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Index of Refraction's Dependence on Pressure and its Effects in Atmospheric Refraction

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Abstract

In this paper, an interferometer allowed for the determination of the refractive index of air, n, for wavelength $\lambda_0 = 632.8$ nm, at different pressures P, and at a temperature of 298 K. This was done by passing one of the light beams through a controlled vacuum pressure cell. From this, the index of refraction at atmosphere was found to be $n_0 = 1.0003 \pm 0.0001$ with the literature value giving 1.000293. This corresponds to an error of 0.07σ and is in excellent agreement. Next, the phenomena of atmospheric refraction was modeled. Atmospheric refraction is the bending of light as photons travel through the atmosphere. This phenomenon must be taken into account when the real angular location of celestial bodies must be known to a great amount of accuracy. Using the results found in the experiment allowed for the deviation caused by the latter to be estimated. The goal of this paper was to investigate if a basic model could account for the corrections required to establish real angular locations.

1 Introduction

1.1 Goal of the Experiment

The refractive index of a material given by $n = c/v$ is the ratio between the light speed in vacuum and the one within the medium. The purpose of the experiment will determine a relation between the index of refraction of air and it's pressure. From this, atmospheric refraction can be modeled. Atmospheric refraction occurs when the refractive index of the atmosphere varies with altitude, displacing the apparent angular locations of celestial objects.

1.2 Theoretical Methods Invoked

The refractive index as a function of air pressure will be determined by use of an interferometer. An interferometer is a device allowing for a precise measurement of the phase difference given by a path length difference between two beams of coherent light. Historically, the interferometer was used in 1881 by Albert Michaelson and Edward Morley and disproved the idea of the Luminiferous Ether [5]. More recently, a high-precision interferometer was used to confirm the existence of gravitational waves at LIGO [2]. The interferometer in this experiment is not of the same caliber but will determine the effective path length difference when one beam of light is subject to a varying pressure cell.

1.3 Path Length Difference by Refractive Index

By changing the index of refraction to n, the original wavelength λ_0 is changed to $\lambda = \lambda_0/n$ within the medium. Now, it is not hard to see how this causes a change of the overall phase of the light when passing through different media. Namely, the light can cycle m times as a function of the pressure within P. The refractive index is then evaluated by,

$$n(P) = m(P)\frac{\lambda_0}{2t} + 1 \tag{1}$$

Here t is the cell depth and a full derivation can be found in Appendix A.

1.4 Total Refraction

To compute the angular corrections, Fermat's principle will be invoked. Fermat's principle states that photons travel in trajectories which minimize time of flight [4] and these trajectories are often non-trivial to find. Using variational calculus leads to the correct equation of motion, which can be found in Appendix B.

If the apparent angular location of an object is at zenith ϕ_0 from the local vertical axis at the surface of the Earth, then the real angular location of the object is ψ from the local axis. The difference between the two angles is the correction required to determine the physical location of a celestial body in the sky. Figure 1 shows the relevant physical quantities.

$$\delta\phi = \psi - \phi_0 \tag{2}$$

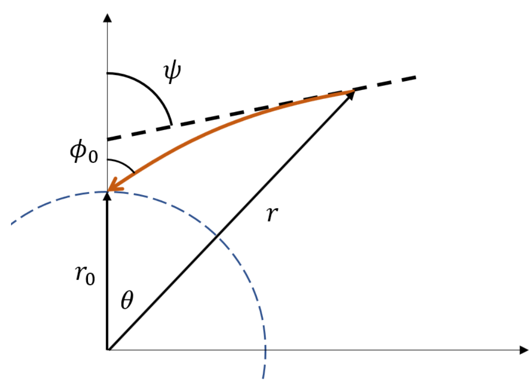

Figure 1: The light trajectory being refracted by the atmosphere. r denotes the distance from the center and θ is the local zenith angle. r_0 denotes the radius of the Earth and ϕ_0 is the apparent location of the object's image. ψ is the physical angular location of the object.

2 Materials and Methods

2.1 Experimental Procedure

The experiment made use of a PASCO interferometer, its accompanied pressure cell, a Neon-Helium light source emitting at a known wavelength of $\lambda_0 = 632.8$ nm, and a screen for viewing. Figure 2 shows a schematic of the apparatus.

