MATHEMATICS _ EXTENDED ESSAY

Title: The application of Group Theory and algorithms in the discovery of God's number

Research question: What is the minimum moves required to solve a 3x3x3 Rubik's Cube in the worst case scenario?

Word Count: 3906

Abstract:

The Rubik's cube was invented about 40 years ago. On the outside, it was just a simple cube, however, it contained many secrets. One of the secrets that make mathematicians tried to discover over 30 years is the question: what is the minimum number of turns required to solve the Rubik's cube? This number was labelled as the God's number, according to Singmaster (1980), and would finally be proved to be 20. In this essay, I will investigate how the God's number was discovered. First, I will investigate how to get the permutations of possible positions of Rubik's cube. Then I investigate how the God's number being reduced to 20.

Word count: 110

Contents:

Introduction

Rubik's Cube is a 3D combination puzzle that was invented in 1974 by a Hungarian architect, Erno Rubik. He designed the Rubik's Cube as a working model in order to explain three-dimension geometry. However, after designing the "magic cube" as he called it, he realized he couldn't actually solve the puzzle until after a month, by using a method of rearranging the corners first, he finally solved the puzzle. The Rubik's Cube was then launched into the market in 1980 and by 1981, it became the best selling toy in many countries.

There is no doubt that this simple puzzle can hold many secrets. For many years, the Rubik's Cube has caught the interest of the mathematicians. And for over 30 years, they have been trying to solve the question: what is the minimum moves required to sove the Rubik's Cube in the worst scenario? Finally, with the understanding of group theory and help from computer programming, the God's number has been discovered and was proved in July 2010 by Tomas Rokicki and his colleges.

Therefore, the main focus on this essay would be:

How was the God's number discovered?

In order to investigate the God's number of Rubik's Cube, I will need to understand the structure of the Rubik's Cube, the system of notations and applications of the group theory. Next, the application of computers will be used to calculate the upper bounds and the lower bounds. Once the upper bounds and lower bounds have been proved to be exactly one number, the God's number will be concluded.

Investigation

The Rubik's Cube

The Rubik's Cube is a cube where six faces (Up, Down, Left, Right, Front, and Back) have a distinct color on each face. The Rubik's cube is built up from 27 smaller cubes called "cubies" and each cubie represents 1 or 2 or 3 colors out of 6 colors. However, it can be easily recognized that there are only 26 cubies, the 27^{th} cubie won't be counted because it is a rotation axis, which located inside the cube, hence the cubie doesn't exist and won't participate in determining the possible positions. And there are 7 cubies that are fixed: the inside cubie (rotation axis) and 6 others cubies which are connected to the rotation axis; the other 20 cubies move.

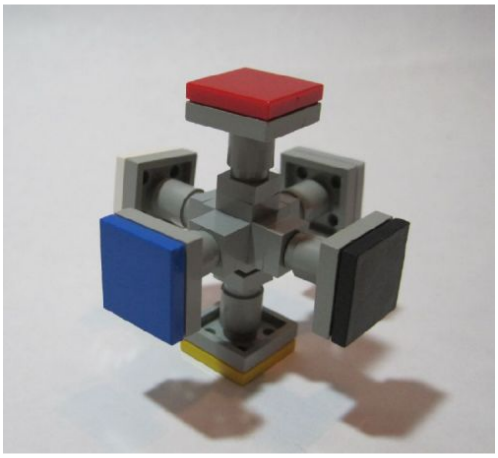

Figure 1: A Rubik Cube structure showing 7 fixed cubies (6 cubies and a rotation axis)

Note. Retrieved from Rubik's Cube Basics, by Sebastian

Let the cubies in the corner be called "corner cubies". A Rubik's Cube will have 8 "corner cubies". Each "corner cubie" has 3 visible faces displaying 3 different colors. Let the cubies at the edge with 2 visible faces be called "edge cubies". There will be 12 "edge cubies" in total. And finally, let the cubies which have only 1 single visible face be called "center cubies". There will be 6 "center cubies".

Throughout the essay, I will use the "Singmaster notation", which was developed by David Singmaster. According to David Singmaster (1981), the letters U, D, L, R, F, B will represent a 90° clockwise turn of the up, down, left, right, front and back faces respectively. A half turn is noted with a square on each notation letter, for instance, U^2 means rotating the U face 180°. Furthermore, a counter-clockwise rotation is noted with a prime symbol ($'$), such as U' represents the 90° anti-clockwise rotation of the Up face.

Permutation of the Rubik's cube

Because there are 8 "corner cubies" and each "corner cubies" visualize 3 distinct colors, hence, I will have a permutation of $(8! \times 3^8)$. Because there are 12 "edge cubies", and each "edge cubie" visualize 2 distinct colors, I will have a permutation of $(12! \times 2^{12})$.

