal{O}_Y)_{\mathrm{Id}}, \mathcal{E}^* \otimes \mathcal{E}) \xrightarrow{\simeq} \mathrm{Hom}_S(S, {}_SS_R \otimes_R S_S)$$

Recall that $\mathbb{S}(\mathcal{E})$ is defined as a quotient of $\mathbb{T}(\mathcal{E})$ by the relations given by the unit morphisms $i_0 \in \mathrm{Hom}_{X\times X}({}_{\mathrm{Id}}(\mathcal{O}_X)_{\mathrm{Id}}, \mathcal{E}\otimes\mathcal{E}^*)$ and $i_1 \in \mathrm{Hom}_{Y\times Y}({}_{\mathrm{Id}}(\mathcal{O}_Y)_{\mathrm{Id}}, \mathcal{E}^*\otimes \mathcal{E})$described in (5.6). Similarly $\Pi_R(S)$ is defined as a quotient of $T_R(S)$ by elements $\eta_0 \in \mathrm{Hom}_R(R, {}_RS_S \otimes_S S_R)$, $\eta_1 \in \mathrm{Hom}_S(S, {}_SS_R \otimes_R S_S)$. Hence we must prove $\zeta_0(i_0) = \eta_0$ and $\zeta_1(i_1) = \eta_1$. To this end, note that there is a commutative diagram of isomorphisms

$$\begin{tikzcd}
\mathrm{Hom}_{X\times Y}(\mathcal{E},\mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Hom}_{X\times X}({}_{\mathrm{Id}}(\mathcal{O}_X)_{\mathrm{Id}}, \mathcal{E}\otimes\mathcal{E}^*) \arrow[d, "\zeta_0"] \\
\mathrm{Hom}_{R\otimes S}({}_RS_S, {}_RS_S) \arrow[r, "\varphi_0"] & \mathrm{Hom}_R(R, {}_RS_S \otimes_S S_R)
\end{tikzcd}$$

where φ_0 is given by the adjunction $(- \otimes_R S_S) \dashv (- \otimes_S S_R) = (-)_R$. Hence $\zeta_0(i_0) = \varphi_0(\mathrm{Id}_{{}_RS_S}) : 1_R \mapsto 1_S \otimes 1_S$and this morphism indeed coincides with η_0. Similarly the existence of the dual bases $(e_i)_i$, $(f_j)_j$ implies there is an adjunction
$- \otimes_S S_R = (-)_R \dashv (-) \otimes_R S_S$ given by

$$\varphi_1 : \mathrm{Hom}_R(M\otimes_S S_R, N) \longrightarrow \mathrm{Hom}_S(M, N\otimes_R S_S) : \psi \mapsto \left(\psi' : m \mapsto \sum_i \psi(me_i) \otimes f_i\right)$$

Where we used the lemma 5.3.19 below to show that the morphisms in the image of φ_1 indeed have an S-module structure. A commutative diagram as above shows that $\zeta_1(i_1) = \varphi_1(\mathrm{Id}_{{}_SS_R}) : 1_S \mapsto \sum_i e_i \otimes f_i$ which coincides with η_1.

$\square$

Lemma 5.3.19. *$\sum_i e_i \otimes f_i$ is central in the S-bimodule $S \otimes_R S$. I.e. for all $a \in S$ we have*

$$\sum_i ae_i \otimes f_i = \sum_i e_i \otimes f_i a$$

Proof. It is sufficient to prove that for all j, k we have

$$\sum_i \lambda(ae_i f_j)\lambda(f_i e_k) = \sum_i \lambda(e_i f_j)\lambda(f_i a e_k)$$

which is clear since both sides are equal to $\lambda(ae_k f_j)$. $\square$

5.3.5 Proof of Theorem 5.3.1

We will now combine everything. As X and Y are noetherian we know that $\mathrm{Qcoh}(X)$ and $\mathrm{Qcoh}(Y)$ are locally noetherian categories and hence there exist collections of noetherian generating objects for these categories, say $\mathcal{N}^X \overset{\text{def}}{=} \{\mathcal{N}_i^X\}_{i\in I}$ and $\mathcal{N}^Y \overset{\text{def}}{=} \{\mathcal{N}_j^Y\}_{j\in J}$. For each $n \in \mathbb{Z}$ we define $\mathcal{N}^n$ in $\mathrm{Qcoh}(X_n)$ as:

$$\mathcal{N}^n = \begin{cases} \mathcal{N}^X & \text{if } n \text{ is even} \\ \mathcal{N}^Y & \text{if } n \text{ is odd} \end{cases}$$

We shall prove that the collection

$$\{\mathcal{N} \otimes e_n \mathbb{S}(\mathcal{E}) \mid n \in \mathbb{Z}, \mathcal{N} \in \mathcal{N}^n\} \tag{5.18}$$

forms a set of noetherian generators for $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$. Note that the collection is easily seen to generate as for each $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$ there is a surjective morphism

$$\bigoplus_{n\in\mathbb{Z}} \mathcal{M}_n \otimes e_n\mathcal{A} \twoheadrightarrow \mathcal{M}$$

and for each $n \in \mathbb{Z}$ there is a surjective morphism

$$\bigoplus_{\alpha} (\mathcal{N}_\alpha^n)^{m_\alpha} \twoheadrightarrow \mathcal{M}_n$$

where $\mathcal{N}_\alpha^n \in \mathcal{N}^n$. Hence we only need to show that the elements of (5.18) are noetherian objects in $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$. By lemma 5.3.7 and Corollary 5.3.8 this can be checked locally for any open cover $X = \bigcup_l U_l$. By theorem 5.3.10 we may hence assume that $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ are smooth affine varieties such that

i) $\mathcal{L} \cong \mathcal{O}_Y \cong \omega_Y$

ii) $\omega_X \cong \mathcal{O}_X$

iii) S/R is relative Frobenius of rank 4.

With these assumptions there are functors

$$\begin{tikzcd} \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \arrow[d, "\Gamma(-)", "\cong"'] \\ \mathrm{Gr}(\Gamma(\mathbb{S}(\mathcal{E}))) \arrow[d, hook, "\text{Proposition 5.3.16}"] \\ \mathrm{Gr}\left(\overline{\Gamma(\mathbb{S}(\mathcal{E}))}\right) \arrow[d, "\text{lemma 5.3.18}", "\cong"'] \\ \mathrm{Gr}(\Pi_R(S)) \end{tikzcd} \tag{5.19}$$

Let $F : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \longrightarrow \mathrm{Gr}(\Pi_R(S))$ be the composition. Then the above diagram shows that F is an exact embedding of categories. Hence $\mathcal{N} \otimes e_n\mathbb{S}(\mathcal{E})$ is a noetherian object in $\mathrm{Gr}(\mathbb{S}(\mathcal{E}))$ if $F(\mathcal{N} \otimes e_n\mathbb{S}(\mathcal{E}))$ is a noetherian object in $\mathrm{Gr}(\Pi_R(S))$. On the other hand, as $\mathcal{N}$ is noetherian in $\mathrm{Qcoh}(X_n)$ there is an $m \in \mathbb{N}$ and a surjection $\mathcal{O}_{X_n}^{\oplus m} \twoheadrightarrow \mathcal{N}$ giving rise to an surjection

$$F(\mathcal{O}_{X_n} \otimes e_n\mathbb{S}(\mathcal{E}))^{\oplus m} \twoheadrightarrow F(\mathcal{N} \otimes e_n\mathbb{S}(\mathcal{E}))$$

Hence it suffices to show that $F(\mathcal{O}_{X_n} \otimes e_n\mathbb{S}(\mathcal{E}))$ is a Noetherian object in $\Pi_R(S)$. This is however obvious as

$$F(\mathcal{O}_{X_n} \otimes e_n\mathbb{S}(\mathcal{E})) = \begin{cases} R \cdot \Pi_R(S)(-n) & \text{if } n \text{ is even} \\ S \cdot \Pi_R(S)(-n) & \text{if } n \text{ is odd} \end{cases}$$

As both $R \cdot \Pi_R(S)$ and $S \cdot \Pi_R(S)$ are direct summands of $\Pi_R(S)$, which is a noetherian ring by theorem 4.5.1, we have proven the theorem. $\square$

5.4 the Homological Properties of $\mathbb{S}(\mathcal{E})$

5.4.1 A Formula for Ext-Groups

Throughout this section $\mathcal{E}$ will be a locally free $X - Y$-bimodule of rank $(4, 1)$ and we let $\mathcal{A} \overset{\text{def}}{=} \mathbb{S}(\mathcal{E})$ denote the associated symmetric sheaf $\mathbb{Z}$-algebra in standard form (see 5.3.5). This section is dedicated to adapting the results in [VdB12], [Nym04a], [Nym04b] and [Mor07] to obtain a formula for the Ext-groups of pulled back sheaves on $\mathrm{Proj}(\mathcal{A})$. To keep the geometric intuition we denote the truncation functors $(\omega(-))_m : \mathrm{Proj}(\mathcal{A}) \longrightarrow \mathrm{Qcoh}(X_m)$ by Π_{m*} (compare with 5.2.23). The left adjoints, which are given explicitly by $p((-)\otimes e_m\mathcal{A})$ following 5.4 and 5.2.6, are in turn denoted by Π_m^*. We shall use the notations X_n and Q_n as in 5.2.11 and (5.7).

If $\mathcal{E} \in \mathrm{bimod}(X - X)$ is locally free of rank (2,2) and $\mathcal{A} = \mathbb{S}(\mathcal{E})$, [Mor07] computes the Euler characteristics $\langle \Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}\rangle$ for two locally free sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. In this section, we perform an analogous calculation in our setting where the bimodule $\mathcal{E} \in \mathrm{bimod}(X - Y)$ is of rank (4,1). Motivated by Proposition 5.2.22 our focus lies on $\langle \Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}\rangle$ with $|n-m| \leq 1$ This section is dedicated to proving the following slightly more general statement:

Theorem 5.4.1. *Let $\mathcal{E} \in \mathrm{bimod}(X, Y)$ be locally free of rank (4,1). Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves on X_m respectively X_n for $m, n \in \mathbb{Z}$ such that $m \geq n-1$. Then*

$$\mathrm{Ext}^i_{\mathrm{Proj}(\mathcal{A})}\left(\Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}\right) \cong \mathrm{Ext}^i_{X_m}\left(\mathcal{F}, \mathcal{G} \otimes \mathbb{S}(\mathcal{E})_{n,m}\right)$$

for all $i \geq 0$.

