AN EVALUATION OF APOLLO POWERED DESCENT GUIDANCE

A Thesis

Presented to the

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In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Aerospace Engineering

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by

Lloyd David Strohl III

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The Undersigned Faculty Committee Approves the

Thesis of Lloyd David Strohl III:

An Evaluation of Apollo

Powered Descent Guidance

Ping Lu, Chair
Department of Aerospace Engineering

Ahmad Bani Younes
Department of Aerospace Engineering

Peiman Naseradinmousavi
Department of Mechanical Engineering

Approval Date

ABSTRACT OF THE THESIS

An Evaluation of Apollo
Powered Descent Guidance
by
Lloyd David Strohl III
Master of Science in Aerospace Engineering with a Concentration in Guidance, Navigation, and Controls
San Diego State University, 2018

This is my abstract which describes my whole thesis.

Many extraterrestrial missions require a powered descent phase. Because this phase is late in the mission its fuel efficiency has an outsized effect on payload capacity. This thesis presents a strategy for optimizing fuel use using well tested guidance algorithms that reduces fuel consumption over conventional strategies by x%.

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GLOSSARY

ACKNOWLEDGMENTS

I would like to thank Dr. Lu for his serendipitous arrival at SDSU and consequent advice and instruction. With his mentoring I have been able to launch an enjoyable and fulfilling career in GN&C, something I could not have achieved otherwise.

CHAPTER 1
INTRODUCTION

1.1 LITERATURE REVIEW

1.2 CONTRIBUTION

1.3 ORGANIZATION

CHAPTER 2

PROBLEM STATEMENT

CHAPTER 3
METHODOLOGY

To investigate a guidance law with careful focus on the time-to-go approach, a law must be developed and implemented in a simulation framework. The Law's derivation is presented here in the context of optimal control, as is the time-to-go approach.

The simulation methodology is also described, including the numerical methods and the aerodynamic model.

3.1 GUIDANCE LAW

The guidance law under investigation is E-Guidance, first presented in Cherry 1964 [6]. E-Guidance was developed empirically by integrating the equations of motion and choosing basis functions for the thrust acceleration input $\boldsymbol{a}_T$ that provided the necessary degrees of freedom to satisfy the terminal constraints in Equations 3.5 and 3.6. With control over thrust acceleration (and therefore total acceleration), satisfying the initial conditions after integration of the equations of motion shows the need for two basis functions with vector coefficients. Cherry developed E-Guidance by considering first one guidance axis at a time, so the acceleration command took the form of Equation 3.1, where $p_1(t)$ and $p_2(t)$ are linearly independent functions of time.

$$\ddot{x} = c_1 p_1(t) + c_2 p_2(t) \tag{3.1}$$

Cherry sets $p_1(t) = 1$ and $p_2(t) = t$, mostly for simplicity while recognizing that it may be suboptimal, arriving at a form identical to the one presented below in Equation 3.12. The Cherry paper also proposes a time-to-go algorithm similar to that in Algorithm 1. It does not, however, do much to consider the optimality of this algorithm or, more critically, to deal with the problem of ignition timing, nor does most of the academic literature since.

Below E-Guidance as a control law will be derived using optimal control theory. An extension that also considers the final orientation will also be presented without claim about its optimality, but its performance will be compared with E-Guidance. The rationale behind choice of time-to-go algorithm and subsequent ignition timing will follow.

3.1.1 Equations of Motion

Derivation of an optimal powered descent guidance law begins with a formulation of the State Equation 3.2

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{3.2}$$

The command input $\boldsymbol{u}$ for this problem is comprised of the thrust magnitude and direction. Aerodynamic effects are not considered for development of the law, though they will be simulated and investigated with regards to performance.