Using these 2 permutation, the number of possible positions can be calculated by:

$8! \times 3^8 \times 12! \times 2^{12} \approx 5.19\times10^{20}$ (about 519 quintillion)

However, these permutations make it sound theoretically possible but in reality, some positions don't exist. For example, if I try to orientate a face of the cube, not only do the "edge cubies" move but the "corner cubies" also move and rearrange themselves. Therefore, the next goal is to:

- Determine the valid positions.

The group theory

Let a group $(G,*)$ consist of a set G and an operation '$*$' such that:

- G is closed under $*$ (this means if $a, b \in G, then\ a * b \in G$)
- '$*$' is associative (this means for any $a, b, c \in G$, $a * (b * c) = (a * b) * c$)
- there is an "identity element" $e \in G$, where $g = e * g = g * e$ for all $e \in G$

- there is an inverse. For instance, for any $g \in G$, there is an element of $h \in G$ where $h * G = G * h = e$

Let treat the set of moves in Rubik's cube as a group, denoted as $(G,*)$. The group operation will be defined as: if there are 3 moves M_1, M_2 and M_3, then $M_1 * M_2 * M_3$means the move M_1 will be done first, following by the move M_2 and M_3. By considering these set of moves as groups, they will have some certain properties:

- G is closed under $*$. This is because if a combination of many moves can also called a move.
- They have an identity element e. e is denoted as an empty move, which means a move that won't change the initial position of the Rubik's cube. For example, the move operation $M * e = M$ because it means first do M then do nothing, which won't change the initial position after move M.
- They have an inverse. It can be easily understood that move M can combine with move M' to convert back to the original position. This mean by doing a 90^{o} clockwise rotation following by a 90^{o} anti-clockwise rotation will give a result of the initial position, or $M * M' = e$
- '$*$' is associative. This mean move $(M_1 * M_2) * M_3 = M_1 * (M_2 * M_3)$. To show this, assuming that C is a cubie that is oriented, $M(C)$ is the cubicle that C will end up after applying M. After applying $(M_1 * M_2) * M_3$, the cubicle which have the oriented cubie C is:

  $$[(M_1 * M_2) * M_3](C) = M_3 * [(M_1 * M_2)](C) = M_3\left(M_2\big(M_1(C\,)\big)\right)$$

  While applying $M_1 * (M_2 * M_3)$, the cubicle which have the oriented C is:

  $$[M_1 * (M_2 * M_3)](C) = (M_2 * M_3) * (M_1(C)) = M_3\left(M_2\big(M_1(C\,)\big)\right)$$

Hence it is proved that $*$ is associative.

Because $(G,*)$ representing the set of moves has the same properties of a group, it is a group.

The Generators

The group G can be operated by moves $M_1, M_2, \ldots, M_n$. Let S be the the set of moves $M_1, M_2, \ldots, M_n$, hence S is a subset of G, or S is a set of generators of G. In other words, group G is generated by the subset S, denote as $G = \langle S \rangle = \langle M_1, M_2, \ldots, M_n \rangle$.

Let σ define as $\sigma: \{1,2, \ldots, n\} \to \{1,2, \ldots, n\}$ where $\sigma(i)$ represents the number that put in slot i. For instance, in figure 2, $\sigma(1) = 1$, $\sigma(2) = 2$ and so on.

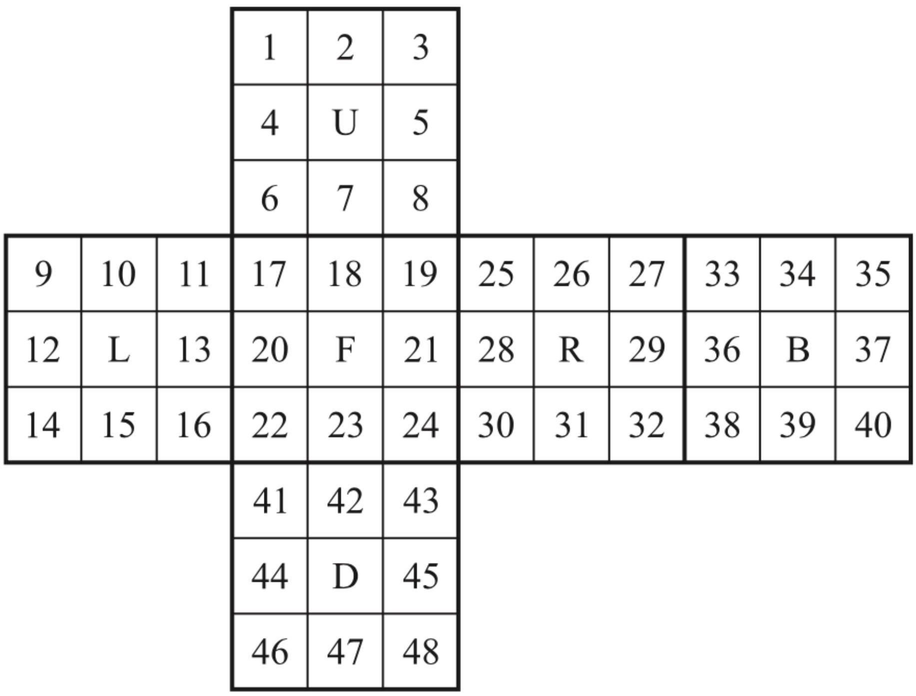

Figure 2: A labeled cube representing the slot number

Note. Adapted from *The College Mathematics Journal* (p. 261), by Joyner D. Copyright by THE MATHEMATICAL ASSOCIATION OF AMERICA

