Chapter 4

Finite Deformation (Hyper-)Elastostatics

4.1 Continuum Mechanics Background

As a point of departure recall the strong form of the boundary-value problem of *linear* elastostatics: Find the displacement vector $\boldsymbol{u}$ such that **abstract notation**

$$(S)\left\{\begin{aligned} \mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f} &= \mathbf{0} && \text{in } \Omega && (\textit{equilibium equations}) \\ \boldsymbol{u} &= \boldsymbol{g} && \text{on } \Gamma_g \\ \boldsymbol{t} := \boldsymbol{\sigma}\cdot\boldsymbol{n} &= \boldsymbol{h} && \text{on } \Gamma_h \end{aligned}\right. \tag{4.1}$$

The stress tensor $\boldsymbol{\sigma}$ is defined in terms of the strain tensor $\boldsymbol{\epsilon}$ by a constitutive equation, the generalized Hooke's law:

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \tag{4.2}$$

(repeated indices imply summation unless otherwise indicated) where $c_{ijkl} = c_{ijkl}(\boldsymbol{x})$ are the elastic coefficients and

$$\epsilon_{kl} = u_{(k,l)} \tag{4.3}$$

This is a special, restricted version of small-deformation *nonlinear* elasticity where there exists a given potential $\Phi(\boldsymbol{\epsilon})$, the strain energy density function, such that:

$$\sigma_{ij} := \frac{\partial \Phi}{\partial \epsilon_{ij}} \tag{4.4}$$

and the tangent moduli $c_{ijkl} = c_{ijkl}(\boldsymbol{x}, \boldsymbol{\epsilon})$ form its Hessian matrix

$$\begin{aligned} c_{ijkl} &= \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} \\ &= \frac{\partial^2 \Phi}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \end{aligned} \tag{4.5}$$

The major symmetry of the tangent moduli thus follows from the existence of a strain energy function.

The weak formulation of the boundary-value problem, sometimes referred to as the principle of virtual work, is:

(W) Find $\boldsymbol{u} \in \boldsymbol{\mathcal{S}}$ (satisfying essential boundary conditions) such that for all $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}$ (satisfying the homogeneous counterpart of essential boundary conditions)

$$\int_\Omega w_{i,j} \sigma_{ij} \, d\Omega = \int_\Omega w_i f_i \, d\Omega + \int_{\Gamma_h} w_i h_i \, d\Gamma \tag{4.6}$$

where, in the nonlinear case, σ_{ij} is no longer defined by the generalized Hooke's law.

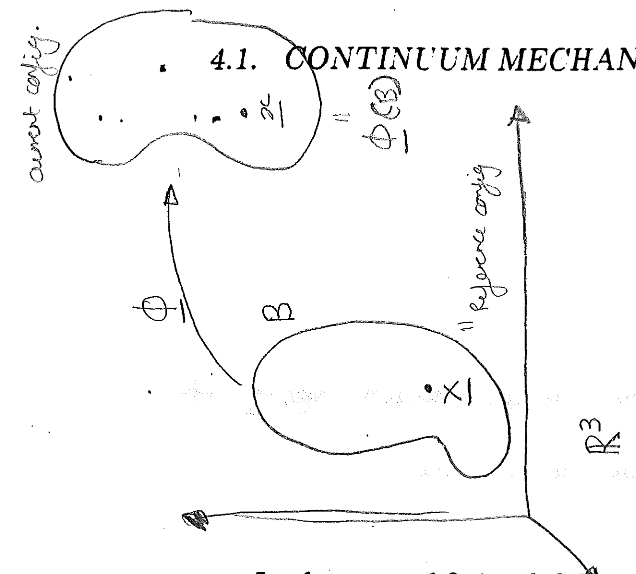


Figure 4.1: [1, Fig. 1.2, p. 3]

$\underline{X}$ = Lagrangian coordinate
" Ref.
" Mat'l.
$\underline{x} = \underline{\phi}(\underline{X})$
= Eulerian
spatial

In the case of finite deformations $\boldsymbol{\epsilon}$ is not a bona fide nonlinear strain measure. To understand the situation we need to introduce some elementary notions from nonlinear continuum mechanics. The following employs notation from [1].

A reference configuration $\mathcal{B}$ is chosen for the body. Points in $\mathcal{B}$, called material points, are described in Lagrangian coordinates $\boldsymbol{X} = (X_1, X_2, X_3)$. A configuration of $\mathcal{B}$ is a mapping $\boldsymbol{\phi}: \mathcal{B} \to \mathbf{R}^3$ representing a deformed state of the body. Points in $\mathbf{R}^3$, called spatial points $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X})$, are described in Eulerian coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$.

Parameterizing the configuration by time (t) leads to a *motion*:

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{\phi}(\boldsymbol{X}, t) \\ &= \boldsymbol{\phi}_t(\boldsymbol{X}) \end{aligned} \tag{4.7}$$

where the second line refers to t fixed. We assume that the initial configuration coincides with the reference configuration, i.e.,

$$\boldsymbol{\phi}_0 = \mathbf{1} \tag{4.8}$$

Identity

so that

$$\boldsymbol{X} = \boldsymbol{\phi}_0(\boldsymbol{X}) \tag{4.9}$$

The *displacement* of material point $\boldsymbol{X}$ is defined by

$$\boldsymbol{U}_t(\boldsymbol{X}) = \underbrace{\boldsymbol{\phi}_t(\boldsymbol{X})}_{\boldsymbol{x}} - \boldsymbol{X} \tag{4.10}$$

The initial displacement is thus

$$\begin{aligned} \boldsymbol{U}_0(\boldsymbol{X}) &= \underbrace{\boldsymbol{\phi}_0(\boldsymbol{X})}_{\boldsymbol{X}} - \boldsymbol{X} \\ &= \boldsymbol{0} \end{aligned} \tag{4.11}$$

(In dynamic applications often $\boldsymbol{U}_0 \neq \boldsymbol{0}$, i.e., the initial configuration differs from the reference configuration.)

A general *rigid motion* of $\mathcal{B} \subset \mathbb{R}^3$ is

$$\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X}, t) := \boldsymbol{Q}(t)\boldsymbol{X} + \boldsymbol{c}(t) \tag{4.12}$$

where $\boldsymbol{Q}$ is a proper orthogonal matrix, representing a rigid rotation ($\boldsymbol{\epsilon} \neq \boldsymbol{0}$ for such a motion, thus disqualifying it as a measure of strain) and $\boldsymbol{c}$ is a rigid translation.

Kinematics. The components of the motion are

$$x_i = \phi_i(\boldsymbol{X}, t) \tag{4.13}$$

The matrix of partial derivatives of $\boldsymbol{\phi}$ is called the *deformation gradient*:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}} = \left[\frac{\partial \phi_i}{\partial X_I}\right] \tag{4.14}$$

Lower case indices i, j, k and l denote spatial quantities, e.g., $\boldsymbol{x} = \{x_i\}$ and capital I, J, K and L are used for material quantities, for example $\boldsymbol{X} = \{X_I\}$.