This formula implies the following facts:

Corollary 5.4.2. *With the above assumptions, one has*

- $\langle \Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}\rangle = \langle \mathcal{F}, \mathcal{G} \otimes \mathbb{S}(\mathcal{E})_{n,m}\rangle$
- *Let $\{\mathcal{F}_1, \ldots, \mathcal{F}_\alpha\}$ and $\{\mathcal{G}_1, \ldots, \mathcal{G}_\beta\}$ be exceptional sequences of locally free sheaves on X_n and X_{n+1} respectively.*
  Then $\Pi_{n+1}^*\mathcal{G}_1, \ldots \Pi_{n+1}^*\mathcal{G}_b, \Pi_n^*\mathcal{F}_1, \ldots, \Pi_n^*\mathcal{F}_b$ is an exceptional sequence on $\mathrm{Proj}(\mathcal{A})$.

The proof of theorem 5.4.1 is based on the existence of an exact sequence (see 5.20 below). To this end, we consider Θ_m defined by

$$(\Theta_m)_n = \begin{cases} 0 & m \neq n \\ \mathcal{O}_{X_m} & n = m \end{cases}$$

Remark 5.4.3. *Note that Θ_m is a right $\mathcal{A}$-module using $\mathcal{A}_{i,i} = \mathcal{O}_{X_i}$*

Theorem 5.4.4. *For each m, there is an exact sequence of locally free objects in* $\operatorname{bimod}(\mathcal{O}_{X_m} - \mathcal{A})$ *(see [VdB12, Section 3.2.] for the definition of this category)*

$$0 \longrightarrow \mathcal{Q}_m \otimes e_{m+2}\mathcal{A} \longrightarrow \mathcal{E}^{*m} \otimes e_{m+1}\mathcal{A} \longrightarrow e_m\mathcal{A} \longrightarrow \Theta_m \longrightarrow 0 \tag{5.20}$$

Proof. By the nature of the relations this sequence is known to be right exact. The proof of the left exactness uses so-called 'point modules' and is deferred to 5.4.2. □

As an immediate corollary of this theorem and its proof we find:

Corollary 5.4.5. *for each $i, j \in \mathbb{Z}$, the bimodule $\mathcal{A}_{i,j}$ is locally free both on the left and on the right. The ranks are given by*

$$\operatorname{rk}(\mathcal{A})_{i,j} \stackrel{def}{=} \begin{cases} (j-i+1, j-i+1) & i \equiv j \, mod \, 2 \\ \left(\dfrac{j-i+1}{2}, 2(j-i+1)\right) & i \, odd, \, j \, even \\ \left(2(j-i+1), \dfrac{j-i+1}{2}\right) & i \, even, \, j \, odd \end{cases}$$

Proof. We have $\operatorname{rk}(\mathcal{E}) = (4,1)$ and $\operatorname{rk}(\mathcal{E}^*) = (1,4)$, $\operatorname{rk}(\mathcal{Q}_m) = (1,1)$. Since the rank is additive on short exact sequence, one can now verify the claim by induction in the three cases on n using the sequences in 5.20 □

This theorem in turn implies the following convenient fact

Lemma 5.4.6. *For each $m \in \mathbb{Z}$, the functor $\Pi_m^* : \operatorname{Qcoh}(X_m) \longrightarrow \operatorname{Proj}(\mathcal{A})$ is an exact functor*

Proof. For each $n \geq m$, $\mathcal{A}_{m,n}$ is locally free by Corollary 5.4.5, hence the functor $- \otimes \mathcal{A}_{m,n} : \mathrm{Qcoh}(X_m) \longrightarrow \mathrm{Qcoh}(X_n)$ is exact. □

As an example application of 5.4.6, we mention the following adjoint formula:

Lemma 5.4.7. *There is a natural isomorphism for all $\mathcal{F} \in \mathrm{Qcoh}(X_m)$ and $\mathcal{M} \in \mathcal{D}^+(\mathrm{Proj}(\mathcal{A}))$:*

$$\mathrm{RHom}_{\mathrm{Proj}(\mathcal{A})}(\Pi_m^* \mathcal{F}, \mathcal{M}) \cong \mathrm{RHom}_{X_m}(\mathcal{F}, \mathrm{R}\,\Pi_{m*}\mathcal{M}) \tag{5.21}$$

Proof. Since Π_m^* is an exact left adjoint to $\Pi_{m,*}$, the latter must preserve injective objects and the result follows. □

For the purposes of proving theorem 5.4.1 we are especially interested in the case where $\mathcal{M} = \Pi_n^* \mathcal{G}$ for a locally free sheaf $\mathcal{G}$ on X_n in the isomorphism (5.21). It follows that we need to understand complexes of the form $R\Pi_{m*}(\Pi_n^* \mathcal{G})$. The strategy for computing the homology of this complex is as follows: by lemma 5.4.9 below, it suffices to give a description the derived functors of the torsion functor $\tau : \mathrm{Gr}(\mathbb{S}(\mathcal{E})) \longrightarrow \mathrm{Tors}(\mathbb{S}(\mathcal{E})$. These in turn follow from the derived functors of an internal Hom-functor $\underline{\mathcal{H}om}$ (lemma 5.4.11).

Lemma 5.4.8. *We have the following facts for the derived functors of the torsion functor $\tau : \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Tors}(\mathcal{A})$:*

i) for $i \geq 1$, there is an isomorphism of functors

$$\mathrm{R}^{i+1}\,\tau \cong (\mathrm{R}^i\,\omega) \circ p$$

ii) For each $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$ there is an exact sequence:

$$0 \longrightarrow \tau(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow \omega(p(\mathcal{M})) \longrightarrow \mathrm{R}^1\,\tau(\mathcal{M}) \longrightarrow 0$$

Proof. By theorem 5.3.1, $\mathrm{Gr}(\mathcal{A})$ is a locally noetherian category. By [Nym04b, lemma 2.12], any essential extension of a torsion module remains a torsion module. In particular, the category $\mathrm{Tors}(\mathcal{A})$ is closed under injective envelopes, the result now follows from [Smi99, theorem 2.14.15]. □

Lemma 5.4.9. *For $i \geq 1$, there is an isomorphism*

$$\mathrm{R}^i\,\Pi_{m*}(\Pi_n^*\mathcal{V}) \cong \mathrm{R}^{i+1}\,\tau(\mathcal{V} \otimes e_n\mathcal{A})_m$$

Proof. As the functors p and $(-)_m$ are exact there is a functorial isomorphism

$$(\mathrm{R}^i\,\Pi_{m*})(p)(-) \cong \mathrm{R}^i\,\omega(p(-))_m. \tag{5.22}$$

Combining this isomorphism with the one in lemma 5.4.8 we obtain for each $i \geq 1$:

$$\mathrm{R}^i\,\Pi_{m*}(\Pi_n^*\mathcal{V}) \stackrel{\mathrm{def}}{=} \mathrm{R}^i\,\Pi_{m*}(p(\mathcal{V}\otimes e_n\mathcal{A})) \cong \mathrm{R}^i\,\omega(p(\mathcal{V}\otimes e_n\mathcal{A}))_m \cong \mathrm{R}^{i+1}\,\tau(\mathcal{V}\otimes e_n\mathcal{A})_m.$$

□

The following is based on [Nym04a, Section 3.2]:
Let $\mathrm{BiMod}(\mathcal{A} - \mathcal{A})$ denote the category whose objects are of the form

$$\{\mathcal{B}_{m,n} \in \mathrm{BiMod}(X_m - X_n)\}_{m,n}$$

such that the left and right multiplications

$$\mathcal{A}_{l,m} \otimes \mathcal{B}_{m,n} \longrightarrow \mathcal{B}_{l,n} \text{ and } \mathcal{B}_{m,n} \otimes \mathcal{A}_{n,l} \longrightarrow \mathcal{B}_{m,l}$$

are compatible in the obvious sense. We denote by $\mathbb{B}$ the subcategory for which all $\mathcal{B}_{m,n}$ are coherent and locally free. There are Hom-functors

$$\underline{\mathcal{H}om} : \mathbb{B}^{op} \times \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Gr}(\mathcal{A}) \text{ and}$$
$$\mathcal{H}om : \mathrm{BiMod}(\mathcal{O}_{X_n} - \mathcal{A}) \times \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Qcoh}(X_n)$$

satisfying the following properties:

Proposition 5.4.10. *i) $\underline{\mathcal{H}om}(\mathcal{B}, \mathcal{M})_m = \mathcal{H}om(e_m\mathcal{B}, \mathcal{M})$ for all $\mathcal{B} \in \mathbb{B}$ and $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$*

ii) $\underline{\mathcal{H}om} : \mathbb{B}^{op} \times \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Gr}(\mathcal{A})$ is a bifunctor, left exact in both its arguments *iii)* $\mathcal{H}om : \mathrm{BiMod}(\mathcal{O}_{X_n} - \mathcal{A}) \times \mathrm{Gr}(\mathcal{A}) \longrightarrow \mathrm{Qcoh}(X_n)$ *is a bifunctor, left exact in both its arguments*

iv) $\mathcal{H}om(\mathcal{Q} \otimes e_m\mathcal{A}, \mathcal{M}) \cong \mathcal{M}_m \otimes \mathcal{Q}^*$ *for all* $\mathcal{M} \in \mathrm{Gr}(\mathcal{A})$ *and locally free* X_m*-bimodules* $\mathcal{Q}$