When I apply move U, then $\sigma(1) = 6$, $\sigma(2) = 4$, $\sigma(3) = 1$ and so on. However, if I apply U 4 times consequently, I will have:

$\sigma(1) = (1,3,8,6)$	$\sigma(3) = (3,8,6,1)$	$\sigma(5) = (5,7,4,2)$	$\sigma(7) = (7,4,2,5)$
$\sigma(2) = (2,5,7,4)$	$\sigma(4) = (4,2,5,7)$	$\sigma(6) = (6,1,3,8)$	$\sigma(8) = (8,6,1,3)$

This is called the cycle. A cycle $(i_1, i_2, \dots, i_k)$ is the element $\tau \in S_n$ such that $\tau(i_1) = i_2, \tau(i_2) = i_3, \dots, \tau(i_{k-1}) = i_k, \tau(i_k) = i_1$. The length of the cycle is denoted as k where k represents the numbers which appear in the cycles. A cycle of length k is also called k-$cycle$.

Two cycles are defined as $disjoint$ if they have no number in common. For instance, when applying U move, I will have 5 disjoint of 4-cycle:

$U = (1,3,8,6)\ (2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)$

Respectively for the following D, L, R, F, B face rotation:

$D = (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)$

$L = (9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)$

$R = (25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)$

$F = (17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)$

$B = (33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)$

By applying these, let G defined as $G = \langle U, D, L, R, F, B \rangle$. This is an important factor because it will be simpler to focus on the properties of 6 basic moves instead of moves required for all the 519 quintillion possible positions.

Valid configurations of the Rubik Cube

Previously, the possible permutations of the Rubik Cube are calculated based on 4 factors:

- The positions of corner cubies
- The orientations of corner cubies
- The positions of edge cubies
- The orientations of edge cubies

Because not all the 519 quintillion positions are valid, therefore, I need to find the valid configurations by eliminating the invalid configurations.

I will start with the corner cubies. Using Janet Chen's notation, I label each face of each corner cubie as in figure 3:

1 on the u face of the ufl cubicle
2 on the u face of the urf cubicle
3 on the u face of the ubr cubicle
4 on the u face of the ulb cubicle
5 on the d face of the dbl cubicle
6 on the d face of the dlf cubicle
7 on the d face of the dfr cubicle
8 on the d face of the drb cubicle

Figure 3: Cubicles with labels from 1 to 8

Notes. Retrieved from *Group Theory and the Rubik's Cube* by Chen J.

If I flatten the Rubik Cube on 2D paper sheet and draw the down face, I have Figure 4:

	f	f	f	
l	d	d	d	r
l	d	d	d	r
l	d	d	d	r
	b	b	b	

Figure 4: A down face of the cube

After labelling each face of each corner cubie, I will have Figure 5:

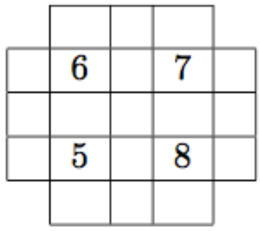

Figure 5: A down face of a labelled cube

By going clockwise, I label the cubie face 0, the first face is 1, the other face is 2. Hence, I have figure 6:

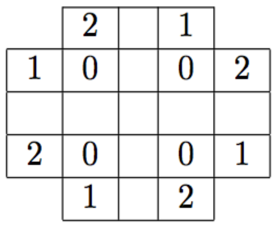

Figure 6: A down face with label of 0,1,2.

By labeling this, when I apply a move, I can identify the number in each cubicle. For instance, if I apply a move R, the cubicles in left face won't be affected. Hence, I will have $x_1 = 0, x_4 = 0, x_5 = 0, x_6 = 0$. While the cubicles in right face will have $x_2 = 1$, $x_3 = 2, x_7 = 2, x_8 = 1$. Therefore, I will have $x =$ (0,1,2,0,00,0,2,1) after applying move R.

Similarly, 12 edge cubicles will be labeled. Hence, it is proved that a move [D, R] will result in $y =$ (0,0,0,0,0,0,0,0,0,0,0,0)

Lets define a map $\phi_{corner}: \mathbb{G} \rightarrow S_8$. Hence, any moves applied on $\mathbb{G}$ will also rearrange the corner cubies. Let $\phi_{corner}(M) = \sigma$. This mean $\phi_{corner}(M)$ is an element that belong to S_8.