The *material (time) derivative*, denoted by a superposed dot, is the derivative with respect to t, holding $\boldsymbol{X}$ fixed. Thus, the *velocity* of a material point $\boldsymbol{X}$ is

$$\boldsymbol{V}(\boldsymbol{X},t) = \dot{\boldsymbol{\phi}}(\boldsymbol{X},t) = \frac{\partial \boldsymbol{\phi}(\boldsymbol{X},t)}{\partial t} \tag{4.15}$$

regarded as a vector emanating from point $\boldsymbol{x}$.

The material *acceleration* of a motion $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{X},t)$ is

$$\boldsymbol{A} = \dot{\boldsymbol{V}} = \ddot{\boldsymbol{\phi}} \tag{4.16}$$

The *spatial velocity* of the motion is $\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{V}(\boldsymbol{X},t)$ where $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X},t)$. We may thus define $\boldsymbol{v}$ by

$$\boldsymbol{v}(\boldsymbol{\phi}_t(\boldsymbol{X}),t) = \boldsymbol{V}(\boldsymbol{X},t) \tag{4.17}$$

i.e., $\boldsymbol{v}_t(\boldsymbol{\phi}_t(\cdot)\,) = \boldsymbol{V}_t(\cdot)$, which defines the composition

$$\boldsymbol{v}_t \circ \boldsymbol{\phi}_t = \boldsymbol{V}_t \tag{4.18}$$

$\boldsymbol{v}$ is the Eulerian representation of the velocity, and $\boldsymbol{V}$ is its Lagrangian representation.

To obtain the *spatial velocity gradient* $\nabla \boldsymbol{v}$ we first consider the material derivative of the deformation gradient

$$\begin{aligned}
\dot{\boldsymbol{F}} &= \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{X}} \boldsymbol{F} \\
&= \frac{\partial}{\partial t}\frac{\partial \boldsymbol{\phi}(\boldsymbol{X},t)}{\partial \boldsymbol{X}} \\
&= \frac{\partial}{\partial \boldsymbol{X}}\frac{\partial \boldsymbol{\phi}}{\partial t} \\
&= \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{X}} \qquad (4.19)
\end{aligned}$$

By (4.18), employing the chain rule

$$\begin{aligned}
\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{X}} &= \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}} \\
&= \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\boldsymbol{F} \qquad (4.20)
\end{aligned}$$

This may be written in components as

$$\begin{aligned}
V_{i,I} &= v_{i,j}\phi_{j,I} \\
&= v_{i,j}F_{jI} \qquad (4.21)
\end{aligned}$$

with implied summation on the index j. Since $\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{X}} = \dot{\boldsymbol{F}}$ and $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \nabla \boldsymbol{v}$

$$\nabla \boldsymbol{v} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} \qquad (4.22)$$

assuming $\boldsymbol{F}^{-1}$ exists.

The Jacobian determinant of the motion is

$$J = \det \boldsymbol{F} \tag{4.23}$$

Note that $J(\boldsymbol{X}, t) > 0$ (corresponding to impenetrability, a property of real motions). ???

The (material or Lagrangian) *Green(-Lagrange) strain tensor* is defined by

$$2\boldsymbol{E} := \boldsymbol{F}^T \boldsymbol{F} - \mathbf{1} \tag{4.24}$$

so that $\boldsymbol{E} = \boldsymbol{E}^T$, and in components

$$2E_{IJ} := F_{iI} F_{iJ} - \delta_{IJ} \tag{4.25}$$

In order to express the Green strain in terms of displacements, recall

$$\begin{aligned} \boldsymbol{F} &= \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}} \\ &= \frac{\partial}{\partial \boldsymbol{X}} (\boldsymbol{X} + \boldsymbol{U}(\boldsymbol{X})) \end{aligned} \tag{4.26}$$

where the second line follows from the definition of the displacement (4.10). In components

$$\begin{aligned} F_{iI} &= \frac{\partial}{\partial X_I} (X_i + U_i) \\ &= \delta_{iI} + U_{i,I} \end{aligned} \tag{4.27}$$

Figure 4.2: Current configuration with outward normal.

since $X_i = \delta_{iJ} X_J$. Components of the Green strain are thus

$$
\begin{aligned}
2E_{IJ} &= (\delta_{iI} + U_{i,I})(\delta_{iJ} + U_{i,J}) - \delta_{IJ} \\
&= \delta_{IJ} + U_{I,J} + U_{J,I} + U_{i,I} U_{i,J} - \delta_{IJ}
\end{aligned}
\tag{4.28}
$$

so that

$$
E_{IJ} = \frac{1}{2}(U_{I,J} + U_{J,I}) + \frac{1}{2} U_{i,I} U_{i,J} \tag{4.29}
$$

with implied summation on i.

Stress. The *first Piola-Kirchhoff stress tensor* is

$$
\boldsymbol{P} = J \boldsymbol{\sigma} \boldsymbol{F}^{-T} \tag{4.30}
$$

$\sigma = \dfrac{P F^T}{J}$

or in components

$$
P_{iI} = J \sigma_{ij} \frac{\partial X_I}{\partial x_j} \tag{4.31}
$$

where $J = \det \boldsymbol{F}$ and the third term is $\left(\boldsymbol{F}^{-T}\right)_{jI} = X_{I,j}$.

The symmetric (true) Cauchy stress tensor $\boldsymbol{\sigma}$ is uniquely determined from Cauchy traction vector $\boldsymbol{t} = \boldsymbol{\sigma n}$ which represents the force per unit *current* area exerted on a surface element with outward normal $\boldsymbol{n}$. Consider an infinitesimal volume element in the current

Figure 4.3: Mapping of reference configuration with $\boldsymbol{N}$ to current with $\boldsymbol{n}$.

configuration $dv = \boldsymbol{\phi}_t(dV)$ (not to be confused with velocities). Similarly, an infinitesimal surface element in the current configuration is $da = \boldsymbol{\phi}_t(dA)$ (not to be confused with acceleration). We claim that there exists a first Piola-Kirchhoff traction vector $\boldsymbol{T} = \boldsymbol{P}\boldsymbol{N}$, representing force per unit *reference* area such that

$$\boldsymbol{t}\, da = \boldsymbol{T}\, dA \tag{4.32}$$

so that the two traction vectors are parallel.

