# Poputchiki: some fellow-traveler properties on finite graphs

Defense, M.S. Thesis in Applied and Computational Mathematics

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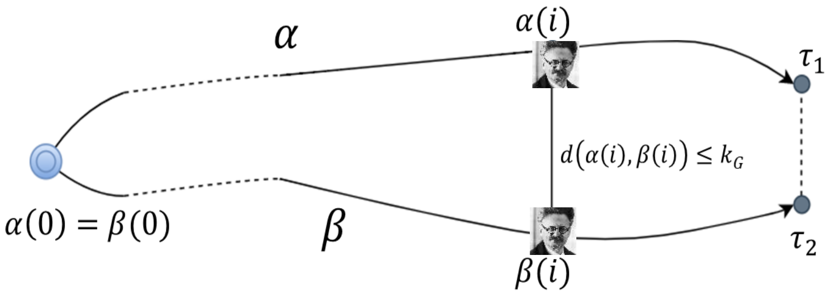
# Leon Trotsky once said…

- Russian Revolutionary Leon Trotsky once coined the term попутчик (*poputchik*, or "one who travels the same path") to refer to those supportive of the Bolshevik revolution but did not join the Soviet Communist Party.
- It has since been taken to mean "a person who hovers within some distance and magnitude of support with respect to a political movement, but does not formally commit to official membership of that movement."

Cassack, (1996).

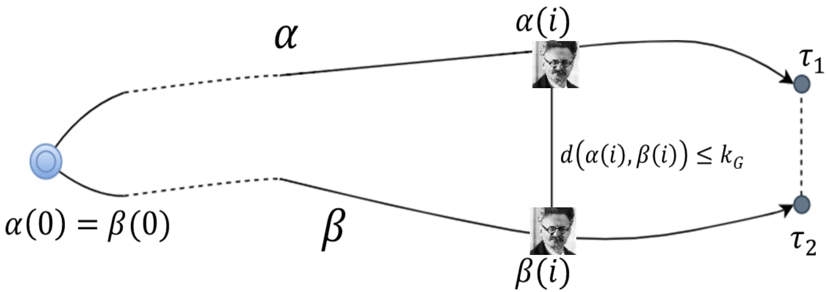
# A similar situation

- Imagine two cars, which leave a Wawa parking lot in Pennsauken, NJ, at exactly the same time.
- Suppose these cars travel exactly the same speed for the entire duration of their journey: car A chooses highway $\alpha$, and car B chooses highway $\beta$.
- We say the highways $\alpha$ and $\beta$ have the synchronous $k$-fellow-traveler property if the distance between the cars is bounded above by a constant $k$, in miles (or kilometers, if those are preferred.)



# A similar situation

- This thesis focuses on an instance of the $k$-fellow-traveler property on finite graphs
  - This instance enforces that the paths must end on either the same vertex or adjacent vertices.
  - Here, we refer to the constant $k$ over a graph $G$ as $k_G$.
- Commonly seen as an invariant on the Cayley graphs of certain groups (both infinite and finite)
  - We seek an application to finite non-Cayley graphs as well



# Outline

- Introduction
- Background
  - Group Theory
  - Graph Theory
- The $k$-fellow-traveler property: an exploration
  - Structural properties
  - Values of $k_G$
- Discussion of results and conclusion

# Preliminaries

# Research Questions

**Question #1:** Can we identify structural properties in finite, undirected, connected simple graphs by just examining the value of $k_G$? Is $k_G$ "interesting" at *all* on such graphs?

**Question #2:** When is $k_G$ equal to the length of the longest geodesic path on $G$? For which graphs is $k_G$ *not* equal to the length of the longest geodesic path, and why?

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**Question #2:** When is $k_G$ equal to the length of the longest geodesic path on $G$? For which graphs is $k_G$ *not* equal to the length of the longest geodesic path, and why?

**Conjecture #1:** Let $G$ be a finite simple graph with $\kappa(G) \geq 2$. Then $diam(G) - 1 \leq k_G \leq diam(G)$.

**Conjecture #2:** Let $G$ be a finite simple graph with $\kappa(G) \geq 2$. Then $k_G = diam(G) - 1$ if and only if $G$ is bipartite with odd diameter.

# Main contributions: short version

- Conjecture #1 is *true.*
- Conjecture #2 is *false* for general graphs with $\kappa(G) \geq 2$, but *true* when $G$ is nontrivial and vertex-transitive.
  - Conjecture #1 is true implies that, if $G$ is a graph with $k_G < diam(G) - 1$, then there is no way that $G$ does *not* have a cut vertex.
  - Conjecture #2 true for nontrivial vertex-transitive graphs implies that, if $\Gamma$ is the (undirected) Cayley graph of some finitely-generated group $\mathcal{G}$, then $k_\Gamma = diam(G) - 1$ implies $\Gamma$ is bipartite with odd diameter (and vice versa.)

