Local Metropolis Dynamics for Signed Multiline Queues at $q = 1$ and a Strongly Lumped Bottom-Word Chain

Abstract

Ben Dali–Williams (BDW) give a signed multiline queue (SMLQ) model whose total weight over SMLQs with fixed bottom word μ equals the interpolation ASEP polynomial $F^*_\mu(\mathbf{x}; 1, t)$, and whose orbit-sum over $\mu \in S_n(\lambda)$ yields the symmetric interpolation Macdonald polynomial $P^*_\lambda(\mathbf{x}; 1, t)$ [1]. Working in the "restricted" setting (distinct parts, a unique 0, and no part 1), and assuming a probabilistic regime in which all BDW SMLQ weights are nonnegative, we construct: (i) a concrete, finite-state Markov chain on the *full* SMLQ state space whose stationary distribution is the Gibbs measure $\Pi(\mathsf{Q}) \propto \mathrm{wt}(\mathsf{Q})$, built from random-scan Metropolis block updates on bounded windows of adjacent rows; crucially, a bottom block update changes the bottom row word using only BDW local admissibility and local weight ratios; and (ii) an (optional) fiber-refresh composition that produces a *strongly lumpable* SMLQ chain whose projection to bottom words is a genuine Markov chain on $\mathrm{supp}(\pi)$ with explicit stochastic kernel $\widetilde{K}$, stationary distribution $\pi(\mu) \propto F^*_\mu(\mathbf{x}; 1, t)$, and no transition probabilities defined from global polynomial evaluations.

Introduction

Fix $n \geq 2$ and a partition $\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0)$ with *distinct parts*, *exactly one part equal to* 0, and *no part equal to* 1. We call this the *restricted case*. Let $S_n(\lambda)$ denote the orbit of λ under coordinate permutations.

BDW define *signed multiline queues* (SMLQs) of type $\mu \in S_n(\lambda)$ and assign each SMLQ Q a (generally signed) weight $\mathrm{wt}(\mathsf{Q})$ [1, Defs. 1.5–1.6, 1.11]. Let $MLQ^{\pm}(\mu)$ be the finite set of SMLQs with bottom word μ. BDW define the weight-generating function

$$F^*_\mu(\mathbf{x}; q, t) \ := \sum_{\mathsf{Q} \in MLQ^{\pm}(\mu)} \mathrm{wt}(\mathsf{Q}) \qquad [1, \text{Def. } 1.14],$$

and prove that $F^*_\mu(\mathbf{x}; q, t)$ equals the interpolation ASEP polynomial $f^*_\mu(\mathbf{x}; q, t)$ [1, Thm. 1.15]. We specialize to $q = 1$ and consider the orbit sum

$$P^*_\lambda(\mathbf{x}; 1, t) \ := \sum_{\mu \in S_n(\lambda)} F^*_\mu(\mathbf{x}; 1, t) \qquad (\text{equivalently } Z^*_\lambda(\mathbf{x}; 1, t) \text{ in [1]}),$$

which is the symmetric interpolation Macdonald polynomial at $q = 1$ [1, Thm. 1.15]. The target bottom-word distribution is

$$\pi(\mu) \ = \ \frac{F^*_\mu(\mathbf{x}; 1, t)}{P^*_\lambda(\mathbf{x}; 1, t)}.$$

The goal is to construct a *genuinely nontrivial* Markov chain on the full SMLQ state space with stationary law proportional to wt, such that (at stationarity) its bottom-word marginal is π, and then (optionally) to produce a *Markov* chain on bottom words via a rigorous strong lumpability argument. The construction must never define transitions using the polynomial values F^*_μ or P^*_λ; only BDW *local* primitives (row/layer admissibility and local weights) may be used. Our local-update design follows the general AMW "multiline diagram dynamics" philosophy [2], but we do *not* assume any AMW-style collapse/row-exchange identity.

1 Concrete, reliable construction on SMLQs

1.1 State space, bottom map, and a positivity hypothesis

Let $L := \lambda_1$ be the largest part. BDW's *enhanced ball systems* have rows labeled

$$1, 1', 2, 2', \ldots, L, L'$$

from bottom to top and columns $1, \ldots, n$ on a cylinder [1, Def. 1.5]. Regular rows r carry labels in $\{0, 1, \ldots, L\}$; signed rows r' carry labels in $\{0, \pm 1, \ldots, \pm L\}$. In the restricted case, 1 never appears in any row word because $m_1 = 0$.

Definition 1.1 (SMLQ state space and bottom map)**.** Let Ω_λ be the finite set of BDW signed multiline queues of content λ, i.e. the union

$$\Omega_\lambda \;=\; \bigcup_{\mu \in S_n(\lambda)} MLQ^{\pm}(\mu) \qquad [1, \text{Def. } 1.6].$$

For $\mathsf{Q} \in \Omega_\lambda$, write $\Phi(\mathsf{Q}) \in S_n(\lambda)$ for its *bottom word* (the label configuration on Row 1), and let $\Omega_\lambda(\mu) = \Phi^{-1}(\mu)$ be the fiber over μ.