To see this recall from calculus and differential geometry that for a volume element

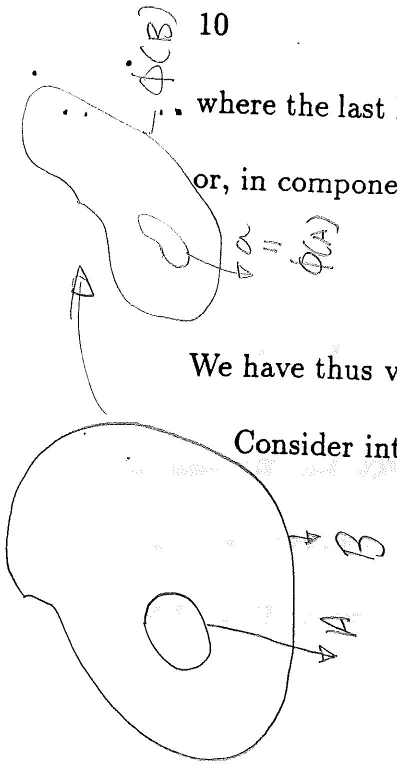
$$dv = J\, dV \tag{4.33}$$

and for a surface element

$$\boldsymbol{n}\, da = J\boldsymbol{F}^{-T}\boldsymbol{N}\, dA \tag{4.34}$$

Therefore

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{n}\, da) &= \boldsymbol{\sigma}\left(J\boldsymbol{F}^{-T}\boldsymbol{N}\, dA\right) \\ &= J\boldsymbol{\sigma}\boldsymbol{F}^{-T}\boldsymbol{N}\, dA \\ &= \boldsymbol{P}\boldsymbol{N}\, dA \end{aligned} \tag{4.35}$$



where the last line follows from the definition of the first Piola-Kirchhoff stress tensor (4.30), or, in components,

$$\sigma_{ij} n_j \, da = P_{iI} N_I \, dA \tag{4.36}$$

We have thus verified (4.32).

Consider integration over a closed surface. For the first Piola-Kirchhoff traction vector

$$\begin{aligned} \int_A T_i \, dA &= \int_A P_{iI} N_I \, dA \\ &= \int_V P_{iI,I} \, dV \end{aligned} \tag{4.37}$$

where the second line follows from the divergence theorem. Likewise, for the Cauchy traction vector

$$\begin{aligned} \int_a t_i \, da &= \int_a \sigma_{ij} n_j \, da \\ &= \int_v \sigma_{ij,j} \, dv \\ &= \int_V \sigma_{ij,j} J \, dV \end{aligned} \tag{4.38}$$

where the last line follows from the change of variables. These integral relations hold for arbitrary closed surfaces which are the boundaries of subsets of v and V (such as $\mathcal{U} \subset V$ and $\boldsymbol{\phi}_t(\mathcal{U}) \subset v$). Thus

Piola identity

$$\sigma_{ij,j} J = P_{iI,I} \tag{4.39}$$

Remark. $\boldsymbol{P}$ is not generally symmetric.

Returning to the concept of strain energy density, a given function of Green strain $\Phi(\boldsymbol{E})$,

Figure 4.4: Mapping of closed surface A in $\mathcal{B}$ to a in $\boldsymbol{\phi}_t(\boldsymbol{B})$.

we have, from thermodynamic considerations, a symmetric stress tensor such that

$$\boldsymbol{S} = \frac{\partial \Phi}{\partial \boldsymbol{E}} \tag{4.40}$$

called the *second Piola-Kirchhoff stress tensor*. In components this material tensor is

$$S_{IJ} = \frac{\partial \Phi}{\partial E_{IJ}} \tag{4.41}$$

The relation to the previously defined stress tensors is

$$\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} \tag{4.42}$$

or

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{P}\boldsymbol{F}^{T} = J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{T} \tag{4.43}$$

The material tangent moduli are defined by

$$C_{IJKL}(\boldsymbol{E}) := \frac{\partial S_{IJ}}{\partial E_{KL}} = \frac{\partial^2 \Phi}{\partial E_{IJ} \partial E_{KL}} \tag{4.44}$$

representing the "slopes" of the $\boldsymbol{S}$-$\boldsymbol{E}$ diagram. The major symmetry of the tangent moduli thus follows from the existence of the strain energy function and the minor symmetries from the symmetry of Green strain. Similarly, the analog of the c_{ijkl} in the small-deformation nonlinear theory considered previously is

$$c_{ijkl}(\boldsymbol{E}) := J^{-1} F_{iI} F_{jJ} F_{kK} F_{lL} C_{IJKL} \tag{4.45}$$

but there will be another term in the consistent tangent (to be seen later).

4.2 Variational (Weighted-residual) Formulations

The weak form survives intact

$$\int_v w_{i,j}\sigma_{ij}\, dv = \int_v w_i f_i\, dv + \int_{a_h} w_i h_i\, da \tag{4.46}$$

where $v = \boldsymbol{\phi}_t(V)$ fills the role of Ω, the current configuration, but the tangent is calculated from the "material description," which hence must be derived. Consider the change of variables

$$\begin{aligned} w_i(\boldsymbol{x}) &= w_i(\boldsymbol{\phi}(\boldsymbol{X})) \\ &=: W_i(\boldsymbol{X}) \end{aligned} \tag{4.47}$$

which is thus defined by the composition

$$W_i = w_i \circ \boldsymbol{\phi} \tag{4.48}$$

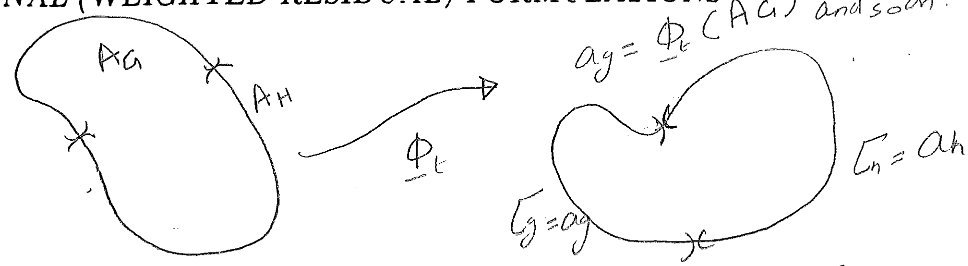


Figure 4.5: Mapping of boundary partition into $A_{\mathcal{G}}$ and $A_{\mathcal{H}}$ to a_g and a_h.

We assume that the boundary conditions can be stated in terms of material coordinates and redefined as follows.