The state equations for the 3-dimensional powered descent guidance problem are as follows

$$\dot{\boldsymbol{r}} = \boldsymbol{V} \qquad\qquad \boldsymbol{r(t_0)} = \boldsymbol{r_0} \tag{3.3}$$

$$\dot{\boldsymbol{V}} = \boldsymbol{g(r)} + \boldsymbol{a_T} \qquad\qquad \boldsymbol{V(t_0)} = \boldsymbol{V_0} \tag{3.4}$$

with terminal constraints at a fixed final time t_f

$$\boldsymbol{r}(t_f) = \boldsymbol{r}_f^* \tag{3.5}$$

$$\boldsymbol{V}(t_f) = \boldsymbol{V}_f^* \tag{3.6}$$

where $\boldsymbol{a}_T$ is the thrust acceleration vector. $\boldsymbol{a}_T$ is limited such that

$$0 < a_{min} \leq ||\boldsymbol{a}_T|| \leq a_{max} \tag{3.7}$$

3.1.2 Performance Index

Fuel consumption is related to the thrust acceleration vector by engine parameters represented by some positive constant k

$$\dot{m} = -k||\boldsymbol{a}_T|| \tag{3.8}$$

A fuel optimal guidance law should therefore use the performance index

$$J = \int_{t_0}^{t_f} ||\boldsymbol{a}_T||dt \tag{3.9}$$

Choosing to minimize the square of the total acceleration $\boldsymbol{a} = \boldsymbol{g} + \boldsymbol{a}_T$ gives a performance index

$$J = \frac{1}{2}\int_{t_0}^{t_f} (\boldsymbol{g} + \boldsymbol{a}_T)^T(\boldsymbol{g} + \boldsymbol{a}_T)dt \tag{3.10}$$

For a constant gravitational acceleration $\boldsymbol{g}$, this performance index attempts to minimize $||\boldsymbol{a}_T||^2$. It is not fuel optimal as in Equation 3.9, but it does provide a cost to large thrust accelerations and might be expected to give good fuel performance.

3.1.3 Guidance solution

Choosing the guidance command $\boldsymbol{u} = \boldsymbol{g} + \boldsymbol{a}_T$ and applying optimal control theory results in the following

$$H = \boldsymbol{p}_r^T \boldsymbol{V} + \boldsymbol{p}_V^T \boldsymbol{u} - \frac{1}{2}\boldsymbol{u}^T \boldsymbol{u} \tag{3.11}$$

$$\dot{\boldsymbol{p}}_r = -\frac{\partial H}{\partial \boldsymbol{r}} = 0 \implies \boldsymbol{p}_r = -\boldsymbol{c}_2$$
$$\dot{\boldsymbol{p}}_V = -\frac{\partial H}{\partial \boldsymbol{V}} = -\boldsymbol{p}_r \implies \boldsymbol{p}_V = \boldsymbol{c}_1 + \boldsymbol{c}_2 t$$
$$\frac{\partial H}{\partial \boldsymbol{u}} = 0 \implies \boldsymbol{u} = \boldsymbol{p}_V = \boldsymbol{c}_1 + \boldsymbol{c}_2 t$$

For convenience, let $\tau = t_f - t$

$$\boldsymbol{u} = \boldsymbol{k}_1 + \boldsymbol{k}_2 \tau \tag{3.12}$$

where $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are constant vectors.

Integrating the equations of motion with $\dot{\boldsymbol{V}} = \boldsymbol{u}$ then gives

$$\int \dot{\boldsymbol{V}}(t)dt = \boldsymbol{k}_1(t - t_0) + \frac{1}{2}\boldsymbol{k}_2(t - t_0)^2 + \boldsymbol{V}(t_0) \tag{3.13}$$

$$\int \dot{\boldsymbol{r}}(t)dt = \frac{1}{2}\boldsymbol{k}_1(t - t_0)^2 + \frac{1}{6}\boldsymbol{k}_2(t - t_0)^3 + \boldsymbol{V}(t_0)(t - t_0) + \boldsymbol{r}(t_0) \tag{3.14}$$

Setting $t = t_f$ and letting $t_{go} = t_f - t_0$ satisfies the terminal constraints from Equations 3.5 and 3.6, resulting in 6 linear equations in 6 unknowns

$$\boldsymbol{k}_1 t_{go} + \frac{1}{2}\boldsymbol{k}_2 t_{go}^2 = \boldsymbol{V}_f^* - \boldsymbol{V}_0 \tag{3.15}$$
$$\frac{1}{2}\boldsymbol{k}_1 t_{go}^2 + \frac{1}{6}\boldsymbol{k}_2 t_{go}^3 = \boldsymbol{r}_f^* - \boldsymbol{r}_0 - \boldsymbol{V}_0 t_{go} \tag{3.16}$$

