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AN OPTIMAL GUIDANCE LAW FOR PLANETARY LANDING

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Abstract

A guidance law which minimizes the commanded acceleration along with the (weighted) final time is developed. This guidance law is a linear function of the states (relative to the landing point) and a nonlinear function of the time-to-go. The time-to-go is obtained as a solution to a quartic equation which is solved analytically. The advantage of this guidance law is that it does not involve any iterations whatsoever. It is the exact solution to the two-point boundary-value problem associated with the first variation necessary conditions. It also satisfies the second variation necessary conditions for a minimum. An example of a lunar landing is given to demonstrate the optimality of this guidance law.

1. Introduction

There has been renewed interest in planetary exploration with the announcement of signs of life on Mars and trapped water at the south lunar pole. In order to confirm these scientific findings, detailed *in situ* analyses will have to be performed. This will necessitate sending spacecraft equipped with landers to carry out these inquiries. It is desirable to land a spacecraft at a particular site of interest using the least amount of fuel. It follows that carrying less propellant on board would allow for a larger scientific payload.

Previous investigations have obtained guidance laws which are based on posing a problem which approximates the optimal control problem [1, 2]. One of the difficulties has been in solving for the time-to-go. These studies were used to evaluate the targeting and divert capabilities of Mars and lunar landers. They analyzed the cases of specified acceleration profiles as well as unspecified acceleration profiles. The landing phase was broken up into three sub-phases: braking, pitchup, and vertical descent/terminal landing. The algorithms were structured similar to the Shuttle Powered Explicit Guidance scheme which is a predictor/corrector algorithm.

This paper will develop a guidance law for a soft planetary landing at a desired target location which minimizes the acceleration (and hence the propellant required). A closed-form, easily mechanized optimal guidance law will be obtained using the Calculus of Variations. This law is a function of the time-to-go which is also obtained in a closed form analytic solution. This guidance law, which solves the associated two-point boundary value problem, does not involve any iterations.

In Section 2 the dynamics and the assumptions associated with the problem at hand are discussed. Section 3 contains the optimal control problem formulation. In Section 4 the first variation necessary conditions are obtained and a closed form guidance law dependent on time-to-go and the current states is obtained. In Section 5 the time-to-go is obtained from the transversality condition on final time which involves the solution of a quartic equation. Section 6 contains the closed form solution of the quartic equation. The second variation necessary conditions are derived in Section 7. An exam-

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ple is presented in Section 8 which involves a lunar landing to a designated site and this is compared with a numerical optimal solution. Finally, Section 9 contains some concluding remarks.

2. The Dynamics

For this analysis the following assumption will be made: $h/R_s << 1$, where h is the altitude above the surface and R_s is the planetary radius. In addition, it will be assumed that the atmospheric forces are significantly less than the gravitational and the propulsive forces, and hence can be neglected. Therefore, the equations of motion in Cartesian coordinates are

$$\dot{x} = u \tag{1}$$
$$\dot{y} = v \tag{2}$$
$$\dot{z} = w \tag{3}$$
$$\dot{u} = a_x \tag{4}$$
$$\dot{v} = a_y \tag{5}$$
$$\dot{w} = a_z + g \tag{6}$$

where x is the downrange position (relative to the target), y is the crossrange position, z is the target-relative altitude, u is the downrange velocity, v is the crossrange velocity, w is the vertical velocity, a_x is the acceleration in the horizontal direction, a_y is the acceleration in the crossrange direction, a_z is the acceleration in the vertical direction, and g is the local gravitational acceleration.