In other words, $\phi_{corner}(M)$ illustrate what move M does to the corner cubies. Similarly, lets define another homomorphism $\phi_{edge}: \mathbb{G} \rightarrow S_{12}$. Let $\phi_{corner}(M) = \Gamma$. This means $\phi_{corner}(M) \in S_{12}$, where it illustrate what move M does to the edge cubies. Finally, let a cube homomorphism define as $\phi_{cube}: \mathbb{G} \rightarrow S_{20}$, the permutations of 20 edge and corner cubies.

Sign homomorphism

Let define even permutation as permutation in S_n which is a product of even number of 2-cycles.

Let define odd permutation as permutation in S_n which is a product of odd number of 2-cycles

Let $p(x_1, \dots, x_n)$ be the polynomial of n variables $x_1, \dots, x_n$

If $n = 1$, $p(x_1)$ is a polynomial of variable x_1, hence, $p(x_1)$ is the sum of terms ax_1^i

If $n = 2$, $p(x_2)$ is a polynomial of variable x_2, and $p(x_2)$ look like $ax_1^i x_2^j$

In general, $p(x_1, \dots, x_n)$ is the sum of terms that look like $ax_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$

The alternative group

A product of 1 even permutation and 1 even permutation is even.

A product of 1 even permutation and 1 odd permutation is odd.

A product of 1 odd permutation and 1 odd permutation is odd.

Inverse of even permutation is even.

Inverse of odd permutation is odd.

Group actions

Assuming a Rubik's Cube has a configuration $C = (\sigma, \tau, x, y)$, a move of $M \in \mathbb{G}$ will put the Rubik's Cube into a new configuration of $C \cdot M$.

For notation, let S_1 and S_2 are two sets, then $S_1 \times S_2$ is the set of ordered pairs (s_1, s_2), in which $s_1 \in S_1$ and $s_2 \in S_2$.

A group action of a group $(G,*)$ on a set A is a map $A \times G \to A$ (which another element of A can be produced and written as $a \cdot g$), which satisfying the following properties:

- $(a \cdot g_1) \cdot g_2 = a \cdot (g_1 * g_2)$ for all $g_1, g_2 \in G$ and $a \in A$
- $a \cdot e = a$ for all $a \in A$.

When G acts on set A, the orbit of $a \in A$ is the set $\{a \cdot g: g \in G\}$

Therefore, when G acts on set configurations A, the orbit of the start configuration under this action is exactly set of valid configurations of the Rubik's Cube.

Valid configurations

A configuration (σ, τ, x, y) of a Rubik's Cube is considered valid if it satisfying these properties:

1. $sgn\sigma = sgn\tau$
2. $\sum x_i \equiv 0(mod3)$ and $\sum y_i \equiv 0(mod2)$

Now I can start to prove each property one by one.

If (σ', τ', x', y')=$(\sigma, \tau, x, y) \cdot M$ where M is one of the 6 basic moves, then $(sgn\sigma)(sgn\tau) = (sgn\sigma')(sgn\tau')$. However, $\sigma' = \sigma\phi_{corner}(M)$ and , $\tau' = \tau\phi_{edge}(M)$, therefore, $(sgn\sigma')(sign\tau') = (sgn\sigma)(sgn\phi_{corner}(M))(sgn\tau)(sgn\phi_{edge}(M))$. Also because M is one of the basic moves, $\phi_{corner}(M)$ and $\phi_{edge}(M)$ are both 4-cycles, so they have sign -1. Thus, $(sgn\sigma')(sgn\tau') = (sgn\sigma)(sgn\tau)$.

Thus, it is proved that if configuration (σ, τ, x, y) valid, then $sgn\sigma = sgn\tau$

Prove the second property: if configuration (σ, τ, x, y) valid, hence $\sum x_i \equiv 0(mod3)$ and $\sum y_i \equiv 0(mod2)$. According to Janet Chen, following steps are required:

1. When (σ, τ, x, y) is a configuration where $sgn\sigma = sgn\tau$, $\sum x_i \equiv 0(mod3)$ and $\sum y_i \equiv 0(mod2)$, it exists a move $M \in \mathbb{G}$ such that $(\sigma, \tau, x, y) \cdot M$ has the form of $(1, \tau', x', y')$ with $sgn\tau' = 1$, $\sum x'_i \equiv 0(mod3)$ and $\sum y'_i \equiv 0(mod2)$. This means all the corner cubies are in the initial position.
2. When $(1, \tau, x, y)$ is a configuration such that $sgn\tau = 1$, $\sum x_i \equiv 0(mod3)$ and $\sum y_i \equiv 0(mod2)$, it exists a move $M \in \mathbb{G}$ such that $(1, \tau, x, y) \cdot M$ has the form of $(1, \tau', 0, y')$ with $sgn\tau' = 1$, and $\sum y'_i \equiv 0(mod2)$. This means all the corner cubies are in the initial orientation.
3. When $(1, \tau, 0, y)$ is a configuration such that $sgn\tau = 1$, $\sum y_i \equiv 0(mod2)$, it exists a move $M \in \mathbb{G}$ such that $(1, \tau, 0, y) \cdot M$ has the form of $(1,1,0, y')$ with $\sum y'_i \equiv 0(mod2)$. This means all the edges can be put in the initial positions and other corner cubies are unaffected.
4. When configuration $(1,1,0, y)$ with $\sum y_i \equiv 0(mod2)$, it exists a move $M \in \mathbb{G}$ such that $(1,1,0, y) \cdot M = (1,1,0,0)$, which is also the initial configuration.

