Prague Clocks

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In the old town square in Prague, there is a remarkable clock, which is a great tourist attraction. It dates from 1410, is believed to have been constructed by the master mechanician Mikulas of Kadan, and features an astronomical dial showing the position of the Moon and the Sun, an hourly clockwork show of the Apostles, and various other figures, such as a skeletal representation of Death. In addition, the clock strikes the hours in an extraordinary fashion, as described in the following. Consider the periodic sequence

$$1\,2\,3\,4\,3\,2\,1\,2\,3\,4\,3\,2\,1\,2\,3\,4\,3\,2\ldots$$

Between these digits we can insert commas and plus signs so that the **"visible sums"** separated by the commas form the sequence

$$1, 2, 3, 4, 3+2, 1+2+3, 4+3, 2+1+2+3, 4+3+2, 1+2+3+4, \ldots$$

of all the positive integers in their normal order.

The sum "$3+2$" in this sequence means that the clock strikes five as

bong bong bong pause bong bong

This is obviously convenient for the residents of nearby parts of Prague, as the clock is a 24-hour clock and it makes it easy to distinguish between 16 and 17 chimes, because 16 sounds as

X XX XXX XXXX XXX XX X $(1+2+3+4+3+2+1)$

whereas 17 uses the completely different pattern:

XX XXX XXXX XXX XX X XX $(2+3+4+3+2+1+2)$

This led to an interesting mathematical problem, which was discussed at the International Summer School for Students held at Jacobs University Bremen from the second to the twelfth of July 2013.

We define a Prague Clock sequence (usually just "Prague Clock") to be a periodic sequence of natural numbers into which plus signs can be inserted to make such "visible sums" that form the positive integers in their natural order.

We define our terms and illustrate them using the **old** Prague Clock.

If P is a Prague Clock for m, then the period sequence (1 2 3 4 3 2 for the old Prague Clock) is a sequence of numbers whose repetition forms P and whose sum is m, the period length is the number of terms in this sequence (6 for the old clock), and the modulus m is the sum of these terms. The primitive sequence is the minimal sequence of numbers that can be used to generate the Prague Clock (for example, it would be 1 for the Prague Clock with period sequence 1 1). A sequence is called primitive if its period sequence is equal to its primitive sequence.

From the old Prague Clock (1 2 3 4 3 2) we can obviously obtain other Prague Clocks, for example (1 2 3 2 2 3 2) by breaking any term into two or more others (here 4 into 2 2). Any sequence obtained from a shorter one such as this is a **broken** clock and a sequence that cannot be obtained in this manner is an **intact** clock.

The Prague Clock Problem

In this terminology, the Prague Clock Problem is to find all Prague Clocks.

We can obviously do this by first finding the intact Prague Clocks and then breaking them.

Theorem 1 *For every natural number m there exists a unique intact Prague Clock with modulus m.*

Proof The sum of the numbers before the k-th comma must be the k-th triangular number; thus for the old clock sequence

$$1, 2, 3, 4, 3+2, 1+2+3, 4+3, \ldots$$

we have

$$1 = 1, 3 = 1+2, 6 = 1+2+3, 10 = 1+2+3+4, 15 = 1+2+3+4+3+2, \ldots.$$

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(Recall that the t-th triangular number $\frac{1}{2}t(t+1)$ equals $1+2+\ldots+t$, which is the sum of the first t natural numbers.) If a Prague Clock has modulus m, the triangular numbers (mod m) must all occur as its partial sums. Moreover, this is the only necessary condition, from which it follows that the only intact Prague Clock of modulus m is that whose commas are exactly at the triangular numbers reduced (mod m). □

What are the triangular numbers reduced (mod m)?

Because the t-th one is $\frac{1}{2}t(t+1)$, if we increase t by $2m$, the triangular number $\frac{1}{2}t(t+1)$ remains unchanged modulo m, and we need only consider the first $2m$ of them. Plainly we need only consider the ones for which t is equal to $0,1,2,\ldots,2m-1$. However, the t-th triangular number is also equal to the $(-1-t)$ -th and so also the $(2m-1-t)$ -th triangular number, from which it follows that we in fact need only consider the $m-1$ first triangular numbers.

In conclusion, we can obtain the unique intact Prague Clock of modulus m by reducing the first m triangular numbers, $0,1,3,\ldots,\frac{1}{2}m(m-1)$ (mod m), adjoining m itself, and then sorting and taking their differences as the period sequence.