3. The Performance Index and Problem Set-up

The minimum control effort (acceleration) problem can be posed as

$$\min J = \Gamma t_f + \frac{1}{2}\int_{t_0}^{t_f} a_x^2 + a_y^2 + a_z^2 \, d\tau \tag{7}$$

subject to the aforementioned dynamics (Eqs. (1)-(6)), with specified initial conditions

$$x_0 = x_{V_0} - x_T \tag{8}$$
$$y_0 = y_{V_0} - y_T \tag{9}$$
$$z_0 = z_{V_0} - z_T \tag{10}$$
$$u_0 = u_{V_0} \tag{11}$$
$$v_0 = v_{V_0} \tag{12}$$
$$w_0 = w_{V_0} \tag{13}$$

and the following terminal constraints

$$\begin{bmatrix} x_f & y_f & z_f & u_f & v_f & w_f \end{bmatrix} = 0^T. \tag{14}$$

Γ is a weighting parameter on the final time (if it is unspecified). This represents a trade-off between the minimum time problem and the minimum control effort problem. It can be set to zero if desired. The Hamiltonian and the Bolza function for this problem, respectively, are

$$\begin{aligned} H &= \frac{a_x^2}{2} + \frac{a_y^2}{2} + \frac{a_z^2}{2} + \lambda_x u + \lambda_y v + \lambda_z w \\ &\quad + \lambda_u a_x + \lambda_v a_y + \lambda_w (a_z + g) \end{aligned} \tag{15}$$
$$\begin{aligned} G &= \Gamma t_f + \nu_x x_f + \nu_y y_f + \nu_z z_f \\ &\quad + \nu_u u_f + \nu_v v_f + \nu_w w_f. \end{aligned} \tag{16}$$

4. The First Variation Conditions

The Euler-Lagrange equations along with the transversality condition on the final state yield the following equations [3, 4]

$$\dot{\lambda}_x = 0 \tag{17}$$
$$\dot{\lambda}_y = 0 \tag{18}$$
$$\dot{\lambda}_z = 0 \tag{19}$$
$$\dot{\lambda}_u = -\lambda_x \tag{20}$$
$$\dot{\lambda}_v = -\lambda_y \tag{21}$$
$$\dot{\lambda}_w = -\lambda_z \tag{22}$$

with the terminal conditions

$$\lambda_{x_f} = \nu_x \tag{23}$$
$$\lambda_{y_f} = \nu_y \tag{24}$$
$$\lambda_{z_f} = \nu_z \tag{25}$$
$$\lambda_{u_f} = \nu_u \tag{26}$$
$$\lambda_{v_f} = \nu_v \tag{27}$$
$$\lambda_{w_f} = \nu_w. \tag{28}$$

If we define

$$t_{go} \triangleq t_f - t \tag{29}$$

the Lagrange multipliers are found to be

$$\lambda_x = \nu_x \tag{30}$$
$$\lambda_y = \nu_y \tag{31}$$
$$\lambda_z = \nu_z \tag{32}$$
$$\lambda_u = \nu_x t_{go} + \nu_u \tag{33}$$
$$\lambda_v = \nu_y t_{go} + \nu_v \tag{34}$$
$$\lambda_w = \nu_z t_{go} + \nu_w. \tag{35}$$

and the control variables, a_x and a_z are

$$a_x = -\lambda_u = \nu_x t_{go} - \nu_u \quad (36)$$
$$a_y = -\lambda_v = \nu_y t_{go} - \nu_v \quad (37)$$
$$a_z = -\lambda_w = \nu_z t_{go} - \nu_w. \quad (38)$$

The states can be written as

$$x = -\frac{\nu_x t_{go}^3}{6} - \frac{\nu_u t_{go}^2}{2} \quad (39)$$
$$y = -\frac{\nu_y t_{go}^3}{6} - \frac{\nu_v t_{go}^2}{2} \quad (40)$$
$$z = -\frac{\nu_z t_{go}^3}{6} - \frac{\nu_w t_{go}^2}{2} + \frac{g t_{go}^2}{2} \quad (41)$$
$$u = \frac{\nu_x t_{go}^2}{2} + \nu_u t_{go} \quad (42)$$
$$v = \frac{\nu_y t_{go}^2}{2} + \nu_v t_{go} \quad (43)$$
$$w = \frac{\nu_z t_{go}^2}{2} + \nu_w t_{go} - g t_{go}. \quad (44)$$

Note that the states in the three directions as well as the associated controls are independent of each other. The final time, if it is unspecified will be obtained by using the transversality condition on the final time (Sections 5 and 6).

