# SD-TSIA204
# Statistical hypothesis testing (for linear model)

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1. Statistical hypothesis testing

Definition

The $p$-value

Tests for linear regression

2. Illustration: forward variable selection

# General principle

## Context

- We observe $X_1, \ldots, X_n$ from a common distribution $\mathcal{P}$
- We are interested in $\theta \in \Theta$, a parameter of $\mathcal{P}$

## Goal

To decide whether an assumption on $\theta$ is likely (or not)

$$\mathcal{H}_0 = \{\theta \in \Theta_0\}$$

against some alternative

$$\mathcal{H}_1 = \{\theta \in \Theta_1\}$$

Call $\mathcal{H}_0$ the null hypothesis, $\mathcal{H}_1$ : the alternative

# General principle

**Means**

Determine a test statistic $T(X_1, \ldots, X_n)$ and a region $R$ such that if

$$T(X_1, \ldots, X_n) \in R \;\Rightarrow\; \text{we reject } \mathcal{H}_0$$

In other words the observed data discriminates between $H_0$ and $H_1$

# Hypothesis testing for "heads or tails"

When flipping a coin the model is a Bernoulli distribution with parameter $p$, $\mathcal{B}(p)$.

**Is the coin fair?**

$$\mathcal{H}_0 = \{p = 0.5\} \quad \text{against} \quad \mathcal{H}_1 = \{p \neq 0.5\}$$

**Is the coin possibly unfair?**

$$\mathcal{H}_0 = \{0.45 \leq p \leq 0.55\} \quad \text{against} \quad \mathcal{H}_1 = \{p \notin [0.45, 0.55]\}$$

# Do we reject or do we accept ?

In most practical situations, $\mathcal{H}_0$ is simple, i.e.,

$$\Theta_0 = \{\theta_0\}$$

and $\Theta_1 = \Theta \backslash \Theta_0$ is large

($\mathcal{H}_0$ is often an hypothesis on which we care particularly, e.g., something acknowledged to be true, easy to formulate)

**We only reject $\mathcal{H}_0$**

If $\mathcal{H}_0$ is not rejected we cannot conclude $\mathcal{H}_0$ is true because $\mathcal{H}_1$ is too general

*e.g.* $\{p \in [0, 0.5[\cup]0.5, 1]\}$ can not be rejected!

# 2 types of error

| | $\mathcal{H}_0$ | $\mathcal{H}_1$ |
|---|---|---|
| $\mathcal{H}_0$ is not rejected | Correct | Wrong (False negative) |
| $\mathcal{H}_0$ is rejected | Wrong (False positive) | Correct |

- Type I: probability of a wrong reject

$$\mathbb{P}(T(X_1, \ldots, X_n) \in R \mid \mathcal{H}_0)$$

- Type II: probability of wrong non-reject

$$\mathbb{P}(T(X_1, \ldots, X_n) \notin R \mid \mathcal{H}_1)$$

# Significance level and power

## Significance level $\alpha$ if

$$\limsup_{n\to+\infty} \mathbb{P}(T(X_1,\ldots,X_n) \in R \mid \mathcal{H}_0) \leq \alpha$$

(We speak of 95%-test when $\alpha$ is 0.05%)

## Consistency

A test statistics (given by $T(X_1,\ldots,X_n)$ and a region $R$) is said to be $\alpha$-consistent if the significant level is $\alpha$ and if the power goes to one, i.e.,

$$\limsup_{n\to+\infty} \mathbb{P}(T(X_1,\ldots,X_n) \in R \mid \mathcal{H}_0) \leq \alpha$$

$$\lim_{n\to\infty} \mathbb{P}(T(X_1,\ldots,X_n) \in R \mid \mathcal{H}_1) = 1$$

# Test statistic and reject region

Goal: to build a $\alpha$-consistent test

(1) Define the test statistic $T(X_1, \ldots, X_n)$ and the level $\alpha$ you wish

(2) Do some maths to determine a reject region $R$ that achieves a significance level $\alpha$

(3) Prove the consistency

(4) Rule decision: reject whenever $\boxed{T_n(X_1, \ldots, X_n) \in R}$

# Famous tests

- Test of the equality of the mean for 1 sample
- Test of the equality of the means between 2 samples
- Chi-square test for the variance
- Chi-square test of independence
- Regression coefficient non-effects test

