- **Weeks 1–2:** **informal introduction**
  - network = path

- **Week 3:** **graph theory**
- **Weeks 4–7:** **models of computing**
  - what can be computed (efficiently)?
- **Weeks 8–11:** **lower bounds**
  - what cannot be computed (efficiently)?
- **Week 12:** **recap**

Week 2

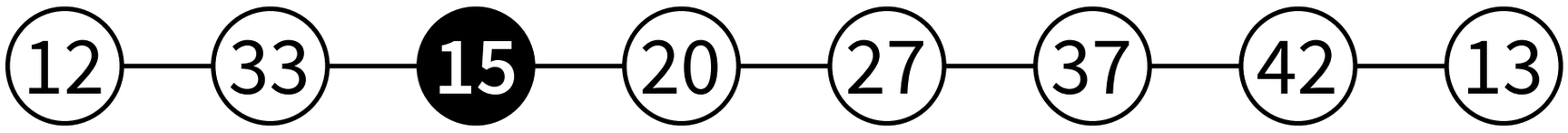
– Warm-up: negative results

Locality

- **Output of a node can only depend on what it knows**
- **After *T* time steps, a node can only know about things up to distance *T***

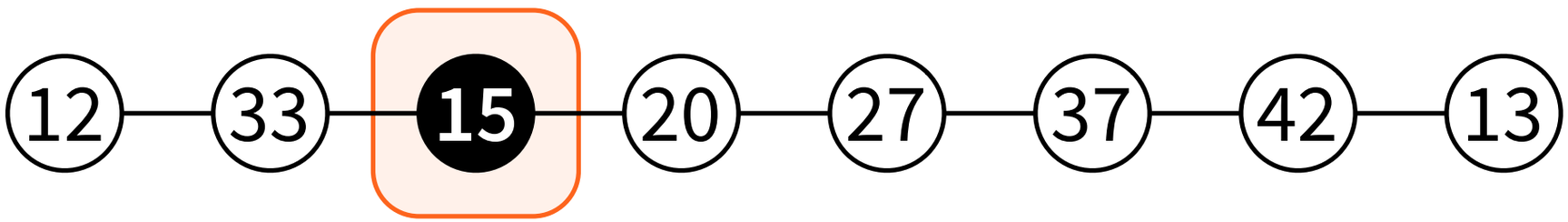
Locality

- **Who knows that node 15 exists?**
- initially, only node 15
- everyone else has to learn it
  by exchanging messages



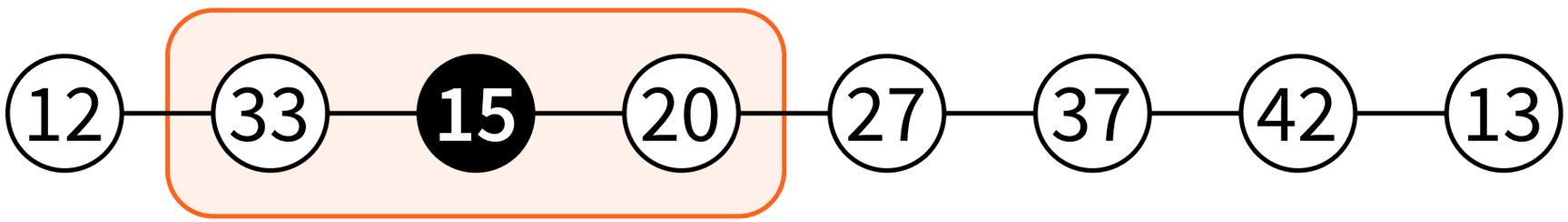
Locality

- **Who knows about node 15 at time $T = 0$?**
- initial state, before we exchange any messages



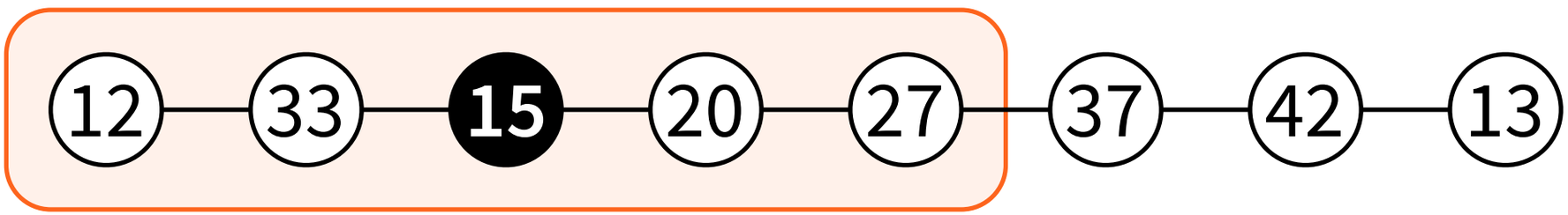
Locality

- **Who knows about node 15 at time $T = 1$?**
- after 1 communication round



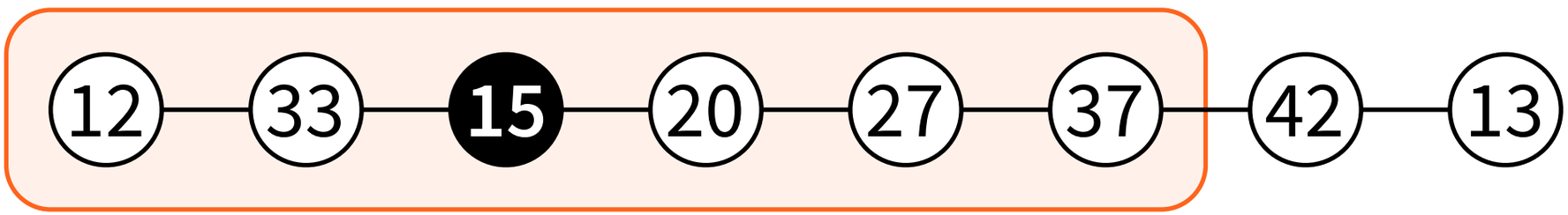
Locality

- **Who knows about node 15 at time $T = 2$?**
- after 2 communication rounds



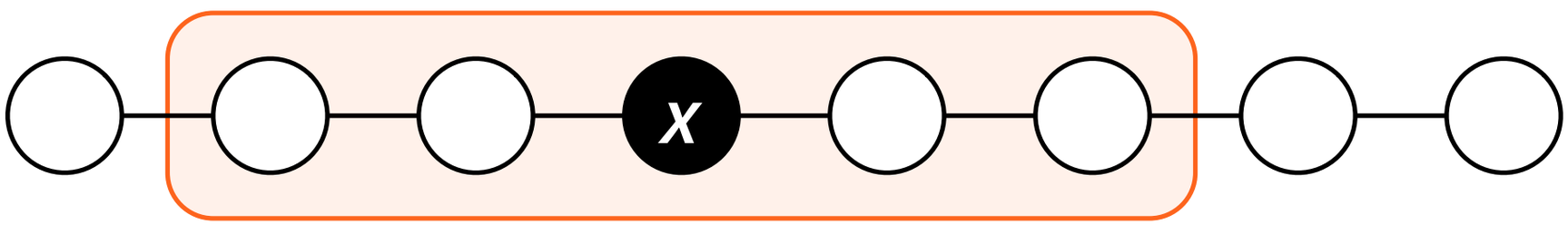
Locality

- **Who knows about node 15 at time $T = 3$?**
- after 3 communication rounds



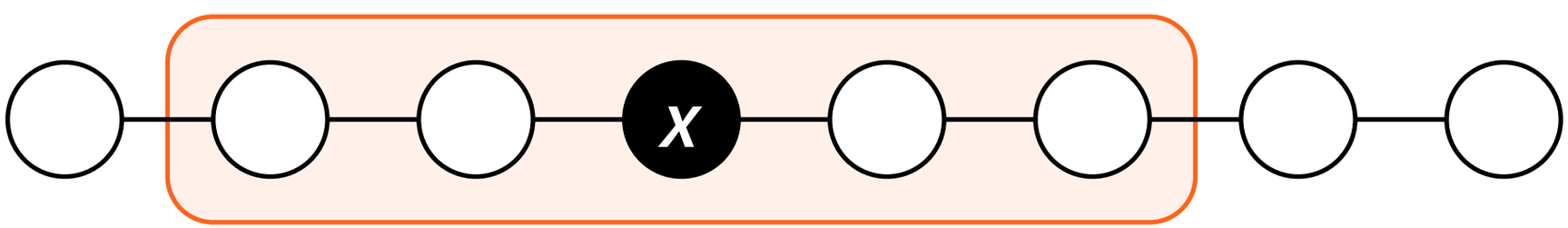
Locality

- **After *T* communication rounds, only nodes up to distance *T* from node *x* can know anything about node *x***
  - distance = “number of hops”



Locality

- **After *T* communication rounds, node *x* can only know about other nodes that are within distance *T* from it**
  - distance = “number of hops”

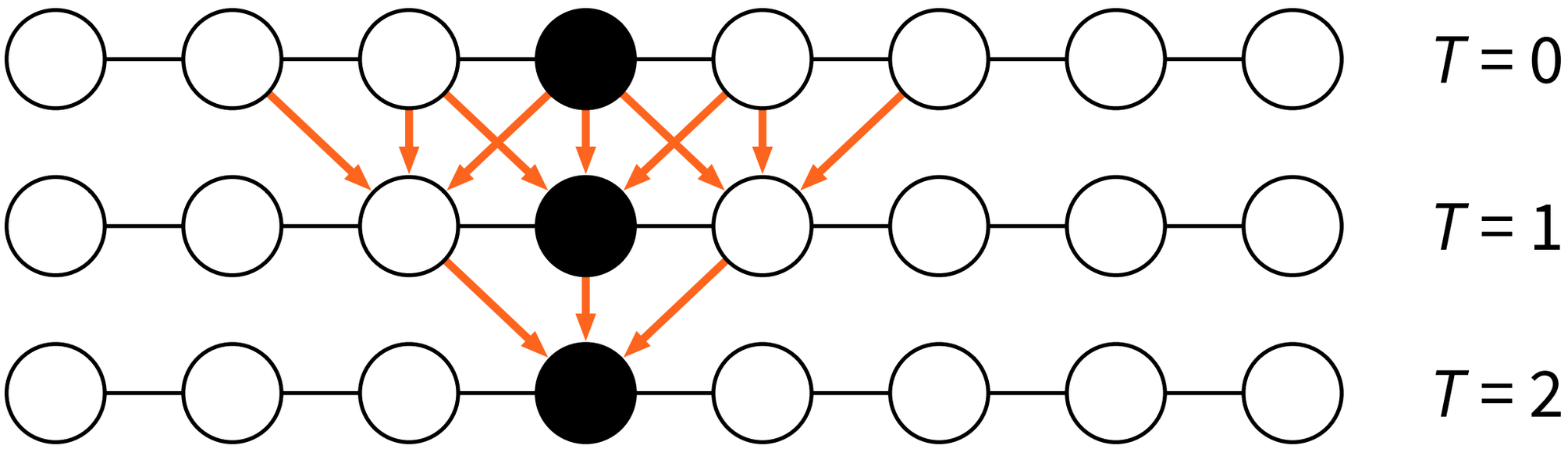


Locality

- **My state at time *T* only depends on:**
  - my state at time $T - 1$, and
  - messages that I received on round T, which only depend on:
    - the state of my neighbours at time $T - 1$