These equations can be separated into sets of two per vector component. Define an inertial guidance frame $\boldsymbol{e} = (\hat{x}, \hat{y}, \hat{z})^T$ such that guidance vector $\boldsymbol{u}$ is composed of components in $\boldsymbol{e}$, $\boldsymbol{u} = (u_x, u_y, u_z)^T$. For the equations in $\hat{x}$ we have

$$\begin{bmatrix} t_{go} & \frac{1}{2}t_{go}^2 \\ \frac{1}{2}t_{go}^2 & \frac{1}{6}t_{go}^3 \end{bmatrix} \begin{pmatrix} k_{1_x} \\ k_{2_x} \end{pmatrix} = \begin{pmatrix} V_{f_x}^* - V_{0_x} \\ r_{f_x}^* - (r_{0_x} + V_{0_x} t_{go}) \end{pmatrix} \tag{3.17}$$

Solving the two-equation system is accomplished by inverting the A matrix, leading to a coefficient matrix E

$$E = \begin{bmatrix} -2/t_{go} & 6/t_{go}^2 \\ 6/t_{go}^2 & -12/t_{go}^3 \end{bmatrix} \tag{3.18}$$

The coefficients in $\hat{x}$ are then

$$\begin{pmatrix} k_{1_x} \\ k_{2_x} \end{pmatrix} = E \begin{pmatrix} V^*_{f_x} - V_{0_x} \\ r^*_{f_x} - (r_{0_x} + V_{0_x} t_{go}) \end{pmatrix} \tag{3.19}$$

It can be shown that the equations in $\hat{y}$ and $\hat{z}$ take the same form. This $2x2$ E matrix is the origin of the name *E-Guidance*, the guidance law used in the Apollo lunar landing missions.

Of some interest is the addition of a final attitude constraint. For a vehicle whose attitude it determined by the thrust acceleration vector, this constraint can be implemented as a final thrust acceleration constraint as in Equation 3.20.

$$\boldsymbol{a}_T(t_f) = \boldsymbol{a}^*_{T_f} \tag{3.20}$$

This vector constraint cannot be satisfied with only two basis functions for the command $\boldsymbol{u}$, so a third linearly independent function must be introduced such that $\boldsymbol{u} = \boldsymbol{c}_1 p_1(t) + \boldsymbol{c}_2 p_2(t) + \boldsymbol{c}_3 p_3(t)$. A tempting choice for the third basis function is $p_3(t) = t^2$ for simplicity, with the other two functions the same as E-Guidance.

After applying the substitution from Equation 3.12, this choice gives a command $\boldsymbol{u} = \boldsymbol{k}_1 + \boldsymbol{k}_2\tau + \boldsymbol{c}_3\tau^2$. This form for the command input is also easily integrable and leads to a similar linear system of 9 equations in $\boldsymbol{k_1}$, $\boldsymbol{k_2}$, and $\boldsymbol{k_3}$. Using the same guidance frame $\hat{e}$ defined for Equation 3.17 we get the system in Equation 3.21

$$\begin{bmatrix} t_{go} & \frac{1}{2}t_{go}^2 \\ \frac{1}{2}t_{go}^2 & \frac{1}{6}t_{go}^3 \end{bmatrix} \begin{pmatrix} k_{1_x} \\ k_{2_x} \end{pmatrix} = \begin{pmatrix} V^*_{f_x} - V_{0_x} \\ r^*_{f_x} - (r_{0_x} + V_{0_x} t_{go}) \end{pmatrix} \tag{3.21}$$

3.1.4 Time-to-go

The E-Guidance solution depends upon a reliable estimate of remaining time-to-go (t_{go}). The Apollo mission's guidance used an estimate that updated continuously using Newton's method, but it was intended to only operate until start of the terminal descent phase at which point guidance switched to a manual vertical descent operation. Updating the t_{go} estimate continuously is attractive since it should be robust; if conditions have to change during the mission a closed-loop (continuously updating) solution will adjust and a new, realistic t_{go} will feed into the guidance solution. This quality was important to the Apollo Guidance solution because it relied upon pilot inputs to define the landing location visually, which meant allowing for landing site redesignations mid-mission. If t_{go} was not recomputed after site redesignation, the guidance law would command unrealizable thrust acceleration commands.

For the purposes of this study, live landing site redesignation was not considered. Without the possibility of landing site redesignation, an open-loop t_{go} solution lends the guidance law more stability in that the performance is less dependent upon specific assumptions and conditions imposed by the t_{go} algorithm. For instance, one closed-loop t_{go} algorithm is implemented in Algorithm 1.