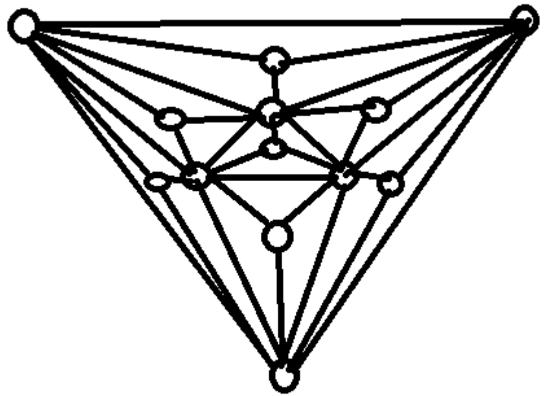
# Background

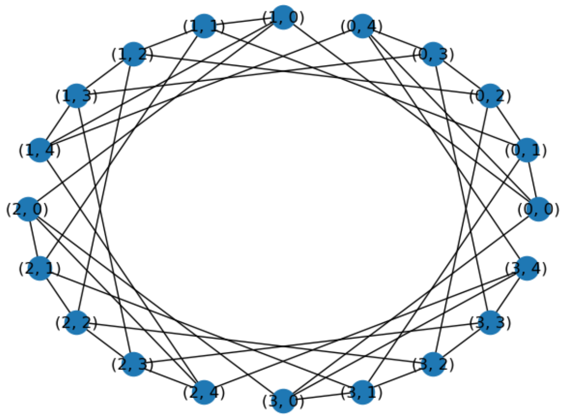
# Background: Relevant Group Theory

- **Definition 2.1.1.** Let $\mathcal{G}$ be a nonempty set and $(\cdot)$ a binary operation on $\mathcal{G}$. A group $(\mathcal{G},\cdot)$ is an ordered pair, where $(\cdot)$ satisfies the following:
  - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathcal{G}$ (Associativity)
  - $\exists e \in \mathcal{G}$ such that, for all $a, \in \mathcal{G}$, we have $a \cdot e = e \cdot a = a$ (Identity)
  - For each $a \in \mathcal{G}$, $\exists a^{-1} \in \mathcal{G}$ such that $a \cdot a^{-1} = a^{-1}\, a = e$ (Invertibility)
- Groups come in handy for all manner of pure and applied situations – cryptography, efficient network design, geometry, number theory, linguistics, topological data analysis, quantum computing, physics…
- **Definition 2.1.7.** (Group Generators). Let $\mathcal{G}$ be a group and let $S \subseteq \mathcal{G}$. We say $S$ **generates** $\mathcal{G}$ if each element of $\mathcal{G}$ can be expressed as a finite combination of the elements of $S$. If $|S| = 1$, then we say that $\mathcal{G}$ is **cyclic.** If $S$ is finite, then $\mathcal{G}$ is **finitely generated.**
- Easy example: the set $\mathbb{Z}$ and the addition operation $+$. We have that $\{1\}$ is a suitable generating set, since any integer $m$ can be expressed as the sum of $m$ ones (or negative ones, if $m$ is negative.)

# Background: Relevant Graph Terminology

- An *undirected graph* on $n$ vertices and $m$ edges is the pair $G = (V, E)$, where $V$ is the set of vertices with cardinality $n$, and $E$ is an edge-set with cardinality $m$ (note $E$ can be empty.) An *edge* $e = uv$ in $E$ connects two vertices, $u, v \in V$. We say that, if $u, v$ are connected by an edge, then they are *adjacent.*

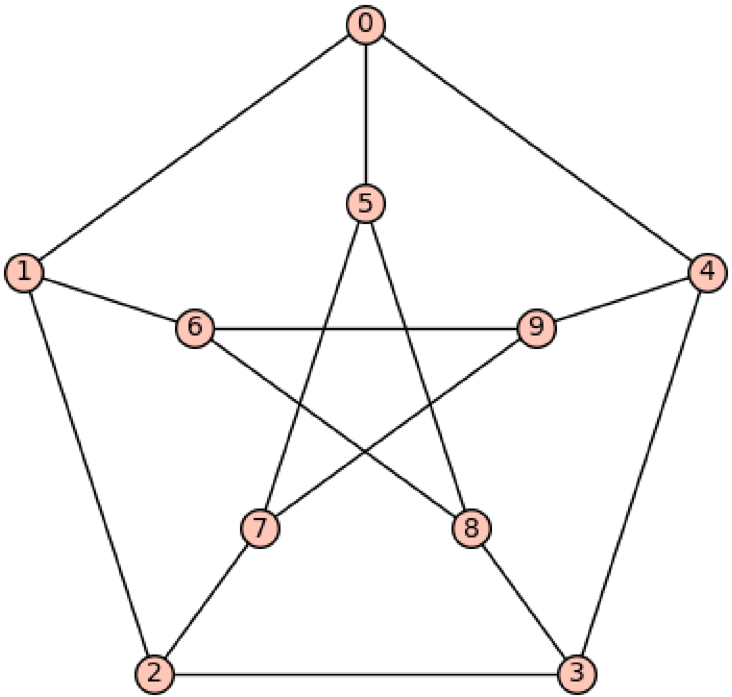





**Left:** torus grid graph $C_4 \times C_5$
**Above:** a planar graph on 12 vertices.

# Background: Relevant Graph Terminology

- If $u, v$ are vertices, then the *shortest-path distance function* between $u, v$ (often denoted by $d(u, v)$), is the length of the shortest $u - v$ path.
- A $u - v$ path whose length is equal to $d(u, v)$ is also referred to as the $u - v$ *geodesic.*
- The length of the longest geodesic path on a graph $G$ is referred to as the *diameter* of $G$, and is denoted by $\mathrm{diam}(G)$.



