1) Prove that the linear combination of 2 D-dimensional Gaussians is a Gaussian. (the 2 Gaussians are independent)

- Suppose you have 2 D-dim Gaussians:

$$p(x|\mu_1, \Sigma_1) = \frac{1}{(2\pi)^{D/2}|\Sigma_1|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_1)^T \Sigma_1^{-1}(x-\mu_1)\right) \qquad \mu_1 = \mathbb{E}(X_1), \ \Sigma_1 = \text{covar}(X_1)$$

$$p(x|\mu_2, \Sigma_2) = \frac{1}{(2\pi)^{D/2}|\Sigma_2|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_2)^T \Sigma_2^{-1}(x-\mu_2)\right) \qquad \mu_2 = \mathbb{E}(X_2), \ \Sigma_2 = \text{covar}(X_2)$$

ie: $X_1 \sim \mathcal{N}_D(\mu_1, \Sigma_1)$
$X_2 \sim \mathcal{N}_D(\mu_2, \Sigma_2)$

We want to prove that $\alpha X_1 + \beta X_2 \sim \mathcal{N}_D(\mu_3, \Sigma_3)$ (for some μ_3, Σ_3)

- We know that a linear transform of a Gaussian is Gaussian,
so $\alpha X_1 \sim \mathcal{N}_D(\alpha\mu_1, \alpha\Sigma_1\alpha^T)$ since $\mathbb{E}(\alpha X_1) = \alpha\mathbb{E}(X_1)$
$\beta X_2 \sim \mathcal{N}_D(\beta\mu_2, \beta\Sigma_2\beta^T)$ and $\text{covar}(\alpha X_1) = \alpha\,\text{covar}(X_1)\alpha^T$

- So now WLOG we can show that $X_1 + X_2 \sim \mathcal{N}_D(\mu_3, \Sigma_3)$
since αX_1 & βX_2 also Gaussians.

using linearity of expected value $\mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2)$
$\Rightarrow \mathbb{E}(X_1 + X_2) = (\mu_1 + \mu_2)$

Using moment generating functions, to show $\Sigma_1 + \Sigma_2 = \Sigma_3$

$m_{X_1} = \mathbb{E}(\exp(x'X_1)) = \exp(x^T\mu_1 + \frac{1}{2}x^T\Sigma_1 x)$ (& similarly m_{X_2})

& the fact that $M_x + M_y = M_x M_y$ for independent R.V.

$$\Rightarrow M_{X_1} + M_{X_2} = \exp(x^T\mu_1 + \tfrac{1}{2}x^T\Sigma_1 x)\exp(x^T\mu_2 + \tfrac{1}{2}x^T\Sigma_2 x)$$

$$= \exp(x^T\mu_1 + x^T\mu_2 + \tfrac{1}{2}x^T\Sigma_1 x + \tfrac{1}{2}x^T\Sigma_2 x)$$

$$= \exp\Big(x^T\underbrace{(\mu_1 + \mu_2)}_{\mu_3} + \tfrac{1}{2}\big[x^T\Sigma_1 x + \tfrac{1}{2}x^T\Sigma_2 x\big]\Big)$$

$x^T\Sigma_1 x + x^T\Sigma_2 x$
(covariance sym. so $\Sigma = \Sigma^T$
$= x^T\Sigma_1^T x + x^T\Sigma_2^T x$
$= (\Sigma_1 x)^T x + (\Sigma_2 x)^T x$
$= ((\Sigma_1 x) + (\Sigma_2 x))^T x$
$= ((\Sigma_1 + \Sigma_2)x)^T x = x^T(\Sigma_1 + \Sigma_2)^T x$
$= x^T(\Sigma_1 + \Sigma_2)x$

$$= \exp\Big(x^T\mu_3 + \tfrac{1}{2}\big[x^T\overbrace{(\Sigma_1 + \Sigma_2)}^{\Sigma_3}x\big]\Big)$$

$\longrightarrow$ moment generating function of a Gaussian $\Rightarrow$ lin. comb. of 2 D-dim Gaussians is a Gaussian. QED.