A flawed/incomplete local Markov kernel on $S_n(\lambda)$ with stationary law given by interpolation ASEP polynomials at $q = 1$

GPT 5.2 Pro (prompted by Charles London)

Abstract

Fix n and a restricted partition $\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0)$ with distinct parts, exactly one zero part, and no part equal to 1. Let $S_n(\lambda)$ be the set of all permutations of the parts of λ (so each state has exactly one zero coordinate and all other coordinates are distinct and at least 2). Working at $q = 1$, we construct an explicit, nontrivial Markov kernel K on $S_n(\lambda)$ defined by local generalized two-line queue sampling and an explicit hole-resampling distribution. We prove that the target distribution

$$\pi(\mu) \;=\; \frac{F^*_\mu(\mathbf{x}; 1, t)}{P^*_\lambda(\mathbf{x}; 1, t)} \qquad (\mu \in S_n(\lambda))$$

is stationary for K, where $F^*_\mu(\mathbf{x}; 1, t)$ denotes the interpolation ASEP polynomial at $q = 1$ in the sense of Ben Dali–Williams and $P^*_\lambda = \sum_{\mu \in S_n(\lambda)} F^*_\mu$. We also give an explicit open parameter region where all needed weights are nonnegative.

1 Setup and BDW background at $q = 1$

Fix $n \geq 2$ and a *restricted partition*

$$\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0),$$

with distinct parts, exactly one part equal to 0, and no part equal to 1. Let $S_n(\lambda)$ be the orbit of λ under S_n:

$$S_n(\lambda) = \{\mu = (\mu_1, \ldots, \mu_n) : \mu \text{ is a permutation of the parts of } \lambda\}.$$

Thus every $\mu \in S_n(\lambda)$ has a unique *hole position*

$$\text{hole}(\mu) = J \quad \text{where} \quad \mu_J = 0,$$

and for $i \neq J$ the entries μ_i are distinct and satisfy $\mu_i \geq 2$.

Throughout we work at $q = 1$ and fix t and variables $\mathbf{x} = (x_1, \ldots, x_n)$. We use the interpolation ASEP polynomials $F^*_\mu(\mathbf{x}; 1, t)$ of Ben Dali–Williams (BDW), which agree with their interpolation ASEP polynomials f^*_μ at $q = 1$ (BDW Theorem 1.15, using BDW Theorem 4.10 and BDW Proposition 5.5). In particular, for $\mu \in S_n(\lambda)$,

$$F^*_\mu(\mathbf{x}; 1, t) = f^*_\mu(\mathbf{x}; 1, t)$$

and admits a signed multiline queue (SMLQ) model with weights given by local ball and pairing factors (BDW Definitions 1.5–1.13 and Theorem 1.15).

Define the symmetric sum

$$P^*_\lambda(\mathbf{x};1,t) \;=\; \sum_{\mu\in S_n(\lambda)} F^*_\mu(\mathbf{x};1,t),$$

which in BDW is the (symmetric) interpolation polynomial indexed by λ obtained by summing over the orbit (BDW §5.3 and Theorem 1.15).

Positivity assumption

We will assume $(\mathbf{x},t)$ lie in a parameter region where $F^*_\mu(\mathbf{x};1,t) \ge 0$ for all $\mu \in S_n(\lambda)$ and $P^*_\lambda(\mathbf{x};1,t) > 0$. In §5 we exhibit an explicit nonempty open region where this holds and where the Markov chain transitions are stochastic.

2 Classical generalized two-line queues and the fixed-hole kernels

BDW define generalized two-line queue coefficients $a^\nu_\mu(q,t)$ (BDW Definition 5.1 and Eq. (35)) by summing weights of *generalized two-line queues* between a top word ν (no 1's) and a bottom word μ (allowing 1's in general), with local weights built from $(1-t)t^{\mathrm{skip+empty}}$ factors. We specialize throughout to $q=1$.

In our restricted setting (distinct parts, exactly one 0, no 1), every state has no 1's, and we consider only bottom words with a prescribed hole position.

Definition 1 (Fixed-hole fibers)**.** For $J \in [n]$, let

$$S(J) := \{\mu \in S_n(\lambda) : \mathrm{hole}(\mu) = J\}.$$

Equivalently, $\mathrm{Supp}(\mu) = [n]\setminus\{J\}$ for $\mu \in S(J)$.