The displacement boundary condition, which is built into the definition of the set of trial solutions

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } a_g \tag{4.49}$$

where $a_g = \boldsymbol{\phi}_t(A_{\mathcal{G}})$ fills the role of Γ_g, is equivalent to

$$\boldsymbol{U} = \boldsymbol{\mathcal{G}} \qquad \text{on } A_{\mathcal{G}} \tag{4.50}$$

Similarly, the traction boundary condition $\underline{t} = \underline{\sigma} \cdot \underline{n} = \underline{h}$ on $\Gamma_h = a_h$

$$\boldsymbol{t} = \boldsymbol{h} \qquad \text{on } a_h \tag{4.51}$$

where $a_h = \boldsymbol{\phi}_t(A_{\mathcal{H}})$ fills the role of Γ_h, is equivalent to

$\underline{T} = \underline{P} \cdot \underline{N} = \underline{H}$

$\underline{h}\, da = \underline{H}\, dA$

$$\boldsymbol{T} = \boldsymbol{\mathcal{H}} \qquad \text{on } A_{\mathcal{H}} \tag{4.52}$$

Note that $\boldsymbol{t} \neq \boldsymbol{T}$ since $\boldsymbol{t}\, da = \boldsymbol{T}\, dA$. Weighting functions satisfy the homogeneous analog of

$\mathcal{U} \subset \mathcal{B}$

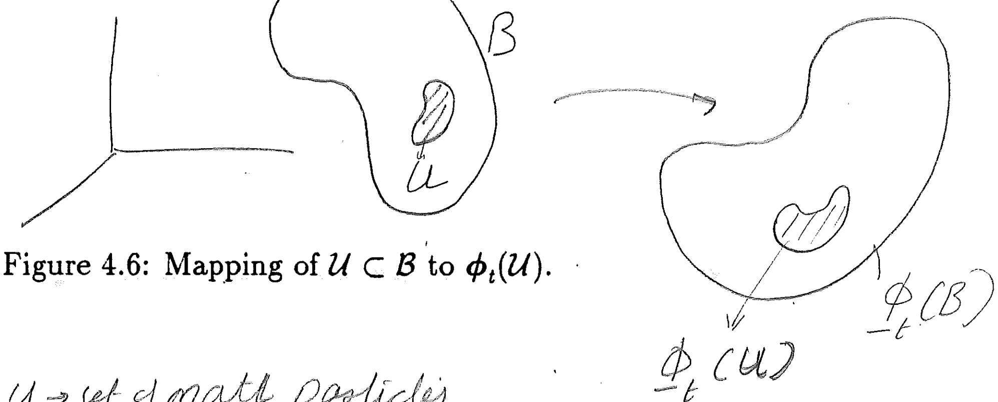


Figure 4.6: Mapping of $\mathcal{U} \subset \mathcal{B}$ to $\phi_t(\mathcal{U})$.

$\mathcal{U} \rightarrow$ set of mat'l particles

essential boundary conditions

$$\boldsymbol{w} = \mathbf{0} \qquad \text{on } a_g \tag{4.53}$$

and

$$\boldsymbol{W} = \mathbf{0} \qquad \text{on } A_G \tag{4.54}$$

Due to the change of variables integration becomes

$$\int_v \cdots \, dv = \int_V \cdots J \, dV \tag{4.55}$$

and for surface integration

$$\int_a \cdots n_i \, da = \int_A \cdots J \left(\boldsymbol{F}^{-T}\right)_{iI} N_I \, dA \tag{4.56}$$

We also need to employ conservation of mass. Consider a set of material points $\mathcal{U} \subset \mathcal{B}$

$$\frac{d}{dt}\left(\int_{\phi_t(\mathcal{U})} \rho_t \, dv_t \right) = 0 \tag{4.57}$$

where $\rho_t(\boldsymbol{x})$ is the mass density of the deformed body at time t. Integrating

$$\int_{\boldsymbol{\phi}_{t_2}(\mathcal{U})} \rho_{t_2}\, dv_{t_2} = \int_{\boldsymbol{\phi}_{t_1}(\mathcal{U})} \rho_{t_1}\, dv_{t_1} \tag{4.58}$$

Taking $t_2 = t$ and $t_1 = 0$ (corresponding to the reference configuration), and noting that $dv_0 = J_0\, dV = dV$ since $J_0 = \det\left(\frac{\partial \boldsymbol{\phi}_0}{\partial \boldsymbol{X}}\right) = 1$, leads to

$$\int_{\boldsymbol{\phi}_t(\mathcal{U})} \rho_t\, dv_t = \int_{\mathcal{U}} \rho_0\, dV \tag{4.59}$$

Employing the formula for change of variables $dv_t = J_t\, dV$ we obtain

$$\int_{\mathcal{U}} \rho_t J_t\, dV = \int_{\mathcal{U}} \rho_0\, dV \tag{4.60}$$

This holds for *any* set of material points $\mathcal{U}$, leading to the *local* form of conservation of mass

Time differentiate to get

$$\rho_{,t} J + \rho_{,x_i} v_i J + \rho J v_{i,i} = 0$$

$$\rho_t J_t = \rho_0 \tag{4.61}$$

$\frac{d\rho}{dx} \,/\, \frac{d\phi}{dt} \,/\, J \,/\, \rho \mathring{J}$ $\neq 0$

$$\Rightarrow \rho_{,t} + v_i \rho_{,i} + \rho v_{i,i} \Rightarrow \rho_{,t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \Rightarrow \forall (x \in \mathcal{U})$$

in terms of the mass density of the reference configuration $\rho_0(\boldsymbol{X})$.

The material description of (4.46) requires that the integrand on the left-hand side be written in a more useful form. Recall that the material description of weighting functions is defined by the composition (4.48). Thus

$$\begin{aligned} \frac{\partial W_i}{\partial X_I} &= \frac{\partial}{\partial X_I}(w_i \circ \boldsymbol{\phi}) \\ &= \frac{\partial w_i}{\partial x_j}\frac{\partial \phi_j}{\partial X_I} \end{aligned} \tag{4.62}$$

where the second line follows from the chain rule. In abstract notation we write $\mathrm{GRAD}\boldsymbol{W} = (\mathrm{grad}\boldsymbol{w})\,\boldsymbol{F}$ or $\mathrm{grad}\boldsymbol{w} = (\mathrm{GRAD}\boldsymbol{W})\,\boldsymbol{F}^{-1}$, which in components is

$$w_{i,j} = W_{i,J}\frac{\partial X_I}{\partial x_j} \tag{4.63}$$

Returning to the weak form (4.46), by change of variables the left-hand side becomes

$$\begin{aligned}\int_v w_{i,j}\sigma_{ij}\,dv &= \int_V w_{i,j}\sigma_{ij}J\,dV \\ &= \int_V W_{i,I}\left(\frac{\partial X_I}{\partial x_j}\sigma_{ij}J\right)dV \\ &= \int_V W_{i,I}P_{iI}\,dV \\ &= \int_V W_{i,I}F_{iJ}S_{JI}\,dV \end{aligned} \tag{4.64}$$

where the third and fourth lines follow from the definitions of the first and second Piola-Kirchhoff stress tensors, (4.30) and (4.42), respectively. The left-hand side of the weak form is left in this form at this stage.