In short, when (σ, τ, x, y) is a configuration with $sgn\sigma = sgn\tau$, $\sum x_i \equiv 0(mod3)$ and $\sum y_i \equiv 0(mod2)$, the orbit (σ, τ, x, y) contains some configurations in form $(1, \tau', x', y')$.

When $(1, \tau, x, y)$ is a configuration with $sgn\tau = 1, \sum x_i \equiv 0(mod3)$ and $\sum y_i \equiv 0(mod2)$, the orbit of $(1, \tau, x, y)$ contains some configurations in form $(1, \tau', 0, y')$.

When $(1, \tau, 0, y)$ is a configuration such that $sgn\tau = 1, \sum y_i \equiv 0(mod2)$, the orbit of $(1, \tau, 0, y)$ contains some configurations in form $(1,1,0, y')$.

When $(1,1,0, y)$ is a configuration with $\sum y_i \equiv 0(mod2)$, the orbit $(1,1,0, y)$ contains the start configuration (1,1,0,0).

After the proof using group theory, it shows that even though there are about 519 quintillion possible positions, on $\frac{1}{12}$ of those are possible. This leads to the permutation of possibilities is:

$$\frac{8! \times 3^8 \times 12! \times 2^{12}}{12} = 42{,}252{,}003{,}274{,}489{,}856{,}000 \approx 4.3 \times 10^{19}$$

Although the number of possible positions has been reduced to 4.3×10^{19}, it cannot be denied that this is still a very big number. Therefore, in order to find the optimum solution, the "God's number", the lower bound and upper bound must be investigated.

The God's number

A face turn is a rotation of a Rubik's Cube face. A quarter turn is a rotation of a face by 90°. A rotation of a face by 90° and 180° will result in a face turn. However, a rotation of a face by 90° will result in a quarter turn while a rotation of a face by 180° will result in 2 quarter turns.

Let the set of quarter turn and face turn denote as:

$$S_{QT} = \{U, U^{-1}, L, L^{-1}, F, F^{-1}, R, R^{-1}, B, B^{-1}, D, D^{-1}\}$$

$$S_{FT} = \{U, U^2, U^{-1}, L, L^2, L^{-1}, F, F^2, F^{-1}, R, R^2, R^{-1}, B, B^2, B^{-1}, D, D^2, D^{-1}\}$$

Because the Rubik's cube has the symmetry characteristic, hence if $s \in S$, $s^{-1} \in S$. This shows that $\{U, L, F, R, B, D\} \subset S_{QT} \subset S_{FT} \subset G$. Therefore, $G = \langle S_{QT} \rangle = \langle S_{FT} \rangle$.

Let g define as $g = s_1, s_2, \dots, s_k$ where g is the list of moves that can be used to solve the Rubik's cube, k is the number of moves and $s_i \in S$. In other words, for each position $g \in G \setminus \{e\}$, it can be solved in k moves.

Lets define "distance" as the minimum number of quarter turn or half turn moves required to restore the Rubik's cube to the initial position. For each position $g \in G$, the "distance" from g to the original position is noted as $d = d_{QT}(g)$ in quarter turn and $d = d_{FT}(g)$ in face turn, where $g = s_1, s_2, \dots, s_k$ and $k \geq 0$. For instance, $d_{QT}(R) = 1$ which means it require 1 move to return the Rubik's Cube back to original position, and that move is R'. However, $d_{FT}(g)$ is different from $d_{QT}(g)$ at which $d_{QT}(R) = 1, d_{QT}(R^2) = 2$ while $d_{FT}(R) = 1, d_{QT}(R^2) = 1$ because $R^2 \in S_{FT}$. Hence:

$$d_{QT}(g) \geq \ d_{FT}(g)$$

where $d_{QT}(g)$ is called "quarter turn metric" and $d_{FT}(g)$ is called the "face turn metric".

The God's number will be the maximum value for these metrics.

By using the Cayley graph, it can be easily visualized $G = \langle S \rangle$ with the graph $\Gamma = (\mathrm{V}, \mathrm{E})$, where:

- The set of V of vertices represents the elements of G
- The set of E of edges is the subset of $G \times G$ defined by $(g, h) \in E$ if and only if there is $s \in S$ such that $g = s * h$.

However, a graph with V verticles that represent the elements of G is incredibly very huge, where there will be 4.3×10^{19} vertices. Hence, it will be simpler to use the Cayley graph of the subset groups instead.