Algorithm 1 Fixed-Point-Iteration t_{go}

procedure FPI(t_{go_1})
  $tol \leftarrow c$
  while $|t_{go_0} - t_{go_1}| \geq tol$ **do**
    $t_{go_0} \leftarrow t_{go_1}$
    $\Delta V \leftarrow \sqrt{(\boldsymbol{V} - \boldsymbol{V_0} + \boldsymbol{g} \cdot t_{go})^T (\boldsymbol{V} - \boldsymbol{V_0} + \boldsymbol{g} \cdot t_{go_0})}$
    $t_{go_1} \leftarrow \frac{m_0}{\dot{m}} \left(e^{\frac{-\Delta V}{v_{ex}}} - 1 \right)$ $\triangleright\ \dot{m} < 0$
  end while
  return t_{go_1}
end procedure

Each guidance update uses the previous update's t_{go} minus clock time as its initial guess t_{go_0}, and the max iterations may be limited to some reasonable number.

This algorithm requires an assumption about a fixed mass flow rate $\dot{m}$ which is not guaranteed by the guidance law. Adjustment of this mass flow rate estimate is very particular to the initial conditions of the mission, resulting in a necessarily conservative t_{go} to account for initial condition dispersion.

One attractive open-loop option is the time to perform a gravity turn landing at maximum thrust. The equation for a gravity turn time-to-go, $t_{go_{GT}}$, was presented first in Cherry 1964 [6]. After engine ignition and initiation of the Powered Descent Guidance, the updated time-to-go is computed as $t_{go_{GT}}$ minus elapsed clock time.

A gravity turn landing does not directly apply to the general powered descent guidance problem under investigation because it does not seek to satisfy the constraints given in Equations 3.5 and 3.6. However, if the magnitude of the terminal velocity target in Equation 3.6 is small, the required time to decelerate from an initial $\boldsymbol{V}_0$ under only the forces of thrust acceleration $\boldsymbol{a}_T$ and gravity $\boldsymbol{g}$ is, at minimum, the gravity turn solution $t_{go_{GT}}$. Since the terminal position $\boldsymbol{r}_f$ is specified, the vehicle necessarily needs more time than given by the gravity turn solution to satisfy it. Assuming a small trajectory error requiring a small diversion to landing site, a small constant factor $c_t \approxeq 1.2$ can be applied to the gravity turn time-to-go to allow for redirection. Because the fuel optimal guidance law would minimize the index given in Equation 3.9, reducing the time of flight should be expected to reduce propellant consumption. This suggests that driving the factor c_t as close to 1 as possible while still landing softly is desirable. The choice of $c_t = 1.2$ will prove to be sufficiently conservative to survive initial condition dispersion, rocket parameter dispersion, and navigation error.

3.1.5 Ignition Timing

The literature has also given very little attention to the topic of ignition timing, or the initiation of powered descent guidance (PDI). Mission planning involves phases and guidance laws like E-Guidance

3.2 SIMULATION

The powered descent simulation (PD Sim) is developed using standard programming techniques and numerical methods. The code is modular to facilitate design and reflect the functions of a mission computer. The simulation has three degrees of freedom (3-DoF) with derived orientation for force modeling.

At the core of the simulation is numerical time integration. The method employed is a 4th order Runge-Kutta (RK4). The RK4 function is called at each time step to progress the simulation forward.

The separate modules called during simulation are the guidance computer which contains the guidance law and computes a commanded thrust acceleration vector, the navigation module which generates a state estimate from simulated instrument measurements for use by the guidance computer, and the aerodynamic module which contains a model of the landing vehicle to compute forces and orientation for use in integration.

The full simulation code is wrapped in a Monte-Carlo script. The Monte-Carlo script calls the PD Sim with simulation settings, vehicle parameters, initial condition, and navigation uncertainty. This allows each individual run to have dispersed conditions without modification of the code.

Further detail for each part of the simulation follows.

