Notes on

Derived Categories

Typed by

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Pisa

My original intention is to collect here some material about derived categories, with applications to algebraic geometry, which will hopefully turn out to be useful for my thesis. I will mainly follow the first chapters of the book "*Fourier-Mukai transforms in algebraic geometry*" by Daniel Huybrects [Huy06] for the part concerning the construction of derived categories. My final goal is to develop enough machinery to understand the proof of [ST00, Lemma 2.18].

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Chapter I.

Triangulated categories

I

While studying algebraic topology and geometry, abelian categories arise as a natural setting to develop homological algebra. In many cases, we come across long exact sequences in homology, e.g. the Mayer-Vietoris sequence for a topological space, which are a powerful tool for the study of our object of interest, starting from some sort of decomposition. Triangulated categories are a special kind of abelian categories, endowed with a *shift functor* which allows us to build long exact sequences of objects. Thanks to this, we will develop new techniques to study (co)homology objects.

Chain homotopy

The ideas in this section and the next are motivated by homotopy theory in topology.

Let $\mathcal{A}$ be an abelian category and consider two cochain complexes $C^\bullet, D^\bullet$ in $\mathcal{A}$. Consider a degree -1 map $s : C^\bullet \to D^\bullet$, that is a collection of morphisms

$$s^n : C^n \longrightarrow D^{n-1}\,, \quad n \in \mathbb{Z}\,.$$

For every $n \in \mathbb{Z}$, set $f^n := d_D^{n-1}s^n + s^{n+1}d_C^n$ to get a morphism $f^n : C^n \to D^n$:

$$\begin{tikzcd} C^{n-1} \arrow[r] & C^n \arrow[r, "d"] \arrow[dl, "s"'] \arrow[d, "f^n"] & C^{n+1} \arrow[dl, "s"] \\ D^{n-1} \arrow[r, "d"] & D^n \arrow[r] & D^{n+1}\,. \end{tikzcd}$$

Then the collection of f^n defines a cochain map $f^\bullet : C^\bullet \to D^\bullet$; indeed, dropping the superscripts for clarity, we compute

$$df = d(ds + sd) = dsd = (ds + sd)d = fd\,.$$

Definition. — A chain map $f^\bullet : C^\bullet \to D^\bullet$ is **null homotopic** if it is of the form $f^\bullet = ds + sd$, for some map s of degree -1. We will call s a **cochain contraction** of $f^\bullet$.

I.1. Lemma. — *If $f^\bullet : C^\bullet \to D^\bullet$ is* ***null homotopic****, then every induced map in cohomology $f^* : H^n(C^\bullet) \to H^n(D^\bullet)$ is zero.*

Proof. If we focus on the n-th level, we have a commutative diagram

$$\begin{tikzcd}
 & \ker d_C^n \arrow[d, hook] & \\
C^{n-1} \arrow[r, "d_C^{n-1}"] \arrow[d] & C^n \arrow[r, "d_C^n"] \arrow[d, "f^n"'] \arrow[dl, "s"'] & C^{n+1} \arrow[d] \arrow[dl, "s"'] \\
D^{n-1} \arrow[r, "d_D^{n-1}"'] \arrow[d, two heads] & D^n \arrow[r, "d_D^n"'] & D^{n+1} \\
\operatorname{im} d_D^{n-1} \arrow[ur, hook] & &
\end{tikzcd}$$

which shows that $\ker d_C^n \hookrightarrow \operatorname{im} d_D^{n-1}$; thus, by passing to the cokernel, the map $\ker d_C^n \to H^n(D^\bullet)$ is zero. Since this holds for every $n \in \mathbb{Z}$, we conclude that $f^* = 0$. □

Notice that this contraction construction gives us a way to "proliferate" cochain maps: indeed, given any morphism $g^\bullet : C^\bullet \to D^\bullet$, then $g^\bullet + (ds + sd)$ is again a cochain map. By the previous **Lemma**, it turns out that $g^\bullet + (ds + sd)$ is in fact not very different from $g^\bullet$, because they induce the same maps in cohomology!

Definition. — Two maps $f^\bullet, g^\bullet : C^\bullet \to D^\bullet$ are **cochain homotopic**, and write $f^\bullet \simeq g^\bullet$, if their difference is null homotopic, that is

$$f^\bullet - g^\bullet = ds + sd\,,$$

for some degree -1 map s. In this case, we call s a (**cochain**) **homotopy** from $f^\bullet$ to $g^\bullet$.

Finally, $f^\bullet : C^\bullet \to D^\bullet$ is a (**cochain**) **homotopy equivalence** if there exists $g^\bullet : D^\bullet \to C^\bullet$ such that

$$g^\bullet f^\bullet \simeq \mathbf{1}_{C^\bullet}\,, \quad f^\bullet g^\bullet \simeq \mathbf{1}_{D^\bullet}\,.$$

I.2. Corollary. — *If $f^\bullet, g^\bullet : C^\bullet \to D^\bullet$ are **cochain homotopic**, then $f^* = g^*$. In particular, if $f^\bullet$ is a homotopy equivalence, then the two complexes have the same cohomology: $H^n(C^\bullet) = H^n(D^\bullet)$.*

Definition. — A morphism of complexes $f^\bullet : A^\bullet \to B^\bullet$ is a **quasi-isomorphism** (shortened **qis**) if, for all $n \in \mathbb{N}$, the induced map

$$H^n(f^\bullet) : H^n(A^\bullet) \xrightarrow{\sim} H^n(B^\bullet)$$

is an isomorphism.

Thus, the previous corollary may be stated as "*homotopy equivalences are quasi-isomorphisms*".

Mapping cones and cylinders

Check sign conventions and computations of the differentials.

Let $\mathcal{A}$ be an abelian category, and consider the category $C^\bullet(\mathcal{A})$ of cochain complexes in $\mathcal{A}$.

Definition. — Let $f^\bullet : A^\bullet \to B^\bullet$ be a map of cochain complexes. The **mapping cone** of $f^\bullet$ is the cochain complex $\mathbf{C}(f^\bullet)$ whose degree n part is

$$\mathbf{C}(f^\bullet)^n := A^{n+1} \oplus B^n ,$$

and the coboundary maps $d_f^n : \mathbf{C}(f^\bullet)^n \longrightarrow \mathbf{C}(f^\bullet)^{n+1}$ are given by the matrices

$$d_f^n := \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix} \tag{I.2.1}$$

Definition. — Let $f^\bullet : A^\bullet \to B^\bullet$ be a map of cochain complexes. The **mapping cylinder** of $f^\bullet$ is the cochain complex $\mathbf{cyl}(f^\bullet)$ whose degree n part is

$$\mathbf{cyl}(f^\bullet)^n := A^n \oplus A^{n+1} \oplus B^n .$$

The coboundary map $d_{\mathbf{cyl}} : \mathbf{cyl}(f^\bullet) \to \mathbf{cyl}(f^\bullet)$ is given on the n-th level by the matrix

$$d_{\mathbf{cyl}}^n := \begin{pmatrix} d_A^n & \mathbf{1}_{A^{n+1}} & 0 \\ 0 & -d_A^{n+1} & 0 \\ 0 & -f^{n+1} & d_B^n \end{pmatrix} .$$

I.3. Exercise. — Show that two cochain maps $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ are cochain homotopic if and only if they extend to a map $(f, s, g) : \mathbf{cyl}(\mathbf{1}_{A^\bullet}) \to B^\bullet$.

Solution. Given a homotopy s from $f^\bullet$ to $g^\bullet$, we may define the map

$$\Psi^\bullet : \mathbf{cyl}(\mathbf{1}_{A^\bullet}) \longrightarrow B^\bullet , \quad \Psi^n(a, a', a'') = f^n(a) + s^{n+1}(a') + g^n(a'') .$$

Then $\Psi^\bullet$ is a cochain map: indeed, if we drop the indices for simplicity, we compute

$$\begin{aligned} \Psi^\bullet d = \Psi^\bullet \begin{pmatrix} d & \mathbf{1} & 0 \\ 0 & -d & 0 \\ 0 & -\mathbf{1} & d \end{pmatrix} &= \Psi^\bullet(d + \mathbf{1}, -d, -\mathbf{1} + d) \\ &= f^\bullet d + f^\bullet - sd - g^\bullet + g^\bullet d \\ &= f^\bullet d + ds + g^\bullet d \\ &= df^\bullet + ds + dg^\bullet = d\Psi^\bullet . \end{aligned}$$

The equations $\Psi^n(a, 0, 0) = f^n(a)$ and $\Psi^n(0, 0, a'') = g^n(a'')$ show that $\Psi^\bullet$ extends both $f^\bullet$ and $g^\bullet$.

Conversely, suppose an extension $\Psi^\bullet : \mathbf{cyl}(\mathbf{1}_{A^\bullet}) \to B^\bullet$ is given and write

$$j_2 = (0, \mathbf{1}, 0) : A^{\bullet+1} \to \mathbf{cyl}(\mathbf{1}_{A^\bullet})$$

for the inclusion. Then $s := \Psi j_2$ defines a homotopy between $f^\bullet$ and $g^\bullet$, indeed

$$ds = d\Psi^\bullet j_2 = \Psi^\bullet d j_2 = \Psi^\bullet \begin{pmatrix} d & \mathbf{1} & 0 \\ 0 & -d & 0 \\ 0 & -\mathbf{1} & d \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{1} \\ 0 \end{pmatrix} = \Psi^\bullet(\mathbf{1}, -d, -\mathbf{1}) = f^\bullet - sd - g^\bullet .$$

$\square$

I.4. Lemma. — *The inclusion map*

$$\iota := 0 \oplus 0 \oplus \mathbf{1}_{B^\bullet} : B^\bullet \to \mathbf{cyl}(f^\bullet)$$

is a quasi-isomorphism.

Proof. The above inclusion fits in the following short exact sequence:

$$\mathbf{0} \longrightarrow B^\bullet \xrightarrow{\ \iota\ } \mathbf{cyl}(f^\bullet) \xrightarrow{\ \pi\ } \mathbf{C}(\mathbf{1}_{A^\bullet}) \longrightarrow \mathbf{0}\,,$$

where π is the projection on the first two components, switched: indeed, the inclusion ι is clearly a cochain map, and so is the projection because for every $n \in \mathbb{Z}$ it holds

$$\begin{aligned} \pi \circ d^n_{\mathbf{cyl}}(x) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} d^n_A & \mathbf{1}_{A^{n+1}} & 0 \\ 0 & -d^{n+1}_A & 0 \\ 0 & -f^{n+1} & d \end{pmatrix} x \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -d^{n+1}_A & 0 \\ \mathbf{1}_{A^{n+1}} & d^n_A \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} x \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} = d^n_{\mathbf{1}} \circ \pi(x) \end{aligned}$$

Thus, by the cohomological version of Theorem **??** we get the long exact sequence

$$\ldots \longrightarrow H^{n-1}\big(\mathbf{C}(\mathbf{1}_{A^\bullet})\big) \longrightarrow H^n(A^\bullet) \longrightarrow H^n\big(\mathbf{cyl}(f^\bullet)\big) \longrightarrow H^n\big(\mathbf{C}(\mathbf{1}_{A^\bullet})\big) \longrightarrow \ldots$$

If we show that the cone has trivial cohomology, then it follows that $H^*(B^\bullet) \simeq H^*\big(\mathbf{cyl}(f^\bullet)\big)$. By the definition of the coboundary maps (I.2.1), any n-cocycle is of the form $(-d_A a, a)$, for some $a \in A^n$, and in fact it is also a coboundary:

$$\begin{pmatrix} -d^n_A(a) \\ a \end{pmatrix} = \begin{pmatrix} -d^n_A & 0 \\ \mathbf{1}_{A^n} & d^{n-1}_A \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = d^n \begin{pmatrix} a \\ 0 \end{pmatrix}\,,$$

hence we conclude $H^*\big(\mathbf{C}(\mathbf{1}_{A^\bullet})\big) \simeq 0$. □

I.5. Exercise. — Show that the map $\beta^\bullet : \mathbf{cyl}(f^\bullet) \to B^\bullet$, defined at the n-th level by

$$\beta^n(a, a', b) = f^n(a) + b\,,$$

defines a cochain map such that $\beta^\bullet \iota = \mathbf{1}_{B^\bullet}$. Then show that the formula

$$s(a, a', b) = (0, a, 0)$$

defines a cochain homotopy from the identity of $\mathbf{cyl}(f^\bullet)$ to $\iota\beta^\bullet$. Conclude that ι is in fact a cochain homotopy equivalence.

Proof. It is clear that $\beta^\bullet \iota = \mathbf{1}_{B^\bullet}$; we show it is a map of complexes:

$$\beta^\bullet d_{\mathbf{cyl}} = \beta^\bullet \begin{pmatrix} d_A & \mathbf{1}_{A^{\bullet+1}} & 0 \\ 0 & -d_A & 0 \\ 0 & -f^\bullet & d_B \end{pmatrix} = f^\bullet d_A + f^\bullet - f^\bullet + d_B = d_B(f^\bullet + \mathbf{1}) = d_B \beta^\bullet\,.$$

To conclude ι is a homotopy equivalence, we show $\mathbf{1_{cyl}} \simeq \iota\beta^\bullet$:

$$\begin{aligned} sd + ds &= (0, d_A + \mathbf{1}_{A^{\bullet+1}}, 0) + (\mathbf{1}_A, -d_A, -f^\bullet) \\ &= (\mathbf{1}_A, \mathbf{1}_{A^{\bullet+1}}, \mathbf{1}_B - (f^\bullet + \mathbf{1}_B)) = \mathbf{1_{cyl}} - \iota\beta^\bullet\,. \end{aligned}$$

□

Bicomplexes

Guarda Rotman per definizioni e esempi.

The homotopy category of complexes

There are many formal similarities between homological algebra and algebraic topology. The Dold-Kan correspondence, for example, provides a dictionary between positive complexes and simplicial theory. The algebraic notions of chain homotopy, mapping cones, and mapping cylinders have their historical origins in simplicial topology.

Let $\mathcal{A}$ be an abelian category and consider the category $C^\bullet(\mathcal{A})$ of cochain complexes in $\mathcal{A}$. We now define the **homotopy category** $\mathbf{K}(\mathcal{A})$ of $C^\bullet(\mathcal{A})$ as follows: we take as objects the same of $C^\bullet(\mathcal{A})$, i.e. cochain complexes, and the morphisms of $\mathbf{K}(\mathcal{A})$ to be *chain homotopy equivalence classes of maps* in $C^\bullet(\mathcal{A})$: for any two cochains $A^\bullet, B^\bullet$, it holds

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{C^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)/\sim\ ,$$

where the relation $\sim$ is given by homotopy equivalence. Notice that the quotient naturally inherits the sum: given $f^\bullet \sim \tilde{f}^\bullet$ and $g^\bullet \sim \tilde{g}^\bullet$, if s is a homotopy between $f^\bullet$ and $\tilde{f}^\bullet$ and t is a homotopy between $g^\bullet$ and $\tilde{g}^\bullet$, then $s+t$ is a homotopy between $f^\bullet + g^\bullet$ and $\tilde{f}^\bullet + \tilde{g}^\bullet$, indeed

$$\begin{aligned} (s+t)d_A^\bullet + d_B^\bullet(s+t) &= (sd_A^\bullet + d_B^\bullet s) + (td_A^\bullet + d_B^\bullet t) \\ &= (f^\bullet - \tilde{f}^\bullet) + (g^\bullet - \tilde{g}^\bullet) \\ &= (f^\bullet + g^\bullet) - (\tilde{f}^\bullet + \tilde{g}^\bullet)\,, \end{aligned}$$

thus $f^\bullet + g^\bullet \sim \tilde{f}^\bullet + \tilde{g}^\bullet$. It follows that $\mathbf{K}(\mathcal{A})$ is an additive category and the quotient functor

$$C^\bullet(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A})$$

is an additive functor.

Sometimes it is useful to consider categories of complexes having special properties: if $\mathcal{C}$ is any *full* subcategory of $C^\bullet(\mathcal{A})$, let $\mathbf{K}(\mathcal{C})$ denote the *full* subcategory of $\mathbf{K}(\mathcal{A})$ whose objects are the cochain complexes in $\mathcal{C}$. Then $\mathbf{K}(\mathcal{A})$ is a quotient category of $\mathcal{C}$; moreover, if $\mathcal{C}$ contains the zero object and it is closed under direct sum $\oplus$, then both $\mathcal{C}$ and $\mathbf{K}(\mathcal{C})$ are additive categories and the quotient $\mathcal{C} \to \mathbf{K}(\mathcal{C})$ is an additive functors.

Definition. — We write $\mathbf{K}^\flat(\mathcal{A}), \mathbf{K}^-(\mathcal{A})$ and $\mathbf{K}^+(\mathcal{A})$ for the full subcategories of $\mathbf{K}(\mathcal{A})$ corresponding to the full subcategories $C^\flat(\mathcal{A}), C^-(\mathcal{A})$ and $C^+(\mathcal{A})$ of bounded, bounded above, and bounded below cochain complexes respectively.

Having introduced the cast of categories, we turn to their properties.

I.6. Lemma. — *For every $n \in \mathbb{Z}$, the n-th cohomology H^n is a well defined functor on the quotient category:*

$$H^n : \mathbf{K}(\mathcal{A}) \longrightarrow \mathcal{A}\,.$$

Proof. We know that, if $f^\bullet \sim g^\bullet$, then $H^n(f^\bullet) = H^n(f^\bullet)$, thus cohomology descends to the quotient. □

The homotopy category $\mathbf{K}(\mathcal{A})$ is characterized by the following

I.7. Proposition (Universal property). — *Let $\mathcal{D}$ be a category and $F : C^\bullet(\mathcal{A}) \to \mathcal{D}$ be any functor that sends chain homotopy equivalences to isomorphisms. Then F factors uniquely through $\mathbf{K}(\mathcal{A})$, that is*

$$\begin{tikzcd} C^\bullet(\mathcal{A}) \arrow[r] \arrow[d, "F"'] & \mathbf{K}(\mathcal{A}) \arrow[dl, dashed, "\exists!"] \\ \mathcal{D}\,. \end{tikzcd}$$

Proof. Let $A^\bullet$ be any cochain complex in $\mathcal{A}$. As we have seen in Exercise I.5, the inclusion $\iota : A^\bullet \to \mathbf{cyl}(\mathbf{1}_{A^\bullet})$ is a homotopy equivalence, hence by assumption $F\iota$ is an isomorphism with inverse $F\beta^\bullet$, where $\beta(a, a', a'') = a'$. Moreover, notice that also the map

$$j : A^\bullet \longrightarrow \mathbf{cyl}(\mathbf{1}_{A^\bullet})\,, \quad a \longmapsto (a, 0, 0)$$

is such that $\beta^\bullet j = \mathbf{1}_{B^\bullet}$, so in particular it holds

$$Fj = F(\iota\,\beta^\bullet)Fj = F\iota\,F(\beta^\bullet j) = F\iota\,.$$

Suppose now there is a cochain homotopy s between $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$; by Exercise I.3, is extends to a map $\gamma^\bullet : \mathbf{cyl}(\mathbf{1}_{A^\bullet}) \to B^\bullet$ such that

$$\gamma^\bullet j = f\,, \quad \gamma^\bullet\iota = g\,,$$

thus in $\mathcal{D}$ we have

$$Ff = F\gamma^\bullet\,, Fj = F\gamma^\bullet\,, F\iota = Fg\,.$$

We conclude that F factors through the quotient $\mathbf{K}(\mathcal{A})$. □

We introduce now a useful operation we can perform on (co)chain complexes: translating indices. This concept of "shift" functor will be taken to define triangulated categories later.

Definition. — Let $A^\bullet \in C^\bullet(\mathcal{A})$ be a cochain complex. For any $p \in \mathbb{Z}$, we define the **p^{th}-translate** $A^\bullet[p]$ of $A^\bullet$ to be the complex whose n-th level objects and differentials are

$$(A^\bullet[p])^n := A^{n+p}\,, \quad d^n_{[p]} = (-1)^p d^{n+p}\,.$$

Dually, if $A_\bullet$ is a chain complex, we define its **p^{th}-translate** as

$$(A_\bullet[p])_n := A_{n+p}\,, \quad d^{[p]}_n = (-1)^p d_{n+p}\,.$$

By shifting indices on (co)chain maps accordingly, the p^{th}-translation defines a functor

$$[p] : C^\bullet(\mathcal{A}) \longrightarrow C^\bullet(\mathcal{A})$$

which is in fact an equivalence of categories (one checks that $[-p]$ is a quasi-inverse). Note that translations shift (co)homology: indeed

$$H^n(A^\bullet[p]) = H^{n+p}(A^\bullet)\,, \quad H_n(A_\bullet[p]) = H_{n-p}(A_\bullet)\,. \tag{I.7.1}$$

Definition. — Let $f^\bullet : A^\bullet \to B^\bullet$ be a cochain map. The mapping cone of $f^\bullet$ fits into a short exact sequence

$$\mathbf{0} \longrightarrow B^\bullet \xrightarrow{g^\bullet} \mathbf{C}(f^\bullet) \xrightarrow{\delta} A[-1]^\bullet \longrightarrow \mathbf{0}\,.$$

The **strict triangle** on $f^\bullet$ is the triple $(f^\bullet, g^\bullet, \delta)$ of maps in $\mathbf{K}(\mathcal{A})$, displayed as

$$\begin{tikzcd} A^\bullet \arrow[rr, "f^\bullet"] & & B^\bullet \arrow[dl, "g^\bullet"] \\ & \mathbf{C}(f^\bullet) \arrow[ul, "\delta"] & \end{tikzcd} .$$

Given three cochain complexes $X^\bullet, Y^\bullet, Z^\bullet \in \mathcal{A}$ and maps

$$u^\bullet : X^\bullet \to Y^\bullet, \quad v^\bullet : Y^\bullet \to Z^\bullet, \quad w^\bullet : Z^\bullet \to X[-1]^\bullet$$

in $\mathbf{K}(\mathcal{A})$, the triple (u, v, w) is an **exact triangle** on $(X^\bullet, Y^\bullet, Z^\bullet)$ if it is "isomorphic" to a strict triangle of $f^\bullet$, for some $f^\bullet : A^\bullet \to B^\bullet$, in the sense that there exists a diagram

$$\begin{tikzcd} X^\bullet \arrow[r, "u^\bullet"] \arrow[d, "\alpha^\bullet"] & Y^\bullet \arrow[r, "v^\bullet"] \arrow[d, "\beta^\bullet"] & Z^\bullet \arrow[r, "w^\bullet"] \arrow[d, "\gamma^\bullet"] & X[-1]^\bullet \arrow[d, "\alpha[-1]^\bullet"] \\ A^\bullet \arrow[r, "f^\bullet"] & B^\bullet \arrow[r] & \mathbf{C}(f^\bullet) \arrow[r] & A[-1]^\bullet \end{tikzcd}$$

which commutes in $\mathbf{K}(\mathcal{A})$, where the vertical arrows are isomorphisms (in $\mathbf{K}(\mathcal{A})$).

I.8. Lemma (LECS). — *Given an exact triangle (u, v, w) on $(A^\bullet, B^\bullet, C^\bullet)$, the long cohomology sequence*

$$\ldots \longrightarrow H^i(A^\bullet) \xrightarrow{u^*} H^i(B^\bullet) \xrightarrow{v^*} H^i(C^\bullet) \xrightarrow{w^*} H^{i+1}(A^\bullet) \longrightarrow \ldots$$

is exact, where we have identified $H^i(A^\bullet[-1]) = H^{i+1}(A^\bullet)$ as in (I.7.1).

Proof. For a strict triangle, the result is clear because H^i is additive and the sequence is split. For any other triangle, exactness follows because H^i is a functor on $\mathbf{K}(\mathcal{A})$, hence it preserves isomorphisms up to homotopy. $\square$

The following two technical results will be crucial for the definition of morphisms in the derived category of $\mathcal{A}$.

