1. (a) True.
   (b) False.
   (c) False.
   (d) True.
   (e) True.
   (f) True.

2. (a) False.
   (b) True.
   (c) True.
   (d) True.
   (e) False.
   (f) False.

3. Following Euclids algorithm:

   (a) Yes.

$$\begin{aligned}
&GCD(10505, 1274)\\
&= GCD(1274, 313)\\
&= GCD(313, 22)\\
&= GCD(22, 5)\\
&= GCD(5, 2)\\
&= GCD(2, 1)\\
&= GCD(1, 1)\\
&= 1
\end{aligned}$$

   (b) No.

$$\begin{aligned}
&GCD(8024, 7289)\\
&= GCD(7289, 740)\\
&= GCD(740, 629)\\
&= GCD(629, 111)\\
&= GCD(111, 74)\\
&= GCD(74, 37)\\
&= 37.
\end{aligned}$$

4.

5. Let $\phi = (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{x} \vee y) \wedge (\bar{x} \vee \bar{y})$. The following truth-table shows that ϕ is not satisfiable.

x	y	$(x \vee y)$	$(x \vee \bar{y})$	$(\bar{x} \vee y)$	$(\bar{x} \vee \bar{y})$	ϕ
0	0	0	1	1	1	0
0	1	1	0	1	1	0
1	0	1	1	0	1	0
1	1	1	1	1	0	0

6. Let $\langle L_i, M_i \rangle$ be a polytime language and decider, such that $L_i = \{w \mid M_i\langle w \rangle \text{ accepts}\}$, and M_i always halts in polynomial time.

   Let $L_\cup = L_i \cup L_j$, $L_\circ = L_i \circ L_j$, $\bar{L_i} = \{w \in \Sigma^* \mid w \notin L_i\}$ be languages. To show that $L_\cup, L_\circ, \bar{L_i} \in$ P, we construct respective polynomial time deciders $M_\cup, M_\circ, \bar{M_i}$.

   $M_\cup$ = "On input w:

   1. Run $M_i\langle w \rangle$. If M_i accepts, *accept*.
   2. Run $M_j\langle w \rangle$. If M_j accepts, *accept*.
   3. If neither $M_i\langle w \rangle$ nor $M_j\langle w \rangle$ accepted, *reject*."

   $M_\circ$ = "On input w:

   1. For each position $k = 0$ to $|w|$, divide w into substrings $w = w_1w_2$, where w_1 is the first k symbols in w.
   2. Run $M_i\langle w_1 \rangle$ and $M_j\langle w_2 \rangle$. If both accept, *accept*.
   3. If no k exists such that $M_i\langle w_1 \rangle$ and $M_j\langle w_2 \rangle$ both *accept*, *reject*."

   $\bar{M_i}$ = "On input w:

   1. Run M_i on w. If M_i accepts *reject*. If M_i rejects, *accept*.

7. Let $\langle L_i, N_i \rangle$ be a language and non-deterministic Turing machine, such that N_i decides L_i in polynomial time.

   Let $L_\cup = L_i \cup L_j$, $L_\circ = L_i \circ L_j$ be languages. To show that $L_\cup, L_\circ \in$ P, we construct respective polynomial time non-deterministic deciders.

   $N_\cup$ = "On input w:

   1. Non-deterministically branch to simulate both $N_i\langle w \rangle$ and $N_j\langle w \rangle$.
   2. If either branch accepts, *accept*, else *reject*."

   $N_\circ$ = "On input w:

```
1. Non-determinsitically branch for each position $k = 0$ to $|w|$, with each branch
   dividing $w$ into substrings $w = w_1 w_2$, where $w_1$ is the first $k$ symbols in $w$.
     2. For each branch, run $N_i\langle w_1\rangle$ and $N_j\langle w_2\rangle$. If both accept, accept.
3. If no branch accepts, reject.”
```

8. To show that CONNECTED $\in P$, we need to show that CONNECTED $\in$ TIME(n^k), where k is some constant. As M decides CONNECTED, we only need to prove that M runs in $O(n^k)$.

   - In step 1, a node is selected and marked. This is done in constant time.
   - In stages 2 and 3, each node in G is scanned and marked if it is a neighbour of a marked node. This is repeated until no new nodes are marked. Assuming we do not visit already marked nodes, this has a worst-case run-time of $n\sum_{i=1}^{n-1} i = \frac{n^2(n-1)}{2}$ steps.
   - In stage 4, each node of G is scanned to determine if it is marked. This takes n steps.

   Combining steps 2,3 and 4, we have a run time of $\frac{n^2(n-1)}{2} + n = \frac{n^3+n^2+2n}{2} = O(n^3)$. This is clearly in P, and the proof is complete.

9. To show that TRIANGLE $\in$ P, we need to show that TRIANGLE $\in$ TIME(n^k), where k is some constant. We define Turing machine M, which operates as follows.

   M = “On input $\langle G\rangle$:

```
1. For each edge $u, v$ in $G$, do
     2. For each vertex $w$, do
          3. If $v, w$ is an edge and $w, u$ is an edge, accept
4. If no edge, vertex combination has accepted, reject.
```

   M runs in $O(|V||E|) = O(n^3)$. This is clearly in P, and the proof is complete.”