

# Existence of a Nontrivial Markov Chain on $S_n(\lambda)$ with Interpolation ASEP Stationary Distribution at $q = 1$ : A Two-Block Gibbs Sampler via BDW Signed Multiline Queues

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## Abstract

Let  $\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0)$  be a partition with distinct parts, exactly one part equal to 0, and no part equal to 1. Under a positivity assumption on the parameters  $(\mathbf{x}, t)$  (Assumption 2.3, for which we identify a conjectured sufficient condition), we prove the existence of a nontrivial Markov chain on the orbit  $S_n(\lambda)$  whose stationary distribution is

$$\pi(\mu) = \frac{F_\mu^*(\mathbf{x}; 1, t)}{P_\lambda^*(\mathbf{x}; 1, t)},$$

where  $F_\mu^*$  and  $P_\lambda^*$  are the interpolation ASEP and interpolation Macdonald polynomials of Ben Dali–Williams (BDW) [2] at  $q = 1$ . The chain is a two-block Gibbs sampler: we introduce the Row 2 word  $\nu \in S_n(\lambda)$  of a BDW signed multiline queue as an auxiliary variable, define a joint distribution  $\tilde{\pi}(\nu, \mu) = Z(\nu, \mu)/P_\lambda^*$  via the BDW weight factorization  $Z(\nu, \mu) = U(\nu) \cdot B(\nu, \mu)$ , and alternate between sampling  $\nu$  given  $\mu$  and sampling  $\mu'$  given  $\nu$ . Stationarity follows from the standard Gibbs-sampler invariance principle; the  $\mu$ -marginal of  $\tilde{\pi}$  equals  $\pi$  by the BDW partition-function theorem. The transition probabilities are defined using only local interface weights  $U(\nu)$  and  $B(\nu, \mu)$ ; the identity  $\sum_\nu U(\nu)B(\nu, \mu) = F_\mu^*$  is a *theorem* invoked only in the proof of stationarity, not in the definition of the chain.

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# 1 Introduction and statement of the problem

## 1.1 The problem

Fix  $n \geq 2$  and a partition  $\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0)$  with distinct parts, exactly one part equal to 0, and no part equal to 1. Let  $S_n(\lambda)$  be the set of all permutations of the parts of  $\lambda$ . The interpolation ASEP polynomial  $F_\mu^*(\mathbf{x}; q, t)$  and the interpolation Macdonald polynomial  $P_\lambda^*(\mathbf{x}; q, t)$  are defined in [2]. At  $q = 1$ , define the probability distribution

$$\pi(\mu) := \frac{F_\mu^*(\mathbf{x}; 1, t)}{P_\lambda^*(\mathbf{x}; 1, t)}, \quad \mu \in S_n(\lambda).$$

**Problem.** Does there exist a nontrivial Markov chain on  $S_n(\lambda)$  whose stationary distribution is  $\pi$ , where “nontrivial” means the transition probabilities are *not* described using the polynomials  $F_\mu^*$ ?

## 1.2 Summary of the solution

We answer the problem affirmatively by constructing an explicit Markov kernel  $K$  on  $S_n(\lambda)$  using a *two-block Gibbs sampler*. The construction proceeds as follows:

- (i) We introduce the Row 2 word  $\nu$  of a BDW signed multiline queue (SMLQ) as an auxiliary variable, so that  $(\nu, \mu)$  ranges over  $S_n(\lambda) \times S_n(\lambda)$ .

- (ii) We define a joint distribution  $\tilde{\pi}(\nu, \mu) = Z(\nu, \mu)/P_\lambda^*$ , where  $Z(\nu, \mu) = U(\nu) \cdot B(\nu, \mu)$  is the two-row refined SMLQ partition function, factored via the BDW layered weight structure.
- (iii) The Markov kernel alternates: sample  $\nu$  given  $\mu$  (proportional to  $U(\nu)B(\nu, \mu)$ ), then sample  $\mu'$  given  $\nu$  (proportional to  $B(\nu, \mu')$ ). This is a systematic-scan Gibbs sampler for  $\tilde{\pi}$ .
- (iv) Stationarity: by the standard Gibbs-sampler invariance principle,  $\tilde{\pi}$  is preserved. The  $\mu$ -marginal of  $\tilde{\pi}$  equals  $\pi$  by the BDW partition-function theorem  $\sum_\nu Z(\nu, \mu) = F_\mu^*$ .
- (v) Nontriviality:  $U(\nu)$  and  $B(\nu, \mu)$  are local queue weights, defined from two-line and signed two-line queue mechanics. The identity  $\sum_\nu U(\nu)B(\nu, \mu) = F_\mu^*$  is a *theorem*, invoked only in the *proof* of stationarity.

### 1.3 Why the simpler AMW approach fails

Ayyer–Martin–Williams [1] construct Markov chains for (non-interpolation) Macdonald polynomials at  $q = 1$  using a single row-exchange step: they define  $K_{\text{AMW}}(\nu \rightarrow \mu) = \rho(J) \cdot a_\mu^\nu(1, t)$ , where  $a_\mu^\nu(1, t)$  are classical two-line queue coefficients and  $\rho(J)$  is an explicit product formula depending on the hole position  $J = \text{hole}(\mu)$ . Stationarity of  $\pi(\mu) \propto f_\mu(\mathbf{x}; 1, t)$  (for the non-interpolation ASEP polynomial  $f_\mu$ ) follows from a “collapse identity”  $\sum_\nu f_\nu \cdot a_\mu^\nu = (P_\lambda/W(J)) \cdot f_\mu$ . This identity exploits a circular symmetry in the (unsigned) multiline queue model.

In the BDW interpolation setting, the bottom interface includes a *signed layer* between signed Row 1' and regular Row 1, so the full bottom-interface weight is

$$B(\nu, \mu) = \sum_\alpha a_{\|\alpha\|}^\nu(1, t) \cdot \text{wt}_{\text{ball}}(\alpha; \mathbf{x}, t) \cdot G_\mu^\alpha(t), \quad (1)$$

where  $\alpha$  ranges over signed permutations,  $\|\alpha\|$  is the unsigned word, and  $G_\mu^\alpha(t)$  is the signed two-line queue weight. This is *not* proportional to  $a_\mu^\nu(1, t)$ : the signed-layer factors  $\text{wt}_{\text{ball}}(\alpha; \mathbf{x}, t) \cdot G_\mu^\alpha(t)$  depend on both  $\alpha$  and  $\mu$ , breaking the proportionality  $B(\nu, \mu) \propto a_\mu^\nu$  that would be needed for the AMW collapse identity. The collapse identity  $\sum_\nu F_\nu^* \cdot a_\mu^\nu \stackrel{?}{=} (P_\lambda^*/W(J)) \cdot F_\mu^*$  is therefore open in the interpolation setting and is expected to fail in general. Our two-block Gibbs sampler circumvents this entirely by treating the Row 2 word as an auxiliary variable and reducing stationarity to the standard Gibbs-sampler principle.

### 1.4 Conventions

All BDW references are to [2] (arXiv:2510.02587). We write  $[n] := \{1, \dots, n\}$ . We work at  $q = 1$  throughout unless stated otherwise.

## 2 Setup: state space and target distribution

**Assumption 2.1** (Standing hypotheses on  $\lambda$ ). Fix  $n \geq 2$  and a partition

$$\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_n \geq 0)$$

with:

- (H1) *Distinct parts*: all  $\lambda_i$  are pairwise distinct.
- (H2) *Exactly one zero*:  $\lambda_n = 0$  and  $\lambda_i > 0$  for  $i < n$ .

(H3) *No part equals 1*:  $\lambda_i \neq 1$  for all  $i$  (equivalently,  $m_1(\lambda) = 0$  in BDW notation, where  $m_j(\lambda) := \#\{i : \lambda_i = j\}$ ).

Set  $L := \lambda_1$ .

**Definition 2.2** (State space and hole map). Let

$$S_n(\lambda) := \{\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n : \mu \text{ is a permutation of the parts of } \lambda\}.$$

Since the parts of  $\lambda$  are distinct,  $|S_n(\lambda)| = n!$ . Each  $\mu \in S_n(\lambda)$  has a unique index  $\text{hole}(\mu) \in [n]$  with  $\mu_{\text{hole}(\mu)} = 0$  (the *hole position*), and  $\mu_i \geq 2$  for  $i \neq \text{hole}(\mu)$ . For  $J \in [n]$ , define the *hole fiber*  $S(J) := \{\mu \in S_n(\lambda) : \text{hole}(\mu) = J\}$ , so  $|S(J)| = (n-1)!$ .

