SPRING BOUNDARY ELEMENTS

TOM

1. Introduction

Calculation of the solution to the inverse elasticity problem in biomechanical imaging requires a forward elasticity solve for each modulus update. The input to the problem is the modulus distribution μ and a measured displacement field $\boldsymbol{u}^m$ on the boundary of the tissue sample, readily providing a source of Dirichlet (displacement) boundary conditions for the forward problem. However, the overall accuracy of the non-axial displacement measurements is poor compared to that of the axial component, and all the measurements are corrupted by noise. Assuming traction-free B.C.'s on the non-axial boundaries and imposing Dirichlet B.C.s on the axial boundary allows for a "hard weighting" of the importance of the B.C.s to the solution.

Another approach is to introduce spring boundary elements (the penalty method). A spring tensor $\boldsymbol{K}$ is introduced as a way of weighting the measured displacement's contribution to the forward solution. A spring with a large spring constant will impose a Dirichlet B.C., while a spring constant of zero imposes a homogenous Neumann (traction-free) B.C. This method allows for a spatial and component based weighting of the boundary data ("flexible weighting"), the application of which hopefully yields "better" forward solutions (and therefore "better" inverse solutions).

2. Energy Functional

The potential energy functional used in the forward problem is :

$$\begin{aligned} \pi[\boldsymbol{u}] &= \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\epsilon}\ d\Omega + \frac{1}{2}\int_\Gamma (\boldsymbol{u}-\boldsymbol{u}^m)\cdot \boldsymbol{K}(\boldsymbol{u}-\boldsymbol{u}^m)\ d\Gamma \\ &= \frac{1}{2}\int_\Omega \sigma_{ij}\epsilon_{ij}\ d\Omega + \frac{1}{2}\int_\Gamma (u_i-u_i^m)K_{ij}(u_j-u_j^m)\ d\Gamma \\ &= \pi_A + \pi_B \qquad \mathit{functional} \end{aligned} \tag{1}$$

The function spaces for the arguments of the functional as as follows

$$\boldsymbol{u} \in \mathcal{S} \equiv \{\boldsymbol{u} \mid u_i \in H^1(\Omega)\} \qquad 3 \tag{2}$$

$$\boldsymbol{w} \in \mathcal{V} \equiv \{\boldsymbol{w} \mid w_i \in H^1(\Omega)\} \qquad 4 \tag{3}$$

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In an isotropic, linear elastic, incompressible plane stress context $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are defined as follows

$$\epsilon_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i) \qquad 7 \tag{4}$$

$$\begin{aligned} \sigma_{ij} &\equiv 2\mu(\delta_{ij}\epsilon_{kk} + \epsilon_{ij}) \\ &= \mu(2\delta_{ij}\partial_k u_k + \partial_i u_j + \partial_j u_i) \\ &= \mu A_{ij} \qquad 6 \end{aligned} \tag{5}$$

$\boldsymbol{K}$ is a spring tensor introduced as a way of enforcing Dirichlet B.C.s through a penalty (the size of the spring constant). In two dimensions $\boldsymbol{K}$ looks like

$$\begin{pmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{pmatrix} \tag{6}$$

A Dirichlet B.C.'s "contribution" to the solution can be altered by making the diagonal components of $\boldsymbol{K}$ larger or smaller. The off-diagonal components represent spring coupling to other degrees of freedom (should they always be set to zero??).

This functional differs from the one currently used to find the forward solution in NLACE by lacking prescribed tractions, and adding a boundary spring potential energy term. The function spaces used here are also "weaker" (no prescribed values on the boundary) than the ones currently employed in the forward problem.

The minimization of the first term of the functional (strain-energy) yields the weak form of the forward elasticity problem over the domain. The stiffness and mass matrices in NLACE are based off of this term. The minimization of the boundary springs term introduces new terms to the rhs and stiffness matrix where the equations for boundary nodes are located.

3. Plane Stress Forward Solution

Let $\pi_A[\boldsymbol{u}]$ be the strain energy density term from the potential energy functional. The principle of minimum potential energy states that the true displacement field of a mechanical system is the one that minimizes (makes stationary) the potential energy functional of the system.