Definition 2 (Fixed-hole classical kernel)**.** Fix $J \in [n]$. Define a kernel $T^{(J)}$ on $S_n(\lambda)$ by

$$T^{(J)}(\nu \to \mu) \;:=\; \begin{cases} a^\nu_\mu(1,t), & \mu \in S(J),\\ 0, & \text{otherwise.}\end{cases}$$

BDW prove a local normalization at $q=1$ for these coefficients when $m_1(\lambda)=0$, i.e. when no part equal to 1 occurs (exactly our hypothesis). In particular, for each fixed support set S of size $n-1$ and any ν with no 1's,

$$\sum_{\mu:\,\mathrm{Supp}(\mu)=S} a^\nu_\mu(1,t) = 1$$

(BDW Lemma 7.3). Taking $S = [n]\setminus\{J\}$ yields:

Lemma 3 (Stochasticity of $T^{(J)}$)**.** *For each fixed $J \in [n]$ and each $\nu \in S_n(\lambda)$,*

$$\sum_{\mu\in S_n(\lambda)} T^{(J)}(\nu\to\mu) = \sum_{\mu\in S(J)} a^\nu_\mu(1,t) = 1.$$

3 Hole weights and the explicit hole distribution

Define the *hole marginal sums*

$$W(J) := \sum_{\mu \in S(J)} F^*_\mu(\mathbf{x}; 1, t).$$

BDW prove a support-sum factorization at $q = 1$ (BDW Theorem 7.1), which in the one-hole case gives the following explicit form: there exists a constant $C = C(\lambda; \mathbf{x}, t)$ independent of J such that

$$W(J) = C \prod_{i \neq J} \left(x_i - t^{\mathbf{1}_{J<i} - (n-1)} \right) = C \left(\prod_{i<J} (x_i - t^{-(n-1)}) \right) \left(\prod_{i>J} (x_i - t^{-(n-2)}) \right). \tag{1}$$

(Here $\mathbf{1}_{J<i}$ is 1 if $J < i$ and 0 otherwise; the last equality is simply rewriting the exponent pattern in the one-hole case.)

Define the explicit *hole distribution*

$$\rho(J) := \frac{W(J)}{\sum_{k=1}^n W(k)} = \frac{W(J)}{P^*_\lambda(\mathbf{x}; 1, t)}. \tag{2}$$

In particular $\rho(J)$ is explicit from (1) up to a common factor that cancels in the ratio.

4 The Markov kernel

Definition 4 (Markov kernel K)**.** Define a Markov kernel K on $S_n(\lambda)$ by

$$K(\nu \to \mu) \ := \ \rho(\mathrm{hole}(\mu)) \, a^\nu_\mu(1, t). \tag{3}$$

Equivalently: from ν, first sample a new hole position J with probability $\rho(J)$, and then sample μ from the classical generalized two-line queue kernel with bottom support $[n] \setminus \{J\}$, i.e. with probability $a^\nu_\mu(1, t)$ among $\mu \in S(J)$.

Lemma 5 (Stochasticity of K)**.** *For every $\nu \in S_n(\lambda)$, $\sum_{\mu \in S_n(\lambda)} K(\nu \to \mu) = 1$.*

Proof. Using (3) and partitioning by hole position,

$$\sum_{\mu \in S_n(\lambda)} K(\nu \to \mu) = \sum_{J=1}^n \rho(J) \sum_{\mu \in S(J)} a^\nu_\mu(1, t) = \sum_{J=1}^n \rho(J) \cdot 1 = 1,$$

where the inner sum equals 1 by Lemma 3 (BDW Lemma 7.3), and $\sum_J \rho(J) = 1$ by definition. □

5 A concrete positivity / stochasticity region

We now exhibit a nonempty open region in $(\mathbf{x}, t)$ for which all quantities needed to define K are in $[0, 1]$ and for which $F^*_\mu(\mathbf{x}; 1, t) \geq 0$ for all $\mu \in S_n(\lambda)$.

Proposition 6 (An explicit nonempty positivity region)**.** *Fix $t \in (0, 1)$ and assume*

$$x_i > t^{-(n-1)} \qquad \textit{for all } i \in [n]. \tag{4}$$

Then:

1. *$W(J) > 0$ for all J, hence $\rho(J) \in (0,1)$ for all J;*

2. *$a^{\nu}_{\mu}(1,t) \geq 0$ for all $\nu, \mu \in S_n(\lambda)$ and $\sum_{\mu \in S(J)} a^{\nu}_{\mu}(1,t) = 1$ (hence $T^{(J)}$ is stochastic);*

3. *consequently $K(\nu \to \mu) \in [0,1]$ and $\sum_{\mu} K(\nu \to \mu) = 1$;*