**Definition 2.3** (Target distribution). Fix parameters  $t$  and  $\mathbf{x} = (x_1, \dots, x_n)$ . Let  $F_\mu^*(\mathbf{x}; q, t)$  and  $P_\lambda^*(\mathbf{x}; q, t)$  denote the interpolation ASEP polynomial and interpolation Macdonald polynomial of [2]. At  $q = 1$ , define

$$P_\lambda^*(\mathbf{x}; 1, t) := \sum_{\mu \in S_n(\lambda)} F_\mu^*(\mathbf{x}; 1, t), \quad \pi(\mu) := \frac{F_\mu^*(\mathbf{x}; 1, t)}{P_\lambda^*(\mathbf{x}; 1, t)}.$$

**Assumption 2.4** (Positivity regime). We assume  $(\mathbf{x}, t)$  lies in a regime where:

- (i)  $t \in (0, 1)$  and  $x_i > 0$  for all  $i$ ;
- (ii) the BDW signed multiline queue weights at  $q = 1$  are nonnegative;
- (iii)  $F_\mu^*(\mathbf{x}; 1, t) > 0$  for all  $\mu \in S_n(\lambda)$  (hence  $P_\lambda^*(\mathbf{x}; 1, t) > 0$ ).

**Remark 2.5** (Existence of a positivity region: status). We conjecture that there exists a nonempty region of  $(\mathbf{x}, t)$  satisfying Assumption 2.4. A conjectured explicit sufficient condition is

$$0 < t < 1, \quad x_i > t^{-(n-1)} \quad \text{for all } i \in [n]. \quad (2)$$

See Section 7 for a heuristic scaling argument supporting this conjecture, as well as a precise identification of the gaps that remain.

### 3 BDW signed multiline queues at $q = 1$ : structural input

We record the BDW results used in the proof. We state them in the precise form needed, with references to [2].

#### 3.1 SMLQ state space and weights

**Definition 3.1** (Signed multiline queues and fibers; [2, Definitions 1.4–1.14]). Let  $\mathcal{Q}(\lambda)$  denote the finite set of BDW signed multiline queues of content  $\lambda$  at  $q = 1$ . Each  $Q \in \mathcal{Q}(\lambda)$  has:

- a *bottom word*  $\text{bot}(Q) \in S_n(\lambda)$  (the labeled word on the bottom regular row);
- a weight  $\text{wt}(Q) = \text{wt}_{\text{ball}}(Q) \cdot \text{wt}_{\text{pair}}(Q)$ , a product of signed-row ball weights and layer pairing weights ([2, Definitions 1.8, 1.11; Eqs. (4)–(7)]), specialized to  $q = 1$ .

For  $\mu \in S_n(\lambda)$ , define the *fiber*  $\mathcal{Q}(\mu) := \{Q \in \mathcal{Q}(\lambda) : \text{bot}(Q) = \mu\}$ .

**Theorem 3.2** (BDW partition-function identity at  $q = 1$ ; [2, Theorem 1.15]). *For each  $\mu \in S_n(\lambda)$ ,*

$$F_\mu^*(\mathbf{x}; 1, t) = \sum_{Q \in \mathcal{Q}(\mu)} \text{wt}(Q), \quad (3)$$

*and consequently*

$$P_\lambda^*(\mathbf{x}; 1, t) = \sum_{Q \in \mathcal{Q}(\lambda)} \text{wt}(Q). \quad (4)$$

### 3.2 Key BDW definitions at $q = 1$

For the reader's convenience, we reproduce the BDW definitions that are essential for our construction, specialized to  $q = 1$ . These are simplified versions of [2, Definitions 1.8, 1.11, 5.1, 5.3].

**Definition 3.3** (Classical two-line queue coefficient  $a_\mu^\nu(1, t)$ ; [2, Definition 5.1]). Let  $\nu, \mu \in S_n(\lambda)$ . The *classical two-line queue* from  $\nu$  (top word) to  $\mu$  (bottom word) is a strand-pairing process. Write the entries of  $\nu$  in positions  $1, \dots, n$  on the top line and the entries of  $\mu$  in positions  $1, \dots, n$  on the bottom line. Strands connect top entries to bottom entries, pairing each label  $\ell$  that appears in both rows. A pairing is *trivial* if a ball of label  $a$  pairs with the ball of the same label in the same column (a straight-down segment); trivial pairings contribute weight 1. For each *nontrivial* pairing, the local weight at  $q = 1$  is  $\frac{(1-t)t^{\text{skip}}}{1-t^{\text{free}}}$  ([2, Definition 1.8, Eq. (4)]), where skip counts unpaired bottom balls the strand passes over, and free counts unpaired bottom balls of the same label ahead of the strand. The classical TLQ weight of a pairing configuration is the product of these local factors over all nontrivial pairings. The total coefficient  $a_\mu^\nu(1, t)$  is the sum of these products over all valid pairing configurations.

By [2, Lemma 7.3], for each hole position  $J$ , the coefficients satisfy  $\sum_{\mu \in S(J)} a_\mu^\nu(1, t) = 1$ , so they define a probability distribution on each hole fiber.

**Definition 3.4** (Signed-row ball weight  $\text{wt}_{\text{ball}}(\alpha; \mathbf{x}, t)$ ; [2, Definition 1.11]). Let  $\alpha$  be a signed word: a word where each entry  $\alpha_i$  carries a sign  $\varepsilon_i \in \{+, -\}$ , with unsigned value  $|\alpha_i| \in \{0, 2, 3, \dots, L\}$ . The signed-row ball weight at  $q = 1$  is

$$\text{wt}_{\text{ball}}(\alpha; \mathbf{x}, t) := \prod_{i=1}^n b_i(\alpha),$$

where

$$b_i(\alpha) := \begin{cases} x_i & \text{if } \varepsilon_i = + \text{ and } |\alpha_i| > 0 \text{ (positive ball),} \\ -t^{-(n-1)} & \text{if } \varepsilon_i = - \text{ and } |\alpha_i| > 0 \text{ (negative ball),} \\ 1 & \text{if } |\alpha_i| = 0 \text{ (no ball at position } i\text{).} \end{cases}$$

Note that negative balls contribute a negative factor, which is the source of potential sign cancellations in the SMLQ weights.

**Definition 3.5** (Signed two-line queue weight  $G_\mu^\alpha(t)$ ; [2, Definition 5.3]). Let  $\alpha$  be a signed word on signed Row 1' and  $\mu \in S_n(\lambda)$  the word on regular Row 1. The *signed two-line queue* from  $\alpha$  to  $\mu$  pairs the unsigned labels of  $\alpha$  with those of  $\mu$  via strands. The weight  $G_\mu^\alpha(t)$  is a product over strand interactions, analogous to the classical case but with additional contributions from the signs in  $\alpha$ .

At  $q = 1$ : if  $\alpha$  has all positive signs and  $\|\alpha\| = \mu$ , then  $G_\mu^\alpha(t) = 1$  (the identity configuration with no crossings; see Lemma 6.4 below). In general,  $G_\mu^\alpha(t)$  is a polynomial in  $t$  that may be zero (if the pairing from  $\alpha$  to  $\mu$  is inadmissible) or a product of crossing weights depending on the strand configuration. The full definition involves a case analysis on strand crossings; see [2, Definition 5.3, Eqs. (6)–(7)] for the complete specification.

**Remark 3.6** (What we use from these definitions). Our proofs use the following properties of these quantities:

- (i)  $a_\mu^\nu(1, t) \geq 0$  and  $\sum_{\mu \in S(J)} a_\mu^\nu(1, t) = 1$  (stochasticity by hole fiber);
- (ii)  $\text{wt}_{\text{ball}}(\alpha; \mathbf{x}, t) > 0$  when all signs are positive and  $x_i > 0$ ;
- (iii)  $G_\mu^{\alpha_0}(t) = 1$  for the identity signed permutation (Lemma 6.4);

(iv) the weight factorization  $\text{wt}(Q) = \text{wt}(Q^\uparrow) \cdot \text{wt}(Q^\downarrow)$  ([2, Lemma 5.6]).

Property (i) follows from the definition of classical TLQ weights and BDW Lemma 7.3. Property (ii) is immediate from Definition 3.4 (each positive ball contributes  $x_i > 0$ , each empty position contributes 1). Property (iii) is Lemma 6.4, proved in Section 6.4; we state it here for reference. Property (iv) is a structural result proved in [2].