Therefore, solving for the Lagrange multipliers in terms of the states yields

$$\nu_x = \frac{6u}{t_{go}^2} + \frac{12x}{t_{go}^3} \quad (45)$$
$$\nu_y = \frac{6v}{t_{go}^2} + \frac{12y}{t_{go}^3} \quad (46)$$
$$\nu_z = \frac{6w}{t_{go}^2} + \frac{12z}{t_{go}^3} \quad (47)$$
$$\nu_u = -\frac{2u}{t_{go}} - \frac{6x}{t_{go}^2} \quad (48)$$
$$\nu_v = -\frac{2v}{t_{go}} - \frac{6y}{t_{go}^2} \quad (49)$$
$$\nu_w = -\frac{2w}{t_{go}} - \frac{6z}{t_{go}^2} + g \quad (50)$$

and the controls can be expressed as

$$a_x = -\frac{4u}{t_{go}} - \frac{6x}{t_{go}^2} \quad (51)$$
$$a_y = -\frac{4v}{t_{go}} - \frac{6y}{t_{go}^2} \quad (52)$$
$$a_z = -\frac{4w}{t_{go}} - \frac{6z}{t_{go}^2} - g. \quad (53)$$

The acceleration can be written more compactly as

$$\mathbf{a} = -\frac{4\mathbf{v}}{t_{go}} - \frac{6\mathbf{r}}{t_{go}^2} - \mathbf{g} \quad (54)$$

where

$$\mathbf{r} \triangleq \begin{bmatrix} x_V - x_T \\ y_V - y_T \\ z_V - z_T \end{bmatrix} \quad (55)$$
$$\mathbf{v} \triangleq \begin{bmatrix} u & v & w \end{bmatrix}^T \quad (56)$$
$$\mathbf{g} \triangleq \begin{bmatrix} 0 & 0 & g \end{bmatrix}^T. \quad (57)$$

Thus, a closed-loop feedback guidance law which is dependent only on the current state, the terminal state, and the time to go has been obtained. The identical solution is obtained using the sweep method (Riccati equations) associated with the terminal control problem and the final time specified. The method of the calculus of variations was used because it allows for a solution of the final time. It is observed that the guidance law obtained above is the linear tangent guidance law [5, 6].

The performance index for the resulting problem is

$$J_{min} = \left(\Gamma + \frac{\mathbf{g}\cdot\mathbf{g}}{2}\right) t_{go} + \mathbf{v}\cdot\left(\mathbf{g} + \frac{2\,\mathbf{v}}{t_{go}}\right) + \frac{6\,\mathbf{r}}{t_{go}^2}\cdot\left(\mathbf{v} + \frac{6\,\mathbf{r}}{t_{go}}\right). \quad (58)$$

5. Solving for the Final Time

The transversality condition from the Euler-Lagrange equations associated with the free final time condition ($H_f = -G_{t_f}$) results in the following equation

$$\frac{a_{x_f}^2}{2} + \frac{a_{y_f}^2}{2} + \frac{a_{z_f}^2}{2} + \lambda_{x_f} u_f + \lambda_{y_f} v_f + \lambda_{z_f} w_f + \lambda_{u_f} a_{x_f} + \lambda_{v_f} a_{y_f} + \lambda_{w_f} a_{z_f} + \lambda_{w_f} g = -\Gamma. \quad (59)$$

A more useful condition is obtained from the first integral (being a constant), since the Hamiltonian is independent of time, which can be expressed as

$$H = -\frac{a_x^2}{2} - \frac{a_y^2}{2} - \frac{a_z^2}{2} + \lambda_x u + \lambda_y v + \lambda_z w + \lambda_w g. \quad (60)$$

This can be expressed in terms of t_{go} as

$$\left(\Gamma + \frac{g^2}{2}\right) t_{go}^4 - 2\left(u^2 + v^2 + w^2\right) t_{go}^2 - 12\left(ux + vy + wz\right) t_{go} - 18\left(x^2 + y^2 + z^2\right) = 0.$$

This equation is to be solved for t_{go}.

