Local Metropolis Dynamics for Signed Multiline Queues at $q = 1$ and a Strongly Lumped Bottom-Word Chain

GPT 5.2 Pro

Abstract

Ben Dali–Williams (BDW) give a signed multiline queue (SMLQ) model whose total weight over SMLQs with fixed bottom word μ equals the interpolation ASEP polynomial $F^*_\mu(\mathbf{x}; 1, t)$, and whose orbit-sum over $\mu \in S_n(\lambda)$ yields the symmetric interpolation Macdonald polynomial $P^*_\lambda(\mathbf{x}; 1, t)$ [1]. Working in the "restricted" setting (distinct parts, a unique 0, and no part 1), and assuming a probabilistic regime in which all BDW SMLQ weights are nonnegative, we construct: (i) a concrete, finite-state Markov chain on the *full* SMLQ state space whose stationary distribution is the Gibbs measure $\Pi(\mathsf{Q}) \propto \mathrm{wt}(\mathsf{Q})$, built from random-scan Metropolis–Hastings block updates on bounded windows of adjacent rows/layers; crucially, the bottom block updates allow the bottom row word to change using only BDW local admissibility and local weight ratios; and (ii) an optional fiber-refresh composition that produces a *strongly lumpable* SMLQ chain whose projection to bottom words is a genuine Markov chain on supported bottom words with explicit stochastic kernel $\widetilde{K}$ and stationary distribution $\pi(\mu) \propto F^*_\mu(\mathbf{x}; 1, t)$, without defining transitions via global polynomial evaluations.

Introduction

Fix $n \geq 2$ and a partition $\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0)$ with *distinct parts*, *exactly one part equal to* 0, and *no part equal to* 1. We call this the *restricted case*. Let $S_n(\lambda)$ denote the orbit of λ under coordinate permutations.

BDW define *signed multiline queues* (SMLQs) of type $\mu \in S_n(\lambda)$ and assign each SMLQ Q a (generally signed) weight $\mathrm{wt}(\mathsf{Q})$ [1, Defs. 1.5–1.6, 1.11]. Let $MLQ^{\pm}(\mu)$ be the finite set of SMLQs with bottom word μ. BDW define the weight generating function

$$F^*_\mu(\mathbf{x}; q, t) \;:=\; \sum_{\mathsf{Q} \in MLQ^{\pm}(\mu)} \mathrm{wt}(\mathsf{Q}) \qquad [1,\ \text{Def. } 1.14],$$

and prove that $F^*_\mu(\mathbf{x}; q, t)$ equals the interpolation ASEP polynomial $f^*_\mu(\mathbf{x}; q, t)$ [1, Thm. 1.15]. We specialize to $q = 1$ and consider the orbit sum

$$P^*_\lambda(\mathbf{x}; 1, t) \;:=\; \sum_{\mu \in S_n(\lambda)} F^*_\mu(\mathbf{x}; 1, t),$$

which equals the interpolation Macdonald polynomial $P^*_\lambda(\mathbf{x}; 1, t)$ [1, Thm. 1.15]. The target bottom-word distribution is

$$\pi(\mu) \;=\; \frac{F^*_\mu(\mathbf{x}; 1, t)}{P^*_\lambda(\mathbf{x}; 1, t)}.$$

The goal is to construct a *genuinely nontrivial* Markov chain on the full SMLQ state space with stationary law proportional to wt, such that (at stationarity) its bottom-word marginal is π, and then (optionally) to produce a *Markov* chain on bottom words via a rigorous strong lumpability argument. Transition rules must be defined only from BDW *local* primitives (local admissibility and local weight factors/ratios), never from global polynomial values. The philosophy is inspired by AMW multiline dynamics [2], but we do *not* assume any AMW-style collapse/row-exchange identity.

1 Concrete, reliable construction on SMLQs

1.1 State space, bottom map, and a positivity hypothesis

Let $L := \lambda_1$ be the largest part. BDW's enhanced ball systems have rows labeled

$$1, 1', 2, 2', \dots, L, L'$$

from bottom to top and columns $1, \dots, n$ on a cylinder [1, Def. 1.5]. Regular rows r carry labels in $\{0, 1, \dots, L\}$; signed rows r' carry labels in $\{0, \pm 1, \dots, \pm L\}$.

Definition 1.1 (SMLQ state space and bottom map)**.** Let Ω_λ be the finite set of BDW signed multiline queues of content λ, i.e.

$$\Omega_\lambda \;=\; \bigcup_{\mu \in S_n(\lambda)} MLQ^{\pm}(\mu) \qquad \text{[1, Def. 1.6]}.$$

For $\mathsf{Q} \in \Omega_\lambda$, write $\Phi(\mathsf{Q}) \in S_n(\lambda)$ for its *bottom word* (the label configuration on Row 1), and let $\Omega_\lambda(\mu) = \Phi^{-1}(\mu)$ be the fiber over μ.

Definition 1.2 (BDW weight at $q = 1$)**.** Fix parameters $(\mathbf{x}, t)$ with $\mathbf{x} = (x_1, \dots, x_n)$. For $\mathsf{Q} \in \Omega_\lambda$, let $\mathrm{wt}(\mathsf{Q})$ be the BDW weight specialized at $q = 1$ [1, Def. 1.11]:

$$\mathrm{wt}(\mathsf{Q}) \;=\; \mathrm{wt}_{\mathrm{ball}}(\mathsf{Q})\, \mathrm{wt}_{\mathrm{pair}}(\mathsf{Q}).$$

Here $\mathrm{wt}_{\mathrm{ball}}$ is the product over all signed rows r' of the shifted ball-weights [1, Eq. (5)–(6)], and $\mathrm{wt}_{\mathrm{pair}}$ is the product of local factors over all nontrivial pairings in all classic and signed layers [1, Eq. (4),(7)], with $q = 1$.