Locality

- **State at time *T* only depends on initial information within distance *T***



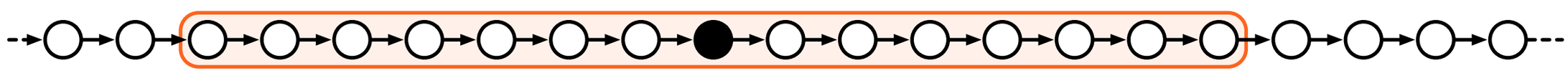
Locality

- **Time = distance**
- **Fast algorithm = “local” algorithm**
  - outputs only depend on local neighbourhoods

Example: 3-colouring

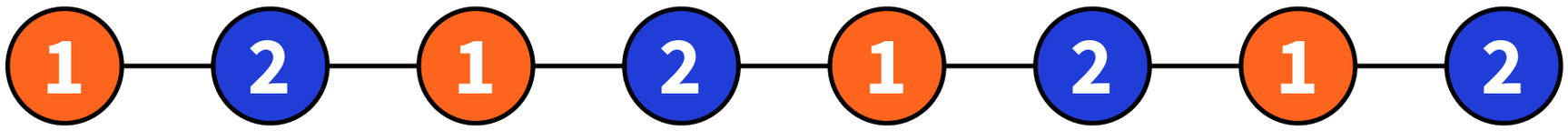
- **Recall: given 128-bit unique identifiers, 3-colouring possible in 7 rounds**

- **Equivalently: each node can pick its colour based on what it sees in its radius-7 neighbourhood**



Using locality to prove lower bounds

- **Example: 2-colouring of a path**
- **Upper bound: possible in time $O(n)$**
- **Lower bound: not possible in time $o(n)$**

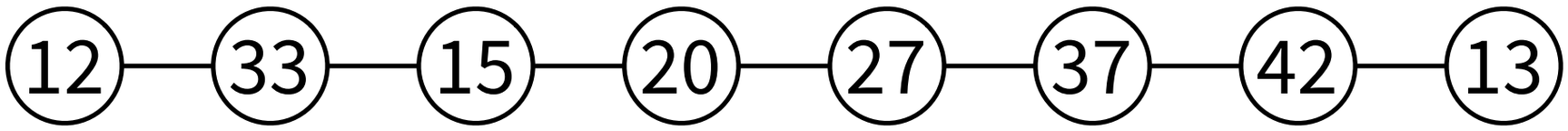


Algorithm for 2-colouring

- **Assumption: path, unique identifiers**
- **Two phases:**
  - find the endpoint with smaller identifier
  - starting from this end, assign colours 1, 2, 1, 2, …

Algorithm for 2-colouring

- **Messages:**
  - “**ID *x***” = there is an endpoint with identifier *x*
  - “**colour *c***” = my colour is *c*



Algorithm for 2-colouring

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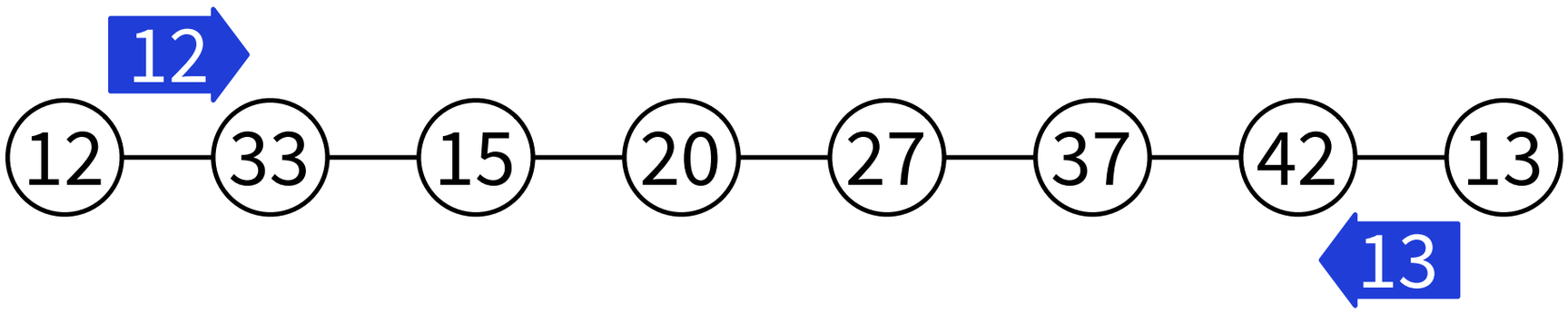

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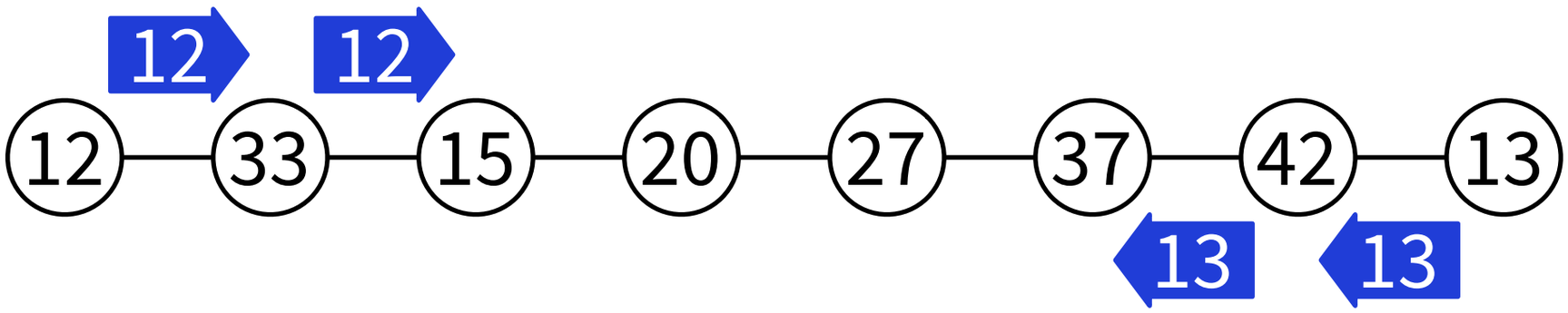

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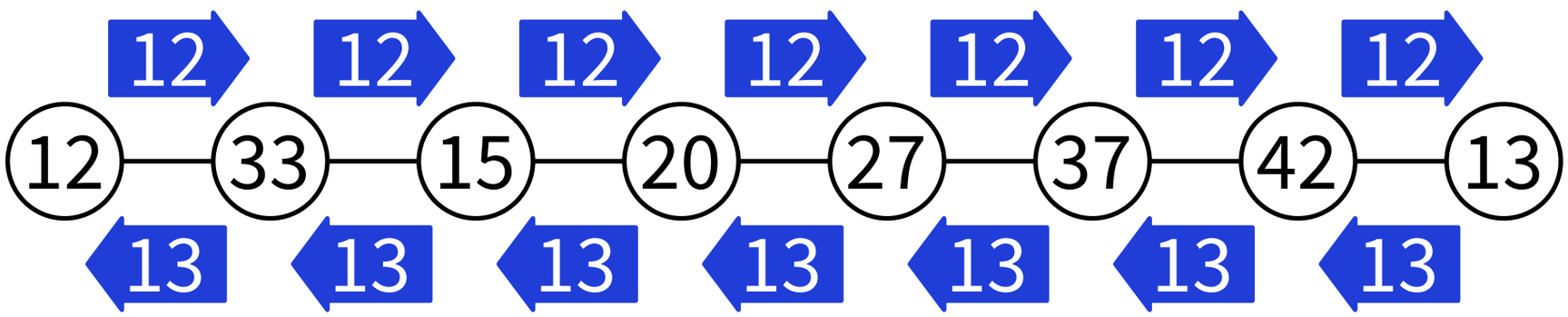

Algorithm for 2-colouring

- **Messages:**
  - “**ID *x***” = there is an endpoint with identifier *x*
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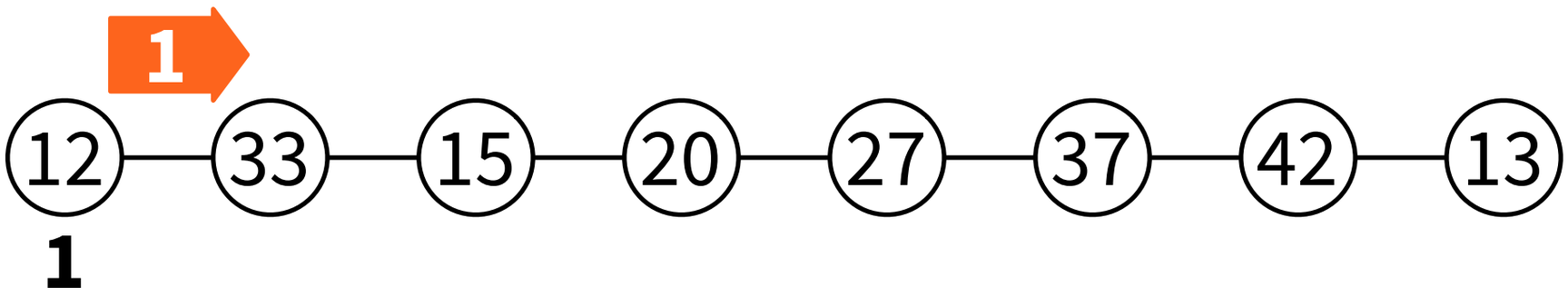

Algorithm for 2-colouring

- **Messages:**
  - “**ID *x***” = there is an endpoint with identifier *x*
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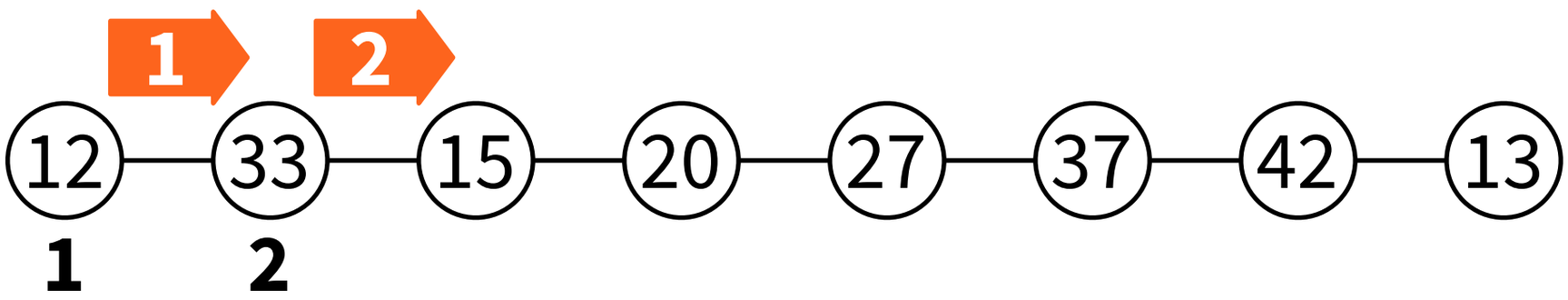

