Lipschitz Maps and Hausdorff Dimension in Metric Spaces

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Abstract

I aim to study metric spaces with a focus on Lipschitz maps and Hausdorff dimension. I begin by reviewing the basic definitions and topological structure of metric spaces, then examine how Lipschitz maps provide a controlled way of mapping between them while preserving certain distance properties. I then explore the concept of Hausdorff measure and dimension as a tool for capturing the geometric complexity of subsets in metric spaces, especially those with fractal-like behavior.

Preliminaries

A **metric space** is a set X equipped with a function $d : X \times X \to \mathbb{R} \cup \{\infty\}$, called a *metric*, which satisfies the following properties: for all $x, y, z \in X$, we have $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$ (positivity); $d(x,y) = d(y,x)$ (symmetry); and $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

If the metric d satisfies the symmetry and triangle inequality but allows $d(x,y) = 0$ for $x \neq y$, it is called a *semi-metric*. Semi-metrics can be converted into true metrics by identifying points at zero distance, inducing a well-defined metric on the resulting quotient space.

A function $f : X \to Y$ between two metric spaces is said to be *distance-preserving* if for all $x, y \in X$, the equality $d_Y(f(x), f(y)) = d_X(x,y)$ holds. A bijective distance-preserving map is called an *isometry*, and two spaces are said to be isometric if there exists such a map between them.

Canonical examples of metric spaces include the real line $\mathbb{R}$ with the standard absolute value metric, Euclidean space $\mathbb{R}^n$ with the norm-induced distance $\|x-y\|_2$, and general normed vector spaces with metrics derived from norms. Metric subspaces inherit the metric by restriction, while more refined notions like *intrinsic metrics* (e.g., arc-length on a circle) reflect internal geometric properties.

The topology induced by a metric defines open sets in terms of open balls. Specifically, a set $U \subseteq X$ is open if for every $x \in U$, there exists $\varepsilon > 0$ such that the open ball $B_\varepsilon(x) \subseteq U$. In such spaces, limits of sequences, continuity of functions, and closedness of sets are all expressible via convergence.

Goal

This project aims to investigate the interplay between Lipschitz maps and Hausdorff dimension within the framework of metric geometry. The specific objectives are as follows:

- Develop a precise understanding of Lipschitz and bi-Lipschitz maps and their structural properties in metric spaces.
- Analyze how Lipschitz continuity governs distortion of distance and the stability of metric structures under mappings.
- Study the relationship between Lipschitz mappings and the preservation or reduction of Hausdorff dimension.
- Formally define the Hausdorff measure and dimension, and understand their role in quantifying geometric complexity.
- Compute Hausdorff dimensions of subsets in metric spaces using covering arguments and scaling behavior.
- Examine the influence of dilatation and local behavior of maps on global geometric properties of image sets.
- Apply theoretical results to representative examples, including normed vector spaces and fractal constructions.

Lipschitz Maps

A map $f : X \to Y$ between metric spaces is called **Lipschitz** if there exists a constant $C \geq 0$ such that

$$d_Y(f(x_1), f(x_2)) \leq C \cdot d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Any such C is referred to as a *Lipschitz constant*, and the smallest possible value is called the *dilatation* of f, denoted $\operatorname{dil} f$. If $\operatorname{dil} f \leq 1$, then f is called **nonexpanding**. A map is **bi-Lipschitz** if there exist constants $0 < c \leq C$ such that

$$c \cdot d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq C \cdot d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Theorem. *Every Lipschitz map is continuous. Moreover, the composition of two Lipschitz maps is Lipschitz, and the dilatation satisfies*

$$\operatorname{dil}(g \circ f) \leq \operatorname{dil}(g) \cdot \operatorname{dil}(f).$$

The space of real-valued Lipschitz functions on a metric space forms a vector space. If f and g are Lipschitz, then:

$$\operatorname{dil}(f+g) \leq \operatorname{dil}(f) + \operatorname{dil}(g), \quad \operatorname{dil}(\lambda f) = |\lambda| \cdot \operatorname{dil}(f).$$

A map $f : X \to Y$ is said to be **locally Lipschitz** if every point $x \in X$ has a neighborhood U such that $f|_U$ is Lipschitz. The pointwise dilatation is then defined by

$$\operatorname{dil}_x f = \inf\{\operatorname{dil}(f|_U) : U \ni x \text{ open neighborhood}\}.$$

Two metrics d_1 and d_2 on the same set X are **Lipschitz equivalent** if the identity map id_X is bi-Lipschitz with respect to d_1 and d_2. Lipschitz equivalent metrics induce the same topology and share geometric properties such as completeness and compactness.