Definition 1.2 (BDW weight at $q = 1$)**.** Fix parameters $(\mathbf{x}, t)$ with $\mathbf{x} = (x_1, \ldots, x_n)$. For $\mathsf{Q} \in \Omega_\lambda$, define $\mathrm{wt}(\mathsf{Q})$ to be the BDW weight specialized at $q = 1$ [1, Def. 1.11]:

$$\mathrm{wt}(\mathsf{Q}) \;=\; \mathrm{wt}_{\mathrm{ball}}(\mathsf{Q})\, \mathrm{wt}_{\mathrm{pair}}(\mathsf{Q}).$$

Here $\mathrm{wt}_{\mathrm{ball}}$ is the product over all signed rows r' of the shifted ball-weights

$$\mathrm{wt}_{\mathrm{ball}}(r') \;=\; \Big(\prod_{i:\alpha_i > 0} x_i\Big)\Big(\prod_{i:\alpha_i < 0} (-t^{n-1})\Big) \quad (\text{since } q^{r-1} = 1 \text{ at } q = 1),$$

where $\alpha \in \mathbb{Z}^n$ is the signed composition on row r' [1, (5)–(6)]. The pairing weight $\mathrm{wt}_{\mathrm{pair}}$ is the product of local factors over all nontrivial pairings in all classic layers and signed layers [1, (4),(7)], with the specialization $q = 1$.

The BDW model is signed in general. The problem statement allows us to work in a regime where the weights become probabilistic.

Assumption 1.3 (Probabilistic regime)**.** We work at $q = 1$ and assume there is a parameter regime (e.g. $t \in (0, 1)$ together with additional inequalities on $\mathbf{x}$) such that:

(i) $\mathrm{wt}(\mathsf{Q}) \geq 0$ for all $\mathsf{Q} \in \Omega_\lambda$;

(ii) $Z := \sum_{\mathsf{Q} \in \Omega_\lambda} \mathrm{wt}(\mathsf{Q}) > 0$.

Definition 1.4 (Positive support and Gibbs measure)**.** Under Assumption 1.3, define

$$\Omega_\lambda^{+} \;:=\; \{\mathsf{Q} \in \Omega_\lambda : \mathrm{wt}(\mathsf{Q}) > 0\}, \qquad \Pi(\mathsf{Q}) \;:=\; \frac{\mathrm{wt}(\mathsf{Q})}{\sum_{\mathsf{R} \in \Omega_\lambda} \mathrm{wt}(\mathsf{R})} \quad (\mathsf{Q} \in \Omega_\lambda^{+}),$$

and $\Pi(\mathsf{Q}) = 0$ for $\mathsf{Q} \notin \Omega_\lambda^{+}$. Define the supported bottom words

$$S_\lambda^{+} \;:=\; \{\mu \in S_n(\lambda) : \Omega_\lambda^{+}(\mu) := \Omega_\lambda^{+} \cap \Omega_\lambda(\mu) \neq \varnothing\}.$$

Remark 1.5 (What is assumed vs. proved about positivity)**.** Assumption 1.3 is an explicit hypothesis: we do not claim here that it holds for all $(\mathbf{x}, t)$, only that we restrict to regimes in which the BDW weight is nonnegative and not identically zero. (In particular examples, e.g. [1, Ex. 1.16], some configurations have negative weights unless parameters and/or allowed sign patterns are restricted.)

1.2 A local/block base kernel U_{base} that changes bottom words

We construct a Metropolis–Hastings chain on Ω_λ^+ built from random-scan block updates on small windows of adjacent rows and their incident layers. The key point is that one update block *includes the bottom row as a variable*, hence can change Φ.

Blocks. We use the row/layer adjacency in BDW's enhanced ball system: regular row r is adjacent to signed row r' (via a signed layer) and to signed row $(r-1)'$ for $r \geq 2$ (via a classic layer); signed row r' is adjacent to regular row r (signed layer) and regular row $r+1$ for $r \leq L-1$ (classic layer). Define the following finite block index set:

$$\mathcal{B} \;:=\; \{\text{bot}\} \;\cup\; \{\text{reg}(r) : 2 \leq r \leq L\} \;\cup\; \{\text{sign}(r) : 2 \leq r \leq L-1\} \;\cup\; \{\text{top}\}.$$

Each block $b \in \mathcal{B}$ specifies a *bounded* collection of adjacent rows and the adjacent layers incident to an "interior" row:

- $b = \text{bot}$: the *bottom window* consisting of Rows $2, 1', 1$ together with the classic layer between 2 and $1'$ and the signed layer between $1'$ and 1. In this block, Rows $1', 1$ and both layers are updated, while Row 2 is held fixed as boundary data. This block *can change the bottom word* Φ.

- $b = \text{reg}(r)$ for $2 \leq r \leq L$: the *regular-row window* consisting of Rows $r', r, (r-1)'$ together with the signed layer between r' and r and the classic layer between r and $(r-1)'$. Here the outer signed rows r' and $(r-1)'$ are held fixed and the middle regular Row r and both incident layers are updated.