I.9. Proposition (Rooves composition). — *Given two morphisms $f^\bullet : A^\bullet \to B^\bullet$ and $g^\bullet : C^\bullet \to B^\bullet$, there exists a commutative diagram in $\mathbf{K}(\mathcal{A})$*

$$\begin{tikzcd} C_0^\bullet \arrow[r, "\tilde{f}^\bullet"] \arrow[d, "\tilde{g}^\bullet"'] & C^\bullet \arrow[d, "g^\bullet"] \\ A^\bullet \arrow[r, "f^\bullet"] & B^\bullet \end{tikzcd},$$

such that:

- *if $f^\bullet$ is a qis, then $\tilde{f}^\bullet$ is a qis too;*
- *if $g^\bullet$ is a qis, then $\tilde{g}^\bullet$ is a qis too.*

Proof. Consider the two following strict triangles:

$$\begin{tikzcd} A^\bullet \arrow[rr, "f^\bullet"] & & B^\bullet \arrow[dl, "\iota"] \\ & \mathbf{C}(f^\bullet) \arrow[ul, "\pi"] & \end{tikzcd}, \qquad \begin{tikzcd} \mathbf{C}(\iota \circ g^\bullet) \arrow[rr, "\varpi"] & & C^\bullet \arrow[dl, "\iota \circ g^\bullet"] \\ & \mathbf{C}(f^\bullet) \arrow[ul, "j"] & \end{tikzcd},$$

and define $\gamma^\bullet : \mathbf{C}(\iota \circ g^\bullet) \to A^\bullet[1]$ on the n-th level as

$$\gamma^n := \begin{pmatrix} 0 & \mathbf{1}_{A^{n+1}} & 0 \end{pmatrix} ;$$

one can check that $\gamma^\bullet$ actually defines a cochain map and, in fact, it makes the following diagram

$$\begin{tikzcd} \mathbf{C}(\iota \circ g^\bullet)[-1] \arrow[r, "\varpi"] \arrow[d, dashed, "\gamma^\bullet[-1]"'] & C^\bullet \arrow[r] \arrow[d, "g^\bullet"] & \mathbf{C}(f^\bullet) \arrow[r, "j"] \arrow[d, equal] & \mathbf{C}(\iota \circ g^\bullet) \arrow[d, dashed, "\gamma^\bullet"'] \\ A^\bullet \arrow[r, "f^\bullet"] & B^\bullet \arrow[r, "\iota"] & \mathbf{C}(f^\bullet) \arrow[r, "\pi"] & A^\bullet[1] \end{tikzcd}$$

commute in $\mathbf{K}(\mathcal{A})$: in fact, the rightmost box commutes in $C^\bullet(\mathcal{A})$ because

$$\gamma^\bullet \circ j = \begin{pmatrix} 0 & \mathbf{1}_{A^\bullet[1]} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathbf{1}_{A^\bullet[1]} & 0 \\ 0 & \mathbf{1}_{B^\bullet} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{A^\bullet[1]} & 0 \end{pmatrix} = \pi ,$$

while the map $h : \mathbf{C}(\iota \circ g^\bullet)[-1] \to B^\bullet[-1]$ given by $h = \begin{pmatrix} 0 & 0 & \mathbf{1}_{B^\bullet[-1]} \end{pmatrix}$ is a homotopy between $g^\bullet \circ \varpi$ and $f^\bullet \circ \gamma^\bullet[-1]$ because

$$\begin{aligned} hd_{-(\iota \circ g)} + d_{B^\bullet[-1]}h &= h \begin{pmatrix} -d_C & 0 \\ -(\iota \circ g^\bullet) & d_f \end{pmatrix} - d_B \begin{pmatrix} 0 & 0 & \mathbf{1}_{B^\bullet[-1]} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \mathbf{1}_{B^\bullet[-1]} \end{pmatrix} \begin{pmatrix} -d_C & 0 & 0 \\ 0 & -d_A & 0 \\ -g^\bullet & f^\bullet & d_B \end{pmatrix} + \begin{pmatrix} 0 & 0 & -d_B \end{pmatrix} \\ &= \begin{pmatrix} -g^\bullet & f^\bullet & 0 \end{pmatrix} = \begin{pmatrix} 0 & f^\bullet & 0 \end{pmatrix} - \begin{pmatrix} g^\bullet & 0 & 0 \end{pmatrix} = f^\bullet \circ \gamma^\bullet[-1] - g^\bullet \circ \varpi . \end{aligned}$$

This shows that the triple $(\gamma^\bullet[-1], g^\bullet, \mathbf{1})$ defines a morphism of triangles in $\mathbf{K}(\mathcal{A})$. Thus, by setting

$$C_0^\bullet := \mathbf{C}(\iota \circ g^\bullet)[-1] , \quad \tilde{f}^\bullet := \varpi , \quad \tilde{g}^\bullet = \gamma^\bullet[-1] ,$$

we get the desired commutative square in $\mathbf{K}(\mathcal{A})$.

Moreover, by applying the cohomology functor we obtain the following commutative diagram in $\mathcal{A}$:

$$\begin{tikzcd}
\dots \arrow[r] & H^{n-1}(\mathbf{C}(f^\bullet)) \arrow[r, "\tilde{f}^*"] \arrow[d, equal] & H^n(C_0^\bullet) \arrow[r] \arrow[d, "\tilde{g}^*"] & H^n(C^\bullet) \arrow[r] \arrow[d, "g^*"] & H^n(\mathbf{C}(f^\bullet)) \arrow[r] \arrow[d, equal] & \dots \\
\dots \arrow[r] & H^{n-1}(\mathbf{C}(f^\bullet)) \arrow[r] & H^n(A^\bullet) \arrow[r, "f^*"] & H^n(B^\bullet) \arrow[r] & H^n(\mathbf{C}(f^\bullet)) \arrow[r] & \dots
\end{tikzcd}$$

where the rows are long exact sequences. Thus, we notice that if $f^\bullet$ is a qis, then $\mathbf{C}(f^\bullet)$ has trivial cohomology, which implies that $\tilde{f}^* : H^*(C_0^\bullet) \simeq H^*(C^\bullet)$, so $\tilde{f}^\bullet$ is a qis. On the other hand, if $g^\bullet$ is a qis, then by the Five Lemma we deduce that also $\tilde{g}^\bullet$ is a qis. $\square$

Exercise. — One might be tempted to define $C_0^\bullet$ in the above Proposition I.9 as the fibered product

$$C_0^\bullet := A^\bullet \times_{B^\bullet} C^\bullet .$$

Show that, in general, this choice does not work and it does not guarantee the nice properties of the $C_0^\bullet$ built above.

An example. Let $B^\bullet$ be the complex

$$\mathbf{0} \longrightarrow B^0 \xrightarrow{d} B^1 \longrightarrow \mathbf{0} ,$$

where d is an epimorphism between non-trivial objets, with non trivial kernel $A := \ker d \neq \mathbf{0}$. Denote by $A^\bullet$ the complex with A concentrated in degree 0: then, there is a natural inclusion $\iota : A^\bullet \hookrightarrow B^\bullet$. Set $C^\bullet := B^\bullet[1]$ and consider the cochain map $\Delta : C^\bullet \to B^\bullet$ given by the diagram

$$\begin{tikzcd}
\mathbf{0} \arrow[r] & \mathbf{0} \arrow[r] \arrow[d] & B^0 \arrow[r, two heads, "d"] \arrow[d, two heads, "d"] & B^1 \arrow[r] \arrow[d] & \mathbf{0} \\
\mathbf{0} \arrow[r] & B^0 \arrow[r, two heads, "d"] & B^1 \arrow[r] & \mathbf{0} \arrow[r] & \mathbf{0} .
\end{tikzcd}$$

One can check that the fibered product $A^\bullet \times_{B^\bullet} C^\bullet$ is given by the complex

$$\mathbf{0} \longrightarrow A \hookrightarrow B^0 \xrightarrow{d} B^1 \longrightarrow \mathbf{0} ,$$

with projection morphisms on $A^\bullet$ and $C^\bullet$ being the obvious ones.

Now consider the 0-th cohomology of these complexes:

$$H^0(B^\bullet) = \operatorname{coker}(\mathbf{0} \to A) \simeq A = H^0(A^\bullet) ,$$

and $H^0(C^\bullet) \simeq H^{-1}(B^\bullet) = \mathbf{0}$. Since the 0-th cohomology object of the fibered product is again A, by applying the functor H^0 to the usual cartesian square one gets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d] & A \arrow[d, equal, "\iota^*"] \\
\mathbf{0} \arrow[r] & A ,
\end{tikzcd}$$

which does not commute in $\mathcal{A}$. This means that

$$\begin{tikzcd} A^\bullet \times_{B^\bullet} C^\bullet \arrow[r] \arrow[d] & A^\bullet \arrow[d, "\iota"] \\ C^\bullet \arrow[r] & B^\bullet \end{tikzcd}$$

does not commute in $\mathbf{K}(\mathcal{A})$. □

Triangulated categories

The notion of triangulated category generalizes the structure that exact triangles give to $\mathbf{K}(\mathcal{A})$. One should think of exact triangles as substitutes for short exact sequences.

Consider a category $\mathcal{D}$ equipped with an equivalence $T : \mathcal{D} \to \mathcal{D}$.

Definition. — A **triangle** on an ordered triple (A, B, C) of objects of $\mathcal{D}$ is a triple (α, β, δ) of morphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\delta} T(A)\,;$$

the triangle (α, β, δ) is usually displayed as follows:

$$\begin{tikzcd} A \arrow[rr, "\alpha"] & & B \arrow[dl, "\beta"] \\ & C \arrow[ul, "\delta"] & \end{tikzcd}\,.$$

A **morphism of triangles** is a triple (f, g, h) forming the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & T(A) \arrow[d, "T(f)"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & T(A')\,. \end{tikzcd}$$

Notation. — When dealing with the automorphism T, it will be convenient to write $A[1] := T(A)$ for any object $A \in \mathcal{D}$; with this idea, if we apply T (or its quasi-inverse) many times, we will write

$$A[n] := T^n(A)\,, \quad n \in \mathbb{Z}\,.$$

Similarly, for any morphism $f \in \mathrm{Hom}_{\mathcal{D}}(A, B)$, we put $f[n] := T^n(f)$.

Definition. — Let $\mathcal{D}$ be an additive category. The structure of a **triangulated category** on $\mathcal{D}$ is given by an additive equivalence $T : \mathcal{D} \longrightarrow \mathcal{D}$, called **shift functor**, and a set of **distinguished triangles** (α, β, δ) which are subject to the following four axioms:

(**TR1**) it consists of three points.

(i) For any $A \in \mathcal{D}$, the triangle $(\mathbf{1}_A, 0, 0)$ is distinguished.

(ii) If (α, β, δ) is a distinguished triangle and $(\alpha', \beta', \delta')$ is isomorphic to it, then $(\alpha', \beta', \delta')$ is distinguished too.

(iii) Any morphism $\alpha : A \to B$ can be completed to a distinguished triangle.

(**TR2**) ***Rotation*** – If (α, β, δ) is a distinguished triangle on (A, B, C), then both its "*rotates*" $(\beta, \delta, -\alpha[1])$ and $(-\delta[-1], \alpha, \beta)$ are distinguished triangles on $(B, C, A[1])$ and $(C[-1], A, B)$, respectively.

(**TR3**) ***Morphisms*** – Suppose there exists a commutative diagram of distinguished triangles with vertical arrows f and g:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "\exists\, h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]\,;
\end{tikzcd}
$$

then there exists a (**not** necessarily unique) morphism $h : C \to C'$ which completes it to a morphism of triangles.

(**TR4**) ***The octahedral axiom*** – Given six objects $A, B, C, A', B', C' \in \mathcal{D}$ and three triangles

$$
\begin{tikzcd}
A \arrow[rr, "\alpha"] & & B \arrow[dl, "i"] \\
& C' \arrow[ul, "\partial"] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
B \arrow[rr, "\beta"] & & C \arrow[dl, "j"] \\
& A' \arrow[ul, "d"] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
A \arrow[rr, "\beta\alpha"] & & C \arrow[dl, "l"] \\
& B' \arrow[ul, "\delta"] &
\end{tikzcd}
\quad ,
$$

then there exists a fourth distinguished triangle

$$
\begin{tikzcd}
C' \arrow[rr, "f"] & & B' \arrow[dl, "g"] \\
& A' \arrow[ul, "{i[1]\circ d}"] &
\end{tikzcd}
\quad ,
$$

such that in the following octahedron

$$
\begin{tikzcd}
& & B \arrow[drr, squiggly, "\beta"] \arrow[ddll, dashed, squiggly, "i"] & & \\
A' \arrow[urr, "d"] \arrow[d, "{i[1]\circ d}"'] & & & & C \arrow[llll, "j"'] \arrow[ddll, squiggly, "l"] \\
C' \arrow[rrrr, dashed, "\partial"] \arrow[drr, squiggly, "f"'] & & & & A \arrow[u, "\beta\alpha"'] \arrow[uull, dashed, "\alpha"] \\
& & B' \arrow[uull, "g"] \arrow[urr, "\delta"'] & &
\end{tikzcd}
\qquad \text{(I.9.1)}
$$

we have:

(i) the four distinguished triangles above forming four of the faces;

(ii) the remaining four faces commute, that boils down to

$$j = g \circ l\,, \quad \partial = \delta \circ f\,;$$

(iii) the most exterior paths connecting B and B' commute, that is

$$f \circ i = l \circ \beta\,, \quad d \circ g = \alpha \circ \delta\,.$$

Remark. — Some comments on axiom (**TR4**) are needed, because it is with no doubt the most mysterious and confusing one. First, we may "unpack" the octahedron (I.9.1) to give a nicer visualization: notice that the triangle $(f, g, i[1] \circ d)$ completes the commutative plane diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, equal] & B \arrow[r, "i"] \arrow[d, "\beta"] & C' \arrow[r, "\partial"] \arrow[d, dashed, "\exists\, f"] & A[1] \arrow[d, equal] \\
A \arrow[r] & C \arrow[r, "l"] \arrow[d, "j"] & B' \arrow[r, "\delta"] \arrow[d, dashed, "\exists\, g"] & A[1] \arrow[d, "\alpha[1]"] \\
& A' \arrow[r, equal] \arrow[d, "d"] & A' \arrow[r, "d"] \arrow[d, dashed, "\exists"] & B[1] \\
& B[1] \arrow[r] & C'[1] & .
\end{tikzcd}$$

We may interpret the ***Octahedral Axiom*** as a generalization of the **Third Isomorphism Theorem** we find in abelian categories: indeed, if we think of distinguished triangles as an analogous of exact sequences in abelian categories, then the three triangles in (**TR4**) tell us that

$$C' \simeq B/A\ , \quad B' \simeq C/A\ , \quad A' \simeq C/B\ ,$$

thus the ***Octahedral Axiom*** states that $(C/A)/(B/A) \simeq C/B$. We may visualize it in the following nice diagram: the three lines passing through A, B and C completely determine the sequence $C' \to B' \to A'$ on the right:

$$\begin{tikzcd}
& & C' \arrow[dr, dashed, "f"] & & \\
& B \arrow[ur, "i"] \arrow[dr, "\beta"] & & B' \arrow[ddr, dashed, "g"] & \\
& & C \arrow[ur, "l"] \arrow[drr, "j"] & & \\
A \arrow[uur, "\alpha"] \arrow[urr] & & & & A'\,.
\end{tikzcd}$$

Remark. — The definition of triangulated category may be modified by changing the forth axiom: for instance, consider the statement

(**TR4'**) given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r, "\delta"] \arrow[d, dashed, "\exists\, h"] & A[1] \arrow[d] \\ A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r, "\delta'"] & A'[1] \end{tikzcd}$$

whose rows are distinguished triangles, there exists a morphism $h : C \to C'$ that makes the diagram commutative and makes the mappying cone a distinguished triangle:

$$B \oplus A' \xrightarrow{\begin{pmatrix} -\beta & 0 \\ g & \alpha' \end{pmatrix}} C \oplus B' \xrightarrow{\begin{pmatrix} -\delta & 0 \\ h & \beta' \end{pmatrix}} A[1] \oplus C' \xrightarrow{\begin{pmatrix} -\alpha[1] & 0 \\ f[1] & \delta' \end{pmatrix}} B[1] \oplus A'[1] \,.$$

Assuming (**TR1**), (**TR2**) and (**TR3**), it turns out that (**TR4'**) is equivalent to (**TR4**). Moreover, it is clear that (**TR4'**) implies (**TR3**) and its strength relies on the fact that it describes the way we can complete a morphism of distinguished triangles. There are many other different (but equivalent) axioms we can choose to modify the ***Octahedral Axiom***: for details, check [Nee01].

Example. — Recall that, given a category $\mathcal{C}$, its opposite category $\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$, but arrows are "reversed", i.e. $f^{op} : B \to A$ is an morphism in $\mathcal{C}^{op}$ if and only if there exists $f : A \to B$ in $\mathcal{C}$. In fact, this shows that we have a bijection

$$(-)^{op} : \mathrm{Hom}_{\mathcal{C}}(A, B) \simeq \mathrm{Hom}_{\mathcal{C}^{op}}(B, A) \,.$$

If $\mathcal{D}$ is a triangulated category, the opposite category $\mathcal{D}^{op}$ is triangulated as well: if T is the shift functor in $\mathcal{D}$, set

$$t : \mathcal{D}^{op} \longrightarrow \mathcal{D}^{op} \,, \quad \left(B \xrightarrow{f^{op}} A \right) \longmapsto \left(T^{-1}B \xrightarrow{(T^{-1}f)^{op}} T^{-1}A \right)$$

to be the shift functor in $\mathcal{D}^{op}$ and say that a sequence

$$C \xrightarrow{g^{op}} B \xrightarrow{f^{op}} A \xrightarrow{h^{op}} tC$$

is a distinguished triangle in $\mathcal{D}^{op}$ if and only if

$$C[-1] \xrightarrow{h} A \xrightarrow{f} B \xrightarrow{g} C$$

is a distinguished triangle in $\mathcal{D}$. It is easy to check that these two notions define a triangulated structure on $\mathcal{D}^{op}$: indeed, all the properties (**TR1**)-(**TR4**) hold because they are true in $\mathcal{D}$, so we can build "opposite" triangles easily; strictly speaking, distinguished triangles in $\mathcal{D}^{op}$ are the same of $\mathcal{D}$, but with reversed indexing.

Example. — For any category $\mathcal{C}$, let $\mathcal{C}^{\mathbb{Z}}$ be the category of graded objects in $\mathcal{C}$, i.e. families $A_* = \{ A_n \}_{n \in \mathbb{Z}}$ indexed on the integers, a morphism $f : A_* \to B_*$ being a family of morphisms $f_n : A_n \to B_n$, for each $n \in \mathbb{Z}$. This category is naturally endowed with a shift functor T given by the translation $TA_* := A[-1]_*$, that is

$$TA_n := A[-1]_n = A_{n-1} \,, \quad n \in \mathbb{Z} \,.$$

We now define distinguished triangles in $\mathcal{C}^{\mathbb{Z}}$ to be triplets (α, β, δ) such that, for every $n \in \mathbb{Z}$, the following sequence is exact:

$$A_n \xrightarrow{\alpha} B_n \xrightarrow{\beta} C_n \xrightarrow{\delta} A_{n-1} \xrightarrow{\alpha} B_{n-1} .$$

Consider now $\mathcal{C} = \mathbf{Ab}$. Then $\mathbf{Ab}^{\mathbb{Z}}$ is an abelian category (it is essentially $C_\bullet(\mathbf{Ab})$ with no boundary maps), and it clearly satisfies both (**TR1**)(i) and (**TR1**)(ii). Note that any morphism $\alpha : A_* \to B_*$ is embedded in a triangle of the form

$$\begin{tikzcd}
A_* \arrow[rr, "\alpha"] & & B_* \arrow[dl] \\
& \mathrm{Coker}(\alpha) \oplus (\ker(\alpha)[-1]) \arrow[ul] &
\end{tikzcd} ,$$

which is given on the n-th level by

$$A_n \xrightarrow{\alpha_n} B_n \xrightarrow{q_n \oplus 0} \mathrm{Coker}(\alpha_n) \oplus (\ker(\alpha_{n-1})) \xrightarrow{0+\iota_{n-1}} A_{n-1} \xrightarrow{\alpha_{n-1}} B_{n-1} ,$$

where q is the quotient, while ι the inclusion. Thus, (**TR1**)(iii) holds true.

If (α, β, δ) is a distinguished triangle, then the n-th level of its rotation $(\beta, \delta, \alpha[-1])$ is given by

$$B_n \xrightarrow{\beta} C_n \xrightarrow{\delta} A_{n-1} \xrightarrow{\alpha} B_{n-1} \xrightarrow{\beta} C_{n-1} ,$$

which is an exact sequence because both

$$A_n \xrightarrow{\alpha} B_n \xrightarrow{\beta} C_n \xrightarrow{\delta} A_{n-1} \xrightarrow{\alpha} B_{n-1}$$

and

$$A_{n-1} \xrightarrow{\alpha} B_{n-1} \xrightarrow{\beta} C_{n-1} \xrightarrow{\delta} A_{n-2} \xrightarrow{\alpha} B_{n-2}$$

are exact. The same holds true for the other rotation, hence we have (**TR2**).

Nevertheless, the axiom (**TR3**) is not valid in $\mathbf{Ab}^{\mathbb{Z}}$: consider any abelian group G to be a graded object concentrated in degree 0; then, the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "\cdot 2"] & \mathbb{Z} \arrow[r] \arrow[d, "\Delta"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbf{0} \arrow[d, equal] \\
\mathbb{Z} \arrow[r, "\Delta"] & \mathbb{Z}^2 \arrow[r, "+"] & \mathbb{Z} \arrow[r] & \mathbf{0} ,
\end{tikzcd}$$

commutes, but it cannot be completed to a morphism of triangles. Thus, the category $\mathbf{Ab}^{\mathbb{Z}}$ is not triangulated.

Check Octahedral Axiom.

I.10. Example. — Let k be an arbitrary field, and $\mathcal{C} = \mathbf{Vect}_k$ be the category of vector spaces over k. Then $\mathbf{Vect}_k^{\mathbb{Z}}$ is triangulated: indeed, axioms (**TR1**) and (**TR2**) hold true for the same reasons as in the previous example.

We now verify (**TR3**): given a commutative diagram

$$\begin{tikzcd} A_n \arrow[r, "\alpha"] \arrow[d, "f_n"] & B_n \arrow[r, "\beta"] \arrow[d, "g_n"] & C_n \arrow[r, "\delta"] & A_{n-1} \arrow[d, "f_{n-1}"] \\ U_n \arrow[r, "u"] & V_n \arrow[r, "v"] & W_n \arrow[r, "w"] & U_{n-1} \end{tikzcd}$$

we can complete it to a morphism of distinguished triangles by taking a basis Λ of C_n and defining $h_n : C_n \to W_n$ via diagram chasing as follows:

$$\begin{tikzcd} C_n \arrow[r, "\delta"] \arrow[d, dashed, "h_n"] & A_{n-1} \arrow[r, "\alpha"] \arrow[d, "f_{n-1}"] & B_{n-1} \arrow[d, "g_{n-1}"] \\ W_n \arrow[r, "w"] & U_{n-1} \arrow[r, "u"] & V_{n-1}\,, \end{tikzcd} \qquad \begin{tikzcd} c \in \Lambda \arrow[r, mapsto] \arrow[d, dashed, "h_n :=" left] & \delta c \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\ u & f_{n-1}\delta c \arrow[l, dashed] \arrow[r, mapsto] & u\, f_{n-1}\delta c = 0\,, \end{tikzcd}$$

for any basis element $c \in \Lambda$, we note that $f\delta c \in \ker u$ by the commutativity of the right box; since the bottom row is exact, we may lift $f\delta c$ to some $u \in U_{n-1}$, so we finally define $h_n c := u$. The diagram commutes by construction.

Finally, we check the ***Octahedral Axiom***: consider three distinguished triangles

$$\begin{tikzcd} A_* \arrow[rr, "\alpha"] & & B_* \arrow[dl, "i"] \\ & W_* \arrow[ul, "\partial"] & \end{tikzcd}, \qquad \begin{tikzcd} B_* \arrow[rr, "\beta"] & & C_* \arrow[dl, "j"] \\ & U_* \arrow[ul, "d"] & \end{tikzcd}, \qquad \begin{tikzcd} A_* \arrow[rr, "\beta\alpha"] & & C_* \arrow[dl, "l"] \\ & V_* \arrow[ul, "\delta"] & \end{tikzcd},$$

and build $f : W_* \to V_*$ as follows: at the n-th level, choose a basis Λ for W_n and, for every $c \in \Lambda$, set

$$f_n : W_n \longrightarrow V_n\,, \quad f_n(c) := \begin{cases} v\,, & \text{if there exists } v \in \delta_n^{-1}(\partial_n(c))\,; \\ 0\,, & \text{otherwise,} \end{cases}$$

and extend it by linearity; by construction, this gives a morphism $f : W_* \to V_*$ such that $\delta f = \partial$. Similarly, one builds $g : V_* \to U_*$ such that $j = gl$.

Exercise. — If (α, β, δ) is an distinguished triangle, show that the compositions $\beta\alpha, \delta\beta$ and $\alpha[1]\,\delta$ are zero in $\mathcal{D}$.

Solution. By (**TR1**)(i), the triangle $(\mathbf{1}_A, 0, 0)$ is distinguished, thus we may compare

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[r] \arrow[d, "\alpha"] & \mathbf{0} \arrow[r] \arrow[d] & A[1] \arrow[d, equal] \\ A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r, "\delta"] & A[1]\,. \end{tikzcd}$$

By the commutativity of the middle box, we deduce $\beta\alpha = 0$. Similarly, if we use the *Rotation axiom* (**TR3**) we find commutative diagrams

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\beta"] & \mathbf{0} \arrow[r] \arrow[d] & B[1] \arrow[d, equal] \\ C \arrow[r, "\delta"] & A[1] \arrow[r, "\alpha[1]"] & B[1] \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r] \arrow[d, "\delta"] & \mathbf{0} \arrow[r] \arrow[d] & C[1] \arrow[d, equal] \\ A[1] \arrow[r, "\alpha[1]"] & B[1] \arrow[r, "\beta[1]"] & C[1]\,, \end{tikzcd}$$

from which we see that $\delta\beta = 0 = \alpha[1]\,\delta$. $\square$

I.11. Exercise (The 5-lemma). — Let (f, g, h) be a morphism of distinguished triangles. If two maps are isomorphisms, then the third is an isomorphism as well.