Algorithm for 2-colouring

- **Messages:**
  - “**ID *x***” = there is an endpoint with identifier *x*
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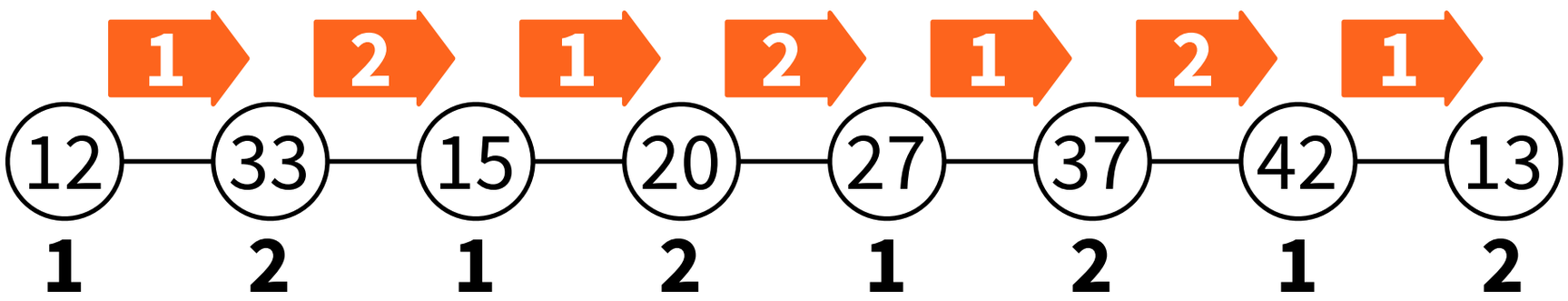
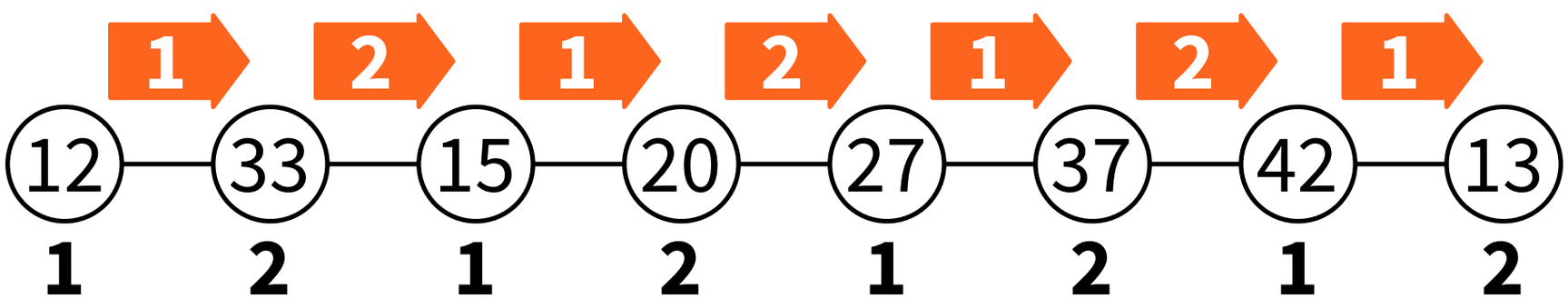


Algorithm for 2-colouring

- **States:** “I have received ID *x* from left and next I will need to send it to right”, …
- **Running time:** $O(n)$ rounds



Algorithm for 2-colouring

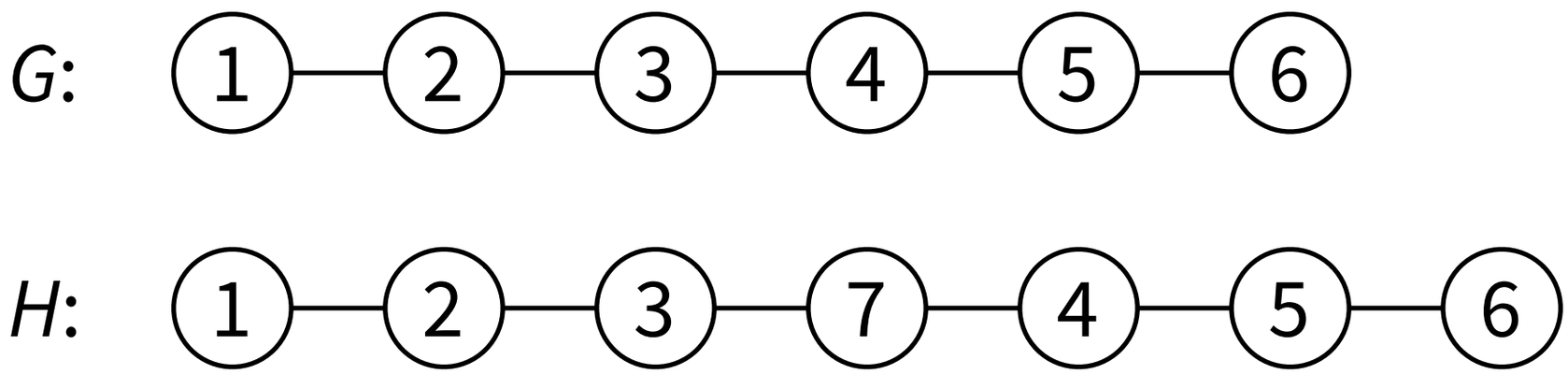
- **2-colouring possible in $O(n)$ rounds**
- **Goal: prove that this is optimal!**
  - there is no algorithm that finds a 2-colouring in time $o(n)$
  - assumptions: path, unique identifiers

Lower bound for 2-colouring

- **Assume: there is an $o(n)$-time algorithm A**
- **For large n, running time $<< n/2$**
- **Idea: construct two possible worlds, show that A fails in one of them**