BDW's model is signed in general. The problem statement allows us to work in a regime where it becomes probabilistic.

Assumption 1.3 (Probabilistic regime)**.** We work at $q = 1$ and assume there is a parameter regime such that:

(i) $\mathrm{wt}(\mathsf{Q}) \geq 0$ for all $\mathsf{Q} \in \Omega_\lambda$;

(ii) $Z := \sum_{\mathsf{Q} \in \Omega_\lambda} \mathrm{wt}(\mathsf{Q}) > 0$.

Definition 1.4 (Positive support and Gibbs measure)**.** Under Assumption 1.3, define

$$\Omega_\lambda^+ \;:=\; \{\mathsf{Q} \in \Omega_\lambda : \mathrm{wt}(\mathsf{Q}) > 0\}, \qquad \Pi(\mathsf{Q}) \;:=\; \frac{\mathrm{wt}(\mathsf{Q})}{\sum_{\mathsf{R} \in \Omega_\lambda} \mathrm{wt}(\mathsf{R})} \quad (\mathsf{Q} \in \Omega_\lambda^+),$$

and $\Pi(\mathsf{Q}) = 0$ for $\mathsf{Q} \notin \Omega_\lambda^+$. Define the supported bottom words

$$S_\lambda^+ \;:=\; \{\mu \in S_n(\lambda) : \Omega_\lambda^+(\mu) := \Omega_\lambda^+ \cap \Omega_\lambda(\mu) \neq \varnothing\}.$$

Remark 1.5 (What Assumption 1.3 buys (and what it does not))**.** Assumption 1.3 guarantees that Π is a genuine probability measure on the finite state space Ω_λ (equivalently on Ω_λ^+). It does *not* by itself assert anything about connectivity of local moves, nor does it logically force $|S_\lambda^+| \geq 2$ without additional input. Note also that positivity is not automatic from $t \in (0, 1)$ and $x_i > 0$: negative signed balls contribute an explicit negative factor in the shifted ball-weight [1, Eq. (5)], and negative-ball pairings in signed layers also carry a negative sign [1, Eq. (7)]. In Section 1.4 we exhibit explicit positive-weight configurations with two distinct bottom words (hence $|S_\lambda^+| \geq 2$) under mild additional conditions (e.g. $t \in (0, 1)$ and $x_i > 0$) by using only *positive* signed balls.

1.2 A local/block base kernel U_{base} that can change bottom words

We now define a Metropolis–Hastings chain on Ω_λ^+ built from random-scan block updates on bounded collections of adjacent rows and their incident layers. The key point is that the *bottom block* makes the bottom row word variable, hence Φ can change by a *local* move.

1.2.1 Blocks as "variable part + fixed boundary"

In an SMLQ, admissibility is enforced *layerwise* between adjacent rows: classic layers must satisfy Definition 1.4 of [1], and signed layers must satisfy Definition 1.6 of [1]. (In classic layers whose lower row is signed, the classic admissibility rule is applied to the *absolute values* of the signed labels, as in BDW's conventions.) Thus, to update locally without disturbing the rest of the configuration, we update a bounded *variable part* while *fixing* the row words (and all other layers) outside it.

Definition 1.6 (Block index set and variable parts)**.** Let $\mathcal{B}$ be the finite set of block indices

$$\mathcal{B} := \{\text{bot}\} \cup \{\text{reg}(r) : 2 \le r \le L\} \cup \{\text{sign}(r) : 2 \le r \le L-1\} \cup \{\text{top}\}.$$

For each $b \in \mathcal{B}$, define its *variable part* $V(b)$ as follows (everything outside $V(b)$ is fixed boundary data):

- $b = \text{bot}$ (bottom block): $V(b)$ consists of the row words on Rows $1', 1$ *and* the two layers adjacent to Row $1'$ (the classic layer between Rows 2 and $1'$ and the signed layer between Rows $1'$ and 1). The row word on Row 2 is fixed boundary.
- $b = \text{reg}(r)$ for $2 \le r \le L$ (regular-row block): $V(b)$ consists of the row word on Row r and its two incident layers: the signed layer between Rows r' and r, and the classic layer between Rows r and $(r-1)'$. The row words on Rows r' and $(r-1)'$ are fixed boundary.
- $b = \text{sign}(r)$ for $2 \le r \le L-1$ (signed-row block): $V(b)$ consists of the row word on Row r' and its two incident layers: the classic layer between Rows $(r+1)$ and r', and the signed layer between Rows r' and r. The row words on Rows $(r+1)$ and r are fixed boundary.
- $b = \text{top}$ (top block): $V(b)$ consists of the row word on Row L' and the signed layer between Rows L' and L. The row word on Row L is fixed boundary.

Remark 1.7 (Overlapping blocks are allowed)**.** Different blocks may overlap (e.g. $V(\text{reg}(L))$ and $V(\text{top})$ both include the signed layer between L' and L). This is harmless: U_{base} will be defined as a convex mixture of per-block Metropolis kernels, and reversibility will be checked *blockwise* (avoiding any false "uniqueness of block" claim).