# Example: Gaussian mean

- Model: $\Theta = \mathbb{R}$, $\mathbb{P}_\theta = \mathcal{N}(\theta, 1)$
- Observe $(X_1, \dots, X_n)$ i.i.d. from this model
- Null hypothesis: $\mathcal{H}_0 : \{\theta = 0\}$
- Under $\mathcal{H}_0$, $\quad T_n(X_1, \dots, X_n) = \frac{1}{\sqrt{n}} \sum_i X_i \sim \mathcal{N}(0, 1)$
- Critical region for $T_n$ ? Gaussian quantile:

$$\mathbb{P}(T_n \in [-1.96, 1.96] \mid \mathcal{H}_0) = 0.95$$

- Take $R = ]-\infty, -1.96[\cup]1.96, +\infty[$.
- **Numerical example**: If $T_n = 1.5$, we do **not** reject $\mathcal{H}_0$ at level $95\%$

# Usage of the $p$-value

- The decision to accept or reject $\mathcal{H}_0$ is subject to the chosen significance level $\alpha$.
- To avoid making this choice in advance, in particular in software, the notion of the $p$-value is used to represent the result of a test.
- **The $p$-value is the probability that, under $\mathcal{H}_0$, the test statistic $T_n$ takes a value at least as extreme as its observed value**.
- Relation to the critical region:
  - If the test is one-sided with $R = \{t \mid t > c\}$ then for the observed $T_n$ the $p$-value is $\mathbb{P}(T > t_0 \mid \mathcal{H}_0)$.
  - If the test is one-sided with $R = \{t \mid t < c\}$ then for the observed $T_n$ the $p$-value is $\mathbb{P}(T < T_n \mid \mathcal{H}_0)$.
  - If the test is two-sided with $R = \{t \mid t \in ]-\infty; c_1) \cup (c_2; +\infty[\}$ then for the observed $T_n$ the $p$-value is $2\mathbb{P}(T < T_n \mid H_0)$ if $T_n$ is smaller than the median, and
    $2\mathbb{P}(T > T_n \mid H_0)$ if $T_n$ is larger than the median.

# Usage of the $p$-value: example

- Model: $\Theta = \mathbb{R}$, $\mathbb{P}_\theta = \mathcal{N}(\theta, 1)$
- Observe $(X_1, \ldots, X_n)$ i.i.d. from this model
- Null hypothesis: $\mathcal{H}_0 : \{\theta \leq 5\}$
- Under $\mathcal{H}_0$, $\quad T_n(X_1, \ldots, X_n) = \frac{\overline{X}_n - 5}{\frac{1}{\sqrt{n}}} \sim \mathcal{N}(0, 1)$

The test decision:

- Reject $\mathcal{H}_0$ if $\overline{X}_n > 5 + z_{1-\alpha}\frac{1}{\sqrt{n}}$.

Using the $p$-value:

- Assume $n = 10$ and $\overline{X}_n = 5.75$.
- The $p$-value equals $\mathbb{P}(\overline{X} > 5.75)$ with $\overline{X} \sim \mathcal{N}(5, \frac{1}{10})$, *i.e.* $\mathbb{P}(Z > 2.3717)$ with $Z \sim \mathcal{N}(0, 1)$, which equals $0.0089$.
- This indicates directly that one should reject at a level $0.05$ and even $0.01$.
- If the test would be two sided, *i.e.* with $\mathcal{H}_0 : \{\theta = 5\}$, the $p$-value for $\overline{X}_n = 5.75$ would be $0.0089 \times 2 = 0.0178$ implying **reject** at a level $0.05$ but **not** $0.01$.

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# Test of no-effect : Gaussian case

## Gaussian Model

$$y_i = \theta_0^\star + \sum_{k=1}^{p} \theta_k^\star x_{i,k} + \varepsilon_i$$

$$x_i^\top = (1, x_{i,1}, \ldots, x_{i,p}) \in \mathbb{R}^{p+1} \text{ (deterministic)}$$

$$\varepsilon_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2), \text{ for } i = 1, \ldots, n$$

## Theorem

Let $X = (x_1, \ldots, x_n)^\top \in \mathbb{R}^{n\times(p+1)}$ of full rank, and $\widehat{\sigma}^2 = \|\mathbf{y} - X\widehat{\boldsymbol{\theta}}\|_2^2/(n-(p+1))$, then

$$\widehat{T}_j = \frac{\widehat{\theta}_j - \theta_j^*}{\widehat{\sigma}\sqrt{(X^\top X)^{-1}_{j,j}}} \sim \mathcal{T}_{n-(p+1)}$$

where $\mathcal{T}_{n-p}$ is a Student law (with $n-(p+1)$ degrees of freedom)

# Test of no-effect : Gaussian case

## Null hypothesis

Aim is to test

$$\mathcal{H}_0 : \theta_j^* = 0$$

equivalently, $\Theta_0 = \{\theta \in \mathbb{R}^p : \theta_j = 0\}$

Under $\mathcal{H}_0$, we know the value of $\widehat{T}_j$ :

$$T_j := \frac{\widehat{\theta}_j}{\widehat{\sigma}\sqrt{(X^\top X)^{-1}_{j,j}}} \sim \mathcal{T}_{n-(p+1)}$$