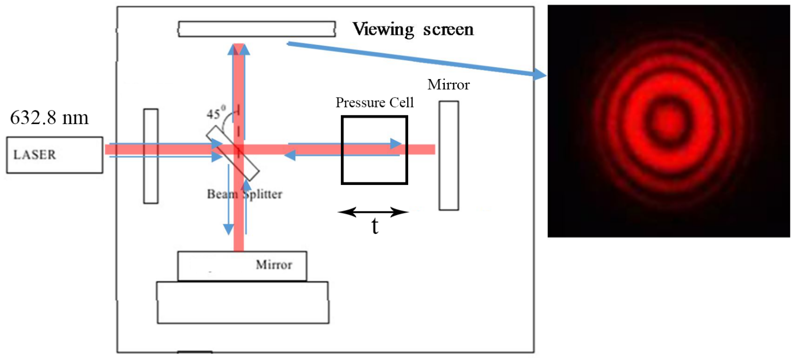

Figure 2: Interferometer used in order to obtain the index of refraction of air. Two light beams are split and recombined while one is subject to a pressure cell. The viewing screen forms concentric interference patterns which are displaced cyclically when the pressure within the cell is varied.

To change the phase shift of the two beams at the screen, the index of refraction within the cell of depth $t = 3.0$ cm was varied by use of the hand pump. A single trial consisted of measuring the relative vacuum pressure P required to displace three fringes δm at the viewing screen. Four trials were measured in the dark by two different observers in order to increase data dimension and reduce systematic errors. The interval size of three was chosen so that the observer would be less likely to miscount while still having sufficient data resolution – a trial generally consisted of six data points. Furthermore, the mirrors were carefully positioned so that the light-reflections were normal to the surface.

2.2 Computational Method

To find the real angle of the celestial object ψ, Eq. 15 was integrated from the initial distance $r = r_0$ to $r = \infty$. Namely, from the surface of the Earth to infinitely far away. However, the experiment only provides a function $n(P)$ 1 as opposed to $n(r)$. The relation $n(r)$ can be found by substituting well known approximations relating the atmospheric pressure $P(r)$ at various distances from Earth's center. From this, the differential equation was integrated using Python's robust SciPy module. The real angle ψ was found as function of the apparent angle ϕ_0. That is,

$$\psi = \int_{r_0}^{\infty} \frac{\sin(\phi_0)}{r\sqrt{\left(\frac{rn(P(r))}{r_0 n_0}\right)^2 - \sin^2(\phi_0)}} dr \tag{3}$$

Namely, the well known barometric formula was used:

$$P(r) = P_0 \exp\left[\frac{-g_0 M(r - r_0)}{R^* T_0}\right] \tag{4}$$

Here, $P_0 = 76$ cmHg, $T_0 = 298$ K is the pressure and temperature at $r_0 = 6371$ km. $R^* = 8.3144598$ J/mol/K, $M = 0.0289644$ kg/mol is the gas constant and molar mass of air.

3 Results

3.1 Index of Refraction as a Function of Pressure

The amount of fringes displaced δm as a function of relative pressure P within the cell was measured. Figure 3 shows the relation between the two quantities.