We now turn to the right-hand side of the weak form (4.46). The body force per unit current volume $\boldsymbol{f}$ can be expressed as $\boldsymbol{f} = \rho\boldsymbol{b}$ where $\boldsymbol{b}$ is the body force per unit mass (e.g., gravity). We express the body force $\boldsymbol{B} = \boldsymbol{b}\circ\boldsymbol{\phi}$ as a function of $\boldsymbol{X}$ and assume that $\boldsymbol{B}(\boldsymbol{X}) = (\boldsymbol{b}\circ\boldsymbol{\phi})(\boldsymbol{X})$, i.e., $\boldsymbol{B}$ does not depend on the motion (this is referred to as dead loading). The contribution of the distributed loading is obtained by the change of variables

$$\begin{aligned}\int_v \boldsymbol{w}\cdot\boldsymbol{f}\,dv &= \int_V \boldsymbol{w}(\boldsymbol{\phi}(\boldsymbol{X}))\cdot(\rho\boldsymbol{b})(\boldsymbol{\phi}(\boldsymbol{X}))J\,dV \\ &= \int_V \boldsymbol{W}\cdot\boldsymbol{B}\rho_0\,dV\end{aligned} \tag{4.65}$$

since $\rho J = \rho_0$ by conservation of mass (4.61). Note that nothing in the second line depends on the motion $\boldsymbol{\phi}$.

The surface traction term is treated in similar fashion. Recall the first Piola-Kirchhoff traction vector $\boldsymbol{T}$ was defined so that $\boldsymbol{t}\,da = \boldsymbol{T}\,dA$, and so, for the traction boundary condition $\boldsymbol{h}\,da = \boldsymbol{\mathcal{H}}\,dA$. The contribution of the traction boundary condition is

$$\int_{a_h} w_i h_i \, da = \int_{A_{\mathcal{H}}} W_i \mathcal{H}_i \, dA \tag{4.66}$$

where we again assume no dependence of the loading on the motion $\mathcal{H}_i = \mathcal{H}_i(\boldsymbol{X})$ (i.e., no follower forces, etc.).

Employing equations (4.64)–(4.66) to the residual, i.e., the balance of internal and external forces, of the weak form (4.46)

$$\boxed{\begin{aligned} 0 &= \int_v w_i f_i \, dv + \int_{a_h} w_i h_i \, da - \int_v w_{i,j} \sigma_{ij} \, dv \\ &= \int_V W_i \rho_0 B_i \, dV + \int_{A_{\mathcal{H}}} W_i \mathcal{H}_i \, dA - \int_V W_{i,I} P_{iI} \, dV \end{aligned}} \tag{4.67}$$

where $v = \Omega$. Note that the two descriptions are identical.

Discretizing in the usual manner $W_i^h(\boldsymbol{X}) = \sum c_{iA} N_A(\boldsymbol{X})$ and $w_i^h(\boldsymbol{x}) = \sum c_{iA} N_A(\boldsymbol{x})$, or locally $W_i^h(\boldsymbol{X}) = \sum c_{ia} N_a(\boldsymbol{X})$ and $w_i^h(\boldsymbol{x}) = \sum c_{ia} N_a(\boldsymbol{x})$ within each element.

We need to calculate ϕ_i, and then F_{iI}, E_{IJ}, $S_{IJ} = S_{IJ}(\boldsymbol{E})$ ($\boldsymbol{\sigma}$ and $\boldsymbol{P}$) and $C_{IJKL}(\boldsymbol{E})$. By change of variables to the parent domain, element integrals become

$$\int_{v^e} \cdots \, dv = \int_{\square} \cdots j \, d\square \tag{4.68}$$

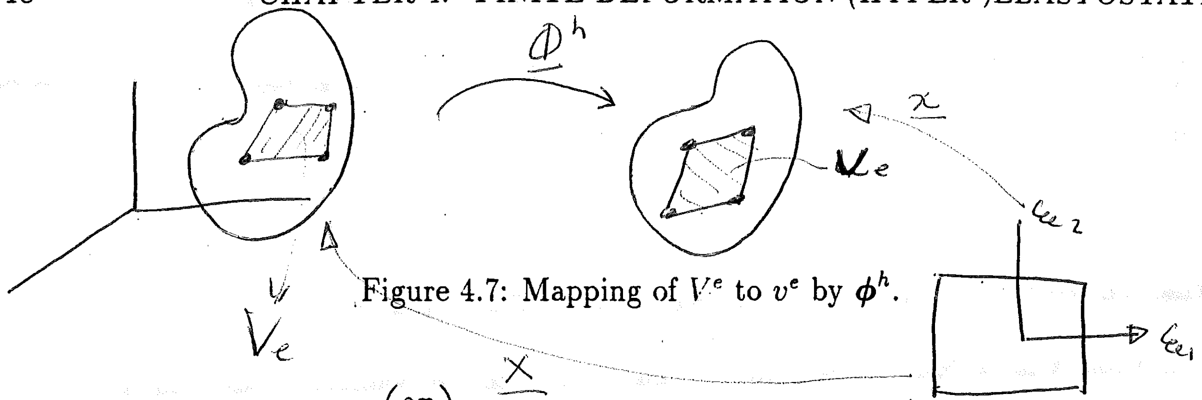


Figure 4.7: Mapping of V^e to v^e by $\boldsymbol{\phi}^h$.

where $v^e = \Omega^e$ and $j = \det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right)$, and in the material description

$$\int_{V^e} \cdots \, dV = \int_{\square} \cdots j_0 \, d\square \tag{4.69}$$

where $j_0 = \det\left(\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\xi}}\right)$. Note that V^e and v^e are sets of the same material points.

This is the Lagrangian description of the motion (in terms of sets of material points). In other applications such as fluid dynamics the Eulerian description is used (in terms of sets of spatial points). Two alternative descriptions of motion are the "updated" Lagrangian (where integrals are calculated in the current configuration) and the "total" Lagrangian (where integrals are calculated in the reference configuration). Both are *identical.*

4.3 Linearized Operators

The variational equations are linearized to obtain the left-hand side operator. The consistent tangent

$$\boldsymbol{K} = \frac{\partial \boldsymbol{N}}{\partial \boldsymbol{d}} \tag{4.70}$$

is obtained by discretization prior to linearization. Alternatively, the variational equations can be directly differentiated (assuming the constitutive algorithm is exact, which is true in depend on ϕ

elasticity). The left-hand side is considered a function of the motion

depends on ϕ

$$
\begin{aligned}
f(\phi) &= \int_v w_{i,j}\sigma_{ij}\, dv \\
&= \int_V W_{i,I}P_{iI}\, dV \\
&= \int_V W_{i,I}F_{iJ}S_{JI}\, dV \\
&= \int_V \frac{\partial W_i(\boldsymbol{X})}{\partial X_I}\frac{\partial \phi_i(\boldsymbol{X})}{\partial X_J} S_{JI}(\boldsymbol{E}(\boldsymbol{X}))\, dV
\end{aligned}
\tag{4.71}
$$

where $v = \Omega$ and $V = \Omega_0$ and the Green strain $\boldsymbol{E}$ depends on the motion by its definition (4.24) and the definition of the deformation gradient $\boldsymbol{F}$ (4.14).