Choosing $R = [-t_{1-\alpha/2}, t_{1-\alpha/2}]^c$ with $t_{1-\alpha/2}$ the $1-\alpha/2$-quantile of $\mathcal{T}_{n-(p+1)}$, we decide to reject $\mathcal{H}_0$ whenever

$$|\widehat{T}_j| > t_{1-\alpha/2}$$

# Test of no-effect : Random-design case

## Random design Model

$$y_i = \theta_0^\star + \sum_{k=1}^{p} \theta_k^\star \mathsf{x}_{i,k} + \varepsilon_i$$

$$\mathsf{x}_i^\top = (1, \mathsf{x}_{i,1}, \dots, \mathsf{x}_{i,p}) \in \mathbb{R}^{p+1}$$

$$(\varepsilon_i, \mathsf{x}_i) \overset{i.i.d}{\sim} (\varepsilon, \mathsf{x}), \text{ for } i = 1, \dots, n$$

$$\mathbb{E}(\varepsilon | \mathsf{x}) = 0, \ \mathbb{V}\mathrm{ar}(\epsilon | \mathsf{x}) = \sigma^2$$

## Theorem

If var$(\mathsf{x})$ has full rank, then

$$\widehat{T}_j = \frac{\widehat{\theta}_j - \theta_j^*}{\widehat{\sigma}\sqrt{(X^\top X)^{-1}_{j,j}}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1)$$

# Test of no-effect : Random-design case

## Null hypothesis

Aim is to test

$$\mathcal{H}_0 : \theta_j^* = 0$$

equivalently, $\Theta_0 = \{\theta \in \mathbb{R}^p : \theta_j = 0\}$

Under $\mathcal{H}_0$, we know the value of $\widehat{T}_j$ :

$$T_j := \frac{\widehat{\theta}_j}{\widehat{\sigma}\sqrt{(X^\top X)^{-1}_{j,j}}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1)$$

Choosing $R = [-z_{1-\alpha/2}, z_{1-\alpha/2}]^c$ with $z_{1-\alpha/2}$ the $1-\alpha/2$-quantile of $\mathcal{N}(0,1)$), we decide to reject $\mathcal{H}_0$ whenever

$$|\widehat{T}_j| > z_{1-\alpha/2}$$

# Link between IC and test

<u>Reminder</u> (Gaussian model):

$$IC_\alpha := \left[\widehat{\theta}_j - t_{1-\alpha/2}\widehat{\sigma}\sqrt{(X^\top X)^{-1}_{j,j}}, \widehat{\theta}_j + t_{1-\alpha/2}\widehat{\sigma}\sqrt{(X^\top X)^{-1}_{j,j}}\right]$$

is a CI at level $\alpha$ for $\theta^*_j$. Stating "$0 \in IC_\alpha$" means

$$|\widehat{\theta}_j| \leq t_{1-\alpha/2}\widehat{\sigma}\sqrt{(X^\top X)^{-1}_{j,j}} \quad \Leftrightarrow \quad \frac{|\widehat{\theta}_j|}{\widehat{\sigma}\sqrt{(X^\top X)^{-1}_{j,j}}} \leq t_{1-\alpha/2}$$

It is equivalent to accepting the hypothesis $\theta^*_j = 0$ at level $\alpha$. The smallest $\alpha$ such that $0 \in IC_\alpha$ is called the ***p*-value**.

<u>Rem</u>: Taking $\alpha$ close to zero $IC_\alpha$ covers the full space, hence one can find (by continuity) an $\alpha$ achieving equality in the aforementioned equations.

1. Statistical hypothesis testing

2. Illustration: forward variable selection

Data set “diabetes”

# “Diabetes” data set

| | age | sex | bmi | bp | Serum measurements | | | | | | Resp |
|---|---|---|---|---|---|---|---|---|---|---|---|
| patient | x1 | x2 | x3 | x4 | x5 | x6 | x7 | x8 | x9 | x10 | y |
| 1 | 59 | 2 | 32.1 | 101 | 157 | 93 | 38 | 4 | 4.9 | 87 | 151 |
| 2 | 48 | 1 | 21.6 | 87 | 183 | 103 | 70 | 3 | 3.9 | 69 | 75 |
| … | … | | | | | | | | | | … |
| … | … | | | | | | | | | | … |
| 441 | 36 | 1 | 30.0 | 95 | 201 | 125 | 42 | 5 | 5.1 | 85 | 220 |
| 442 | 36 | 1 | 19.6 | 71 | 250 | 133 | 97 | 3 | 4.6 | 92 | 57 |

$n = 442$ patients having diabetes, $p = 10$ variables “baseline” body mass index (bmi), average blood pressure (bp), *etc.*. . . have been measured.
**Goal**: predict disease progression one year in advance after the “baseline” measurement [EHJT04].