(7) $D_{\boldsymbol{u}}\pi_A \cdot \boldsymbol{w} \overset{set}{=} 0$ 8

(8) $D_{\boldsymbol{u}}\pi_A \cdot \boldsymbol{w} = \dfrac{d}{d\alpha}\pi_A[\boldsymbol{u} + \alpha\boldsymbol{w}]\Big|_{\alpha=0}$ 26

(9) $\pi_A[u_i + \alpha w_i] = \dfrac{1}{2}\displaystyle\int_\Omega \mu\{(2\delta_{ij}\partial_k(u_k + \alpha w_k) + \partial_j(u_i + \alpha w_i) + \partial_i(u_j + \alpha w_j))$ 27

$$\frac{1}{2}(\partial_i(u_j + \alpha w_j) + \partial_j(u_i + \alpha w_i))\}\ d\Omega$$

(10) $\dfrac{d}{d\alpha}(9) = \dfrac{1}{2}\displaystyle\int_\Omega \mu\{(2\delta_{ij}\partial_k w_k + \partial_j w_i + \partial_i w_j)\frac{1}{2}(\partial_i(u_j + \alpha w_j) + \partial_j(u_i + \alpha w_i))$ 28

$$+\ (2\delta_{ij}\partial_k(u_k + \alpha w_k) + \partial_j(u_i + \alpha w_i) + \partial_i(u_j + \alpha w_j))\frac{1}{2}(\partial_i w_j + \partial_j w_i)\}\ d\Omega$$

(11) $(10)\Big|_{\alpha=0} = \dfrac{1}{2}\displaystyle\int_\Omega \mu\{(2\delta_{ij}\partial_k w_k + \partial_j w_i + \partial_i w_j)\frac{1}{2}(\partial_i u_j + \partial_j u_i)$ 29

$$+\ \ (2\delta_{ij}\partial_k u_k + \partial_j u_i + \partial_i u_j)\frac{1}{2}(\partial_i w_j + \partial_j w_i)\}\ d\Omega$$

$$= \frac{1}{2}\int_\Omega \mu\{\delta_{ij}\partial_k w_k(\partial_i u_j + \partial_j u_i) +\ \ (\partial_j w_i + \partial_i w_j)(\partial_i u_j + \partial_j u_i)$$

$$+\ \delta_{ij}\partial_k u_k(\partial_i w_j + \partial_j w_i)\}\ d\Omega$$

(12) Note: $(\partial_j w_i + \partial_i w_j)(\partial_i u_j + \partial_j u_i) = 2\partial_j w_i(\partial_i u_j + \partial_j u_i)$ by symmetry 30

(13) Application of relation (12) and the δ_{ij}s gives 31

$$(11) = \frac{1}{2}\int_\Omega \mu(4\partial_k u_k \partial_l w_l + 2\partial_j w_i(\partial_i u_j + \partial_j u_i))\}\ d\Omega$$

$$= \int_\Omega \mu\ \partial_j w_i(2\delta_{ij}\partial_k u_k + \partial_i u_j + \partial_j u_i)\ d\Omega = 0$$

(14) Equation (13) can be written in symbolic form as 10

$$\int_\Omega \nabla\boldsymbol{w} : \mu\boldsymbol{A}\ d\Omega = 0\ \forall\ \boldsymbol{w} \in \mathcal{V}$$

4. Boundary Spring Element Contribution

Let $\pi_B[\boldsymbol{u}]$ be the boundary spring term. Minimization yields

(15) $D_{\boldsymbol{u}}\pi_B \cdot \boldsymbol{w} \overset{set}{=} 0$ 1

(16) $\pi_B[\boldsymbol{u} + \alpha\boldsymbol{w}] = \dfrac{1}{2}\displaystyle\int_\Gamma (\boldsymbol{u} + \alpha\boldsymbol{w} - \boldsymbol{u}^m) \cdot \boldsymbol{K}(\boldsymbol{u} + \alpha\boldsymbol{w} - \boldsymbol{u}^m)\ d\Gamma$ 33

(17) $\dfrac{d}{d\alpha}(16) = \dfrac{1}{2}\displaystyle\int_\Gamma \boldsymbol{w} \cdot \boldsymbol{K}(\boldsymbol{u} + \alpha\boldsymbol{w} - \boldsymbol{u}^m) + (\boldsymbol{u} + \alpha\boldsymbol{w} - \boldsymbol{u}^m) \cdot \boldsymbol{K}\boldsymbol{w}\ d\Gamma$ 34