1.2.2 Local admissible fillings and the finite catalogue

Definition 1.8 (Block neighborhood $\mathcal{S}_b(\mathsf{Q})$)**.** Fix $b \in \mathcal{B}$ and $\mathsf{Q} \in \Omega_\lambda$. Let $\mathcal{S}_b(\mathsf{Q})$ be the set of all $\mathsf{Q}' \in \Omega_\lambda$ such that:

(i) Q' agrees with Q on every row word and every layer *outside* the variable part $V(b)$ (equivalently, Q' differs from Q only on $V(b)$);

(ii) the restriction of Q' to the layers in $V(b)$ is BDW-admissible (classic layers satisfy [1, Def. 1.4] and signed layers satisfy [1, Def. 1.6]), and the row word(s) in $V(b)$ have the prescribed row-content (each variable row word is a signed/unsigned permutation of the appropriate multiset $\lambda^{(r)}$ [1, Def. 1.5]).

Lemma 1.9 (Finite local catalogue)**.** *For every $\mathsf{Q} \in \Omega_\lambda$ and every block $b \in \mathcal{B}$, the set $\mathcal{S}_b(\mathsf{Q})$ is finite and contains Q.*

Proof. Nonemptiness holds because $\mathsf{Q} \in \mathcal{S}_b(\mathsf{Q})$: it agrees with itself outside $V(b)$ and is globally admissible. Finiteness holds because $V(b)$ contains only finitely many rows (at most two row words are variable) and finitely many adjacent layers; each variable row word must be a (signed) permutation of a fixed finite multiset [1, Def. 1.5], hence there are finitely many candidate row words, and for each such choice there are finitely many admissible pairing patterns in finitely many layers. Therefore $\mathcal{S}_b(\mathsf{Q})$ is finite. □

Lemma 1.10 (Boundary-determined proposal sets). *Fix $b \in \mathcal{B}$. If $\mathsf{Q}' \in \mathcal{S}_b(\mathsf{Q})$, then $\mathcal{S}_b(\mathsf{Q}') = \mathcal{S}_b(\mathsf{Q})$ (hence $|\mathcal{S}_b(\mathsf{Q}')| = |\mathcal{S}_b(\mathsf{Q})|$).*

Proof. By definition, Q' and Q agree on all row words and layers outside $V(b)$, i.e. they have the same fixed boundary data for the block. The set $\mathcal{S}_b(\cdot)$ is exactly the set of all global SMLQs obtained by choosing an admissible filling on $V(b)$ consistent with that fixed boundary. Since the boundary data are identical for Q and Q', the allowable completions are identical, so $\mathcal{S}_b(\mathsf{Q}') = \mathcal{S}_b(\mathsf{Q})$. □

1.2.3 Per-block Metropolis kernels and the base chain

Fix positive block-selection probabilities $(p_b)_{b\in\mathcal{B}}$ with $\sum_{b\in\mathcal{B}} p_b = 1$ and $p_b > 0$ for each b.

Definition 1.11 (Per-block Metropolis kernels U_b). For each $b \in \mathcal{B}$, define a Markov kernel U_b on Ω_λ^+ as follows. Given $\mathsf{Q} \in \Omega_\lambda^+$:

Step 1: Propose $\mathsf{Q}^{\mathrm{prop}}$ uniformly from the finite set $\mathcal{S}_b(\mathsf{Q})$.

Step 2: If $\mathrm{wt}(\mathsf{Q}^{\mathrm{prop}}) = 0$ (i.e. $\mathsf{Q}^{\mathrm{prop}} \notin \Omega_\lambda^+$), *reject* (set $\mathsf{Q}^+ = \mathsf{Q}$). Otherwise accept with probability

$$\alpha_b(\mathsf{Q} \to \mathsf{Q}^{\mathrm{prop}}) \;=\; \min\Big\{1,\; \frac{\mathrm{wt}(\mathsf{Q}^{\mathrm{prop}})}{\mathrm{wt}(\mathsf{Q})}\Big\},$$

and set $\mathsf{Q}^+ = \mathsf{Q}^{\mathrm{prop}}$ if accepted, $\mathsf{Q}^+ = \mathsf{Q}$ if rejected.

Definition 1.12 (The base kernel U_{base}). Define U_{base} on Ω_λ^+ by choosing a block $b \in \mathcal{B}$ with probability p_b and then applying U_b:

$$U_{\mathrm{base}} \;:=\; \sum_{b\in\mathcal{B}} p_b\, U_b.$$

Remark 1.13 (Local computability of MH ratios). By BDW, $\mathrm{wt}(\mathsf{Q})$ factors over signed rows (ball weights) and over layers (pairing weights) [1, Def. 1.11]. A b-block proposal changes only the finitely many local factors supported in $V(b)$. Hence $\mathrm{wt}(\mathsf{Q}^{\mathrm{prop}})/\mathrm{wt}(\mathsf{Q})$ is computable from local data in $V(b)$ and the fixed boundary, without invoking any global partition function.

1.3 Stationarity and aperiodicity for U_{base}

Theorem 1.14 (Per-block reversibility and stationarity of U_{base}). *Under Assumption 1.3:*

(i) for each $b \in \mathcal{B}$, the kernel U_b is reversible with respect to Π on Ω_λ^+;

(ii) consequently, the mixture $U_{\mathrm{base}} = \sum_b p_b U_b$ is reversible with respect to Π, hence Π is stationary for U_{base}.

Proof. Fix $b \in \mathcal{B}$. For $\mathsf{Q}, \mathsf{Q}' \in \Omega_\lambda^+$, the proposal probability under U_b is

$$q_b(\mathsf{Q} \to \mathsf{Q}') \;=\; \begin{cases} \frac{1}{|\mathcal{S}_b(\mathsf{Q})|}, & \mathsf{Q}' \in \mathcal{S}_b(\mathsf{Q}), \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathsf{Q}' \in \mathcal{S}_b(\mathsf{Q})$, then Lemma 1.10 gives $\mathcal{S}_b(\mathsf{Q}') = \mathcal{S}_b(\mathsf{Q})$, hence $q_b(\mathsf{Q} \to \mathsf{Q}') = q_b(\mathsf{Q}' \to \mathsf{Q})$.