An example of a Cayley graph is shown in figure 7:

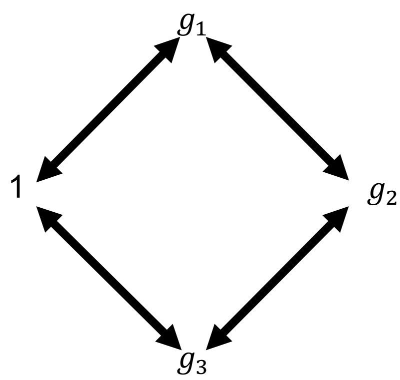

Figure 7: The Cayley graph representing a subgroup generated by a single move

According to David Joyner, the Cayley graph of Γ_{QT} will has 12 edges incident to each vertex. While the Cayley graph of Γ_{FT} will has 18 edges to each vertex.

Note that the God's number is the distance from the original state of Rubik's cube, or in the Cayley Graph, the God's number is the diameter of the graph Γ. This Cayley graph has become the most important factor that helped the mathematicians to discover the God's number.

Lower bound and Upper bound

A lower bound is the minimum number of moves required to solve a Rubik's Cube

An upper bound is the maximum number of possible moves required to solve a Rubik's Cube without repeating previous moves.

Hence, to find the God's number, the lower bound and upper bound have to be calculated and proved to be the exact same number.

In July 1981, Thistlethwaite had discovered the algorithm which allow him to solve the Rubik's cube by the order of groups. In his algorithm, he defined that:

$G_0 = \langle L, R, F, B, U, D \rangle$

$G_1 = \langle L, R, F, B, U^2, D^2 \rangle$

$G_2 = \langle L, R, F^2, B^2, U^2, D^2 \rangle$

$G_3 = \langle L^2, R^2, F^2, B^2, U^2, D^2 \rangle$

$G_4 = \{1\}$

Where he noted:

- G_o contain all the possible positions of Rubik's cube
- G_1 contain all the possible position that apply only quarter turn for left, up, front, back face but only half turn for up and down face.
- G_2 contain restricted so positions with only half turn for front, back, up and down face while quarter turn for only left and right face
- G_3 contain positions that can only be solved using only half turn
- G_4 is the position of the Rubik's cube in which it is solved.

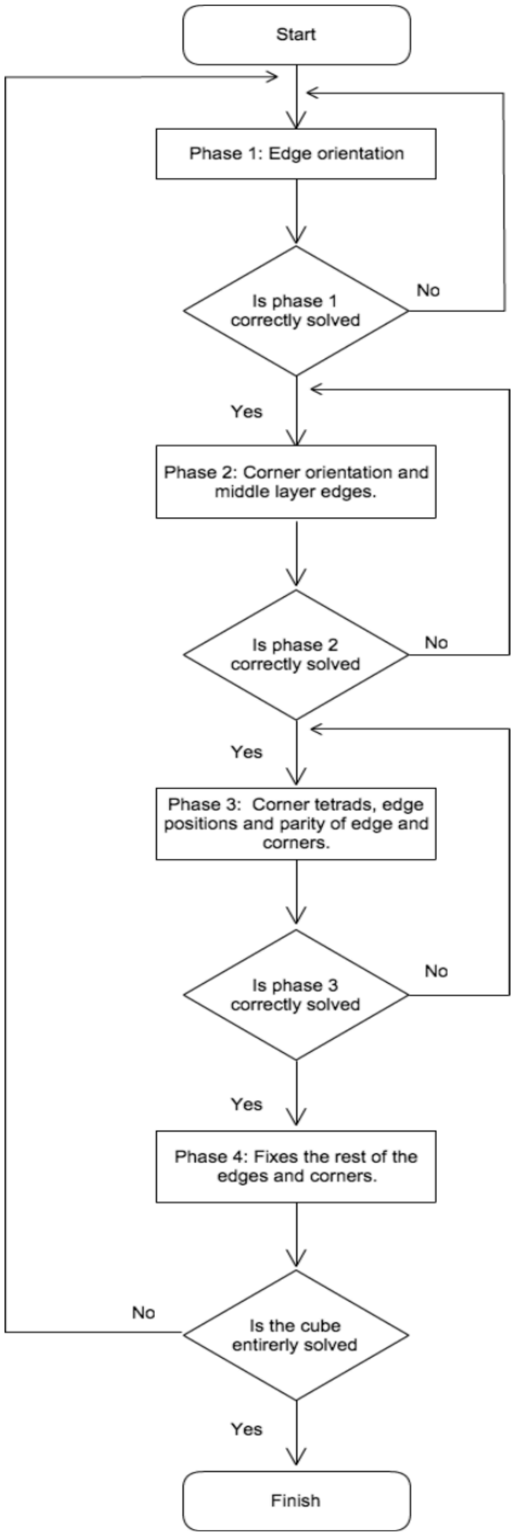

Figure 8: Thistlethwaite's algorithm

Note. Retrieved from *Algorithms for solving the Rubik's cube* by Kaur, H., 2015

By solving phase by phase, Thistlethwaite had put the first lower bound and upper bound as 18 and 52 respectively.