Solution. Up to rotating triangles, we may assume without loss of generality that both f and g are isomorphism. We show that h is an isomorphism too: for any object $X \in \mathcal{D}$, we apply the functor $\mathrm{Hom}_{\mathcal{D}}(-, X)$ to the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal, "f"] & B \arrow[r] \arrow[d, equal, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, equal, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]\,, \end{tikzcd}$$

so that we obtain the following commutative diagram in $\mathbf{Ab}$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(B'[1], X) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{D}}(A'[1], X) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{D}}(C', X) \arrow[r] \arrow[d, "-\circ h"] & \mathrm{Hom}_{\mathcal{D}}(B', X) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{D}}(A', X) \arrow[d, equal] \\ \mathrm{Hom}_{\mathcal{D}}(B[1], X) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(A[1], X) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(C, X) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(B, X) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(A, X)\,. \end{tikzcd}$$

By the classic 5**-lemma** for abelian groups, we know the central arrow is an isomorphism; thus, when $X = C$, we deduce there exists $k : C' \to C$ such that $kh = \mathbf{1}_C$. Moreover, if we plug $X = C'$, we see that $(\mathbf{1}_{C'} - hk) \circ h = h - h(kh) = 0$, so we conclude $\mathbf{1}_{C'} = hk$ because precomposition is an isomorphism. $\square$

Remark. — As a consequence, we see that the completion of any $\alpha : A \to B$ to a triangle, as declared in the axiom (**TR1**)(iii), is unique up to isomorphism: taken triangles (α, β, δ) on (A, B, C) and $(\alpha, \beta', \delta')$ on (A, B, C'), then by (**TR3**) one has

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, equal] & B \arrow[r, "\beta"] \arrow[d, equal] & C \arrow[r, "\delta"] \arrow[d, dashed, "\exists\, h"] & A[1] \arrow[d, equal] \\ A \arrow[r, "\alpha"] & B \arrow[r, "\beta'"] & C' \arrow[r, "\delta'"] & A[1]\,, \end{tikzcd}$$

and the 5-lemma I.11 tells us h is an isomorphism. This means that every distinguished triangle is determined, up to isomorphism, by just one of its maps! In particular, the data of the ***Octahedral Axiom*** are completely determined by $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$.

I.12. Exercise (The Split Lemma)**.** — Let $A \to B \to C \to A[1]$ be a distinguished triangle in a triangulated category $\mathcal{D}$. Suppose that $C \to A[1]$ is trivial. Show that then the triangle is split, i.e. is given by a direct sum decomposition $B \simeq A \oplus C$.

Solution. Consider the commutative diagram

$$\begin{tikzcd}
C[-1] \arrow[r, "0"] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, dashed, "\exists\, h"] & C \arrow[d, equal] \\
C[-1] \arrow[r, "0"] & A \arrow[r, "j_A"] & A \oplus C \arrow[r, "\pi_C"] & C\,.
\end{tikzcd}$$

By axiom (**TR3**), there exists an arrow h which completes the diagram to a morphism of triangles. Then by the 5-lemma we conclude h must be an isomorphism. ☐

I.13. Proposition. — *Given an abelian category $\mathcal{A}$, the category $\mathbf{K}(\mathcal{A})$ is triangulated.*

Proof. We show that a triangulated structure on $\mathbf{K}(\mathcal{A})$ is given by letting the shift functor $TA^\bullet := A^\bullet[1]$ to be the translation and the family of distinguished triangles to be given by exact triangles.

It is easy to check that both axioms (**TR1**) and (**TR2**) hold. Without loss of generality, it is enough to check (**TR3**) on strict triangles: the diagram

$$\begin{tikzcd}
A^\bullet \arrow[r, "u"] \arrow[d, "f"] & B^\bullet \arrow[r] \arrow[d, "g"] & \mathbf{C}(u) \arrow[r] \arrow[d, dashed, "\exists\, h"] & A^\bullet[1] \arrow[d, "f[1]"] \\
\tilde{A}^\bullet \arrow[r, "w"] & \tilde{B}^\bullet \arrow[r] & \mathbf{C}(w) \arrow[r] & \tilde{A}^\bullet[1]
\end{tikzcd}$$

can be completed by defining the cochain map

$$h := (f[1], g) : \mathbf{C}(u) \longrightarrow \mathbf{C}(w)\,.$$

It remains to prove the ***Octahedral Axiom*** (**TR4**). As before, we may assume the given triangles are strict:

$$\begin{tikzcd}
A^\bullet \arrow[rr, "\alpha"] & & B^\bullet \arrow[dl, "j_B"] \\
& \mathbf{C}(\alpha) \arrow[ul, "\pi_{A[1]}"] &
\end{tikzcd}\ , \qquad
\begin{tikzcd}
B^\bullet \arrow[rr, "\beta"] & & C^\bullet \arrow[dl, "j_C"] \\
& \mathbf{C}(\beta) \arrow[ul, "\pi_{B[1]}"] &
\end{tikzcd}\ , \qquad
\begin{tikzcd}
A^\bullet \arrow[rr, "\beta\alpha"] & & C^\bullet \arrow[dl, "j_C"] \\
& \mathbf{C}(\beta\alpha) \arrow[ul, "\pi_{A[1]}"] &
\end{tikzcd}\ ,$$

thus we may define

$$\begin{tikzcd}
\mathbf{C}(\alpha) \arrow[rr, "(\mathbf{1}_{A[1]}{,}\beta)"] & & \mathbf{C}(\beta\alpha) \arrow[dl, "(\alpha[1]{,}\mathbf{1}_C)"] \\
& \mathbf{C}(\beta) \arrow[ul, "j_{B[-1]}\pi_{B[1]}"] &
\end{tikzcd}\ .$$

By construction, the octahedron described in (**TR4**) commutes, so we conclude if we show that the above triangle is a distinguished triangle. Set $f := (\mathbf{1}_{A[1]}, \beta)$ and consider $\mathbf{C}(f)$; on degree n it is

$$\mathbf{C}(f)^n = \mathbf{C}(\alpha)^{n+1} \oplus \mathbf{C}(\beta)^n = (A^{n+2} \oplus B^{n+1}) \oplus (A^{n+1} \oplus C^n)\,,$$

thus we can embed $\mathbf{C}(\beta)^n = B^{n+1} \oplus C^n$ in it. This gives us a natural inclusion ι which fits in the commutative diagram

$$\begin{array}{ccccccc}
\mathbf{C}(\alpha) & \xrightarrow{f} & \mathbf{C}(\beta\alpha) & \longrightarrow & \mathbf{C}(\beta) & \longrightarrow & \mathbf{C}(\alpha)[1] \\
\| & & \| & & \downarrow{\scriptstyle\iota} & & \| \\
\mathbf{C}(\alpha) & \xrightarrow{f} & \mathbf{C}(\beta\alpha) & \longrightarrow & \mathbf{C}(f) & \longrightarrow & \mathbf{C}(\alpha)[1]\,,
\end{array}$$

in which ι is a homotopy equivalence: define

$$\varphi^\bullet : \mathbf{C}(f) \longrightarrow \mathbf{C}(\beta)\,, \quad \varphi^n\big(a', b, a, c\big) = (\alpha^{n+1}(a) + b, c)\,,$$

and note that $\varphi^\bullet\iota(b, c) = \varphi(0, b, 0, c) = (b, c)$, i.e. it holds $\varphi^\bullet\iota = \mathbf{1}_{\mathbf{C}(\beta)}$. On the other hand, the map

$$s : \mathbf{C}(f)^n \longrightarrow \mathbf{C}(f)^{n-1}\,, \quad s(a', b, a, c) = (a, 0, 0, 0)$$

defines a homotopy between $\iota\varphi^\bullet$ and $\mathbf{1}_{\mathbf{C}(f)}$: the coboundary map of the mappying cone $\mathbf{C}(f)$ is represented by the matrix

$$d_{\mathbf{C}(f)} = \begin{pmatrix} -d_{\mathbf{C}(\alpha)} & 0 \\ f^\bullet & d_{\mathbf{C}(\beta\alpha)} \end{pmatrix} = \begin{pmatrix} d_A & 0 & 0 & 0 \\ \alpha & -d_B & 0 & 0 \\ \mathbf{1} & 0 & -d_A & 0 \\ 0 & -\beta & \beta\alpha & d_C \end{pmatrix}\,,$$

hence we compute

$$ds + sd = \begin{pmatrix} d_A \\ \alpha \\ -\mathbf{1}_A \\ 0 \end{pmatrix} + \begin{pmatrix} -d_A - \mathbf{1}_A \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\mathbf{1}_A \\ \alpha \\ -\mathbf{1}_A \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha + \mathbf{1}_B \\ 0 \\ \mathbf{1}_C \end{pmatrix} - \begin{pmatrix} \mathbf{1}_A \\ \mathbf{1}_B \\ \mathbf{1}_A \\ \mathbf{1}_C \end{pmatrix} = \iota\varphi^\bullet - \mathbf{1}_{\mathbf{C}(f)}\,.$$

We conclude that ι is an isomorphism in $\mathbf{K}(\mathcal{A})$. □

I.14. Corollary. — *Let $\mathcal{C} \subset C^\bullet(\mathcal{A})$ be a full subcategory and $\mathbf{K}(\mathcal{C})$ be its corresponding quotient category. Suppose that $\mathcal{C}$ is an additive category and is closed under translations and under contruction of mapping cones. Then $\mathbf{K}(\mathcal{C})$ is a triangulated category. In particular, $\mathbf{K}^\flat(\mathcal{A}), \mathbf{K}^-(\mathcal{A})$ and $\mathbf{K}^+(\mathcal{A})$ are triangulated.*

Definition. — An additive functor $F : \mathcal{C} \to \mathcal{D}$ between triangulated categories with shift functors $T_\mathcal{C}$, and $T_\mathcal{D}$ respectively, is called **exact** (or **triangulated**) if it satisfies the following:

(**EF1**) there exists an isomorphism of functors

$$F \circ T_\mathcal{C} \xrightarrow{\sim} T_\mathcal{D} \circ F\,;$$

(**EF2**) any distinguished triangle $A \to B \to C \to A[1]$ in $\mathcal{C}$ is mapped to a distinguished triangle

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(A)[1]$$

in $\mathcal{D}$, where we use the isomorphism $F(A[1]) \simeq F(A)[1]$ given by (**EF1**).

Remark. — Once again, the notions of a triangulated category and of an exact functor have to be adjusted when one is interested in additive categories over a field k. In this case, the shift functor should be k-linear and one usually considers only k-linear exact functors.

Example. — Recall that in Exercise I.10 we proved that $\mathbf{Vect}_k^{\mathbb{Z}}$ is a triangulated category. The total cohomology $H^* : \mathbf{K}(\mathbf{Vect}_k) \to \mathbf{Vect}_k^{\mathbb{Z}}$ is a morphism of triangulated categories: indeed, H^* is additive and for every $n \in \mathbb{Z}$ and every object $V^\bullet$ it holds

$$(H^*TV^\bullet)^n = H^n(V^\bullet[1]) = H^{n-1}(V^\bullet) = (H^*(V^\bullet)[1])^n = (TH^*V^\bullet)^n \,,$$

so H^* commutes with shifts; axiom **(EF2)** is the Long Exact Sequence **??** induced in cohomology.

Definition. — Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $F : \mathcal{D} \to \mathcal{A}$ is called a **covariant cohomological functor** if, whenever (α, β, δ) is a distinguished triangle on (A, B, C), the long sequence

$$\ldots \longrightarrow H(A[n]) \xrightarrow{H\alpha} H(B[n]) \xrightarrow{H\beta} H(C[n]) \xrightarrow{H\delta} H(A[n+1]) \longrightarrow \ldots$$

is exact in $\mathcal{A}$. We often write $H^n(A) := H(A[n])$.

A **contravariant cohomological functor** on $\mathcal{D}$ is a covariant cohomological functor $F : \mathcal{D}^{op} \to \mathcal{A}$ (remember $\mathcal{D}^{op}$ is triangulated).

Example. — The **zero-th cohomology** $H^0 : \mathbf{K}(\mathcal{A}) \to \mathcal{A}$ is a cohomological functor.

Example. — Let $\mathcal{D}$ be a triangulated category. For any $X \in \mathcal{D}$, the functor

$$\mathrm{Hom}_{\mathcal{D}}(X, -) : \mathcal{D} \longrightarrow \mathbf{Ab}$$

is cohomological: indeed, given a distinguished triangle (α, β, δ) on (A, B, C), the sequence

$$\mathrm{Hom}_{\mathcal{D}}(X, A) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathcal{D}}(X, B) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{D}}(X, C)$$

is exact because $\beta\alpha = 0$ implies $\operatorname{im} \alpha^* \subset \ker \beta^*$, and conversely, whenever $g \in \mathrm{Hom}_{\mathcal{D}}(X, B)$ is such that $\alpha_* g = \alpha \circ g = 0$, by **(TR3)** applied to the rotated triangle we have

$$\begin{tikzcd}
X[-1] \arrow[r] \arrow[d] & \mathbf{0} \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d, dashed, "\exists f"] & X \arrow[r] \arrow[d, "g"] & \mathbf{0} \arrow[d] \\
B[-1] \arrow[r] & C[-1] \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \,,
\end{tikzcd}$$

thus $g = \alpha_* f$ and we conclude that $\operatorname{im} \alpha^* = \ker \beta^*$. Finally, by shifting the triangle thanks to **(TR2)**, it follows that the sequence is exact everywhere else. In a similar fashion, one proves that $\mathrm{Hom}_{\mathcal{D}}(-, X)$ is a contravariant cohomological functor.

Moreover, if we additionally assume that $\mathcal{D}$ is k-linear, for some field k, then for any $X \in \mathcal{D}$ yields a functor

$$\mathrm{Hom}_{\mathcal{D}}(X, -) : \mathcal{D} \longrightarrow \mathbf{Vect}_k$$

which is cohomological and gives rise to a long exact sequence of vector spaces over k. The same holds true for the contravariant functor $\mathrm{Hom}_{\mathcal{D}}(-, X)$.

Remark. — Once again, the notions of a triangulated category and of an exact functor have to be adjusted when one is interested in *additive categories over a field k*. In this case, the shift functor should be k-linear and one usually considers only k-linear exact functors.

I.15. Proposition. — *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories. If $F \dashv H$, then $H : \mathcal{D}' \to \mathcal{D}$ is exact. Similarly, if $G \dashv F$, then $G : \mathcal{D}' \to \mathcal{D}$.*

Proof. Let T, respectively T', denote the shift functor of $\mathcal{D}$, resp. $\mathcal{D}'$. We first show that H satisfies (**EF1**): by exactness of F, there is a natural isomorphism $FT \simeq T'F$, which yields $(T')^{-1}F \simeq FT^{-1}$. Hence, for every $A, B \in \mathcal{D}$ it holds

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{D}}(A, HT'(B)) &\simeq \operatorname{Hom}_{\mathcal{D}'}(F(A), T'(B)) \\
&\simeq \operatorname{Hom}_{\mathcal{D}'}((T')^{-1}F(A), B) \\
&\simeq \operatorname{Hom}_{\mathcal{D}'}(FT^{-1}(A), B) \\
&\simeq \operatorname{Hom}_{\mathcal{D}}(T^{-1}(A), H(B)) \\
&\simeq \operatorname{Hom}_{\mathcal{D}}(A, TH(B))\,,
\end{aligned}$$

thus, the Yoneda's Lemma yields an isomorphism $HT' \simeq TH$.

Consider now a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

in $\mathcal{D}'$; we need to show that H maps it in a distinguished triangle in $\mathcal{D}$. By (**EF1**) we know that $H(A[1]) \simeq H(A)[1]$, hence by the axiom (**TR1**)(iii) we can complete $H(A) \to H(B)$ to a distinguished triangle

$$H(A) \longrightarrow H(B) \longrightarrow C_0 \longrightarrow H(A)[1]\,.$$

Usign the counit $\varepsilon : FH \to \mathbf{1}_{\mathcal{D}'}$ and the assumption that F is exact, we get the following commutative diagram, whose rows are distinguished triangles:

$$\begin{tikzcd}
FH(A) \arrow[r] \arrow[d, "\varepsilon_A"] & FH(B) \arrow[r] \arrow[d, "\varepsilon_B"] & F(C_0) \arrow[r] \arrow[d, dashed, "h"] & FH(A)[1] \arrow[d, "\varepsilon_A[1]"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1]\,,
\end{tikzcd}$$

thus, by (**TR3**) it can be completed to a morphism $(\varepsilon_A, \varepsilon_B, h)$ of distinguished triangles. If we apply H to this diagram, we can "attach" the first triangle to it by using the unit $\eta : \mathbf{1}_{\mathcal{D}} \to HF$ in the following way:

$$\begin{tikzcd}
H(A) \arrow[r] \arrow[d, "\eta_{H(A)}"] \arrow[dd, bend right=60, equal] & H(B) \arrow[r] \arrow[d, "\eta_{H(B)}"] \arrow[dd, bend right=60, equal] & C_0 \arrow[r] \arrow[d, "\eta_{C_0}"] & H(A)[1] \arrow[d] \arrow[dd, bend left=60, equal] \\
HFH(A) \arrow[r] \arrow[d, "\varepsilon_A"] & HFH(B) \arrow[r] \arrow[d, "\varepsilon_B"] & HF(C_0) \arrow[r] \arrow[d, "H(h)"] & HFH(A)[1] \arrow[d] \\
H(A) \arrow[r] & H(B) \arrow[r] & H(C) \arrow[r] & H(A)[1]\,,
\end{tikzcd}$$

where the compositions $\varepsilon \circ \eta_{H(-)}$ are the identities by Exercise **??**. Unfortunately, we cannot conclude by applying the **5-lemma** because the bottom triangle is not distinguished. Nevertheless, remember that the functor $\mathrm{Hom}_{\mathcal{D}}(X,-)$ is cohomological, for any $X \in \mathcal{D}$; thus, by using the adjunction we obtain the diagram of abelian groups

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}'}(F(X),A) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{D}'}(F(X),B) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{D}}(X,C_0) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}'}(F(X),A[1]) \arrow[d, equal] \\
\mathrm{Hom}_{\mathcal{D}'}(F(X),A) \arrow[r] & \mathrm{Hom}_{\mathcal{D}'}(F(X),B) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(X,H(C)) \arrow[r] & \mathrm{Hom}_{\mathcal{D}'}(F(X),A[1])\,,
\end{tikzcd}$$

so by applying the classical 5 lemma for abelian groups we deduce that $\mathrm{Hom}_{\mathcal{D}'}(X,C_0) \simeq \mathrm{Hom}_{\mathcal{D}'}(X,H(C))$, thus from the Yoneda's Lemma it follows $C_0 \simeq H(C)$. Finally, by (**TR1**)(ii) we see that $H(A) \to H(B) \to H(C) \to H(A)[1]$ is distinguished, so we conclude that H satisfies (**EF2**). $\square$

Definition. — Given a triangulated category $\mathcal{D}$, a subcategory $\mathcal{D}' \subset \mathcal{D}$ is a **triangulated subcategory** if $\mathcal{D}'$ admits a structure of triangulated category such that the inclusion is exact.

Remark. — (i) Notice in particular that the family of distinguished triangles of $\mathcal{D}'$ must be included in the distinguished triangles of $\mathcal{D}$.

(ii) If $\mathcal{D}' \subset \mathcal{D}$ is full, then it is a triangulated subcategory if and only if $\mathcal{D}'$ is invariant under the shift functor and, for every distinguished triangle $A \to B \to C \to A[1]$ in $\mathcal{D}$, with $A, B \in \mathcal{D}'$, then C is isomorphic to some object in $\mathcal{D}'$: indeed, $A \to B$ is a morphism in $\mathcal{D}'$, thus it can be completed to a distinguished triangle $A \to B \to C' \to A[1]$ in $\mathcal{D}'$, hence we have $C \simeq C'$ by the **5-lemma**.

Definition. — A triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is called **admissible** if the inclusion has a right adjoint $\pi : \mathcal{D} \to \mathcal{D}'$, that we will call **orthogonal projection**. The (**right**) **orthogonal complement** of $\mathcal{D}'$ is the full subcategory $\mathcal{D}'^{\perp} \subset \mathcal{D}$ of those objects $C \in \mathcal{D}$ such that

$$\mathrm{Hom}_{\mathcal{D}}(A,C) = 0\,, \quad \text{for all } A \in \mathcal{D}'\,.$$

Similarly, one can define ${}^{\perp}\mathcal{D}'$ as the full subcategory of those objects $C \in \mathcal{D}$ such that

$$\mathrm{Hom}_{\mathcal{D}}(C,A) = 0\,, \quad \text{for all } A \in \mathcal{D}'\,.$$

Note that the orthogonal projection π is an exact functor by Proposition I.15.

Remark. — The orthogonal complement of a triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ naturally inherits a triangulated structure from $\mathcal{D}$: indeed, we see that $T|_{\mathcal{D}'^{\perp}}$ has values in $\mathcal{D}'^{\perp}$ because, for every $A \in \mathcal{D}'$ and $n \in \mathbb{Z}$, it holds

$$\mathrm{Hom}_{\mathcal{D}}(A,C[n]) \simeq \mathrm{Hom}_{\mathcal{D}}(A[-n],C) = 0\,;$$

moreover, given an exact triangle

$$C \longrightarrow C' \longrightarrow B \longrightarrow C[1]$$

in $\mathcal{D}$, where $C, C' \in \mathcal{D}'^{\perp}$, then by applying $\mathrm{Hom}_{\mathcal{D}}(A, -)$ with $A \in \mathcal{D}'$ we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}}(A, B) \longrightarrow 0\,,$$

from which we deduce that $B \in \mathcal{D}'^{\perp}$.

The concept of orthogonality in triangulated category is inspired by the familiar orthogonality on vector spaces that we know from linear algebra. We know that, is $W \subset V$ is a vector subspace, the the total space admits a decomposition $V = W \oplus W^{\perp}$. Similarly, whenever a full admissible subcategory $\mathcal{D}' \subset \mathcal{D}$ exists, every object in $\mathcal{D}$ fits in a sequence between a component in $\mathcal{D}'$ and a component in its orthogonal complement

I.16. Lemma (Semi-orthogonal decomposition)**.** — *A full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is admissible if and only if, for all $B \in \mathcal{D}$, there exist a distinguished triangle*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]\,,$$

with $A \in \mathcal{D}'$ and $C \in \mathcal{D}'^{\perp}$.

Proof. Assume $\mathcal{D}'$ is admissible: call π its orthogonal projection and set $A := \pi(B) \in \mathcal{D}'$. Using the adjunction, the identity $\mathbf{1}_A$ corresponds to a map $A \to B$ in $\mathcal{D}$, which can be completed to a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

note that for any $A' \in \mathcal{D}'$, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(A', A) = \mathrm{Hom}_{\mathcal{D}'}(A', A) = \mathrm{Hom}_{\mathcal{D}'}(A', \pi(B)) \simeq \mathrm{Hom}_{\mathcal{D}}(A', B)\,,$$

hence, if we apply the functor $\mathrm{Hom}_{\mathcal{D}}(A', -)$ on the triangle, then we get $\mathrm{Hom}_{\mathcal{D}}(A', C) = 0$, so $C \in \mathcal{D}'^{\perp}$.

Conversely, for any $B \in \mathcal{D}$, define $\pi(B)$ to be an object in $\mathcal{D}'$ that fits in a distinguished triangle

$$\pi(B) \longrightarrow B \longrightarrow C \longrightarrow \pi(B)[1]\,,$$

with $C \in \mathcal{D}'^{\perp}$. If $A' \to B \to C' \to A'[1]$ is another triangle of this form, then for every $X \in \mathcal{D}'$ we have

$$\mathrm{Hom}_{\mathcal{D}'}(X, \pi(B)) = \mathrm{Hom}_{\mathcal{D}}(X, \pi(B)) \simeq \mathrm{Hom}_{\mathcal{D}}(X, B) \simeq \mathrm{Hom}_{\mathcal{D}}(X, A') = \mathrm{Hom}_{\mathcal{D}}(X, A')\,,$$

so by the Yoneda's Lemma we deduce that $A' \simeq \pi(B)$ in $\mathcal{D}'$. This shows that π is defined on objects up to isomorphism. Finally, we can define π on morphisms too: given $f : B \to B'$ in $\mathcal{D}$, we define $\pi(f)$ to be the image of f via the composition

$$\mathrm{Hom}_{\mathcal{D}}(B, B') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\pi(B), B') \simeq \mathrm{Hom}_{\mathcal{D}}(\pi(B), \pi(B')) = \mathrm{Hom}_{\mathcal{D}'}(\pi(B), \pi(B'))\,,$$

where the isomorphism inbetween follows from the fact that $\mathrm{Hom}_{\mathcal{D}}(\pi(B), C') = 0$, for any $C' \in \mathcal{D}'^{\perp}$ in the decomposition of B'. One can check that this defines a functor $\pi : \mathcal{D} \to \mathcal{D}'$ which is right adjoint to the inclusion $\mathcal{D}' \subset \mathcal{D}$, by construction. $\square$

As it turns out, Serre functors and triangulated structures are always compatible. In the geometric situation considered later, this will be obvious, for the Serre functors there will by construction be exact.