This algebraic equation for t_{go} can be written in more compact form as

$$\left(\Gamma + \frac{g^2}{2}\right) t_{go}^4 - 2\mathbf{v}\cdot\mathbf{v}t_{go}^2 - 12\,\mathbf{v}\cdot\mathbf{r} - t_{go} \\ -18\,\mathbf{r}\cdot\mathbf{r} = 0. \tag{61}$$

It should be noted that for $\mathbf{v}\cdot\mathbf{r} > 0$, there is only one positive real solution for t_{go}. This can be observed from the Routh-Hurwitz criterion [7] (or the Descartes rule). If $\mathbf{v}\cdot\mathbf{r} < 0$, as will most likely be the case, there will be three solutions with positive real parts, two of them most likely being complex. A simple check can easily select the valid t_{go}.

It should also be noted that a guidance law which provides for a minimum time to landing can be obtained quite easily by setting Γ to a large positive number. In most cases, however, when it is desired to minimize the acceleration (and thus the propellant consumed), Γ can be set to zero.

It follows that when if this law is used in environments with atmospheres of low densities, the trajectory will be different than the equivalent vacuum trajectory. The difference will become more and more pronounced for larger dynamic pressures.

6. The Analytic Solution for t_{go}

The equation for t_{go} can be solved analytically in the following manner [8]. First transform the equation to the following form

$$t_{go}^4 + at_{go}^2 + bt_{go} + c = 0 \tag{62}$$

where

$$a = -\frac{2\,\mathbf{v}\cdot\mathbf{v}}{\Gamma + \frac{g^2}{2}} \tag{63}$$

$$b = -\frac{12\,\mathbf{v}\cdot\mathbf{r}}{\Gamma + \frac{g^2}{2}} \tag{64}$$

$$c = -\frac{18\,\mathbf{r}\cdot\mathbf{r}}{\Gamma + \frac{g^2}{2}}. \tag{65}$$

This equation can be reduced to a cubic resolvant of the form

$$\eta^3 + 2a\eta^2 + \left(a^2 - 4c\right)\eta - b^2 = 0 \tag{66}$$

which can be transformed into the following equation

$$Z^3 + \alpha Z + \beta = 0 \tag{67}$$

where

$$Z = \eta + \frac{2a}{3} \tag{68}$$

$$\alpha = \frac{1}{3}\left[3\left(a^2 - 4c\right) - 4a^2\right] \tag{69}$$

$$\beta = \frac{1}{27}\left[16a^3 - 18a\left(a^2 - 4c\right) - 27b^2\right]. \tag{70}$$

The one real solution of the equation for Z is

$$Z = \sqrt[3]{-\frac{\beta}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{\beta}{2} - \sqrt{\Delta}} \tag{71}$$

where

$$\Delta = \frac{\alpha^3}{27} + \frac{\beta^2}{4}. \tag{72}$$

Once the solution for Z has been obtained, η can be obtained from Eq.(68). In order to find the candidate solutions for t_{go}, the following auxillary quantities are computed:

$$\zeta = -\frac{b}{2\eta} \tag{73}$$

$$\xi = \frac{a + \eta}{2} \tag{74}$$

and the four solutions for t_{go} are

$$t_{go_{1,2}} = \frac{\sqrt{\eta} \pm \sqrt{\eta - 4\left(\xi - \sqrt{\eta}\zeta\right)}}{2} \tag{75}$$

$$t_{go_{3,4}} = \frac{-\sqrt{\eta} \pm \sqrt{\eta - 4\left(\xi + \sqrt{\eta}\zeta\right)}}{2}. \tag{76}$$

7. Second Variation Necessary Conditions

The Weierstrass condition for strong variations can be stated as

$$H\left(\mathbf{x}, \mathbf{u}^*, \lambda, t\right) - H\left(\mathbf{x}, \mathbf{u}, \lambda, t\right) \geq 0 \tag{77}$$

where $\mathbf{x}$ is the state vector, $\mathbf{u}^*$ is the vector of admissible comparison controls, $\mathbf{u}$ is the optimal control vector, λ is the costate (Lagrange multiplier) vector, and t is the time. Applying this condition to the problem at hand results in

$$H\left(\mathbf{x},\mathbf{u}^*,\lambda,t\right) - H\left(\mathbf{x},\mathbf{u},\lambda,t\right) = \frac{\left(a_x^* - a_x\right)^2 + \left(a_y^* - a_y\right)^2 + \left(a_z^* - a_z\right)^2}{2} \quad (78)$$

which always satisfies the Weierstrass condition for a strong relative minimum.