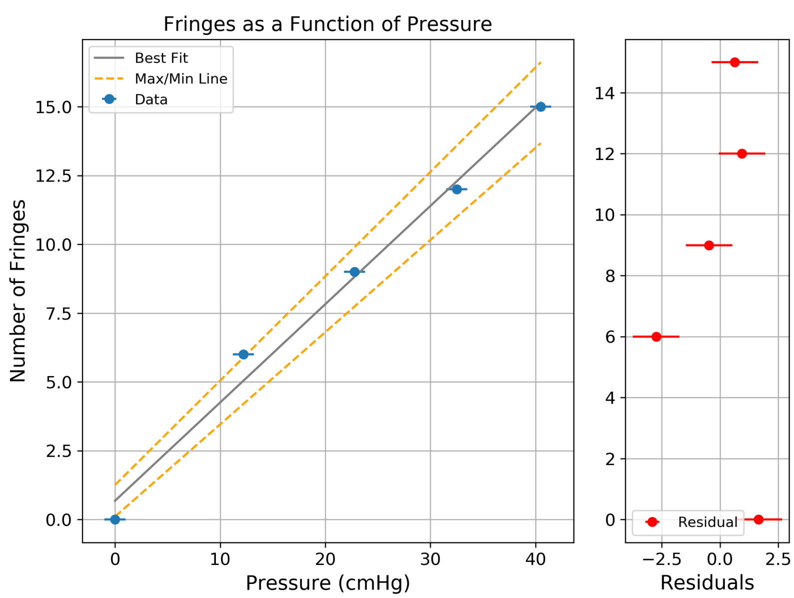

Figure 3: The relation between fringe count displacement δm and relative vacuum pressure P shows a linear trend. The least squares performed a fit of $\delta m(P) = (0.36 \pm 0.02)P + (0.7 \pm 0.6)$. The uncertainty in each data point was the same with a value of $\pm$ 1 cmHg.

A linear regression was performed using the least-squares method and the χ^2 was found to be 11.75. From this, the index of refraction as a function of temperature was found by setting the appropriate boundary condition at absolute pressure. Namely, that $n(P = 0) = 1$.

From this the pressure dependence of the refractive index could be estimated:

$$n = 1 + \frac{\lambda_0}{2t} m(P) = 1 + \frac{632.8\text{nm}}{2 \cdot 3.0\text{cm}} \left[(0.36 \pm 0.02)(P) + (0.7 \pm 0.6)\right] \tag{5}$$

The refractive index of air was found by setting $n(P = 76\text{cmHg}) = 1.0003 \pm 0.0001$ in Equation 5.

3.2 Atmospheric Refraction

The correction $\delta\phi$ due to atmospheric refraction was computed by integration of Equation 3, with Equation 4 as the barometric formula. The difference between the predicted angle ψ and the observed zenith angle ϕ_0 was tabulated for various initial angle ϕ_0 in Table 4 and compared to experimental data from [6].

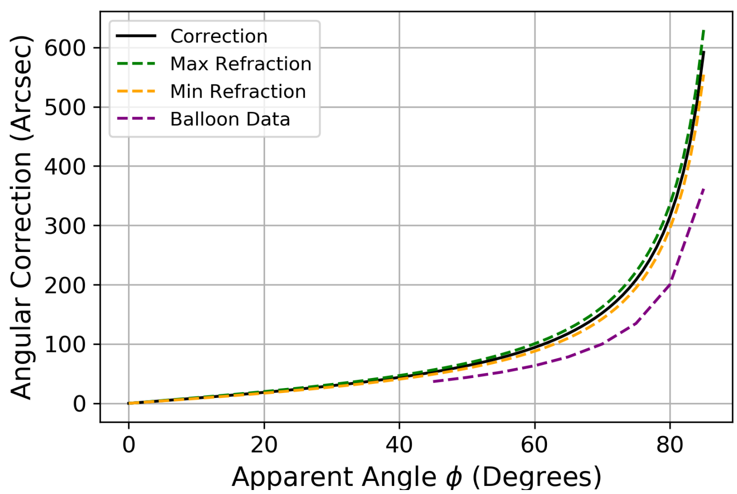

Figure 4: The correction of angular location $\delta\phi$ as a function of the apparent zenith angle ϕ_0. The model is then compared to experimental data found in [6]

Table 1: The correction of angular location $\delta\phi$ as a function of the apparent zenith angle ϕ_0 compared to experimental data. The first column is the apparent location of an object in the sky, the second column is the experimental correction found by other experimental methods [6] and the last three columns are the corrections predicted by the model derived in this paper.

Observed ϕ_0	**Observed** $\delta\phi$	**Model** $\delta\phi$	**Model** $\delta\phi_{min}$	**Model** $\delta\phi_{max}$
[degrees]	[arcsec]	[arcsec]	[arcsec]	[arcsec]
45.00	36.70	52.68	49.04	56.33
50.00	43.72	63.43	59.08	67.77
55.00	52.35	76.82	71.61	82.03
60.00	63.41	94.17	87.87	100.48
65.00	78.35	117.83	110.03	125.63
70.00	100.00	152.38	142.42	162.34
75.00	134.70	208.19	194.75	221.64
80.00	200.00	314.50	294.41	334.61
85.00	361.80	591.81	554.16	629.52

