$$\begin{cases} \nabla\cdot\sigma = 0 & \text{on } \Omega \\ + \text{B.C.} \\ J - 1 = 0 & \text{on } \Omega \end{cases}$$

$$\sigma = \frac{1}{J} F S F^T$$

$$S = -pJF^{-T}F^{-1} + 2\frac{\partial W}{\partial C} \qquad \frac{\partial S_{IJ}}{\partial C_{KL}} = \frac{1}{2}\mathbb{C}_{IJKL} \qquad D_{\delta u} J = J\,tr(\nabla \delta u F)$$

$$C = F^T F \qquad D_{\delta u} C = 2F^T F \nabla \delta u F \qquad \frac{\partial J}{\partial C} = \frac{1}{2} J C^{-1}$$

$$F = (\underline{1} - \nabla \underline{u})^{-1}. \qquad \boxed{D_{\delta u} F = F \nabla \delta u F} \qquad \frac{\partial J}{\partial F} = J F^{-T}$$

$$\int_\Omega \nabla w : \frac{1}{J} F S F^T d\Omega + \int_\Omega q(J-1)\, d\Omega + \sum^{nel} \int_{\Omega^e} \tau \nabla q \nabla p \, d\Omega^e = 0$$

$$\underbrace{\int_\Omega w_{i,j} \frac{1}{J} F_{iI} S_{IJ} F_{jJ}\, d\Omega}_{①} + \underbrace{\int_\Omega q(J-1)\, d\Omega}_{②} + \underbrace{\sum^{nel}\int_{\Omega^e} \tau\, q_{,i} P_{,i}\, d\Omega^e}_{③} = 0$$

Linearize ① on $\frac{1}{J}$ in δu:

$$\int_\Omega \nabla w : \frac{\partial J^{-1}}{\partial J} \frac{\partial J}{\partial \varepsilon}\Big|_{\varepsilon \to 0} F S F^T d\Omega = \int_\Omega \nabla w : \left(-\frac{1}{J^2}\right) J\, tr(\nabla \delta u F)\, F S F^T d\Omega$$

$$= -\int_\Omega \nabla w : \frac{1}{J} tr(\nabla \delta u F) F S F^T d\Omega = -\int_\Omega \nabla w : tr(\nabla \delta u F)\, \sigma\, d\Omega$$

$$= -\int_\Omega w_{i,j}\, \delta u_{q,\ell} F_{\ell q} \sigma_{ij}\, d\Omega = -\int_\Omega w_{i,j} F_{\ell q} \sigma_{ij}\, \delta u_{q,\ell}\, d\Omega$$

$$- N^A_{,j} F_{\ell q} \sigma_{ij} N^B_{,\ell} = k_{iAqB}$$

Linearize ① on F in δu:

$$\int_\Omega \nabla w : \frac{1}{J} \frac{dF}{d\varepsilon}\Big|_{\varepsilon \to 0} S F^T d\Omega = \int_\Omega \nabla w : \frac{1}{J} F \nabla \delta u F S F^T d\Omega$$

$$= \int_\Omega \nabla w : F \nabla \delta u\, \sigma\, d\Omega = \int_\Omega w_{i,j} F_{iq}\, \delta u_{q,\ell}\, \sigma_{\ell j}\, d\Omega = \int_\Omega w_{i,j} F_{iq} \sigma_{\ell j}\, \delta u_{q,\ell}\, d\Omega$$

$$N^A_{,j} F_{iq} \sigma_{\ell j} N^B_{,\ell} = k_{iAqB}$$

Linearize ① on S in δu:

$$\int_\Omega w_{i,j} \frac{1}{J} F_{iI} \frac{\partial S_{IJ}}{\partial C_{KL}} \frac{\partial C_{KL}}{\partial \varepsilon}\Big|_{\varepsilon\to 0} F_{jJ}\, d\Omega = \int_\Omega w_{i,j} \frac{1}{J} F_{iI} \frac{1}{2} \breve{C}_{IJKL}\, 2 F_{kK} F_{kq}\, \delta u_{q,\ell} F_{\ell L} F_{jJ}\, d\Omega$$

$$= \int_\Omega w_{i,j} \frac{1}{J} F_{iI} F_{jJ} F_{kK} F_{\ell L} \breve{C}_{IJKL} F_{kq}\, \delta u_{q,\ell}\, d\Omega$$

$$= \int_\Omega w_{i,j}\, \breve{c}_{ijk\ell} F_{kq}\, \delta u_{q,\ell}\, d\Omega, \quad \text{where } \breve{c}_{ijk\ell} = \frac{1}{J} F_{iI} F_{jJ} F_{kK} F_{\ell L} \breve{C}_{IJKL}$$

Linearize ① on F^T in δu:

$$\int_\Omega \nabla w : \frac{1}{J} F S \frac{dF^T}{\partial \varepsilon}\Big|_{\varepsilon\to 0} d\Omega = \int_\Omega \nabla w : \frac{1}{J} F S (F \nabla \delta u F)^T d\Omega$$

$$= \int_\Omega \nabla w : \frac{1}{J} F S F^T \nabla\delta u^T F^T d\Omega = \int_\Omega \nabla w : \sigma \nabla \delta u^T F^T d\Omega = \int_\Omega \nabla w : (F \nabla\delta u\, \sigma)^T d\Omega.$$

$$= \int_\Omega w_{i,j} F_{jq}\, \delta u_{q,\ell}\, \sigma_{\ell i}\, d\Omega = \int_\Omega w_{i,j} F_{jq}\, \sigma_{\ell i}\, \delta u_{q,\ell}\, d\Omega.$$

$N^A_{,j} F_{jq} \sigma_{\ell i} N^B_{,\ell} = k_{iAqB}.$

∴ Linearization of ①

$$\int_\Omega w_{i,j} (-F_{\ell q} \sigma_{ij} + F_{iq}\sigma_{\ell j} + \breve{c}_{ijk\ell} F_{kq} + F_{jq}\sigma_{\ell i})\, \delta u_{q,\ell}\, d\Omega$$
$$+ \int_\Omega w_{i,j} (-\delta P\cdot)\, d\Omega$$

Linearization of ②

$$\int_\Omega q \frac{\partial (J-1)}{\partial \varepsilon} d\Omega = \int_\Omega q J\, \mathrm{tr}(\nabla\delta u F)\, d\Omega = \int_\Omega q J\, \delta u_{q,\ell} F_{\ell q}\, d\Omega.$$

$N^A J N^B_{,\ell} F_{\ell q}$

Linearization of ③.

$$\frac{d \int_{\Omega^e} \tau \nabla q \nabla P\, d\Omega^e}{d\varepsilon} = \int_{\Omega^e} \tau \nabla q \nabla \delta P\, d\Omega^e = \int_{\Omega^e} \tau\, q_{,i}\, \delta P_{,i}\, d\Omega^e.$$

$\tau N^A_{,i} N^B_{,i}$