(18) $(17)\Big|_{\alpha=0} = \dfrac{1}{2}\displaystyle\int_\Gamma (\boldsymbol{K}\boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{K}\boldsymbol{w} \cdot \boldsymbol{u} - \boldsymbol{K}\boldsymbol{u}^m \cdot \boldsymbol{w} - \boldsymbol{K}\boldsymbol{w} \cdot \boldsymbol{u}^m)\ d\Gamma$ 2

$\boldsymbol{K}$ is a symmetric tensor by construction. Let $\boldsymbol{K}^S$ and $\boldsymbol{K}^A$ be the symmetric and anti-symmetric parts of $\boldsymbol{K}$, respectively.

(19) $$\boldsymbol{K} = \boldsymbol{K}^{(\boldsymbol{S})} + \boldsymbol{K}^{(\boldsymbol{A})} \qquad 17$$

(20) $$\boldsymbol{K}^{(\boldsymbol{S})} = \frac{1}{2}(\boldsymbol{K} + \boldsymbol{K}^T) \qquad 18$$

(21) $$\boldsymbol{K}^{(\boldsymbol{A})} = \frac{1}{2}(\boldsymbol{K} - \boldsymbol{K}^T) \qquad 19$$

Substitution of (19) into π_B gives the following expressions.

(22) $$\pi_B = \frac{1}{2}\int_\Gamma (\boldsymbol{u} - \boldsymbol{u}^m) \cdot (\boldsymbol{K}^{(\boldsymbol{S})} + \boldsymbol{K}^{(\boldsymbol{A})})(\boldsymbol{u} - \boldsymbol{u}^m)\; d\Gamma \qquad 20$$

Substituting (21) into the anti-symmetric part of (22) produces

(23) $$\begin{aligned} &\frac{1}{2}\int_\Gamma (\boldsymbol{u} - \boldsymbol{u}^m) \cdot \boldsymbol{K}^{(\boldsymbol{A})}(\boldsymbol{u} - \boldsymbol{u}^m)\; d\Gamma \qquad 21 \\ &= \frac{1}{2}\int_\Gamma (\boldsymbol{u} - \boldsymbol{u}^m) \cdot (\boldsymbol{K}\boldsymbol{u} - \boldsymbol{K}\boldsymbol{u}^m - \boldsymbol{K}^T\boldsymbol{u} + \boldsymbol{K}^T\boldsymbol{u}^m)\; d\Gamma \\ &= \frac{1}{2}\int_\Gamma \{(\boldsymbol{u}\cdot\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}\cdot\boldsymbol{K}\boldsymbol{u}^m - \boldsymbol{u}\cdot\boldsymbol{K}^T\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{K}^T\boldsymbol{u}^m) + \\ &(-\boldsymbol{u}^m\cdot\boldsymbol{K}\boldsymbol{u} + \boldsymbol{u}^m\cdot\boldsymbol{K}\boldsymbol{u}^m + \boldsymbol{u}^m\cdot\boldsymbol{K}^T\boldsymbol{u} - \boldsymbol{u}^m\cdot\boldsymbol{K}^T\boldsymbol{u}^m)\}\; d\Gamma \end{aligned}$$

(24) Definition of a tensor transpose: $\boldsymbol{a}\cdot\boldsymbol{K}\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{K}^T\boldsymbol{a}$ 32

(25) Application of (24) yields (23) $= 0$ 23

Therefore, $\boldsymbol{K}$ is a symmetric tensor, and (18) simplifies to

(26) $$\int_\Gamma \boldsymbol{K}\boldsymbol{u}\cdot\boldsymbol{w} - \boldsymbol{K}\boldsymbol{u}^m\cdot\boldsymbol{w}\; d\Gamma = 0\ \forall\ \boldsymbol{w} \in \mathcal{V} \qquad 12$$

The next step is to determine the discrete form of (26). The stationary conditions derived above hold for all functions $\boldsymbol{w}$ in $\mathcal{V}$. We choose a subset of $\mathcal{V}$, C^0 smooth shape functions, for substitution into (26). These functions are defined to be 1 over their assigned node, and they decrease linearly to zero at neighboring nodes. In this way a vector of nodal values defines a linearly interpolated displacement field, the samples of which are the nodal values.