Proof. i) This follows immediately by checking the precise definitions in [Nym04a, Section 3.2]

ii) see [Nym04a, Proposition 3.11, theorem 3.16(1)]

iii) see [Nym04a, theorem 3.16(3)]

iv) see [Nym04a, theorem 3.16(4)] □

By ii. and iii. in the above proposition one cans define the right derived functors $\underline{\mathcal{E}xt}^i$ and $\mathcal{E}xt^i$ for all $i \geq 0$. Moreover we use the notation $\mathcal{A}_{\geq l}$ to denote the object in $\mathbb{B}$ given by

$$(\mathcal{A}_{\geq l})_{m,n} = \begin{cases} \mathcal{A}_{m,n} & \text{if } n-m \geq l \\ 0 & \text{else} \end{cases}$$

and $\mathcal{A}_0 \stackrel{\text{def}}{=} \mathcal{A}/\mathcal{A}_{\geq 1}$. Then we have the following relation between the derived functors of τ and the $\underline{\mathcal{E}xt}^i$:

Lemma 5.4.11. $R^i\tau(-) \cong \lim\limits_{l\to\infty} \underline{\mathcal{E}xt}^i_{\mathrm{Gr}(\mathcal{A})}(\mathcal{A}/\mathcal{A}_{\geq l}, -)$

Proof. By [Nym04b, Proposition 3.19], we have an isomorphism of functors

$$\tau \cong \lim_{l\to\infty} \underline{\mathcal{H}om}_{\mathrm{Gr}(\mathcal{A})}(\mathcal{A}/\mathcal{A}_{\geq l}, -)$$

Applying this to the injective resolution and subsequently taking homology yields the required result □

Lemma 5.4.12. *Let* $\mathcal{B} \in \mathbb{B}$ *be concentrated in degree* $l \geq 0$ *(i.e.* $\mathcal{B}_{m,n} = 0$ *whenever* $m + l \neq n$*) and* $\mathcal{V}$ *a locally free sheaf. Then for* $n - l - 1 \leq m$ *and for all* $i \geq 0$*:*

$$\underline{\mathcal{E}xt}^i(\mathcal{B}, \mathcal{V} \otimes e_n\mathcal{A})_m = 0$$

Proof. By [Nym04a, cor. 4.6], there is an isomorphism

$$\underline{\mathcal{E}xt}^i(\mathcal{B}, \mathcal{V} \otimes e_n\mathcal{A})_m \cong \underline{\mathcal{E}xt}^i(\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A})_{m+l} \otimes \mathcal{B}^*_{m,m+l}$$

which easily reduces the proof to the case $\mathcal{B} = \mathcal{A}_0$ and in particular $l = 0$. By Proposition 5.4.10(4) we see that the exact sequence from theorem 5.4.4 forms a resolution of $e_m\mathcal{A}_0 = \Theta_m$ through $\mathcal{H}om(-, \mathcal{V} \otimes e_n\mathcal{A})$-acyclic sheaves. In particular we can calculate $\underline{\mathcal{E}xt}^i(\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A})_m = \mathcal{E}xt^i(e_m A_0, \mathcal{V} \otimes e_n\mathcal{A})$ by taking homology of the complex

$$\begin{aligned} &0 \longrightarrow \mathcal{H}om(e_m\mathcal{A}, \mathcal{V} \otimes e_n\mathcal{A}) \xrightarrow{d_0} \mathcal{H}om(\mathcal{E}^{*m} \otimes e_{m+1}\mathcal{A}, \mathcal{V} \otimes e_n\mathcal{A}) \\ &\xrightarrow{d_1} \mathcal{H}om(\mathcal{Q}_m \otimes e_{m+2}\mathcal{A}, \mathcal{V} \otimes e_n\mathcal{A}) \longrightarrow 0 \end{aligned}$$

using Proposition 5.4.10(iv), this complex becomes

$$0 \longrightarrow \mathcal{V} \otimes \mathcal{A}_{n,m} \xrightarrow{d_0} \mathcal{V} \otimes \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1} \xrightarrow{d_1} \mathcal{V} \otimes \mathcal{A}_{n,m+2} \otimes \mathcal{Q}^*_m \longrightarrow 0 \quad (5.23)$$

Hence we have

- $\mathcal{E}xt^{\,0}(e_m\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A}) = \ker(d_0)$
- $\mathcal{E}xt^{\,1}(e_m\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A}) = \ker(d_1)/\operatorname{im}(d_0)$
- $\mathcal{E}xt^{\,2}(e_m\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A}) = \operatorname{coker}(d_1)$
- $\mathcal{E}xt^i(e_m\mathcal{A}_0, \mathcal{V} \otimes e_n\mathcal{A}) = 0$ for all $i \geq 3$

To show the exactness of (5.23), we first note that the explicit nature of the isomorphisms in [Nym04a] yield that (5.23) is obtained from the sequence

$$0 \longrightarrow \mathcal{A}_{n,m} \longrightarrow \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1} \longrightarrow \mathcal{A}_{n,m+2} \otimes Q^*_m \longrightarrow 0 \quad (5.24)$$

by tensoring with $\mathcal{V}$. Since $\mathcal{V}$ is locally free, it preserves exactness and it suffices to verify that (5.24) is exact. Next, we tensor with the invertible bimodule Q_m to obtain

$$0 \longrightarrow \mathcal{A}_{n,m} \otimes Q_m \xrightarrow{d_0} \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1} \otimes Q_m \xrightarrow{d_1} \mathcal{A}_{n,m+2} \longrightarrow 0 \quad (5.25)$$

We can replace the middle term in 5.25 to obtain:

$$0 \longrightarrow \mathcal{A}_{n,m} \otimes Q_m \xrightarrow{d_0} \mathcal{A}_{n,m+1} \otimes \mathcal{E}^{*m+1} \xrightarrow{d_1} \mathcal{A}_{n,m+2} \longrightarrow 0 \tag{5.26}$$

A similar but tedious computation as in [Nym04a, §7.5] shows that this sequence coincides with is the exact sequence 5.4.4 in degree n for left modules. We conclude the result by the same argument as for 5.4.4

□

Lemma 5.4.13. $\underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{V} \otimes e_n\mathcal{A})_m = 0$ *for* $m \geq n-1$ *and* $i \geq 0$

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathcal{A}_{\geq l}/\mathcal{A}_{\geq l+1} \longrightarrow \mathcal{A}/\mathcal{A}_{\geq l+1} \longrightarrow \mathcal{A}/\mathcal{A}_{\geq l} \longrightarrow 0$$

Applying $\underline{\mathcal{H}om}(-, \mathcal{V} \otimes e_n\mathcal{A})$ gives rise to a long exact sequence for each $m \geq n-1$

$$\begin{aligned} \ldots \longrightarrow \underline{\mathcal{E}xt}^i(\mathcal{A}_{\geq l}/\mathcal{A}_{\geq l+1}, \mathcal{V} \otimes e_n\mathcal{A})_m &\longrightarrow \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l+1}, \mathcal{V} \otimes e_n\mathcal{A})_m \\ \longrightarrow \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{V} \otimes e_n\mathcal{A})_m &\longrightarrow \underline{\mathcal{E}xt}^{i+1}(\mathcal{A}_{\geq l}/\mathcal{A}_{\geq l+1}, \mathcal{V} \otimes e_n\mathcal{A})_m \longrightarrow \ldots \end{aligned}$$

As $m \geq n-1$ it follows from lemma 5.4.12 that for each $i \geq 0$ we have an exact sequence

$$0 \longrightarrow \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l+1}, \mathcal{V} \otimes e_n\mathcal{A})_m \longrightarrow \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{V} \otimes e_n\mathcal{A})_m \longrightarrow 0$$

Hence

$$\underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{V} \otimes e_n\mathcal{A})_m \cong \underline{\mathcal{E}xt}^i(\mathcal{A}/\mathcal{A}_{\geq 0}, \mathcal{V} \otimes e_n\mathcal{A})_m = \underline{\mathcal{E}xt}^i(0, \mathcal{V} \otimes e_n\mathcal{A})_m = 0$$

□

We can now finish the proof of theorem 5.4.1

Proof. of theorem 5.4.1
Take $m, n \in \mathbb{Z}$ with $m \geq n-1$. Let $\mathcal{F}$ be locally free on X_m and $\mathcal{G}$ locally free on X_n, then by Corollary 5.4.7:

$$\begin{aligned} \operatorname{Ext}^i_{\operatorname{Proj}(\mathcal{A})}(\Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}) &= h^i\left(\operatorname{R}\operatorname{Hom}_{\operatorname{Proj}(\mathcal{A})}(\Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G})\right) \\ &\cong h^i\left(\operatorname{R}\operatorname{Hom}_{X_m}(\mathcal{F}, \operatorname{R}\Pi_{m*}\Pi_n^*\mathcal{G})\right) \end{aligned}$$

Now for $i \geq 1$ we have

$$\begin{aligned} \mathrm{R}^i\, \Pi_{m*}\Pi_n^*\mathcal{G} &\cong \mathrm{R}^{i+1}\,\tau(\mathcal{G}\otimes e_n\mathcal{A})_m \\ &\cong \lim_{l\to\infty} \underline{\mathcal{E}xt}^{i+1}(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{G}\otimes e_n\mathcal{A})_m \\ &= 0 \end{aligned}$$

by lemmas 5.4.9, 5.4.11 and 5.4.13 respectively.
In particular the complex $\mathrm{R}\,\Pi_{m*}\Pi_n^*\mathcal{G}$ is quasi-isomorphic to the complex that is equal to $\Pi_{m*}\Pi_n^*\mathcal{G}$ concentrated in position zero. Finally we can conclude by noticing that $\Pi_{m*}\Pi_n^*\mathcal{G} = (\omega p(\mathcal{G}\otimes e_n\mathcal{A}))_m$ and by lemma 5.4.8 there is an exact sequence

$$0 = \tau(\mathcal{G}\otimes e_n\mathcal{A})_m \longrightarrow \mathcal{G}\otimes\mathcal{A}_{n,m} \xrightarrow{\cong} \omega(p(\mathcal{G}\otimes e_n\mathcal{A}))_m \longrightarrow \mathrm{R}^1\,\tau(\mathcal{G}\otimes e_n\mathcal{A})_m = 0$$

where the first term equals zero because $\mathcal{G}\otimes e_n\mathcal{A}$ is torsion free and the last term is zero because $\mathrm{R}^1\,\tau(\mathcal{G}\otimes e_n\mathcal{A})_m \cong \lim_{l\to\infty} \underline{\mathcal{E}xt}\;^1(\mathcal{A}/\mathcal{A}_{\geq l}, \mathcal{G}\otimes e_n\mathcal{A})_m = 0$.
Hence we can conclude that for $m \geq n-1$ we have

$$\begin{aligned} \mathrm{Ext}^i_{\mathrm{Proj}(\mathcal{A})}(\Pi_m^*\mathcal{F}, \Pi_n^*\mathcal{G}) &\cong h^i\left(\mathrm{R}\,\mathrm{Hom}_{X_m}(\mathcal{F}, \mathrm{R}\,\Pi_{m*}\Pi_n^*\mathcal{G})\right) \\ &\cong h^i\left(\mathrm{R}\,\mathrm{Hom}_{X_m}(\mathcal{F}, \mathcal{G}\otimes\mathcal{A}_{n,m})\right) \\ &= \mathrm{Ext}^i_{X_m}(\mathcal{F}, \mathcal{G}\otimes\mathcal{A}_{n,m}) \quad \square \end{aligned}$$