I.17. Theorem (Bondal, Kapranov). — *Any Serre functor on a k-linear triangulated category is exact.*

Proof. See [Huy06, Proposition 1.46]. □

Equivalences of triangulated categories

In this section we discuss criteria that allow us to decide whether a given exact functor is fully faithful or even an equivalence.

Definition. — Two triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ are **equivalent** if there exists an exact equivalence $T : \mathcal{D} \to \mathcal{D}'$, i.e. an equivalence which is an exact functor, whose inverse is exact. The set $\mathrm{Aut}(\mathcal{D})$ of isomorphism classes of equivalences $F : \mathcal{D} \to \mathcal{D}$ is called **group of autoequivalences** of $\mathcal{D}$.

In many geometric situations, we will encounter the following notion:

Definition. — Let $\mathcal{D}$ be a triangulated category. A **spanning class** for $\mathcal{D}$ is a collection Ω of objects in $\mathcal{D}$ such that, for any $B \in \mathcal{D}$, the following two conditions hold:

(i) if for every $n \in \mathbb{Z}$ and $A \in \Omega$ we have $\mathrm{Hom}_{\mathcal{D}}(A, B[n]) = 0$, then $B \simeq \mathbf{0}$;

(ii) if for every $n \in \mathbb{Z}$ and $A \in \Omega$ we have $\mathrm{Hom}_{\mathcal{D}}(B[n], A) = 0$, then $B \simeq \mathbf{0}$.

Remark. — If $\mathcal{D}$ is also k-linear, endowed with a Serre functor S, then the two conditions in the definition are equivalent: assume (i) to be true. If for every $n \in \mathbb{Z}$ and $A \in \Omega$ we have

$$0 = \mathrm{Hom}_{\mathcal{D}}(B[n], A) \simeq \mathrm{Hom}_{\mathcal{D}}(A, S(B[n]))^*\,,$$

since by the Bondal-Kapranov Theorem we have $S(B[n]) \simeq S(B)[n]$, from (i) we deduce that $S(B) \simeq \mathbf{0}$, and hence $B \simeq \mathbf{0}$ because S is additive. Similarly one proves (ii) $\implies$ (i).

Spanning classes give a sufficient collections of objects on which we can check whether an exact functor is fully faithful or not.

I.18. Proposition. — *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories with both left and right adjoints: $G \dashv F \dashv H$. Suppose Ω is a spanning class such that, for every $A, B \in \Omega$, the natural maps*

$$F : \mathrm{Hom}_{\mathcal{D}}(A, B[n]) \longrightarrow \mathrm{Hom}_{\mathcal{D}'}(F(A), F(B)[n])$$

are bijections for every $n \in \mathbb{Z}$. Then F is fully faithful.

Proof. For any pair of objects $A, B \in \mathcal{D}$, we have the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(A,B) \arrow[r, "\eta_B\circ -"] \arrow[d, "-\circ\varepsilon_A"'] \arrow[dr, "F"] & \mathrm{Hom}_{\mathcal{D}}(A,HF(B)) \arrow[d, equal] \\ \mathrm{Hom}_{\mathcal{D}}(GF(A),B) \arrow[r, equal] & \mathrm{Hom}_{\mathcal{D}'}(F(A),F(B))\,. \end{tikzcd} \tag{I.18.1}$$

We show that the counit ε_A is an isomorphism, for $A \in \Omega$: complete it to a distinguished triangle

$$GF(A) \xrightarrow{\varepsilon_A} A \longrightarrow A' \longrightarrow GF(A)[1]\,,$$

where we have used that $GF(A[1]) \simeq GF(A)[1]$ by exactness of both F and G (see Proposition I.15). Apply $\mathrm{Hom}_{\mathcal{D}}(-, B[n])$ to it and get

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(A',B[n]) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(A,B[n]) \arrow[r, "-\circ\varepsilon"] \arrow[dr, "F"'] & \mathrm{Hom}_{\mathcal{D}}(GF(A),B[n]) \arrow[d, equal] \\ & & \mathrm{Hom}_{\mathcal{D}'}(F(A),F(B)[n])\,. \end{tikzcd}$$

If we take $B \in \Omega$, then F is bijective, so $- \circ \varepsilon$ is an isomorphism, from which we deduce that $\mathrm{Hom}_{\mathcal{D}}(A', B[n]) = 0$; since this holds for every $B \in \Omega$ and $n \in \mathbb{Z}$, then $A' \simeq \mathbf{0}$ because Ω spans $\mathcal{D}$, so we conclude that $\varepsilon_A : GF(A) \simeq A$.

It follows that $- \circ \varepsilon_A$ is an isomorphism, for $A \in \Omega$ and any $B \in \mathcal{D}$, and hence all the maps in the diagram (I.18.1) are isomorphisms. Now complete η_B to a distinguished triangle

$$B \xrightarrow{\eta_B} HF(B) \longrightarrow B' \longrightarrow B[1]\,,$$

and apply $\mathrm{Hom}_{\mathcal{D}}(A, -) \circ [n]$, for all $n \in \mathbb{Z}$ and $A \in \Omega$ to see that $\mathrm{Hom}_{\mathcal{D}}(A, B') = 0$, and hence $B' \simeq \mathbf{0}$. This means that $\eta_B : B \simeq HF(B)$, for any $B \in \mathcal{D}$. Thus, the arrows in (I.18.1) are isomorphisms for all $A, B \in \mathcal{D}$, in particular

$$F : \mathrm{Hom}_{\mathcal{D}}(A,B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}'}(F(A),F(B))\,.$$

□

Suppose we already know that the functor is fully faithful. What do we need to know in order to be able to decide whether it is in fact an equivalence? The following lemma provides a first criterion, whose assumption however is difficult to check.

I.19. Lemma. — *Let $F : \mathcal{D} \to \mathcal{D}'$ be a fully faithful exact functor between triangulated categories and suppose that F has a right adjoint $F \dashv H$. Then F is an equivalence if and only if, for every $C \in \mathcal{D}'$, $H(C) \simeq \mathbf{0}$ implies $C \simeq \mathbf{0}$.*

The same holds true if F has a left adjoint $G \dashv F$ with the above property.

Proof. By Corollary **??**, we know that $\eta_A : A \xrightarrow{\sim} HF(A)$ is an isomorphism, for any $A \in \mathcal{D}$. We prove that also $\varepsilon_B : FH(A) \to B$ is an isomorphism, and hence H is a quasi-inverse of F.

Given any $B \in \mathcal{D}'$, complete ε_B to a distinguished triangle

$$FH(B) \xrightarrow{\varepsilon_B} B \longrightarrow B' \longrightarrow FH(B)[1]\,;$$

since H is exact by Proposition I.15, it gives a distinguished triangle

$$HFH(B) \xrightarrow{H\varepsilon_B} H(B) \longrightarrow H(B') \longrightarrow HFH(B)[1]\,.$$

From Exercise **??** we know that $H\varepsilon_B$ is an isomorphism, hence $H(B') \simeq \mathbf{0}$. By assumption $B' \simeq \mathbf{0}$ and we get the thesis. $\square$

Definition. — A triangulated category $\mathcal{D}$ is **decomposed into triangulated subcategories** $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{D}$ if the following three conditions are satisfied:

(i) both $\mathcal{D}_1$ and $\mathcal{D}_2$ contain objects non-isomorphic to $\mathbf{0}$;

(ii) for all $A \in \mathcal{D}$, there exists a distinguished triangle

$$A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow A_1[1]\,,$$

with $A_1 \in \mathcal{D}_1$ and $A_2 \in \mathcal{D}_2$;

(iii) the two subcategories are "disjoint" in the sense that, for every $X_1 \in \mathcal{D}_1$ and $X_2 \in \mathcal{D}_2$, it holds

$$\mathrm{Hom}_{\mathcal{D}}(X_1, X_2) = \mathrm{Hom}_{\mathcal{D}}(X_2, X_1) = 0\,.$$

A triangulated category which cannot be decomposed is called **indecomposable**.

Remark. — Notice that, in the presence of (iii), the property (ii) states that A is a direct sum $A \simeq A_1 \oplus A_2$ (see the Split Lemma). Thus, condition (ii) is symmetric, despite the chosen order in the statement.

I.20. Proposition. — *Let $F : \mathcal{D} \to \mathcal{D}'$ be a fully faithful exact functor between triangulated categories. Suppose that $\mathcal{D}$ contains no objects isomorphic to $\mathbf{0}$ and that $\mathcal{D}'$ is indecomposable. Then F is an equivalence if and only if it admits both right and left adjoints $G \dashv F \dashv H$ such that, for any $B \in \mathcal{D}$, $H(B) \simeq \mathbf{0}$ implies $G(B) \simeq \mathbf{0}$.*

Proof. The "only if part is clear" because $G \simeq H \simeq F^{-1}$. So we need to prove the "if" implication: our goal is to show that H is a quasi-inverse of F.

First, we introduce two full triangulated subcategories $\mathcal{D}'_1, \mathcal{D}'_2 \subset \mathcal{D}'$ defined as follows: let $\mathcal{D}'_1$ the full subcategory of objects $B \in \mathcal{D}'$ such that $B \simeq F(A)$, for some $A \in \mathcal{D}$; let $\mathcal{D}'_2$ be the full subcategory consisting of all objects $C \in \mathcal{D}'$ such that $H(C) \simeq \mathbf{0}$. One can easily check that both $\mathcal{D}'_1$ and $\mathcal{D}'_2$ are triangulated subcategories of $\mathcal{D}'$.

The proof of Lemma I.19 shows that, for every $B \in \mathcal{D}'_1$, the counit $FH(B) \simeq B$ is an isomorphism and also that every $B \in \mathcal{D}'$ sits in a distinguished triangle of the form

$$B_1 \longrightarrow B \longrightarrow B_2 \longrightarrow B_1[1]\,,$$

with $B_1 \in \mathcal{D}'_1$ and $B_2 \in \mathcal{D}'_2$. Since by assumption $H(B_2) \simeq \mathbf{0}$ implies $G(B_2) \simeq \mathbf{0}$, then for all $B_1 \in \mathcal{D}'_1$ and $B_2 \in \mathcal{D}'_2$ it holds

$$\mathrm{Hom}_{\mathcal{D}'}(B_1, B_2) \simeq \mathrm{Hom}_{\mathcal{D}'}(FH(B_1), B_2) \simeq \mathrm{Hom}_{\mathcal{D}}(H(B_1), H(B_2)) \simeq \mathrm{Hom}_{\mathcal{D}}(H(B_1), \mathbf{0}) = 0\,,$$
$$\mathrm{Hom}_{\mathcal{D}'}(B_2, B_1) \simeq \mathrm{Hom}_{\mathcal{D}'}(B_2, FH(B_1)) \simeq \mathrm{Hom}_{\mathcal{D}}(G(B_2), H(B_1)) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathbf{0}, H(B_1)) = 0\,,$$

so $\mathcal{D}'_1$ and $\mathcal{D}'_2$ decompose $\mathcal{D}'$. As $\mathcal{D}'$ is indecomposable, either $\mathcal{D}'_1$ or $\mathcal{D}'_2$ is trivial, i.e. contains only objects isomorphic to $\mathbf{0}$.

Suppose $\mathcal{D}'_1$ is trivial. Then for every $A \in \mathcal{D}$, the image $F(A)$ is trivial, and hence $A \simeq HF(A) \simeq \mathbf{0}$ for F is fully faithful: this contradicts the non-triviality of $\mathcal{D}$. Thus $\mathcal{D}'_2$ must be trivial, which implies that $\mathcal{D}'_1 \subset \mathcal{D}'$ is an equivalence, i.e. for every $B \in \mathcal{D}'$, it holds $FH(B) \simeq B$, and H is a quasi-inverse of F. □

The following proposition gives a criterion for equivalences of triangulated endowed with Serre functors.

I.21. Corollary. — *Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories, endowed with Serre functors $S_{\mathcal{D}}$, resp. $S_{\mathcal{D}'}$. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor which admits both right and left adjoints $G \dashv F \dashv H$. Assume there is Ω a spanning class of $\mathcal{D}$ satisfying the following three properties:*

(i) *for all $A, B \in \Omega$, the natural morphisms*

$$\mathrm{Hom}_{\mathcal{D}}(A, B[n]) \longrightarrow \mathrm{Hom}_{\mathcal{D}'}(F(A), F(B)[i])$$

are bijective for all $n \in \mathbb{Z}$;

(ii) *Serre functors commute with F on the spanning class, that is*

$$F \circ S_{\mathcal{D}}(A) \simeq S_{\mathcal{D}'} \circ F(A)\,, \quad A \in \Omega\,;$$

(iii) *the category $\mathcal{D}'$ is indecomposable and $\mathcal{D}$ is non-trivial.*

Then F is an equivalence.

Proof. Condition (i) ensures that F is fully faithful by Proposition I.18. Now we want to prove F is an equivalence by verifying the conditions in Proposition I.20.

Suppose $H(B) \simeq \mathbf{0}$, for $B \in \mathcal{D}'$. For every $A \in \Omega$, using adjunction and condition (ii), one finds

$$\begin{aligned} 0 = \mathrm{Hom}_{\mathcal{D}}(A, H(B)) &\simeq \mathrm{Hom}_{\mathcal{D}'}(F(A), B) \simeq \mathrm{Hom}_{\mathcal{D}'}(B, S_{\mathcal{D}'}F(A)))^* \\ &\simeq \mathrm{Hom}_{\mathcal{D}'}(B, FS_{\mathcal{D}}(A)))^* \simeq \mathrm{Hom}_{\mathcal{D}}(G(B), S_{\mathcal{D}}(A)))^* \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(A, G(B))\,, \end{aligned}$$

thus $G(B) \simeq \mathbf{0}$ because Ω spans $\mathcal{D}$; more precisely, this argument shows that $G \simeq H$ by the Yoneda's Lemma. □

Recall that in a k-linear category endowed with a Serre functor, the existence of an adjoint functor implies the existence of the other one; hence, the previous Corollary may be stated assuming the existence of H only.

Exceptional sequences and orthogonal decompositions

In the geometric context, the derived categories in question will usually be indecomposable. However, there are geometrically relevant situations where one can decompose the derived category in a weaker sense. This leads to the abstract notion of semi-orthogonal decompositions of a triangulated category, the topic of this section. Any full exceptional sequence yields such a semi-orthogonal decomposition, so we will discuss this notion first.

Definition. — Let $\mathcal{D}$ be k-linear triangulated category. An object $E \in \mathcal{D}$ is **exceptional** if

$$\mathrm{Hom}_{\mathcal{D}}(E, E[n]) = \begin{cases} k\,, & \text{if } n = 0\,; \\ 0\,, & \text{if } n \neq 0\,. \end{cases}$$

An **exceptional sequence** is a sequence $E_1, E_2, \dots, E_m$ of exceptional objects such that for all i, j it holds

$$\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[n]) = \begin{cases} k\,, & \text{if } i = j, n = 0\,; \\ 0\,, & \text{if } i > j \text{ or if } i = j, n \neq 0\,. \end{cases}$$

An exceptional sequence is **full** if $\mathcal{D}$ is generated by $\{ E_i \}$, i.e. any full triangulated subcategory of $\mathcal{D}$ containing all objects E_i is equivalent to $\mathcal{D}$ via the inclusion.

I.22. Lemma. — *Let $\mathcal{D}$ be a k-linear triangulated category such that, for any $A, B \in \mathcal{D}$ the vector space $\bigoplus_i \mathrm{Hom}_{\mathcal{D}}(A, B[i])$ is finite-dimensional. If $E \in \mathcal{D}$ is exceptional, then the objects $\bigoplus_i E[i]^{\oplus j_i}$ form an admissible subcategory $\langle E \rangle$ of $\mathcal{D}$.*

Proof. It is easy to check that $\langle E \rangle$ inherits the structure of a triangulated category: direct sums behave well in forming distinguished triangles; moreover, non-trivial endomorphisms of the exceptional object are automorphisms. In order to see that it is admissible, we use the tensor product $\otimes_{\mathcal{D}}$: for every $n \in \mathbb{Z}$, consider the morphism induced by shifting $(-n)$ times

$$[-n] : \mathrm{Hom}_{\mathcal{D}}(E, A[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(E[-n], A)\,;$$

by adjunction, it gives an evaluation morphism

$$\mathbf{ev}_n : \mathrm{Hom}_{\mathcal{D}}(E, A[n]) \otimes_{\mathcal{D}} E[-n] \longrightarrow A\,,$$

for each $n \in \mathbb{Z}$; take the direct sum of these morphisms and complete it to a distinguished triangle

$$\bigoplus_n \big(\mathrm{Hom}_{\mathcal{D}}(E, A[n]) \otimes_{\mathcal{D}} E[-n] \big) \longrightarrow A \longrightarrow B\,.$$

For every $n \in \mathbb{Z}$, let $d_n := \dim \mathrm{Hom}_{\mathcal{D}}(E, A[n])$ and notice it is finite by assumption; then it holds

$$\mathrm{Hom}_{\mathcal{D}}(E, A[n]) \otimes_{\mathcal{D}} E[-n] \simeq k^{d_n} \otimes_{\mathcal{D}} E[-n] \simeq \bigoplus_{m=1}^{d_n} E[-n]\,,$$

hence, if we apply $\mathrm{Hom}_{\mathcal{D}}(E[-i], -)$ to the above triangle, by exceptionality of E we get

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}} \left(E[-i], \bigoplus_n \big(\mathrm{Hom}_{\mathcal{D}}(E, A[n]) \otimes_{\mathcal{D}} E[-n] \big) \right) &\simeq \bigoplus_n \left(\mathrm{Hom}_{\mathcal{D}}(E[-i], E[-n]) \right)^{d_n} \\ &\simeq \left(\mathrm{Hom}_{\mathcal{D}}(E[-i], E[-i]) \right)^{d_i} \\ &\simeq k^{d_i} \simeq \mathrm{Hom}_{\mathcal{D}}(E, A[i]) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(E[-i], A)\,, \end{aligned}$$

thus, $\mathrm{Hom}_{\mathcal{D}}(E, B[i]) \simeq \mathrm{Hom}_{\mathcal{D}}(E[-i], B) \simeq 0$. Since this holds for every $i \in \mathbb{Z}$, then one has $B \in \langle E \rangle^{\perp}$, so we conclude by Lemma I.16. $\square$

Now we generalize the concept of exceptional sequence:

Definition. — A sequence of full admissible triangulated subcategories

$$\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m \subset \mathcal{D}$$

is **semi-orthogonal** if for all $i < j$ it holds $\mathcal{D}_i \subset \mathcal{D}_j^{\perp}$.

We say that $\mathcal{D}_1, \ldots, \mathcal{D}_m$ is a **semi-orthogonal decomposition** of $\mathcal{D}$ if $\mathcal{D}$ is generated by the $\mathcal{D}_i$, i.e. the smallest full triangulated subcategory containing all the $\mathcal{D}_i$ is equivalent to $\mathcal{D}$ via the inclusion.

Example. — Let $\mathcal{D}' \subset \mathcal{D}$ be an admissible full triangulated subcategory. Then

$$\mathcal{D}_1 := \mathcal{D}'^{\perp}\,, \quad \mathcal{D}_2 := \mathcal{D}'\,,$$

defines a semi-orthogonal decomposition of $\mathcal{D}$.

Example. — Let $E_1, \ldots, E_m$ be an exceptional sequence in $\mathcal{D}$. Then the admissible trangulated subcategories generated by these objects

$$\mathcal{D}_1 := \langle E_1 \rangle\,, \quad \ldots \quad \mathcal{D}_m := \langle E_m \rangle\,,$$

form a semi-orthogonal sequence. If the sequence is *full*, then $\mathcal{D}_1, \ldots, \mathcal{D}_m$ is a semi-orthogonal decomposition of $\mathcal{D}$.

I.23. Lemma. — *Any semi-orthogonal sequence of full admissible triangulated subcategories $\mathcal{D}_1, \ldots, \mathcal{D}_m \subset \mathcal{D}$ defines a semi-orthogonal decomposition for $\mathcal{D}$ if and only if any object $A \in \mathcal{D}$ such that $A \in \mathcal{D}_i$, for all $i = 1, 2, \ldots, m$, is then trivial, i.e. $A \simeq \mathbf{0}$.*

Proof. Suppose $\mathcal{D}_1, \ldots, \mathcal{D}_m$ is a semi-orthogonal decomposition of $\mathcal{D}$. If $A_0 \in \bigcap \mathcal{D}_i^{\perp}$, then each $\mathcal{D}_i \subset {}^{\perp}A_0$; by assumption ${}^{\perp}A_0 = \mathcal{D}$ and in particular $A_0 \in {}^{\perp}A_0$, thus $\mathrm{Hom}_{\mathcal{D}}(A_0, A_0) = 0$. This means $A_0 \simeq \mathbf{0}$, indeed $\mathbf{1}_{A_0} = 0$.

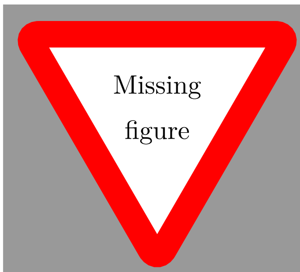
Conversely, assume $\bigcap \mathcal{D}_i^{\perp} = \{\, \mathbf{0} \,\}$. For simplicity, we consider the case $m = 2$: given $A_0 \in \mathcal{D}$, we want to show A_0 is in the triangulated subcategory generated by $\mathcal{D}_1$ and $\mathcal{D}_2$. As $\mathcal{D}_2$ is admissible, by Lemma I.16 there is a distinguished triangle

$$A \longrightarrow A_0 \longrightarrow A' \longrightarrow A[1]\,,$$

with $A \in \mathcal{D}_2$ and $A' \in \mathcal{D}_2^{\perp}$. Now, using that $\mathcal{D}_1$ is admissible, we can decompose A' as

$$B \longrightarrow A' \longrightarrow B' \longrightarrow B[1]\,,$$

with $B \in \mathcal{D}_1$ and $B' \in \mathcal{D}_1^{\perp}$. As the sequence is semi-orthogonal we have $B \in \mathcal{D}_1 \subset \mathcal{D}_2^{\perp}$, and since $A' \in \mathcal{D}_2^{\perp}$, we deduce that $B' \in \mathcal{D}_2^{\perp}$ because it is a full triangulated subcategory. So $B' \in \mathcal{D}_1^{\perp} \cap \mathcal{D}_2^{\perp}$ implies $B' \simeq \mathbf{0}$, from which we deduce that $B \simeq A'$. Then A_0 has a semi-orthogonal decomposition with $A \in \mathcal{D}_2$ and $B \in \mathcal{D}_1$. (The general case follows by applying inductively this argument). $\square$



Do exercises on Exceptional sequences.

Chapter II.

Derived categories

II

The derived category of an abelian category

We begin by stating the existence of the derived category as a theorem, and explain the technical features, necessary for any calculation, later on. In the sequel, we will mostly be interested in the derived category of the abelian category of (coherent) sheaves or of modules over a ring.

Let $\mathcal{A}$ be an abelian category. Recall that in the chapter of Abelian Categories, we have defined the category $C^\bullet(\mathcal{A})$ of cochain complexes and cohomology functors H^n over it.

Definition. — A morphism of complexes $f^\bullet : A^\bullet \to B^\bullet$ is a **quasi-isomorphism** (shortened **qis**) if, for all $n \in \mathbb{N}$, the induced map

$$H^n(f^\bullet) : H^n(A^\bullet) \xrightarrow{\sim} H^n(B^\bullet)$$

is an isomorphism.

The central idea for the definition of the derived category is this: quasi-isomorphic complexes should become isomorphic objects in the derived category. We shall begin our discussion with the following existence theorem.

II.1. Theorem/Definition. — *Given an abelian category $\mathcal{A}$, there exists a category $\mathbf{D}(\mathcal{A})$, called the **derived category** of $\mathcal{A}$, and a functor*

$$Q : C^\bullet(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{A})$$

that satisfy the following two conditions:

(i) *if $f^\bullet$ is a qis, then $Q(f^\bullet)$ is an isomorphism;*

(ii) ***universal property**: if a functor $F : C^\bullet(\mathcal{A}) \to \mathcal{D}$ satisfies* (i)*, then it factorizes uniquely through Q, i.e. there exists a unique (up to isomorphism) functor $G : \mathbf{D}(\mathcal{A}) \to \mathcal{D}$ such that $F \simeq GQ$:*

$$\begin{tikzcd} C^\bullet(\mathcal{A}) \arrow[rr, "Q"] \arrow[dr, "F"'] & & \mathbf{D}(\mathcal{A}) \arrow[dl, dashed, "\exists!"', "G"] \\ & \mathcal{D} & \end{tikzcd} .$$

Construction. In order to be able to work with the derived category, we have to understand which objects become isomorphic under $Q : C^\bullet(A) \to \mathbf{D}(A)$ and, more complicated, how to represent morphisms in the derived category. Explaining this, will at the same time provide a proof the theorem. Recall that $\mathbf{K}(\mathcal{A})$ denotes the homotopy category of complexes.

