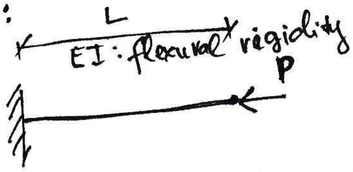
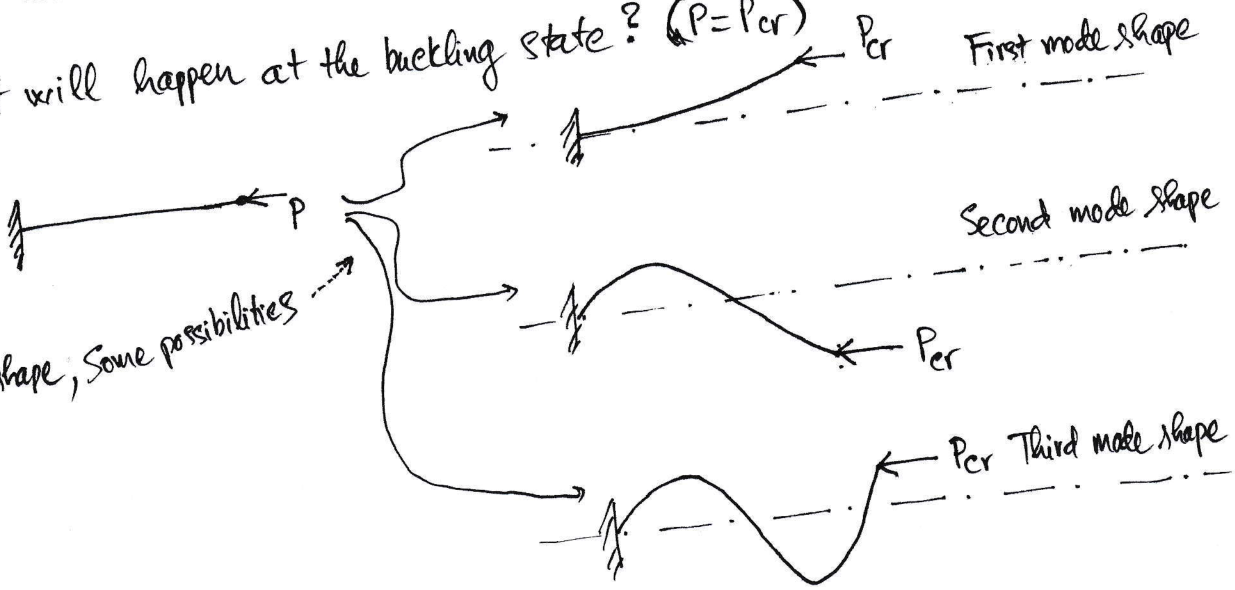
TOPIC 1. BUCKLING OF BEAMS

1. Definition:
   - Buckling is instability that leads to a failure mode
   - It happens when the structure or elements are subjected to a compressive load or stress, especially when the structure has slender members such as a beam

2. Phenomenon description:



   - Assume that the axial rigidity of the beam (EA) is infinitely large, so the beam does not undergo any stretching
   - Increase P continuosly until $P = P_{cr}$: buckling phenomenon occurs
   - $P_{cr} = \frac{\pi^2 EI}{4L^2}$: critical buckling load $\therefore L\uparrow, P\downarrow$
   - What will happen at the buckling state? ($P = P_{cr}$)



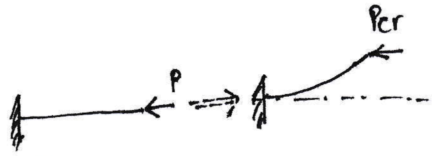
→ The beam suddenly changes its shape, Some possibilities

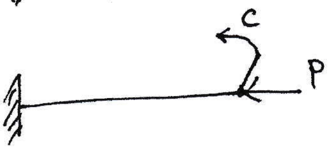
3. Target of this topic:
   - Understand buckling phenomenon
   - Can calculate P_{cr} and predict buckling mode shape of beam or column

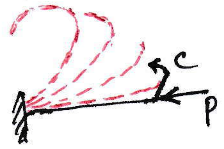
4. End shortening due to compressive load: (Key section)

- Classify two problems:

Ⓐ 

: buckling analysis 

Ⓑ 

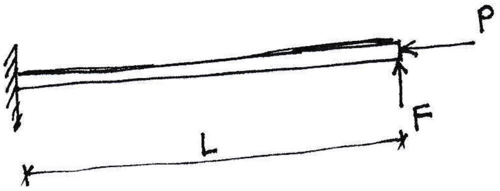
: bending analysis 

- Buckling analysis: increase P until $P = P_{cr}$, the beam suddenly changes its shape
- Bending analysis: increase P and C ~~are constants~~, the beam continuosly changes its shape right after applying the load.
- Problem A (buckling analysis) is a special case of problem B when $C = 0$
- Problem A (buckling analysis) is a special case of problem B when vertical forces are zeros we

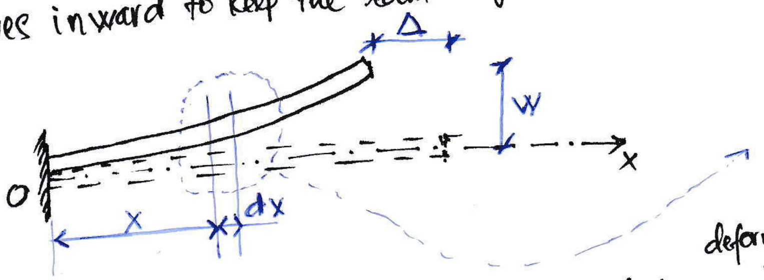
$\Rightarrow$ we establish formulation ~~of~~ of problem B, when vertical forces are zeros we can obtain formulation of problem A.

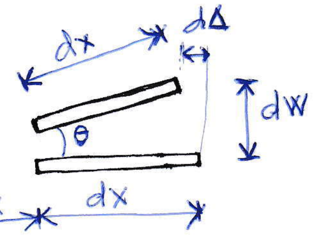
- Now, we calculate end shortening of beam/column:

⊛ Assume: the beam does not undergo any stretching due to high axial rigidity



Under P and F, the tip moves inward to keep the beam length constant:





Instead of looking at deformation of whole structure we look at the deformation of an infinitesimal length dx

End shortening of an element, $\Delta_e = ?$

θ is rotation of element

$$dx = dx\cos\theta + d\Delta$$

$$\begin{aligned} d\Delta &= dx - dx\cos\theta \\ &= dx(1-\cos\theta) \\ &= dx \cdot 2\sin^2\frac{\theta}{2} \\ &= 2dx\sin^2\frac{\theta}{2} \qquad (1) \end{aligned}$$

We use the assumption: small rotation and large deflection. Therefore:

Rotation: $\theta \approx \sin\theta \approx \dfrac{dw}{dx}$

$$\frac{\theta}{2} \approx \sin\frac{\theta}{2} = \frac{1}{2}\cdot\frac{dw}{dx}$$

$$\sin^2\frac{\theta}{2} = \frac{1}{4}\left(\frac{dw}{dx}\right)^2 \qquad (2)$$

$(2) \xrightarrow{\text{into}} (1)$: $\quad d\Delta = 2dx\cdot\dfrac{1}{4}\left(\dfrac{dw}{dx}\right)^2$

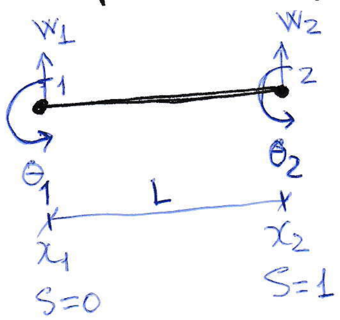
$= \dfrac{1}{2}\left(\dfrac{dw}{dx}\right)^2 dx$: end shortening of an infinitesimal length dx

For whole ~~beam~~ element: $\Delta_e = \displaystyle\int_0^L d\Delta = \frac{1}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx$; L: length of ~~beam/column~~ element

End shortening of ~~beam~~ an element due to compressive load:

$$\boxed{\Delta_e = \frac{1}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx} \qquad (3)$$

Remind: Finite element interpolation using Hermite functions



$$s = \frac{x - x_1}{L}$$

$$L\,ds = dx$$

$$w(s) = N_1(s)w_1 + N_2(s)\theta_1 + N_3(s)w_2 + N_4(s)\theta_2$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}$$

$$w(s) = \mathbf{N}\mathbf{w}_e \qquad (4)$$

$$\frac{dw}{ds} = \begin{bmatrix} N_1' & N_2' & N_3' & N_4' \end{bmatrix}\mathbf{w}_e$$

$$= \mathbf{N}'\mathbf{w}_e \qquad (5)$$

N_1, N_2, N_3, N_4 are Hermite functions of s

Interpolate end shortening Δ_e using Hermite functions :