The Metropolis acceptance in Definition 1.11 is the standard symmetric-proposal MH rule, with the explicit convention that proposals with $\mathrm{wt} = 0$ are rejected (equivalently acceptance probability 0). Therefore, for all $\mathsf{Q}, \mathsf{Q}' \in \Omega_\lambda^+$ we have the detailed balance identity

$$\Pi(\mathsf{Q})\, U_b(\mathsf{Q} \to \mathsf{Q}') = \Pi(\mathsf{Q}')\, U_b(\mathsf{Q}' \to \mathsf{Q}),$$

since $\Pi(\mathsf{Q}) \propto \mathrm{wt}(\mathsf{Q})$ and $\alpha_b(\mathsf{Q} \to \mathsf{Q}') = \min\{1, \mathrm{wt}(\mathsf{Q}')/\mathrm{wt}(\mathsf{Q})\}$. Thus U_b is Π-reversible. Since U_{base} is a convex combination of Π-reversible kernels, it is also Π-reversible, hence Π-stationary. $\square$

Proposition 1.15 (Aperiodicity with an explicit self-loop)**.** *For every* $\mathsf{Q} \in \Omega_\lambda^+$,

$$U_{\mathrm{base}}(\mathsf{Q} \to \mathsf{Q}) \;\geq\; \sum_{b \in \mathcal{B}} p_b \cdot \frac{1}{|\mathcal{S}_b(\mathsf{Q})|} \;>\; 0.$$

In particular, U_{base} *is aperiodic on each of its communicating classes.*

Proof. Fix Q and $b \in \mathcal{B}$. By Lemma 1.9, $\mathsf{Q} \in \mathcal{S}_b(\mathsf{Q})$, so with probability $p_b \cdot (1/|\mathcal{S}_b(\mathsf{Q})|)$ the proposal equals the current state and is accepted with probability 1. Since $|\mathcal{S}_b(\mathsf{Q})| \geq 1$, this contribution is strictly positive. Summing over b yields the bound. $\square$

Remark 1.16 (Stationarity vs. ergodicity)**.** The preceding results establish that Π is stationary for U_{base}, and that U_{base} is aperiodic on each communicating class. Global irreducibility (and hence convergence to Π from arbitrary starts) may require additional connectivity input; the optional Doeblinization in Section 1.6 is included if one wants an unconditional ergodicity statement on Ω_λ^+.

1.4 Genuine bottom-word motion from local moves

We now prove that the local base dynamics can change the bottom word, addressing the critique that a local chain might be fiber-confined. The mechanism is the bottom block $b = \mathrm{bot}$, whose variable part includes Row 1.

Definition 1.17 (A diagonal positive SMLQ with prescribed bottom word)**.** Fix $\mu \in S_n(\lambda)$. Define an SMLQ $\mathsf{Q}^{\mathrm{diag}}(\mu)$ by:

- In Row 1 (bottom regular row), place the word μ.
- For each label $a \in \{\lambda_1, \dots, \lambda_n\} \setminus \{0\}$, let $i(a)$ be the unique column with $\mu_{i(a)} = a$. For every regular row r with $1 \leq r \leq a$, place a regular ball labeled a in column $i(a)$; for $r > a$ that position is empty. For every signed row r' with $1 \leq r \leq a$, place a *positive* signed ball labeled $+a$ in column $i(a)$.
- In each signed layer $r' \to r$, pair each $+a$ trivially to the a directly below. In each classic layer $r \to (r-1)'$ $(r \geq 2)$, pair each a trivially to the $+a$ directly below.

Lemma 1.18 (Diagonal configuration is a valid SMLQ and has positive weight)**.** *For each* $\mu \in S_n(\lambda)$*, the object* $\mathsf{Q}^{\mathrm{diag}}(\mu)$ *is a BDW signed multiline queue in* $\Omega_\lambda(\mu)$ *[1, Defs. 1.4,1.6]. Moreover, at* $q = 1$*, if* $t \in (0,1)$ *and* $x_i > 0$ *for all* i*, then* $\mathrm{wt}(\mathsf{Q}^{\mathrm{diag}}(\mu)) > 0$.

Proof. Row contents: by construction, each regular row r contains exactly the labels $a \geq r$ (once each, since λ has distinct parts) and 0 elsewhere, hence is a permutation of $\lambda^{(r)}$ [1, Def. 1.5(a)]; each signed row r' is the same with all positive signs, hence a signed permutation of $\lambda^{(r)}$ [1, Def. 1.5(b)].

Layer admissibility: in every classic layer $r \to (r-1)'$, the lower row is signed and we apply [1, Def. 1.4] to the absolute values of signed labels; here each ball a has directly below it a ball with absolute label $|+a| = a$, so by [1, Def. 1.4] it *must* be trivially paired, which we do. In every signed layer $r' \to r$, each positive ball $+a$ has directly below it a ball with label $a' = a$, so by [1, Def. 1.6(b')] it *must* be trivially paired, which we do. Hence $\mathsf{Q}^{\text{diag}}(\mu) \in \Omega_\lambda(\mu)$.