4 Discussion

4.1 Index of Refraction as a Function of Pressure

Our results show that the index of refraction is linearly dependent on the pressure within the medium. This was confirmed since $\chi^2 = 11.75$. By setting $P = 76$ cmHg, the atmospheric pressure, the index of refraction on the surface of the Earth was found to be $n_0 = 1.0003 \pm 0.0001$ while the literature value gives 1.000293. This corresponds to an error of 0.07σ and is in excellent agreement. The dominating errors in the experiment were found to be the pressure gauge and the fringe count measurements. The pressure gauge was mechanical and its highest precision was 1 cmHg given by half a division. To reduce this uncertainty, a high precision pressure gauge could be used that can have a 0.02% error in relative pressure [1] instead of 2.5% from the gauge. The fringe count measured by human eye could have an uncertainty of 1 count. This can be completely removed by use of an automated digital method such as a photometer. With these suggestions, the overall relative uncertainty in our fit can be narrowed down by at least one order of magnitude. Of course, a more precisely manufactured interferometer would also improve accuracy and precision but this is beyond what was needed.

4.2 Atmospheric Refraction

The predicted corrections required to accurately determine the angular location of astrological bodies in terms of the zenith angle ϕ_0 was tabulated in Table 1. The literature data does not lie within the maximum and minimum lines of our model. That is, within the uncertainties of our experiment, our model is not in agreement. However, the model presented is instructive since the general trend of the corrections needed is demonstrated. Namely, its monotonicity and concavity. In addition, the model provides an order of magnitude estimate for the real corrections. Using a more sophisticated barometric model could lead to a more accurate prediction. Furthermore, Eq. 5 is only valid for constant temperature. Another common formula relating both the refractive index and temperature of the gas, such as the equation found in [3] could be used. It is of the form,

$$n(P,T) = 1 + \alpha \frac{P}{P_0} \frac{T_0}{T} \tag{6}$$

Where P, T is the pressure and temperature and the subscripts denote atmospheric conditions and α is a small constant of proportionality.

Indeed, the model presented in this paper is very simple. However, it determines an order of magnitude estimate for the corrections needed for ground-based telescope observations.

5 Conclusions

Our work shows that an interferometer can be used to determine a relationship between the index of refraction as a function of pressure. From the latter, using an atmospheric model for determining pressure at different altitudes, the index of refraction was related to different heights above the Earth's surface. Furthermore, the angular corrections required to account for atmospheric refraction could subsequently be determined using the derived, albeit simple, model. Developing an accurate model for calculating angular deviations from the apparent location of celestial objects is an important procedure in ground based astronomy, which is limited by the optical distortions due to Earth's atmosphere.

6 Acknowledgements

Luca D'Angelo was responsible for the data analysis and writing the conclusion section of this paper.

Vincent Van Duong was responsible writing the abstract, introduction, methods, results and discussion sections of this paper.

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A Derivation of Fringe Count Relation

A cell of depth t and index of refraction n will change the beam's original wavelength to $\lambda = \lambda_0/n$. So the effective phase difference α with a path length difference of Δr is given by,

$$\alpha = \frac{2\pi}{\lambda}\Delta r = \frac{2\pi n}{\lambda}2t \tag{7}$$

Here we have substituted the total path length within the cell and adjusted for its local index of refraction. If we have provoked a fringe to displace by δm integer times by changing the refractive index by δn, then we know that,

$$\delta\alpha = 2\pi\delta m = \frac{2\pi\delta n}{\lambda_0}2t \tag{8}$$

For which it is easy to see that,

$$\delta n = \delta m\frac{\lambda_0}{2t} \iff n = m\frac{\lambda_0}{2t} + 1 \tag{9}$$

Where the last step is obtained by integration and setting the boundary condition that $n(m = 0) = 1$. However, the change of index of refraction is due to changing the pressure inside P. So it is more natural to write the previous equation as,

$$n(P) = m(P)\frac{\lambda_0}{2t} + 1 \tag{10}$$

B Derivation of Light Trajectory Based on Fermat's Principle

The goal is to determine the trajectory of light that minimizes time traveled. The relevant coordinate system and trajectory can be visualized in Fig. 1.