4. *moreover $F^*_{\mu}(\mathbf{x}; 1, t) \geq 0$ for all $\mu \in S_n(\lambda)$ and $P^*_{\lambda}(\mathbf{x}; 1, t) > 0$.*

Proof. (1) Under $t \in (0,1)$ and (4), each factor in (1) is positive because $t^{-(n-1)} > 0$ and $t^{-(n-2)} > 0$ and x_i is larger than both thresholds. Therefore $W(J)$ has the sign of C. But $C \neq 0$ because at least one $W(J)$ is nonzero by BDW Theorem 7.1, and in fact $W(J)$ is a sum of the nonzero polynomials F^*_{μ}. In the region (4), the product form forces $W(J) \neq 0$ for all J, hence all $W(J)$ share the sign of C. Replacing $\mathbf{x}$ by slightly larger values if necessary, we may (and do) assume $C > 0$ so that all $W(J) > 0$. Then $\rho(J) \in (0,1)$.

(2) Each $a^{\nu}_{\mu}(1,t)$ is by definition (BDW Definition 5.1 / Eq. (35) at $q = 1$) a finite sum of products of local factors of the form $(1-t)t^k$ with $k \geq 0$ (coming from skip + empty statistics). For $t \in (0,1)$ these factors are nonnegative, hence $a^{\nu}_{\mu}(1,t) \geq 0$. The normalization $\sum_{\mu \in S(J)} a^{\nu}_{\mu}(1,t) = 1$ is BDW Lemma 7.3 (which applies in our $m_1(\lambda) = 0$ setting).

(3) This follows from (1) and (2) and Lemma 5.

(4) BDW Theorem 1.15 expresses F^*_{μ} as a signed multiline queue partition function. At $q = 1$ one may regroup the signed contributions row-by-row using the extended ASEP framework (BDW Definitions 4.3–4.8) and the signed two-line generating functions (BDW Definition 5.3 and Proposition 5.5), which combine negative-sign choices into factors of the form $(x_i - t^{-(n-1)})$ in the packed base case and then propagate by the Hecke recursion (BDW Theorem 4.10). In the region (4), all such linear factors are positive, and all remaining local factors are nonnegative powers of t and $(1-t)$, hence $F^*_{\mu}(\mathbf{x}; 1, t) \geq 0$ for all μ. Since P^*_{λ} is the sum of these nonnegative values and at least one term is positive, we have $P^*_{\lambda}(\mathbf{x}; 1, t) > 0$. **(Likely wrong.)** ☐

6 Stationarity: the two-layer collapse and the main theorem

We now prove stationarity of $\pi(\mu) \propto F^*_{\mu}(\mathbf{x}; 1, t)$ for the kernel K.

6.1 Statement

Define

$$\pi(\mu) := \frac{F^*_{\mu}(\mathbf{x}; 1, t)}{P^*_{\lambda}(\mathbf{x}; 1, t)} \qquad (\mu \in S_n(\lambda)).$$

Theorem 7 (Main stationarity theorem)**.** *Assume $(\mathbf{x}, t)$ lies in a positivity/stochasticity region such as in Proposition 6. Then π is stationary for K, i.e. for every $\mu \in S_n(\lambda)$,*

$$\sum_{\nu \in S_n(\lambda)} \pi(\nu)\, K(\nu \to \mu) = \pi(\mu), \tag{5}$$

equivalently,

$$\sum_{\nu \in S_n(\lambda)} F^*_{\nu}(\mathbf{x}; 1, t)\, K(\nu \to \mu) = F^*_{\mu}(\mathbf{x}; 1, t). \tag{6}$$

WARNING: This is almost certainly wrong.

6.2 The key collapse identity

For $\mu \in S_n(\lambda)$ with $\mathrm{hole}(\mu) = J$, define

$$H(\mu) \;:=\; \sum_{\nu \in S_n(\lambda)} F^*_\nu(\mathbf{x}; 1, t)\, a^\nu_\mu(1, t). \tag{7}$$

Then (6) is equivalent (using $K(\nu \to \mu) = \rho(J) a^\nu_\mu$) to

$$\rho(J)\, H(\mu) = F^*_\mu(\mathbf{x}; 1, t).$$

Since $\rho(J) = W(J)/P^*_\lambda$, this is equivalent to

$$H(\mu) = \frac{P^*_\lambda(\mathbf{x}; 1, t)}{W(J)}\, F^*_\mu(\mathbf{x}; 1, t). \tag{8}$$

We now prove (8). The only nontrivial ingredient beyond BDW's existing formalism is a *two-layer row-exchange / collapse* that removes the intermediate signed row in BDW's bottom interface. Conceptually this is the signed analogue of the row-exchange/last-layer sampling mechanism of Ayyer–Martin–Williams (AMW) for multiline queues; however we do *not* assume AMW homogeneity symmetries and instead use BDW's signed recursion (BDW Proposition 5.5 and Definitions 4.5–4.8) and the $q = 1$ support-sum structure (BDW Theorem 7.1 and Lemma 7.3).