Weight positivity: there are no negative signed balls, so every shifted ball-weight factor is a product of some $x_i > 0$ (and no negative factor from [1, Eq. (5)]). All pairings are trivial, so $\mathrm{wt}_{\text{pair}} = 1$. Therefore $\mathrm{wt}(\mathsf{Q}^{\text{diag}}(\mu)) > 0$. □

Lemma 1.19 (A local bottom move exists (swap the hole with the maximum label))**.** *Let $\mu \in S_n(\lambda)$ and let $M := \lambda_1$ be the unique maximum part. Let J be the unique hole position in μ ($\mu_J = 0$), and let I be the unique position with $\mu_I = M$. Let $\mu^{\text{swap}} \in S_n(\lambda)$ be obtained by swapping entries at I and J. Assume $q = 1$, $t \in (0,1)$, and $x_i > 0$ for all i.*

Then the bottom block set $\mathcal{S}_{\text{bot}}\big(\mathsf{Q}^{\text{diag}}(\mu)\big)$ contains a configuration Q^{swap} such that

$$\Phi(\mathsf{Q}^{\text{swap}}) = \mu^{\text{swap}} \qquad \textit{and} \qquad \mathrm{wt}(\mathsf{Q}^{\text{swap}}) > 0.$$

Consequently, U_{base} changes the bottom word with positive probability.

Proof. Construction inside the bottom block. Start from $\mathsf{Q}^{\text{diag}}(\mu)$ and modify *only* the variable part $V(\text{bot})$ (Definition 1.6):

- Set Row 1 to be μ^{swap}.
- Set Row $1'$ to be the positive signed word $+\mu^{\text{swap}}$ (same absolute values in the same columns).
- In the signed layer $1' \to 1$, pair each $+a$ trivially to a directly below.
- In the classic layer $2 \to 1'$, keep all trivial pairings except for the label M: Row 2 (fixed boundary) contains M in column I, while Row $1'$ now contains $+M$ in column J. Pair the top M at column I to the bottom M at column J with a shortest admissible strand traveling left-to-right (wrapping around the cylinder if necessary), as permitted in [1, Def. 1.4]. If there is a tie between two shortest choices on the cylinder, fix once and for all a deterministic tie-break (e.g. prefer the non-wrapping strand when available).

All other rows and layers remain unchanged. Denote the resulting global configuration by Q^{swap}.

Admissibility checks. We check explicitly the two relevant BDW constraints:

(a) Signed layer $1' \to 1$. Every ball in Row $1'$ is positive. At column k, Row $1'$ has $+\mu_k^{\text{swap}}$ and Row 1 has μ_k^{swap}. Thus if $\mu_k^{\text{swap}} = a > 0$, the ball below is $a' = a \geq a$ and by [1, Def. 1.6(b')] the pairing must be trivial; we chose it trivial. If $\mu_k^{\text{swap}} = 0$, there is no ball. Hence the signed layer is admissible.

(b) Classic layer $2 \to 1'$. Here the lower row is signed, and we apply [1, Def. 1.4] to the absolute values in Row $1'$. For every label $a \neq M$, Row 2 has a in column $i(a)$ and Row $1'$ has $+a$ in the same column (because swapping only affects the positions of 0 and M). Hence below a sits $a' = |+a| = a$, so by [1, Def. 1.4] it must be trivially paired; we keep it trivial.

For the label M in Row 2 at column I, the position directly below in Row $1'$ is now 0 (empty) because $\mu_I^{\text{swap}} = 0$. This is allowed by the second bullet of [1, Def. 1.4] (a ball may have an empty spot below). The MLQ/SMLQ requirement that each ball in the upper row is paired to a ball of the same label in the lower row is met by our nontrivial pairing from column I to column J, where Row $1'$ has $+M$ (absolute value M). The strand travels left-to-right and is chosen shortest (with the fixed tie-break if needed), as required by [1, Def. 1.4]. Thus the classic layer is admissible.

Therefore $\mathsf{Q}^{\mathrm{swap}} \in \Omega_\lambda$ and differs from $\mathsf{Q}^{\mathrm{diag}}(\mu)$ only on $V(\mathrm{bot})$, so $\mathsf{Q}^{\mathrm{swap}} \in \mathcal{S}_{\mathrm{bot}}(\mathsf{Q}^{\mathrm{diag}}(\mu))$. By construction its bottom word is μ^{swap}.

Positivity of the weight. All signed balls in Row $1'$ are positive, hence the shifted ball-weight factor contributed by Row $1'$ is a product of $x_i > 0$ (no negative factor from [1, Eq. (5)]). All signed-layer pairings in $1' \to 1$ are trivial, hence contribute no pairing factors.

The only nontrivial pairing is the classic pairing p of label M in the classic layer $2 \to 1'$. Its weight is given by [1, Eq. (4)]. Specializing [1, Eq. (4)] to $q = 1$ yields

$$\mathrm{wt}_{\mathrm{pair}}(p) = \frac{(1-t)\,t^{\mathrm{skip}(p)}}{1-t^{\mathrm{free}(p)}},$$

since the possible wrap factor q^{a-r+1} becomes 1 at $q = 1$. Here $\mathrm{skip}(p) \geq 0$, so $t^{\mathrm{skip}(p)} > 0$. Moreover, by definition of $\mathrm{free}(p)$ in [1, Def. 1.8, preceding Eq. (4)], the ball in the lower row that p pairs to is free immediately before placing p, hence $\mathrm{free}(p) \geq 1$. Therefore $1 - t^{\mathrm{free}(p)} > 0$ for $t \in (0,1)$, and $\mathrm{wt}_{\mathrm{pair}}(p) > 0$.

All other pairings are trivial, hence contribute weight 1. It follows that $\mathrm{wt}(\mathsf{Q}^{\mathrm{swap}}) > 0$.