First, we set the objects of the derived category to be cochain complexes, so

$$\mathrm{Obj}(\mathbf{D}(\mathcal{A})) := \mathrm{Obj}(\mathbf{K}(\mathcal{A})) = \mathrm{Obj}(C^\bullet(\mathcal{A}))\,.$$

Now we describe how morphisms should behave. Since the derived category is built in such a way that quasi-isomorphisms become isomorphisms, if $C^\bullet \to A^\bullet$ is an isomorphism, then any morphism of complexes $C^\bullet \to B^\bullet$ will have to count as a morphism $A^\bullet \to B^\bullet$ in $\mathbf{D}(\mathcal{A})$. Thus, given two complexes $A^\bullet, B^\bullet$, a representative of a morphism $A^\bullet \to B^\bullet$ in $\mathbf{D}(\mathcal{A})$ is given by a diagram

$$\begin{tikzcd} & C^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd},$$

where $C^\bullet \to A^\bullet$ is a quasi-isomorphism; we will call such a diagram a **roof** over $A^\bullet$ and $B^\bullet$. Two roofs over $A^\bullet$ and $B^\bullet$ are said to be **equivalent** if they are dominated by a third roof in $\mathbf{K}(\mathcal{A})$, i.e. there is a diagram

$$\begin{tikzcd}
& & C^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] \arrow[ddll, bend right, dashed, "\text{qis}"'] & & \\
& C_1^\bullet \arrow[dl, "\text{qis}"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "\text{qis}"] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}$$

which commutes in $\mathbf{K}(\mathcal{A})$, i.e. compositions are homotopy equivalent. This property defines, in fact, an *equivalence relation* on roofs:

- **reflextivity**: a roof is equivalent to itself because we have the diagram

$$\begin{tikzcd}
& & C^\bullet \arrow[dl, equal] \arrow[dr, equal] & & \\
& C^\bullet \arrow[dl, "\text{qis}"'] \arrow[drrr] & & C^\bullet \arrow[dlll, "\text{qis}"] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd};$$

- **symmetry**: given an equivalence of roofs

$$\begin{tikzcd}
& & C^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] \arrow[ddll, bend right, dashed, "\text{qis}"'] & & \\
& C_1^\bullet \arrow[dl, "\text{qis}"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "\text{qis}"] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd},$$

we notice that the map $C^\bullet \to C_2^\bullet$ is, in fact, a qis: indeed, by passing to cohomology objects, in $\mathcal{A}$ we have a commutative diagram

$$\begin{tikzcd}
H^*(C^\bullet) \arrow[rr, "\simeq"] \arrow[dr] & & H^*(A^\bullet) \\
& H^*(C_2^\bullet) \arrow[ur, "\simeq"] &
\end{tikzcd},$$

implying that $H^*(C^\bullet) \simeq H^*(C_2^\bullet)$. Thus, the "equivalence" diagram is symmetric;

- **transitivity**: given two diagrams

$$\begin{tikzcd}
& & C^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & \\
& C_1^\bullet \arrow[dl, "\text{qis}"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "\text{qis}"] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}, \qquad
\begin{tikzcd}
& & D^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & \\
& C_2^\bullet \arrow[dl, "\text{qis}"'] \arrow[drrr] & & C_3^\bullet \arrow[dlll, "\text{qis}"] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd},$$

by Proposition I.9 we can build a diagram

$$\begin{tikzcd}
& & C_0^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & \\
& C^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & D^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\
C_1^\bullet & & C_2^\bullet & & C_3^\bullet
\end{tikzcd}$$

which commutes up to homotopy; thus, we obtain the diagram

$$\begin{tikzcd}
& & C_0^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & \\
& C_1^\bullet \arrow[dl, "\text{qis}"'] \arrow[drrr] & & C_3^\bullet \arrow[dlll, "\text{qis}"] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd},$$

which shows that the roof dominated by $C_1^\bullet$ is equivalent to the roof dominated by $C_3^\bullet$.

Finally, we can define $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$: a morphism $A^\bullet \to B^\bullet$ in the derived category is the equivalence class of a roof over $A^\bullet$ and $B^\bullet$. Given two morphisms $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ in $\mathbf{D}(\mathcal{A})$, we describe the composition in the derived category: taken two representatives

$$\begin{tikzcd}
& C_1^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\
A^\bullet & & B^\bullet & & C^\bullet
\end{tikzcd},$$

the composition $A^\bullet \to B^\bullet \to C^\bullet$ is the equivalence class of a roof on top of the other two, in such a way that the following

$$\begin{tikzcd}
 & & C_0^\bullet \arrow[dl, "\mathrm{qis}"'] \arrow[dr] & & \\
 & C_1^\bullet \arrow[dl, "\mathrm{qis}"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "\mathrm{qis}"'] \arrow[dr] & \\
A^\bullet & & B^\bullet & & C^\bullet
\end{tikzcd}$$

is a commutative diagram in $\mathbf{K}(\mathcal{A})$: Proposition I.9 ensures that such a diagram always exists; moreover, the equivalence class of the appearing roof is unique: indeed, for two choices $C_0^\bullet$ and $D_0^\bullet$ on the top of the above diagram, one can use Proposition I.9 again to show that the two choices are equivalent. Thus, the composition of two morphisms in $\mathbf{D}(\mathcal{A})$ is well defined and one can show it is associative, so that $\mathbf{D}(\mathcal{A})$ defines a category.

Notice that, for any given qis $f^\bullet : A^\bullet \to B^\bullet$, its image $Q(f^\bullet)$ is an isomorphism in the category $\mathbf{D}(\mathcal{A})$ just described: indeed, it can be represented by the roof $A^\bullet = A^\bullet \to B^\bullet$, whose inverse is the equivalence class of

$$\begin{tikzcd}
 & A^\bullet \arrow[dl, "f^\bullet"'] \arrow[dr, equal] & \\
B^\bullet & & A^\bullet ,
\end{tikzcd}$$

as we can see from the diagrams:

$$\begin{tikzcd}
 & & A^\bullet \arrow[dl, equal] \arrow[dr, equal] & & \\
 & A^\bullet \arrow[dl, equal] \arrow[dr, "f^\bullet"] & & A^\bullet \arrow[dl, "f^\bullet"'] \arrow[dr, equal] & \\
A^\bullet & & B^\bullet & & A^\bullet ,
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A^\bullet \arrow[dl, equal] \arrow[dr, equal] & & \\
 & A^\bullet \arrow[dl, "f^\bullet"'] \arrow[dr, equal] & & A^\bullet \arrow[dl, equal] \arrow[dr, "f^\bullet"] & \\
B^\bullet & & A^\bullet & & B^\bullet ,
\end{tikzcd}$$

where the second composition is indeed in the equivalence class of the identity of $B^\bullet$ because

$$\begin{tikzcd}
 & & A^\bullet \arrow[dl, equal] \arrow[dr, "f^\bullet"] & & \\
 & A^\bullet \arrow[dl, "f^\bullet"'] \arrow[drrr, "f^\bullet"' near end] & & B^\bullet \arrow[dlll, equal, crossing over] \arrow[dr, equal] & \\
B^\bullet & & & & B^\bullet .
\end{tikzcd}$$

Hence, one can easily verify the ***universal property*** holds by setting $G(f : A^\bullet \to B^\bullet) := (F(f^\bullet) : F(A^\bullet) \to F(B^\bullet))$, for any representative $f^\bullet$ of the morphism f. $\square$

Remark. — Under the functor Q, we identify objects in $\mathbf{D}(\mathcal{A})$ with objects in $C^\bullet(\mathcal{A})$: hence, we may speak of complexes $A^\bullet, B^\bullet, ... \in \mathbf{D}(\mathcal{A})$. As a consequence, the cohomology objects $H^n(A^\bullet)$ of $A^\bullet \in \mathbf{D}(\mathcal{A})$ are well-defined objects of the abelian category $\mathcal{A}$ because of the universal property of the derived category. Moreover, given two homotopic maps $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$, the diagram

$$\begin{tikzcd}
& & A^\bullet \arrow[dl, equal] \arrow[dr, equal] & & \\
& A^\bullet \arrow[dl, equal] \arrow[drrr, "f^\bullet"'] & & A^\bullet \arrow[dr, "g^\bullet"] & \\
A^\bullet \arrow[urrr, equal] & & & & B^\bullet .
\end{tikzcd}$$

commutes in $\mathbf{K}(\mathcal{A})$, so we deduce that $Q(f^\bullet) = Q(g^\bullet)$. By the **universal property of the homotopy category**, it follows that there exists a unique factorization

$$\begin{tikzcd}
C^\bullet(\mathcal{A}) \arrow[rr] \arrow[dr, "Q"'] & & \mathbf{K}(\mathcal{A}) \arrow[dl] \\
& \mathbf{D}(\mathcal{A}) & .
\end{tikzcd}$$

This means that, for every $n \in \mathbb{Z}$, the n-th cohomology H^n is a well-defined functor on the derived category.

Remark. — Viewing any object $A \in \mathcal{A}$ as a complex concentrated in degree zero yields an equivalence between $\mathcal{A}$ and the full subcategory of $\mathbf{D}(A)$ that consists of all complexes $B^\bullet$ with $H^n(B^\bullet) = \mathbf{0}$, for $n \neq 0$.

II.2. Theorem. — *The inclusion $\mathcal{A} \hookrightarrow \mathbf{D}(\mathcal{A})$ which sends an object A to the complex concentrated in degree zero $A[0]$ yields an equivalence with the full subcategory $\mathbf{D}^0(\mathcal{A})$ of $\mathbf{D}(\mathcal{A})$ formed by complexes $B^\bullet$ such that $H^n(B^\bullet) \simeq \mathbf{0}$, for $n \neq 0$.*

Proof. Guarda Manin - III.5.Prop2. □

II.3. Exercise. — Show that $A^\bullet$ is isomorphic to $\mathbf{0}$ in $\mathbf{D}(\mathcal{A})$ if and only if $H^n(A^\bullet) \simeq \mathbf{0}$ in $\mathcal{A}$, for every $n \in \mathbb{Z}$. On the other hand, find an example of a complex morphism $f^\bullet : A^\bullet \to B^\bullet$ which is trivial in cohomology, but $Q(f^\bullet) \neq 0$.

Solution. We have already noticed that cohomology is a well defined functor on the derived category, so if $A^\bullet \simeq \mathbf{0}$ in $\mathbf{D}(\mathcal{A})$, then $H^*(A^\bullet)$ is trivial. Conversely, assume $A^\bullet$ has trivial cohomology: then the zero morphism $A^\bullet \to \mathbf{0}$ is a qis, hence an isomorphism in $\mathbf{D}(\mathcal{A})$.

As an example of a non-trivial morphism, with trivial cohomology, consider the following commutative diagram in $\mathcal{A} = \mathbf{Mod}_{\mathbb{Z}}$:

$$\begin{tikzcd}
A^\bullet : \arrow[d, "f^\bullet"'] & \mathbf{0} \arrow[r] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "q"] & \mathbb{Z}/4\mathbb{Z} \arrow[r] \arrow[d] & \mathbf{0} \\
B^\bullet : & \mathbf{0} \arrow[r] & \mathbf{0} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbf{0} \arrow[r] & \mathbf{0}
\end{tikzcd} ;$$

since $H^0(A^\bullet) \simeq \mathbf{0}$, then it is clear that $f^* = 0$. Nevertheless, $f^\bullet$ in not null-homotopic: indeed, if there existed a homotopy h, then for each $x \in \mathbb{Z}/4\mathbb{Z}$ we would have

$$q(x) = h(2x) = 2h(x) = 0\,,$$

which is a contradiction. In particular, this means that in $\mathbf{K}(\mathbf{Mod}_\mathbb{Z})$ there is no commutative diagram of the form

$$\begin{tikzcd}
& & C_0^\bullet \arrow[dl, "\mathrm{qis}"'] \arrow[dr] & & \\
& A^\bullet \arrow[dl, equal] \arrow[drrr, "f^\bullet" near end] & & C^\bullet \arrow[dlll, "\mathrm{qis}"' near start] \arrow[dr, "0"] & \\
A^\bullet & & & & B^\bullet\,,
\end{tikzcd}$$

so $Q(f^\bullet) \neq 0$. Why? Guarda esercizi Danilo. □

Exercise. — Check that the derived category $\mathbf{D}(\mathcal{A})$ is additive.

Thus, if $\mathcal{A}$ is an abelian category, its derived category $\mathbf{D}(\mathcal{A})$ is additive, but in general not abelian; nevertheless, it is triangulated.

II.4. Proposition. — *The category $\mathbf{D}(\mathcal{A})$ is triangulated and the canonical functor*

$$\mathbf{K}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{A})$$

is an exact functor of triangulated categories.

Proof. Check Gelfand-Manin IV.2 □

II.5. Exercise. — Suppose

$$0 \longrightarrow A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \longrightarrow 0$$

is a short exact sequence in an abelian category $\mathcal{A}$. Show that under the full embedding into the homotopy category $\mathcal{A} \hookrightarrow \mathbf{K}(\mathcal{A})$ (or the one in the derived category $\mathcal{A} \hookrightarrow \mathbf{D}(\mathcal{A})$), this becomes a distinguished triangle

$$A \longrightarrow B \longrightarrow C \xrightarrow{\;\delta\;} A[1]\,,$$

where δ is the composition of the inverse of the qis $\mathbf{C}(f) \to C$ with the projection $\mathbf{C}(f) \to A[1]$. Conversely, if $A, B, C \in \mathcal{A}$ form a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]\,,$$

then $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$.

Solution. Notice that the cone $\mathbf{C}(f)$ is given by the complex

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0\,,$$

thus $H^{-1}(\mathbf{C}(f)) = \ker f$ and $H^{0}(\mathbf{C}(f)) = \operatorname{Coker} f$. Since the short sequence is exact, it follows that

$$H^{-1}(\mathbf{C}(f)) = 0\,, \quad H^{0}(\mathbf{C}(f)) \simeq C\,,$$

hence the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

defines a qis $\mathbf{C}(f) \to C$. Then, by passing in $\mathbf{K}(\mathcal{A})$ (or equivalently in $\mathbf{D}(\mathcal{A})$), we end up with the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d] & A[1] \arrow[d, equal] \\ A \arrow[r, "f"] & B \arrow[r] & \mathbf{C}(f) \arrow[r] & A[1]\,, \end{tikzcd}$$

which is an isomorphism of triangles by the 5-lemma I.11.

Conversely, if three objects $A, B, C \in \mathcal{A}$ form a distinguished triangle $A \to B \to C \to A[1]$, then the induced LECS **??** is a short exact sequence in $\mathcal{A}$. $\square$

II.6. Exercise. — Suppose $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle in the derived category $\mathbf{D}(\mathcal{A})$. Show that it naturally induces a long exact sequence

$$\ldots \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow H^{i+1}(A^\bullet) \longrightarrow \ldots$$

Solution. LALALA. $\square$

By definition, complexes in the categories $\mathbf{K}(A)$ and $\mathbf{D}(A)$ are unbounded, but often it is more convenient to work with bounded ones, especially in the algebro-geometric context.

Definition. — Given an abelian category $\mathcal{A}$, let $C^*(\mathcal{A})$, with $* \in \{+, -, b\}$, be the category of complexes $A^\bullet$ with $A^n = 0$ for $n << 0, n >> 0$, respectively $|n| >> 0$.

By dividing out first by homotopy equivalence and then by qis one obtains the categories $\mathbf{K}^*(\mathcal{A})$ and $\mathbf{D}^*(\mathcal{A})$, with $* \in \{+, -, b\}$. Let us consider the natural functors $\mathbf{D}^*(A) \to \mathbf{D}(A)$ given by just forgetting the boundedness condition.

II.7. Proposition. — *The natural functor $\mathbf{D}^+(\mathcal{A}) \hookrightarrow \mathbf{D}(\mathcal{A})$, defines an equivalence of $\mathbf{D}^+(\mathcal{A})$ with the full triangulated subcategory of all complexes $A^\bullet \in \mathbf{D}(\mathcal{A})$ with $A^n = 0$ for $n << 0$. Analogous statements hold true for $\mathbf{D}^-(\mathcal{A})$ and $\mathbf{D}^b(\mathcal{A})$.*

Proof. KASHIWARA SHAPIRA. $\square$

II.8. Exercise. — Let $A^\bullet$ be a complex with $H^n(A^\bullet) = 0$, for $n > m$. Show that $A^\bullet$ is quasi-isomorphic (and hence isomorphic as an object in $\mathbf{D}(\mathcal{A})$) to a complex $B^\bullet$, with $B^n = 0$ for $n > m$.

Solution. AAAAAAAH □

Exercise. — Let $A^\bullet$ be a complex with $m := \{ n \in \mathbb{Z} \mid H^n(A^\bullet) \neq 0 \} < \infty$. Show there exists a morphism

$$\varphi : A^\bullet \longrightarrow H^m(A^\bullet)[-m]$$

in $\mathbf{D}(\mathcal{A})$ such that $H^m(\varphi) : H^m(A^\bullet) \to H^m(A^\bullet)$ is the identity.

Solution. AAAAAAAAAA □

Exercise. — Suppose $H^n(A^\bullet) = 0$ for $n < n_0$. Show there exists a distinguished triangle

$$H^{n_0}(A^\bullet)[-n_0] \longrightarrow A^\bullet \xrightarrow{\varphi} B^\bullet \longrightarrow H^{n_0}(A^\bullet)[1-n_0]$$

in $\mathbf{D}(\mathcal{A})$ with $H^n(B^\bullet) = 0$ for $n \leq n_0$ and φ inducing isomorphisms $H^n(A^\bullet) \simeq H^n(B^\bullet)$ for $n > n_0$.

Solution. AAAAAAAAA. □

Splitting and derived categories in Gelfand Manin.

Injective and projective resolutions

Fix this chapter structure. Follow the notes of Tamas.

Due to the very construction of the derived category, it is sometimes quite cumbersome to do explicit calculations there. Often, however, it is possible to work with a very special class of complexes for which morphisms in the derived category and in the homotopy category are the same thing. Depending on the kind of functors one is interested in, the notion of injective, respectively, projective objects will be crucial. As always, let $\mathcal{A}$ denote an abelian category.

II.9. Lemma. — *A map $f : A \to B$ is a monomorphism if and only if* $\ker f \simeq \mathbf{0}$. *Dually, f is an epimorphism if and only if* $\operatorname{Coker} f \simeq \mathbf{0}$.

Proof. If $f : A \to B$ is mono, then by the commutativity of the diagram

$$\begin{tikzcd} \ker f \arrow[r, bend left] \arrow[r, bend right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

we deduce that the canonical map $\ker f \to A$ factors through the zero object, hence $\ker f \simeq \mathbf{0}$. Conversely, if $\ker f$ is the zero object, for any two morphisms $g, h : C \to A$ such that $gf = hf$ it holds $(g-h)f = 0$, which means we have a factorization

$$\begin{tikzcd} & \ker f \simeq \mathbf{0} \arrow[dr] & & \\ C \arrow[ur] \arrow[rr, "g-h"'] & & A \arrow[r, "f"] & B \end{tikzcd},$$

thus $g = h$ and f is a monomorphism. Dually, one proves the statement for epimorphisms. □

The previous lemma justifies the following notation, already used in the setting of abelian groups: a map $f : A \to B$ is a monomorphism if and only if the sequence

$$\mathbf{0} \longrightarrow A \xrightarrow{\;f\;} B$$

is exact, thus it will be convenient to represent monomorphisms in terms of this sequence, whose exactness won't be stressed further on. In a similar fashion, if $f : A \to B$ is an epimorphism, it will be drawn as

$$A \xrightarrow{\;f\;} B \longrightarrow \mathbf{0}\,.$$

Definition. — An object $P \in \mathcal{A}$ is called **projective** if, given any epimorphism $A \to B$, every map $P \to B$ can be lifted to a map $P \to A$, making the following diagram commute

$$\begin{tikzcd} P \arrow[d, dashed, "\exists"'] \arrow[dr] & & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}. \tag{II.9.1}$$

Recall that, given any object $X \in \mathcal{A}$, the covariant functor

$$\mathrm{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \longrightarrow \mathbf{Ab}$$

is *left exact*, i.e. by applying it to any short exact sequence

$$\mathbf{0} \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \mathbf{0}\,,$$

we obtain the following exact sequence of abelian groups:

$$\mathbf{0} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, A) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, B) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, C)\,.$$

It is easy to see that projectivity is characterized by the exactness of this functor:

II.10. Proposition. — *An object $P \in \mathcal{A}$ is projective if and only if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is an exact functor.*

Proof. Since we already know that the Hom functor is left exact, it will suffice to check exactness on the right. Thus, we notice that any diagram of the form (II.9.1) can be filled with a vertical arrow $P \to A$ if and only if the induced homomorphism $\mathrm{Hom}_{\mathcal{A}}(X, A) \to \mathrm{Hom}_{\mathcal{A}}(X, B)$ is surjective. □

Remark. — It will be useful to consider the following slightly more general version of the diagram (II.9.1): if P is projective, from the exactness of $\mathrm{Hom}_{\mathcal{A}}(P, -)$ we deduce that, given any commutative diagram of the form

$$\begin{tikzcd}
& P \arrow[dl, dashed, "\exists"'] \arrow[d] \arrow[dr, "0"] & \\
A \arrow[r] & B \arrow[r] & C\,,
\end{tikzcd}$$

whose bottom row is exact, there exists a map $P \to A$ which fits in it.

The dual notion of the projective property is given by **injectivity**:

Definition. — An object $I \in \mathcal{A}$ is called **injective** if, given any monomorphism $A \to B$, every map $A \to I$ can be extended to a map $B \to I$, in such a way that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{0} \arrow[r] & A \arrow[r] \arrow[dr] & B \arrow[d, dashed, "\exists"'] \\
& & I\,.
\end{tikzcd}$$

Remark. — Note that injectivity and projectivity are indeed dual notions, in the sense that an object $P \in \mathcal{A}$ is projective if and only if P is injective in the opposite category $\mathcal{A}^{op}$. From this, we deduce a similar characterization of injectivity in terms of Hom functors: an object $I \in \mathcal{A}$ is injective if and only if the contravariant functor

$$\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \longrightarrow \mathbf{Ab}$$

is exact.

Example. — The direct sum of two injective objects is again injective: assume $I, J \in \mathcal{A}$ are injective and $A \to B$ is a monomorphism; then we have the following diagram

$$\begin{tikzcd}
\mathbf{0} \arrow[r] & A \arrow[rr] \arrow[d] \arrow[dr] & & B \arrow[dll, dashed, "\exists"'] \arrow[d, dashed, "\exists"'] \\
& I \arrow[drr] & I \oplus J \arrow[l, two heads] \arrow[r, two heads] \arrow[dr, equal] & J \arrow[d, hook] \\
& & & I \oplus J\,,
\end{tikzcd}$$

which shows that any map $A \to I \oplus J$ extends to a morphism $B \to I \oplus J$. In other words, we use that

$$\mathrm{Hom}_{\mathcal{A}}(I \oplus J, -) \simeq \mathrm{Hom}_{\mathcal{A}}(I, -) \oplus \mathrm{Hom}_{\mathcal{A}}(J, -)$$

is a sum of exact functors, and hence exact.

Dually, one checks that a sum of projective objects is again projective.

Definition. — We say that an abelian category $\mathcal{A}$ **has enough injective** (resp. **projective**) **objects** if, for any $A \in \mathcal{A}$, there exists a monomorphism $\mathbf{0} \to A \to I$, with $I \in \mathcal{A}$ injective (resp. an epimorphism $P \to A \to \mathbf{0}$, with $P \in \mathcal{A}$ projective).

An **injective resolution** of an object $A \in \mathcal{A}$ is an exact sequence

$$\mathbf{0} \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots,$$

with all $I^i \in \mathcal{A}$ injective. Similarly, a **projective resolution** of $A \in \mathcal{A}$ is an exact sequence

$$\dots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow A \longrightarrow \mathbf{0},$$

where all $P^i \in \mathcal{A}$ are projective.

In other words, an injective resolution of $A \in \mathcal{A}$ consists of a quasi isomorphism $A[0] \to I^\bullet$, where $I^\bullet$ is a complex of injective objects, such that $I^i = \mathbf{0}$, for $i < 0$. In an analogous way, a projective resolution of A is a particular quasi-isomorphism $P^\bullet \to A[0]$, where $P^\bullet$ is a complex with non-positive terms, which are projective. It follows that $A[0], P^\bullet$ and $I^\bullet$ are isomorphic in $\mathbf{D}(\mathcal{A})$. In fact, the idea of the derived category as we understand it today, is that an object $A \in \mathcal{A}$ should be *identified with all its resolutions.* We can actually show that resolutions are unique in some sense.

II.11. Proposition. — *Let $f : A \to B$ be a morphism in $\mathcal{A}$ and two projective resolutions*

$$P^\bullet \longrightarrow A[0], \quad Q^\bullet \longrightarrow B[0].$$

Then there exists a morphism of resolutions $R(f) : P^\bullet \to Q^\bullet$ which extends f, i.e. the following diagram in $\mathcal{A}$ commutes:

$$\begin{tikzcd} P^0 \arrow[r] \arrow[d, "R(f)^0"'] & A \arrow[d, "f"] \\ Q^0 \arrow[r] & B; \end{tikzcd}$$

moreover, any two such extensions $R(f)$ and $R'(f)$ are homotopic.