### 3.3 Row 2 takes values in the same orbit

This is the key structural fact that makes the auxiliary-variable construction possible, and it is where the hypothesis  $m_1(\lambda) = 0$  is essential.

**Lemma 3.7** (Row 2 orbit). *Under Assumption 2.1 (in particular  $m_1(\lambda) = 0$ ), for every  $Q \in \mathcal{Q}(\lambda)$  the Row 2 word  $\text{row}_2(Q)$  lies in  $S_n(\lambda)$ .*

*Proof.* In the BDW SMLQ model ([2, §1.2, Definition 1.3]), regular Row  $r$  carries labels obtained from  $\lambda$  by replacing each  $\lambda_i < r$  with 0. Formally, the content at level  $r$  is

$$\lambda^{(r)} := \langle L^{m_L}, (L-1)^{m_{L-1}}, \dots, r^{m_r}, 0^{m_{r-1}+\dots+m_0} \rangle.$$

In particular:

$$\begin{aligned} \lambda^{(1)} &= \langle L^{m_L}, \dots, 2^{m_2}, 1^{m_1}, 0^{m_0} \rangle, \\ \lambda^{(2)} &= \langle L^{m_L}, \dots, 2^{m_2}, 0^{m_1+m_0} \rangle. \end{aligned}$$

Under our hypothesis  $m_1(\lambda) = 0$  (no part equals 1), the  $1^{m_1}$  term in  $\lambda^{(1)}$  vanishes, so  $\lambda^{(1)} = \lambda^{(2)}$ . Both Row 1 and Row 2 carry the same multiset of labels, namely  $\{\lambda_1, \dots, \lambda_{n-1}, 0\}$  (the parts of  $\lambda$ ). Since the parts are distinct, any word with this content is a permutation of  $\lambda$ , hence lies in  $S_n(\lambda)$ .  $\square$

**Remark 3.8** (Why  $m_1(\lambda) = 0$  is needed). If  $\lambda$  had a part equal to 1, then  $\lambda^{(2)}$  would replace each 1 by a 0, and the multiset of labels on Row 2 would *not* be a permutation of  $\lambda$ . The Gibbs-sampler construction below requires both rows to range over  $S_n(\lambda)$ , so  $m_1(\lambda) = 0$  is essential.

**Remark 3.9** (BDW row words and our notation). In BDW ([2, Definition 1.3]), the ball system is an  $L \times n$  array:  $L = \lambda_1$  rows (one per level) and  $n$  columns (arranged on a cylinder, labeled  $1, \dots, n$ ). A row word at level  $r$  is a word of length  $n$  whose entries form a permutation of the content  $\lambda^{(r)}$  ([2, Definition 1.3]). Under Assumption 2.1, the content at levels  $r \in \{1, 2\}$  is a permutation of  $(\lambda_1, \dots, \lambda_{n-1}, 0)$ —i.e., the  $n$  parts of  $\lambda$ —so each row word is an element of  $S_n(\lambda)$ . When we write “Row 2 word  $\nu \in S_n(\lambda)$ ,” this refers directly to BDW’s length- $n$  row word. Columns whose entry is 0 (the hole position) carry no ball in BDW’s terminology ([2, Definition 1.4]), but they are genuine columns in the  $n$ -column cylinder and participate in the geometry (e.g., strand wrapping).

### 3.4 The bottom-interface cut and weight factorization

The BDW SMLQ weight factors over interfaces between adjacent rows. We isolate the factorization across the “bottom interface” — the data between regular Row 2 and regular Row 1.

**Definition 3.10** (Bottom interface). For  $Q \in \mathcal{Q}(\lambda)$  with  $\text{bot}(Q) = \mu$  and  $\text{row}_2(Q) = \nu$ , the *bottom interface* of  $Q$  consists of:

- (a) the *classical layer* from regular Row 2 (word  $\nu$ ) to signed Row 1' ([2, Definition 5.1]);

- (b) the *signed-row ball weights* on Row 1' ([2, Definition 1.11, Eqs. (5)–(6)]);
- (c) the *signed layer* from signed Row 1' to regular Row 1 (word  $\mu$ ) ([2, Definition 5.3]).

Note that the signed-row ball weights on Row 1' are entirely contained in the bottom interface and do not depend on any upper-layer data. The *upper part* of  $Q$  is everything above the bottom interface: rows  $2, 2', 3, 3', \dots, L, L'$  and their interface data.

**Definition 3.11** (Upper weight  $U(\nu)$  and bottom-interface weight  $B(\nu, \mu)$ ). For  $\nu, \mu \in S_n(\lambda)$ , define:

$$U(\nu) := \sum_{\substack{\text{upper configs } Q^\uparrow \\ \text{with Row 2 word } \nu}} \text{wt}(Q^\uparrow), \quad (5)$$

$$B(\nu, \mu) := \sum_{\substack{\text{bottom-interface configs } Q^\downarrow \\ \text{Row 2 word } \nu, \text{ Row 1 word } \mu}} \text{wt}(Q^\downarrow). \quad (6)$$

Here  $\text{wt}(Q^\uparrow)$  and  $\text{wt}(Q^\downarrow)$  are the products of local BDW weights (ball and pairing) restricted to the upper part and bottom interface respectively.

**Proposition 3.12** (Weight factorization across the bottom-interface cut). *For every  $Q \in \mathcal{Q}(\lambda)$  with  $\text{bot}(Q) = \mu$  and  $\text{row}_2(Q) = \nu$ ,*

$$\text{wt}(Q) = \text{wt}(Q^\uparrow) \cdot \text{wt}(Q^\downarrow), \quad (7)$$

where  $Q^\uparrow$  and  $Q^\downarrow$  are the upper and bottom-interface parts of  $Q$ . Consequently, defining the two-row refined partition function

$$Z(\nu, \mu) := \sum_{\substack{Q \in \mathcal{Q}(\lambda): \\ \text{row}_2(Q) = \nu, \text{ bot}(Q) = \mu}} \text{wt}(Q), \quad (8)$$

we have

$$Z(\nu, \mu) = U(\nu) \cdot B(\nu, \mu). \quad (9)$$

*Proof.* The BDW SMLQ weight is a product of local factors assigned to each interface (classical or signed layer) and each signed-row ball weight ([2, §1.2–§1.3]). BDW Lemma 5.6 ([2, Lemma 5.6]) establishes that the map  $(Q^\uparrow, Q^\downarrow) \mapsto Q$  is a weight-multiplicative bijection between pairs of compatible configurations sharing the Row 2 word  $\nu$  and full SMLQs  $Q$  with  $\text{row}_2(Q) = \nu$  and  $\text{bot}(Q) = \mu$ . More precisely: every valid upper configuration  $Q^\uparrow$  with Row 2 word  $\nu$  can be glued to every valid bottom-interface configuration  $Q^\downarrow$  with Row 2 word  $\nu$  and Row 1 word  $\mu$  to produce a unique SMLQ  $Q$  with  $\text{wt}(Q) = \text{wt}(Q^\uparrow) \cdot \text{wt}(Q^\downarrow)$ , and conversely every such  $Q$  decomposes uniquely into such a pair. The two parts share only the boundary word  $\nu$ ; in particular, the signed-row ball weights on Row 1' belong entirely to  $Q^\downarrow$  and are independent of upper-layer data.

Summing over all  $Q$  with fixed  $(\nu, \mu)$  and using the bijective independence of the upper and lower configuration spaces yields

$$Z(\nu, \mu) = \left( \sum_{Q^\uparrow} \text{wt}(Q^\uparrow) \right) \cdot \left( \sum_{Q^\downarrow} \text{wt}(Q^\downarrow) \right) = U(\nu) \cdot B(\nu, \mu). \quad \square$$

### 3.5 Local computability of $B(\nu, \mu)$ and $U(\nu)$

**Proposition 3.13** ( $B(\nu, \mu)$  is computed from local BDW primitives). *The bottom-interface weight  $B(\nu, \mu)$  is a finite sum of products of local BDW two-line queue weights. Explicitly, summing over signed permutations  $\alpha$  of the content (the signed-Row 1' word):*

$$B(\nu, \mu) = \sum_{\alpha} a_{\|\alpha\|}^{\nu}(1, t) \cdot \text{wt}_{\text{ball}}(\alpha; \mathbf{x}, t) \cdot G_{\mu}^{\alpha}(t), \quad (10)$$

where:

- $\|\alpha\|$  is the unsigned word obtained from  $\alpha$ ;
- $a_{\|\alpha\|}^{\nu}(1, t)$  is the classical two-line queue coefficient ([2, Definition 5.1]), giving the weight of the classical layer from Row 2 (word  $\nu$ ) to signed Row 1' (unsigned word  $\|\alpha\|$ );
- $\text{wt}_{\text{ball}}(\alpha; \mathbf{x}, t)$  is the signed-row ball weight ([2, Definition 1.11]);
- $G_{\mu}^{\alpha}(t)$  is the signed two-line queue weight ([2, Definition 5.3]), giving the weight of the signed layer from signed Row 1' (word  $\alpha$ ) to regular Row 1 (word  $\mu$ ).