Bottom-word motion under U_{base}. In a bot-update from $\mathsf{Q}^{\mathrm{diag}}(\mu)$, the proposal distribution is uniform on the finite set $\mathcal{S}_{\mathrm{bot}}(\mathsf{Q}^{\mathrm{diag}}(\mu))$ containing $\mathsf{Q}^{\mathrm{swap}}$. Hence the proposal probability is $p_{\mathrm{bot}} \cdot 1/|\mathcal{S}_{\mathrm{bot}}(\mathsf{Q}^{\mathrm{diag}}(\mu))| > 0$, and the Metropolis acceptance probability is strictly positive since both weights are positive. Thus U_{base} changes Φ with positive probability. □

1.5 Bottom marginal identification (polynomials appear only here)

Theorem 1.20 (Bottom-word marginal of Π equals π). *Under Assumption 1.3, for every $\mu \in S_n(\lambda)$,*

$$\Pi(\Phi = \mu) \;=\; \frac{\sum_{\Phi(\mathsf{Q})=\mu} \mathrm{wt}(\mathsf{Q})}{\sum_{\mathsf{Q}\in\Omega_\lambda} \mathrm{wt}(\mathsf{Q})} \;=\; \frac{F^*_\mu(\mathbf{x};1,t)}{P^*_\lambda(\mathbf{x};1,t)}.$$

Proof. By Definition 1.4, Π is the normalized BDW weight measure on Ω_λ (equivalently on Ω_λ^+ since weights are nonnegative). Thus

$$\Pi(\Phi = \mu) = \frac{\sum_{\mathsf{Q}:\Phi(\mathsf{Q})=\mu} \mathrm{wt}(\mathsf{Q})}{\sum_{\mathsf{Q}} \mathrm{wt}(\mathsf{Q})}.$$

The numerator is exactly $F^*_\mu(\mathbf{x};1,t)$ by BDW Definition 1.14 at $q = 1$ [1]. The denominator is $\sum_{\nu\in S_n(\lambda)} F^*_\nu(\mathbf{x};1,t) = P^*_\lambda(\mathbf{x};1,t)$ by definition of the orbit sum and [1, Thm. 1.15]. This yields the claim. □

1.6 Optional: a nonlocal Doeblinization U_{ind} for irreducibility

The base chain U_{base} is intentionally *local*. Proving its global irreducibility on Ω_λ^+ can be intricate. If one wants an *existence* proof of an ergodic chain without a connectivity lemma, one may add a purely technical nonlocal "independence" move. This is *not* the only bottom-word-moving mechanism (Lemma 1.19); it is included solely to force irreducibility/aperiodicity.

Definition 1.21 (Nonlocal independence MH kernel U_{ind}). Assume the finite set Ω_λ has been enumerated (global preprocessing). Let g be uniform on Ω_λ. Given $\mathsf{Q} \in \Omega_\lambda^+$, propose $\mathsf{Q}^{\mathrm{prop}} \sim g$ and accept with

$$\alpha_{\mathrm{ind}}(\mathsf{Q} \to \mathsf{Q}^{\mathrm{prop}}) = \begin{cases} 0, & \mathrm{wt}(\mathsf{Q}^{\mathrm{prop}}) = 0, \\ \min\left\{1, \frac{\mathrm{wt}(\mathsf{Q}^{\mathrm{prop}})}{\mathrm{wt}(\mathsf{Q})}\right\}, & \mathrm{wt}(\mathsf{Q}^{\mathrm{prop}}) > 0, \end{cases}$$

which is valid because $g(\mathsf{Q}) = g(\mathsf{Q}^{\mathrm{prop}})$.

Proposition 1.22 (Ergodic mixture kernel). *Fix $\varepsilon \in (0,1)$ and set*

$$U \;:=\; (1-\varepsilon)\,U_{\mathrm{base}} + \varepsilon\,U_{\mathrm{ind}}.$$

Then U is Π-reversible and Π-stationary on Ω_λ^+. Moreover, U is irreducible and aperiodic on Ω_λ^+.

Proof. Both U_{base} and U_{ind} are Metropolis kernels with symmetric proposals, hence Π-reversible and Π-stationary. A convex combination of Π-reversible kernels is Π-reversible and Π-stationary. Irreducibility holds because, for any $\mathsf{Q}, \mathsf{Q}' \in \Omega_\lambda^+$, the independence proposal chooses Q' with probability $1/|\Omega_\lambda| > 0$ and acceptance is strictly positive since $\mathrm{wt}(\mathsf{Q}'), \mathrm{wt}(\mathsf{Q}) > 0$. Aperiodicity follows from Proposition 1.15 and the mixture. □

2 Fiber refresh and strong lumpability

Section 1 provides a concrete SMLQ chain with the desired Gibbs stationary law and genuine bottom-word motion. However, the projection $\Phi(\mathsf{Q}_t)$ of U_{base} (or of U) need not itself be Markov. In this section we add an (optional) "fiber refresh" scaffold to enforce *strong lumpability* and thereby obtain an explicit Markov chain on bottom words.

2.1 The ideal fiber refresh kernel

Definition 2.1 (Fiber refresh kernel). For $\mu \in S_\lambda^+$, define the conditional (fiber) Gibbs distribution

$$\Pi_\mu(\mathsf{Q}) \;:=\; \Pi(\mathsf{Q} \mid \Phi = \mu) \;=\; \frac{\mathrm{wt}(\mathsf{Q})}{\sum_{\mathsf{R} \in \Omega_\lambda(\mu)} \mathrm{wt}(\mathsf{R})} \qquad (\mathsf{Q} \in \Omega_\lambda(\mu)).$$

Define a Markov kernel $\mathcal{R}_\mu$ on $\Omega_\lambda^+(\mu)$ by

$$\mathcal{R}_\mu(\mathsf{Q} \to \mathsf{Q}') \;:=\; \Pi_\mu(\mathsf{Q}') \qquad (\mathsf{Q}, \mathsf{Q}' \in \Omega_\lambda^+(\mu)),$$

i.e. $\mathcal{R}_\mu$ ignores its input and outputs an exact fiber sample.