(27) $$\boldsymbol{w}^h = \sum_A \boldsymbol{w}^A N_A \qquad 13$$

(28) $$\boldsymbol{u}^h = \sum_B \boldsymbol{u}^B N_B \qquad 14$$

In (27) and (28) A and B represent global node locations, and N_A is the shape function for node A . It is important to remember that $\boldsymbol{u}$ and $\boldsymbol{w}$ have multiple spatial components (represented using the indices i and j). Inserting these functions into (26) leads to the following expressions (using indices)

$$\begin{aligned}\int_\Gamma \sum_A \sum_B \cancel{w_i^A} K_{ij} N_A N_B u_j^B \; d\Gamma \qquad 25 \\ = \int_\Gamma \sum_A \sum_B \cancel{w_i^A} K_{ij} N_A N_B u_j^{Bm} \; d\Gamma\end{aligned} \tag{29}$$

In (29) u_j^B is the unknown vector of nodal displacements while u_j^{Bm} is a given vector of measured displacements. These vectors are multiplied by the mass and stiffness matricies. The boundary integral can be split into a sum of integrals over each boundary element. The resulting expressions for the elemental stiffness matrix and force vector are

$$k_{ai,bj} = \int_{\Gamma^e} K_{ij} N_a N_b \; d\Gamma^e \qquad 15 \tag{30}$$

$$f_{ai} = \sum_{b,j} k_{ai,bj} u_j^{bm} \qquad 16 \tag{31}$$

Where a and b refer to local node numbers

5. Objective Function and Gradient

After a displacement field is found for a given modulus distribution (the predicted displacement $\boldsymbol{u}$) we would like to solve for a new modulus distribution that minimizes the mismatch between $\boldsymbol{u}$ and $\boldsymbol{u}^m$. The solution is found through an optimization routine that requires an evaluation of the data mismatch functional and its gradient in the direction of μ.

$$\pi_d[\mu] = \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}^m\|_\Omega \qquad 40 \tag{32}$$

The dependence of μ in (32) is implicitly defined through $\boldsymbol{u}$. The gradient is calculated as follows

$$D_\mu \pi_d[\mu] \cdot \delta\mu = \frac{d}{d\alpha} \pi_d[\mu + \alpha\delta\mu]\Big|_{\alpha=0} \stackrel{set}{=} 0 \qquad 41 \tag{33}$$

$$D_\mu \pi_d[\mu] \cdot \delta\mu = (\delta\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^m)_\Omega \tag{34}$$

$$\delta\boldsymbol{u} \equiv \frac{d}{d\alpha} \boldsymbol{u}(\boldsymbol{x}; \mu + \alpha\delta\mu)\Big|_{\alpha=0} \qquad 43 \tag{35}$$

In order to determine (33) we use the weak form of the forward solution. The subscripts t, s, and e below stand for total, spring, and equilibrium.

$$a_t(\boldsymbol{w}, \boldsymbol{u}; \mu) = l_s(\boldsymbol{w}) \qquad 44 \tag{36}$$

$$a_t(\boldsymbol{w}, \boldsymbol{u}; \mu) = a_e(\boldsymbol{w}, \boldsymbol{u}; \mu) + a_s(\boldsymbol{w}, \boldsymbol{u}) \qquad 45 \tag{37}$$

$$a_e(\boldsymbol{w}, \boldsymbol{u}; \mu) = \int_\Omega \nabla \boldsymbol{w} : \mu \boldsymbol{A} \; d\Omega \qquad 46 \tag{38}$$

$$a_s(\boldsymbol{w}, \boldsymbol{u}) = \int_\Gamma \boldsymbol{K}\boldsymbol{u} \cdot \boldsymbol{w} \; d\Gamma \qquad 47 \tag{39}$$

$$l_s(\boldsymbol{w}) = \int_\Gamma \boldsymbol{K}\boldsymbol{u}^m \cdot \boldsymbol{w} \; d\Gamma \qquad 48 \tag{40}$$