3.2.1 Numerical Integration

The numerical time integration is done via a standard Runge-Kutta formulation, presented here as in Ferziger and Perić [8]. The state vector $\boldsymbol{\phi}$ is passed to the equations of motion $\boldsymbol{f}(\boldsymbol{\phi}, t)$ several times as in Equation 3.22.

$$\boldsymbol{\phi}^*_{n+\frac{1}{2}} = \boldsymbol{\phi}_n + \frac{\Delta t}{2}\boldsymbol{f}(t_n, \boldsymbol{\phi}_n) \tag{3.22}$$

$$\boldsymbol{\phi}^{**}_{n+\frac{1}{2}} = \boldsymbol{\phi}^n + \frac{\Delta t}{2}\boldsymbol{f}(t_{n+\frac{1}{2}}, \boldsymbol{\phi}^*_{n+\frac{1}{2}}) \tag{3.23}$$

$$\boldsymbol{\phi}^*_{n+1} = \boldsymbol{\phi}^n + \Delta t\boldsymbol{f}(t_{n+\frac{1}{2}}, \boldsymbol{\phi}^{**}_{n+\frac{1}{2}}) \tag{3.24}$$

$$\boldsymbol{\phi}_{n+1} = \boldsymbol{\phi}^n + \frac{\Delta t}{6}[\boldsymbol{f}(t_n, \boldsymbol{\phi}_n) + 2\boldsymbol{f}(t_{n+\frac{1}{2}}, \boldsymbol{\phi}^*_{n+\frac{1}{2}}) \tag{3.25}$$

$$+ 2\boldsymbol{f}(t_{n+\frac{1}{2}}, \boldsymbol{\phi}^{**}_{n+\frac{1}{2}}) + \boldsymbol{f}(t_{n+1}, \boldsymbol{\phi}^*_{n+1})] \tag{3.26}$$

3.2.2 Guidance Computer

3.2.3 Navigation Module

3.2.4 Aerodynamic Model

The Aerodynamic model is comprised of Mars atmosphere data from (insert atmosphere data reference), and the vehicle model was developed by Cerimele et. al [5]

3.2.4.1 Atmosphere Model

3.2.4.2 Lift and Drag

3.2.4.3 Orientation

3.2.5 Monte Carlo

CHAPTER 4

RESULTS AND DISCUSSION

CHAPTER 5
CONCLUSIONS AND FUTURE WORK

CHAPTER 6
REFERENCING

Below a list of references are provided in the acceptable format for Master's thesis submission. References are to be numbered and should appear either alphabetically or in the order of appearance in the text. (LaTeX does the former for the student.) For students using LaTeX these are obtained using the plain style with BibTeX. The Department of Mathematics and Statistics will accept either the plain style or the SIAM style. (For the SIAM style, get a copy of the SIAM.BST file from your graduate adviser or the Mathematical Sciences computer system.) There are references for journal `articles` [1], `books` and `booklets` [4, 20], `inbooks`, `incollections`, and `inproceedings` [3, 7, 18]. *Note that when you have more than one citation in a single bracket they must be in increasing numerical order!* Other sources may be `proceedings` [2], technical reports (`techreport`) [16], theses (`mastersthesis`, or `PhDthesis`) [9], or `unpublished` material [17]. This should provide a fairly comprehensive list for any material that the student may encounter. For additional assistance, see the graduate adviser in your area of concentration. LaTeX source codes are available for copying.

Style note

If you cite a website [15] and you can't find the year on the website, you should put ”n.d.” (not dated) at the end. (this is true for other reference also.) It must also has the word ”accessed” and the month and year you access the website. You can change how things with no author(s) are sorted in the bibliography by supplying a `key` entry (see `thbib.bib`), *e.g.* this news release [19] will be sorted under “U,” the leading letter of the publishing agency (as preferred by the thesis publisher).

This [10] is an example of a patent. ***Notice:*** how the `month` and `year` fields in `thbib.bib` have been abused to force the “correct” format.

