7.1 (a) True.
(b) False.
(c) False.
(d) True.
(e) True.
(f) True.

7.2 (a) False.
(b) True.
(c) True.
(d) True.
(e) False.
(f) False.

7.3 Following Euclids algorithm:

(a) Yes.

$$\begin{aligned}
&GCD(10505, 1274)\\
&= GCD(1274, 313)\\
&= GCD(313, 22)\\
&= GCD(22, 5)\\
&= GCD(5, 2)\\
&= GCD(2, 1)\\
&= GCD(1, 1)\\
&= 1
\end{aligned}$$

(b) No.

$$\begin{aligned}
&GCD(8024, 7289)\\
&= GCD(7289, 740)\\
&= GCD(740, 629)\\
&= GCD(629, 111)\\
&= GCD(111, 74)\\
&= GCD(74, 37)\\
&= 37.
\end{aligned}$$

7.4

7.5 Let $\phi = (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{x} \vee y) \wedge (\bar{x} \vee \bar{y})$. The following truth-table shows that $\phi$ is not satisfiable.

| x | y | $(x \vee y)$ | $(x \vee \bar{y})$ | $(\bar{x} \vee y)$ | $(\bar{x} \vee \bar{y})$ | $\phi$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 |

7.6 Let $\langle L_i, M_i \rangle$ be a polytime language and decider, such that $L_i = \{w \mid M_i\langle w \rangle \text{ accepts}\}$, and $M_i$ always halts in polynomial time.

Let $L_\cup = L_i \cup L_j$, $L_\circ = L_i \circ L_j$, $\bar{L_i} = \{w \in \Sigma^* \mid w \notin L_i\}$ be languages. To show that $L_\cup, L_\circ, \bar{L_i} \in$ P, we construct respective polynomial time deciders $M_\cup, M_\circ, \bar{M_i}$.

$M_\cup$ = "On input $w$:

1. Run $M_i\langle w \rangle$. If $M_i$ accepts, *accept*.
2. Run $M_j\langle w \rangle$. If $M_j$ accepts, *accept*.
3. If neither $M_i\langle w \rangle$ nor $M_j\langle w \rangle$ accepted, *reject*."

$M_\circ$ = "On input $w$:

1. For each position $k = 0$ to $|w|$, divide $w$ into substrings $w = w_1 w_2$, where $w_1$ is the first $k$ symbols in $w$.
2. Run $M_i\langle w_1 \rangle$ and $M_j\langle w_2 \rangle$. If both accept, *accept*.
3. If no $k$ exists such that $M_i\langle w_1 \rangle$ and $M_j\langle w_2 \rangle$ both *accept*, *reject*."

$\bar{M_i}$ = "On input $w$:

1. Run $M_i$ on $w$. If $M_i$ accepts *reject*. If $M_i$ rejects, *accept*.

7.7 Let $\langle L_i, N_i \rangle$ be a language and non-deterministic Turing machine, such that $N_i$ decides $L_i$ in polynomial time.

Let $L_\cup = L_i \cup L_j$, $L_\circ = L_i \circ L_j$ be languages. To show that $L_\cup, L_\circ \in$ P, we construct respective polynomial time non-deterministic deciders.

$N_\cup$ = "On input $w$:

1. Non-deterministically branch to simulate both $N_i\langle w \rangle$ and $N_j\langle w \rangle$.
2. If either branch accepts, *accept*, else *reject*."

$N_\circ$ = "On input $w$:

1. Non-determinsitically branch for each position $k = 0$ to $|w|$, with each branch dividing $w$ into substrings $w = w_1 w_2$, where $w_1$ is the first $k$ symbols in $w$.
2. For each branch, run $N_i\langle w_1\rangle$ and $N_j\langle w_2\rangle$. If both accept, *accept*.
3. If no branch accepts, *reject*."

7.8 To show that CONNECTED $\in P$, we need to show that CONNECTED $\in$ TIME$(n^k)$, where $k$ is some constant. As $M$ decides CONNECTED, we only need to prove that $M$ runs in $O(n^k)$.

- In step 1, a node is selected and marked. This is done in constant time.
- In stages 2 and 3, each node in $G$ is scanned and marked if it is a neighbour of a marked node. This is repeated until no new nodes are marked. Assuming we do not visit already marked nodes, this has a worst-case run-time of $n\sum_{i=1}^{n-1} i = \frac{n^2(n-1)}{2}$ steps.
- In stage 4, each node of $G$ is scanned to determine if it is marked. This takes $n$ steps.

Combining steps 2,3 and 4, we have a run time of $\frac{n^2(n-1)}{2} + n = \frac{n^3+n^2+2n}{2} = O(n^3)$. This is clearly in P, and the proof is complete.