- $b = \text{sign}(r)$ for $2 \leq r \leq L-1$: the *signed-row window* consisting of Rows $(r+1), r', r$ together with the classic layer between $(r+1)$ and r' and the signed layer between r' and r. Here the outer regular rows $(r+1)$ and r are held fixed and the middle signed Row r' and both incident layers are updated.

- $b = \text{top}$: the *top window* consisting of Rows L', L together with their signed layer. Here Row L is held fixed and Row L' and the signed layer are updated.

Local admissible fillings. For a full SMLQ $\mathsf{Q} \in \Omega_\lambda$, each block b comes with a finite set of alternative local configurations inside the rows/layers of b that are BDW-admissible and consistent with the fixed boundary rows outside the block.

Definition 1.6 (Block fiber $\mathcal{S}_b(\mathsf{Q})$)**.** Fix $b \in \mathcal{B}$ and $\mathsf{Q} \in \Omega_\lambda$. Let $\mathcal{S}_b(\mathsf{Q})$ be the set of all $\mathsf{Q}' \in \Omega_\lambda$ such that:

(i) Q' agrees with Q on every row word and every layer *not* belonging to the window b;

(ii) the restriction of Q' to the rows and layers in b is BDW-admissible (i.e. satisfies the forbidden-configuration rules for classic and signed layers [1, Defs. 1.4, 1.6]) and has the correct row-contents (each row word in the window is a signed/unsigned permutation of the prescribed multiset $\lambda^{(r)}$ for that row [1, Def. 1.5]).

Lemma 1.7 (Finite local catalogue: $\mathcal{S}_b(\mathsf{Q})$ is finite and nonempty)**.** *For every $\mathsf{Q} \in \Omega_\lambda$ and every block $b \in \mathcal{B}$, the set $\mathcal{S}_b(\mathsf{Q})$ is finite and contains Q.*

Proof. Nonemptiness holds because $\mathsf{Q} \in \mathcal{S}_b(\mathsf{Q})$ by construction. Finiteness holds because the rows inside b have length n and each such row must be a permutation (signed or unsigned) of a fixed finite multiset $\lambda^{(r)}$ [1, Def. 1.5]; hence there are finitely many row-word possibilities. For each such choice, the number of possible BDW-admissible pairing patterns on finitely many adjacent layers is finite. Therefore $\mathcal{S}_b(\mathsf{Q})$ is finite. $\square$

The proposal and acceptance rule. Fix positive block-selection probabilities $(p_b)_{b\in\mathcal{B}}$ with $\sum_{b\in\mathcal{B}} p_b = 1$ and $p_b > 0$ for each b.

Definition 1.8 (The base kernel U_{base})**.** Define U_{base} on Ω_λ^+ by the following random-scan Metropolis block update:

Step 1: Choose a block $b \in \mathcal{B}$ with probability p_b.

Step 2: Propose $\mathsf{Q}^{\mathrm{prop}}$ uniformly from $\mathcal{S}_b(\mathsf{Q})$.

Step 3: If $\mathrm{wt}(\mathsf{Q}^{\mathrm{prop}}) = 0$ (equivalently $\mathsf{Q}^{\mathrm{prop}} \notin \Omega_\lambda^+$), *reject* (i.e. set $\mathsf{Q}^+ = \mathsf{Q}$). Otherwise accept with Metropolis probability

$$\alpha(\mathsf{Q} \to \mathsf{Q}^{\mathrm{prop}}) \;=\; \min\left\{1,\; \frac{\mathrm{wt}(\mathsf{Q}^{\mathrm{prop}})}{\mathrm{wt}(\mathsf{Q})}\right\},$$

and set $\mathsf{Q}^+ = \mathsf{Q}^{\mathrm{prop}}$ if accepted, $\mathsf{Q}^+ = \mathsf{Q}$ if rejected.

Remark 1.9 (Local computability of the acceptance ratio)**.** By BDW's definition, $\mathrm{wt}(\mathsf{Q})$ factors over signed rows (ball weights) and over layers (pairing weights) [1, Def. 1.11]. A block update changes only the finitely many local factors supported inside the chosen window. Hence the ratio $\mathrm{wt}(\mathsf{Q}^{\mathrm{prop}})/\mathrm{wt}(\mathsf{Q})$ can be computed using only local information inside the chosen window and its fixed boundary rows; no global partition function appears.

1.3 Stationarity for U_{base}

Theorem 1.10 (Reversibility and stationarity of U_{base})**.** *Under Assumption 1.3, the kernel U_{base} is reversible with respect to Π on Ω_λ^+, and therefore Π is stationary for U_{base}.*

Proof. Fix $\mathsf{Q}, \mathsf{Q}' \in \Omega_\lambda^+$. If Q' is not reachable from Q in one block update, then both one-step probabilities are zero and detailed balance is trivial.