Linearizing this function

$$
\left(\frac{d}{d\varepsilon} f(\boldsymbol{\phi} + \varepsilon\, \Delta \boldsymbol{U})\right)\bigg|_{\varepsilon=0} = Df(\boldsymbol{\phi}) \cdot \Delta \boldsymbol{U} \tag{4.72}
$$

where ε is a real parameter and the right-hand side represents the action of a linear operator $Df(\boldsymbol{\phi})$ on the displacement increment $\Delta\boldsymbol{U}$ defined by the change of variables $\Delta\boldsymbol{U}(\boldsymbol{X}) = \Delta\boldsymbol{u}(\boldsymbol{\phi}(\boldsymbol{x}))$ or the composition $\Delta\boldsymbol{U} = \Delta\boldsymbol{u} \circ \boldsymbol{\phi}$. Discretizing the displacement increment as $\sum N_a \Delta d_{ia}$ within each element leads to $\boldsymbol{K}(\boldsymbol{d})\Delta\boldsymbol{d}$. Linearizing the left-hand side of the variational equation

$$
\begin{aligned}
\left(\frac{d}{d\varepsilon} f(\boldsymbol{\phi} + \varepsilon\, \Delta \boldsymbol{U})\right)\bigg|_{\varepsilon=0} &= \left(\frac{d}{d\varepsilon}\left(\int_V W_{i,I}\frac{\partial(\phi_i + \varepsilon \Delta U_i)}{\partial X_J} S_{JI}(\boldsymbol{E}(\boldsymbol{\phi} + \varepsilon \Delta \boldsymbol{U}))\, dV\right)\right)\bigg|_{\varepsilon=0} \\
= \int_V \left(W_{i,I}\frac{\partial(\Delta U_i)}{\partial X_J} S_{JI}(\boldsymbol{E}(\boldsymbol{\phi})) \right. &+ \left. W_{i,I}\frac{\partial \phi_i}{\partial X_J}\frac{\partial S_{JI}}{\partial E_{KL}}\left(\frac{dE_{KL}(\boldsymbol{\phi} + \varepsilon \Delta\boldsymbol{U})}{d\varepsilon}\right)\bigg|_{\varepsilon=0}\right) dV
\end{aligned}
\tag{4.73}
$$

where $\frac{\partial S_{JI}}{\partial E_{KL}} = C_{JIKL}$. Employing the definition of Green strain

$$2\boldsymbol{E}(\boldsymbol{\phi} + \varepsilon\Delta\boldsymbol{U}) = \left(\frac{\partial(\boldsymbol{\phi} + \varepsilon\Delta\boldsymbol{U})}{\partial\boldsymbol{X}}\right)^T \left(\frac{\partial(\boldsymbol{\phi} + \varepsilon\Delta\boldsymbol{U})}{\partial\boldsymbol{X}}\right) - \mathbf{1} \tag{4.74}$$

leads to

$$\left(\frac{d(2\boldsymbol{E}(\boldsymbol{\phi} + \varepsilon\Delta\boldsymbol{U}))}{d\varepsilon}\right)\Bigg|_{\varepsilon=0} = \left(\frac{\partial\Delta\boldsymbol{U}}{\partial\boldsymbol{X}}\right)^T \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{X}} + \left(\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{X}}\right)^T \frac{\partial\Delta\boldsymbol{U}}{\partial\boldsymbol{X}} \tag{4.75}$$

and in components

$$\begin{aligned} 2\left(\frac{dE_{KL}}{d\varepsilon}\right)\Bigg|_{\varepsilon=0} &= \Delta U_{i,K}\phi_{i,L} + \phi_{i,K}\Delta U_{i,L} \\ &= \Delta U_{i,K}F_{i,L} + F_{i,K}\Delta U_{i,L} \end{aligned} \tag{4.76}$$

Thus

$$\begin{aligned} \left(\frac{d}{d\varepsilon}f(\boldsymbol{\phi} + \varepsilon\,\Delta\boldsymbol{U})\right)\Bigg|_{\varepsilon=0} &= \int_V \left(W_{i,I}\Delta U_{i,J}S_{JI} + W_{i,I}F_{iJ}C_{JIKL}\frac{1}{2}(\Delta U_{j,K}F_{jL} + F_{j,K}\Delta U_{j,L})\right) dV \\ &= \int_V (W_{i,I}\Delta U_{i,J}S_{JI} + W_{i,I}F_{iJ}C_{JIKL}\Delta U_{j,K}F_{jL})\, dV \end{aligned} \tag{4.77}$$

where the second line follows from the minor symmetry on indices KL of the tangent moduli.

Pushing forward to the current configuration we employ $dv = J\,dV$, where $J = \det \boldsymbol{F}$, $W_{i,I} = w_{i,j}F_{jI}$, where $F_{jI} = \frac{\partial x_j}{\partial X_I}$, and $\Delta U_{i,J} = \Delta u_{i,j}F_{jJ}$. The first term is

$$\begin{aligned} \int_V W_{i,I}\Delta U_{i,J}S_{JI}\,dV &= \int_V w_{i,j}F_{jI}\Delta u_{i,k}F_{kJ}S_{JI}\,dV \\ &= \int_v w_{i,j}F_{jI}S_{JI}F_{kJ}\Delta u_{i,k}\frac{1}{J}\,dv \\ &= \boxed{\int_v w_{i,j}\sigma_{kj}\Delta u_{i,k}\,dv} \end{aligned} \tag{4.78}$$

where the last line follows from (4.43). This is thus an initial stress contribution. The second term is

$$\begin{aligned}\int_V W_{i,I}F_{iJ}C_{JIKL}\Delta U_{j,K}F_{jL}\,dV &= \int_V W_{i,I}F_{iJ}F_{jL}C_{JIKL}\Delta U_{j,K}\,dV \\ &= \int_v w_{i,k}F_{kI}F_{iJ}F_{jL}C_{JIKL}\Delta u_{j,l}F_{lK}\frac{1}{J}\,dv \\ &= \int_v w_{j,i}F_{iI}F_{jJ}F_{lL}F_{kK}C_{JIKL}\Delta u_{l,k}\frac{1}{J}\,dv \\ &= \int_v w_{(j,i)}c_{ijkl}\Delta u_{(l,k)}\,dv \qquad (4.79)\end{aligned}$$

where dummy indices are exchanged on the third line and the last line is obtained by the definition of the spatial tangent moduli (4.45) and their symmetries. This term is similar to the one leading to stiffness in the linear theory, giving rise to the standard $\int \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}$. The strain-displacement matrix $\boldsymbol{B}$ is identical to the one in linear theory. The matrix $\boldsymbol{D}$ is derived from $c_{ijkl}(\boldsymbol{F})$.