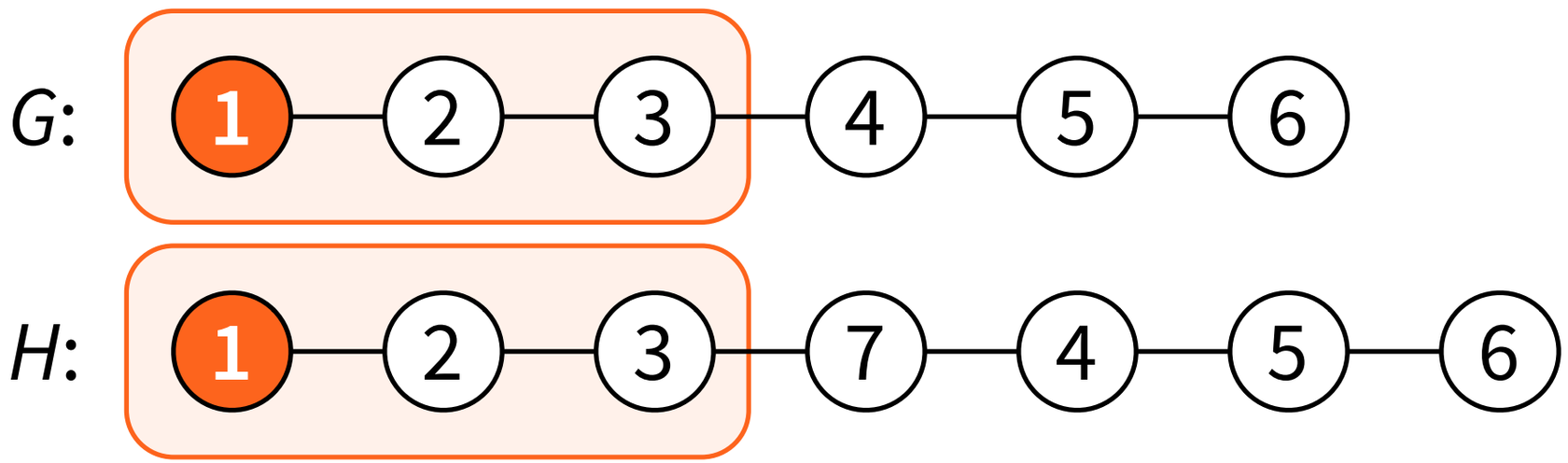
Lower bound for 2-colouring

- **Long paths with $2k$ and $2k+1$ nodes, algorithm runs in $\leq k-1$ rounds**



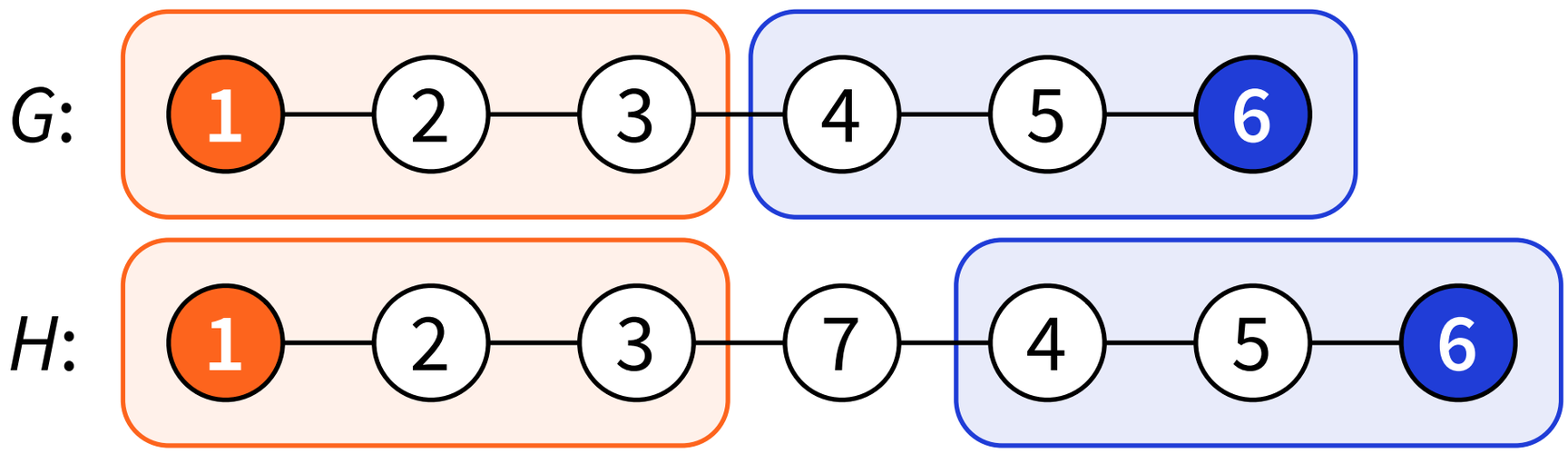
Lower bound for 2-colouring

- **Same ($k-1$)-neighbourhood, same output**



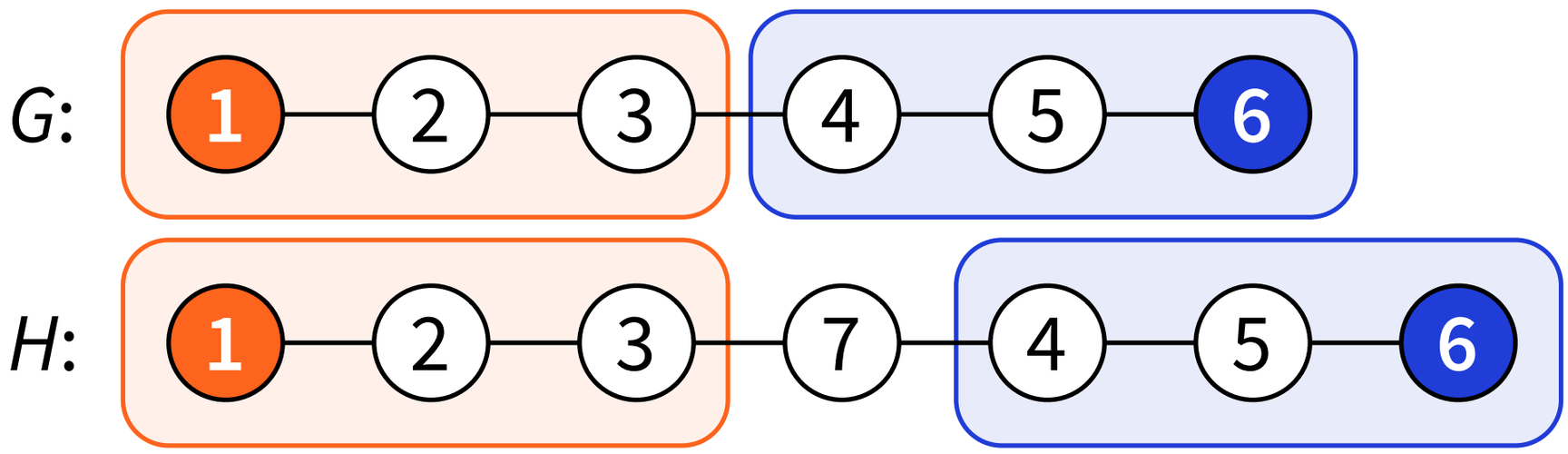
Lower bound for 2-colouring

- **Same ($k-1$)-neighbourhood, same output**



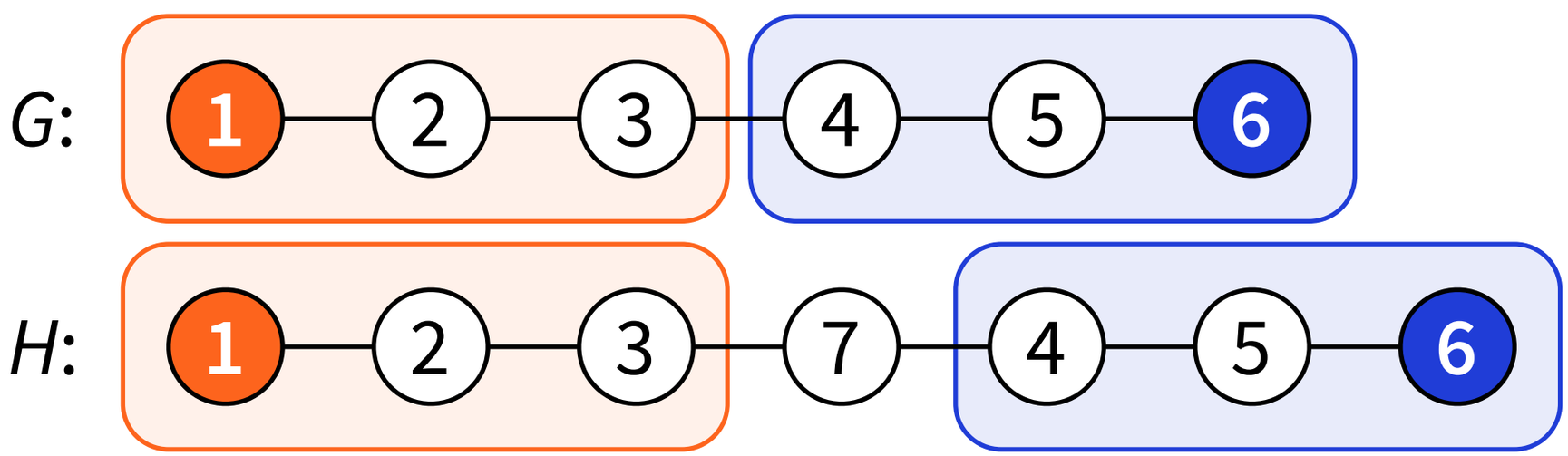
Lower bound for 2-colouring

- **Contradiction — why?**



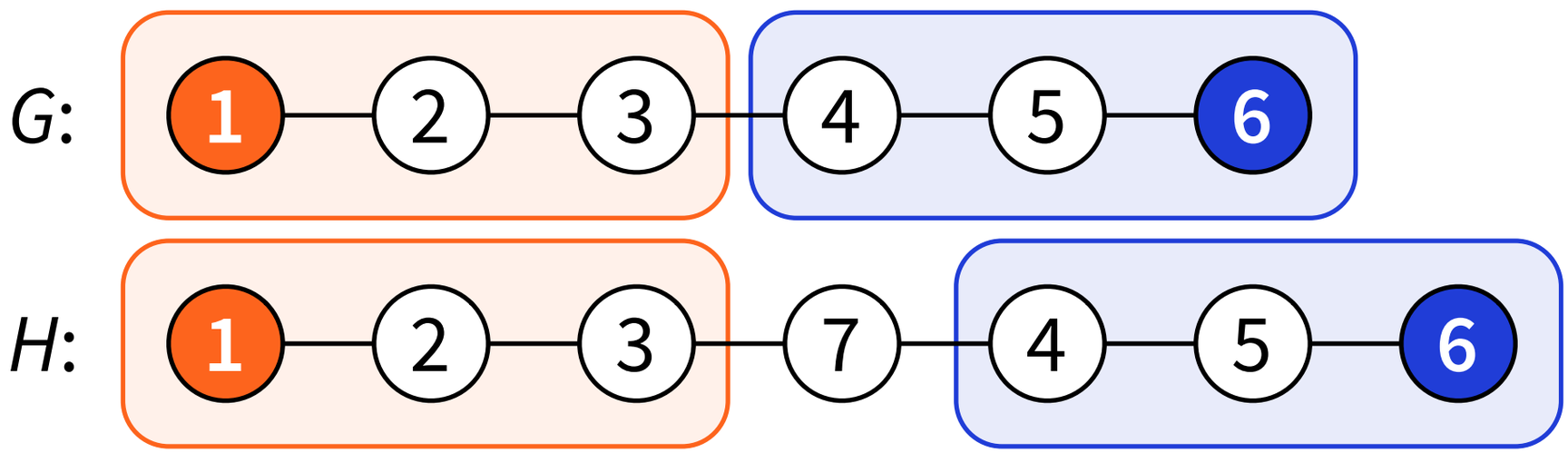
Lower bound for 2-colouring

- ***G*: nodes 1 and 6 must have different colours**
- ***H*: nodes 1 and 6 must have the same colour**



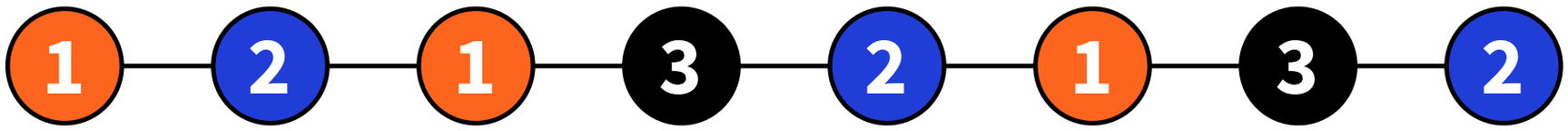
Lower bound for 2-colouring

- **Conclusion: there is no algorithm that finds a 2-colouring of a path in time $o(n)$**



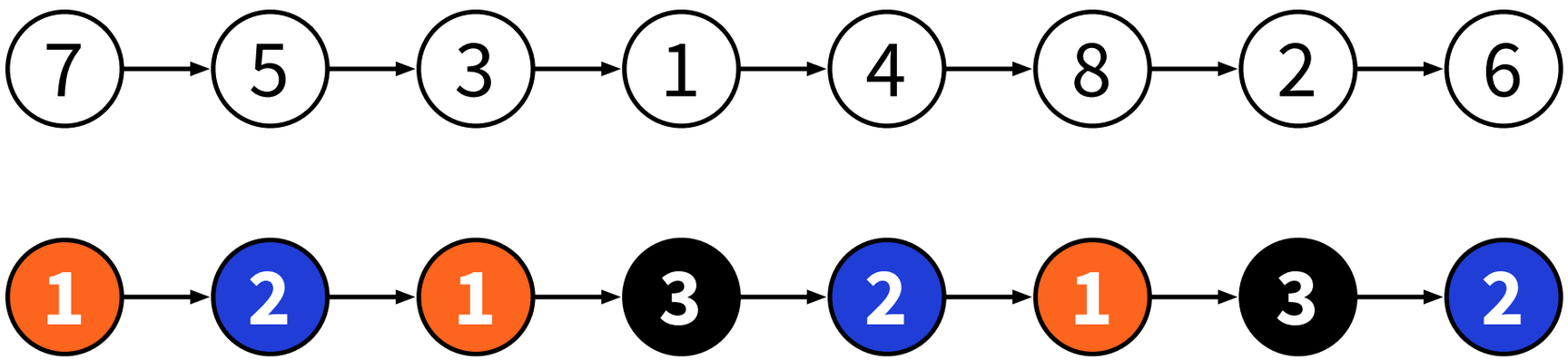
Using locality to prove lower bounds

- **Example: 3-colouring of a path**
- **Upper bound: possible in time $O(\log^* n)$**
- **Lower bound: not possible in time $o(\log^* n)$**



Lower bound for 3-colouring

- **Given: directed path with *n* nodes, identifiers are a permutation of {1, 2, …, *n*}**



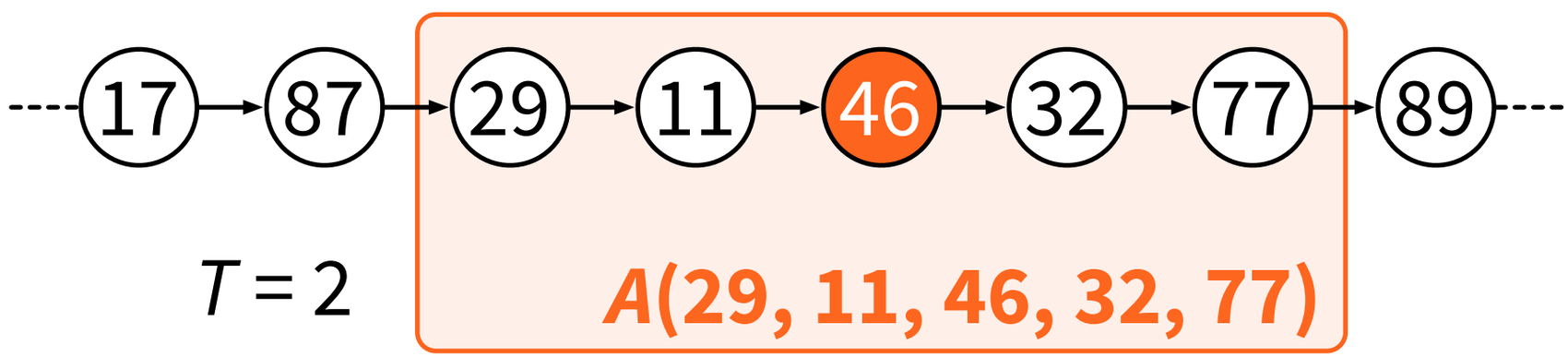
Lower bound for 3-colouring

- **Given: directed path with *n* nodes, identifiers are a permutation of {1, 2, …, *n*}**
- **Assume: there is an algorithm *A* that finds a 3-colouring in time *T***
- **Goal: prove that $T \geq \frac{1}{2} \log^*(n) - 1$**