All three factors are defined from local strand-pairing statistics. No global partition function ( $F_{\mu}^*$  or  $P_{\lambda}^*$ ) appears.

*Proof.* This is the content of the BDW bottom-interface decomposition ([2, Definitions 5.1, 5.3, and Lemma 5.6]). The signed Row 1' word  $\alpha$  is the intermediate data between the classical layer (Row 2  $\rightarrow$  Row 1') and the signed layer (Row 1'  $\rightarrow$  Row 1). Summing over all compatible  $\alpha$  gives (10).  $\square$

**Proposition 3.14** ( $U(\nu)$  is computed by a local transfer-matrix recursion). *The upper weight  $U(\nu)$  can be computed by a finite transfer-matrix recursion. For  $r = L, L-1, \dots, 2$ , define partial partition functions  $U_r(w)$  as the total weight of all upper configurations whose Row  $r$  word is  $w$ . Then  $U_L(\cdot)$  is determined by the (fixed) top boundary, and*

$$U_r(w) = \sum_v U_{r+1}(v) \cdot B_r(v, w)$$

where  $B_r(v, w)$  is the interface partition function at level  $r$ , computed from local two-line and signed two-line queue weights. Finally,  $U(\nu) = U_2(\nu)$ . No evaluation of  $F_{\mu}^*$  or  $P_{\lambda}^*$  is needed.

*Proof.* This follows from the layered structure of the BDW SMLQ model ([2, §1.2–§1.3; Lemma 5.6]): the weight of the upper part factors over interfaces, and summing over internal row words yields a standard transfer-matrix contraction.  $\square$

### 3.6 Recovering $F_{\mu}^*$ from the refined partition function

**Proposition 3.15** (Marginal identities). *For all  $\mu \in S_n(\lambda)$ ,*

$$\sum_{\nu \in S_n(\lambda)} Z(\nu, \mu) = F_{\mu}^*(\mathbf{x}; 1, t), \quad (11)$$

and

$$\sum_{\nu, \mu \in S_n(\lambda)} Z(\nu, \mu) = P_{\lambda}^*(\mathbf{x}; 1, t). \quad (12)$$

*Proof.* Partition  $\mathcal{Q}(\mu)$  by the value of  $\text{row}_2(Q)$ :

$$\sum_{\nu} Z(\nu, \mu) = \sum_{\nu} \sum_{\substack{Q \in \mathcal{Q}(\mu): \\ \text{row}_2(Q) = \nu}} \text{wt}(Q) = \sum_{Q \in \mathcal{Q}(\mu)} \text{wt}(Q) = F_{\mu}^*(\mathbf{x}; 1, t),$$

where the last equality is Theorem 3.2. Summing over  $\mu$  gives (12).  $\square$

## 4 The Markov chain: a two-block Gibbs sampler

### 4.1 The joint distribution

**Definition 4.1** (Joint distribution on pairs). Define a function  $\tilde{\pi}$  on  $S_n(\lambda) \times S_n(\lambda)$  by

$$\tilde{\pi}(\nu, \mu) := \frac{Z(\nu, \mu)}{P_\lambda^*(\mathbf{x}; 1, t)} = \frac{U(\nu) \cdot B(\nu, \mu)}{P_\lambda^*(\mathbf{x}; 1, t)}. \quad (13)$$

**Lemma 4.2** (Well-defined probability distribution). *Under Assumption 2.4,  $\tilde{\pi}$  is a well-defined probability distribution:  $\tilde{\pi}(\nu, \mu) \geq 0$  for all  $(\nu, \mu)$ , and  $\sum_{\nu, \mu} \tilde{\pi}(\nu, \mu) = 1$ .*

*Proof.* Nonnegativity: under Assumption 2.4, all SMLQ weights are nonnegative, so  $Z(\nu, \mu) \geq 0$ . Normalization: by (12),  $\sum_{\nu, \mu} Z(\nu, \mu) = P_\lambda^* > 0$ .  $\square$

**Lemma 4.3** ( $\mu$ -marginal of  $\tilde{\pi}$  equals  $\pi$ ). *For all  $\mu \in S_n(\lambda)$ ,*

$$\sum_{\nu \in S_n(\lambda)} \tilde{\pi}(\nu, \mu) = \pi(\mu).$$

*Proof.* By (11),  $\sum_{\nu} \tilde{\pi}(\nu, \mu) = \sum_{\nu} Z(\nu, \mu) / P_\lambda^* = F_\mu^* / P_\lambda^* = \pi(\mu)$ .  $\square$

### 4.2 Conditionals of the joint distribution

**Lemma 4.4** (Full conditionals of  $\tilde{\pi}$ ). *The conditional distributions of  $\tilde{\pi}$  are:*

(a)  $\nu$  given  $\mu$ :

$$\tilde{\pi}(\nu \mid \mu) = \frac{Z(\nu, \mu)}{\sum_{\nu'} Z(\nu', \mu)} = \frac{U(\nu) B(\nu, \mu)}{\sum_{\nu'} U(\nu') B(\nu', \mu)}. \quad (14)$$

(b)  $\mu$  given  $\nu$ :

$$\tilde{\pi}(\mu \mid \nu) = \frac{Z(\nu, \mu)}{\sum_{\kappa} Z(\nu, \kappa)} = \frac{B(\nu, \mu)}{\sum_{\kappa} B(\nu, \kappa)}. \quad (15)$$

In (15), the common factor  $U(\nu)$  cancels.

*Proof.* Both follow from the factorization  $Z(\nu, \mu) = U(\nu) B(\nu, \mu)$  (Proposition 3.12). For part (b), the  $U(\nu)$  factor appears in both numerator and denominator and cancels.  $\square$

### 4.3 Definition of the Markov kernel

**Definition 4.5** (Two-block Gibbs sampler on  $S_n(\lambda)$ ). Define a Markov kernel  $K$  on  $S_n(\lambda)$  by the following two-step procedure from the current state  $\mu$ :

**Step 1** (*Resample Row 2 given Row 1*). Sample  $\nu \in S_n(\lambda)$  with probability

$$\mathbb{P}(\nu \mid \mu) := \frac{U(\nu) B(\nu, \mu)}{\sum_{\nu' \in S_n(\lambda)} U(\nu') B(\nu', \mu)}. \quad (16)$$

**Step 2** (*Resample Row 1 given Row 2*). Given  $\nu$ , sample the next state  $\mu' \in S_n(\lambda)$  with

$$\mathbb{P}(\mu' \mid \nu) := \frac{B(\nu, \mu')}{\sum_{\kappa \in S_n(\lambda)} B(\nu, \kappa)}. \quad (17)$$

The transition kernel is

$$K(\mu, \mu') = \sum_{\nu \in S_n(\lambda)} \mathbb{P}(\nu \mid \mu) \mathbb{P}(\mu' \mid \nu). \quad (18)$$

**Remark 4.6** (Well-defined denominators on the support). The denominators in Steps 1 and 2 are positive for all states that arise with positive probability. Indeed, if  $\mu$  has  $\pi(\mu) > 0$  then  $\sum_{\nu'} U(\nu')B(\nu', \mu) = F_\mu^*(\mathbf{x}; 1, t) > 0$  by Assumption 2.4. If  $\nu$  arises in Step 1 with positive probability, then  $U(\nu) > 0$ , and Lemma 6.5 together with Lemma 6.4 implies  $B(\nu, \nu) > 0$ , hence  $\sum_{\kappa} B(\nu, \kappa) \geq B(\nu, \nu) > 0$ .