Without loss of generality, suppose the light particle is constrained to a plane in polar coordinates. Then we seek to minimize the time of travel between two points (r_1, θ_1) and (r_2, θ_2) – the r coordinate is the radial distance from the center of the planet and θ denotes the local zenith angle. Furthermore, let us assume the refractive index $n(r)$ is explicitly dependent on the r coordinate so that the speed of travel of light is given by $v(r) = c/n(r)$. From this we can compute the functional which finds the total time of travel.

$$ds = vdt = \sqrt{dr^2 + r^2 d\theta^2} \implies dt = \frac{n(r)}{c}\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}dr \implies T = \int_{r_1}^{r_2} \frac{n(r)}{c}\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}dr \tag{11}$$

We identify the integrand as Lagrange's function and subsequently apply Lagrange's equation to determine the path which yields a stationary value of T. To this end,

$$L = \frac{n(r)}{c}\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \tag{12}$$

$$\frac{d}{dr}\left(\frac{\partial L}{\partial \theta'}\right) = \frac{\partial L}{\partial \theta} = 0 \implies Q = \frac{rn(r)}{c}\frac{r\theta'}{\sqrt{1+\theta'^2}} = \frac{n(y)}{c} r \sin\psi \tag{13}$$

In the previous step, we used the fact that Lagrange's function is explicitly independent of θ to obtain an integral of the motion. Moreover, ψ denotes the zenith angle of the trajectory from the vertical. This is in fact Snell's law of refraction for an index of refraction depending on r only. From this, we can calculate the trajectory of the particle through separation of variables,

$$Q = \frac{rn(r)}{c}\frac{r\theta'}{\sqrt{1+r^2\theta'^2}} \implies \frac{d\theta}{dr} = \frac{Q}{r\sqrt{\left(\frac{rn(r)}{c}\right)^2 - Q^2}} \tag{14}$$

Knowing that we have one integral of the motion Q we can invoke a boundary condition. At the origin we have zenith angle ϕ_0 and index of refraction n_0 at r_0 so that we can write the following equation:

$$\frac{d\theta}{dr} = \frac{\sin(\phi_0)}{r\sqrt{\left(\frac{rn(r)}{r_0 n_0}\right)^2 - \sin^2(\phi_0)}} \tag{15}$$

The trajectory follows directly by integration with a known function $n(r)$ and applying the known boundary condition at the origin.

C Sample Calculations

The atmospheric refractive index was found by substituting the atmospheric pressure into Eq. 5:

$$n_{air} = 1+\frac{\lambda_0}{2t}m(P_{air}) = 1+\frac{(632.8 \pm 0.1)\text{nm}}{2 \cdot (0.03 \pm 0.01)\text{m}}\left[(0.36 \pm 0.02)(76 \pm 1) + (0.7 \pm 0.6)\right] = 1.0003 \pm 0.0001 \tag{16}$$

The relative error for the refractive index of air was simply found by taking the z-score,

$$z = \frac{\bar{x} - \mu}{\sigma} = \frac{1.0003 - 1.000293}{0.0001} = +0.07 \tag{17}$$

To compute the χ^2 of the linear fit obtained in Fig. 3, the following was computed within Python:

$$\chi^2 = \sum_{i}^{6} \frac{(m_i - P(m_i))^2}{\Delta P^2} = 11.75 \tag{18}$$

Here, we are using the line of best fit for the relative pressure P in terms of the total number of fringes displaced m. For all measurements, the uncertainty in the pressure measurement ΔP was 1 cmHg and is one half a division.