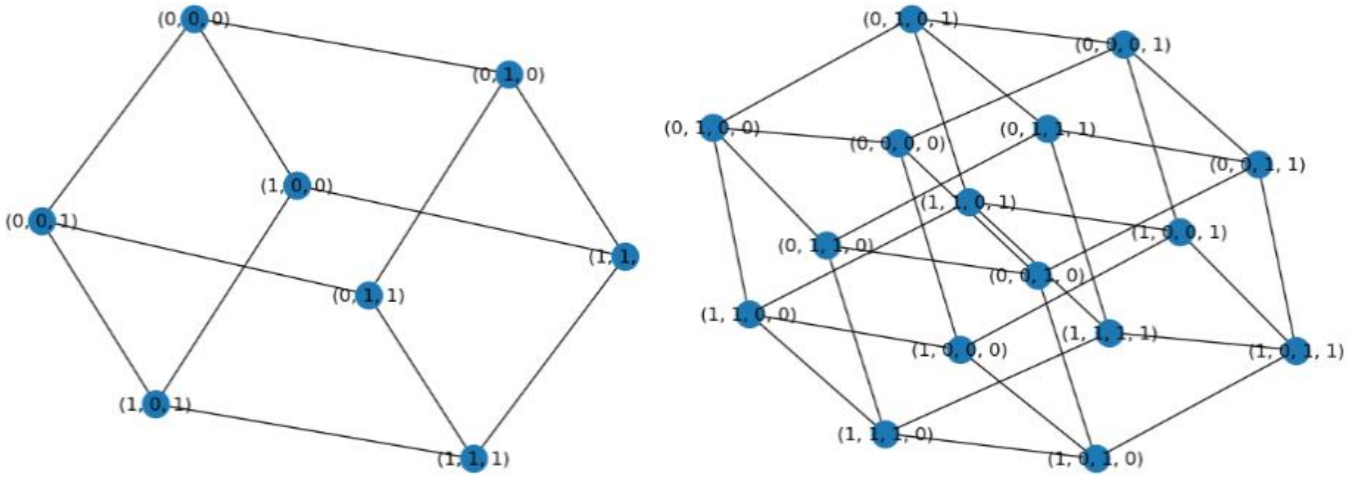

The Petersen graph has diameter $2$.

# Background: Relevant Graph Terminology

**Definition 2.2.2.** *Let $G$ be a graph. We say that $G$ is* **bipartite** *if $V(G)$ can be partitioned into two sets $U$ and $W$ (called partite sets) so that every edge of $G$ joins a vertex of $U$ and a vertex of $W$.*

**Theorem 2.2.3.** *A graph $G$ is bipartite if and only if it contains no odd cycles.*

# Background: Relevant Graph Terminology



(a) $n = 3$ (b) $n = 4$

Figure 2.2: An $n$-cube on $2^n$ vertices and $n2^{n-1}$ edges, shown here with $n = 3$ and $n = 4$, is both $n$-regular and bipartite.

# Background: Relevant Graph Terminology

Intuitively, a vertex transitive graph is one such that, given two vertices $u, v \in V(G)$, $u$ is not "distinguishable" from $v$ on $G$ by examining local graphical features (connections, neighbors, and degrees.)

A *Cayley graph* is a vertex-transitive graph, which is used to encode information about a group with respect to a group generator $X$.

- Vertex-set: the elements of $\mathcal{G}$
- Edge-set: if $a \in \mathcal{G}$ and $b \in X$, and $ab = c$ for some other group element $b \in \mathcal{G}$, then there exists a (directed) edge between $a$ and $c$.

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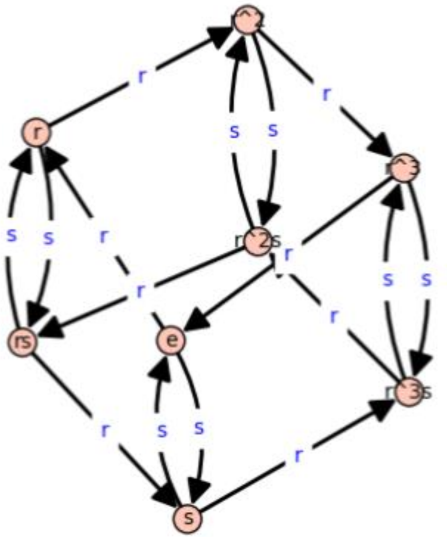
This subset may or may not contain inverses. A generating set $\Gamma$ for which, if $s \in \Gamma$ then $s^{-1} \in \Gamma$, is referred to as **symmetric**, and is isomorphic to an undirected graph.

# Background: Relevant Graph Terminology




(b) The Cayley graph for the dihedral group $D_4 = \langle r, s \mid r^4 = s^2, srs = r^{-1} \rangle$ with generating set $\{r, s\}$.