#### 4.4 Stochasticity

**Lemma 4.7** (Stochasticity of  $K$ ). *For every  $\mu \in S_n(\lambda)$ ,  $K(\mu, \mu') \geq 0$  for all  $\mu'$ , and  $\sum_{\mu'} K(\mu, \mu') = 1$ .*

*Proof.* Nonnegativity is immediate from Assumption 2.4 (all weights are nonnegative). For stochasticity:

$$\sum_{\mu'} K(\mu, \mu') = \sum_{\nu} \mathbb{P}(\nu \mid \mu) \sum_{\mu'} \mathbb{P}(\mu' \mid \nu) = \sum_{\nu} \mathbb{P}(\nu \mid \mu) \cdot 1 = 1. \quad \square$$

### 5 Stationarity

**Theorem 5.1** (Main theorem: stationarity of  $\pi$ ). *Under Assumptions 2.1 and 2.4, the Markov kernel  $K$  of Definition 4.5 has stationary distribution*

$$\pi(\mu) = \frac{F_\mu^*(\mathbf{x}; 1, t)}{P_\lambda^*(\mathbf{x}; 1, t)}.$$

That is, for every  $\mu \in S_n(\lambda)$ ,

$$\sum_{\nu \in S_n(\lambda)} \pi(\nu) K(\nu, \mu) = \pi(\mu). \quad (19)$$

*Proof.* The proof uses the standard Gibbs-sampler / auxiliary-variable argument, applied to the joint distribution  $\tilde{\pi}$  of Definition 4.1.

**Step A: The chain is a Gibbs sampler for  $\tilde{\pi}$ .**

By Lemma 4.4, Step 1 of Definition 4.5 samples  $\nu$  from  $\tilde{\pi}(\nu \mid \mu)$ , and Step 2 samples  $\mu'$  from  $\tilde{\pi}(\mu' \mid \nu)$ . These are exactly the two full-conditional updates of a systematic-scan two-block Gibbs sampler for  $\tilde{\pi}$ .

**Step B: The Gibbs sampler preserves  $\tilde{\pi}$ .**

We verify this directly. Suppose  $\mu \sim \pi$ .

*After Step 1:* The joint distribution of  $(\nu, \mu)$  is

$$\mathbb{P}(\nu, \mu) = \pi(\mu) \cdot \tilde{\pi}(\nu \mid \mu) = \frac{F_\mu^*}{P_\lambda^*} \cdot \frac{Z(\nu, \mu)}{F_\mu^*} = \frac{Z(\nu, \mu)}{P_\lambda^*} = \tilde{\pi}(\nu, \mu), \quad (20)$$

where we used  $\sum_{\nu'} Z(\nu', \mu) = F_\mu^*$  (Proposition 3.15).

*After Step 2:* We replace  $\mu$  by  $\mu' \sim \tilde{\pi}(\cdot \mid \nu)$ . The new pair  $(\nu, \mu')$  has joint distribution

$$\begin{aligned} \mathbb{P}(\nu, \mu') &= \mathbb{P}(\nu) \cdot \tilde{\pi}(\mu' \mid \nu) \\ &= \left( \sum_{\kappa} \tilde{\pi}(\nu, \kappa) \right) \cdot \frac{Z(\nu, \mu')}{\sum_{\kappa} Z(\nu, \kappa)} \\ &= \frac{\sum_{\kappa} Z(\nu, \kappa)}{P_\lambda^*} \cdot \frac{Z(\nu, \mu')}{\sum_{\kappa} Z(\nu, \kappa)} = \frac{Z(\nu, \mu')}{P_\lambda^*} = \tilde{\pi}(\nu, \mu'). \end{aligned} \quad (21)$$

Therefore  $(\nu, \mu') \sim \tilde{\pi}$ .

**Step C: Marginal stationarity of  $\pi$ .**

Since  $(\nu, \mu') \sim \tilde{\pi}$ , the  $\mu'$ -marginal is

$$\mathbb{P}(\mu' = \mu_0) = \sum_{\nu} \tilde{\pi}(\nu, \mu_0) = \pi(\mu_0),$$

by Lemma 4.3. Therefore, if  $\mu \sim \pi$  then after one step of the chain  $\mu' \sim \pi$ . This is exactly stationarity (19).  $\square$

**Remark 5.2** (Where BDW’s partition-function theorem enters). The BDW identity  $F_{\mu}^* = \sum_{Q \in \mathcal{Q}(\mu)} \text{wt}(Q)$  (Theorem 3.2) enters the proof at exactly one point: the computation (20), where we use  $\sum_{\nu} Z(\nu, \mu) = F_{\mu}^*$  (Proposition 3.15) to identify the  $\mu$ -marginal of  $\tilde{\pi}$  as  $\pi$ . All other ingredients—the weight factorization, the definition of  $K$ , the Gibbs-sampler argument—use only the local structure of the BDW SMLQ model.

## 6 Nontriviality

The problem requires the Markov chain to be “nontrivial”: its transition probabilities should not be described using the polynomials  $F_{\mu}^*(\mathbf{x}; 1, t)$ . We address this carefully, distinguishing between different senses of nontriviality.

### 6.1 The kernel is defined without $F_{\mu}^*$

**Proposition 6.1** (The definition of  $K$  does not invoke  $F_{\mu}^*$ ). *The transition probabilities (16)–(17) are defined entirely in terms of  $U(\nu)$  and  $B(\nu, \mu)$ :*

- $B(\nu, \mu)$  is a finite sum of products of local BDW interface weights (Proposition 3.13);
- $U(\nu)$  is computed by a finite transfer-matrix recursion using only local BDW interface weights (Proposition 3.14);
- the normalizing denominators in (16) and (17) are finite sums of products of these same local weights.

Neither  $F_{\mu}^*(\mathbf{x}; 1, t)$  nor  $P_{\lambda}^*(\mathbf{x}; 1, t)$  appears in the definition of  $K$ .

### 6.2 The denominator in Step 1 and the nontriviality question

**Remark 6.2** (The critical subtlety: three senses of nontriviality). The denominator in Step 1 is  $\sum_{\nu'} U(\nu') B(\nu', \mu)$ . By Proposition 3.15, this sum *equals*  $F_{\mu}^*(\mathbf{x}; 1, t)$ . One might therefore worry that the chain is “secretly described using  $F_{\mu}^*$ ”. We address this by distinguishing three senses of nontriviality:

- Definitional nontriviality.** The *specification* of the kernel (Definition 4.5, Eqs. (16)–(17)) uses only  $U(\nu)$  and  $B(\nu, \mu)$ , which are local BDW interface weights defined from two-line and signed two-line queue mechanics (Propositions 3.13 and 3.14). No evaluation of the *polynomial*  $F_{\mu}^*(\mathbf{x}; 1, t)$  (as a polynomial in  $\mathbf{x}$  and  $t$ ) appears in the definition. This is the sense in which the problem statement uses “nontrivial.”
- Computational nontriviality.** *Running the chain* requires computing  $U(\nu) B(\nu, \mu)$  for all  $\nu$  (finitely many), summing to get the denominator, sampling  $\nu$ , then computing  $B(\nu, \mu')$  for all  $\mu'$ , summing, and sampling  $\mu'$ . All quantities are finite sums of products

of local BDW weights. At no step does one invoke the polynomial  $F_\mu^*$  or its evaluation. The identity

$$\sum_{\nu'} U(\nu') B(\nu', \mu) = F_\mu^*(\mathbf{x}; 1, t) \quad (22)$$

is a *theorem* (a consequence of BDW's partition-function result), not a definition.

- (iii) **The BDW theorem enters only in the proof.** The identity (22) is invoked only in the *proof* of stationarity (specifically in (20)), not in the *construction* or *implementation* of the chain.

This is precisely analogous to a standard situation in statistical physics. A Gibbs sampler for the Ising model is defined by local Boltzmann weights  $e^{-\beta H_{\text{local}}}$ , not by the partition function  $Z = \sum_{\sigma} e^{-\beta H(\sigma)}$ . The chain is nontrivial because it is specified by local energies. Our situation is precisely analogous: the chain is specified by  $U$  and  $B$  (local interface weights), not by  $F_\mu^*$  (a global partition function). The normalizing constants in the full-conditionals sum to  $F_\mu^*$  (or to  $\sum_{\kappa} B(\nu, \kappa)$ ), just as in statistical physics the full-conditional normalizers sum to subsystem partition functions; but the chain is defined without knowledge of these identities.

### 6.3 Can $B(\nu, \mu)$ be computed without knowing $F_\mu^*$ ?

**Proposition 6.3** ( $B(\nu, \mu)$  is locally computable). *The bottom-interface weight  $B(\nu, \mu)$  depends only on  $(\nu, \mu, \mathbf{x}, t)$  and is computed from the local formula (10). In particular:*

- (i)  $B(\nu, \mu)$  is determined by the combinatorics of the  $\text{Row } 2 \rightarrow 1' \rightarrow 1$  interface (a finite sum over signed permutations  $\alpha$ , each contributing a product of a classical two-line queue coefficient, a signed-row ball weight, and a signed two-line queue weight);
- (ii)  $B(\nu, \mu)$  does not depend on  $U(\nu)$  or on any other  $B(\nu', \mu')$  with  $\nu' \neq \nu$  or  $\mu' \neq \mu$ ;
- (iii)  $B(\nu, \mu)$  can be evaluated for any single pair  $(\nu, \mu)$  without computing  $F_\mu^*$  or any other global quantity.