Remark 2.2 (Implementability (global preprocessing)). Since $\Omega_\lambda(\mu)$ is finite, $\mathcal{R}_\mu$ can be implemented after *global* preprocessing by enumerating $\Omega_\lambda(\mu)$ and sampling by table lookup using the explicit BDW local weight formula [1, Def. 1.11]. This uses no evaluations of F_μ^* as a polynomial, but is computationally heavy and not "local".

2.2 A refreshed SMLQ chain and its stationarity

Let U be any Π-stationary kernel on Ω_λ^+ (e.g. $U = U_{\mathrm{base}}$ or the ergodic mixture of Proposition 1.22).

Definition 2.3 (Refreshed chain K). Define K on Ω_λ^+ by the two-step update: given current Q with $\mu = \Phi(\mathsf{Q})$,

(i) refresh: draw $\widetilde{\mathsf{Q}} \sim \Pi_\mu$ (i.e. apply $\mathcal{R}_\mu$);

(ii) move: apply one step of U from $\widetilde{\mathsf{Q}}$.

Equivalently, in kernel/matrix multiplication convention,

$$K \;=\; \mathcal{R}U, \qquad \text{meaning} \qquad K(\mathsf{Q} \to \mathsf{R}) = \sum_{\widetilde{\mathsf{Q}}} \mathcal{R}(\mathsf{Q} \to \widetilde{\mathsf{Q}})\, U(\widetilde{\mathsf{Q}} \to \mathsf{R}),$$

where $\mathcal{R}$ is the fiberwise kernel $\mathcal{R}(\mathsf{Q} \to \cdot) = \mathcal{R}_{\Phi(\mathsf{Q})}(\mathsf{Q} \to \cdot)$.

Proposition 2.4 (Π is stationary for K). *If U is Π-stationary on Ω_λ^+, then so is K.*

Proof. The refresh step preserves Π because it samples from the conditional distribution given Φ; formally, for any test function f,

$$\mathbb{E}_{\mathsf{Q}\sim\Pi}\big[f(\widetilde{\mathsf{Q}})\big] = \mathbb{E}_{\mu\sim\Pi\circ\Phi^{-1}}\Big[\mathbb{E}_{\widetilde{\mathsf{Q}}\sim\Pi_\mu}\big[f(\widetilde{\mathsf{Q}})\big]\Big] = \mathbb{E}_{\mathsf{Q}\sim\Pi}\big[f(\mathsf{Q})\big].$$

Thus Π is stationary for the refresh kernel $\mathcal{R}$. Since Π is stationary for U by assumption, it is stationary for the composition $K = \mathcal{R}U$. □

2.3 Strong lumpability and the explicit lumped kernel

Definition 2.5 (Strong lumpability criterion). Let K be a Markov kernel on a finite space Ω and $\Phi : \Omega \to \Sigma$ a surjection. We say K is *strongly lumpable* with respect to Φ if for all $\sigma, \sigma' \in \Sigma$ and all $\omega, \omega' \in \Phi^{-1}(\sigma)$,

$$\sum_{\eta\in\Phi^{-1}(\sigma')} K(\omega \to \eta) \;=\; \sum_{\eta\in\Phi^{-1}(\sigma')} K(\omega' \to \eta).$$

In this case the lumped kernel $\widetilde{K}$ on Σ is defined by

$$\widetilde{K}(\sigma \to \sigma') \;:=\; \sum_{\eta\in\Phi^{-1}(\sigma')} K(\omega \to \eta) \quad \text{for any } \omega \in \Phi^{-1}(\sigma),$$

and is well-defined and stochastic.

Theorem 2.6 (Strong lumpability of the refreshed chain). *Let K be the refreshed chain of Definition 2.3. Then K is strongly lumpable with respect to $\Phi : \Omega_\lambda^+ \to S_\lambda^+$. Moreover, for $\mu, \mu' \in S_\lambda^+$ the lumped kernel is*

$$\widetilde{K}(\mu \to \mu') = \sum_{\widetilde{\mathsf{Q}}\in\Omega_\lambda^+(\mu)} \Pi_\mu(\widetilde{\mathsf{Q}}) \sum_{\mathsf{R}\in\Omega_\lambda^+(\mu')} U(\widetilde{\mathsf{Q}} \to \mathsf{R}), \tag{1}$$

and $\widetilde{K}$ is a stochastic matrix on S_λ^+.

Proof. Fix $\mu, \mu' \in S_\lambda^+$ and two states $\mathsf{Q}, \mathsf{Q}' \in \Omega_\lambda^+(\mu)$. By definition of K,

$$K(\mathsf{Q} \to \mathsf{R}) = \sum_{\widetilde{\mathsf{Q}}\in\Omega_\lambda^+(\mu)} \mathcal{R}_\mu(\mathsf{Q} \to \widetilde{\mathsf{Q}})\, U(\widetilde{\mathsf{Q}} \to \mathsf{R}) = \sum_{\widetilde{\mathsf{Q}}\in\Omega_\lambda^+(\mu)} \Pi_\mu(\widetilde{\mathsf{Q}})\, U(\widetilde{\mathsf{Q}} \to \mathsf{R}),$$

which depends on Q only through $\mu = \Phi(\mathsf{Q})$. Summing over all $\mathsf{R} \in \Omega_\lambda^+(\mu')$ yields (1), which is therefore the same for Q and Q'. This proves strong lumpability and identifies the lumped kernel.