5.4.2 Point Modules in the Rank $(4,1)$ Case

We remain in the setting where $\mathcal{A} = \mathbb{S}(\mathcal{E})$ denotes a symmetric sheaf $\mathbb{Z}$-algebra in standard form with $\mathcal{E} \in \mathrm{bimod}(X-Y)$ locally free of rank (4,1), given in the form of $\mathcal{E} = {}_f(\mathcal{L})_{\mathrm{Id}}$ for a finite flat morphism $f : Y \longrightarrow X$ as in lemma 5.3.3. Furthermore, X, Y are smooth varieties over the algebraically closed field $\Bbbk$ and denote by $\alpha : X \longrightarrow \mathrm{Spec}(\Bbbk)$ and $\beta : Y \longrightarrow \mathrm{Spec}(\Bbbk)$ be the structure morphisms. Extending our convention, 5.2.11 we will write

$$(X_n, \alpha_n) = \begin{cases} (X, \alpha) & \text{if } n \text{ is even} \\ (Y, \beta) & \text{if } n \text{ is odd} \end{cases}$$

We say $P_n \in \mathrm{coh}(X_n)$ is locally free over $\Bbbk$ of rank l if the support of P_n is finite over $\Bbbk$ and $\dim_{\Bbbk} \alpha_{n,*} P_n = l$.

A module $P \in \mathrm{Gr}(\mathcal{A})$ is said to be generated in degree m if $P_n = 0$ for all $n < m$ and $P_m \otimes \mathcal{A}_{m,n} \longrightarrow P_n$ is surjective for all $n \geq m$. As $\mathcal{A}$ is generated in degree one as an algebra, we have surjectivity of $P_{n_1} \otimes \mathcal{A}_{n_1,n_2} \longrightarrow P_{n_2}$ for all $n_2 \geq n_1 \geq m$ by the following commuting diagram

$$\begin{tikzcd}
P_m \otimes \mathcal{A}_{m,n_1} \otimes \mathcal{A}_{n_1,n_2} \arrow[r, two heads] \arrow[d, two heads] & P_{n_1} \otimes \mathcal{A}_{n_1,n_2} \arrow[d] \\
P_m \otimes \mathcal{A}_{m,n_2} \arrow[r, two heads] & P_{n_2}
\end{tikzcd}$$

Remark 5.4.14. *An obvious example of a module generated in degree m is $e_m\mathcal{A}$. The above diagram implies that the maps $\mathcal{A}_{m,n} \otimes e_n\mathcal{A} \longrightarrow e_m\mathcal{A}$ are surjective for all $m \geq n$.*

An m-shifted point-module over $\mathcal{A}$ is defined in [VdB12] as an object $P \in \mathrm{Gr}(\mathcal{A})$ such that P is generated in degree m and for which P_n is locally free of rank one over $\Bbbk$ for all $n \geq m$. As the next lemma shows, this concept however is not very useful in our setting:

Lemma 5.4.15. *Let $i \in \mathbb{Z}$ and $P \in \mathrm{Gr}(\mathcal{A})$ generated in degree $2i$ such that P_{2i} and P_{2i+1} are locally free of rank one over $\Bbbk$. Then $P_n = 0$ for all $n \geq 2i+2$.*

Proof. Recall that the following composition

$$P_{2i} \longrightarrow P_{2i} \otimes \mathcal{E}^{*2i} \otimes \mathcal{E}^{*2i+1} \longrightarrow P_{2i+1} \otimes \mathcal{E}^{*2i+1} \longrightarrow P_{2i+2}$$

must be zero as it represents the action of $\mathcal{Q}_{2i}$. By [VdB12, lemma 4.3.2.] this composition equals

$$P_{2i} \xrightarrow{\varphi_{2i}^*} P_{2i+1} \otimes \mathcal{E}^{*2i+1} \xrightarrow{\varphi_{2i+1}} P_{2i+2}$$

where φ_{2i}^* is obtained by adjointness from $\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \longrightarrow P_{2i+1}$ and $\mathcal{E}^{*2i+1}$ has rank $(1,4)$. Since P_{2i} and $P_{2i+1} \otimes \mathcal{E}^{*2i+1}$ are locally all free of rank one over

$\Bbbk$ we obtain that φ_{2i}^* is either an isomorphism or the zero morphism. Similarly φ_{2i+1} is either injective or zero. Hence the only way the composition can be zero is if $\varphi_{2i}^* = 0$ or $\varphi_{2i+1} = 0$. The first doesn't occur as $\varphi_{2i} \neq 0$ (because P is generated in degree $2i$ and $P_{2i+1} \neq 0$). Hence we have $\varphi_{2i+1} = 0$. However φ_{2i+1} is surjective (because P is generated in degree $2i$), implying that $P_{2i+2} = 0$. Using surjectivity of $P_{2i+2} \otimes \mathcal{A}_{2i+2,n} \longrightarrow P_n$ for all $n \geq 2i+2$ the result follows. $\square$

We thus propose the following variation of the above definition, better suited to our needs:

Definition 5.4.16. A shifted point module is an object $P \in \mathrm{Gr}(\mathcal{A})$ which is generated in degree $2i$ for some integer i and such that for all $n \geq 2i$, P_n is locally free over $\Bbbk$ of rank one if n is even and rank two if n is odd. We will often use the short hand notation $\dim_{\Bbbk}(P_n) = \dim_{\Bbbk}(\alpha_{n,*}(P_n))$ whenever the latter is finite. So we could say P is a shifted point module if is generated in degree $2i$ and:

$$\dim_{\Bbbk}(P_n) = \begin{cases} 0 & \text{if } n < 2i \\ 1 & \text{if } n \geq 2i \text{ is even} \\ 2 & \text{if } n > 2i \text{ is odd} \end{cases}$$

The following lemma shows that this new definition of point modules is better behaved than the naive one:

Lemma 5.4.17. *Let $P \in \mathrm{Gr}(\mathcal{A})$ be a graded module and $i \in \mathbb{Z}$ such that:*

- *P is generated in degree $2i$*
- $\dim_{\Bbbk}(P_{2i}) = 1$
- $\dim_{\Bbbk}(P_{2i+1}) = 2$

Then for all $n \geq 2i+2$ fixed, we have

$$\dim_{\Bbbk}(P_n) \leq \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 2 & \textit{if } n \textit{ is odd} \end{cases} \tag{5.27}$$

Moreover if equality holds in (5.27)*, then P_n is characterized up to unique isomorphism by the data $\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \longrightarrow P_{2i+1}$.*
If on the other hand (5.27) *is a strict inequality for some n, then $P_l = 0$ for all $l > n$.*

Proof. We prove all facts by induction on n. So suppose (5.27) and the subsequent claims hold for $n = 2i, \ldots, m$. We distinguish several cases depending on whether the inequalities are in fact equalities or not.

Case 1: Equality holds in (5.27) for $n = 2i, \ldots, m$.
The following composition is zero:

$$P_{m-1} \overset{\varphi^*_{m-1}}{\longrightarrow} P_m \otimes \mathcal{E}^{*m} \overset{\varphi_m}{\longrightarrow} P_{m+1}$$

φ_m is surjective, using the fact that the ranks are $(4,1)$ or $(1,4)$ depending on the parity of m, on easily verifies that (5.27) holds for $n = m+1$ if φ^*_{m-1} is injective. Moreover the same reasoning show that if the equality holds for $\dim_{\Bbbk}(P_{m+1})$, then $P_{m+1} \cong \operatorname{coker}(\varphi^*_{m-1})$ and is hence defined up to unique isomorphism.

Case 1a: m is odd
$\dim_{\Bbbk}(P_{m-1}) = 1$ hence it suffices to prove $\varphi^*_{m-1} \neq 0$ and this holds because $\varphi_{m-1} \neq 0$
Case 1b: m is even
If φ^*_{m-1} is not injective, then there is with $W \subset P_{m-1}$, of $\dim_{\Bbbk}(W) = 1$ such that the composition

$$W \hookrightarrow P_{m-1} \overset{\varphi^*_{m-1}}{\longrightarrow} P_m \otimes \mathcal{E}^{*m}$$

or equivalently the composition

$$W \otimes \mathcal{E}^{*m-1} \hookrightarrow P_{m-1} \otimes \mathcal{E}^{*m-1} \longrightarrow P_m$$

is zero. This implies that there is a $\overline{W} \in \operatorname{Gr}(\mathcal{A})$ given by $\overline{W}_{m-1} = W$ and $\overline{W}_l = 0$ for $l \neq m-1$ such that there is an embedding $\chi : \overline{W} \hookrightarrow P_{\geq m-2}$. Let

$C = \operatorname{coker}(\chi)_{\geq m-2}$. Then C is generated in degree $m-2$ (which is even!) and $\deg_k(C_{m-2}) = \deg_k(C_{m-1}) = \deg_k(C_m) = 1$ contradicting lemma 5.4.15.