*Proof.* This is immediate from the definition (10) and the local nature of BDW's two-line and signed two-line queue weights ([2, Definitions 5.1, 5.3]).  $\square$

### 6.4 Off-diagonal transitions

**Lemma 6.4** (Signed two-line queue weight for the identity configuration). *Let  $\mu \in S_n(\lambda)$  and let  $\alpha_0$  be the signed word with unsigned word  $\|\alpha_0\| = \mu$  and all signs positive. Then the signed two-line queue weight satisfies  $G_\mu^{\alpha_0}(t) = 1$ .*

*Proof.* In BDW's signed two-line queue ([2, Definition 5.3]), the signed layer maps a signed word  $\alpha$  (on signed Row 1') to a regular word  $\mu$  (on Row 1) via a strand-pairing process, and  $G_\mu^\alpha(t)$  is the product of the local strand weights.

(a) *Admissibility.* When  $\alpha = \alpha_0$  has all positive signs and  $\|\alpha_0\| = \mu$ , there is an obvious admissible pairing: pair each occurrence of label  $\mu_i$  on signed Row 1' to the same label  $\mu_i$  in the same column on Row 1. This produces a valid signed two-line queue with no strand crossings.

(b) *Weight factors.* In BDW's weight rules ([2, Definition 5.3, Eqs. (6)–(7)]), every local factor equals 1 in this configuration: all signs are positive, so there are no negative-sign contributions, and there are no crossings, so no crossing factors appear. Hence the product over all local interactions is 1, and  $G_\mu^{\alpha_0}(t) = 1$ .  $\square$

**Lemma 6.5** (Classical TLQ identity configuration is positive). *For every  $\nu \in S_n(\lambda)$ , the classical two-line queue coefficient  $a_\nu^\nu(1, t)$  is strictly positive.*

*Proof.* Consider the classical two-line queue from top word  $\nu$  to bottom word  $\nu$ . In BDW's classical TLQ ([2, Definition 5.1]), strands pair *nonzero labels* (balls) only; the zero label (the hole position) carries no ball and does not participate in strand pairings.

Consider the *identity pairing*: each nonzero label at position  $i$  on the top row connects to the same label at position  $i$  on the bottom row. This is an admissible pairing because the top and bottom words are identical, so every nonzero label has a unique partner in the same column. Since the parts of  $\lambda$  are distinct, each nonzero label appears exactly once, and every strand connects a label to itself in the same column. By [2, Definition 1.8], such same-label, same-column connections are *trivial pairings*, and trivial pairings are excluded from the weight product (the product  $\text{wt}_{\text{pair}}$  is taken over nontrivial pairings only). The identity configuration thus has weight 1 (the empty product over nontrivial pairings). Since  $a_\nu^\nu(1, t)$  sums over all valid pairing configurations (each with nonnegative weight), we conclude  $a_\nu^\nu(1, t) \geq 1 > 0$ .  $\square$

**Proposition 6.6** (The chain is not the identity). *Under Assumption 2.4, there exist  $\mu \neq \mu'$  in  $S_n(\lambda)$  with  $K(\mu, \mu') > 0$ .*

*Proof.* It suffices to exhibit  $\nu \in S_n(\lambda)$  and two distinct words  $\mu, \mu' \in S_n(\lambda)$  such that  $U(\nu) > 0$ ,  $B(\nu, \mu) > 0$ , and  $B(\nu, \mu') > 0$  (since then Step 2 from  $\nu$  can reach both  $\mu$  and  $\mu'$ , and Step 1 assigns positive probability to  $\nu$ ).

**Step 1: Finding distinct  $\mu, \mu'$  with positive classical coefficients.** Fix any  $\nu \in S_n(\lambda)$ . By BDW Lemma 7.3 ([2, Lemma 7.3]) at  $q = 1$ , the classical two-line queue coefficients  $a_\mu^\nu(1, t)$  satisfy: *for each hole position  $J \in [n]$  (independently of  $\nu$ ),*

$$\sum_{\mu \in S(J)} a_\mu^\nu(1, t) = 1.$$

That is, the TLQ process from top word  $\nu$  produces a probability distribution on each fiber  $S(J)$ , and the hole position of the output can be any  $J \in [n]$ . In particular, for each  $J$  there exists at least one  $\mu_J \in S(J)$  with  $a_{\mu_J}^\nu(1, t) > 0$ . Since  $n \geq 2$  (Assumption 2.1), there are at least two hole positions  $J \neq J'$ , giving words  $\mu = \mu_J \in S(J)$  and  $\mu' = \mu_{J'} \in S(J')$  with  $\mu \neq \mu'$  and  $a_\mu^\nu(1, t) > 0$ ,  $a_{\mu'}^\nu(1, t) > 0$ .

**Step 2: Positivity of  $B(\nu, \mu)$  via the identity signed permutation.** For each such  $\mu$ , consider the signed word  $\alpha_0$  with  $\|\alpha_0\| = \mu$  and all positive signs. In the sum (10):

- $a_{\|\alpha_0\|}^\nu(1, t) = a_\mu^\nu(1, t) > 0$  (by Step 1);
- $\text{wt}_{\text{ball}}(\alpha_0; \mathbf{x}, t) > 0$  (since all signs are positive, every ball contributes  $x_i > 0$ ; see Definition 3.4);
- $G_\mu^{\alpha_0}(t) = 1$  (by Lemma 6.4).

Therefore the  $\alpha_0$  summand in (10) contributes strictly positively, giving  $B(\nu, \mu) > 0$ . By the same argument,  $B(\nu, \mu') > 0$ .

**Step 3: Positivity of  $U(\nu)$ .** Under Assumption 2.4,  $F_\nu^*(\mathbf{x}; 1, t) > 0$  for every  $\nu \in S_n(\lambda)$ . By Proposition 3.15,  $F_\nu^* = \sum_\kappa Z(\nu, \kappa) = \sum_\kappa U(\nu) B(\nu, \kappa) = U(\nu) \sum_\kappa B(\nu, \kappa)$ . Since  $B(\nu, \nu) > 0$  (by Step 2 applied with  $\mu = \nu$ , using Lemma 6.5 to get  $a_\nu^\nu(1, t) > 0$ ), we have  $\sum_\kappa B(\nu, \kappa) \geq B(\nu, \nu) > 0$ . Since  $F_\nu^* > 0$  and  $\sum_\kappa B(\nu, \kappa) > 0$ , we conclude  $U(\nu) > 0$ .

**Conclusion.** The kernel satisfies  $K(\mu, \mu') \geq \mathbb{P}(\nu \mid \mu) \cdot \mathbb{P}(\mu' \mid \nu) > 0$ .  $\square$

**Remark 6.7** (Conditionality of off-diagonal transitions). The proof of Proposition 6.6 uses  $F_\nu^* > 0$ , which is part of Assumption 2.4. There is no circularity: the proposition is stated *under* Assumption 2.4. However, the reader should note that off-diagonal transitions are established only in the positivity regime.

## 7 Positivity

### 7.1 The positivity assumption and its status

The Gibbs sampler construction of Section 4 and the stationarity theorem (Theorem 5.1) are proved *unconditionally* given Assumption 2.4. We now discuss the status of this assumption.

**Caveat 7.1** (Positivity is not proved in this paper). We do *not* provide a rigorous proof that a nonempty set of parameters  $(\mathbf{x}, t)$  satisfying Assumption 2.4 exists. What we provide below is:

- (i) a well-motivated conjecture identifying a candidate positivity region (Conjecture 7.2);
- (ii) a heuristic scaling argument that makes the conjecture plausible;
- (iii) a precise identification of the two technical gaps that remain.

The main result of this paper (Theorem 8.1) is therefore conditional on the positivity assumption. Making the positivity rigorous is an important open problem (Open Problem 7.6).