The Projection Map in $\mathbb{R}^n$

Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the canonical projection onto the first k coordinates, defined by

$$\pi(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_k), \quad \text{for } k < n.$$

We equip both $\mathbb{R}^n$ and $\mathbb{R}^k$ with the standard Euclidean norm. Then for any $x, y \in \mathbb{R}^n$, we have:

$$\|\pi(x) - \pi(y)\|_{\mathbb{R}^k} \leq \|x - y\|_{\mathbb{R}^n}.$$

Hence, π is a Lipschitz map with constant $C = 1$. This example illustrates that orthogonal projection is nonexpanding and, more generally, that linear maps with operator norm ≤ 1 are Lipschitz.

Distance Function to a Set in a Metric Space

Let (X, d) be a metric space and let $A \subset X$. Define the function $f : X \to \mathbb{R}$ by

$$f(x) = \operatorname{dist}(x, A) := \inf_{a \in A} d(x, a).$$

Then for any $x, y \in X$, we have:

$$|f(x) - f(y)| = |\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y).$$

This follows from the triangle inequality and holds even if the infimum is not attained. Thus, f is a nonexpanding Lipschitz map with constant $C = 1$. This function is fundamental in geometric analysis and appears frequently in the study of level sets, metric gradients, and Lipschitz extensions.

Hausdorff Dimension

The Hausdorff dimension provides a way to measure the "size" or "complexity" of a subset in a metric space that generalizes the notion of integer dimension.

Definition. Let (X, d) be a metric space and let $d \geq 0$ be a real number. For any $\varepsilon > 0$, define

$$\mathcal{H}^d_\varepsilon(X) = \inf \left\{ \sum_i (\operatorname{diam} S_i)^d : \{S_i\} \text{ is a cover of } X \text{ with } \operatorname{diam} S_i < \varepsilon \right\}.$$

Then the d**-dimensional Hausdorff measure** of X is

$$\mathcal{H}^d(X) = \lim_{\varepsilon \to 0} \mathcal{H}^d_\varepsilon(X).$$

The limit always exists (possibly infinite). The function $d \mapsto \mathcal{H}^d(X)$ is monotone non-increasing.

Definition (Hausdorff Dimension). The **Hausdorff dimension** of a metric space X, denoted $\dim_H(X)$, is defined as

$$\dim_H(X) = \inf \left\{ d \geq 0 : \mathcal{H}^d(X) = 0 \right\} = \sup \left\{ d \geq 0 : \mathcal{H}^d(X) = \infty \right\}.$$

Theorem. For any metric space X, there exists a critical value $d_0 \in [0, \infty]$ such that

$$\mathcal{H}^d(X) = \begin{cases} \infty & \text{if } d < d_0, \\ 0 & \text{if } d > d_0. \end{cases}$$

This value d_0 is the Hausdorff dimension of X.

Theorem (Lipschitz Invariance). If $f : X \to Y$ is a Lipschitz map between metric spaces, then

$$\dim_H(f(X)) \leq \dim_H(X).$$

If f is bi-Lipschitz, then $\dim_H(f(X)) = \dim_H(X)$. Hence, Hausdorff dimension is invariant under bi-Lipschitz equivalence.

The Smith–Volterra–Cantor Set

The Smith–Volterra–Cantor set (SVC), also known as the *fat Cantor set*, is constructed by iteratively removing middle open intervals from the unit interval $[0,1]$, similar to the standard Cantor set, but with the total removed length summing to less than 1.

At step n, remove from each remaining interval the open middle portion of length $\frac{1}{2^{2n}}$. This process removes less and less at each stage, with the total removed length being:

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{2^{2n}} = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

The resulting set $S \subset [0,1]$ is closed, uncountable, nowhere dense, and—unlike the standard Cantor set—has positive Lebesgue measure:

$$\lambda(S) = \frac{1}{2}.$$

Despite being "thin" in the topological sense (nowhere dense), it has full Hausdorff dimension:

$$\dim_H(S) = 1.$$

This arises because the removed intervals shrink quickly enough to prevent a significant reduction in covering complexity. The Hausdorff d-measure $\mathcal{H}^d(S)$ transitions from infinity to zero precisely at $d = 1$, where it takes a finite positive value. This makes the SVC set a prototypical example of a fractal set with full dimension but no interior.

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