Algorithm for 3-colouring paths

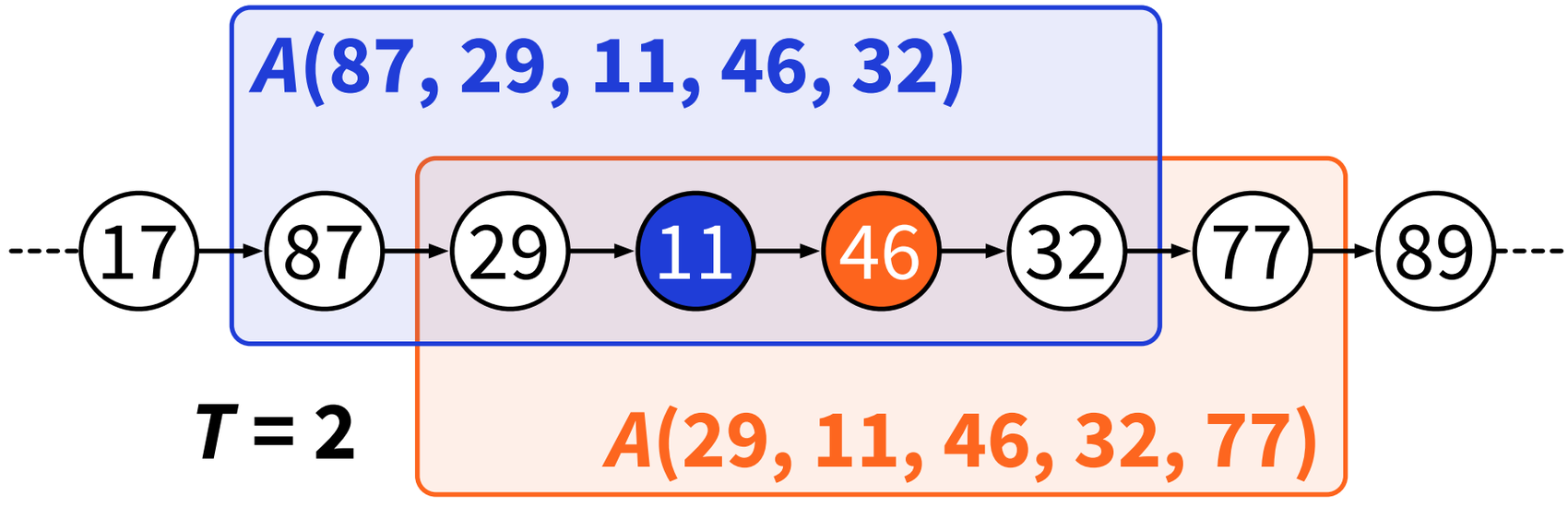
- **Running time T = output only depends on radius-T neighbourhood of the node**

- **Algorithm = k-ary function where $k = 2T+1$**



Algorithm for 3-colouring paths

$A(87, 29, 11, 46, 32) \neq A(29, 11, 46, 32, 77)$



Algorithm for c-colouring paths

- $A(x_1, \ldots, x_k) \in \{1, \ldots, c\}$
  for all distinct $x_1, \ldots, x_k \in \{1, \ldots, n\}$

- $A(x_1, \ldots, x_k) \neq A(x_2, \ldots, x_{k+1})$
  for all distinct $x_1, \ldots, x_{k+1} \in \{1, \ldots, n\}$

Definition: “***k*-ary *c*-colouring function”**

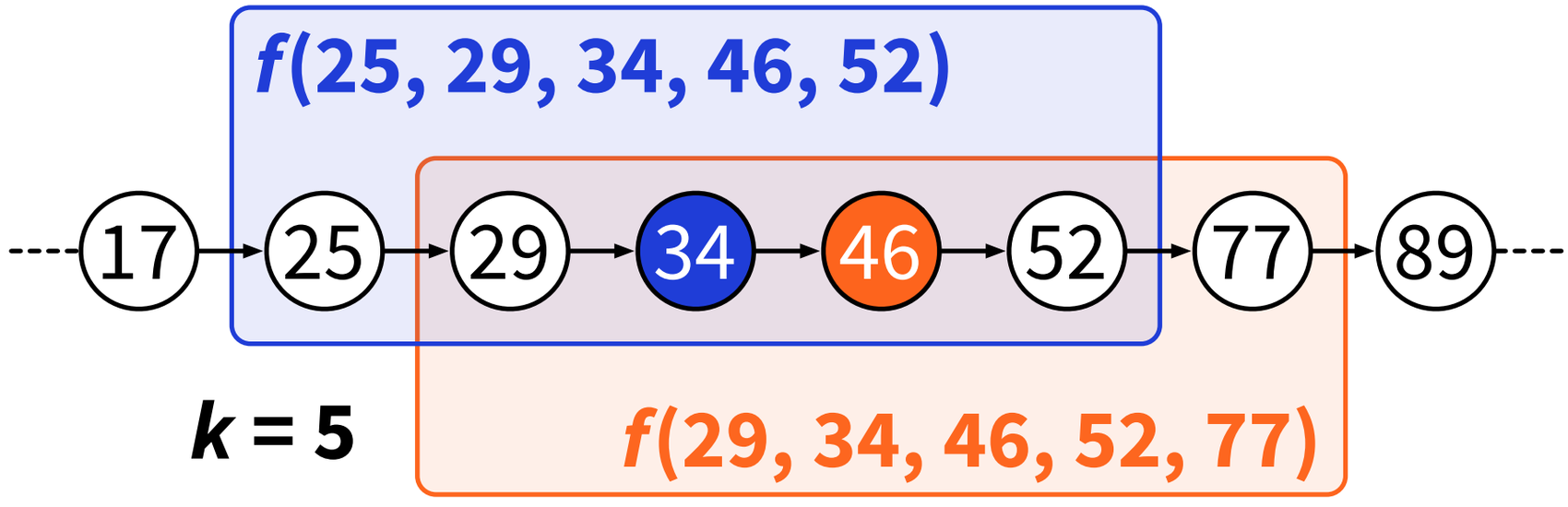
- $f(x_1, \ldots, x_k) \in \{1, \ldots, c\}$
  for all $1 \leq x_1 < \ldots < x_k \leq n$

- $f(x_1, \ldots, x_k) \neq f(x_2, \ldots, x_{k+1})$
  for all $1 \leq x_1 < \ldots < x_{k+1} \leq n$

- **We only care what happens with increasing identifiers**

k-ary *c*-colouring function

$f(25, 29, 34, 46, 52) \neq f(29, 34, 46, 52, 77)$



k-ary *c*-colouring function

- **Assume: *A* is a distributed algorithm that finds a 3-colouring in directed *n*-cycles in time *T***
- **Then: *A* is also a *k*-ary 3-colouring function for $k = 2T + 1$**
- **Plan: show that $k + 1 \geq \log^* n$**

Lemma 1

- **If *f* is a 1-ary *c*-colouring function, then $c \geq n$**

- **Proof:**
  - pigeonhole principle
  - if $c < n$, there is a collision $f(x) = f(y)$ for some $1 \leq x < y \leq n$, contradiction

Lemma 2

- **If *f* is a *k*-ary *c*-colouring function, then we can construct a ($k - 1$)-ary 2^c-colouring function *g***

- **Proof:**
  - $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
  - $g(x_1, \ldots, x_{k-1}) = h(g'(x_1, \ldots, x_{k-1}))$
  - h = bijection that maps sets to colours

Lemma 2 (continued)

- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
- $g(x_1, \ldots, x_{k-1}) = h(g'(x_1, \ldots, x_{k-1}))$
- h = bijection that maps sets to colours
- **By construction:** $g(x_1, \ldots, x_{k-1}) \in \{1, \ldots, 2^c\}$
- **Need to show:** $g(x_1, \ldots, x_{k-1}) \neq g(x_2, \ldots, x_k)$ for all $1 \leq x_1 < \ldots < x_k \leq n$

Lemma 2 (continued)

- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
- $g(x_1, \ldots, x_{k-1}) = h(g'(x_1, \ldots, x_{k-1}))$
- h = bijection that maps sets to colours

- **Need to show:** $g(x_1, \ldots, x_{k-1}) \neq g(x_2, \ldots, x_k)$ for all $1 \leq x_1 < \ldots < x_k \leq n$

Lemma 2 (continued)

- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
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- **Need to show:** $g'(x_1, \ldots, x_{k-1}) \neq g'(x_2, \ldots, x_k)$ for all $1 \leq x_1 < \ldots < x_k \leq n$

Lemma 2 (continued)

- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$

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Lemma 2 (continued)

- $1 \leq x_1 < x_2 < \ldots < x_k \leq n$
- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
- $g'(x_2, \ldots, x_k) = \{f(x_2, \ldots, x_k, z) : z > x_k\}$
- $f(x_1, \ldots, x_{k-1}, x_k) \in g'(x_1, \ldots, x_{k-1})$
- $f(x_1, \ldots, x_{k-1}, x_k) \notin g'(x_2, \ldots, x_k)$
- $g'(x_1, \ldots, x_{k-1}) \neq g'(x_2, \ldots, x_k)$

Lemma 2 (continued)