Proof. We build the morphism $R(f)$ by induction on its degree: to build the first step, consider the diagram

$$\begin{tikzcd} P^0 \arrow[r] \arrow[d, dashed, "R(f)^0"'] \arrow[dr] & A \arrow[d, "f"] & \\ Q^0 \arrow[r] & B \arrow[r] & \mathbf{0}; \end{tikzcd}$$

since $Q^0 \to B$ is an epimorphism and P^0 is projective, then there exists $R(f)^0 : P^0 \to Q^0$, which makes the above diagram commutative.

Assume we already have $R(f)^{-j} : P^{-j} \to Q^{-j}$, for every $0 \leq j < i$, such that

$$d_Q^{-j} \circ R(f)^{-j} = R(f)^{-j+1} \circ d_P^{-j};$$

then, consider the diagram

$$\begin{tikzcd} P^{-i} \arrow[d, dashed, "\exists"'] \arrow[dr, "R(f)^{-i+1}\circ d_P^{-i}"] & & \\ Q^{-i} \arrow[r] & Q^{-i+1} \arrow[r] & Q^{-i+2}\,, \end{tikzcd}$$

whose bottom row is exact; as P^{-i} is projective and

$$d_Q^{-i+1} \circ R(f)^{-i+1} \circ d_P^{-i} = R(f)^{-i+2} \circ d_P^{-i-1} \circ d_P^{-i-2} = 0\,,$$

we deduce there exists $R(f)^{-i} : P^{-i} \to Q^{-i}$ which makes the diagram commute.

Now assume $R(f)$ and $R'(f)$ are two morphisms of resolutions that extend $f : A \to B$. As in the previous part, we build a homotopy s between $R(f)$ and $R'(f)$ inductively: start by setting $s^1 : A \to Q^0$ to be the zero morphism, and then consider the diagram

$$\begin{tikzcd} & P^0 \arrow[dl, dashed, "\exists"'] \arrow[d, "R(f)^0-R'(f)^0"] & \\ Q^{-1} \arrow[r] & Q^0 \arrow[r] & B\,, \end{tikzcd}$$

whose bottom row is exact; we see that

$$d_Q^0 \circ (R(f)^0 - R'(f)^0) = (f - f) \circ d_P^0 = 0\,,$$

thus there exists $s^0 : P^0 \to Q^{-1}$ that makes the diagram commute because P^0 is projective. Notice that at the 0-th level it holds

$$R(f)^0 - R'(f)^0 = s^0 d_Q^{-1} + d_P^0 s^1\,.$$

Similarly, to build the $(-i)$-th level of the homotopy, once $s^{-j} : P^{-j} \to Q^{-i-1}$ are given for all $0 \le j < i$, consider the commutative diagram

$$\begin{tikzcd} & P^{-i} \arrow[dl, dashed, "\exists"'] \arrow[d, "R(f)^{-i}-R'(f)^{-i}-s^{-i+1}d_P^{-i}"] & \\ Q^{-i-1} \arrow[r] & Q^{-i} \arrow[r] & Q^{-i+1} \end{tikzcd}$$

and notice that

$$\begin{aligned} & d_Q^{-i} \circ \big(R(f)^{-i} - R'(f)^{-i} - s^{-i+1} \circ d_P^{-i}\big) \\ =& (R(f)^{-i+1} - R'(f)^{-i+1}) \circ d_P^{-i} - (d_Q^{-i} \circ s^{-i+1}) \circ d_P^{-i} \\ =& (R(f)^{-i+1} - R'(f)^{-i+1}) \circ d_P^{-i} - \big(R(f)^{-i+1} - R'(f)^{-i+1} - s^{-i+2} \circ d_P^{-i+1}\big) \circ d_P^{-i} \\ =& - s^{-i+2} \circ d_P^{-i+1} \circ d_P^{-i} = 0\,, \end{aligned}$$

hence by projectivity of P^{-i} we conclude that there exists $s^{-i} : P^{-i} \to Q^{-i-1}$ such that

$$R(f)^{-i} - R'(f)^{-i} = s^{-i+1} \circ d_P^{-i} + d^{-i-1} \circ s^{-i}\,.$$

□

II.12. Corollary. — *Any two projective resolutions of an object are homotopy equivalent.*

Proof. Using the notation of Proposition II.11, let $B = A$, $f = \mathbf{1}_A$ and then consider $R(\mathbf{1}_A) : P^\bullet \to Q^\bullet$ and $R'(\mathbf{1}_A) : Q^\bullet \to P^\bullet$ extensions of the identity. Since both $R'(\mathbf{1}_A) \circ R(\mathbf{1}_A)$ and $\mathbf{1}_{P^\bullet}$ are cochain maps $P^\bullet \to P^\bullet$ that extend the identity of A, by the last part of Proposition II.11 we conclude that $R'(\mathbf{1}_A) \circ R(\mathbf{1}_A) \sim \mathbf{1}_{P^\bullet}$. Similarly, we deduce that $R(\mathbf{1}_A) \circ R'(\mathbf{1}_A) \sim \mathbf{1}_{Q^\bullet}$. □

Remark. —

1. In the proof of Proposition II.11 we never used that $Q^\bullet$ was a projective resolution: indeed, Proposition II.11 remains valid even if we assume that $Q^\bullet \to B \to 0$ is an exact complex.
2. By reversing arrows and replacing projective resolutions with injective ones, we obtain a dual version of Proposition II.11, as well as a dual version of Corollary II.12.
3. The meaning of Corollary II.12 is that a projective resolution of an object is *unique up to homotopy equivalence*, and the same holds true for an injective resolution. This means that, for any object $A \in \mathcal{A}$, in the homotopy category $\mathbf{K}(\mathcal{A})$ (and hence in $\mathbf{D}(\mathcal{A})$) there is at most one projective resolution $P^\bullet$ of A and at most one injective resolution $I^\bullet$ of A *up to isomorphism.*

The fundamental idea of the derived category, as we understand it today, is that any object of the abelian category should be identified with all of its resolutions. In fact, a more general result holds true: given an abelian category $\mathcal{A}$, let $\mathcal{I}$ be the full subcategory of its injective objects and consider $\mathbf{K}^+(\mathcal{I})$, the full subcategory of $\mathbf{K}^+(\mathcal{A})$ spanned by complexes with injective terms. Consider the natural immersion $\mathbf{K}^+(\mathcal{I}) \hookrightarrow \mathbf{K}^+(\mathcal{A})$ with $Q : \mathbf{K}^+(\mathcal{A}) \to \mathbf{D}^+(\mathcal{A})$.

II.13. Theorem. — *The functor $\mathbf{K}^+(\mathcal{I}) \to \mathbf{D}^+(\mathcal{A})$ is an equivalence of $\mathbf{K}^+(\mathcal{I})$ with a full subcategory of $\mathbf{D}^+(\mathcal{A})$. Moreover, if $\mathcal{A}$* ***has enough injectives****, the above functor is an equivalence of $\mathbf{K}^+(\mathcal{I})$ and $\mathbf{D}^+(\mathcal{A})$.*

Proof. Check [**gelfand**]. The proof is rather long and technical, although not very complicated, and passes through the definition of the derived category as a *localization* of the homotopy category. □

II.14. Corollary. — *Suppose $\mathcal{A}$ is an abelian category with enough injectives. Any complex $A^\bullet$ with $H^n(A^\bullet) = \mathbf{0}$ for $n << 0$ is isomorphic in $\mathbf{D}(\mathcal{A})$ to a complex $I^\bullet$ of injective objects I^i, such that $I^i = \mathbf{0}$ for $i << 0$.*

Proof. Suppose $\mathcal{A}$ is an abelian category with enough injectives. By Exercise II.8 we may assume $A^i = \mathbf{0}$, for $i << 0$, thus $A^\bullet \in \mathbf{D}^+(\mathcal{A})$ and by the equivalence $\mathbf{K}^+(\mathcal{I}) \simeq \mathbf{D}^+(\mathcal{A})$ we have $A^\bullet \simeq I^\bullet$, for some injective resolution $I^\bullet$. □

As one might expect, a similar statement holds true for projective resolutions: if $\mathcal{P}$ is the full subcategory of projective objects of $\mathcal{A}$, then $\mathbf{K}^-(\mathcal{P})$ is the full subcategory of $\mathbf{K}^-(\mathcal{A})$ spanned by complexes with projective terms, and hence we have the following

II.15. Theorem. — *The functor $\mathbf{K}^-(\mathcal{P}) \to \mathbf{D}^-(\mathcal{A})$ is an equivalence of $\mathbf{K}^-(\mathcal{P})$ with a full subcategory of $\mathbf{D}^-(\mathcal{P})$. Moreover, if $\mathcal{A}$* ***has enough projectives****, the above functor is an equivalence of $\mathbf{K}^-(\mathcal{P})$ and $\mathbf{D}^-(\mathcal{A})$.*

It follows that every complex $A^\bullet$ with $H^n(A^\bullet) = \mathbf{0}$ for $n >> 0$ is isomorphic in $\mathbf{D}(\mathcal{A})$ to a complex $P^\bullet$ of projective objects P^i, such that $P^i = \mathbf{0}$ for $i >> 0$.

We now try to relate morphisms between complexes in the derived category and morphisms between their resolutions, in order to better understand how to do computations in $\mathbf{D}(\mathcal{A})$.

II.16. Lemma. — *Let $A^\bullet \to B^\bullet$ be a qis in $\mathbf{K}^+(\mathcal{A})$. Then for every complex of injective objects $I^\bullet$ which is bounded below, the induced map*

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(B^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A^\bullet, I^\bullet)\,,$$

is an isomorphism.

Proof. Since $\mathbf{K}^+(\mathcal{A})$ is triangulated, we can complete the qis to a distinguished triangle

$$A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1]\,,$$

and if we apply the cohomological functor $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(-, I^\bullet)$ we obtain a LECS; thus, it is enough to show that $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(C^\bullet, I^\bullet) = \mathbf{0}$ whenever the complex $C^\bullet \in \mathbf{K}^+(\mathcal{A})$ is acyclic, i.e. any cochain map $f^\bullet : C^\bullet \to I^\bullet$ is null homotopic.

Indeed, by following the same procedure as in the proof of Proposition II.11, one can build a homotopy by induction: first, we may assume the first non-zero term of $C^\bullet$ is C^0 and set $s^0 : C^0 \to I^{-1}$ to be the zero map. Since I^0 is injective, the morphism f^0 extends to a map $s^1 : C^1 \to I^0$ such that

$$f^0 = s^1 \circ d_C^0 = s^1 \circ d_C^0 + d_I^{-1} \circ s^0\,.$$

If we have already built s^j for $0 \leq j < i$ such that

$$f^{j-1} = s^j \circ d_C^{j-1} + d_I^{j-2} \circ s^{j-1}\,,$$

then consider the commutative diagram

$$\begin{tikzcd}
C^{i-2} \arrow[r] \arrow[d] & C^{i-1} \arrow[r] \arrow[d, "\alpha"'] & C^i \arrow[dl, dashed, "\exists"', "s^i"] \\
\mathbf{0} \arrow[r] & I^{i-1} &
\end{tikzcd}\,,$$

with $\alpha = f^{i-1} - d_I^{i-2} \circ s^{i-1}$; since the top row is exact and I^{i-1} is projective, there is an extension $s^i : C^i \to I^{i-1}$ such that

$$f^{i-1} = s^i \circ d_C^{i-1} + d_I^{i-2} \circ s^{i-1}\,,$$

so the s^i all together form the desired homotopy. □

II.17. Lemma. — *Given $A^\bullet, I^\bullet \in \mathbf{K}^+(\mathcal{A})$, with all I^i injective objects, then*

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, I^\bullet)\,.$$

Proof. Clearly the canonical functor $Q : \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ induces a map

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, I^\bullet)\,, \qquad \left(A^\bullet \xrightarrow{f^\bullet} I^\bullet\right) \longmapsto \left(A^\bullet = A^\bullet \xrightarrow{f^\bullet} I^\bullet\right),$$

and hence we have to show that, for every roof

$$\begin{tikzcd} & B^\bullet \arrow[dl, "\mathrm{qis}"'] \arrow[dr] & \\ A^\bullet \arrow[rr, dashed, "\exists!"] & & I^\bullet\,, \end{tikzcd}$$

there exists a unique morphism $A^\bullet \to I^\bullet$ which makes the diagram commute *up to homotopy*. This is equivalent to saying that, once a qis $B^\bullet \to A^\bullet$ is fixed, there is a bijection $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(B^\bullet, I^\bullet)$, but this is the content of Lemma II.16. $\square$

Qui ho fatto un po' un casino per l'ordine, perché ho messo il teorema sull'equivalenza prima, aiuto. Rivedere un po' l'ordine di questa parte, magari seguire Tamas.

Derived functors

Let $F : \mathcal{A} \to \mathcal{B}$ be an *additive* functor between abelian categories; it naturally extends to a unique functor between the associated categories of cochain complexes $C^\bullet(\mathcal{A}) \to C^\bullet(\mathcal{B})$ and hence, by dividing out by homotopy equivalence, it induces a well-defined additive functor $\tilde{F} : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$: indeed, if s is a homotopy between two cochain maps $f^\bullet, g^\bullet : A^\bullet \to \tilde{A}^\bullet$, then the identity

$$\begin{aligned}\tilde{F}(f^\bullet) - \tilde{F}(g^\bullet) &= \tilde{F}(f^\bullet - g^\bullet)\\ &= \tilde{F}\left(sd_A + d_{\tilde{A}}s\right) = \tilde{F}(s)\tilde{F}\left(d_A\right) + \tilde{F}\left(d_{\tilde{A}}\right)\tilde{F}(s) = \tilde{F}(s)d_{F(A)} + d_{F(\tilde{A})}F(s)\end{aligned}$$

shows that $\tilde{F}(s)$ is a homotopy between $\tilde{F}(f^\bullet)$ and $\tilde{F}(g^\bullet)$. Moreover, we can check that $\tilde{F} : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ is an exact functor between triangulated categories:

(**EF1**) since the functor $\tilde{F}$ sends a complex $A^\bullet \in \mathbf{K}(\mathcal{A})$ to the complex

$$\dots \longrightarrow F(A^{i-1}) \xrightarrow{F(d_A^{i-1})} F(A^i) \xrightarrow{F(d_A^i)} F(A^{i+1}) \longrightarrow \dots$$

it is clear that $\tilde{F}$ commutes with the shift functor:

$$\left(\tilde{F}(A^\bullet)[1]\right)^n = \tilde{F}(A^\bullet)^{n-1} = F(A^{n-1}) = F\left((A^\bullet[1])^n\right) = \tilde{F}(A^\bullet[1])^n\,;$$

(**EF2**) recall that the image of a direct sum via an additive functor is the direct sum of the images: in particular, given a map $f^\bullet : A^\bullet \to \tilde{A}^\bullet$ in $\mathbf{K}(\mathcal{A})$, then $\tilde{F}\big(\mathbf{C}(f^\bullet)\big) \simeq \mathbf{C}\big(\tilde{F}(f^\bullet)\big)$, and hence any distinguished triangle of the form

$$A^\bullet \xrightarrow{f^\bullet} B^\bullet \longrightarrow \mathbf{C}(f^\bullet) \longrightarrow A^\bullet[1]$$

is sent to a distinguished triangle

$$\tilde{F}(A^\bullet) \xrightarrow{\tilde{F}(f^\bullet)} \tilde{F}(B^\bullet) \longrightarrow \mathbf{C}\big(\tilde{F}(f^\bullet)\big) \longrightarrow \tilde{F}(A^\bullet)[1]\,;$$

by definition of distinguished triangles in the homotopy category, this is enough to conclude that $\tilde{F}$ preserves all triangles.

Now consider the following question: given the diagram

$$\begin{tikzcd} \mathbf{K}^*(\mathcal{A}) \arrow[r] \arrow[d, "\tilde{F}"'] & \mathbf{D}^*(\mathcal{A}) \\ \mathbf{K}(\mathcal{B}) \arrow[r] & \mathbf{D}(\mathcal{B}) \end{tikzcd}$$

there exists a triangulated functor $\mathbf{D}^*(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ making the diagram commute? When $F : \mathcal{A} \to \mathcal{B}$ is an exact functor of abelian categories, then the answer is yes: indeed, $\tilde{F}$ sends any acyclic complex $A^\bullet$ to an acyclic complex because, for every $n \in \mathbb{Z}$, the short exact sequence

$$\mathbf{0} \longrightarrow \ker d_A^n \longrightarrow A^n \xrightarrow{d_A^n} \ker d_A^{n+1} \longrightarrow \mathbf{0}$$

is sent to the sequence

$$\mathbf{0} \longrightarrow \ker d_{F(A)}^n \longrightarrow \tilde{F}(A^\bullet)^n \xrightarrow{d_{F(A)}^n} \ker d_{F(A)}^{n+1} \longrightarrow \mathbf{0}$$

which is exact in $\mathcal{B}$, and hence $H^n(\tilde{F}(A^\bullet) = \mathbf{0}$. Thus, the hypothesis of the following lemma are satisfied:

II.18. Lemma. — *Let $G : \mathbf{K}^*(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ be an exact functor of triangulated categories. Then G naturally induces a commutative diagram*

$$\begin{tikzcd} \mathbf{K}^*(\mathcal{A}) \arrow[r, "G"] \arrow[d] & \mathbf{K}(\mathcal{B}) \arrow[d] \\ \mathbf{D}^*(\mathcal{A}) \arrow[r] & \mathbf{D}(\mathcal{B}) \end{tikzcd}$$

if one of the two following conditions is satisfied:

(i) *the image of a qis under G is a qis;*

(ii) *the functor G sends acyclic complexes to acyclic complexes.*

Proof. If condition (i) holds, then the composition $\mathbf{K}^*(\mathcal{A}) \to \mathbf{K}(\mathcal{B}) \to \mathbf{D}(\mathcal{B})$ sends each qis to an isomorphism, hence by the universal property of the derived category of $\mathcal{A}$, it factors through $\mathbf{D}^*(\mathcal{A})$ as in the diagram above.

Assume now that condition (ii) holds true and consider a qis $f^\bullet : A^\bullet \to B^\bullet$; the cone $\mathbf{C}(f^\bullet)$ is acyclic, so by hypothesis $\mathbf{C}\big(G(f^\bullet)\big)$ is acyclic too, which implies that $G(f^\bullet)$ is a quasi-isomorphism. Thus, G sends quasi-isomorphisms to quasi-isomorphisms, and we conclude by part (i). $\square$

Remark. — In the Lemma above, the functor G need not come from a functor between the abelian categories!

If F is not exact, the image of an acyclic complex in A, i.e. one that becomes trivial in $\mathbf{D}(\mathcal{A})$, is not, in general, acyclic. Thus, the naive extension of F to a functor between the derived categories $\mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ does not make sense and in this case a more complicated construction is needed in order to induce a natural functor between the derived categories. The new functor, called the *derived functor*, will not produce a commutative diagram as in Lemma II.18, but it has the advantage to encode more information even when applied to an object in the abelian category. Roughly, it explains why the original functor fails to be exact. In order to ensure existence of the derived functor, we will always have to assume some kind of exactness: for a **left exact** functor $F : \mathcal{A} \to \mathcal{B}$, one constructs the **right derived functor**

$$RF : \mathbf{D}^+(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{B}) \,,$$

and for a **right exact** functor $G : \mathcal{A} \longrightarrow \mathcal{B}$ one constructs the **left derived** functor

$$LG : \mathbf{D}^-(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{B}) \,.$$

They can be defined by a universal property:

Definition. — Let $F : \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ be a triangulated functor. A **right derived functor** for F is a triangulated functor

$$RF : \mathbf{D}^+(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{B}) \,,$$

together with a morphism of functors $\varepsilon : Q \circ F \to RF \circ Q$ such that, for any pair (Φ, η) of a triangulated functor $\Phi : \mathbf{D}^+(\mathcal{A}) \to \mathbf{D}^+(\mathcal{B})$ and natural transformation $\eta : Q \circ F \to \Phi \circ Q$, there exists a unique morphism of functors $\alpha : RF \to \Phi$ such that $\eta = \alpha_{Q(-)} \circ \varepsilon$; more explicitly, for every $A^\bullet \in \mathbf{K}^+(\mathcal{A})$, in $\mathbf{D}(\mathcal{B})$ we have the following commutative diagram:

$$\begin{tikzcd} Q \circ F(A^\bullet) \arrow[rr, "\eta_{A^\bullet}"] \arrow[dr, "\varepsilon_{A^\bullet}"'] & & \Phi \circ Q(A^\bullet) \\ & RF \circ Q(A^\bullet) \arrow[ur, "\alpha_{Q(A^\bullet)}"'] & \end{tikzcd} .$$

Similarly, if $G : \mathbf{K}^-(\mathcal{A}) \to \mathbf{K}^-(\mathcal{B})$ is a triangulated functor, a **left derived functor** for G is a triangulated functor

$$LG : \mathbf{D}^-(\mathcal{A}) \longrightarrow \mathbf{D}^-(\mathcal{B}) \,,$$

together with a morphism of functors $\varepsilon : LG \circ Q \to Q \circ F$ such that, for any pair (Φ, η) of a triangulated functor $\Phi : \mathbf{D}^-(\mathcal{A}) \to \mathbf{D}^-(\mathcal{B})$ and natural transformation $\eta : \Phi \circ Q \to Q \circ G$, there exists a unique morphism of functors $\beta : \Phi \to LG$ such that $\eta = \varepsilon \circ \beta_{Q(-)}$; more explicitly, for every $A^\bullet \in \mathbf{K}^-(\mathcal{A})$, in $\mathbf{D}^-(\mathcal{B})$ we have the following commutative diagram:

$$\begin{tikzcd} \Phi \circ Q(A^\bullet) \arrow[rr, "\eta_{A^\bullet}"] \arrow[dr, "\beta_{Q(A^\bullet)}"'] & & Q \circ G(A^\bullet) \\ & LG \circ Q(A^\bullet) \arrow[ur, "\varepsilon_{A^\bullet}"'] & \end{tikzcd} .$$

Of course, when LG or RF exists, it is unique up to unique isomorphism, for it is defined by a universal property. A sufficient condition for derived functors to exist is the existence of enough injectives and projectives.

II.19. Proposition. — *If $F : \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ is a triangulated functor and $\mathcal{A}$ has enough injective objects, then the right derived functor RF exists.*

Similarly, if $\mathcal{A}$ has enough projectives and $G : \mathbf{K}^-(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ is triangulated, then LG exists.

Proof. We do the case of the right derived functor: if $\mathcal{I}_\mathcal{A}$ is the full subcategory of injective objects of $\mathcal{A}$, then consider $\psi : \mathbf{D}^+(\mathcal{A}) \to \mathbf{K}^+(\mathcal{I}_\mathcal{A})$ a quasi-inverse of the equivalence stated in Theorem II.13 and define RF to be the composition $RF := Q \circ F|_{\mathbf{K}^+(\mathcal{I}_\mathcal{A})} \circ \psi$, i.e.

$$\begin{tikzcd}
\mathbf{K}^+(\mathcal{I}_\mathcal{A}) \arrow[r, hook] \arrow[d, "Q"] & \mathbf{K}^+(\mathcal{A}) \arrow[r, "F"] & \mathbf{K}(\mathcal{B}) \arrow[d, "Q"] \\
\mathbf{D}^+(\mathcal{A}) \arrow[u, bend left, "\psi"] \arrow[rr, dashed, "RF"] & & \mathbf{D}(\mathcal{B}) .
\end{tikzcd}$$

To define $\varepsilon : Q \circ F \to RF \circ Q$ on objects, let $A^\bullet$ be a complex bounded below and consider a qis $A^\bullet \to I^\bullet$ in $\mathbf{K}^+(\mathcal{A})$, where $I^\bullet \in \mathbf{K}^+(\mathcal{I}_\mathcal{A})$, which exists because of the equivalence $\mathbf{D}^+(\mathcal{A}) \simeq \mathbf{K}^+(\mathcal{I}_\mathcal{A})$. By applying Q to the induced morphism $F(A^\bullet) \to F(I^\bullet)$, one gets a morphism

$$\varepsilon_{A^\bullet} : Q \circ F(A^\bullet) \longrightarrow Q \circ F(I^\bullet) \simeq Q \circ F \circ \psi\big(Q(I^\bullet)\big) \simeq RF \circ Q(A^\bullet) ,$$

where we used that $Q(I^\bullet) \simeq Q(A^\bullet)$. One can check that ε is natural because, given any map $f^\bullet : A^\bullet \to B^\bullet$ and quasi-isomorphisms $A^\bullet \to I^\bullet$ and $B^\bullet \to J^\bullet$, there is a unique (up to homotopy) induced map $I^\bullet \to J^\bullet$ in $\mathbf{K}^+(\mathcal{I}_\mathcal{A})$, making the following diagram in $\mathbf{K}^+(\mathcal{A})$ commute:

$$\begin{tikzcd}
A^\bullet \arrow[r, "\mathrm{qis}"] \arrow[d, "f^\bullet"'] & I^\bullet \arrow[d, dashed, "\exists!"] \\
B^\bullet \arrow[r, "\mathrm{qis}"] & J^\bullet .
\end{tikzcd}$$

Consider now any (Φ, η) as in the definition of derived functor, and we show that η factors through ε: as above, let $A^\bullet \simeq I^\bullet$, where $I^\bullet$. ☐

The above result actually holds in a more general setting: given a functor F between homotopy categories, the right and the left derived functors exist whenever we find a class of objects adapted to the functor.