Using equation (36)

$$a_t(\boldsymbol{w}, \boldsymbol{u} + \alpha\delta\boldsymbol{u}; \mu + \alpha\delta\mu) = l_s(\boldsymbol{w}) \qquad 55 \tag{41}$$

$$\begin{aligned} a_e(\boldsymbol{w}, \boldsymbol{u}; \mu) + \alpha a_e(\boldsymbol{w}, \delta\boldsymbol{u}; \mu) + \alpha a_e(\boldsymbol{w}, \boldsymbol{u}; \delta\mu) \qquad 56 \\ + \alpha^2 a_e(\boldsymbol{w}, \delta\boldsymbol{u}; \delta\mu) + a_s(\boldsymbol{w}, \boldsymbol{u}) + \alpha a_s(\boldsymbol{w}, \delta\boldsymbol{u}) = l_s(\boldsymbol{w}) \end{aligned} \tag{42}$$

$$\frac{d}{d\alpha}(42)\bigg|_{\alpha=0} = a_e(\boldsymbol{w}, \delta\boldsymbol{u}; \mu) + a_s(\boldsymbol{w}, \delta\boldsymbol{u}) + a_e(\boldsymbol{w}, \boldsymbol{u}; \delta\mu) = 0 \qquad 58 \tag{43}$$

$$a_t(\boldsymbol{w}, \delta\boldsymbol{u}; \mu) = -a_e(\boldsymbol{w}, \boldsymbol{u}; \delta\mu) \;\forall\; \boldsymbol{w} \in \mathcal{V} \qquad 57 \tag{44}$$

The calculation of the gradient also requires the solution of the dual problem, i.e. find a $\bar{\boldsymbol{w}} \in \mathcal{V}$ that satisfies

$$a_t(\bar{\boldsymbol{w}}, \boldsymbol{v}; \mu) = (\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{u}^m)_\Omega \;\forall\; \boldsymbol{v} \in \mathcal{V} \qquad 53 \tag{45}$$

We use shape functions to represent $\bar{\boldsymbol{w}}$ and $\boldsymbol{v}$ to formulate the discrete version of (45).

$$\boldsymbol{v}^h = \sum_A \boldsymbol{v}^A N_A \qquad 60 \tag{46}$$

$$\bar{\boldsymbol{w}}^h = \sum_B \bar{\boldsymbol{w}}^B N_B \qquad 61 \tag{47}$$

Substitution of the above expressions into (45) yields

$$\begin{aligned} \int_\Gamma \sum_A \sum_B \cancel{v_i^A} K_{ij} N_A N_B \bar{w}_j^B \; d\Gamma + \int_\Omega \sum_A \sum_B \cancel{v_i^A} \nabla N_A A_{ij}(N_B) \mu \bar{w}_j^B \; d\Omega \qquad 62 \\ = \int_\Omega \sum_A \sum_B \cancel{v_i^A} N_A N_B (u_j^B - u_j^{Bm}) \; d\Omega \end{aligned} \tag{48}$$

The global stiffness matrix from (48) (the dual problem) is the transpose of the stiffness matrix from the forward problem. However, the stiffness matrix from the forward problem is symmetric, so the stiffness matrix for the primal and dual problem is identical. On the other hand, the residual (right hand side) is differs. In the dual problem the residual is proportional to the difference between the predicted and measured displacement, while in the primal problem the residual is a forcing term from the boundary springs.

We can replace the $\boldsymbol{v}$ in (45) with $\delta\boldsymbol{u}$ because they are both contained in $\mathcal{V}$, and the gradient can be found using $\bar{\boldsymbol{w}}$ from the dual problem.

(49) $$a_t(\bar{\boldsymbol{w}}, \delta\boldsymbol{u}; \mu) = (\delta\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^m)_\Omega$$ 49

(50) $$a_t(\bar{\boldsymbol{w}}, \delta\boldsymbol{u}; \mu) = -a_e(\bar{\boldsymbol{w}}, \boldsymbol{u}; \delta\mu)$$ 50

(51) $$(\delta\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^m)_\Omega = -a_e(\bar{\boldsymbol{w}}, \boldsymbol{u}; \delta\mu)$$ 51

(52) $$D_\mu \pi_d[\mu] \cdot \delta\mu = -a_e(\bar{\boldsymbol{w}}, \boldsymbol{u}; \delta\mu)$$ 52