In (3) : $\Delta_e = \frac{1}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx$

$$= \frac{1}{2}\int_0^1 \left(\frac{dw}{L\,ds}\right)^2 L\,ds$$

$$= \frac{1}{2}\cdot\frac{1}{L}\int_0^1 \left(\frac{dw}{ds}\right)^2 ds$$

$$= \frac{1}{2}\cdot\frac{1}{L}\int_0^1 (\mathbf{N}'\mathbf{w}_e)(\mathbf{N}'\mathbf{w}_e)\,ds$$

$$= \frac{1}{2}\cdot\frac{1}{L}\int_0^1 (\mathbf{w}_e^T\mathbf{N}'^T)(\mathbf{N}'\mathbf{w}_e)\,ds$$

$$= \frac{1}{2}\cdot\mathbf{w}_e^T\cdot\frac{1}{L}\int_0^1 \mathbf{N}'^T\mathbf{N}'\,ds\cdot\mathbf{w}_e$$

$$\Rightarrow \boxed{\Delta_e = \frac{1}{2}\mathbf{w}_e^T\cdot\mathbf{K}_{e,inc}\cdot\mathbf{w}_e} \qquad (6)$$

$$\mathbf{K}_{e,inc} = \frac{1}{L}\int_0^1 \mathbf{N}'^T\mathbf{N}'\,ds$$

$$= \frac{1}{L}\int_0^1 \begin{bmatrix} N_1' \\ N_2' \\ N_3' \\ N_4' \end{bmatrix}\begin{bmatrix} N_1' & N_2' & N_3' & N_4' \end{bmatrix} ds \qquad (7)$$

Home work 1: Calculate all elements of $\mathbf{K}_{e,inc}$ in (7) (perform 16 intergrations)

Result:

$$\mathbf{K}_{e,inc} = \frac{1}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$$

- Calculate energies:
  - Strain energy : $\mathbf{U}$
  - External work done by F : $\mathbf{W}$
  - External work done by $\mathbf{P}$: $\mathbf{V}$

External work of element done by P

$$V_e = P_e \cdot \Delta_e$$

$$= P_e \cdot \frac{1}{2} \mathbf{w}_e^T \mathbf{K}_{e,inc} \mathbf{w}_e$$

$$= \frac{P_e}{P_{cr}} \cdot P_{cr} \cdot \frac{1}{2} \mathbf{w}_e^T \mathbf{K}_{e,inc} \mathbf{w}_e \text{ ; } P_{cr} \text{ : critical buckling load}$$

$$V_e = P_{cr} \cdot \frac{1}{2} \mathbf{w}_e^T \frac{P_e}{P_{cr}} \mathbf{K}_{e,inc} \mathbf{w}_e$$

Assemble elements:

$$\mathbf{V} = \sum_{e=1}^{N} V_e = \sum_{e=1}^{N} P_{cr} \cdot \frac{1}{2} \mathbf{w}_e^T \frac{P_e}{P_{cr}} \mathbf{K}_{e,inc} \mathbf{w}_e$$

$$V = \frac{1}{2} P_{cr} \cdot \mathbf{w}^T \mathbf{K}_{inc} \mathbf{w}$$

$$\mathbf{K}_{inc} = \sum \frac{P_e}{P_{cr}} \cdot \mathbf{K}_{e,inc}$$

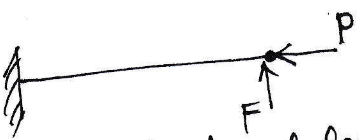
Total ~~strain~~ potential energy of an element:

$$\Pi = U - W - V$$

$$\Pi = \frac{1}{2} \mathbf{w}^T \mathbf{K} \mathbf{w} - \mathbf{w}^T \mathbf{f} - \frac{1}{2} \mathbf{w}^T P_{cr} \mathbf{K}_{inc} \mathbf{w}$$

FEM solutions are obtained when Lagrangean principle is satisfied:

$$\frac{\partial \Pi}{\partial \mathbf{w}} = \mathbf{0} \Leftrightarrow \boxed{(\mathbf{K} - P_{cr} \mathbf{K}_{inc}) \mathbf{w} = \mathbf{f}}$$: for vertical and axial loads (compressive)



if P is tensile load: $\boxed{(\mathbf{K} + P_{cr} \mathbf{K}_{inc}) \mathbf{w} = \mathbf{f}}$

if F=0 (no vertical load): $\boxed{(\mathbf{K} - P_{cr} \mathbf{K}_{inc}) \mathbf{w} = \mathbf{0}}$: buckling analysis
- critical buckling load
- mode shapes

if P=0 : $\boxed{\mathbf{K} \mathbf{w} = \mathbf{f}}$