Otherwise, there exists a unique block b such that Q' differs from Q only inside b; in particular $\mathsf{Q}' \in \mathcal{S}_b(\mathsf{Q})$. Because the boundary outside b is unchanged in either direction, we also have $\mathsf{Q} \in \mathcal{S}_b(\mathsf{Q}')$ and $|\mathcal{S}_b(\mathsf{Q})| = |\mathcal{S}_b(\mathsf{Q}')|$. Hence the proposal probabilities satisfy

$$q(\mathsf{Q} \to \mathsf{Q}') \;=\; \frac{p_b}{|\mathcal{S}_b(\mathsf{Q})|} \;=\; \frac{p_b}{|\mathcal{S}_b(\mathsf{Q}')|} \;=\; q(\mathsf{Q}' \to \mathsf{Q}).$$

Metropolis acceptance then gives the standard identity

$$\Pi(\mathsf{Q})\, q(\mathsf{Q} \to \mathsf{Q}')\, \alpha(\mathsf{Q} \to \mathsf{Q}') = \Pi(\mathsf{Q}')\, q(\mathsf{Q}' \to \mathsf{Q})\, \alpha(\mathsf{Q}' \to \mathsf{Q}),$$

because $\Pi(\mathsf{Q}) \propto \mathrm{wt}(\mathsf{Q})$ and $\alpha(\mathsf{Q} \to \mathsf{Q}') = \min\{1, \mathrm{wt}(\mathsf{Q}')/\mathrm{wt}(\mathsf{Q})\}$. Thus detailed balance holds, implying stationarity. □

Proposition 1.11 (Aperiodicity via an explicit self-loop)**.** *For every $\mathsf{Q} \in \Omega_\lambda^+$,*

$$U_{\mathrm{base}}(\mathsf{Q} \to \mathsf{Q}) \;\geq\; \sum_{b\in\mathcal{B}} p_b \cdot \frac{1}{|\mathcal{S}_b(\mathsf{Q})|} \;>\; 0.$$

In particular, U_{base} is aperiodic on each of its communicating classes.

Proof. For each block b, Lemma 1.7 implies $\mathsf{Q} \in \mathcal{S}_b(\mathsf{Q})$, so with probability $p_b \cdot (1/|\mathcal{S}_b(\mathsf{Q})|)$ the proposal equals the current block configuration. The Metropolis acceptance is then 1. Summing over b yields the bound. □

1.4 Genuine bottom-word motion from local moves

The following lemma addresses the key critique: the base dynamics must *move bottom words by local BDW primitives*, not only via a global independence move.

Definition 1.12 (A "diagonal" positive SMLQ with prescribed bottom word)**.** Fix $\mu \in S_n(\lambda)$. Define an SMLQ $\mathsf{Q}^{\mathrm{diag}}(\mu)$ by:

- In Row 1 (bottom regular row), place the word μ.
- For each label $a \in \{\lambda_1, \dots, \lambda_n\} \setminus \{0\}$, let $i(a)$ be the unique column with $\mu_{i(a)} = a$. For every regular row r with $1 \le r \le a$, place a ball labeled a in column $i(a)$; for $r > a$ that position is empty. Do the same in each signed row r' with $1 \le r \le a$, and give every signed ball the *positive* sign.
- Pair every ball to the ball of the same absolute value directly beneath it (straight down), in both signed and classic layers. (All pairings are trivial.)

Lemma 1.13 (The diagonal configuration is a valid SMLQ and has positive weight)**.** *For each $\mu \in S_n(\lambda)$, the object $\mathsf{Q}^{\mathrm{diag}}(\mu)$ is a BDW signed multiline queue in $\Omega_\lambda(\mu)$ [1, Def. 1.6]. Moreover, if $x_i > 0$ for all i, then $\mathrm{wt}(\mathsf{Q}^{\mathrm{diag}}(\mu)) > 0$ for all $t \in (0, 1)$.*

Proof. By construction, each row r (and r') contains exactly the labels $a \ge r$ (once each, since parts are distinct) and is 0 elsewhere, hence is a permutation (signed or unsigned) of $\lambda^{(r)}$ [1, Def. 1.5]. Every ball has a ball of the same absolute value directly below whenever such a ball exists in the next row, so the "trivial vertical" pairings are well-defined.

We must check layer admissibility. In every signed layer $r' \to r$, each signed ball is positive and sits above the equal regular label in the same column, so the signed-layer constraint [1, Def. 1.6(b')] holds and the trivial pairing is required and used. In every classic layer $r \to (r-1)'$ (for $r \ge 2$), each regular ball sits above the equal absolute value in the same column, so the classic-layer constraint [1, Def. 1.4] holds and trivial pairing is required and used. Therefore $\mathsf{Q}^{\mathrm{diag}}(\mu) \in \Omega_\lambda(\mu)$.