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APPENDIX A
MORE INFORMATION ON EQUATIONS

MORE INFORMATION ON EQUATIONS

To demonstrate how an appendix should be inserted into the thesis we have provided two appendices. This first appendix illustrates some more advanced techniques to improve the appearance of your equations. Below is a system of partial differential equations from a model for cellular control by an external nutrient. The equations are complicated and LaTeX tends to allow them to run into each other. To prevent this we have used the `\vrule` command to separate them. Note this is an ordinary TeX command and is not in L. Lamport's book [13]. Furthermore, we have some complicated boundary conditions that we needed to align, so we used the array command, but to get the equations looking right we also needed the `\dfrac` command instead of the `\frac` command. The equations for our model are as follows:

$$\begin{aligned}
\dot{U}_1(t) &= \tilde{f}(W_1(t-T)) - U_1(t) + \gamma_1 U_2(R\sigma, t), \\
\dot{W}_1(t) &= -\hat{b}_3 W_1(t) + \gamma_3 W_2(R\sigma, t), \\
\frac{\partial U_2}{\partial t} &= D_1 \nabla^2 U_2 - U_2 - \tilde{f}(W_1(t-T)) - \gamma_1 U_2(R\sigma, t), \qquad \text{(A.1)} \\
\frac{\partial V_2}{\partial t} &= D_2 \nabla^2 V_2 - b_2 V_2 + c_0\big(U_2 + U_1(t)\big), \\
\frac{\partial W_2}{\partial t} &= D_3 \nabla^2 W_2 - b_3 W_2 + (\hat{b}_3 - b_3) W_1 - \gamma_3 W_2(R\sigma, t) \\
&\quad + k\left[\left[\left(\frac{D_3}{r^2}\right)\frac{d}{dr}\left(r^2\frac{dh}{dr}\right) - b_3 h\right] V_2(R,t) - h\dot{V}_2(R,t)\right],
\end{aligned}$$

for $t > 0$ and $R\sigma < r < R$ and with the boundary conditions:

$$\begin{array}{rclcrcl}
\dfrac{\partial U_2(R\sigma,t)}{\partial r} &=& \beta_1 U_2(R\sigma,t), & \qquad & \dfrac{\partial U_2(R,t)}{\partial r} &=& 0, \\[2ex]
\dfrac{\partial V_2(R\sigma,t)}{\partial r} &=& 0, & & \dfrac{\partial V_2(R,t)}{\partial r} &=& 0, \\[2ex]
\dfrac{\partial W_2(R\sigma,t)}{\partial r} &=& \beta_3 W_2(R\sigma,t), & & \dfrac{\partial W_2(R,t)}{\partial r} &=& 0.
\end{array}$$

Notice that the system is numbered only once by (A.1) and that this is centered as best we can on one line. All other lines have the *nonumber* command.

A.1 THEOREMS

The appendix can also include technical theorems and lemmas which are call in the same manner as before. For example,

Theorem A.1. *The system of equations (A.1) can exhibit periodic solutions for certain parameter values.*

Proof. The argument uses Hopf bifurcation techniques and is very complicated. See Mahaffy *et al* [14]. $\square$

APPENDIX B

LISTS AND QUOTATIONS

LISTS AND QUOTATIONS

The thesis will rarely use list environments, but they are valuable for résumés. For more information on creating a résumé you may want to see the author of this document (you also need to learn quite a bit about `\parbox` commands). To create a list you will want to use one of `itemize, enumerate,` or `description`. For example:

continuous A function f is **continuous** at x if and only if for every $\varepsilon > 0$ there exists a $\delta(x) > 0$ such that whenever $|y - x| < \delta$, $|f(y) - f(x)| < \varepsilon$.

uniformily continuous A function f is **uniformly continuous** if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $|y - x| < \delta$, $|f(y) - f(x)| < \varepsilon$ independent of x and y.

equicontinuous A family of functions f_n is **equicontinuous** at a point x if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $|y - x| < \delta$, $|f_n(y) - f_n(x)| < \varepsilon$ for all functions f_n.

LaTeX provides an environment for block quotations. To agree with the thesis manual follow the format below for a quotation exceeding four lines. From Lewis Carrol's *Hunting of the Snark* we hear the Bellman tell his crew:

> The Bellman himself they all praised to the skies–
> Such a carriage, such ease and such grace!
> Such solemnity, too! One could see he was wise,
> The moment one looked in his face!
>
> He had bought a large map representing the sea,
> Without the least vestige of land:
> And the crew were much pleased when they found it to be
> A map they could all understand.
>
> "What's the good of Mercator's, North Poles and Equators,
> Tropics, Zones, and Meridian Lines?"
> So the Bellman would cry: and the crew would reply,
> "They are merely conventional signs!"
>
> "Other maps are such shapes, with their islands and capes!
> But we've got our brave Captain to thank"
> (So the crew would protest) "that he's bought us the best–
> A perfect and absolute blank!"