Stochasticity: for fixed μ,

$$\sum_{\mu'\in S_\lambda^+} \widetilde{K}(\mu \to \mu') = \sum_{\widetilde{\mathsf{Q}}\in\Omega_\lambda^+(\mu)} \Pi_\mu(\widetilde{\mathsf{Q}}) \sum_{\mathsf{R}\in\Omega_\lambda^+} U(\widetilde{\mathsf{Q}} \to \mathsf{R}) = \sum_{\widetilde{\mathsf{Q}}\in\Omega_\lambda^+(\mu)} \Pi_\mu(\widetilde{\mathsf{Q}}) \cdot 1 = 1.$$

□

Corollary 2.7 (Stationary distribution of the lumped chain). *Let π be the pushforward of Π under Φ, i.e. $\pi(\mu) = \Pi(\Phi = \mu)$ for $\mu \in S_\lambda^+$. Then π is stationary for the lumped bottom-word chain $\widetilde{K}$.*

Proof. Since Π is stationary for K (Proposition 2.4) and $\widetilde{K}$ is the strong lumping of K under Φ, the pushforward measure π is stationary for $\widetilde{K}$: if $\mathsf{Q}_0 \sim \Pi$ then $\mathsf{Q}_1 \sim \Pi$, hence $\Phi(\mathsf{Q}_0) \sim \pi$ and $\Phi(\mathsf{Q}_1) \sim \pi$ with transition kernel $\widetilde{K}$. □

3 Final word-chain consequences

3.1 A nontrivial Markov chain on bottom words with stationary π

Combining Theorem 1.20 and Theorem 2.6 yields the requested (supported) bottom-word chain.

Theorem 3.1 (A nontrivial word chain with stationary π and no polynomial-defined transitions)**.** *Assume Assumption 1.3 and take $U = U_{\text{base}}$ (or any Π-stationary kernel built from BDW local blocks and MH ratios, e.g. the ergodic mixture of Proposition 1.22). Let K be the refreshed chain of Definition 2.3 and $\widetilde{K}$ its strong lumping to S_λ^+ given by (1). Then:*

(i) $\widetilde{K}$ is a well-defined stochastic Markov kernel on S_λ^+.

(ii) Its stationary distribution is

$$\pi(\mu) \;=\; \frac{F_\mu^*(\mathbf{x};1,t)}{P_\lambda^*(\mathbf{x};1,t)} \qquad (\mu \in S_\lambda^+).$$

(iii) The kernel $\widetilde{K}$ is nontrivial *provided $|S_\lambda^+| \geq 2$; in the restricted setting with $t \in (0,1)$ and $x_i > 0$, Lemma 1.19 supplies explicit distinct μ, μ' with positive-weight configurations and hence positive transition probability.*

(iv) Neither U nor K nor $\widetilde{K}$ is defined using evaluations of F_μ^* or P_λ^*; only BDW admissibility rules and BDW local weight factors/ratios enter the transition rules.

Proof. Items (i) and the explicit kernel follow from Theorem 2.6. Item (ii) follows from Corollary 2.7 and Theorem 1.20. Item (iii) follows from Lemma 1.19: there exist explicit positive-weight configurations with bottom words μ and $\mu^{\text{swap}} \neq \mu$ connected by a single bottom-block move, hence $\widetilde{K}(\mu \to \mu^{\text{swap}}) > 0$ by (1). Item (iv) is by construction. □

3.2 Optional extension to all of $S_n(\lambda)$

The lumped chain is naturally defined on the support S_λ^+. If one insists on a chain on *all* of $S_n(\lambda)$, one may extend $\widetilde{K}$ by declaring every $\mu \notin S_\lambda^+$ absorbing (or by adding any stochastic transitions among the zero-mass states), which does not affect stationarity of π on $S_n(\lambda)$.

Conclusion

We constructed a concrete, local Metropolis–Hastings Markov chain U_{base} on the *full* BDW signed multiline queue state space Ω_λ^+ with stationary distribution $\Pi(\mathsf{Q}) \propto \mathrm{wt}(\mathsf{Q})$. The block definitions are stated consistently as "variable parts with fixed boundary data", ensuring that local updates do not disturb adjacent layers. A bottom block update changes the bottom word using only local BDW admissibility and local BDW weights, establishing genuine bottom-word motion without any global independence move. BDW's partition-function theorem then identifies the stationary bottom-word marginal as $\pi(\mu) \propto F_\mu^*(\mathbf{x};1,t)$. Finally, adding an ideal fiber refresh yields a strongly lumpable chain whose projection to bottom words is a genuine Markov chain $\widetilde{K}$ on $\mathrm{supp}(\pi)$ with stationary distribution π and transitions defined without invoking F_μ^* or P_λ^* values. If one also wants a fully unconditional ergodicity statement on Ω_λ^+ (rather than stationarity on communicating classes), one may use the optional Doeblinization U_{ind} from Section 1.6.

References

[1] H. Ben Dali and L. K. Williams, *A combinatorial formula for Interpolation Macdonald polynomials*, arXiv:2510.02587 (v2, Oct. 22, 2025).

[2] A. Ayyer, J. Martin, and L. K. Williams, *Multiline diagrams and t-PushTASEP / Macdonald measures at $q = 1$*, arXiv:2403.10485 (2024).