For the weight, all signed balls are positive so the shifted ball-weight contributes only factors $x_i > 0$ [1, (5)–(6)]. All pairings are trivial, hence there are no nontrivial pairing factors in $\mathrm{wt}_{\mathrm{pair}}$ [1, (4),(7)], so $\mathrm{wt}_{\mathrm{pair}} = 1$. Thus $\mathrm{wt}(\mathsf{Q}^{\mathrm{diag}}(\mu)) > 0$. □

Lemma 1.14 (A local bottom move exists (swap the hole with the maximum label))**.** *Let $\mu \in S_n(\lambda)$ and let $M := \lambda_1$ be the unique maximum part. Let J be the unique hole position in μ ($\mu_J = 0$), and let I be the unique position with $\mu_I = M$. Define $\mu^{\mathrm{swap}} \in S_n(\lambda)$ by swapping entries at I and J. Then the bottom block set $\mathcal{S}_{\mathrm{bot}}\big(\mathsf{Q}^{\mathrm{diag}}(\mu)\big)$ contains a configuration $\mathsf{Q}^{\mathrm{swap}}$ such that $\Phi(\mathsf{Q}^{\mathrm{swap}}) = \mu^{\mathrm{swap}}$ and $\mathrm{wt}(\mathsf{Q}^{\mathrm{swap}}) > 0$ (for $t \in (0, 1)$ and $x_i > 0$). Consequently U_{base} changes the bottom word with positive probability.*

Proof. Start from $\mathsf{Q}^{\mathrm{diag}}(\mu)$ and modify *only* the bottom window bot as follows:

- Set Row 1 to be μ^{swap}.
- Set Row $1'$ to be the *positive* signed word $+\mu^{\mathrm{swap}}$ (same absolute values in the same columns).
- In the signed layer $1' \to 1$, pair each $+a$ straight down to a (trivial).
- In the classic layer $2 \to 1'$, keep all trivial vertical pairings except for the label M: Row 2 (unchanged from $\mathsf{Q}^{\mathrm{diag}}(\mu)$) has M in column I, while Row $1'$ now has M in column J; pair the top M at column I to the bottom M at column J with the (unique) shortest right-moving strand, wrapping around the cylinder if needed (allowed in classic layers [1, Def. 1.4]). At column I, the ball M in Row 2 sits above an empty position (since $\mu^{\mathrm{swap}}_I = 0$), which is permitted by the classic-layer rule.

All other rows and layers remain unchanged. By the preceding checks, the resulting configuration is globally BDW-admissible: we have only modified two adjacent layers and their shared row, and we have done so in a way consistent with the fixed boundary Row 2 and with BDW forbidden configurations. Hence this defines a valid $\mathsf{Q}^{\mathrm{swap}} \in \mathcal{S}_{\mathrm{bot}}(\mathsf{Q}^{\mathrm{diag}}(\mu))$ with bottom word μ^{swap}.

For positivity: all signed balls in Row $1'$ are positive so their ball-weight contribution is $\prod_{i:\mu_i^{\mathrm{swap}}>0} x_i > 0$. The only potentially nontrivial pairing is the classic pairing of M; at $q = 1$ its local weight is a positive function of t (it is a specialization of [1, (4)] with $q = 1$), and all other pairings remain trivial, hence contribute no factors. Thus $\mathrm{wt}(\mathsf{Q}^{\mathrm{swap}}) > 0$ for $t \in (0, 1)$ and $x_i > 0$.

Finally, in a bot-update from $\mathsf{Q}^{\mathrm{diag}}(\mu)$, the proposal distribution is uniform on the finite set $\mathcal{S}_{\mathrm{bot}}(\mathsf{Q}^{\mathrm{diag}}(\mu))$ which contains $\mathsf{Q}^{\mathrm{swap}}$. Therefore the proposal probability is $p_{\mathrm{bot}} \cdot 1/|\mathcal{S}_{\mathrm{bot}}(\mathsf{Q}^{\mathrm{diag}}(\mu))| > 0$, and the Metropolis acceptance probability is strictly positive because both weights are positive. Hence U_{base} changes Φ with positive probability. □

1.5 Bottom marginal identification (polynomials appear only here)

Theorem 1.15 (Bottom-word marginal of Π equals π)**.** *Under Assumption 1.3, for every $\mu \in S_n(\lambda)$,*

$$\Pi(\Phi = \mu) \;=\; \frac{\sum_{\mathsf{Q}\in\Omega_\lambda(\mu)} \mathrm{wt}(\mathsf{Q})}{\sum_{\mathsf{Q}\in\Omega_\lambda} \mathrm{wt}(\mathsf{Q})} \;=\; \frac{F^*_\mu(\mathbf{x};1,t)}{P^*_\lambda(\mathbf{x};1,t)}.$$

Proof. By Definition 1.4, Π is the normalized weight measure on Ω_λ (equivalently on Ω_λ^+ since weights are nonnegative). Thus

$$\Pi(\Phi = \mu) = \frac{\sum_{\mathsf{Q}:\Phi(\mathsf{Q})=\mu} \mathrm{wt}(\mathsf{Q})}{\sum_{\mathsf{Q}} \mathrm{wt}(\mathsf{Q})}.$$

The numerator is exactly $F^*_\mu(\mathbf{x}; 1, t)$ by BDW Definition 1.14 specialized at $q = 1$ [1]. The denominator is $\sum_{\nu\in S_n(\lambda)} F^*_\nu(\mathbf{x}; 1, t) = P^*_\lambda(\mathbf{x}; 1, t)$ by definition of the orbit sum and [1, Def. 1.14, Thm. 1.15]. This yields the claim. □

1.6 Optional: a nonlocal Doeblinization U_{ind} for irreducibility/ergodicity

The base chain U_{base} is intentionally *local*; proving its global irreducibility on Ω^+_λ can be intricate. If one wants ergodicity without any connectivity lemma, one may add a purely technical nonlocal "independence" move. This is *not* the only bottom-word-moving mechanism (Lemma 1.14), but it guarantees irreducibility/aperiodicity on Ω^+_λ.