Case 2: There is an integer $n \in \{2i+2, \dots, m\}$ such that there is a strict inequality for $\dim_{\Bbbk}(P_n)$ in (5.27)
Let n_0 be the smallest such n. We have to show $P_l = 0$ for all $l > n_0$.
Assume that $P_{n_0} = 0$, then $P_l = 0$ by surjectivity of $P_{n_0} \otimes \mathcal{A}_{n_0,l} \longrightarrow P_l$.
The only nontrivial case is when n_0 is odd and $\dim_{\Bbbk}(P_{n_0}) = 1$. In this case $\dim_{\Bbbk}(P_{n_0-1}) = 1$ as well and the result follows from lemma 5.4.15. □

Remark 5.4.18. *The proof of the above lemma also shows that any data $\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \twoheadrightarrow P_{2i+1}$ with $\dim_{\Bbbk}(P_{2i}) = 1$ and $\dim_{\Bbbk}(P_{2i+1}) = 2$ can be extended to a shifted point module which is unique up to a unique isomorphism.*

From now on we use the following short hand notation:

$$L_{n,p} \stackrel{\text{def}}{=} \mathcal{O}_p \otimes e_n\mathcal{A} \tag{5.28}$$

where p is any point on X_n.

Proof. of theorem 5.4.4
Exactness of the sequence (5.20) can be checked for each degree n separately:

$$0 \longrightarrow \mathcal{Q}_m \otimes \mathcal{A}_{m+2,n} \longrightarrow \mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n} \longrightarrow \mathcal{A}_{m,n} \longrightarrow 0 \tag{5.29}$$

As all terms in this sequence are elements of $\operatorname{bimod}(X_m - X_n)$, applying the pushforward of the projection $X_m \times X_n \longrightarrow X_m$, $\pi_{m,*}$ yields a sequence of coherent sheaves on X_m:

$$0 \longrightarrow \pi_{m,*}(\mathcal{Q}_m \otimes \mathcal{A}_{m+2,n}) \longrightarrow \pi_{m,*}(\mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n}) \longrightarrow \pi_{m,*}(\mathcal{A}_{m,n}) \longrightarrow 0 \tag{5.30}$$

and (5.30) is exact if and only if (5.29) is since the support of these bimodules is finite. The structure of the relations on $\mathcal{A}$ implies that (5.20) and hence also (5.29) and (5.30) are right exact. Now for any point $p \in X_m$ the following complex will be right exact as well:

$$\begin{aligned} 0 \to \mathcal{O}_p \otimes \pi_{m,*}(\mathcal{Q}_m \otimes \mathcal{A}_{m+2,n}) &\to \mathcal{O}_p \otimes \pi_{m,*}(\mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n}) \to \\ &\to \mathcal{O}_p \otimes \pi_{m,*}(\mathcal{A}_{m,n}) \to 0 \end{aligned} \tag{5.31}$$

As all terms (5.31) are locally free over $\Bbbk$, its left exactness can be checked numerically. Hence in order to prove the lemma we show that the terms in (5.31) have the 'correct' constant dimension (see (5.36) for each point p. From this it follows that (5.29) is exact and its terms are locally free on the left. The locally freeness on the right then follows from [VdB12, Proposition 3.1.6.]. .)

So we are left with finding the length of the objects in (5.31). Any object in bimod$(X_m - X_n)$ is of the form ${}_u\mathcal{U}_v$ for finite maps u and v. As taking the direct image through a finite morphism does not change the length of sheaves, we have for such a bimodule:

$$\begin{aligned} \dim_\Bbbk(\mathcal{O}_p \otimes \pi_{m,*}({}_u\mathcal{U}_v)) &= \dim_\Bbbk(\mathcal{O}_p \otimes u_*\mathcal{U}) \\ &= \dim_\Bbbk(u_*(u^*(\mathcal{O}_p) \otimes \mathcal{U})) \\ &= \dim_\Bbbk(u^*(\mathcal{O}_p) \otimes \mathcal{U}) \\ &= \dim_\Bbbk(v_*(u^*(\mathcal{O}_p) \otimes \mathcal{U})) \\ &= \dim_\Bbbk(\mathcal{O}_p \otimes {}_u\mathcal{U}_v)) \end{aligned}$$

Hence the length of the terms in (5.31) can be calculated from

$$0 \to \mathcal{O}_p \otimes \mathcal{Q}_m \otimes \mathcal{A}_{m+2,n} \to \mathcal{O}_p \otimes \mathcal{E}^{*m} \otimes \mathcal{A}_{m+1,n} \to \mathcal{O}_p \otimes \mathcal{A}_{m,n} \to 0 \tag{5.32}$$

In the case where $m = 2i-1$, the fact that $\dim_\Bbbk(\mathcal{O}_p \otimes \mathcal{E}^{*2i-1}) = 1$, implies that there must be a point $q \in X_{2i}$ such that $\mathcal{O}_p \otimes \mathcal{E}^{*2i-1} = \mathcal{O}_q$. Similarly, in the case where $m = 2i$, we have $\dim_\Bbbk(\mathcal{O}_p \otimes \mathcal{E}^{*2i}) = 4$, and there must be points $\widetilde{q^a} \in X_{2i+1}$, $a = 1, \ldots, 4$ such that $\mathcal{O}_p \otimes \mathcal{E}^{*2i}$ is an extension of the $\mathcal{O}_{\widetilde{q^a}}$. Put

$$M_{2i+1,p} = \mathcal{O}_p \otimes_{X_{2i}} \mathcal{E}^{*2i} \otimes_{X_{2i+1}} e_{2i+1}\mathcal{A}.$$

Then $M_{2i+1,p}$ is an extension of the $L_{2i+1,\tilde{q^a}}$. The sequence (5.32) now gives rise to the following right exact sequences

$$L_{2i+1,p} \longrightarrow L_{2i,q} \longrightarrow L_{2i-1,p} \longrightarrow 0 \tag{5.33}$$

$$L_{2i+2,p} \longrightarrow M_{2i+1,p} \longrightarrow L_{2i,p} \longrightarrow 0 \tag{5.34}$$

Finally there also is a right exact sequence:

$$L_{2i+1,p'} \longrightarrow L_{2i-1,p} \longrightarrow P_p \longrightarrow 0 \tag{5.35}$$

where the morphism $L_{2i+1,p'} \longrightarrow L_{2i-1,p}$ comes from the fact that $\dim_{\Bbbk}(\mathcal{O}_p \otimes \mathcal{A}_{2i-1,2i+1}) = 3 > 0$ so that there is a $p' \in X_{2i+1}$ with a nonzero morphism $\mathcal{O}_{p'} \longrightarrow \mathcal{O}_p \otimes \mathcal{A}_{2i-1,2i+1}$. P_p is defined as the cokernel of this morphism.

We now prove the following by induction on j (simultaneously for all p and all i):

$$\begin{aligned}
\dim_{\Bbbk}((P_p)_{2i+2j}) &= 1 \\
\dim_{\Bbbk}((P_p)_{2i+2j+1}) &= 2 \\
\dim_{\Bbbk}((L_{2i,p})_{2i+2j}) &= 2j+1 \\
\dim_{\Bbbk}((L_{2i,p})_{2i+2j+1}) &= 4j+4 \\
\dim_{\Bbbk}((L_{2i-1,p})_{2i+2j}) &= j+1 \\
\dim_{\Bbbk}((L_{2i-1,p})_{2i+2j+1}) &= 2j+3
\end{aligned} \tag{5.36}$$

It is easy to see that these claims hold for $j = 0$. So by induction we suppose they hold for $j = 0, \ldots, l$, for all p and for all $i \in \mathbb{Z}$. We prove that the claims also hold for $j = l + 1$.

By (5.34) we see:

$$\begin{aligned}
\dim_{\Bbbk}((L_{2i,p})_{2i+2l+2}) &\geq \dim_{\Bbbk}((M_{2i+1,p})_{2i+2l+2}) - \dim_{\Bbbk}((L_{2i+1,p})_{2i+2l+2}) \\
&= \sum_{a=1}^{4} \dim_{\Bbbk}((L_{2i+2,\widetilde{q^a_{2i+1}}})_{2i+2l+2}) - \dim_{\Bbbk}((L_{(2i+2,p})_{2i+2l+2}) \\
&= 4 \cdot (l+1) - (2l+1) \\
&= 2l+3
\end{aligned}$$

where the last equality follows from the induction hypothesis. This can be written schematically as:

$$\begin{array}{cccccccccc}
0 & & & & & & & & & \\
\uparrow & & & & & & & & & \\
L_{2i,p} & 0 & 1 & 4 & 3 & \dots & 2l+1 & 4l+4 & \underline{2l+3} & \underline{4l+8} \\
\uparrow & & & & & & & & & \\
M_{2i+1,p} & 0 & 0 & 4 & 4 & \dots & 4l & 8l+4 & 4l+4 & 8l+12 \\
\uparrow & & & & & & & & & \\
L_{2i+2,\widetilde{p}} & 0 & 0 & 0 & 1 & \dots & 2l-1 & 4l & 2l+1 & 4l+4
\end{array} \tag{5.37}$$

Where the numbers on the right of a module signifies $\dim_{\Bbbk}((-)_x)$ for $x = 2i-1, \dots, 2i+2l+3$ and an underlined number implies a lower bound for $\dim_{\Bbbk}$. Similarly we write $\overline{N}$ to denote an upperbound for a certain $\dim_{\Bbbk}$.