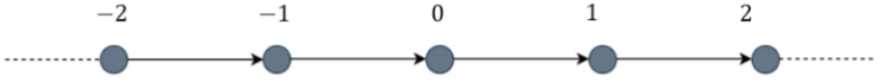
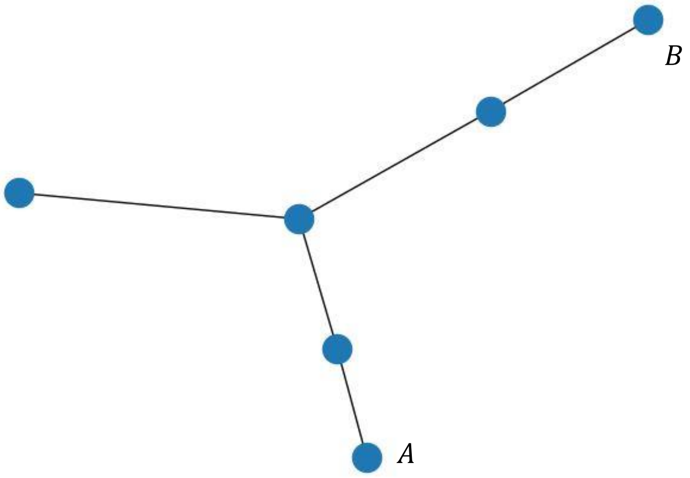



Figure 3.1: $\mathrm{Cay}(\mathbb{Z}, \{1\})$.

# $k$-fellow-traveling: full definition

- General definition: if $\alpha$ and $\beta$ are two paths in a metric space parameterized by arc-length, the **synchronous $k$-fellow traveler property** simply states that $d\big(\alpha(i), \beta(i)\big) \leq k$ for all $i$.
- Not that interesting on finite graphs.
- This thesis uses a special case, adapted from Holt, et. al (2017):
  - Let $G$ be a graph and let $\alpha, \beta$ be paths on $G$ of length $n$, which begin at the same "base point" (origin vertex) $\mathcal{O}$ and terminate on a single vertex $\tau$ or a pair of adjacent vertices $\tau_1, \tau_2$.
  - Here, $\alpha, \beta$ *synchronously* $k$*-fellow-travel* if $d\big(\alpha(i), \beta(i)\big) \leq k$ for some constant $k$ and all $0 \leq i \leq n$.



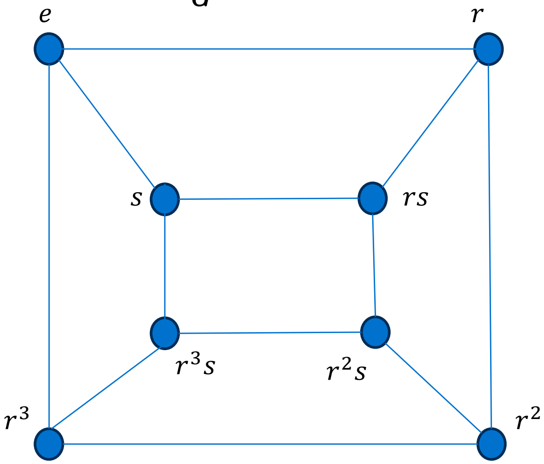

A tree. Under the second definition, $k_G = 0$, but under the first, $k_G = diam(G) = 4$.

# Fellow-traveling origins: why interesting?

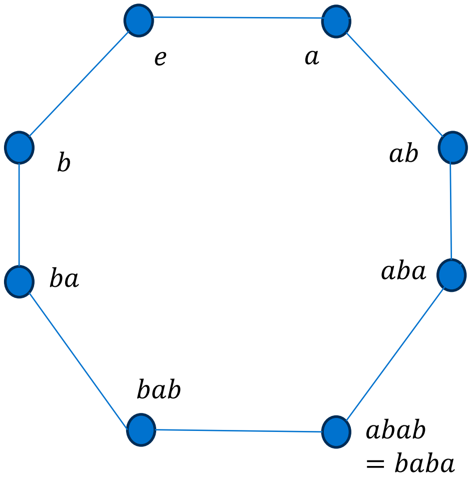
- Historically defined in terms of geodesic paths on the Cayley graph of a group (finite and infinite)
- In some groups, this property is present on *all* Cayley graphs of $\mathcal{G}$, independent of the generating set $X$
  - Aka, a group property and not a graph property
- All finite groups have Cayley graphs $G$ which possess the fellow-traveler property, since $d(u,v) \leq \mathrm{diam}(G)\ \forall u,v \in V(G)$.
  - The precise value of the constant $k_G$ can differ based on generating set (see the following slide), which indicates the constant value is a graphical property
- All finite connected graphs *also* have $d(u,v) \leq \mathrm{diam}(G)$, and, therefore, the fellow-traveler property.
- **This thesis: examine the structural implications of fellow-traveling, including on graphs which are *not* graphs of a group**

# Examples of $k_G$ on different Cayley graphs of the same group $D_4$

$k_G = 2$




$k_G = 4$




Generating sets:
{r,s} (rotation and symmetry, left)
and {a,b} (2 adjacent reflections, right)

Nicol, A (2014). What is…a Cayley Graph? https://math.osu.edu/sites/math.osu.edu/files/Cayley.pdf