The Legendre-Clebsch condition ($H_{\mathbf{uu}} \geq 0$), which is a special case of the Weierstrass condition for weak variations, is found to be satisfied easily since

$$H_{\mathbf{aa}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (79)$$

which is positive definite.

Hence, the optimal solution obtained above satisfies the necessary conditions for a (strong and weak) relative minimum.

8. Example: Lunar Landing

In order to demonstrate the use of the algorithm, a powered descent from an altitude of 50,000 ft and a distance of 500,000 ft downrange and 100,000 ft crossrange was investigated. The horizontal velocity was 3000 ft/sec. A new guidance command was computed every second and was held constant over that interval of time. This was done in a closed loop guidance simulation. The resulting trajectory was compared to an open-loop simulation which solved the two-point boundary value problem using the same initial conditions using a shooting method. Figures 1 and 2 contain the comparison of the two trajectory profiles. Figures 3 and 4 contain, respectively, the acceleration components and the acceleration magnitudes using the two algorithms. As can be seen from the figures, there is no discernable difference between the two algorithms. For this case, with $\Gamma = 0$ and and $t_f = 404$ seconds, $J_{min} = 18989$. For comparison, a trajectory whose guidance law penalized the final time ($\Gamma = 100$) was obtained, The resulting flight time for this trajectory was 301 seonds with the performance index $J_{min} = 52429$. The corresponding acceleration components and magnitude are in Figures 5 and 6. Obviously, as Γ increases, the flight time decreases correspondingly as does the acceleration required to obtain that trajectory. Many other cases