Combining these results and exploiting minor symmetries of the second term yields

$$\left(\frac{d}{d\varepsilon}f(\boldsymbol{\phi}+\varepsilon\,\Delta\boldsymbol{U})\right)\bigg|_{\varepsilon=0} = \int_v (w_{i,j}\sigma_{jk}\Delta u_{i,k} + w_{i,j}c_{ijkl}\Delta u_{k,l})\,dv \qquad (4.80)$$

The second expression may be written in terms of strain due to the minor symmetries, and the major symmetry leads to symmetry of the stiffness matrix. We wish to investigate the symmetries of the first term, starting with the major one. For this purpose we write

$$w_{i,j}\sigma_{jl}\Delta u_{i,l} = w_{i,j}(\sigma_{jl}\delta_{ik})\Delta u_{k,l} \qquad (4.81)$$

The expression in the parentheses represents effective moduli due to current stress leading to initial-stress stiffness, as mentioned. The first pair of indices ij may indeed be replaced by the second pair kl in this expression since

$$\sigma_{jl}\delta_{ik} = \sigma_{lj}\delta_{ki} \tag{4.82}$$

so that major symmetry does exist and symmetry of the stiffness matrix follows. This is expected by the existence of a potential, the strain energy density function.

Looking at minor symmetries, we need to ascertain whether i and j can be switched, and whether k and l can be switched. However

$$\begin{aligned} \sigma_{jl}\delta_{ki} &\neq \sigma_{ij}\delta_{kj} \\ \sigma_{jl}\delta_{ki} &\neq \sigma_{jk}\delta_{li} \end{aligned} \tag{4.83}$$

(Consider, for example, $i = k = 1$ and $j = l = 2$. This leads to $\delta_{11} = 1$ on the left-hand side in both cases, whereas on the right-hand side we have $\delta_{12} = \delta_{21} = 0$, so that equality cannot hold.) This expression does not possess minor symmetries, so that the additional term depends on *all* the information contained in the deformation gradient $\boldsymbol{F}$, and cannot be written in terms of strain (the symmetric part) alone. To demonstrate this point consider the polar decomposition [1, pp. 51–55] of the deformation gradient

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \tag{4.84}$$

where $\boldsymbol{R}$ is a proper-orthogonal tensor (i.e., $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{1}$ and $\det \boldsymbol{R} = 1$) representing local rotation and $\boldsymbol{U}$ is a symmetric, positive definite tensor representing local stretching (not to be confused with displacement of material points).

$$\begin{aligned} \boldsymbol{F}^T\boldsymbol{F} &= (\boldsymbol{RU})^T\boldsymbol{RU} \\ &= \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{RU} \\ &= \boldsymbol{U}^T\boldsymbol{U} \\ &= \boldsymbol{U}^2 \end{aligned} \tag{4.85}$$

For the Lagrangian strain measure $\boldsymbol{E}$

$$\begin{aligned} 2\boldsymbol{E} &= \boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{1} \\ &= \boldsymbol{U}^2 - \boldsymbol{1} \end{aligned} \tag{4.86}$$

so that it does not contain information related to rotation. The linearized left-hand side in its spatial and material representations is

$$\int_v w_{i,j}(\delta_{ik}\sigma_{jl} + c_{ijkl})\Delta u_{k,l}\, dv = \int_V W_{i,J}(\delta_{ik}S_{JL} + F_{iI}F_{kK}C_{IJKL})\Delta U_{k,L}\, dV \tag{4.87}$$

Both versions are *identical*. The expressions in the parentheses represent the effective moduli, which, due to the first term in each which leads to initial-stress stiffness, do not possess minor symmetries.

4.4 Discretization

Coding. As previously mentioned, the term containing material moduli $\int_v w_{i,j} c_{ijkl} \Delta u_{k,l}\, dv$ leads to the standard stiffness term $\int_{v^e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}\, dv$. The strain-displacement matrix $\boldsymbol{B}$ is identical to the one in the case of small deformations. The matrix $\boldsymbol{D}$ is derived from $c_{ijkl}(\boldsymbol{F})$.

The geometric stiffness term which results from initial stresses (and may not always be included in the tangent stiffness)

(sums on A, B a, b need to be indicated.)

$$\int_v w^h_{i,j} \delta_{ik} \sigma_{jl} \Delta u^h_{k,l}\, dv = \int_v c_{iA} N_{A,j} \delta_{ik} \sigma_{jl} \Delta d_{kB} N_{B,l}\, dv \tag{4.88}$$

leads to the following contribution to the element stiffness matrix

$$k^e_{iakb} = \int_{v^e} N_{a,j} \delta_{ik} \sigma_{jl} N_{b,l}\, dv \tag{4.89}$$

in which the Kronecker delta can be taken outside of the integration.

Consider, for example, a four-noded linear tetrahedron in three-dimensional analysis (Fig. 4.8) with three degrees of freedom per node for a total of 12 element equations. The 12×12 element stiffness matrix is made up of 4×4 nodal blocks of size 3×3 each. Due to the Kronecker delta, initial stresses contribute only diagonal terms in each nodal block

$$\begin{aligned} k^e_{iaib} &= \int_{v^e} N_{a,j} \sigma_{jl} N_{b,l}\, dv \qquad \text{(no sum on } i\text{)} \\ &= \int_{\Box} (\nabla N_a)^T \boldsymbol{\sigma} \nabla N_b j\, d\Box \end{aligned} \tag{4.90}$$

where $j = \det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right)$. In each block these three diagonal entries are equal. This is similar

Figure 4.8: Linear tetrahedron.

Figure 4.9: Block-diagonal contribution of initial stress to stiffness.

to the case of heat conduction. The stress matrix $\boldsymbol{\sigma}$ need not be positive, with the effect of destabilizing the stiffness as in the case of buckling under compressive loads.

The small-deformation nonlinear case may be recaptured from this general nonlinear formulation by

1. Not adding displacements to the initial coordinates (i.e., not updating the geometry).

2. Not adding the quadratic terms in the strain calculation.

3. Not computing and accounting for the initial stress stiffness.

Furthermore, the linear elasticity case may be similarly obtained if, in addition, the elastic coefficients c_{ijkl} are fixed as constants.

To recapitulate the finite deformation case

$$\boldsymbol{x}_{n+1}^{(i)} = \boldsymbol{X} + \boldsymbol{d}_{n+1}^{(i)} \tag{4.91}$$

where $\boldsymbol{X} = \boldsymbol{x}_0$ is used to define the current element geometry. Displacements are updated by

$$\boldsymbol{K}\Delta\boldsymbol{d}^{(i)} = \boldsymbol{F}^{\text{ext}} - \boldsymbol{N}(\boldsymbol{d}^{(i)}_{n+1}) \tag{4.92}$$

where the right-hand side $\int_{v^e} N_a f_i \, dv + \int_{a^e_h} N_a h_i \, da - \boldsymbol{e}_i^T \int_{v^e} \boldsymbol{B}_a^T \boldsymbol{\sigma}^{\text{vect}} \, dv$ is identical to the small-deformation case and the left-hand side stiffness $\boldsymbol{K} = \boldsymbol{K_C} + \boldsymbol{K_\sigma}$ is obtained by contributions from material moduli stiffness and initial stress stiffness.