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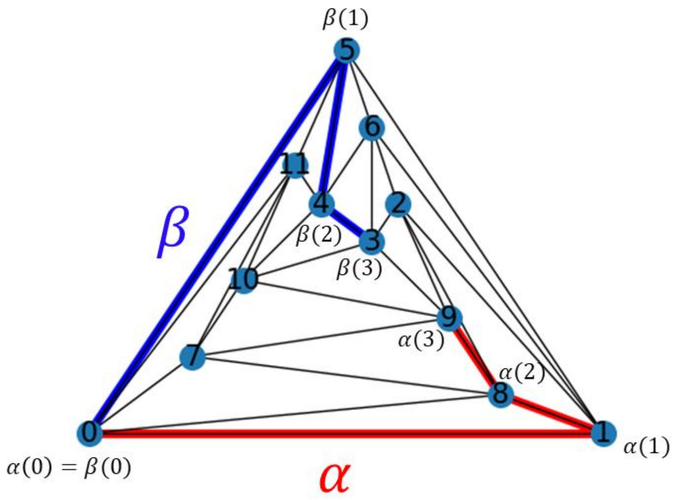
# Results: applications to finite undirected graphs

# Results: Assumptions

- We assume the following about a graph $G$:
  - $G$ is **finite** (with nonempty vertex set)
  - $G$ is **connected** (every vertex on $G$ is accessible by a path)
  - $G$ is **simple** (no multiedges or self-loops)
  - $G$ is **undirected** (no directed edges)
- The lengths of the fellow-traveling paths $\alpha, \beta$ need **not** be geodesic.

# Results: Format

- Split into two parts: *structural properties* and *explicit values of* $k_G$.
- Notation:
  - $\alpha, \beta$: synchronous fellow-traveling paths
  - $\mathcal{O}$: origin point $\alpha(0) = \beta(0)$
  - $\tau$ or $\tau_1, \tau_2$: terminal point $\alpha(n) = \beta(n)$ or adjacent terminal point(s) $\alpha(n) \neq \beta(n)$, $\alpha(n)\beta(n) \in E(G)$
  - $\alpha(i)$: $i^{th}$ vertex on $\alpha$ (and similarly for $\beta$.)




Fellow-travelers on an icosahedron.

Given a graph $G$ and a pair of vertices $u, v \in V(G)$, how do we know if there exist a pair of fellow-traveling paths $\alpha, \beta$ for which $u, v$ correspond?

i.e., $u = \alpha(i), v = \beta(i)$?

# General strategy for identifying whether $u, v$ fellow-travel with $u \neq v$

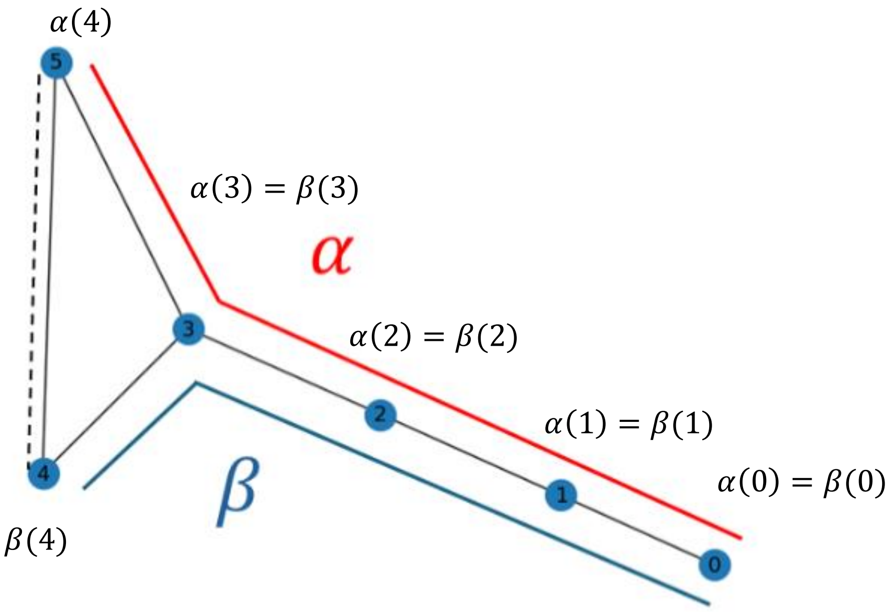



Figure 3.6: An acceptable set of fellow-traveling paths, where $d(u, v) \geq 0$ for each pair of $i^{\text{th}}$ vertices $(u, v)$, with $u \in \alpha$ and $v \in \beta$. Here, $\alpha$ and $\beta$ begin on '0' and terminate on the adjacent vertices '4' and '5'.

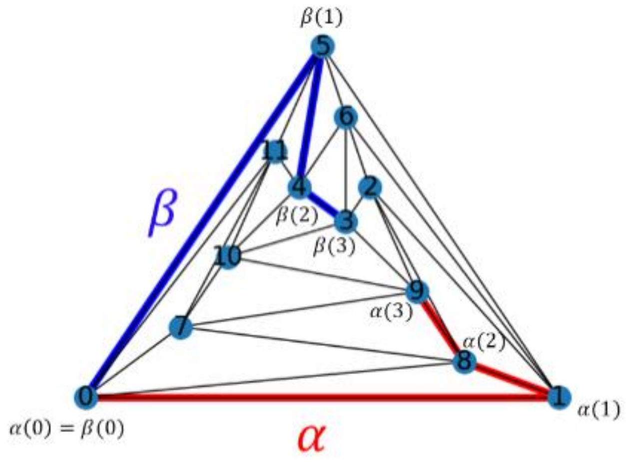



Figure 3.2: Fellow-traveling paths applied to an icosahedron.