- $1 \leq x_1 < x_2 < \ldots < x_k \leq n$
- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
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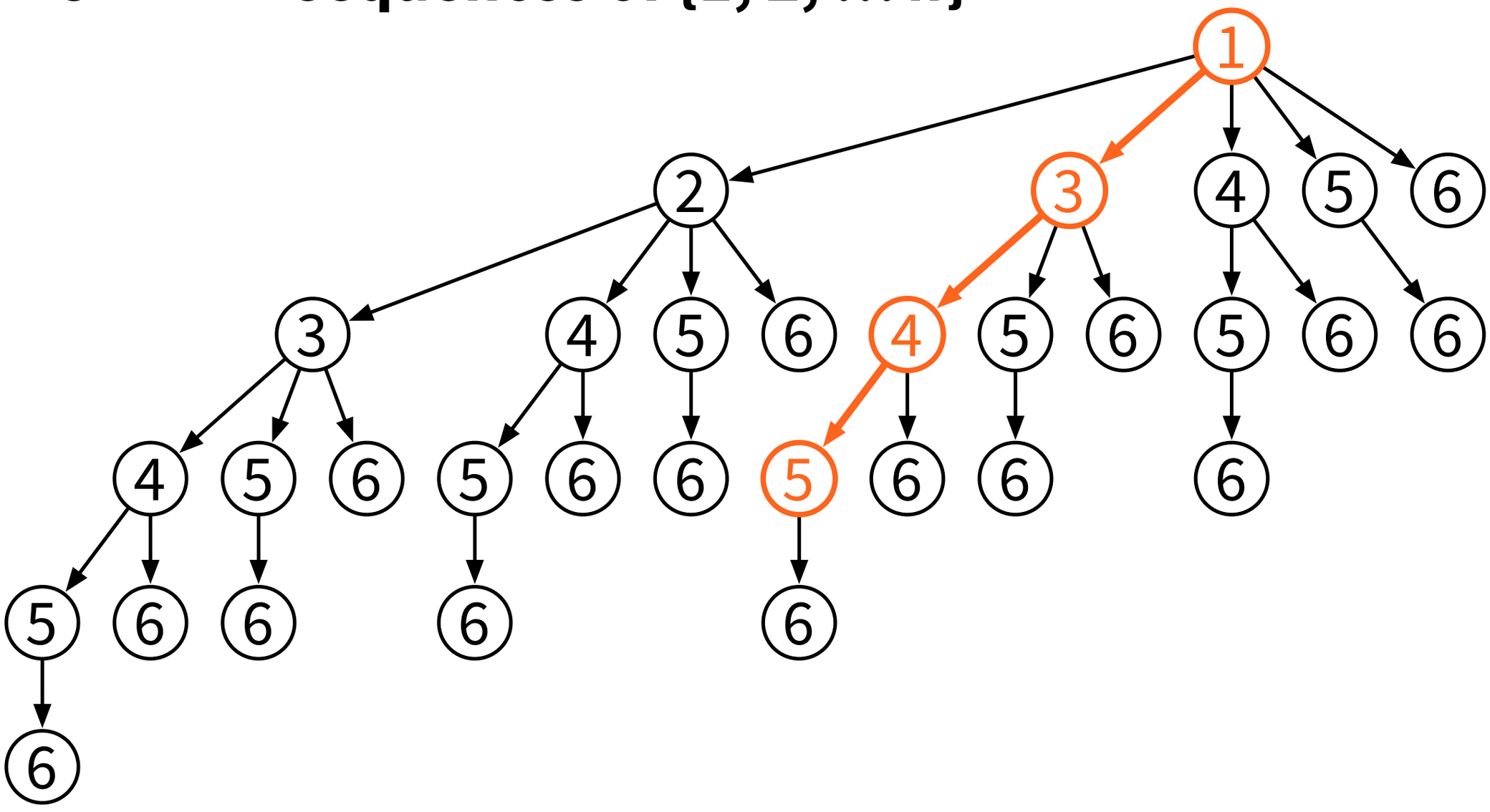
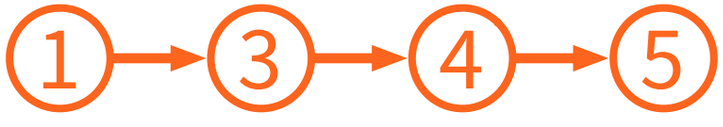
- $g(x_1, \ldots, x_{k-1}) \neq g(x_2, \ldots, x_k)$