Element arrays. Given $\boldsymbol{d}^{(i)}_{n+1}$, $\boldsymbol{x}^{(i)}_{n+1}$ and $\boldsymbol{X}$, the above quantities are localized. Shape functions and derivatives are calculated at the quadrature points. Derivatives $N_{a,i}$ are needed for spatial integrals containing $\boldsymbol{B}_a$. To calculate $\boldsymbol{\sigma}$, derivatives $N_{a,I}$ are needed for $\boldsymbol{F}$, since

$$\begin{aligned} \boldsymbol{F} &= \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}} \\ &= \mathbf{1} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} \end{aligned} \tag{4.93}$$

where displacements are approximated in the usual way inside an element

$$\boldsymbol{u}^h = \sum_{a=1}^{n_{en}} N_a \boldsymbol{d}_a \tag{4.94}$$

The Jacobian determinants $j_0 = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\xi}}$ and $j = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}$ are also obtained from these shape function derivatives. The calculation of $\boldsymbol{F}$ is thus completed. Next the Green strain is calculated

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \mathbf{1}) \tag{4.95}$$

The second Piola-Kirchhoff stress tensor

$$\boldsymbol{S} = \frac{\partial \Phi}{\partial \boldsymbol{E}} \tag{4.96}$$

is calculated by a formula that is encoded and from it Cauchy stress

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T \tag{4.97}$$

where $J = \det \boldsymbol{F}$. The material moduli $\boldsymbol{C}$ are also obtained by an encoded formula and then $\boldsymbol{c}$ is calculated from

$$c_{ijkl} = \frac{1}{J}F_{iI}F_{jJ}F_{kK}F_{lL}C_{IJKL} \tag{4.98}$$

Principle of Minimum Potential Energy

Consider the functional of a motion $\boldsymbol{\phi}$ under "dead" loads

$$\Pi(\boldsymbol{\phi}) = \int_V \Phi(\boldsymbol{E}(\boldsymbol{\phi}))\, dV - \int_V \rho_0 \boldsymbol{B} \cdot \boldsymbol{\phi}\, dV - \int_{A_{\mathcal{H}}} \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\phi}\, dA \tag{4.99}$$

where Φ is the strain energy density (per unit undeformed volume), $\boldsymbol{B}$ is the body force per unit mass in the initial configuration and $\boldsymbol{\mathcal{H}}$ is the traction per unit undeformed (reference) surface area.

Consider "variations" of $\boldsymbol{\phi}$ (i.e., weighting functions) W_i that satisfy $W_i = 0$ on $A_{\mathcal{G}}$. The stationary value is obtained by forming $\Pi(\boldsymbol{\phi} + \varepsilon \boldsymbol{W})$, where ε is a real parameter, and taking the variational derivative (see [2, p. 188])

$$
\begin{aligned}
0 &= \delta\Pi \\
&= D\Pi(\boldsymbol{\phi}) \cdot \overbrace{\boldsymbol{W}}^{\delta\boldsymbol{\phi}} \\
&= \left.\left(\frac{d}{d\varepsilon}\Pi(\boldsymbol{\phi} + \varepsilon \boldsymbol{W})\right)\right|_{\varepsilon=0} \\
&= \left.\left(\frac{d}{d\varepsilon}\left(\int_V \Phi(\boldsymbol{E}(\boldsymbol{\phi} + \varepsilon\boldsymbol{W}))\, dV - \int_V \rho_0 \boldsymbol{B}\cdot(\boldsymbol{\phi} + \varepsilon\boldsymbol{W})\, dV - \int_{A_{\mathcal{H}}} \boldsymbol{\mathcal{H}} \cdot (\boldsymbol{\phi} + \varepsilon\boldsymbol{W})\, dA\right)\right)\right|_{\varepsilon=0} \\
&= \int_V \underbrace{\frac{\partial\Phi(\boldsymbol{E}(\boldsymbol{\phi}))}{\partial E_{IJ}}}_{S_{IJ}} \left.\left(\frac{\partial \boldsymbol{E}(\boldsymbol{\phi} + \varepsilon\boldsymbol{W})}{\partial\varepsilon}\right)\right|_{\varepsilon=0} dV - \int_V \rho_0 \boldsymbol{B}\cdot\boldsymbol{W}\, dV - \int_{A_{\mathcal{H}}} \boldsymbol{\mathcal{H}}\cdot\boldsymbol{W}\, dA \qquad (4.100)
\end{aligned}
$$

Recall from the calculation of the linear operator

$$
\left.\left(\frac{\partial \boldsymbol{E}(\boldsymbol{\phi} + \varepsilon\Delta\boldsymbol{u})}{\partial\varepsilon}\right)\right|_{\varepsilon=0} = \frac{1}{2}(\Delta u_{i,I}F_{iJ} + \Delta u_{i,J}F_{iI}) \qquad (4.101)
$$

+ive definite.

and by symmetry of this term the stationary value is found from

$\frac{\partial^2 \phi}{\partial E\, \partial E}$

$$
\int_V S_{IJ} W_{i,I} F_{i,J}\, dV - \int_V \rho_0 \boldsymbol{B}\cdot\boldsymbol{W}\, dV - \int_{A_{\mathcal{H}}} \boldsymbol{\mathcal{H}}\cdot\boldsymbol{W}\, dA = 0 \qquad (4.102)
$$

This is the equation of virtual work (or the weak form) in the material configuration. For verification, push forward to the current configuration, employing $W_{i,I} = w_{i,j}F_{jI}$ and $\boldsymbol{\mathcal{H}}\, dA = \boldsymbol{h}\, da$.

$$
\int_v w_{i,j} \underbrace{F_{jI} S_{IJ} F_{i,J} \frac{1}{J}}_{\sigma_{ij}} dv - \int_v \rho \boldsymbol{b}\cdot\boldsymbol{w}\, dv - \int_{a_h} \boldsymbol{h}\cdot\boldsymbol{w}\, da \doteq 0 \qquad (4.103)
$$

and since $\rho = \rho_0/J$, the spatial version is obtained.

Remark. The virtual work equation $f(\boldsymbol{\phi}) = 0$ is a nonlinear variational equation that is written in this form for linearization, by taking $\boldsymbol{\phi} \leftarrow \boldsymbol{\phi} + \varepsilon \boldsymbol{W}$

$$f(\boldsymbol{\phi}) = D\Pi(\boldsymbol{\phi}) \cdot \boldsymbol{W} \tag{4.104}$$

is linear in $\boldsymbol{W}$.

$$Df(\boldsymbol{\phi}) \cdot \Delta\boldsymbol{u} = D^2\Pi(\boldsymbol{\phi}) \cdot (\boldsymbol{W}, \Delta\boldsymbol{u}) \tag{4.105}$$

is linear in both $\boldsymbol{W}$ and $\Delta\boldsymbol{u}$, and is symmetric by the symmetry of the second derivative, leading to $\boldsymbol{K} = \boldsymbol{K}^T$.

Bibliography

[1] J.E. Marsden and T.J.R. Hughes, *Mathematical Foundations of Elasticity* (Prentice-Hall, Englewood Cliffs, New Jersey, 1983).

[2] T.J.R. Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis* (Prentice-Hall, Englewood Cliffs, New Jersey, 1987).