Now consider the module $P_{p,\geq 2i+2l}$. It is generated in degree $2i+2l$ because P_p is a quotient of $L_{2i-1,p}$. Moreover $\dim_{\Bbbk}((P_p)_{2i+2l}) = 1$ and $\dim_{\Bbbk}((P_p)_{2i+2l+1}) = 2$, so lemma 5.4.17 implies $\dim_{\Bbbk}((P_p)_{2i+2l+2}) \leq 1$ and $\dim_{\Bbbk}((P_p)_{2i+2l+3}) \leq 2$. Together with the right exact sequence (5.35) this gives us the following upper bounds:

$$\begin{array}{ccccccccc}
0 & & & & & & & & \\
\uparrow & & & & & & & & \\
P & 1 & 1 & 2 & \dots & 1 & 2 & \overline{1} & \overline{2} \\
\uparrow & & & & & & & & \\
L_{2i-1,p} & 1 & 1 & 3 & \dots & l+1 & 2l+3 & \overline{l+2} & \overline{2l+5} \\
\uparrow & & & & & & & & \\
L_{2i+1,p'} & 0 & 0 & 1 & \dots & l & 2l+1 & l+1 & 2l+3
\end{array} \tag{5.38}$$

Combining the bounds found in (5.37) and (5.38) and using (5.33) we have found

$$\begin{array}{ccccccccc}
0 & & & & & & & & \\
\uparrow & & & & & & & & \\
L_{2i-1,p} & 1 & 1 & 4 & \dots & l+1 & 2l+3 & \overline{l+2} & \overline{2l+5} \\
\uparrow & & & & & & & & \\
L_{2i,q} & 0 & 1 & 4 & \dots & 2l+1 & 4l+4 & \underline{2l+3} & \underline{4l+8} \\
\uparrow & & & & & & & & \\
L_{2i+1,\widetilde{p}} & 0 & 0 & 1 & \dots & l-1 & 2l+1 & l+1 & 2l+3
\end{array} \tag{5.39}$$

Right exactness of (5.33) implies that the bounds in (5.39) are in fact equalities, because for example we find the upper bound

$$\begin{aligned}
\dim_{\Bbbk}((L_{2i,q})_{2i+2l+2}) &\leq \dim_{\Bbbk}((L_{2i-1,p})_{2i+2l+2}) + \dim_{\Bbbk}((L_{2i+1,\widetilde{p}})_{2i+2l+2}) \\
&\leq l+2+l+1 \\
&= 2l+3
\end{aligned}$$

which equals the already known lower bound for $\dim_{\Bbbk}((L_{2i,q})_{2i+2l+2}$. Hence we have found exact values for $\dim_{\Bbbk}((L_{2i+1,q})$. A priori the above right exact sequence only gives those exact value for the points $q \in X_{2i}$ for which there is a $p \in X_{2i-1}$ such that $\mathcal{O}_p \otimes \mathcal{E}^{*2i-1} = \mathcal{O}_q$. But as $\mathcal{E}^{*2i-1}$ is of the form ${}_{\mathrm{Id}}(\mathcal{L}_{i-1})_f$ as in (5.14) we have $q = f(p)$ and surjectivity of f implies that q runs through all points of X_{2i} as p runs through all points of X_{2i-1}. With the same reasoning we now obtain from (5.31) the exact values for $\dim_k(L_{2i-1})_{2i+2l+2}$ and

$\dim_k(L_{2i-1})_{2i+2l+3}$..

Hence we have proven (5.36) for all $i, j \in \mathbb{Z}$ and for all points p. As these values do not depend on p and X is a smooth variety, it follows from [Har97, ex. II, §5, no.8] that the terms in (5.30) are locally free on the left (and hence also on the right). Filling in these values for (5.31), the theorem follows. $\square$

5.4.3 the Main Full Exceptional Sequence

We have done the preparatory work needed to compute the dimensions of the Ext-groups of the exceptional sequence 5.1. We will prove that the sequence 5.1 is full through the following series of lemmas:

Lemma 5.4.19. *Let $\mathcal{T}$ be a $\Bbbk$-linear triangulated category. Assume that $E_1, \ldots, E_n$ is a collection of objects in $\mathcal{T}$ such that*

(a) $\sum_j \dim \mathrm{Hom}^j_{\mathcal{T}}(E_i, T) < \infty$ *for all i and for all $T \in \mathrm{Ob}(\mathcal{T})$.*

(b) $(E_i)_i$ *satisfies the conditions for an exceptional sequence (see definition 0.2.1), except that we do not require* Hom*-finiteness of $\mathcal{T}$.*

(c) we have

$$((E_i)_i)^{\perp} \stackrel{def}{=} \{Y \in \mathcal{T} \,|Hom^i(E_m, Y) = 0\, \forall m\} = 0$$

Then

1. *$E_1, \ldots, E_n$ generate $\mathcal{T}$ as a triangulated category and*
2. *$\mathcal{T}$ is* Ext*-finite.*

Proof. Let $T \in \mathcal{T}$. We have to prove that T is in the triangulated subcategory of $\mathcal{T}$ generated by $E_1, \ldots, E_n$. We put $T_n = T$ and define T_{i-1} inductively by $L_{E_i} T_i$ for $i = n, n-1, \ldots, 1$, i.e.

$$T_{i-1} = \mathrm{cone}(\mathrm{Hom}^{\bullet}_{\mathcal{T}}(E_i, T_i) \otimes_k E_i \longrightarrow T_i)$$

(see 3.5) Then T_i is in the triangulated subcategory of $\mathcal{T}$ generated by T_{i-1} and E_i. Furthermore

$$\mathrm{Hom}^{\bullet}_{\mathcal{T}}(E_j, T_i) = 0 \text{ for } j > i$$

It follows that $T_0 = 0$. Hence we are done.

For (2) we have to prove that if $T_1, T_2 \in \mathcal{T}$ then $\sum_j \dim \mathrm{Hom}^j_{\mathcal{T}}(T_1, T_2) < \infty$. Since T_1 is in the triangulated category generated by $(E_i)_i$ we may assume $T_1 = E_i$ for some i. But then the claim is part of the hypotheses. $\square$

Lemma 5.4.20. *Let X, Y be smooth varieties over $\Bbbk$ and let $\mathcal{E}$ be a locally free $X - Y$-bimodule of rank $(4,1)$. Put $\mathcal{A} = \mathbb{S}(\mathcal{E})$. Then for all $m \in \mathbb{Z}$ one has*

1. *the cohomological dimension of $\Pi_{m,*}$ satisfies*[4]

$$\mathrm{cd}\, \Pi_{m,*} \leq 1.$$

2. *If $\mathcal{F}$ is a noetherian object then $R^i \Pi_{m,*} \mathcal{F}$ is a coherent sheaf for all i.*

Proof. 1. By (5.22), we have

$$\mathrm{R}^i\, \Pi_{m,*}(p(-)) = \mathrm{R}^i\, \omega(p(-))_m$$

which reduces the claim to $\mathrm{cd}\, \omega = 1$. From lemma 5.4.8 we in turn obtain

$$\mathrm{R}^i\, \omega(p(-)) \cong \mathrm{R}^{i+1}\, \tau$$

and the claim now reduces to $\mathrm{cd}\, \tau = 2$. This is proved as in [Nym04a, cor 4.10] using the exact sequence (5.20) instead of the exact sequence (4.1) in loc. cit.

[4]It is easy to see that in (1) the cohomological dimension is exactly one, but we do not need it and leave it out for clarity

2. Since $\mathrm{Gr}(\mathcal{A})$ is locally noetherian we may construct a left resolution of $\mathcal{F}$ by objects which are finite direct sums of objects of the form

$$p(\mathcal{G} \otimes_{\mathcal{O}_{X_n}} e_n\mathcal{A}) = \Pi_n^*(\mathcal{G})$$

for $\mathcal{G} \in \mathrm{coh}(X_n)$. Using that $\Pi_{m,*}$ has finite cohomological dimension we reduce to the case $\mathcal{F} = \Pi_n^*(\mathcal{G})$.

Tensoring (5.20) (with m replaced by n) on the left with $\mathcal{G} \in \mathrm{coh}(X_n)$ we obtain exact sequences in $Z = \operatorname{Proj} \mathcal{A}$

$$0 \longrightarrow \Pi_{n+2}^*(\mathcal{G}) \longrightarrow \Pi_{n+1}^*(\mathcal{G} \otimes_{X_n} \mathcal{E}^{*n}) \longrightarrow \Pi_n^*(\mathcal{G}) \longrightarrow 0 \tag{5.40}$$

Hence repeatedly using such exact sequences we may reduce to the case $\mathcal{F} = \Pi_n^*(\mathcal{G})$ for $n \leq m$. When $n \leq m$ it is shown in the proof of theorem 5.4.1 that

$$R^i\Pi_{m,*}\Pi_n^*\mathcal{G} = \begin{cases} \mathcal{G} \otimes_{X_n} \mathcal{A}_{n,m} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

This is indeed coherent. ☐

Lemma 5.4.21. *Assume $X = \mathbb{P}^1$ and let $f : Y \longrightarrow X$ be a morphism of degree 4. Put $\mathcal{E} = {}_f(\mathcal{O}_X)_{\mathrm{Id}}$ and $\mathcal{A} = \mathbb{S}(\mathcal{E})$. Then the right orthogonal to the subcategory generated by*

$$E = (\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1)))$$

in $\mathscr{D}(\operatorname{Proj} \mathcal{A})$ is zero.

Proof. Assume that $A \in \operatorname{Proj} \mathcal{A}$ is right orthogonal to E. Using the exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a+1)^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a+2) \longrightarrow 0$$

and the right exactness of Π_m^* (Lemma 5.4.7) we find that A is right orthogonal to $\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(a))$ for $m = 0, 1$ and all a.

From (5.40) we obtain exact sequences in $\operatorname{Proj} \mathcal{A}$

$$0 \longrightarrow \Pi^*_{m+2}(\mathcal{O}_{\mathbb{P}^1}(a)) \longrightarrow \Pi^*_{m+1}(\mathcal{O}_{\mathbb{P}^1}(a) \otimes_{X_m} \otimes E^{*m}) \longrightarrow \Pi^*_m(\mathcal{O}_{\mathbb{P}^1}(a)) \longrightarrow 0$$

Since $\mathcal{O}_{\mathbb{P}^1}(a) \otimes_{X_m} \mathcal{E}^{*m}$, being locally free, is isomorphic to a sum of $\mathcal{O}_{\mathbb{P}}(b)$ we conclude by induction that A is right orthogonal to $\Pi^*_m(\mathcal{O}_{\mathbb{P}^1}(a))$ for all m, a.