# General strategy for identifying whether $u, v$ fellow-travel with $u \neq v$

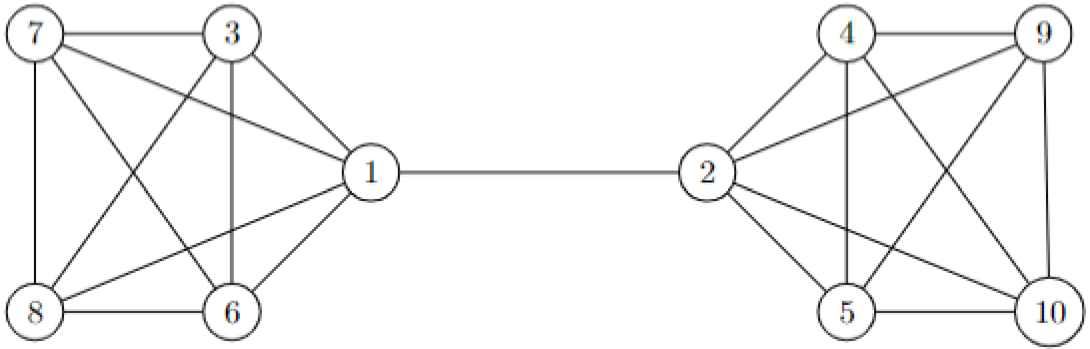



Figure 1: A barbell graph, possessing two $K_5$-cliques.

# Structural Properties

- **Theorem 3.3.7.** Let $G$ be a graph. Let $u, v \in V(G)$ with $u \neq v$. Then there exist $n$-length fellow-traveling paths $\alpha, \beta$ for which $u = \alpha(i), v = \beta(i)$, $0 \leq i \leq n$ (or $0 \leq i < n$) if and only if there exist **two internally vertex-disjoint paths between $u$ and $v$, with at least one path having even length.**

What is the intuition? How to prove?

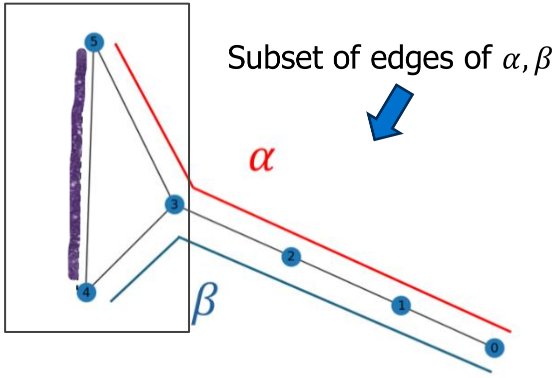
# General strategy for identifying whether $u, v$ fellow-travel with $u \neq v$ (forward direction sketch)

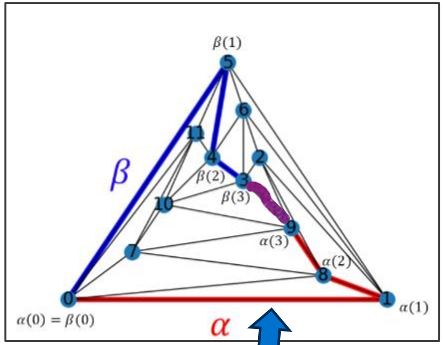
- Consider arbitrary vertices $u, v \in V(G)$
- Find two internally vertex-disjoint $u - v$ paths, such that at least one has even length
- Set the middle-vertex of the even path to be $\mathcal{O}$
- If the other path is even, set the middle-vertex of that path to be $\tau$
- If odd, then set the $\frac{(L-1)}{2}$th vertex  to be $\tau_1$ and the $\frac{L+1}{2}$th vertex to be $\tau_2$  (WLOG).
- If no such paths exist, then $u$ and $v$ cannot possibly correspond on any fellow-traveling paths on $G$.

# General strategy for identifying whether $u, v$ fellow-travel with $u \neq v$ (converse direction sketch)

- **Lemma 3.3.6.** Let $G$ be a graph, and let $u, v \in V(G)$ with $u \neq v$. Let $\alpha, \beta$ be fellow-traveling paths of length $n$ on $G$. If $u = \alpha(i)$ and $v = \beta(i)$ for $1 \leq i < n$ (or $1 \leq i \leq n$ if these paths terminate on adjacent vertices), **there exists a cycle containing $u, v$, which either contains all of $\alpha, \beta$ or a subset of the edges of $\alpha, \beta$.**