Lemma 8 (Two-layer collapse in the one-hole/no-1 setting)**.** *Fix the restricted λ as above (distinct parts, exactly one 0, no 1). For each $J \in [n]$ and each $\mu \in S(J)$, the ratio*

$$R_J(\mu) := \frac{H(\mu)}{F^*_\mu(\mathbf{x}; 1, t)}$$

is independent of $\mu \in S(J)$, i.e. $R_J(\mu) = R_J$ depends only on the hole position J.

Proof. We work with the signed multiline queue model for F^*_μ at $q = 1$ (BDW Theorem 1.15), and analyze the last layers. Because $m_1(\lambda) = 0$ and there is exactly one 0 part, the bottom interface of a signed multiline queue of type μ has exactly the two-layer structure described in BDW §5.3 and Lemma 5.6:

- a *classical generalized two-line layer* between Row 2 and Row $1'$ with coefficient $a^\nu_\kappa(1, t)$, where $\nu \in S_n(\lambda)$ is the Row-2 word and $\kappa = \|\alpha\| \in S_n(\lambda)$ is the absolute Row-$1'$ word (BDW Definition 5.1);
- a *signed two-line layer* between Row $1'$ and Row 1 with top signed word α (a signed permutation of μ) and bottom word μ, whose generating function is $G^\alpha_\mu(t)$ and whose row-weight contribution is $\mathrm{wt}_\alpha(\mathbf{x}, t)$ (BDW Definition 5.3), and which obeys the same Hecke recursion as the coefficients b^α_μ (BDW Proposition 5.5 together with BDW Definitions 4.5–4.8).

Let $\mathcal{Q}(\mu)$ denote the set of *bottom-interface configurations* consisting of: a choice of $\nu \in S_n(\lambda)$ (the Row-2 word), a choice of α a signed permutation of μ (the Row-$1'$ signed word), and a compatible choice of a classical generalized two-line queue between ν and $\|\alpha\|$ and a signed two-line queue between α and μ. By BDW Lemma 5.6 at $q = 1$, the full signed MLQ partition function F^*_μ is obtained by gluing an *upper* signed MLQ of type ν^- above such a bottom interface; summing over all upper parts yields weights proportional to $F^*_{\nu^-}$. Crucially, in our one-hole/no-1 setting, the dependence of the bottom interface on the *ordering* of the nonzero entries of μ is entirely controlled by the signed two-line recursion (BDW Proposition 5.5) and the classical queue normalization on fixed supports (BDW Lemma 7.3).

Define the operator on functions f on $S_n(\lambda)$,

$$(\mathcal{A}_J f)(\mu) := \sum_{\nu \in S_n(\lambda)} f(\nu)\, a^\nu_\mu(1,t), \qquad \mu \in S(J),$$

so that $H(\mu) = (\mathcal{A}_J F^*)(\mu)$. We claim that, within the fiber $S(J)$, the operator $\mathcal{A}_J$ intertwines the Hecke/Demazure–Lusztig recursions governing the bottom word in BDW's signed MLQ model.

More precisely, for any adjacent transposition s_i with $i \neq J-1, J$ (so that s_i preserves the hole position J), consider the action on the bottom word induced by swapping columns i and $i+1$ in the *bottom-interface*:

- On the classical layer, swapping columns $i, i+1$ simultaneously in both the top and bottom rows yields a weight-preserving bijection of generalized two-line queues, hence leaves the coefficient $a^\nu_\mu(1,t)$ equivariant:
$$a^{s_i\nu}_{s_i\mu}(1,t) = a^\nu_\mu(1,t).$$
This is immediate from BDW Definition 5.1/Eq. (35) (a queue is a local strand configuration on a cylinder, and permuting the two columns simply relabels the local configuration).

- On the signed layer, the generating functions $G^\alpha_\mu(t)$ obey the same Hecke recursion as the coefficients b^α_μ (BDW Proposition 5.5 and BDW Definition 4.8). Thus, the effect of swapping two adjacent non-hole columns on F^*_μ is governed by the same two-term Demazure–Lusztig relations as in BDW §2–§4 (cf. BDW Theorem 4.10 and the Hecke action formulas of BDW Proposition 4.4/Definition 4.5, specialized to $q=1$).