Now $(\Pi^*_m(\mathcal{O}_{\mathbb{P}^1}(a)))_{m,a}$ is a collection of generators for $\operatorname{Proj} \mathcal{A}$ as a Grothendieck category. From this it is easy to see that the right orthogonal to $(\Pi^*_m(\mathcal{O}_{\mathbb{P}^1}(a)))_{m,a}$ in $\mathscr{D}(\operatorname{Proj} \mathcal{A})$ is zero. This finishes the proof. □

We are now ready to prove the main theorem of this chapter:

Theorem 5.4.22. *Let $\mathcal{E}$ be a $\mathbb{P}^1$-bimodule of rank $(4,1)$. Let $\mathbb{S}(\mathcal{E}))$ be the associated symmetric sheaf $\mathbb{Z}$-algebra and put $Z = \operatorname{Proj}(\mathbb{S}(\mathcal{E}))$. Let $\mathscr{D}$ denote the triangulated subcategory of objects in $\mathscr{D}(Z)$ with bounded noetherian cohomology. Then $\mathscr{D}$ is* Ext*-finite and*

$$\left(\Pi^*_1(\mathcal{O}_{\mathbb{P}^1}), \Pi^*_1(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi^*_0(\mathcal{O}_{\mathbb{P}^1}), \Pi^*_0(\mathcal{O}_{\mathbb{P}^1}(1)\right)$$

is a full strong exceptional sequence in $\mathscr{D}$ for which the Gram matrix of the Euler form is given by

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Proof. The computation of the Gram matrix, the strongness and exceptionality is an immediate application of 5.4.1:

$$\operatorname{Ext}^i_Z(\Pi^*_n\mathcal{F}, \Pi^*_n\mathcal{G}) = \operatorname{Ext}^i_{\mathbb{P}^1}(\mathcal{F}, \mathcal{G} \otimes \mathbb{S}(\mathcal{E})_{n,n}) = \operatorname{Ext}^i_{\mathbb{P}^1}(\mathcal{F}, \mathcal{G})$$

proving the claim for the subsequences

$$\left(\Pi^*_1(\mathcal{O}_{\mathbb{P}^1}), \Pi^*_1(\mathcal{O}_{\mathbb{P}^1}(1))\right) \text{ and } \left(\Pi^*_0(\mathcal{O}_{\mathbb{P}^1}), \Pi^*_0(\mathcal{O}_{\mathbb{P}^1}(1)\right)$$

There are no backward Hom's by the formula 5.4.1 once again.
There are four remaining cases. Since they are all very similar, we pick one out and leave the other three to the reader:

$$\begin{aligned}
\mathrm{Ext}^i_Z\left(\Pi_1^*(\mathcal{O}_{\mathbb{P}^1}), \Pi_0^*(\mathcal{O}_{\mathbb{P}^1}(1))\right) &= \mathrm{Ext}^i_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{S}(\mathcal{E})_{0,1}) \\
&= \mathrm{Ext}^i_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1) \otimes {}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{Id}}) \\
&= \mathrm{H}^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \otimes {}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{Id}}) \\
&\overset{(5.3)}{=} \mathrm{H}^i(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^1}(1)) \\
&= \mathrm{H}^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))
\end{aligned}$$

which is indeed only nonzero for $i \neq 0$, in which case it is 5-dimensional over $\mathbb{k}$.

To show that the sequence is full, We have to verify conditions (a)(b)(c) of lemma 5.4.19. Condition (a) follows from Lemma 5.4.20, which implies that $R\Pi_{m,*}\mathcal{G}$ lives in $D^b_{\mathrm{coh}}(\mathrm{Qcoh}(X_m))$, combined with the fact that by Lemma 5.4.8 we have $\mathrm{Ext}^i_{\mathrm{Proj}\,\mathcal{A}}(\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(a)), \mathcal{G}) = \mathrm{Ext}^i_{X_m}(\mathcal{O}_{\mathbb{P}^1}(a), R\Pi_{m,*}\mathcal{G})$.
Condition (b) is proven above. Finally, condition (c) follows from Lemma 5.4.21 $\square$

Bibliography

[AIR] C. Amiot, O. Iyama, and I. Reiten. Stable categories of Cohen-Macaulay modules and cluster categories. arXiv:1104.3658.

[AM69] M.F. Atiyah and I.G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley publishing company, 1969.

[Ami09] C. Amiot. Cluster categories for algebras of global dimension 2 and quivers with potential. *Ann. Inst. Fourier*, 59.6(2525-2590), 2009.

[AO14] C. Amiot and S. Oppermann. Cluster equivalence and graded derived equivalence. *Documenta. Math.*, 19:1155–1206, 2014.

[Aus78] M. Auslander. *Functors and morphisms determined by objects*, volume 37 of *Representation theory of algebras (Proc. Conf. Temple Univ.)*, pages 1–244. Dekker, 1978.

[Aus86] M. Auslander. Rational singularities and almost split sequences. *Trans. AMS.*, 23(2):511–531, 1986.

[AZ94] M. Artin and J. Zhang. Noncommutative projective schemes. *Adv. Math.*, 109(2):228–287, 1994.

[Bas68] H. Bass. *Algebraic K-Theory.* W.A.Benjamin,Inc., 1968.

[Bei90] A. Beilinson. Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra. *Funktsional. Anal. I. Prilozhen*, 12(3):68–69, 1990.

[BIRS33] A. Buan, O. Iyama, I. Reiten, and D. Smith. Mutation of cluster-tilting objects and potentials. *Amer. J. Math.*, 2011(4):835–887, 133.

[BK89] A. Bondal and M. Kapranov. Representable functors, Serre functors and mutations. *Izv. Akad. Nauk. SSSR Ser. Mat.*, 53(6):1183–1205, 1989.

[BK12] I. Burban and M. Kalck. Relative singularity category of a non-commutative resolution of singularities. *Adv. Math.231*, 1(414-435), 2012.

[BMR+06] A. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov. Tilting theory and cluster combinatorics. *Adv. Math.*, 204(2):572–618, 2006.

[Boc08] R. Bocklandt. Graded Calabi-Yau algebras of dimension 3. *Journal of Pure and Applied Algebra*, 212(1):14–32, 2008.

[BP94] A. Bondal and A. Polishchuk. Homological properties of associative algebras: the method of helices. *Russian Acad. Sci. Izv. Math.*, 42(2):219–260, 1994.

[BS10] T. Bridgeland and D. Stern. Helices on Del Pezzo surfaces and tilting Calabi-Yau algebras. *Adv. Math.*, 224(4):1672–1716, 2010.

[Buc] R-O. Buchweitz. Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings. *unpublished.*

[BVdB03] A. Bondal and M. Van den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. Journ.*, 3(1):1–36, 2003.

[BWS10] R. Bocklandt, M. Wemyss, and T. Schedler. Superpotentials and higher order derivations. *Journal of Pure and Applied Algebra*, 214(9):1501–1522, 2010.

[CBH98] W. Crawley-Boevey and M. Holland. Non-commutative deformations of Kleinian singularities. *Duke Math. J.*, 92:605–635, 1998.

[CDT07] A. Craw, Maclaganm D., and R. Thomas. Moduli of quiver representations II: Groebner basis techniques. *Journal of Algebra*, 316(2):514–535, 2007.

[CFT02] A. Cattaneo, G. Felder, and L. Tomassini. From local to global deformations quantization of Poisson manifolds. *Duke Math. J.*, 115(239-352), 2002.

[CR11] D. Calaque and A. Rossi. Compatibility with cap-procuts in Tsygan's formality and homological Duflo isomorphism. *Lett. Math. Phys*, 95(2):135–209, 2011.

[Cra] Alastair Craw. Explicit methods for derived categories of sheaves. lecture notes.

[CRVdB10] D. Calaque, A. Rossi, and M. Van den Bergh. Hochschild (co)homology for Lie algebroids. *Int. Math. Res. Not,*, 21(4098-4136), 2010.

[CS99] M. Chas and D. Sullivan. String topology. arXiv:9911159, 1999.

[Dav12] B. Davison. Superpotential algebras and manifolds. *Adv. Math.*, 231(2):879–912, 2012.

[Dol06] V. Dolgushev. A formality theorem for Hochschild chains. *adv. Math.*, 200(1):51–101, 2006.

[Dol08] V. Dolgashev. Derived categories. lecture notes, 2008.

[Dol09] V. Dolgushev. The Van den Bergh duality and the modular symmetry of a Poisson variety. *Selecta Math.*, 14(2):199–228, 2009.

[DT99] Y. Daletskii and B. Tsygan. Operations on Hochschild and cyclic complexes. *Methods Funct. Anal. Top.*, 5(4):62–86, 1999.

[dTdVdB10] Louis de Thanhoffer de Völcsey and Michel Van den Bergh. Explicit models for some stable categories of maximal Cohen-Macaulay modules. arXiv:1006.2021, 2010.

[dTdVdBar] L. de Thanhoffer de Volcsey and M. Van den Bergh. Numerical classification of exceptional collections of length 4 on Del Pezzo surfaces. in preparation, to appear.

[dTdVP14] L. de Thanhoffer de Volcsey and D. Presotto. Some generalizations of Preprojective algebras and their properties. arXiv:1412.6899, 2014.

[dTdVP15] Louis de Thanhoffer de Völcsey and Dennis Presotto. Homological properties of a certain noncommutative Del Pezzo surface. arXiv:1503.03992, 2015.

[dTdVVdB13] L. de Thanhoffer de Volcsey and M. Van den Bergh. Some new examples of nondegenerate quiver potentials. *Int. Math. Res. Not.*, 20(4672-4686), 2013.

[DTT08] V. Dolgushev, D. Tamarkin, and B. Tsygan. Formality of the homotopy calculus algebra of Hochschild cochains. arXiv:0807.5117, 2008.

[DWZ08] H. Derksen, J. Weyman, and A. Zelevinsky. Quivers with potentials and their representations. *Selecta. Math.*, 14(1):59–119, 2008.

[EE07] P. Etingof and C. Eu. Koszulity and the Hilbert series of preprojective algebras. *Math. Research Letters*, 14(4):589–596, 2007.