Definition 1.16 (Nonlocal independence MH kernel U_{ind})**.** Assume the finite set Ω_λ has been enumerated (global preprocessing). Define a proposal distribution g that is uniform on Ω_λ. Given $\mathsf{Q} \in \Omega^+_\lambda$, propose $\mathsf{Q}^{\mathrm{prop}} \sim g$ and accept with

$$\alpha_{\mathrm{ind}}(\mathsf{Q} \to \mathsf{Q}^{\mathrm{prop}}) = \begin{cases} 0, & \mathrm{wt}(\mathsf{Q}^{\mathrm{prop}}) = 0, \\ \min\Big\{1, \frac{\mathrm{wt}(\mathsf{Q}^{\mathrm{prop}})}{\mathrm{wt}(\mathsf{Q})}\Big\}, & \mathrm{wt}(\mathsf{Q}^{\mathrm{prop}}) > 0, \end{cases}$$

which is valid because $g(\mathsf{Q}) = g(\mathsf{Q}^{\mathrm{prop}})$.

Proposition 1.17 (Ergodic mixture kernel)**.** *Fix $\varepsilon \in (0, 1)$ and set*

$$U \;:=\; (1-\varepsilon)\, U_{\mathrm{base}} + \varepsilon\, U_{\mathrm{ind}}.$$

Then U is Π-reversible and Π-stationary on Ω^+_λ. Moreover, U is irreducible and aperiodic on Ω^+_λ.

Proof. Both U_{base} and U_{ind} are Metropolis kernels with symmetric proposals, hence Π-reversible and Π-stationary. A convex combination of Π-reversible kernels is Π-reversible and Π-stationary. Irreducibility holds because, for any $\mathsf{Q}, \mathsf{Q}' \in \Omega_\lambda^+$, the independence proposal chooses Q' with probability $1/|\Omega_\lambda| > 0$ and acceptance is strictly positive since $\mathrm{wt}(\mathsf{Q}'), \mathrm{wt}(\mathsf{Q}) > 0$. Aperiodicity follows from the positive self-loop of U_{base} (Proposition 1.11) and the mixture. $\square$

Remark 1.18 (Local vs. nonlocal)**.** The kernel U_{ind} is *nonlocal* and exists only to force ergodicity without a connectivity proof. The local dynamics U_{base} is the "intended" chain: it already moves bottom words (Lemma 1.14) using only local BDW primitives and local weight ratios.

2 Fiber refresh and strong lumpability

Section 1 provides a concrete SMLQ chain with the desired Gibbs stationary law and genuine bottom-word motion. However, the projection $\Phi(\mathsf{Q}_t)$ of U_{base} (or U) need not be Markov. In this section we add a standard "fiber refresh" scaffold to enforce *strong lumpability* and to extract an explicit Markov chain on bottom words.

2.1 The ideal fiber refresh kernel

Definition 2.1 (Fiber refresh kernel)**.** For $\mu \in S_\lambda^+$, define the conditional (fiber) Gibbs distribution

$$\Pi_\mu(\mathsf{Q}) \;:=\; \Pi(\mathsf{Q} \mid \Phi = \mu) \;=\; \frac{\mathrm{wt}(\mathsf{Q})}{\sum_{\mathsf{R} \in \Omega_\lambda(\mu)} \mathrm{wt}(\mathsf{R})} \qquad (\mathsf{Q} \in \Omega_\lambda(\mu)).$$

Define a Markov kernel $\mathcal{R}_\mu$ on $\Omega_\lambda^+(\mu)$ by

$$\mathcal{R}_\mu(\mathsf{Q} \to \mathsf{Q}') \;:=\; \Pi_\mu(\mathsf{Q}') \qquad (\mathsf{Q}, \mathsf{Q}' \in \Omega_\lambda^+(\mu)),$$

i.e. $\mathcal{R}_\mu$ ignores its input and outputs an exact fiber sample.

Remark 2.2 (Implementability)**.** Since $\Omega_\lambda(\mu)$ is finite, $\mathcal{R}_\mu$ can be implemented after *global* preprocessing by enumerating $\Omega_\lambda(\mu)$ and sampling by table lookup using the explicit local weight formula [1, Def. 1.11]. This uses no evaluations of F_μ^* as a polynomial, but it is computationally heavy and not "local".

2.2 A refreshed SMLQ chain and its stationarity

Let U be any Π-stationary kernel on Ω_λ^+ (e.g. $U = U_{\text{base}}$ or the ergodic mixture of Proposition 1.17).

Definition 2.3 (Refreshed chain K)**.** Define K on Ω_λ^+ by the two-step update: given current Q with $\mu = \Phi(\mathsf{Q})$,

(i) refresh: draw $\widetilde{\mathsf{Q}} \sim \Pi_\mu$ (i.e. apply $\mathcal{R}_\mu$);

(ii) move: apply one step of U from $\widetilde{\mathsf{Q}}$.