Definition. — Given a triangulated functor $F : \mathbf{K}^*(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$, a triangulated subcategory $\mathcal{K}_F \subset \mathbf{K}^*(\mathcal{A})$ is ***F*-adapted** if it satisfies the following two conditions:

(i) if $A^\bullet \in \mathcal{K}_F$ is acyclic, then $F(A^\bullet)$ is acyclic;

(ii) any $A^\bullet \in \mathbf{K}^*(\mathcal{A})$ is quasi-isomophic to a complex in $\mathcal{K}_F$.

We can define the notion of adapted class already on the level of the abelian category $\mathcal{A}$.

Definition. — Let $F : \mathcal{A} \to \mathcal{B}$ be a left (resp. right) exact functor between abelian categories. A class of objects $\mathcal{I}_F \subset \mathcal{A}$ is ***F*-adapted** if the following conditions hold true:

(i) the class $\mathcal{I}_F$ is closed under finite sums, i.e. given $A, B \in \mathcal{I}_F$, then $A \oplus B \in \mathcal{I}_F$;

(ii) if $A^\bullet \in \mathbf{K}^+(\mathcal{I}_F)$ (resp. $\mathbf{K}^-(\mathcal{I}_F)$) is acyclic, then $F(A^\bullet)$ is acyclic too;

(iii) any object of $\mathcal{A}$ can be embedded into an object of $\mathcal{I}_F$.

One can prove that, whenever $\mathcal{I}_F$ is an adapted class for a left exact functor F, then the localization of $\mathbf{K}^+(\mathcal{I}_F)$ by the class of quasi-isomorphisms is equivalent to the derived category $\mathbf{D}^+(\mathcal{A})$; for a complete description of this fact, check [**gelfand**]: the idea is that F sends qis of complexes in $\mathcal{I}_F$ to qis by condition (ii), while condition (iii) ensures that we can identify any object $A \in \mathcal{A}$ with an object of $\mathbf{K}^+(\mathcal{I}_F)$, as it happens with injetive resolutions. In fact, this is not by chance: whenever $\mathcal{A}$ has enough injectives, the class $\mathcal{I}_{\mathcal{A}}$ spanned by injective objects is F-adapted for *any* left exact functors, indeed we know that $I \oplus J$ is injective whenever $I, J \in \mathcal{I}$; left exactness guarantees (ii) and (iii) is the definition of injective resolution.

Whenever $\mathcal{I}_F \subset \mathcal{A}$ is an F-adapted class with respect to a left exact functor $F : \mathcal{A} \to \mathcal{B}$, one can prove that the full subcategory $\mathbf{K}^+(\mathcal{I}_F)$ is an adapted with respect to the induced homotopy functor $\tilde{F} : \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$, thus there exists the right derived functor, that we will write as

$$RF : \mathbf{D}^+(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{B})\,.$$

An analogous statement is true for right exact functors and the class of projective objects; for a complete treatment, see [**gelfand**]. By assuming these general constructions, we can deduce the following

II.20. Corollary. — *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.*

- *If $\mathcal{A}$ has enough injective objects and F is left exact, then the right derived functor $RF : \mathbf{D}^+(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ exists.*
- *If $\mathcal{A}$ has enough projective objects and F is right exact, then the left derived functor $LF : \mathbf{D}^-(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ exists.*

Definition. — Whenever the right derived functor RF of a left exact functor $F : \mathcal{A} \to \mathcal{B}$ exists, for every $i \in \mathbb{Z}$ we define

$$R^iF : \mathbf{D}^+(\mathcal{A}) \longrightarrow \mathcal{B}\,, \qquad A^\bullet \longmapsto H^i\big(RF(A^\bullet)\big)\,.$$

By precomposing with the canonical embedding $\mathcal{A} \subset \mathbf{D}^+(\mathcal{A})$, we can define, for every $i \in \mathbb{Z}$, the ***i*-th higher derived functor** $R^iF : \mathcal{A} \longrightarrow \mathcal{B}$.

The proof of Proposition II.19 shows a way to compute the higher derived functors of F: indeed, whenever $\mathcal{A}$ has enough injectives, for any $A \in \mathcal{A}$ we can pick an injective resolution $\mathbf{0} \to A \to I^\bullet$ and, by the the natural isomorphism $\varepsilon : Q \circ F \simeq RF \circ Q$, we have

$$RF^i(A) = H^i\big(RF(A[0])\big) \simeq H^i\big(F(I^\bullet)\big)\,.$$

It follows that, if $i < 0$, the higher derived functors are $RF^i = 0$ and $R^0F(A) \simeq F(A)$ by left exactness:

$$R^0F(A) = H^0\big(F(I^\bullet)\big) = \ker\big(F(I^0) \to F(I^1)\big) \simeq F(A)$$

II.21. Corollary. — *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor, where $\mathcal{A}$ has enough injectives. Then every short exact sequence in $\mathcal{A}$*

$$\mathbf{0} \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \mathbf{0}$$

gives rise in $\mathcal{B}$ to the long exact sequence

$$\mathbf{0} \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow R^1F(A) \longrightarrow R^1F(B) \longrightarrow \dots$$

$$\dots \longrightarrow R^iF(A) \longrightarrow R^iF(B) \longrightarrow R^iF(C) \longrightarrow R^{i+1}F(A) \longrightarrow R^{i+1}F(B) \longrightarrow \dots$$

Proof. According to Exercise II.5, any short exact sequence in $\mathcal{A}$ gives a distinguished triangle $A \to B \to C \to A[1]$ in $\mathbf{D}^+(\mathcal{A})$. Since RF is triangulated, the triangle $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$ is distinguished in $\mathbf{D}^+(\mathcal{B})$, thus it induces a LECS by Exercise II.6, which is the desired long exact sequence in $\mathcal{B}$. □

Exercise. — Let F be a left exact functor and $\mathcal{I}_F$ an F-adapted class in $\mathcal{A}$. We say that $A \in \mathcal{A}$ is F**-acyclic** if $R^iF(A) \simeq \mathbf{0}$ for all $i \neq 0$. Show that we obtain an F-adapted class by enlarging $\mathcal{I}_F$ by all F-acyclic objects.

Proof. It is enough to check the three conditions in the definition of F-adapted class:

(i) by additivity of the functors R^iF, it holds

$$R^iF(A \oplus B) \simeq R^iF(A) \oplus R^iF(B)\,,$$

thus the sum of finitely many F-acyclic objects is still F-acyclic. Nevertheless, the sum $I \oplus A$ of some $I \in \mathcal{I}_F$ and A an F-acyclic object needs not be neither in $\mathcal{I}_F$, nor F-acyclic. Thus, when we say "enlarge", we mean we need to consider also direct sums of these objects;

(ii) consider

(iii) any objects of $\mathcal{A}$ can be embedded into some object of $\mathcal{I}_F$, so it still remains true if we enlarge the class.

□

Example (Kernel). — Let $\mathcal{A}$ be an abelian category. Let $\mathcal{A}^{\{*\to*\}}$ be the category whose objects are maps $f : A \to A'$ in $\mathcal{A}$; a morphism Φ from $f : A \to A'$ to $g : B \to B'$ is a pair (φ, φ') of arrows $\varphi : A \to B$ and $\varphi' : A' \to B'$ such that the following square commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ A' \arrow[r, "\varphi'"] & B'\,. \end{tikzcd}$$

Given any two morphisms $\Phi = (\varphi, \varphi') : f \to g$ and $\Psi = (\psi, \psi') : g \to h$, set $\Psi \circ \Phi$ to be the componentwise composition, i.e.

$$\Psi \circ \Phi := (\psi \circ \varphi, \psi' \circ \varphi')\,.$$

One can check this composition is associative, making $\mathcal{A}^{\{*\to *\}}$ into a category. Moreover, $\mathcal{A}^{\{*\to *\}}$ is abelian:

(**A1**) if we write $\mathbf{0}$ for the zero object of $\mathcal{A}$, then the zero morphism $0 : \mathbf{0} \to \mathbf{0}$ is the zero object of $\mathcal{A}^{\{*\to *\}}$;

(**A2**) the existence of finite products and coproducts in $\mathcal{A}^{\{*\to *\}}$ follows by their existence in $\mathcal{A}$: indeed, given $f : A \to A'$ and $g : B \to B'$, their product is given by $f \oplus g : A \oplus B \to A' \oplus B'$ because, for any object $t : C \to C'$ and any pair of maps $t \to f, t \to g$, we have the factorization

$$\begin{tikzcd}
& C \arrow[dl] \arrow[dr, dashed, "\exists!"'] \arrow[drr] \arrow[dd, "t"] & & \\
A \arrow[dd, "f"'] & & A \oplus B \arrow[ll] \arrow[r] \arrow[dd, dashed, "\exists!"'] & B \arrow[dd, "g"] \\
& C' \arrow[dl] \arrow[dr, dashed, "\exists!"'] \arrow[drr] & & \\
A' & & A' \oplus B' \arrow[ll] \arrow[r] & B' ,
\end{tikzcd}$$

which can be rewritten as the following commutative diagram in $\mathcal{A}^{\{*\to *\}}$

$$\begin{tikzcd}
& t \arrow[dl] \arrow[d, dashed, "\exists!"'] \arrow[dr] & \\
f & f \oplus g \arrow[l] \arrow[r] & g .
\end{tikzcd}$$

The coproduct of f and g is given again by $f \otimes g$ because $A \times B \simeq A \amalg B$ in $\mathcal{A}$, which is indeed the direct sum;

(**A3**) we already know that $\mathcal{A}$ has kernels, and hence for any $\Phi : f \to g$ in $\mathcal{A}^{\{*\to *\}}$ we have

$$\begin{tikzcd}
\ker \varphi \arrow[r, hook, "i"] \arrow[d, dashed, "\exists!"', "k"] & A \arrow[r, "\varphi"] \arrow[d, "f"] & B \arrow[d, "g"] \\
\ker \varphi' \arrow[r, hook] & A' \arrow[r, "\varphi'"] & B' ,
\end{tikzcd}$$

where the vertical arrow k is the unique map described by the universal property of $\ker \varphi'$, which exists by $\varphi' \circ f \circ i = g \circ \varphi \circ i = 0$. We claim that $k = \ker \Phi$: given any $t : C \to C'$ and $\Psi : t \to f$ such that $\Phi \circ \Psi = 0$, then in $\mathcal{A}$ we have the commutative diagram

$$\begin{tikzcd}
C \arrow[dr, dashed, "\exists!"'] \arrow[drr] \arrow[drrr, "0"] \arrow[dd, "t"'] & & & \\
& \ker \varphi \arrow[r, hook] \arrow[dd, "k"'] & A \arrow[r, "\varphi"'] \arrow[dd, "f"'] & B \arrow[dd, "g"] \\
C' \arrow[dr, dashed, "\exists!"'] \arrow[drr] \arrow[drrr, "0"] & & & \\
& \ker \varphi' \arrow[r, hook] & A' \arrow[r, "\varphi'"'] & B' ,
\end{tikzcd}$$

which can be written in $\mathcal{A}^{\{*\to*\}}$ as the commutative diagram

$$\begin{tikzcd} t \arrow[d, dashed] \arrow[dr, "\Psi"'] \arrow[drr, "0"] & & \\ k \arrow[r] & f \arrow[r, "\Phi"'] & g\,, \end{tikzcd}$$

which proves $k = \ker\Phi$. Similarly, one shows that $\operatorname{Coker}\Phi$ is given by the induced map on cokernels;

(**A4**) consider $\Phi = (\varphi, \varphi') : f \to g$ in $\mathcal{A}^{\{*\to*\}}$. Since axiom (**A4**) holds in $\mathcal{A}$, then $\operatorname{coim}\varphi \simeq \operatorname{im}\varphi$ and $\operatorname{coim}\varphi' \simeq \operatorname{im}\varphi'$, which means that $\operatorname{coim}\Phi \simeq \operatorname{im}\Phi$ beacuse isomorphisms in $\mathcal{A}^{\{*\to*\}}$ are given by pairs of isomorphisms.

We now define the functor

$$\ker : \mathcal{A}^{\{*\to*\}} \longrightarrow \mathcal{A}\,, \qquad \big(f \to g\big) \longmapsto \big(\ker f \to \ker g\big)\,.$$

This is an additive functor between abelian categories, and we claim it is left exact: given a short exact sequence $0 \to f \to g \to h \to 0$ in $\mathcal{A}^{\{*\to*\}}$, in $\mathcal{A}$ we get a commutative diagram of the form

$$\begin{tikzcd} & \ker f \arrow[r, dashed, "i"] \arrow[d, hook] & \ker g \arrow[r, dashed, "j"] \arrow[d, hook] & \ker h \arrow[d, hook] & \\ \mathbf{0} \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & \mathbf{0} \\ \mathbf{0} \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \mathbf{0}\,. \end{tikzcd}$$

Notice that $i : \ker f \to \ker g$ is a monomorphism, for given $t : W \to \ker f$ such that $i \circ t = 0$, then the composition $W \to A \to B$ is zero again, so that t factors through $\ker\big(A \to B\big) \simeq \mathbf{0}$. Thus $\ker i \simeq \mathbf{0}$ and this shows ker is exact on the left. To show exactness in $\ker g$, noticce that the exactness of the middle row implies $j \circ i = 0$, so that we have an induced map $\operatorname{im} i \to \ker j$. Conversely, we can find a map $\ker j \to \operatorname{im} i$ which will be the inverse of the above $\ker j \to \operatorname{im} i$ because of the uniqueness guaranteed by the universal property. To get the desired map, one sees that the composition

$$\ker j \to \ker g \to B \overset{\psi}{\hookrightarrow} C$$

is the zero map, and hence we get a factorization through $\ker j \to \ker\varphi$; by exactness, $\operatorname{im}\varphi \simeq \ker\psi$, and by composing $\ker j \to \operatorname{im}\varphi \to \operatorname{im}\varphi'$ we get the zero morphism (because it commutes with $\ker j \to \ker g \to B \to B'$), and hence we can lift $\ker j \to \operatorname{im} i$.

$$
\begin{tikzcd}
& \operatorname{im} i \arrow[ddd] & \ker j \arrow[l, dashed, "\exists!"'] \arrow[dr] \arrow[ddd, dashed] \arrow[rr, bend left, "0"] & & \\
\ker f \arrow[ur, dashed] \arrow[rrr, "i"] \arrow[d, hook] & & & \ker g \arrow[d, hook] & \\
A \arrow[rrr, "\varphi"] \arrow[dr, dashed] \arrow[dd, "f"] & & & B \arrow[r, "\psi"] \arrow[dd, "g"] & C \arrow[dd, "h"] \\
& \operatorname{im} \varphi \arrow[r, equal] \arrow[dd] & \ker \psi \arrow[ur, hook] & & \\
A' \arrow[rrr, "\varphi'"] \arrow[dr, dashed] & & & B' \arrow[r] & C' \\
& \operatorname{im} \varphi' \arrow[urr, hook] & & & .
\end{tikzcd}
$$

Now that we know ker is a left exact functor, assume $\mathcal{A}$ has enough injective objects; it follows that $\mathcal{A}^{\{* \to *\}}$ has enough injectives too: indeed, a map $I \to J$ between injective objects in $\mathcal{A}$ is injective as an object of $\mathcal{A}^{\{* \to *\}}$, and by the dual version of Proposition II.11 we can always find an injective resolution

$$
\begin{tikzcd}
\mathbf{0} \arrow[r] & A \arrow[r] \arrow[d, "f"] & I^0 \arrow[r] \arrow[d, "j^0"] & I^1 \arrow[r] \arrow[d, "j^1"] & I^2 \arrow[r] \arrow[d, "j^2"] & \dots \\
\mathbf{0} \arrow[r] & A' \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \dots
\end{tikzcd}
\tag{II.21.1}
$$

for any object $f \in \mathcal{A}^{\{* \to *\}}$. This means that we can compute ker's higher right derived functors: given any $f : A \to A'$, we already know that $R^0 \ker f \simeq \ker f$, so now consider an injective resolution $f \to j^\bullet$ as in (II.21.1).

Finish this exercise.

Chapter III.

Differential graded algebras

III

For the whole chapter, fix a commutative ring Λ with unit $1 \neq 0$. We are going to study new additional structures on Λ-algebras, namely *differential* and *graded* structures; the aim of this is to define and understand their main properties, and after that we will be able to approach the related concept of graded modules over these algebras. In the geometric setting, our main interest will be the special case of the semisimple algebra $\Lambda = k$, for some field and $m \leq 1$.

The differential structure on these new objects allows us to talk about *cohomology*. It is well known that isomorphic complexes yield the same information in cohomology, while the converse is false in general; thus, we are going to focus on a class of very well-behaved graded algebras, *intrinsically formal* algebras: strictly speaking, these objects are determined by their cohomology.

Intrinsic formality of DGA-algebras can be characterized in terms of vanising of some *Hochschild cohomology groups*, thus we safe a section to define this notion.

DG-algebras

Every Λ-algebra in this chapter is associative, not necessarily commutative, with unit $1 \neq 0$.

Definition. — A Λ-algebra A is **graded algebra** if there exist submodules $\{ A^k \mid k \in \mathbb{Z} \}$, such that $1 \in A^0$ and, for every $i, j \in \mathbb{Z}$, we have multiplications

$$A^i \otimes_\Lambda A^j \longrightarrow A^{i+j}, \quad x \otimes y \longmapsto xy,$$

such that $A = \bigoplus_{k \in \mathbb{Z}} A^k$. An element of A^k is said **homogeneous of degree** k; sometimes we will simply write $|x|$ instead of k, for a homogeneous element $x \in A^k$.

A graded Λ-algebra is **graded-commutative** (or **anti-**, sometimes called **super-**commutative), if it holds

$$yx = (-1)^{kh} xy, \quad \text{for every } x \in A^k, y \in A^h.$$

The concept of differential graded algebra A is not new: it boils down to a cochain complex whose differential must satisfy some compatibility axiom with respect to the multiplication.

Definition. — A **differential** on a graded algebra A is a Λ-linear endomorphism $d : A \to A$ such that $d^2 = 0$ and, for every $k \in \mathbb{Z}$, it holds $d(A^k) \subset A^{k+1}$; moreover, d must satisfy the following **graded Liebniz rule**:

$$d(xy) = (dx)y + (-1)^k x(dy), \quad \text{for } x \in A^k. \tag{III.0.1}$$

A **DG-algebra** $\mathcal{A} = (A, d)$ is a graded Λ-algebra A endowed with a differential d.

Remark. — Notice that (III.0.1) implies that $d(1) = 0$.

The condition $d(A^k) \subset A^{k+1}$ in the above definition allows us to interpret a DG-algebra $\mathcal{A} = (A, d)$ as a sequence

$$\begin{tikzcd} \dots \arrow[r] & A^{k-1} \arrow[r, "d"] \arrow[rr, bend right, "0"'] & A^k \arrow[r, "d"] & A^{k+1} \arrow[r] & \dots \end{tikzcd}$$

thus, any DG-algebra is a cochain complex $\mathcal{A} = A^\bullet$ of Λ-modules, whose coboundary map is the same d at each level; this means we can compute its cohomology $H^*(\mathcal{A})$. The interesting fact is that this is not just a module: indeed, the relation (III.0.1) implies that the multiplication induced on $H^*(\mathcal{A})$ is well defined and moreover

$$H^k(\mathcal{A}) \otimes H^h(\mathcal{A}) \longrightarrow H^{k+h}(\mathcal{A}), \quad [x] \cdot [y] = [xy]$$

shows that the cohomology inherits a graded Λ-algebra structure $H^*(\mathcal{A}) = \oplus_k H^k(\mathcal{A})$. By endowing it with a trivial differential $d = 0$, we conclude that the cohomology $H^*(\mathcal{A})$ of a DG-algebra $\mathcal{A}$ is again a DG-algebra.

We define $\mathbf{DG}_\Lambda$ to be the category whose objects are DG-algebras over Λ, and morphisms $f : \mathcal{A} \to \mathcal{B}$ between them given by homomorphisms of unital Λ-algebras

$$f : A \longrightarrow B, \quad \text{such that } f(1_A) = 1_B,$$

which are also maps of complexes, that is $f(A^k) \subset B^k$ and $f \circ d = \delta \circ f$. One can check that $\mathbf{DG}_\Lambda$ is an abelian category (we might expect this because it is a category of modules).

Definition. — Given a DG-algebra $\mathcal{A} = (A, d)$, we define its **opposite algebra** $\mathcal{A}^{op} = (A^{op}, d)$ as the DG-algebra whose elements and differential are the same of $\mathcal{A}$, but we consider a new multiplication

$$a \cdot^{op} b := (-1)^{|a|\,|b|} ba,$$

where ba denotes the usual multiplication in $\mathcal{A}$. Notice that $\mathcal{A}$ is graded-commutative if and only if $\mathcal{A} = \mathcal{A}^{op}$.

Example. — Let X be a smooth real manifold. The algebra $\Omega^\bullet(X)$ of smooth differential forms endowed with the *exterior derivative* is a DG-algebra and $\Omega^\bullet$ determines a functor

$$\Omega^\bullet : \mathbf{Man} \longrightarrow \mathbf{DG}_\mathbb{R},$$

where **Man** is the category of smooth manifolds and smooth maps.

Example. — Taking cohomology defines a covariant functor $H^* : \mathbf{DG}_\Lambda \to \mathbf{DG}_\Lambda$.

Example. — Given a finite dimensional k-vector space V, define $T^0 := k$ and for each $k \geq 1$ set

$$T^k(V) := T^{\otimes k} = \underbrace{V \otimes_k \cdots \otimes_k V}_{k \text{ times}}.$$

Then $T(V) := \bigoplus_{k\geq 0} T^k(V)$ naturally inherits an associative multiplication

$$T^k(V) \otimes_k T^h(V) \longrightarrow T^{k+h}(V)\,;$$

from the canonical isomorphism $k \otimes_k V \simeq V$ we deduce that $1 \in T^0(V)$ is the unit of the multiplication, hence $T(V)$ is a graded algebra over k, called **tensor algebra** of V.

Let $\{e_1, \ldots, e_n\}$ be a basis of V, and set $A^{-k} := T^k(V)$. We can define a differential $d : T(V) \to T(V)$ by setting on basis elements

$$d(e_i) = (-1)^i\,, \quad i \in \{\,1, 2, \ldots, n\,\}\,,$$

and then extending it componentwise to a map

$$d : A^{-k} \longrightarrow A^{-k+1}\,, \quad d(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j=1}^{k} e_{i_1} \otimes \cdots \otimes d(e_{i_j}) \otimes \cdots \otimes e_{i_k}\,.$$

We write $\iota_{\mathcal{A}} : \Lambda \to A$ to be the structure map $\iota(\lambda) := \lambda \cdot 1_A$.

Definition. — An **augmentation** on a DG-algebra $\mathcal{A}$ is a morphism $\varepsilon : \mathcal{A} \to \Lambda$ of unital Λ-algebras such that $\varepsilon \circ d = 0$ and $\varepsilon \circ \iota_{\mathcal{A}} = \mathbf{1}_\Lambda$. Its kernel is a two-sided ideal of $\mathcal{A}$, called the **augmentation ideal** $\mathcal{A}^+ := \ker \varepsilon$.

Notice that the condition $\varepsilon d = 0$ implies that there is a well defined aumentation on $H^*(\mathcal{A})$. Since ε is a retraction on Λ, we may identify the ground ring Λ with a subring of A; in particular, notice that Λ is a direct summand of A by the Split Lemma. Thus, an augmentation is determined by its restriction to A^0.

From now on, we will consider DG-algebras with terms of non-negative degree, that is $A^k = \mathbf{0}$ for $k < 0$. In this case, the augmetation ideal coincides with the submodule of positive degree elements $\mathcal{A}^+ = \bigoplus_{k\geq 0} A^k$. If moreover $\mathcal{A}$ is **connected**, that is $A^k = \mathbf{0}$ for $k < 0$ and $\iota_{\mathcal{A}} : \Lambda \to A^0$ is an isomorphism, then $\mathcal{A}$ has a unique augmentation map given by ι_Λ^{-1}.

Definition. — A DG-algebra $\mathcal{A}$ endowed with an augmentation ε will be called **DGA-algebra** (which stands for **differential augmented graded algebra**). Given two DGA-algebras $\mathcal{A} = (A, d, \varepsilon)$ and $\mathcal{A}' = (A', d', \varepsilon')$, a **DGA-homomorphism** is a morphism of DG-algebras compatible with augmentations, that is $\varepsilon' \circ f = \varepsilon$.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & A' \arrow[dl, "\varepsilon'"] \\ & \Lambda & \end{tikzcd}\,.$$

We will write $\mathbf{DGA}_\Lambda$ for the category of DGA-algebras over Λ. Notice that a DGA-homomorphism induces a morphism $f* : H^*(\mathcal{A}) \to H^*(\mathcal{A}')$ of DGA-algebras, hence cohomology defines a covariant functor $H^* : \mathbf{DGA}_\Lambda \to \mathbf{DGA}_\Lambda$.