[Eis95] D. Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry.* Graduate Texts in Mathematics. Springer, 1995.

[Ful98] W. Fulton. *Intersection Theory.* Springer, 2nd edition, 1998.

[FZ02] S. Fomin and A. Zelevinsky. Cluster algebras i. foundations. *J. Amer. Math. Soc.*, 497-529(2), 2002.

[GD71] A. Grothendieck and J. Dieudonné. *Éléments de géometrie algebrique III: Étude cohomologique des faiscaux coherents, premiere partie (EGA 3a)*, volume 11 of *Lecture Notes in Mathematics.* IHES, 1971.

[Get92] E. Getzler. Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology. In *Quantum deformations and their representations*, volume 7, pages 96–78, Ramat Gan, 1991-1992. Bar-Ilan Univ.

[Gin] V. Ginzburg. Calabi-Yau algebras. arXiv:0612139.

[GK04] A. Gorodentsev and S. Kuleshov. Helix theory. *Moscow Mathematical Journal*, 4(2):377–440, 2004.

[Gro71] A. Grothendieck. *Rêvetement étales et groupe fondamental (SGA 1)*, volume 224 of *Lecture Notes in Mathematics.* Springer, 1971.

[Guo10] L. Guo. Cluster tilting objects in generalized higher cluster categories. arXiv:1005.3564, 05 2010.

[GV95] M. Gerstenhaber and A. Voronov. Higher order operations on the Hochschild complex. *Funktsional. Anal. I. Prilozhen*, 29(1):6–96, 1995.

[Har97] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 8 edition, 1997.

[IO09] Osamu Iyama and Steffen Oppermann. Stable categories of higher preprojective algebras. arXiv:0912.3412, 2009.

[IR08] O. Iyama and I. Reiten. Fomin-Zelevsinsky mutations and tilting modules over Calabi-Yau algebras. *Amer. J. Math.*, 130(4):1087–1149, 2008.

[IT10] Osamu Iyama and Ryo Takahashi. Tilting and cluster tilting for quotient singularities. arXiv:1012.5954, 12 2010.

[IW14] O. Iyama and M. Wemyss. Maximal modifications and Auslander-Reiten duality for non-isolated singularities. *Invent. Math.*, 197(3):521–586, 2014.

[IY08] O. Iyama and Y. Yoshino. Mutation in triangulated categories and rigid Cohen-Macauley modules. *Invent. Math.*, 172(1):117–168, 2008.

[Kar77] M. Karoubi. *K-theory: an introduction*. Number 226 in Grundlehren der mathematische wissenschaften. Springer, 1977.

[Kel03] B. Keller. Derived invariance and higher structures on the Hochschild complex. *preprint*, `http://www.math.jussieu.fr/ keller/publ/dih.dvi`, 2003.

[Kel08] B. Keller. Cluster algebras, quiver representations and triangulated categories. arXiv:0807.1960, 2008.

[Kel11] B. Keller. Deformed Calabi-Yau completions. *J. Reine. Ang. Math. (Crelle's Journal)*, 654(125-180), 2011.

[KKP08] L. Katzarkov, M. Kontsevich, and T. Pantev. Hodge theoretic aspects of mirror symmetry. In Amer. Math. Soc, editor, *For Hodge theory to integravility and TQFT*, volume 78, pages 87–114, Providence, RI., 2008.

[KN] K. Kurano and S. Nishi. Gorenstein isolated quotient singularities of odd prime dimension are cyclic. arXiv:0903.3270.

[KN98] B. Karpov and D. Nogin. Three-block exceptional collections over Del Pezzo surfaces. *Izv. Math.*, 62(3):429–463, 1998.

[KO95] A. Kuleshov and D. Orlov. Exceptional sheaves on Del Pezzo surfaces. *Russian Acad. Sci. Izv. Math.*, 44(3):479, 1995.

[Kon03] M. Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys*, 66(3):157–216, 2003.

[KS09] M. Kontsevich and Y. Soibelman. *Notes on A_∞-algebra, A_∞-categories and noncommutative geometry*, volume 757 of *Lecture Notes in Phys.*, chapter Homological Mirror Symmetry, pages 153–219. Springer, 2009.

[KY11] B. Keller and D. Yang. Derived equivalences from mutations of quivers with potential. *Adv. Math.*, 226(1):2118–2168, 2011.

[Lam07] T.Y. Lam. *Exercises in modules and rings.* Problem books in mathematics. Springer, 2007.

[Len69] H. Lenzing. Nilpotente Elementen in Ringen von Endlicher Globale Dimension. *Math. Z.*, 108:313–324, 1969.

[Leu11] Graham J. Leuschke. Non-commutative crepant resolutions: scenes from categorical geometry. arXiv:1103.5380, 03 2011.

[Lod98] J.-L. Loday. *Cyclic Homology*, volume 301 of *Grundlehren der mathematische wissenschaften*. Springer, 1998.

[LV99] J.-L. Loday and B. Vallette. *Algebraic operads*. 1999.

[Man04] M. Manetti. Lectures on deformations of complex manifolds (deformations from a differential graded viewpoint). *Rend. Mat. Appl.*, 24(1):1–183, 2004.

[Man09] M. Manetti. Differential graded lie algebras and formal deformation theory. In Amer. Math. Soc, editor, *Proc. Symp. Pure Math.*, volume 80, pages 785–810, 2009.

[Mar79] A. Markov. Sur les formes quadratiques binaires indéfinies. *Math. Ann.*, 15(3):281–406, 1879.

[Men09] L. Menichi. Batalin-Vilkovisky algebra stuctures on Hochschild cohomology. *Bull. Soc. Math. France*, 137(2):277–295, 2009.

[Mor07] I. Mori. Intersection theory over quantum ruled surfaces. *Journal of Pure and Applied Algebra*, 211(1):25–41, 2007.

[Nog90] D. Nogin. Spirals of period 4 and equations of Markov type. *Izv. Acad. Nauk. USSR*, 54(4):862–878, 1990.

[Nog94] D. Nogin. Helices on some fano threefolds: constructivity of semiorthogonal bases of K_0. *Ens. Ser*, 27(2):129–172, 1994.

[Nym04a] A. Nyman. Serre duality for noncommutative $\mathbb{P}^1$-bundles. *Trans. AMS.*, 357(4):1349–1416, 2004.

[Nym04b] A. Nyman. Serre finiteness and Serre vanishing for noncommutative $\mathbb{P}^1$-bundles. *Journal of Algebra*, 278(1):32–42, 2004.

[Orl92] D. Orlov. Projective bundles, monoidal transformations and derived categories of coherent sheaves. *Izv. Ross. Akad. Nauk. Ser. Mat.*, 56(4):852–862, 1992.

[Orl46] D. Orlov. Triangulated categories of singularities and D-branes in Landau-Ginzberg models. *Trudy Steklov Math. Inst.*, 2004(240-262), 246.

[Poo08] B. Poonen. Isomorphism types of commutative algebras of finite rank over an algebraically closed field. In *Computational arithmetic geometry*, volume 463 of *Contemp. Math.*, pages 111–120. Amer. Math. Soc., Providence, RI, 2008.

[Rin98] C.M. Ringel. The preprojective algebra of a quiver. *Can. Math. Soc. Conf. Proc.*, (24), 1998.

[RVdB89] I. Reiten and M. Van den Bergh. Two-dimensional tame and maximal orders of finite representation type. *Memoirs of the American Mathematical Society*, 80(408), 1989.

[Sch06] Vadim Schechtman. Remarks on formal deformations and Batalin-Vilkovisky algebras. arXiv:980.2006, 2006.

[Ser55] J-P. Serre. Faisceaux algebriques coherents. *Annals of Mathematics*, 61(2):197–278, 1955.

[Sho03] B. Shoikhet. A proof of the Tsygan formality conjecture for chains. *Adv. Math.*, 179(1):7–37, 2003.

[Smi99] Paul Smith. Noncommutative algebraic geometry. lecture notes, 1999.

[ST08] G. Sharygin and D. Talalaev. On the Lie-formality of Poisson manifolds. *J. K-theory*, 2(2):361–384, 2008.

[Ter06] Deformations of associative algebras with inner product. *Homology, Homotopy Appl.*, 8(2):115–131, 2006.

[Ter08] J. Terilla. Smoothness theorem for differential BV algebras. *J. Topol.*, 1(3):693–702, 2008.

[Tsy99] B. Tsygan. *Differential Topology, infinite dimensional Lie algebras and applications*, volume 2 of *Math. Soc. Transl. Ser.*, chapter Formality conjecture for chains, pages 261–274. 1999.

[TT05] D. Tamarkin and B. Tsygan. The ring of differential operators on forms in noncommutative calculus. In Amer. Math. Soc, editor, *Graphs and patterns in mathematics and theoretical physics*, volume 73, pages 105–131, Providence, RI., 2005.

[VB08] Alexander Quintero Velez and Alex Boer. Noncommutative resolutions of ADE fibered Calabi-Yau threefolds. 06 2008.

[VdB02a] M. Van den Bergh. Noncommutative crepant resolutions. In Springer, editor, *the Legacy of Niels Hendrik Abel*, pages 749–770, 2002.

[VdB02b] M. Van den Bergh. A relation between Hochschild homology and cohomology for Gorenstein rings. *Proc. Amer. Math. Soc*, 130:2809–2810, 2002.

[VdB10] M. Van den Bergh. Calabi-Yau algebras and superpotentials. arXiv:1008.0599, 2010.

[VdB12] M. Van den Bergh. Noncommutative $\mathbb{P}^1$ bundles over commutative schemes. *Trans. AMS.*, 364(12):6279–6313, 2012.

[Wei94] C. Weibel. *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, 1994.

[Wil08] Thomas Willwacher. Formality of cyclic chains. arXiv:0804.3887, 2008.

[Yek05] A. Yekutieli. Deformation quantization in algebraic geometry. *Adv. Math.*, 198(1):383–432, 2005.

[Yek06] A. Yekutieli. Continous and twisted L_∞-morphisms. *J. Pure Appl. Algebra*, 207:575–606, 2006.