In December 1990, Hans Kloosterman improved the upper bound to 42 moves.

In May 1992, Michael Reid improved the upper bound to 39 moves. The following day, Dik Winter improved the upper bound to 37 moves.

In January 1995, by using Kociemba's two phase algorithm, Michael Reid had given a revolution to the upper bound and lower bound. The Kociemba's two phase algorithm is based on Thistlethwaite's algorithm, however, it only used two phase: first phase move is from G_o to $G_1 = \langle L, R, F^2, B^2, U^2, D^2 \rangle$, and the second phase is from G_1 to $G_2 = 1$. While two phase were too large for a table to be built, with the help from programming, the phases had actually been calculated and their maximum length was found to be between 12 and 18 moves, hence the cube can be solved in maximum of 30 moves using half turn metric. However, it was also proved that the positions that required 29 and 30 moves can be avoided, leading to conclusion that any possible arrangements can be solved in 28 moves. This discovery allowed the upper bound to be improved to 28 moves.

By that time, Michael Reid also discovered "super flip" position, which allow the lower bound to be improved. The "super flip" position can be modified by a set of moves:

$$R, L, U^2, F, U', D, F^2, R^2, B^2, L, U^2, F', B', U, R^2, D, F^2, U, R^2,$$

The "super flip" position is the position where all the 20 movable cubies are in the correct position, the corner cubies and the edge cubies are in the right position, however, the visible face of these edge was "flipped". The "super flip" position is shown in figure 9:

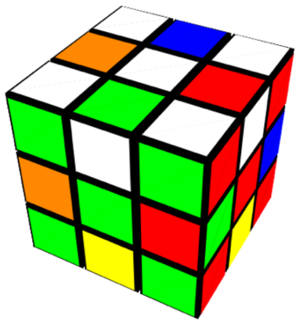

Figure 9: The "super flip" position

This "super flip" position was the first position that was found in as the position which required no less than 20 moves. This made the lower bound to be improved to 20 moves.

On December 2005, Silviu Radu proved that the upper bound required only 28 moves. On the following year, he continued to improve the upper bound to 27 moves.

By May 2007, Dan Kunkle and Gene Cooperman had proved the upper bound to be 26 moves.

In March 2008, Tomas Rokicki designed a program that allowed him to find solutions of distance 20 or less at the rate of more than 16 million positions per second. This cut the upper bound to 25 moves. By the following month, Tomas and his colleague, John Welborn, succeeded in improving the upper bound to 23 moves. By August, they improved the upper bound and proved that the Rubik's cube can be solved in less than 22 moves. And finally, by July 2010, Tom Rokicki and his colleagues proved the upper bound is 20, leaving the gap between lower bound and upper bound equals zero.

A table summary the discovery of the bounds is shown in Table 1:

Date	Lower bound	Upper bound	Mathematician
July, 1981	18	52	Morwen Thistlethwaite
December, 1990	18	42	Hans Kloosterman
May, 1992	18	39	Michael Reid
May, 1992	18	37	Dik Winter
January, 1995	18	29	Michael Reid
January, 1995	20	29	Micheal Reid
December, 2005	20	28	Silviu Radu
April, 2006	20	27	Silviu Radu
May, 2007	20	26	Dan Kunkle and Gene Cooperman
March, 2008	20	25	Tomas Rokicki
April, 2008	20	23	Tomas Rokicki and John Welborn
August, 2008	20	22	Tomas Rokicki and John Welborn
July, 2010	20	20	Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge

Table 1: A table showing the history of improvement of lower bound and upper bound

Note. Rokicki, T., Kociemba, H., Davidson, M., Dethridge, J., Garron, L., (n.d) cube 20, adapted from http://cube20.org

The improvement of upper bound and lower bound also displayed in figure 10:

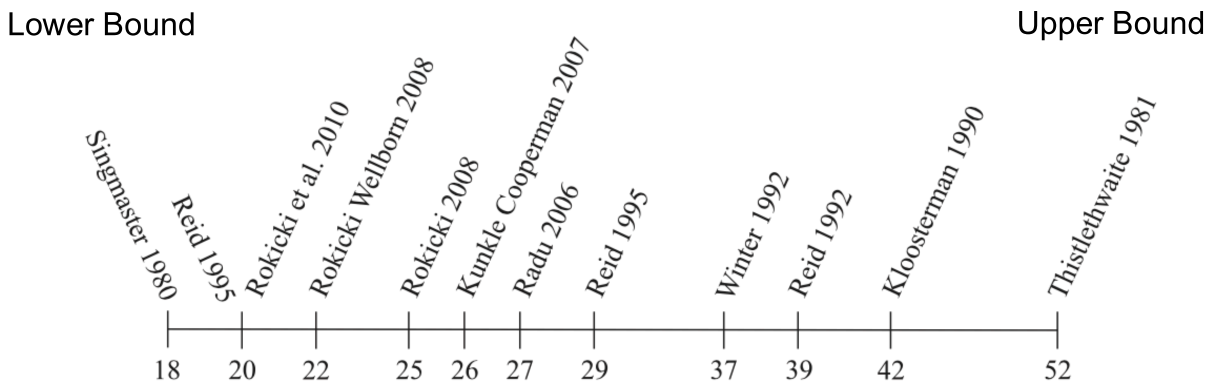

Figure 10: Names and date of progress of improving the lower bound and upper bound