Equivalently, $K = \mathcal{R} \circ U$ where $\mathcal{R}$ is the fiberwise kernel that applies $\mathcal{R}_{\Phi(\mathsf{Q})}$ at state Q.

Proposition 2.4 (Π is stationary for K)**.** *If U is Π-stationary on Ω_λ^+, then so is K.*

Proof. The refresh step preserves Π because it samples from the conditional distribution given Φ; formally, for any test function f,

$$\mathbb{E}_{\mathsf{Q}\sim\Pi}\big[f(\widetilde{\mathsf{Q}})\big] = \mathbb{E}_{\mu\sim\Pi\circ\Phi^{-1}}\Big[\mathbb{E}_{\widetilde{\mathsf{Q}}\sim\Pi_\mu}\big[f(\widetilde{\mathsf{Q}})\big]\Big] = \mathbb{E}_{\mathsf{Q}\sim\Pi}\big[f(\mathsf{Q})\big].$$

Thus Π is stationary for the refresh kernel $\mathcal{R}$. Since Π is stationary for U by assumption, it is stationary for the composition $K = \mathcal{R} \circ U$. $\square$

2.3 Strong lumpability and the explicit lumped kernel

We now prove strong lumpability with respect to Φ.

Definition 2.5 (Strong lumpability criterion)**.** Let K be a Markov kernel on a finite space Ω and $\Phi : \Omega \to \Sigma$ a surjection. We say K is *strongly lumpable* with respect to Φ if for all $\sigma, \sigma' \in \Sigma$ and all $\omega, \omega' \in \Phi^{-1}(\sigma)$,

$$\sum_{\eta \in \Phi^{-1}(\sigma')} K(\omega \to \eta) \;=\; \sum_{\eta \in \Phi^{-1}(\sigma')} K(\omega' \to \eta).$$

In this case the lumped kernel $\widetilde{K}$ on Σ is defined by

$$\widetilde{K}(\sigma \to \sigma') \;:=\; \sum_{\eta \in \Phi^{-1}(\sigma')} K(\omega \to \eta) \quad \text{for any } \omega \in \Phi^{-1}(\sigma),$$

and is well-defined and stochastic (see e.g. classical lumping theory in Markov chains).

Theorem 2.6 (Strong lumpability of the refreshed chain)**.** *Let K be the refreshed chain of Definition 2.3. Then K is strongly lumpable with respect to $\Phi : \Omega_\lambda^+ \to S_\lambda^+$. Moreover, for $\mu, \mu' \in S_\lambda^+$ the lumped kernel is*

$$\widetilde{K}(\mu \to \mu') = \sum_{\widetilde{\mathsf{Q}} \in \Omega_\lambda^+(\mu)} \Pi_\mu(\widetilde{\mathsf{Q}}) \sum_{\mathsf{R} \in \Omega_\lambda^+(\mu')} U(\widetilde{\mathsf{Q}} \to \mathsf{R}), \tag{1}$$

and $\widetilde{K}$ is a stochastic matrix on S_λ^+.

Proof. Fix $\mu, \mu' \in S_\lambda^+$ and two states $\mathsf{Q}, \mathsf{Q}' \in \Omega_\lambda^+(\mu)$. By definition of K, the one-step transition from Q to any R is

$$K(\mathsf{Q} \to \mathsf{R}) = \sum_{\widetilde{\mathsf{Q}} \in \Omega_\lambda^+(\mu)} \mathcal{R}_\mu(\mathsf{Q} \to \widetilde{\mathsf{Q}})\, U(\widetilde{\mathsf{Q}} \to \mathsf{R}) = \sum_{\widetilde{\mathsf{Q}} \in \Omega_\lambda^+(\mu)} \Pi_\mu(\widetilde{\mathsf{Q}})\, U(\widetilde{\mathsf{Q}} \to \mathsf{R}),$$

which depends on Q only through $\mu = \Phi(\mathsf{Q})$. Summing over all $\mathsf{R} \in \Omega_\lambda^+(\mu')$ yields (1), which is therefore the same for Q and Q'. This proves strong lumpability and identifies the lumped kernel.

Finally, stochasticity: for fixed μ,

$$\sum_{\mu' \in S_\lambda^+} \widetilde{K}(\mu \to \mu') = \sum_{\widetilde{\mathsf{Q}} \in \Omega_\lambda^+(\mu)} \Pi_\mu(\widetilde{\mathsf{Q}}) \sum_{\mathsf{R} \in \Omega_\lambda^+} U(\widetilde{\mathsf{Q}} \to \mathsf{R}) = \sum_{\widetilde{\mathsf{Q}} \in \Omega_\lambda^+(\mu)} \Pi_\mu(\widetilde{\mathsf{Q}}) \cdot 1 = 1,$$

since U is stochastic and Π_μ is a probability distribution. □

Corollary 2.7 (Stationary distribution of the lumped chain)**.** *Let π be the pushforward of Π under Φ, i.e. $\pi(\mu) = \Pi(\Phi = \mu)$ for $\mu \in S_\lambda^+$. Then π is stationary for the lumped bottom-word chain $\widetilde{K}$.*