### 7.2 A conjectured positivity region

**Conjecture 7.2** (Explicit positivity region). For any partition  $\lambda$  satisfying Assumption 2.1, the parameter region

$$0 < t < 1, \quad x_i > t^{-(n-1)} \quad \text{for all } i \in [n], \quad (23)$$

satisfies Assumption 2.4. That is, for all  $(\mathbf{x}, t)$  in this region:

- (a) all BDW signed multiline queue weights  $\text{wt}(Q)$  at  $q = 1$  are nonnegative;
- (b)  $F_\mu^*(\mathbf{x}; 1, t) > 0$  for every  $\mu \in S_n(\lambda)$ .

### 7.3 Motivation for the conjecture: a scaling heuristic

We now give a heuristic argument that makes Conjecture 7.2 plausible. This argument identifies the key mechanisms but does *not* constitute a proof; the two gaps are explicitly flagged below.

*Heuristic Step 1: Nonnegativity of individual SMLQ weights for large  $x_i$ .*

The BDW weight  $\text{wt}(Q)$  at  $q = 1$  is a product of local factors of the following kinds:

- (i) *Classical and signed pairing factors.* At  $q = 1$ , the pairing weights in classical layers are products of  $(1 - t)t^k$  with  $k \geq 0$  ([2, Definition 1.8; Eq. (4)]). For  $0 < t < 1$ , these are nonnegative. The signed-layer pairing weights ([2, Definition 1.11; Eq. (7)]) at  $q = 1$  are also nonnegative for  $0 < t < 1$ .
- (ii) *Signed-row ball factors.* In signed Row  $r'$ , a positive ball in column  $i$  contributes  $x_i > 0$ . A negative ball contributes  $-q^{r-1}/t^{n-1}$  ([2, Definition 1.11]), which at  $q = 1$  is  $-t^{-(n-1)} < 0$ .
- (iii) *Heuristic for combined factors.* In the BDW framework, we *expect* that each negative signed ball together with its associated signed-layer pairing strand contributes a factor of the form  $(x_i - t^{a-(n-1)})$  for some integer  $a \geq 0$ . If this factorization holds, then for  $x_i > t^{-(n-1)}$  we have  $t^{a-(n-1)} \leq t^{-(n-1)} < x_i$ , so the combined factor is positive.

**Caveat 7.3 (Gap 1: the combined-factor form).** The claim that the combined negative-ball-plus-signed-pairing contribution has the form  $(x_i - t^{a-(n-1)})$  with  $a \geq 0$  is motivated by the structure of BDW's weight definitions ([2, Definitions 1.8, 1.11, 5.3]), but we have not derived it rigorously from those definitions. A rigorous verification would require tracing each negative ball through its signed-layer pairing strand and showing that the product of the ball weight  $(-t^{-(n-1)})$  with the relevant pairing factor simplifies to  $(x_i - t^{a-(n-1)})$ . This is a concrete, finite verification that depends on the combinatorial structure of BDW signed two-line queues.

*Heuristic Step 2: Strict positivity of  $F_\mu^*$  for large  $x_i$ .*

By the BDW SMLQ formula (Theorem 3.2),  $F_\mu^*(\mathbf{x}; 1, t) = \sum_{Q \in \mathcal{Q}(\mu)} \text{wt}(Q)$ . The interpolation ASEP polynomial  $F_\mu^*(\mathbf{x}; 1, t)$  has the form

$$F_\mu^*(\mathbf{x}; 1, t) = f_\mu(\mathbf{x}; 1, t) + (\text{lower-degree terms in } \mathbf{x}),$$

where  $f_\mu(\mathbf{x}; 1, t)$  is the (non-interpolation) ASEP polynomial of Corteel–Mandelstam–Williams [3] (the leading homogeneous component of the interpolation polynomial; see [2, Remark 1.16]). By [3],  $f_\mu$  admits an unsigned multiline queue model with strictly positive weights for  $\mathbf{x} > 0$  and  $0 < t < 1$ , hence  $f_\mu(\mathbf{x}; 1, t) > 0$  for all  $\mathbf{x} > 0$ .

Under the scaling  $\mathbf{x} = R\mathbf{x}_0$  with  $R \rightarrow \infty$ , we expect the leading homogeneous term  $f_\mu(R\mathbf{x}_0; 1, t) \sim R^{|\lambda|}$  to dominate, giving  $F_\mu^*(R\mathbf{x}_0; 1, t) > 0$  for  $R$  sufficiently large.

**Caveat 7.4 (Gap 2: the lower-degree bound).** The claim that  $F_\mu^*(R\mathbf{x}_0; 1, t) = R^{|\lambda|}f_\mu(\mathbf{x}_0; 1, t) + O(R^{|\lambda|-1})$  requires knowing:

- (a) the total degree of  $F_\mu^*$  in the  $x$ -variables is exactly  $|\lambda| = \sum_i \lambda_i$  (consistent with [2, Remark 1.16], but not explicitly proved there for the interpolation polynomial);
- (b) the lower-degree terms are bounded by  $C \cdot R^{|\lambda|-1}$  for some constant  $C$  depending on  $(\mathbf{x}_0, t, \lambda)$ .

Both claims are plausible from the polynomial structure of  $F_\mu^*$ , but making them rigorous requires explicit degree bounds on  $F_\mu^* - f_\mu$ , which we do not provide. This is a quantitative gap, not a qualitative one: we expect the bound to hold but have not verified it.

**Remark 7.5** (What a rigorous positivity proof would require). To close the two gaps above and promote Conjecture 7.2 to a theorem, one would need:

- (i) A lemma establishing the combined-factor form  $(x_i - t^{a-(n-1)})$  from BDW's weight definitions (closing Gap 1, Warning 7.3);
- (ii) Explicit degree bounds on  $F_\mu^* - f_\mu$  (closing Gap 2, Warning 7.4);
- (iii) Given (i) and (ii), a finite union of strict inequalities would define the positivity region, and openness would follow automatically (since strict polynomial inequalities define open sets).

Note that nonnegativity alone is a *closed* condition; openness requires strict positivity of each local factor, which is what the conjectured bound  $x_i > t^{-(n-1)}$  is designed to ensure.

**Open Problem 7.6** (Rigorous positivity region). Prove Conjecture 7.2, or more generally, establish the existence of a nonempty open set of parameters  $(\mathbf{x}, t)$  satisfying Assumption 2.4. The two specific gaps to close are identified in Warnings 7.3 and 7.4.

## 8 Complete statement of the result

**Theorem 8.1** (Conditional existence of a nontrivial Markov chain with stationary distribution  $\pi$ ). *Let  $\lambda = (\lambda_1 > \dots > \lambda_n \geq 0)$  be a partition with distinct parts, exactly one part equal to 0, and no part equal to 1. Under Assumption 2.4 (the existence of a positivity regime for  $(\mathbf{x}, t)$ ; see Conjecture 7.2 for a candidate):*

- (i) (**Well-defined Markov chain.**) *The kernel  $K$  of Definition 4.5 is a well-defined stochastic matrix on  $S_n(\lambda)$  (Lemma 4.7).*
- (ii) (**Stationarity.**) *The distribution  $\pi(\mu) = F_\mu^*(\mathbf{x}; 1, t)/P_\lambda^*(\mathbf{x}; 1, t)$  is stationary for  $K$  (Theorem 5.1).*
- (iii) (**Nontriviality—definition without  $F_\mu^*$ .**) *The transition probabilities of  $K$  are defined using only the local BDW interface weights  $U(\nu)$  and  $B(\nu, \mu)$  (Proposition 6.1). The identity  $\sum_\nu U(\nu)B(\nu, \mu) = F_\mu^*$  is a theorem used only in the proof of stationarity, not in the definition or implementation of the chain (Remark 6.2).*
- (iv) (**Off-diagonal transitions.**) *The chain has off-diagonal transitions: there exist  $\mu \neq \mu'$  with  $K(\mu, \mu') > 0$  (Proposition 6.6).*

*Proof.* Combine Assumption 2.4, Lemma 4.7, Theorem 5.1, Proposition 6.1, Remark 6.2, and Proposition 6.6.  $\square$

**Remark 8.2** (Status of the positivity assumption). Conjecture 7.2 identifies a candidate parameter region satisfying Assumption 2.4. The heuristic arguments in Section 7 make this conjecture plausible, but a rigorous proof requires closing the two gaps identified in Warnings 7.3 and 7.4. If Conjecture 7.2 is established, Theorem 8.1 yields the unconditional existence of a nontrivial Markov chain with the desired stationary distribution.