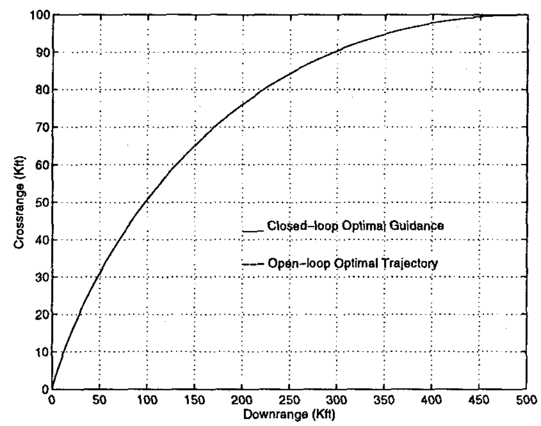

Figure 1: Horizontal Trajectory Profile

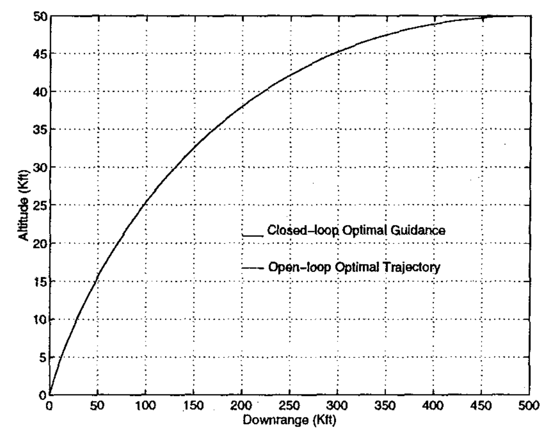

Figure 2: Vertical Trajectory Profile

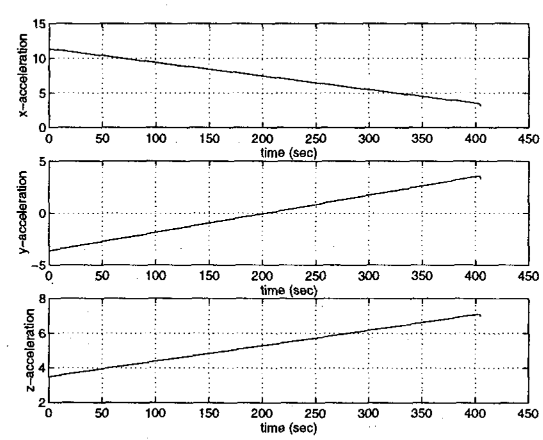

Figure 3: Components of Acceleration for $\Gamma = 0$

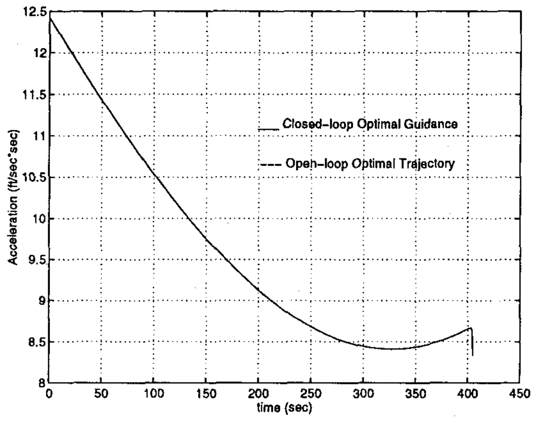

Figure 4: Acceleration Magnitude for $\Gamma = 0$

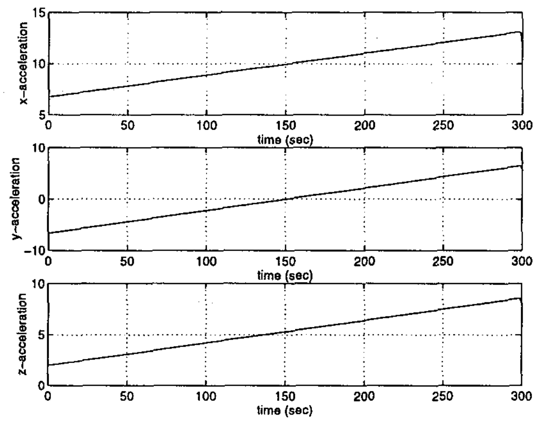

Figure 5: Components of Acceleration for $\Gamma = 100$

were investigated, and the trajectories and acceleration profiles generated by the optimal guidance law matched the open-loop optimal trajectories to within .1%.

9. Conclusions

An optimal control law was obtained which minimized the acceleration which required to effect a soft planetary landing. In addition, a minimum time trajectory can be easily obtained by setting Γ to a large number. The resulting guidance law was linear in the states and was a function of time-to-go. A solution of time-to-go was obtained which involved the solution of a quartic equation in time-to-go which was solved analytically. The guidance law obtained in this investigation was simple, easily mechanized and was the solution of the two-

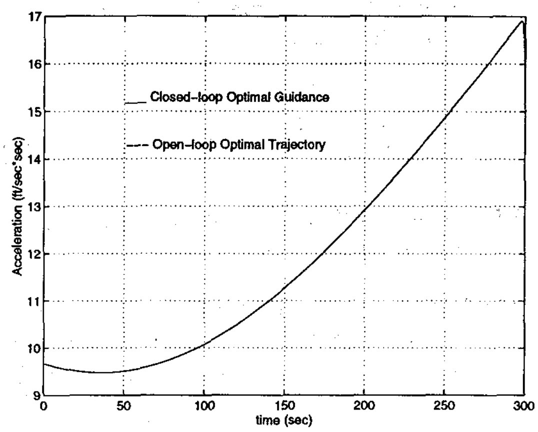

Figure 6: Acceleration Magnitude for $\Gamma = 100$

point boundary-value problem. This algorithm is extremely versatile and does not rely upon a nominal trajectory to be generated and followed. Instead it computes the optimal path from the current position, which could be perturbed by unmodeled gravity accelerations or as could be the case during atmospheric flight, unmodeled lift and drag. It shows great promise in dealing with robustness to unknown environment and vehicle parameter variations.

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