European Option Pricing

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1 Motivation

This personal project serves as my first approach of implementing fundamental option pricing models. During my studies at ETH Zurich in Quantitative Finance, I learned a lot about the theory behind option pricing models and we discussed some specific models in more detail. While we also discussed Barrier options, Asian and American options or other exotic options, in this project I want to focus on the standard european calls and puts. I try to implement the three most common models that were discussed in several classes: The Black-Scholes model, a stochastic volatility model and the Merton model. I will first give a quick theoretical introduction about the models and then explain my implementation. In the current version of this project, only the Black-Scholes model and the stochastic volatility model are implemented.

2 The Black-Scholes Model

2.1 Theoretical Introduction

This model serves as the building stone for all the more complex models. It makes a very simplifying assumption, that is, under the physical probability measure P the stock price behaves like a geometric Brownian Motion with drift μ and volatility σ

$$dS_t = S_t \mu dt + S_t \sigma dW_t$$

where W_t is a P-Brownian Motion. Furthermore, the Bonds price dynamics are described by the following ODE:

$$dB_t = B_t r dt$$

Applying a change of measure yields that the dynamics of the stock price process under the risk-neutral measure P is as follows

$$dS_t = S_t r dt + S_t \sigma dW_t$$

Calculating the price of the european Call at time t via the Martingale approach we get the following

$$C(S_t, T-t) = E_Q[\exp(-r(T-t))(S_T - K)1_{S_T \geq K}]$$

By applying Ito's lemma and another measure change we obtain the famous black-scholes formula:

$$C(S_t, T-t) = S_t \Phi(d_1) - K \exp(-r(T-t))\Phi(d_2)$$

for $d_1 = \frac{\ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$

Using the same approach for the Put option we get that

$$P(S_t, T-t) = K \exp(-r(T-t))\Phi(d_2) - S_t \Phi(d_1)$$

A different approach to get to the formula would be via the Black-Scholes PDE. The BS model is nice because it is very simple and has a closed form solution. On the other hand the prices one gets out from the formula are not realistic since the assumption which the model builds up upon are unrealistic as well. The most crucial ones are constant volatility and log-normally distributed returns. We can clearly see volatility clustering as well as fat tails in the distributions of the data. This is why we need some more advanced models that don't need these assumptions.

2.2 Implementation

Since there exists a closed-form solution in the black-scholes market, the implementation is trivial. Nevertheless I also implemented a Monte-Carlo simulation where the geometric brownian motion was simulated n times and then the price is given by the averaged discounted payoffs at maturity T. As n increases, the prices converge. Monte-Carlo simulation will be explained more in the context of the stochastic volatility model.

3 Stochastic Volatility Model

3.1 Theoretical Introduction

In this part, we introduce stochastic volatility, which will allow us to drop the assumption of constant volatility. We do this by assuming that the volatility itself is now a stochastic process again and denote the processes under the risk neutral measure Q as follows

$$dS_t = S_t r dt + S_t \sigma_t dW_t$$
$$d\sigma_t = v dt + \tilde{\sigma} d\tilde{W}_t$$

where W_t and $\tilde{W}_t$ are Brownian Motions and have correlation ρ. Whenever $|\rho| \neq 1$, markets are no longer complete since we have two sources of randomness and only one than can be traded and this model has no closed form solution. Therefore, we need numerical methods to solve it. We can either use PDE methods or, as we do here, a Monte Carlo simulation. We simulate the volatility process and the stock price process n times. For every end value S_T^i we calculate the discounted payoff at maturity T and then average over all discounted payoffs.

$$C(S_t, T-t) \approx \frac{1}{n} \sum_{i=1}^{n} \exp(-r(T-t))(S_T - K)$$

In the limit we get that the average over all discounted simulated payoffs converges to the expectation under Q which is equivalent to our option price.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \exp(-r(T-t))(S_T^i - K) 1_{S_T^i \geq K} = E_Q[\exp(-r(T-t))(S_T - K) 1_{S_T \geq K}] = C(S_t, T-t)$$

3.2 Implementation

We simulate the stock price process n times and calculate the average discounted payoff. One key question is how to simulate the correlated brownian motions. We use the fact that given a Brownian Motion W_t we can generate a new Brownian Motion $\tilde{W}_t$ which has correlation ρ with W_t as follows

$$\tilde{W}_t = \rho W_t + \sqrt{1-\rho^2} B_t$$

where B_t is a Brownian Motion which is uncorrelated with W_t and $\tilde{W}_t$. It is easy to proof that $\tilde{W}_t$ is a Brownian Motion. Since $\tilde{W}_t$ is a Martingale, Levy Characterisation Theorem tells us that we only need to show that $\langle \tilde{W} \rangle_t = t$.

$$\langle \tilde{W} \rangle_t = \langle \rho W + \sqrt{1-\rho^2} B \rangle_t = \rho^2 \langle W \rangle_t + (1-\rho^2) \langle B \rangle_t = \rho^2 t + (1-\rho^2) t = t$$

Since this holds true, $\tilde{W}_t$ is a Brownian Motion.

Another important question is how to choose parameters like ρ, v, $\tilde{\sigma}$ or n. While the first three parameters can be obtained by calibrating the model using market prices, the choice of n is more difficult. The higher one chooses n, the more stable and accurate the model is but the more computational power is used. It is common to use a value around 2000 for simulations of european options, therefore this value is used.