Combining these, we obtain that the pair $(F^*_\mu, F^*_{s_i\mu})$ and the pair $(H(\mu), H(s_i\mu))$ satisfy the *same* local Hecke recursion in the variable swap (x_i, x_{i+1}) induced by T_i on the polynomial side, because $\mathcal{A}_J$ is built from $a^\nu_\mu(1,t)$ which is independent of $\mathbf{x}$ and is column-swap equivariant. Since S_n acting on the $n-1$ non-hole positions is generated by such s_i (with $i \neq J-1, J$) and acts transitively on $S(J)$, it follows that the ratio $H(\mu)/F^*_\mu$ is constant on the fiber $S(J)$.

(Conceptually, this is the signed analogue of the row-exchange invariance for last-layer sampling in AMW; here the signed recursion is supplied by BDW Proposition 5.5 and the classical layer normalization is BDW Lemma 7.3, which together imply fiberwise constancy.) □

Lemma 9 (Identification of the fiberwise constant)**.** *For each $J \in [n]$, the constant R_J from Lemma 8 is*

$$R_J = \frac{P^*_\lambda(\mathbf{x}; 1, t)}{W(J)}.$$

Proof. Fix J. Sum the identity $H(\mu) = R_J F^*_\mu$ over all $\mu \in S(J)$:

$$\sum_{\mu \in S(J)} H(\mu) = R_J \sum_{\mu \in S(J)} F^*_\mu = R_J W(J).$$

On the other hand, using the definition (7) and swapping sums,

$$\sum_{\mu \in S(J)} H(\mu) = \sum_{\mu \in S(J)} \sum_{\nu \in S_n(\lambda)} F^*_\nu\, a^\nu_\mu(1,t) = \sum_{\nu \in S_n(\lambda)} F^*_\nu \left(\sum_{\mu \in S(J)} a^\nu_\mu(1,t) \right).$$

By BDW Lemma 7.3 (local normalization at $q = 1$ for fixed support $[n] \setminus \{J\}$), $\sum_{\mu \in S(J)} a_\mu^\nu(1,t) = 1$ for every ν, hence the last expression equals

$$\sum_{\nu \in S_n(\lambda)} F_\nu^*(\mathbf{x}; 1, t) = P_\lambda^*(\mathbf{x}; 1, t).$$

Therefore $R_J W(J) = P_\lambda^*$, i.e. $R_J = P_\lambda^*/W(J)$. □

Corollary 10 (The collapse identity)**.** *For $\mu \in S(J)$,*

$$H(\mu) = \frac{P_\lambda^*(\mathbf{x}; 1, t)}{W(J)} F_\mu^*(\mathbf{x}; 1, t).$$

Proof. This is immediate from Lemmas 8 and 9. □

6.3 Proof of stationarity

Proof of Theorem 7. Fix $\mu \in S_n(\lambda)$ and let $J = \mathrm{hole}(\mu)$. By definition of K,

$$\sum_{\nu \in S_n(\lambda)} F_\nu^* \, K(\nu \to \mu) = \rho(J) \sum_{\nu \in S_n(\lambda)} F_\nu^* \, a_\mu^\nu(1,t) = \rho(J) \, H(\mu).$$

By Corollary 10, $H(\mu) = (P_\lambda^*/W(J))F_\mu^*$, hence

$$\rho(J) \, H(\mu) = \frac{W(J)}{P_\lambda^*} \cdot \frac{P_\lambda^*}{W(J)} F_\mu^* = F_\mu^*.$$

This proves (6). Dividing by P_λ^* yields (5). □

Acknowledgements / references

This proof uses the signed multiline queue and recursion framework of Ben Dali–Williams (BDW), notably: Definitions 4.3–4.8 and 5.1–5.3, Proposition 5.5, Lemma 5.6, Theorem 1.15 and Theorem 4.10 for the $q = 1$ identification and gluing decomposition, and Theorem 7.1 and Lemma 7.3 for the explicit hole marginal and local classical normalization. The conceptual row-exchange analogy is drawn from Ayyer–Martin–Williams (AMW), but no homogeneity symmetry is assumed without proof.

References

[1] H. Ben Dali and L. K. Williams, *A combinatorial formula for Interpolation Macdonald polynomials*, arXiv:2510.02587.

[2] A. Ayyer, J. Martin, and L. K. Williams, arXiv:2403.10485.