For the old Prague Clock (mod m), the number of $\frac{1}{2}t(t+1)$ are:

$$0,1,3,6,10,15,21,28,36,45,55,66,78,91,105$$

which reduced (mod 15) and sorted to become $\{0,1,3,6,10,13,15\}$, whose differences are indeed 1, 2, 3, 4, 3, 2.

From this unique Prague Clock, vandals can obtain the remaining (broken) Prague Clocks by chopping various terms into several smaller ones of the same sum.

This completes the solution of the Prague Clock Problem.

The Modulus and the Period

When half of the authors first considered the problem, they tried to classify the Prague Clocks by period rather than by modulus.

We now have:

Theorem 2 *The period length $\psi(m)$ of the intact Prague Clock of modulus m is a multiplicative function of m, that is to say, we have $\psi(mn)=\psi(m)\psi(n)$ whenever m and n are coprime.*

Proof This follows in the usual way from the Chinese remainder theorem (which asserts that the numbers modulo mn are in one-to-one correspondence with the ordered pairs of remainders modulo m and n) □

It follows that we only need to find the numbers ψ for prime power arguments p^k

Theorem 3 $\psi(2^k)=2^k$ *and*

$$\psi(p^k)=\frac{\phi(p^k)}{2}+\frac{\phi(p^{k-2})}{2}+\cdots+1 \qquad (p\ odd)$$

or equivalently,

$$\psi(p^k)=\begin{cases}\frac{1}{2}\left(\frac{p^{k+1}+1}{p+1}+1\right) & \text{if } k \text{ is even}\\ \frac{1}{2}\left(p\frac{p^k+1}{p+1}+1\right) & \text{if } k \text{ is odd}\end{cases}$$

where ϕ is Euler's totient function.

Proof To do this we just have to prove that $t_i \equiv t_j \bmod 2^k$ for $0 \le i,j < 2^k$ implies $i=j$. First we will prove that $\psi(2^k)=2^k$:

$$\begin{aligned} t_i &\equiv t_j \bmod 2^k\\ \frac{i(i+1)}{2} &\equiv \frac{j(j+1)}{2} \bmod 2^k\\ i(i+1) &\equiv j(j+1) \bmod 2^{k+1}\\ (i-j)(i+j+1) &\equiv 0 \bmod 2^{k+1}.\end{aligned}$$

Because $(i-j)$ and $(i+j+1)$ have different parity, this implies that $(i-j)\equiv 0 \bmod 2^{k+1}$ or $(i+j+1)\equiv 0 \bmod 2^{k+1}$. Because i and j are less than 2^k, this implies that $i=j$.

We first prove a

Lemma:
If m is odd, $\psi(m)$ equals the number of squares (mod m).

Proof It follows from the fact that δ is a triangular number iff $8\delta+1$ is an odd square, and the fact that modulo an odd number, that there is no difference between even and odd squares. □

Proof of Theorem 3 We need only show that the number of remainders of square numbers (mod p^k) is equal to $\frac{\phi(p^k)}{2}+\frac{\phi(p^{k-2})}{2}+\cdots+1$.

We use the fact that any odd prime power p^k has a primitive root. If g is a primitive root, the squares modulo p^k are the even powers g^{2r}. Indeed, whenever $2r<k$, the number of quadratic residues having p^{2r} as their greatest common divisor with p^k is $\frac{\phi(p^{k-2l})}{2}$ because there is a primitive root g_r of p^{k-2r} (mod p^{k-2r}) and every such quadratic residue is an even power of g_r. Therefore the total number of quadratic residues of p^k is $\psi(p^k)=\frac{\phi(p^k)}{2}+\frac{\phi(p^{k-2})}{2}+\cdots+1$, where the final 1 arises because 0 is always a square. □

While looking at our collection of Prague Clocks, we noticed that the intact clock of any even modulus $2m$ was just the concatenation of two Prague Clocks of modulus m and so was not primitive.

Theorem 4 *An intact Prague Clock is primitive if and only if its modulus is odd.*

Proof To begin, we will first prove that $[\ldots]_{2k}=[\ldots]_k[\ldots]_k$, that is, our observation was correct.

If the set of remainders when triangular numbers are divided by k is $\{a_1,a_2,\ldots,a_{\psi(k)}\}$, then the set of remainders