$n = 6$

$k = 3$

Tree that contains all increasing sequences of {1, 2, … *n*}

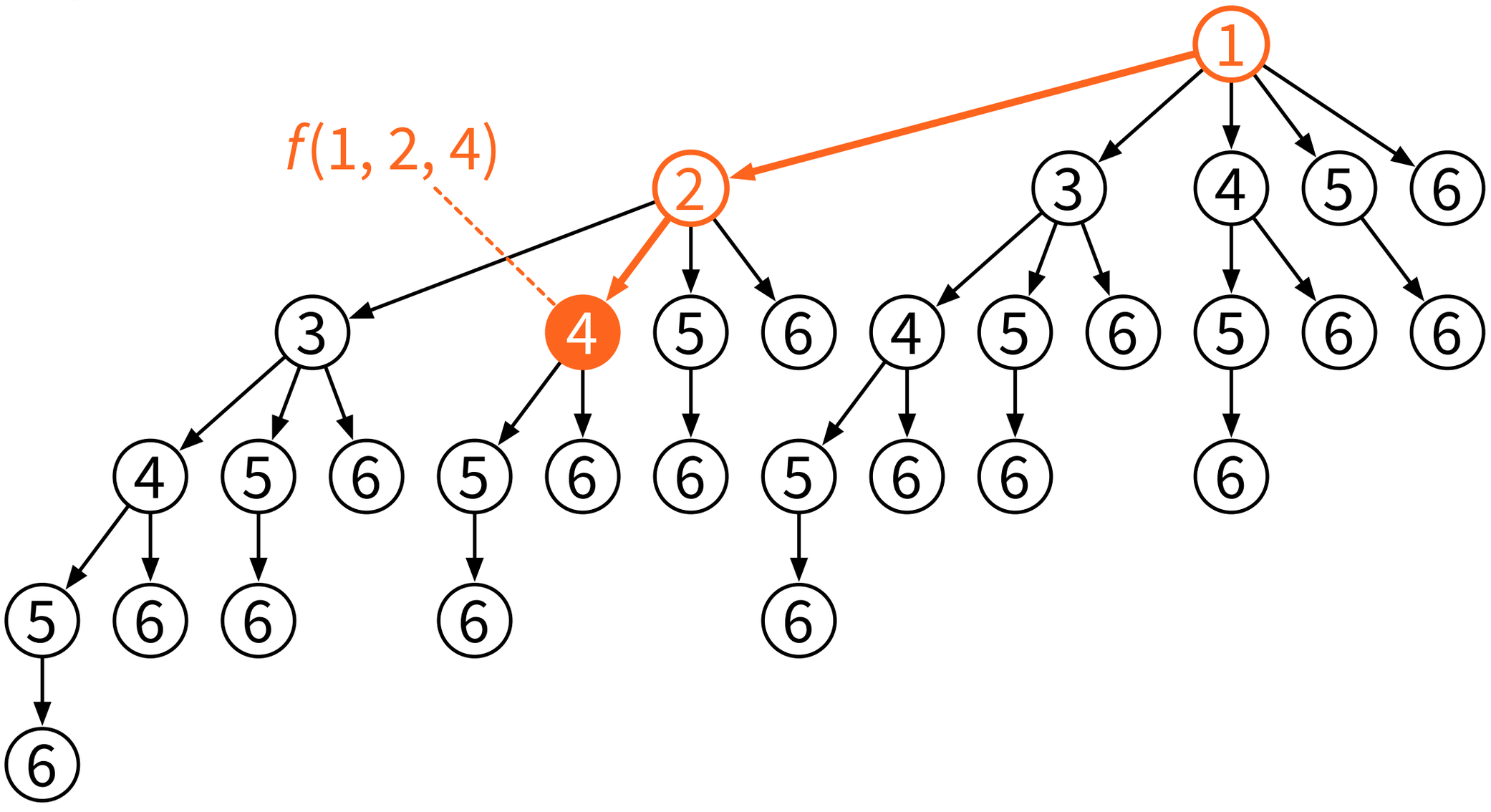




$n = 6$

$k = 3$

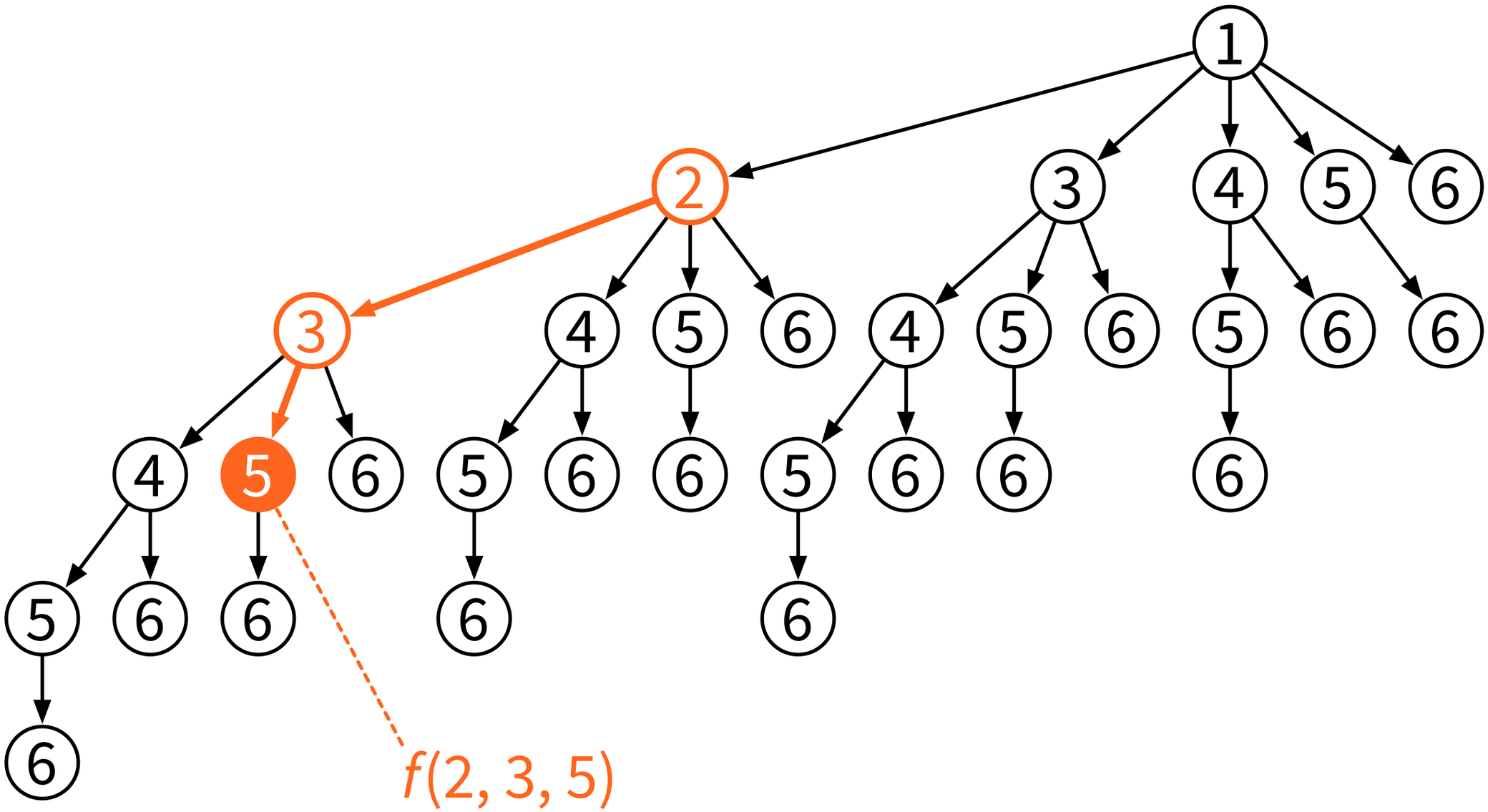
Colour of a node = value of *f*



$n = 6$

$k = 3$

Colour of a node = value of *f*

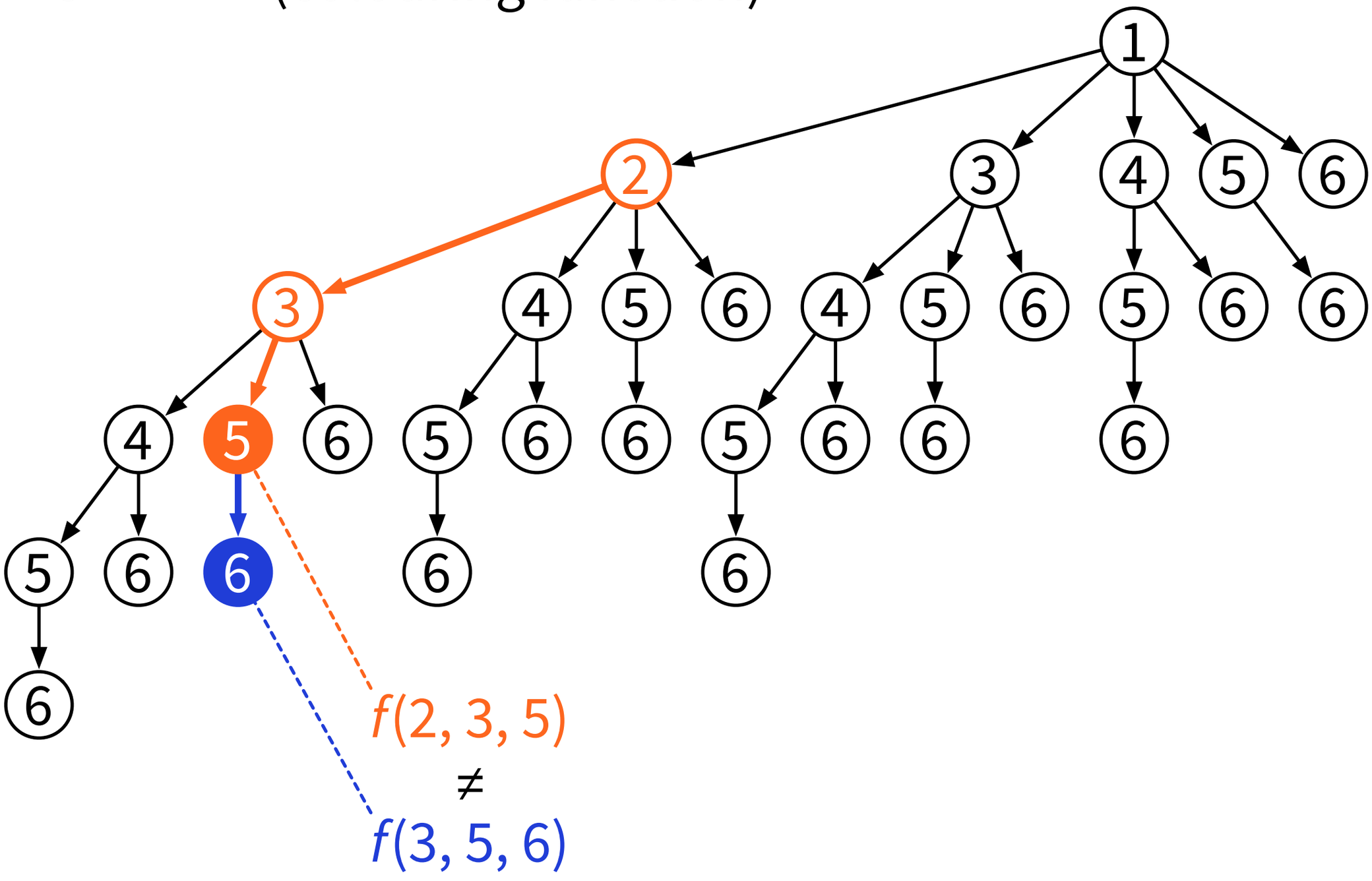


$n = 6$

$k = 3$

Colour of a node = value of *f*

(colouring function)

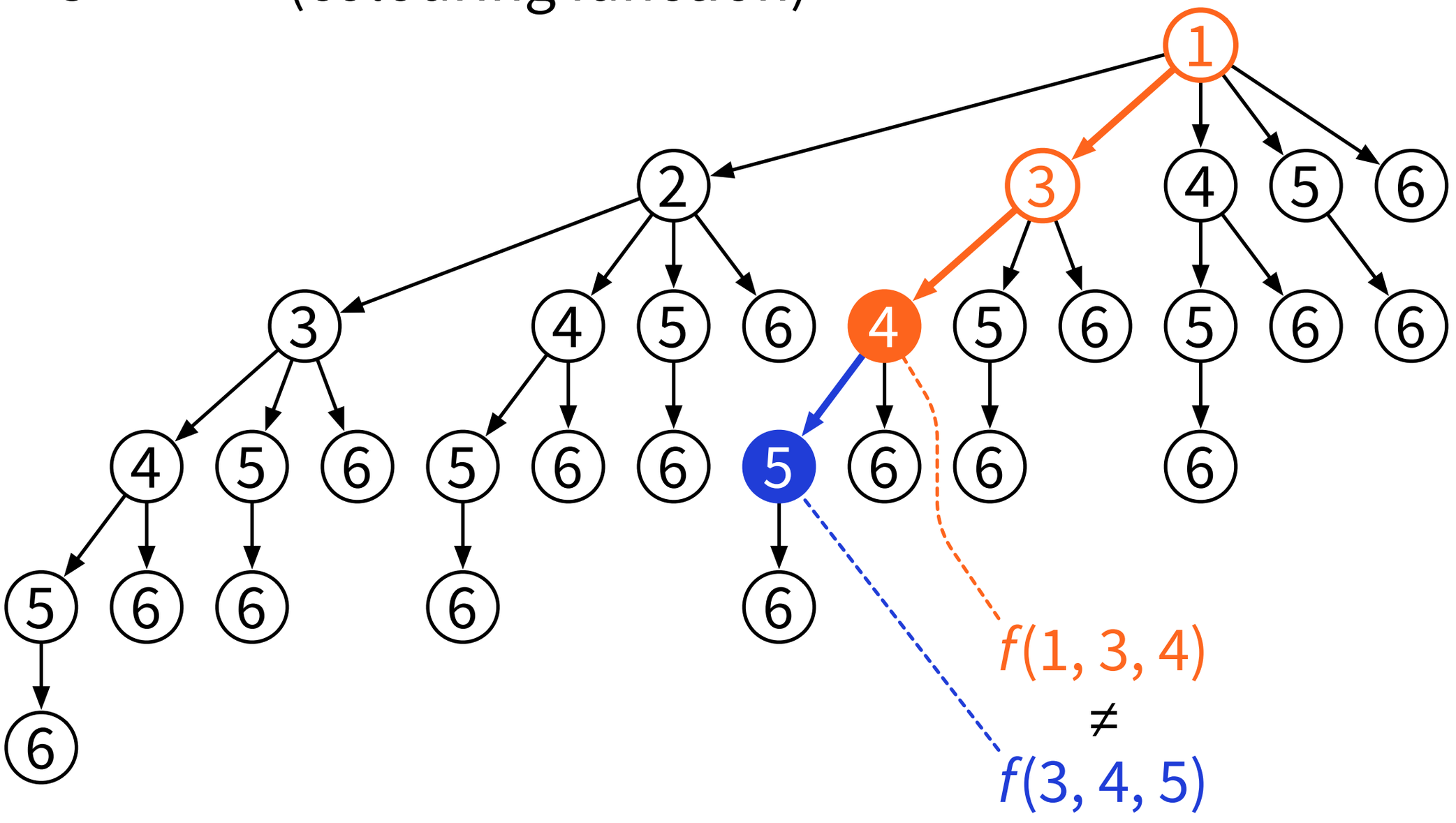


$n = 6$

$k = 3$

Colour of a node = value of *f*

(colouring function)

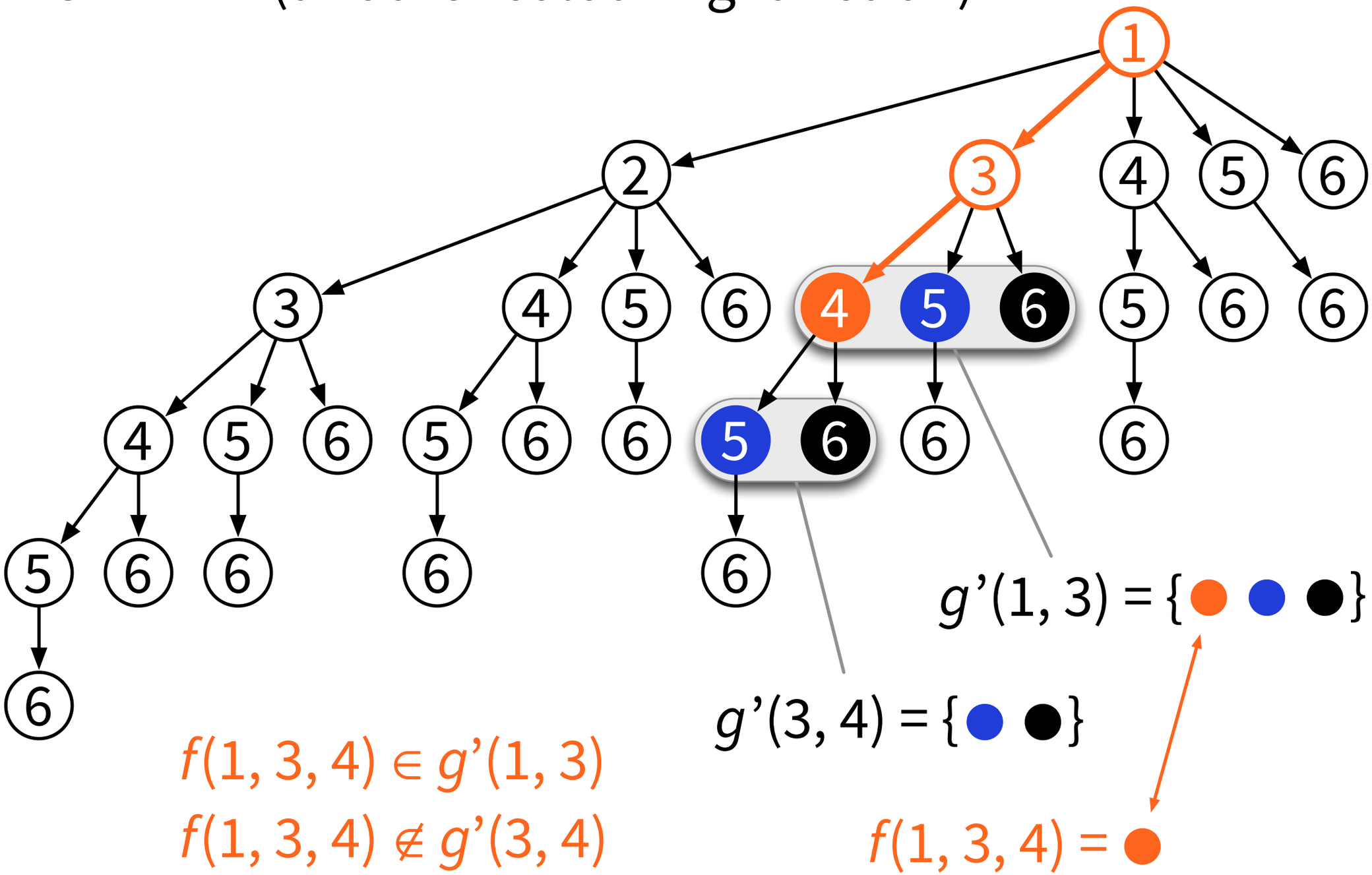


$n = 6$

$k = 3$

Colours of all children = value of *g*'