Subset of edges of $\alpha, \beta$




All of $\alpha, \beta$

# General strategy for identifying whether $u, v$ fellow-travel with $u \neq v$ (converse direction sketch)

- If $\alpha, \beta$ vertex-disjoint, we are done by the reasoning on the previous slide.
- By Lemma 3.3.6, $u, v$ belong to a cycle containing a subset of the edges of $\alpha, \beta$ or all $\alpha, \beta$
  - B/c $u, v$ belong to a cycle, then there exist two internally vertex-disjoint $u - v$ paths $P, Q$; assume none are even
  - If all $\alpha, \beta$ contained on $C$ then $\mathcal{O}$ belongs on $C$; the $u - v$ path $P_1$ which contains $\mathcal{O}$ must be even
  - Because $C$ a cycle, there is another $u - v$ path $P_2$ containing $\tau$ (or $\tau_1$, $\tau_2$) internally vertex-disjoint from $P_1$
- If just a subset of $\alpha, \beta$:
- We have that $\mathcal{O}$ does not belong to $C$, since then there is an even $u - v$ path on $C$
- Then $u \neq v$ implies there is a vertex $w = \alpha(j) = \beta(j), 0 \leq j < i$, along some $u - v$ path $P$ on $C$ (Lemma 3.3.6)
  - Why? We know that $u \neq v$ implies that, because $\alpha, \beta$ not internally vertex-disjoint, there exists a point $w = \alpha(j) = \beta(j),\ \ 0 \leq j < i,$ where $\alpha$ and $\beta$ "branch off"; otherwise, $\alpha(i)\alpha(i-1) = \beta(i)\beta(i-1)$, which implies that $u = v$.
- $P$ has even length because $u - w$ and $v - w$ paths are of the same length $(j - i)$; the $u - v$ path containing $w$ on $C$ has length $2(j - i)$

What are the graphs for which $k_G \neq diam(G)$ and why?

# Explicit values of $k_G$ (sketches)

- **Proposition 3.3.9.** If $G$ has $\kappa(G) \geq 2$ and $G$ is bipartite with even diameter, then $k_G = \mathrm{diam}(G)$.
- **Proposition 3.3.10.** If $G$ has $\kappa(G) \geq 2$ and $G$ is bipartite with odd diameter, then $k_G = \mathrm{diam}(G) - 1$.

- There exist no even paths between vertices an odd distance apart on a bipartite graph $G$.
  - There is no way $u, v$ can fellow-travel if $diam(G)$ is odd and $d(u, v) = diam(G)$.
- Applying Whitney's theorem to a pair of vertices $r, s$ an even distance apart to obtain at least 2 internally vertex-disjoint paths with *every* $r - s$ path even gives the result for bipartite graphs having odd diameter with $\kappa(G) \geq 2$.
- Similar logic applies if $diam(G)$ is even: all vertices with $d(u, v) = diam(G)$ fellow-travel if $G$ is bipartite with $\kappa(G) \geq 2$.

# Explicit values of $k_G$

- **Example 3.3.11.** Bipartiteness with odd diameter is *not* a necessary condition for $k_G = \operatorname{diam}(G) - 1$.

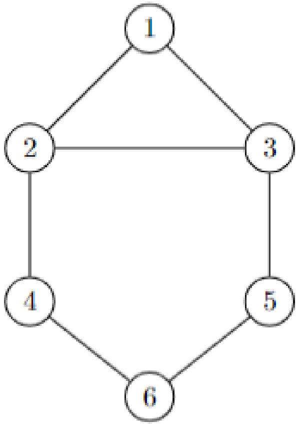



Figure 3.8: A non-bipartite graph $G$ with $\kappa(G) \geq 2$ such that $k_G = \operatorname{diam}(G) - 1$.

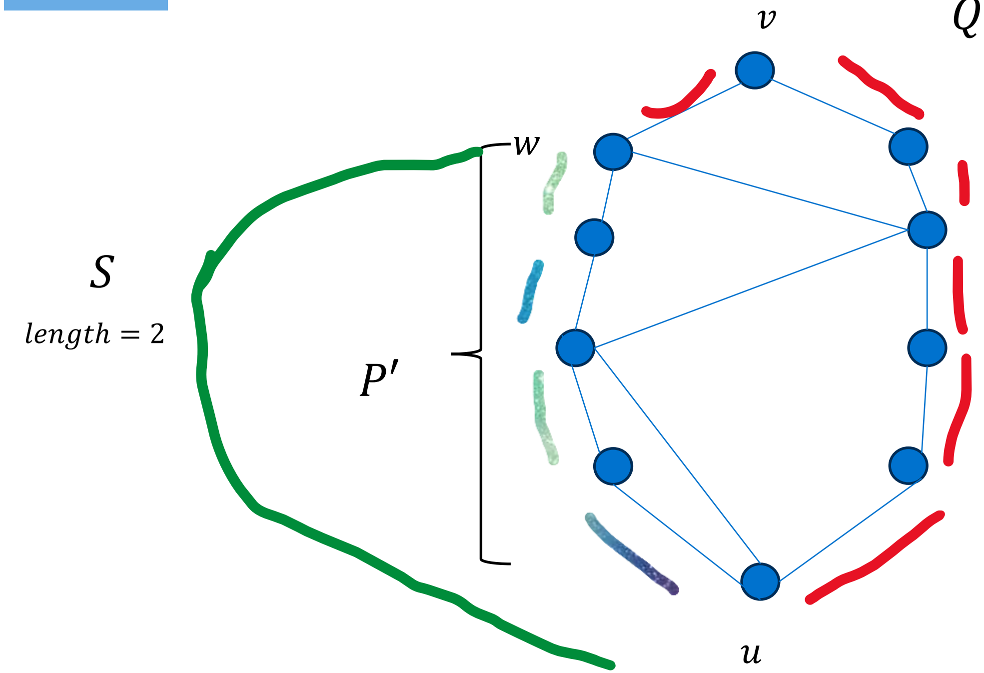
# Explicit values of $k_G$ (sketches)