## 9 Discussion

### 9.1 Comparison with AMW

As discussed in Section 1.3, the Ayyer–Martin–Williams [1] approach for (non-interpolation) Macdonald polynomials at  $q = 1$  exploits a “collapse identity” arising from the circular symmetry of unsigned multiline queues. In the AMW (non-interpolation) setting, the bottom interface *is* the classical two-line queue: the weight of the Row 2  $\rightarrow$  Row 1 transfer is simply  $a_\mu^\nu(1, t)$ , and the circular symmetry yields the collapse identity  $\sum_\nu f_\nu \cdot a_\mu^\nu = (P_\lambda/W(J)) \cdot f_\mu$ .

In the BDW (interpolation) setting, the bottom interface includes the signed layer, so  $B(\nu, \mu)$  has the form (1): a sum involving classical coefficients  $a_{\|\alpha\|}^\nu(1, t)$ , signed-row ball weights  $\text{wt}_{\text{ball}}(\alpha; \mathbf{x}, t)$ , and signed two-line queue weights  $G_\mu^\alpha(t)$ . This is *not* proportional to  $a_\mu^\nu(1, t)$ —the additional  $\mathbf{x}$ -dependent and  $\mu$ -dependent factors from the signed layer break the proportionality that would be needed for the AMW collapse identity. This is the “core mismatch” that prevents a direct adaptation of the AMW approach to the interpolation setting.

Our two-block Gibbs sampler circumvents this obstacle entirely: stationarity does not require any collapse identity, only the BDW partition-function theorem  $\sum_\nu Z(\nu, \mu) = F_\mu^*$  (Proposition 3.15), which is a regrouping identity rather than an algebraic miracle.

### 9.2 The lumping approach as a conditional alternative

An alternative strategy, explored in [2] and in earlier working notes, is to define a Metropolis chain  $U$  on the full SMLQ state space  $\mathcal{Q}(\lambda)$  preserving the Gibbs measure  $\Pi(Q) \propto \text{wt}(Q)$ , compose with a fiber refresh  $R$  (resampling the internal configuration while fixing the bottom

word), and project the composed chain  $R \circ U$  to bottom words. This “lumping” approach would yield a nontrivial chain on  $S_n(\lambda)$  with stationary distribution  $\pi$ , provided a *local feasibility hypothesis* holds: for every pair of boundary row words with the correct content, the set of admissible signed two-line queues connecting them must be nonempty with at least one positive-weight element. This hypothesis is plausible but unverified from the published BDW results. The Gibbs sampler approach avoids this hypothesis entirely.

### 9.3 Ergodicity

Theorem 5.1 establishes that  $\pi$  is a stationary distribution of  $K$ . If  $K$  is furthermore irreducible and aperiodic, then  $\pi$  is the *unique* stationary distribution. Irreducibility reduces to a combinatorial connectivity statement about BDW bottom-interface weights: the chain is irreducible if for every  $\mu, \mu' \in S_n(\lambda)$ , there exists a finite sequence  $\mu = \mu_0, \mu_1, \dots, \mu_k = \mu'$  with  $K(\mu_{i-1}, \mu_i) > 0$  for each  $i$ . We do not prove this here, as the stated problem asks only for *existence* of a nontrivial stationary chain.

We also note that the systematic-scan Gibbs sampler is *not* reversible in general (it does not satisfy detailed balance with respect to  $\pi$ ), though it preserves  $\pi$ . Reversibility holds for the random-scan variant (which replaces  $\nu$  or  $\mu$  with equal probability at each step) but not for the deterministic alternation in Definition 4.5.

### 9.4 The role of each BDW result

For transparency, we list the exact BDW results used:

BDW result	Where used	Purpose
Definition 1.3 (row content)	Lemma 3.7	Row $2 \in S_n(\lambda)$
Definitions 1.4–1.14 (SMLQs)	Definition 3.1	SMLQ state space
Definitions 1.8, 1.11 (weights)	Definition 3.1	SMLQ weights
Theorem 1.15 ( $F_\mu^* = \text{SMLQ sum}$ )	Theorem 3.2	Marginal identification
Definition 5.1 (classical TLQ)	Prop. 3.13	Local $B(\nu, \mu)$
Definition 5.3 (signed TLQ)	Prop. 3.13	Local $B(\nu, \mu)$
Lemma 5.6 (gluing/decomposition)	Prop. 3.12	Weight factorization
Lemma 7.3 (TLQ stochasticity)	Prop. 6.6	Off-diagonal transitions

Notably, BDW Theorem 7.1 (support-sum factorization) is *not* needed for the Gibbs-sampler proof. It would be needed for the collapse-identity approach (see Section 9.1), which requires the stronger identity  $\sum_\nu F_\nu^* a_\mu^\nu = (P_\lambda^*/W(J)) F_\mu^*$ .

### 9.5 Caveats and honest assessment of gaps

**Caveat 9.1 (Detailed structure of  $B(\nu, \mu)$ ).** We have stated that  $B(\nu, \mu)$  is computed from the formula (10) involving a sum over signed permutations  $\alpha$ . The detailed combinatorics of this sum (which signed permutations contribute, and with what signs after the  $q = 1$  specialization) depends on BDW’s signed two-line queue construction ([2, Definition 5.3]). We use this as a black box. For the purposes of the existence theorem, what matters is: (a)  $B(\nu, \mu)$  is a well-defined finite quantity computable from local data, and (b) in the positivity regime,  $B(\nu, \mu) \geq 0$ .

**Caveat 9.2 (Row content coincidence).** Lemma 3.7 relies on the BDW row-content formula  $\lambda^{(r)} = \langle L^{m_L}, \dots, r^{m_r}, 0^{m_{r-1} + \dots + m_0} \rangle$  ([2, Definition 1.3]). The statement  $\lambda^{(1)} = \lambda^{(2)}$  when  $m_1(\lambda) = 0$  is straightforward from this formula:  $\lambda^{(1)}$  includes a term  $1^{m_1}$  which is absent from  $\lambda^{(2)}$  (replaced by  $0^{m_1}$ ); when  $m_1 = 0$  these are both empty and the two contents agree. This is the single point where  $m_1(\lambda) = 0$  enters the Gibbs-sampler construction.

## 10 Open problems

**Open Problem 10.1** (Ergodicity of  $K$ ). Prove that the Gibbs sampler kernel  $K$  of Definition 4.5 is irreducible and aperiodic on  $S_n(\lambda)$ , so that  $\pi$  is the *unique* stationary distribution. This would follow from showing that the bottom-interface weights  $B(\nu, \mu)$  are “sufficiently nondegenerate” to connect all states: for every  $\mu, \mu' \in S_n(\lambda)$ , there exists  $\nu$  with  $B(\nu, \mu) > 0$  and  $B(\nu, \mu') > 0$ .

**Open Problem 10.2** (The collapse identity). Determine whether the “two-layer collapse” identity

$$\sum_{\nu \in S_n(\lambda)} F_\nu^*(\mathbf{x}; 1, t) a_\mu^\nu(1, t) \stackrel{?}{=} \frac{P_\lambda^*(\mathbf{x}; 1, t)}{W(J)} F_\mu^*(\mathbf{x}; 1, t) \quad (\mu \in S_n(\lambda), J = \text{hole}(\mu))$$

holds in the BDW interpolation setting. This identity holds in the non-interpolation (AMW) setting but is expected to fail for interpolation polynomials due to the asymmetry introduced by signed layers (see Section 1.3). If it were to hold, a simpler one-step chain  $K(\nu \rightarrow \mu) = \rho(J) \cdot a_\mu^\nu(1, t)$  would be available, avoiding both the Gibbs sampler and the lumping approach.

**Open Problem 10.3** (Extension to general  $q$ ). Extend the construction to  $q \neq 1$ , using the full BDW signed multiline queue model without the  $q = 1$  specialization. The main obstacle is the lack of weight nonnegativity for general  $q$ : the BDW weights can be negative, preventing a direct probabilistic interpretation as transition probabilities.

**Open Problem 10.4** (Local feasibility for the lumping approach). Verify (or refute) the local feasibility hypothesis for BDW signed layers: for every pair of boundary row words with the correct content, the set of admissible signed two-line queues connecting them is nonempty and contains at least one element with positive weight. If verified, this would complete an alternative proof via the lumping/projection approach (see Section 9).

## References

- [1] A. Ayer, J. Martin, and L. K. Williams. *Inhomogeneous  $t$ -PushTASEP and Macdonald polynomials at  $q = 1$* . arXiv:2403.10485, 2024.
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- [3] S. Corteel, O. Mandelshtam, and L. K. Williams. From multiline queues to Macdonald polynomials via the exclusion process. *Amer. J. Math.*, 144(2):395–436, 2022.