(another colouring function)


1
2
3
4
5
6
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6
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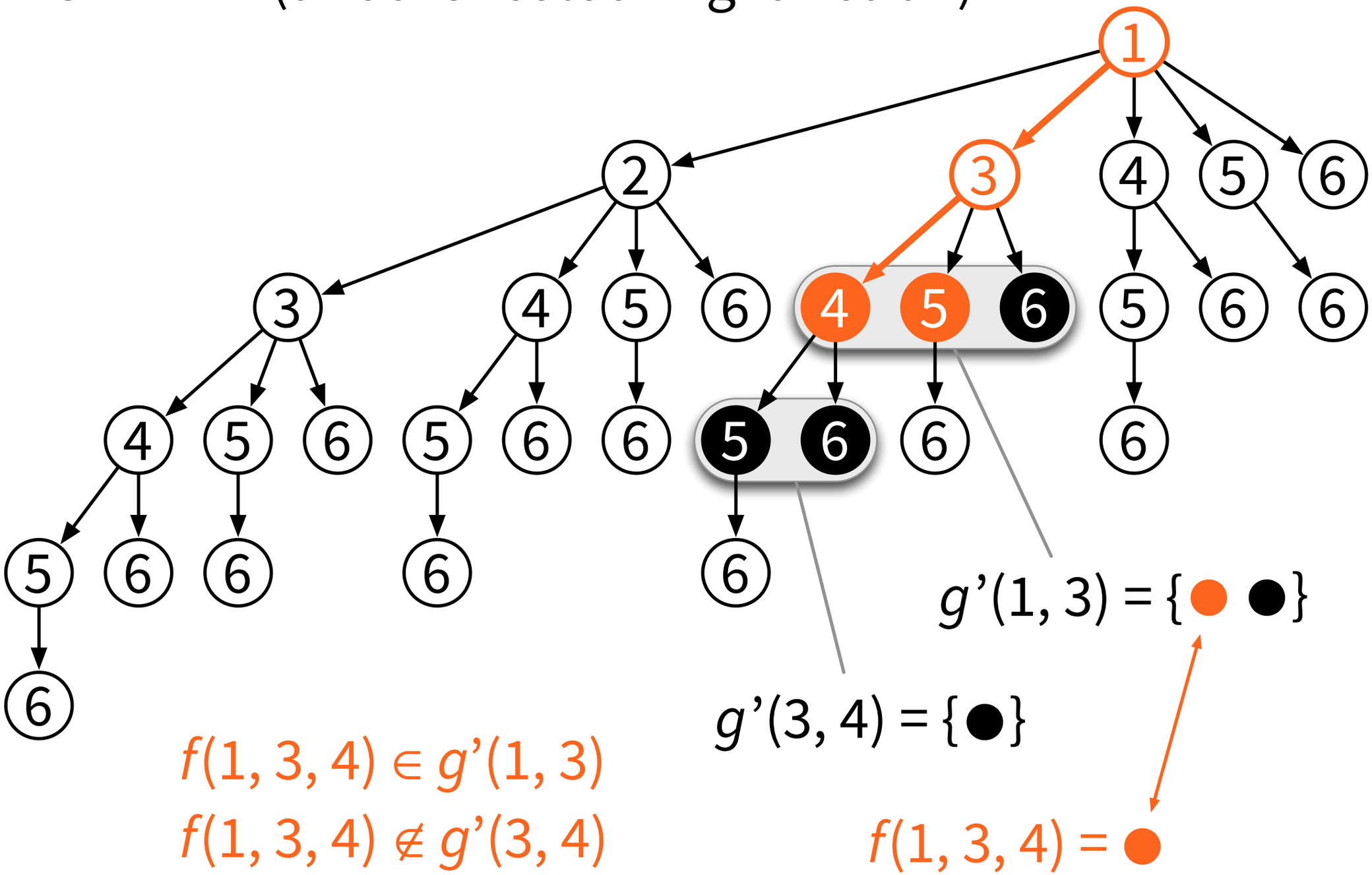
6

$n = 6$

$k = 3$

Colours of all children = value of *g*'

(another colouring function)

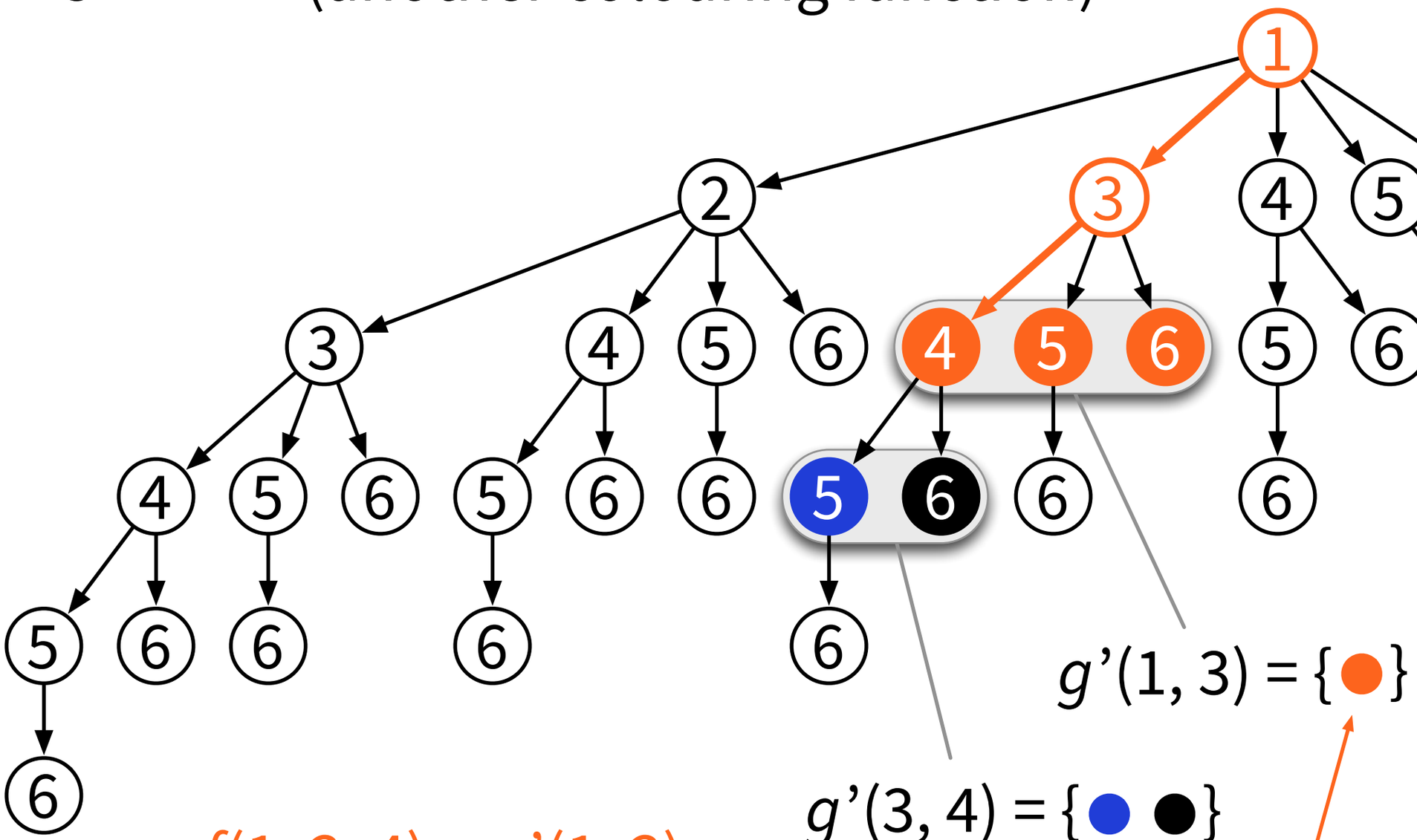


$n = 6$

$k = 3$

Colours of all children = value of g'

(another colouring function)



$f(1, 3, 4) \in g'(1, 3)$

$f(1, 3, 4) \notin g'(3, 4)$

Lemma 2

- **If *f* is a *k*-ary *c*-colouring function, then we can construct a ($k - 1$)-ary 2^c-colouring function *g***

- **Proof:**
  - $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
  - $g(x_1, \ldots, x_{k-1}) = h(g'(x_1, \ldots, x_{k-1}))$
  - h = bijection that maps sets to colours

Iterate Lemma 2

$${}^{i}2 = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \ (i \text{ twos})$$

- ***k*-ary 3-colouring function →**

***k*-ary $^{2}2$-colouring function →**

(*k* − 1)-ary $^{3}2$-colouring function →

(*k* − 2)-ary $^{4}2$-colouring function →

(*k* − 3)-ary $^{5}2$-colouring function →

…

1-ary $^{k+1}2$-colouring function

Lemma 1 + Lemma 2

$${}^{i}2 = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\ (i \text{ twos})$$

- **Lemma 2:**
  - k-ary **3**-colouring function →
    1-ary $^{k+1}2$-colouring function

- **Lemma 1:**
  - $^{k+1}2 \geq n$ (that is, $k + 1 \geq \log^* n$)

Lower bound for 3-colouring

- **Assume: *A* is a distributed algorithm that finds a 3-colouring in directed *n*-cycles in time *T***

- **Then: *A* is also a *k*-ary 3-colouring function for $k = 2T + 1$**

- **Then: $k + 1 \geq \log^* n$, therefore: $T \geq \frac{1}{2} \log^*(n) - 1$**

Conclusions: tight bounds

- **2-colouring paths:**
  - possible in time $O(n)$
  - not possible in time $o(n)$
- **3-colouring paths:**
  - possible in time $O(\log^* n)$
  - not possible in time $o(\log^* n)$

Assuming: directed path, unique IDs = $\{1, 2, \ldots, n\}$

Conclusions: tight bounds

- **2-colouring paths:**
  - possible in time $O(n)$
  - not possible in time $o(n)$
- **3-colouring paths:**
  - possible in time $O(\log^* n)$ — Richard Cole and Uzi Vishkin (1986)
  - not possible in time $o(\log^* n)$ — Nathan Linial (1992)

- **Weeks 1–2:** **informal introduction**
  - network = path

- **Week 3:** **graph theory**
- **Weeks 4–7:** **models of computing**
  - what can be computed (efficiently)?
- **Weeks 8–11:** **lower bounds**
  - what cannot be computed (efficiently)?
- **Week 12:** **recap**