- Each variable of the data set from *sklearn* has been previously standardized.
- We apply an “expensive” version of the **forward variable selection** method (see, *e.g.*, [Zha09])

# "Diabetes" data set

- We define a vector of covariates with intercept $\tilde{X} = (\mathbb{1}, \mathbf{x}_1, \ldots, \mathbf{x}_{10})$.

**Step 0**

- for each variable $\tilde{X}_k$, $k = 1, \ldots, 11$, we consider the model

$$\mathbf{y} \simeq \beta_k \mathbf{x}_k$$

- we test whether its regression coefficient equals zero, *i.e.*

$$H_0 : \beta_k = 0$$

  using the statistic $\frac{\widehat{\beta}_k}{\widehat{s}_k}$ with $\widehat{s}_k$ being the estimated standard deviation.

- we compare all of the $p$-values, and keep the one possessing the smallest $p$-value. We save the residuals in the vector $\mathbf{r}_0$.

# “Diabetes” data set

## Step $\ell$

We have selected $\ell$ variable(s) : $\tilde{X}^{(\ell)} \in \mathbb{R}^{\ell}$. Those not selected are noted $\tilde{X}^{(-\ell)} \in \mathbb{R}^{p-\ell}$. We possess the vector of residuals $\mathbf{r}_{\ell-1}$ calculated on the previous step.

- for each variable $\mathbf{x}_k$ in $\tilde{X}^{(-\ell)}$, we consider the model

$$\mathbf{r}_{\ell-1} \simeq \beta_k \mathbf{x}_k$$

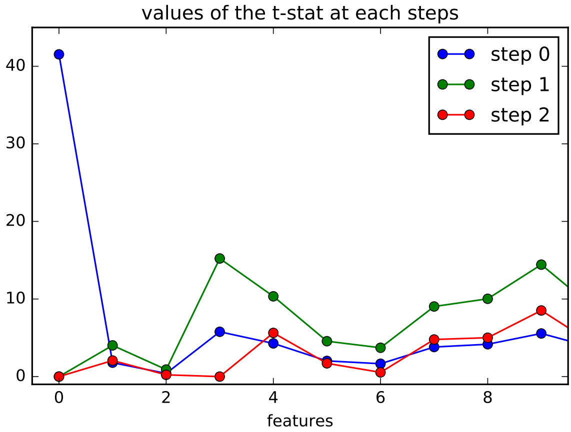
- we test if its regression coefficient equal zero, *i.e.*

$$H_0 : \beta_k = 0$$

  using the test statistic $\frac{\widehat{\beta}_k}{\widehat{s}_k}$ with $\widehat{s}_k$ being the estimated standard deviation.

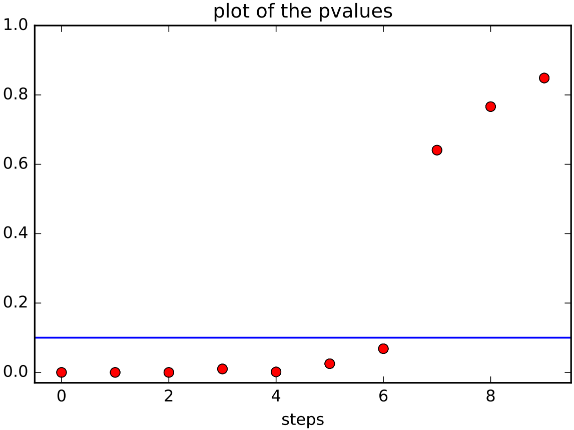
- we compare all of the $p$-values, and keep the one possessing the smallest $p$-value. We save the residuals in the vector $\mathbf{r}_{\ell}$.

# Values of the test statistics at each step




- The test statistic of the selected variable is $0$ on the following steps.
- The intercept is the first selected variable, then $x_3$, *etc...*

# Values of the test statistics at each step




- Sequence of the selected variables wit the test size 0.1 :

`[ 0, 3, 9, 5, 4, 2, 7 ]`

# References I

[EHJT04] B. Efron, T. Hastie, I. M. Johnstone, and R. Tibshirani. Least angle regression. *Ann. Statist.*, 32(2):407–499, 2004. With discussion, and a rejoinder by the authors.

[Zha09] Tong Zhang. Adaptive forward-backward greedy algorithm for sparse learning with linear models. In *Advances in Neural Information Processing Systems*, pages 1921–1928, 2009.

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