- **Proposition 3.3.12.** If $G$ is a nontrivial *vertex-transitive* graph which is not bipartite, then $k_G = \text{diam}(G)$.
  - Non-bipartite implies odd cycle $C$ on $G$
  - $G$ is vertex transitive implies all vertices belong to a copy of $C$!
  - If all internally vertex-disjoint $u, v$ paths are odd between $u, v$ with $d(u, v) = diam(G)$, then ∃ vertices on $G$ that do not belong to $C$, contradiction.
- **Theorem 3.3.13.** Let $G$ be a nontrivial *vertex-transitive* graph. Then $k_G = \text{diam}(G) - 1$ if and only if $G$ is bipartite with odd diameter.
  - Forward direction is immediate (already shown if $G$ is bipartite with odd diameter and $\kappa(G) \geq 2$)
  - Converse direction: Assume $k_G = diam(G) - 1$. If $G$ non-bipartite, by Proposition 3.3.12, $k_G = diam(G)$. Hence, $G$ is bipartite. But if $G$ has even diameter, $k_G = diam(G)$, so it follows that $diam(G)$ is odd.

# Explicit values of $k_G$ (sketches)

- **Proposition 3.3.14.** Let $G$ be a graph. If $k_G < \mathrm{diam}(G) - 1$, then $\kappa(G) < 2$; i.e., $G$ has a cut vertex.
  - Suppose $\kappa(G) \geq 2$ and $k_G < diam(G) - 1$.
  - $G$ can't be bipartite, so $G$ must be non-bipartite.
  - Take vertices $u, v \in V(G)$ s.t. $d(u, v) = diam(G)$. Because $\kappa(G) \geq 2$, there are two internally vertex-disjoint $u - v$ paths: $P, Q$. Because $k_G \neq diam(G)$, both must be odd in length; i.e., $u, v$ do not fellow-travel.
  - WLOG there exists an even $u - w$ path $P' \subset P$ s.t. $|P'| = |P| - 1$ (e.g., $w, v$ are adjacent.) Furthermore, $|P'| \geq diam(G) - 1$ since $|P| \geq diam(G)$. (Equality only if odd diameter.)
  - $P'$ is internally vertex-disjoint from the $u - w$ path $Q' = Q \cup (P - P')$, so by Theorem 3.3.7, $u, w$ fellow-travel.
  - Because $k_G < diam(G) - 1$, there must be some other $u - w$ path $S$ (internally vertex disjoint from $P'$) having length at most $diam(G) - 2$.
  - Then there is a $u - w$ path of length at most $diam(G) - 2$, which implies the existence of a $u - v$ path of length $diam(G) - 2 + 1 = diam(G) - 1$. But then $d(u, v) \neq diam(G)$.
  - Hence, if $\kappa(G) \geq 2$, then $k_G \geq diam(G) - 1$.

# Visualization




$Q' = Q \cup (P - P')$

$diam(G) = 4$

# End result:

- **Theorem 3.3.15.** Let $G$ be a graph with $\kappa(G) \geq 2$. Then $diam(G) - 1 \leq k_G \leq diam(G)$.
- If $G$ bipartite, we are done.
- If not bipartite, then $\kappa(G) < 1$ if $k_G < diam(G) - 1$, so it follows that $k_G \geq diam(G) - 1$.

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# Summary and Conclusions

# Recall…

- **Question #1:** Can we identify structural properties in finite, undirected, connected simple graphs by just examining the value of $k_G$? Is $k_G$ "interesting" at *all* on such graphs?

- **Question #2:** When is $k_G$ equal to the length of the longest geodesic path on $G$? For which graphs is $k_G$ *not* equal to the length of the longest geodesic path, and why?

# Recall…

- **Question #1:** Can we identify structural properties in finite, undirected, connected simple graphs by just examining the value of $k_G$? Is $k_G$ "interesting" at *all* on such graphs?
  - **A:** Yes! We can say whether $\kappa(G) < 2$, whether $G$ is bipartite (if vertex-transitive) and whether $G$ is a tree. The value of $k_G$ turns out to be highly dependent on the structure of $G$.
- **Question #2:** When is $k_G$ equal to the length of the longest geodesic path on $G$? For which graphs is $k_G$ *not* equal to the length of the longest geodesic path, and why?
  - **A:** If $d(u, v) = diam(G)$ AND there are no vertex-disjoint $u - v$ paths on $G$ for which at least one path is even, for any such $u, v$, then $k_G < diam(G)$. If $\kappa(G) \geq 2$, then $k_G = diam(G) - 1$.

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# Questions?

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