Proof. Since Π is stationary for K (Proposition 2.4) and $\widetilde{K}$ is the strong lumping of K under Φ, the pushforward measure π is stationary for $\widetilde{K}$ by standard projection-of-stationarity arguments: if $\mathsf{Q}_0 \sim \Pi$, then $\mathsf{Q}_1 \sim \Pi$, hence $\Phi(\mathsf{Q}_0) \sim \pi$ and $\Phi(\mathsf{Q}_1) \sim \pi$ with the Markov transition law $\widetilde{K}$. □

Remark 2.8 (Nontriviality survives lumping)**.** If there exists $\mu \neq \mu'$ and a state $\mathsf{Q} \in \Omega_\lambda^+(\mu)$ with $\sum_{\mathsf{R} \in \Omega_\lambda^+(\mu')} U(\mathsf{Q} \to \mathsf{R}) > 0$, then $\widetilde{K}(\mu \to \mu') > 0$ because $\Pi_\mu(\mathsf{Q}) > 0$. Lemma 1.14 provides such Q for $U = U_{\text{base}}$, hence the lumped chain is genuinely nontrivial.

3 Final word-chain consequences

3.1 The induced Markov chain on supported bottom words

Combining Theorem 1.15 and Theorem 2.6 gives the requested Markov chain on bottom words with stationary distribution π.

Theorem 3.1 (A nontrivial word chain with stationary π and no polynomial-defined transitions)**.** *Assume Assumption 1.3 and take $U = U_{\mathrm{base}}$ (or any Π-stationary kernel built from BDW local blocks and MH ratios, e.g. the ergodic mixture of Proposition 1.17). Let K be the refreshed chain of Definition 2.3 and $\widetilde{K}$ its strong lumping to S_λ^+ given by (1). Then:*

(i) $\widetilde{K}$ is a well-defined stochastic Markov kernel on S_λ^+.

(ii) Its stationary distribution is

$$\pi(\mu) \;=\; \frac{F_\mu^*(\mathbf{x};1,t)}{P_\lambda^*(\mathbf{x};1,t)} \qquad (\mu \in S_\lambda^+).$$

(iii) The kernel $\widetilde{K}$ is nontrivial*: there exist $\mu \neq \mu'$ with $\widetilde{K}(\mu \to \mu') > 0$.*

(iv) Neither U nor K nor $\widetilde{K}$ is defined using evaluations of F_μ^* or P_λ^*; only BDW admissibility rules and BDW local weight factors/ratios enter the transition rules.

Proof. Items (i) and the explicit kernel follow from Theorem 2.6. Item (ii) follows from Corollary 2.7 and Theorem 1.15. For nontriviality (iii), apply Lemma 1.14 to obtain $\mathsf{Q} \in \Omega_\lambda^+(\mu)$ and $\mu' \neq \mu$ such that U_{base} (hence U) moves from Q to some state in $\Omega_\lambda^+(\mu')$ with positive probability, implying $\widetilde{K}(\mu \to \mu') > 0$ by (1). Finally, (iv) is by construction: the only inputs are (a) finite admissibility sets of local blocks and (b) Metropolis ratios of products of BDW local weights [1, Def. 1.11]; polynomial identities appear only in the marginal identification. □

3.2 Extension to all of $S_n(\lambda)$ (optional formal completion)

If some $\mu \in S_n(\lambda)$ has $\pi(\mu) = 0$ (equivalently $\mu \notin S_\lambda^+$ in the nonnegative regime), the word chain is naturally defined on the support S_λ^+. If one insists on a chain on *all* of $S_n(\lambda)$, one may extend $\widetilde{K}$ by declaring every $\mu \notin S_\lambda^+$ absorbing (or by adding any stochastic transitions among the zero-mass states), which does not affect stationarity of π on the full set.

Conclusion

We constructed a concrete, local Metropolis Markov chain U_{base} on the *full* BDW signed multiline queue state space Ω_λ^+ with stationary distribution $\Pi(\mathsf{Q}) \propto \mathrm{wt}(\mathsf{Q})$. A bottom window update changes the bottom word using only local BDW admissibility and local BDW weights, ensuring nontrivial bottom-word motion without any global independence step. BDW's partition-function theorem then identifies the stationary bottom-word marginal as $\pi(\mu) \propto F_\mu^*(\mathbf{x};1,t)$. Finally, adding an ideal fiber refresh yields a strongly lumpable chain whose projection to bottom words is a genuine Markov chain $\widetilde{K}$ on $\mathrm{supp}(\pi)$ with stationary distribution π and transitions defined without invoking F_μ^* or P_λ^* values.

References

[1] H. Ben Dali and L. K. Williams, *A combinatorial formula for Interpolation Macdonald polynomials*, arXiv:2510.02587 (v2, Oct. 22, 2025).

[2] A. Ayyer, J. Martin, and L. K. Williams, *Multiline diagrams and t-PushTASEP / Macdonald measures at $q = 1$*, arXiv:2403.10485 (2024).