Note. Adapted from *The College Mathematics Journal* (p. 264), by Joyner D. Copyright by THE MATHEMATICAL ASSOCIATION OF AMERICA

Another table show the number of positions that correspond with the distance is shown in table 2:

Distance	Count of Positions
0	1
1	18
2	243
3	3,240
4	43,239
5	574,908
6	7,618,438
7	100,803,036
8	1,332,343,288
9	17,596,479,795
10	232,248,063,316
11	3,063,288,809,012
12	40,374,425,656,248
13	531,653,418,284,628
14	6,989,320,578,825,358
15	91,365,146,187,124,313
16	about 1,100,000,000,000,000,000
17	about 12,000,000,000,000,000,000
18	about 29,000,000,000,000,000,000
19	about 1,500,000,000,000,000,000
20	about 490,000,000

Table 2: Number of positions correspond with each distance

Note. Rokicki, T., Kociemba, H., Davidson, M., Dethridge, J., Garron, L., (n.d) cube 20. Retrieved from http://cube20.org

Conclusion:

"When you learn to solve it the first time you either get the bug or not."

— Billy Jeffs (2014)

Through out my research, it is proved that a position of a Rubik's cube can be solved in many different moves. An optimum move is a move that won't repeat any cycles and positions. Hence, if an optimum move is not used, a repetition of a similar position will be achieved. Once this happen, it was seen as "get a bug" and therefore, the player needs to go back to previous position to apply another move. This is the technique that many mathematicians and programmers, including Thomas Rokicki, had used to calculated the the God's number.

In a programming language, a move is known as "distance" where in a particular position, a "distance" of 1 means that position requires one move to solve the position. By calculating the positions which correspond with the "distance", the mathematicians could write a program that made computers examine through all the possible positions to find out the "shortest distance". Once the computer defined that it wasn't the "shortest distance", the computer would "get a bug" and did the examination again with another possible positions. After years of investigation and trials using computers, the answer for my research question, the God's number, is finally identified as 20 moves.

Evaluation

Throughout the research, I have found out that most of the mathematics used in finding the minimum moves required to solve the Rubik's Cube aren't applied in calculating the distance, since the number of distances are too large for a human to discover. Hence a program will be typed in order for a computer to calculate the "distance". On the other hand, most of the mathematics are used in finding the possible positions and in proving those positions to be valid. Permutations will be used to find possible configurations and group theory will be used to find valid configurations.

References:

Castella, T. (2014, April 28). The people who are still addicted to the Rubik's Cube. *BBC News Magazine.* Retrieve from http://www.bbc.com/news/magazine-27186297

Chen J. *Group Theory and the Rubik's Cube*, Notes.

Joyner, D. (2014). The Man Who Found God's Number, *The College Mathematics Journal, 45,* 258-266. Retrieved from https://www.maa.org/sites/default/files/pdf/upload_library/2/Joyner-CMJ-2015.pdf

Kaur, H. (2015). *Thistlewaite's algorithm* [Image]. Retrieved from http://www.diva-portal.org/smash/get/diva2:816583/FULLTEXT01.pdf

Reid, M. (1995, January 18). Superflip requires 20 face turns. Retrieved from http://www.math.rwth-aachen.de/~Martin.Schoenert/Cube-Lovers/michael_reid__superflip_requires_20_face_turns.html

Rokicki, T. (2008, April 29). Twenty-Three Moves Suffice. Retrieved from http://cubezzz.dyndns.org/drupal/?q=node/view/117

Rokicki, T. (2008, March 24). Twenty-Five Moves Suffice for Rubik's Cube. Retrieved from https://arxiv.org/abs/0803.3435

Rubik's Cube Basic [Image]. (2012). Retrieved from http://images.mocpages.com/user_images/36781/12666362052_SPLASH.jpg

Scherphuis, J. (n.d.). Computer Puzzling, *Japp's Puzzle Page.* Copyright 1999-2015. Retrieved from https://www.jaapsch.net/puzzles/compcube.htm#kocal

Silviu. (2005, December 22). Solving Rubik's cube in 28 face turns. Retrieved from http://cubezzz.dyndns.org/drupal/?q=node/view/37

Silviu. (2006, April 01). Rubik can be solved in 27f. Retrieved from http://cubezzz.dyndns.org/drupal/?q=node/view/53

Singmaster, D. (1982, Deccember 23-30). Moral and mathematical lessons from a Rubik cube. *New Scientist,* 786-791