D Data

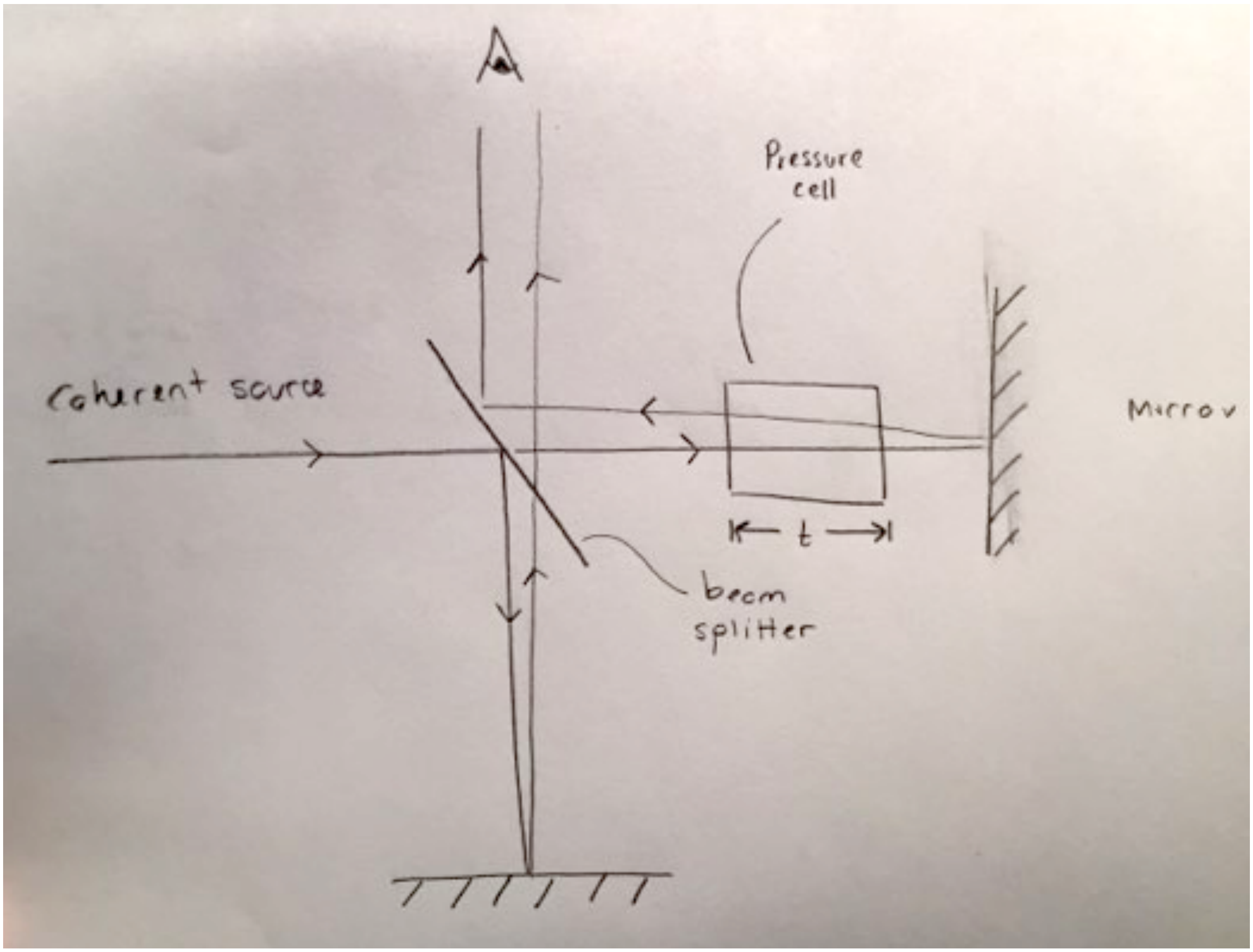
	Fringes	Pressure
0	0	0.0
1	6	12.0
2	9	20.0
3	12	30.0
4	0	0.0
5	6	11.5
6	9	22.5
7	12	32.5
8	15	38.5
9	0	0.0
10	6	12.0
11	9	24.0
12	0	0.0
13	6	13.0
14	9	22.0
15	0	0.0
16	6	12.0
17	9	23.0
18	12	32.0
19	15	41.0
20	0	0.0
21	6	13.0
22	9	26.0
23	12	34.0
24	0	0.0
25	6	12.0
26	9	22.0
27	12	34.0
28	15	42.0

Figure 5: Aggregate data from the interferometer experiment.

Lab Project

March 21, 2018

- Setup the interferometer
- Used a wooden block to stabilize the laser beam
- Zeroed pressure cell at given pressure before starting to count
- Counted with lights off, in intervals of 8 fringes
  - Alternated between lab partners for each trial
- Recorded data on laptop for each trial so we could plot data as experiment progressed to determine if appropriate linear trend was acquired.
- Did 8 trials in total
- Uncertainty in pressure was +/- 1 cmHg


Pressure cell
Coherent source
Mirror
t
beam splitter