Example. — Given a topological space X, let $S_\bullet(X)$ denote its singular simplicial complex. For each pair of spaces X, Y, one can define a chain map $\sigma : S_\bullet(X) \otimes_{\mathbb{Z}} S_\bullet(Y) \to S_\bullet(X \times Y)$. Now, assume X is a topological group, with a continuous multiplication

$$\mu : X \times X \longrightarrow X\,,$$

with neutral element $e \in X$; then the composition

$$S_\bullet(X) \otimes_{\mathbb{Z}} S_\bullet(X) \xrightarrow{\sigma} S_\bullet(X \times X) \xrightarrow{\mu_\bullet} S_\bullet(X)$$

defines an associative multiplication on $S_\bullet(X)$, whose neutral element is the 0-simplex e, called **Pontrjagin product**. Moreover, there exists an augmentation $\varepsilon : S(X) \to \mathbb{Z}$ which sends each 0-simplex (i.e. a point of X) to $1 \in \mathbb{Z}$, and vanishes on higher dimensional simplices; this makes $S_\bullet(X)$ into a DGA-algebra[1].

III.1. Proposition. — *The category* $\mathbf{DGA}_\Lambda$ *is a monoidal category.*

Proof. Consider any two DGA-algebras $\mathcal{A} = (A^\bullet, d, \varepsilon)$ and $\mathcal{B} = (B^\bullet, \delta, \eta)$. We define their tensor product $\mathcal{A} \otimes \mathcal{B}$ to be the cochain complex $A^\bullet \otimes B^\bullet$ as defined in Example **??**, i.e. the complex whose degree n elements are

$$(A^\bullet \otimes B^\bullet)^n = \bigoplus_{k \in \mathbb{Z}} A^k \otimes_\Lambda B^{n-k}\,,$$

equipped with the differential D defined by

$$D(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes \delta b\,.$$

We can define the multiplication

$$(a \otimes b) \cdot (a' \otimes b') := (-1)^{|a|\,|a'|}(aa') \otimes (bb')\,,$$

in such a way that $(A^\bullet \otimes B^\bullet, D)$ becomes a DG-algebra: indeed, it holds

$$\begin{aligned}
D\big((a \otimes b)\cdot(a' \otimes b')\big) &= (-1)^{\deg(a)\deg(a')} D\big((aa') \otimes (bb')\big) \\
&= (-1)^{\deg(a)\deg(a')}\big(d(aa') \otimes bb' + (-1)^{\deg(aa')} aa' \otimes \delta(bb')\big) \\
&= (-1)^{\deg(a)\deg(a')}\Big((da)a' \otimes bb' + (-1)^{\deg(a)} a(da') \otimes bb' \\
&\quad + (-1)^{\deg(a)+\deg(a')}\big(aa' \otimes (\delta b)b' + (-1)^{\deg(b)} aa' \otimes b(\delta b')\big)\Big) \\
&= \big((-1)^{\deg(a')}(da \otimes b + (-1)^{\deg(a)} a \otimes \delta b) \cdot (a' \otimes b')\big) \\
&\quad + \big((a \otimes b) \cdot (da' \otimes b' + a' \otimes \delta b')\big) \\
&= (-1)^{\deg(a')}\big(D(a \otimes b) \cdot (a' \otimes b') + \big) \\
&= D(a \otimes b) \cdot (a' \otimes b') + (-1)^{\deg(a \otimes b)}(a \otimes b) \cdot D(a' \otimes b')
\end{aligned}$$

Please fix this. so the differential satisfies the Liebniz rule (III.0.1). Finally, the map α defined on every simple tensor $a \otimes b$ by

$$\alpha(a \otimes b) := \varepsilon(a)\eta(b) = 0\,,$$

is an augmentation on $\mathcal{A} \otimes \mathcal{B}$ because it clearly vanishes on elements of $(\mathcal{A} \otimes \mathcal{B})^0 = A^0 \otimes B^0$, and by composing it with the differential one gets

$$\alpha \circ D = (\varepsilon \circ d)\eta \pm \varepsilon(\eta \circ \delta) = 0\,.$$

This shows that $\mathcal{A} \otimes \mathcal{B} = (A^\bullet \otimes B^\bullet, D, \alpha)$ is a DGA-algebra. One can check that $\mathbf{DGA}_\Lambda$ equipped with $\otimes$ becomes a monoidal category by an analgous argument as for $C^\bullet(\mathbf{Mod}_\Lambda)$. □

[1] In this case, $S_\bullet(X)$ is a DGA-algebra whose differential follows the *homological* convention, while our definition is based on the cochain complex convention.

DG-modules

Definition. — Let $\mathcal{A} = (A, d)$ be a DG-algebra. A **left DG-module** $\mathcal{M}$ over $\mathcal{A}$, or simply **$\mathcal{A}$-DG-module**, is a cochain[2] complex $\mathcal{M} = (M^\bullet, d_M)$ in $C^\bullet(\mathbf{Mod}_A)$ such that:

- the multiplication by an homogeneous element $a \in A^k$ is a group homomorphism of degree k, that is
$$A^k \otimes_{\mathcal{A}} M^h \longrightarrow M^{k+h}\,, \quad a \otimes m \longmapsto am\,;$$
- the differential d_M satisfies the following Liebniz rule:
$$d_M(am) = (da)m + (-1)^{|a|}a(d_M m)\,, \quad \text{for } a \in \mathcal{A}, m \in \mathcal{M}\,. \tag{III.1.1}$$

A **morphism of $\mathcal{A}$-DG-modules** $f : \mathcal{M} \to \mathcal{N}$ is simply a cochain map $f : M^\bullet \to N^\bullet$ of complexes of A-modules.

One defines a **right DG-module** $\mathcal{M}$ over $\mathcal{A}$ (in short a **DG-$\mathcal{A}$-module**) in an analogous way, but the Liebniz rule for the right multiplication becomes
$$d_M(ma) = (d_M m)a + (-1)^{|m|}m(da)\,, \quad \text{for } a \in \mathcal{A}, m \in \mathcal{M}\,.$$

Example. — Left DG-modules over a fixed DG-algebra $\mathcal{A}$, together with maps of complexes, form a category called ${}_{\mathcal{A}}\mathbf{DGMod}$. If $\mathcal{A} = A^0$, then there is no differential structure on the ground ring, thus an $\mathcal{A}$-DG-modules is just a cochain complex of A^0-modules. In particular, ${}_{\mathbb{Z}}\mathbf{DGMod} = C^\bullet(\mathbf{Ab})$.

Example. — Given any $\mathcal{A}$-DG-module $\mathcal{M}$, its cohomology $H^*(\mathcal{M})$ is naturally a cochain complex (with trivial differential), which has a natural module structure over $H^*(\mathcal{A})$.

Definition. — Given a DG-module $\mathcal{M}$ over a DGA-algebra $\mathcal{A} = (A, d, \varepsilon)$, an **augmentation** on $\mathcal{M}$ is a Λ-linear homomorphism $\varepsilon_{\mathcal{M}} : \mathcal{M} \to \Lambda$ such that:

(i) $\varepsilon_M \circ d_M = 0$;
(ii) vanishes in positive degree, i.e. if $|m| > 0$, then $\varepsilon_M(m) = 0$;
(iii) it is compatible with augmentation on $\mathcal{A}$, that is for every $a \in \mathcal{A}$ and every $m \in \mathcal{M}$, it holds $\varepsilon_{\mathcal{M}}(am) = \varepsilon(a)\varepsilon_{\mathcal{M}}(m)$.

An $\mathcal{A}$-DG-module $\mathcal{M}$ endowed with an augmentation is called **$\mathcal{A}$-DGA-module**.

Example. — Let G be a topological group acting (on the left) on a topological space X. The action $G \times X \to X$ induces a morphism on singular complexes
$$S_\bullet(G) \otimes_{\mathbb{Z}} S_\bullet(X) \longrightarrow S_\bullet(X)\,,$$
which makes $S_\bullet(X)$ into a left DGA-module over $S_\bullet(G)$.

Definition. — Consider a homomorphism $f : (\mathcal{A}, \varepsilon) \to (\mathcal{B}, \eta)$ of DGA-algebras over Λ, an $\mathcal{A}$-DGA-module $\mathcal{M}$ and a $\mathcal{B}$-DGA-module $\mathcal{N}$. A **DGA-homomorphism compatible** with f is a Λ-linear map of complexes $g : \mathcal{M} \to \mathcal{N}$ of degree 0 such that:

[2] Some authors, for instance Henri Cartan, use the homological indexing, which means they consider a differential of degree -1. These two concepts are basically the same: from our definition

(i) it is comptible with restriction of scalars, i.e. for every $a \in \mathcal{A}$ and every $m \in \mathcal{M}$ it holds $g(am) = f(a)g(m)$;

(ii) it preserves augmentations, that is $\eta_{\mathcal{N}} \circ g = \varepsilon_{\mathcal{M}}$.

From now on we borrow all the terminology used for complexes of modules. For example, we say that a $\mathcal{A}$-DG-module $\mathcal{M}$ is **acyclic** if $H^*(\mathcal{M}) = \mathbf{0}$; we say that a morphism $f : \mathcal{M} \to \mathcal{N}$ of $\mathcal{A}$-DG-modules is a **quasi-isomorphism** if it induces isomorphisms $f^* : H^*(\mathcal{M}) \simeq H^*(\mathcal{N})$, and so on and so forth.

When a DG-module $\mathcal{M}$ is endowed with an augmentation, its structure becomes "enough rigid": in fact, whenever we have a base of homogeneous elements of $\mathcal{M}$ (e.g. if the ground ring $\Lambda = k$ is a field), we can always define a DGA-homomorphism from $\mathcal{M}$ to an acyclic module $\mathcal{N} \to \Lambda \to 0$; moreover, the quasi-isomorphism class of this map is *unique*. The following result by Henri Cartan explains more precisely what it means.

III.2. Theorem. — *Let $\mathcal{A}$ be DGA-algebra and $\mathcal{M}$ be a DGA-module over it, with a free basis of homogeneous elements; similarly, consider $\mathcal{M}'$ to be a DGA-algebra over the DGA-algebra $\mathcal{A}'$, and assume $\mathcal{M}'$ has a homogoeneous free basis. Given a DGA-homomorphism $f : \mathcal{A} \to \mathcal{A}'$, if both $\mathcal{M}$ and $\mathcal{M}'$ are acyclic, then there exists a map $g : \mathcal{M} \to \mathcal{M}'$ compatible with f; moreover, if $f* : H_*(\mathcal{A}) \to H_*(\mathcal{A}')$ is an isomorphism, then also $g_* : H_*(\mathcal{M}) \to H_*(\mathcal{M}')$ is an isomorphism.*

Proof. For details, look at [Car55]. Notice that Cartan uses the homological convention. The proof shows that a map g as above is defined on a basis $\{m_i\}$ of $\mathcal{M}$ by the formula

$$g\left(\sum_i a_i m_i\right) := \sum_i (f a_i) m'_i \,,$$

where each homogeneous element m_i of degree k is sent to an element $m'_i \in \mathcal{M}'$ of the same degree, which is built by induction on k by following the rules:

- for each $m_i \in M_0$, consider $m'_i \in M'_0$ such that $\varepsilon_{\mathcal{M}}(m_i) = \varepsilon_{\mathcal{M}'}(m'_i)$;
- if $|m_i| \geq 1$, then pick m'_i such that $d' m'_i = g(dm_i)$.

This construction is obviously non-unique, but it can be shown that any two homomorphisms built this way are *homotopic*. $\square$

By considering the case $f = \mathbf{1}_{\mathcal{A}}$, one obtains the following

III.3. Corollary. — *Any two acyclic free DGA-modules $\mathcal{M}$ and $\mathcal{M}'$ over a DGA-algebra $\mathcal{A}$ are quasi-isomorphic.*

Definition. — We define the **translation functor** $[1] : {}_{\mathcal{A}}\mathbf{DGMod} \to {}_{\mathcal{A}}\mathbf{DGMod}$ by sending any $\mathcal{A}$-DG-module $\mathcal{M}$ to the graded module

$$(\mathcal{M}[1])^n := M^{n+1}\,, \quad d_{M[1]} := -d_M\,,$$

in which the A-module structure on $M[1]$ is **twisted**, i.e. we define the scalar multiplication as

$$a * m := (-1)^{|a|} am\,,$$

where am is the multiplication in M.

At this point we have enough structure to talk about *triangles*: in fact, our next goal is to develop enough theory to be able to state and prove that the homotopy category $\mathcal{K}(\mathcal{A})$ is a triangulated category.

Definition. — Two morphisms $f, g : \mathcal{M} \to \mathcal{N}$ in ${}_{\mathcal{A}}\mathbf{DGMod}$ are **homotopic** if there exists a homotopy $s : \mathcal{M} \to \mathcal{N}[-1]$ of A-modules (possibly **not** of $\mathcal{A}$-DG-modules) such that

$$f - g = sd_M + d_N s\,,$$

in which case we write $f \sim g$.

Since null-homotopic morphisms form a 2-sided ideal in $\mathrm{Hom}_{{}_{\mathcal{A}}\mathbf{DGMod}}(\mathcal{M}, \mathcal{N})$, we may quotient by the equivalence relation $f \sim g$ given by homotopy to obtain the **homotopy category** $\mathcal{K}(\mathcal{A})$, whose objects are $\mathcal{A}$-DG-modules and morphisms between them are

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(\mathcal{M}, \mathcal{N}) := \mathrm{Hom}_{{}_{\mathcal{A}}\mathbf{DGMod}}(\mathcal{M}, \mathcal{N})/\sim\ .$$

Given a morphism $f : \mathcal{M} \to \mathcal{N}$ in ${}_{\mathcal{A}}\mathbf{DGMod}$, we may build the **cone** of f in the usual way, namely $\mathbf{C}(f) := N \oplus M[1]$ with differential $(d_N + f, -d_M)$; this naturally inherits a structure of $\mathcal{A}$-DG-module, thus we may define a **strict triangle**

$$\mathcal{M} \xrightarrow{\ f\ } \mathcal{N} \longrightarrow \mathbf{C}(f) \longrightarrow \mathcal{M}[1]\,.$$

An **exact triangle** in $\mathcal{K}(\mathcal{A})$ is a diagram $\mathcal{M}' \to \mathcal{N}' \to \mathcal{C}' \to \mathcal{M}'[1]$ which is isomorphic (in he homotopy category) to a strict triangle as above: explicitly, there is a diagram

$$\begin{tikzcd}
\mathcal{M}' \arrow[r] \arrow[d, "\simeq"] & \mathcal{N}' \arrow[r] \arrow[d, "\simeq"] & \mathcal{C}' \arrow[r] \arrow[d, "\simeq"] & \mathcal{M}'[1] \arrow[d, "\simeq"] \\
\mathcal{M} \arrow[r, "f"] & \mathcal{N} \arrow[r] & \mathbf{C}(f) \arrow[r] & \mathcal{M}[1]\,,
\end{tikzcd}$$

which commutes up to homotopy, in which vertical maps are homotopy equivalences.

As it happens for the homotopy category $\mathbf{K}(\mathcal{B})$ of an abelian category $\mathcal{B}$, it turns out that $\mathcal{K}(\mathcal{A})$ becomes a triangulated category if we equip it with the translation functor $[1]$ and take the exact triangles as the family of distinguished triangles. In this category, quasi-isomorphisms of $\mathcal{A}$-DG-modules form a *multiplicative system*, and hence we can build the **derived category** $\mathcal{D}(\mathcal{A})$ by formally inverting quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$: as it happens for $\mathbf{D}(\mathcal{B})$, morphisms are pictured as *roofs*, and $\mathcal{D}(\mathcal{A})$ also inherits a triangulated structure.

Remark. — Even though $\mathcal{K}(\mathcal{A})$, resp. $\mathcal{D}(\mathcal{A})$, is called the homotopy category of $\mathcal{A}$-DG-modules, resp. the derived category, the reader must be careful that these constructions are *not* the usual ones described in Section II for abelian categories, for $\mathcal{K}(\mathcal{A})$ is not isomorphic to $\mathbf{K}({}_{\mathcal{A}}\mathbf{DGMod})$ in general (thus, we use a different symbol). Indeed, objects in $\mathcal{K}(\mathcal{A})$ are DG-modules seen already as complexes, while objects in $\mathbf{K}({}_{\mathcal{A}}\mathbf{DGMod})$ are complexes of DG-modules, which can be interpreted as bicomplexes. This explains why we should morally prove again that $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ are triangulated, as it happens in [Jos94, Part II], but in fact constructions are analogous for the classic homotopy and derived categories.

Let $\mathcal{A} = (A, d)$ be a DG-algebra. Given two left DG-modules $\mathcal{M}$ and $\mathcal{N}$ over $\mathcal{A}$, we define the **internal hom** to be the cochain complex $Hom^\bullet(\mathcal{M}, \mathcal{N})$ of abelian groups

$$Hom^n(\mathcal{M}, \mathcal{N}) := \mathrm{Hom}_A(M, N[n]) = \{ f : M \to N[n] \mid f \text{ morphism of } A\text{-modules} \} ,$$

where the differential d is defined on $f \in Hom^n(\mathcal{M}, \mathcal{N})$ by

$$df := d_\mathcal{N} \circ f - (-1)^n f \circ d_\mathcal{M} .$$

We define the tensor product of $\mathcal{A}$-DG-modules in a similar fashion as for DG-algebras: given $\mathcal{M}$ and $\mathcal{N}$ left DG-modules over $\mathcal{A}$, we define their **tensor product** $\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ to be the total complex associated to $M^\bullet \otimes_A \mathcal{N}^\bullet$, that is the A-module $M \otimes_A N$ with the differential

$$d(m \otimes n) := (d_\mathcal{M} m) \otimes n + (-1)^{|m|} m(d_\mathcal{N} n) .$$

Example. — A 0-cocycle of $Hom^\bullet(\mathcal{M}, \mathcal{N})$ is a map of complexes: indeed, by definition $f : \mathcal{M} \to \mathcal{N}$ is such that $d_\mathcal{N} f - f d_\mathcal{M}$. Since 0-coboundaries are null homotopic maps, one deduces that

$$H^0 \left(Hom^\bullet(\mathcal{M}, \mathcal{N}) \right) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(\mathcal{M}, \mathcal{N}) .$$

If $\mathcal{A}$ is a graded-commutative DG-algebra, then both $Hom^\bullet(\mathcal{M}, \mathcal{N})$ and $\mathcal{M} \otimes_\mathcal{A} \mathcal{N}$ have a natural $\mathcal{A}$-DG-module structure, in which scalar multiplications are defined for every $a \in \mathcal{A}$ by

$$(a \cdot f) : m \mapsto a f(m) , \quad a \cdot (m \otimes n) := (-1)^{|a|\,|m|} am \otimes n .$$

One can check that both $Hom^\bullet(\mathcal{N}, -)$ and $- \otimes_\mathcal{A} \mathcal{N}$ are endofunctors of ${}_\mathcal{A}\mathbf{DGMod}$ which send null-homotopic maps to null-homotopic maps, thus they descend to additive functors

$$- \otimes_\mathcal{A} \mathcal{N} , Hom^\bullet(\mathcal{N}, -) : \mathcal{K}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{A}) ,$$

which are in fact *triangulated* functors. Moreover, given any three $\mathcal{A}$-DG-modules $\mathcal{M}, \mathcal{N}$ and $\mathcal{P}$, there exist functorial isomorphisms

$$\begin{aligned} (\mathcal{M} \otimes_\mathcal{A} \mathcal{N}) \otimes_\mathcal{A} \mathcal{P} &\simeq \mathcal{M} \otimes_\mathcal{A} (\mathcal{N} \otimes_\mathcal{A} \mathcal{P}) , \\ Hom^\bullet(\mathcal{M}, Hom^\bullet(\mathcal{N}, \mathcal{P})) &\simeq Hom^\bullet(M \otimes_\mathcal{A} \mathcal{N}, \mathcal{P}) \\ \mathrm{Hom}_{{}_\mathcal{A}\mathbf{DGMod}}(\mathcal{M}, Hom^\bullet(\mathcal{N}, \mathcal{P})) &\simeq \mathrm{Hom}_{{}_\mathcal{A}\mathbf{DGMod}}(M \otimes_\mathcal{A} \mathcal{N}, \mathcal{P}) , \\ \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(\mathcal{M}, Hom^\bullet(\mathcal{N}, \mathcal{P})) &\simeq \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(M \otimes_\mathcal{A} \mathcal{N}, \mathcal{P}) . \end{aligned}$$

One can sum up all these properties by saying that ${}_\mathcal{A}\mathbf{DGMod}$ is a *symmetrical monoidal closed category*, where the monoidal structure is given by the tensor product $\otimes_\mathcal{A}$, with $\mathcal{A}$ as a neutral element, and the internal hom as its right adjoint. For details, see for instance [**stacksDGA**].

As it happens for the classical notion of modules, whenever we have a homomorphsim of DG-algebras $f : \mathcal{A} \to \mathcal{B}$, we can give $\mathcal{B}$ a structure of $\mathcal{A}$-DG-module by setting the multiplication $a \cdot b := f(a)b$. Hence, we can relate the category of DG-modules over $\mathcal{A}$ and the ones over $\mathcal{B}$ via two functors:

- by **restriction of scalars**, which consists in considering any $\mathcal{B}$-DG-module as a left module over $\mathcal{A}$, via the induced multiplication defined above:

$$f_* : {}_\mathcal{B}\mathbf{DGMod} \longrightarrow {}_\mathcal{A}\mathbf{DGMod} , \quad \mathcal{M} \longmapsto f_*\mathcal{M} ;$$

- by **extension of scalars**, given by the assignment

$$f^* : {}_{\mathcal{A}}\mathbf{DGMod} \longrightarrow {}_{\mathcal{B}}\mathbf{DGMod}\,, \quad \mathcal{M} \longmapsto \mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}\,.$$

The two functors are adjoint to each other, namely for every $\mathcal{A}$-DG-module $\mathcal{M}$ and every $\mathcal{B}$-DG-module $\mathcal{N}$, there are isomorphisms

$$\mathrm{Hom}_{{}_{\mathcal{B}}\mathbf{DGMod}}(f^*\mathcal{M}, \mathcal{N}) \simeq \mathrm{Hom}_{{}_{\mathcal{A}}\mathbf{DGMod}}(\mathcal{M}, f_*\mathcal{N})\,,$$

and one can check that they preserve homotopy equivalences, thus induce adjoint functors between the homotopy categories. If both $\mathcal{A}$ and $\mathcal{B}$ are graded-commutative, then for every $\mathcal{M}, \mathcal{M}'$ DG-modules over $\mathcal{A}$ it holds

$$\mathcal{B} \otimes_{\mathcal{A}} (\mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}') \simeq (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}) \otimes_{\mathcal{B}} (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}')\,.$$

Definition. — A left DG-module $\mathcal{P}$ over $\mathcal{A}$ is called **$\mathcal{K}$-projective** if one of the two following conditions holds:

(i) $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(\mathcal{P}, -) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{P}, -)$;

(ii) for every acyclic $\mathcal{A}$-DG-module $\mathcal{C}$, then the complex $Hom^\bullet(\mathcal{P}, \mathcal{C})$ is acyclic.

The conditions i) and ii) are equivalent (see [Jos94, Lemma 10.12.2.2]). Given any DG-module $\mathcal{M}$ over $\mathcal{A}$, there exists a canonical construction, called the **bar construction**, of a $\mathcal{K}$-projective module $B(\mathcal{M})$ together with a quasi-isomorphism $B(\mathcal{M}) \to \mathcal{M}$. Thanks to this, whenever $\mathcal{A}$ is graded-commutative one may define the right derived functor

$$RHom^\bullet(\mathcal{M}, -) : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})\,, \quad \mathcal{N} \longmapsto RHom^\bullet(\mathcal{M}, \mathcal{N}) := Hom^\bullet(B(\mathcal{M}), \mathcal{N})$$

and the left derived functor of the tensor product, i.e.

$$- \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{M} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})\,, \quad \mathcal{N} \longmapsto \mathcal{N} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{M} := \mathcal{N} \otimes_{\mathcal{A}} B(\mathcal{M})\,,$$

thus, for any homomorphism $f : \mathcal{A} \to \mathcal{B}$ of DG-algebras, one can define extension of scalars between derived categories; this induces an adjunction

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(f^*\mathcal{M}, \mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{M}, f_*\mathcal{N})\,.$$

III.4. Theorem. — *If $f : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism of DG-algebras, then the extension of scalars induces an exact equivalence $f^* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ of triangulated categories.*

Proof. The complete proof can be found in [Jos94, Theorem 10.12.5.1]. □

Hochschild (co)homology

For now, let Λ be a commutative unital ring and consider A an associative Λ-algebra with unit 1_A, which is not necessarily commutative. As always, we identify the two multiplicative identities via the structure map $\lambda \in \Lambda \mapsto \lambda \cdot 1_A$.

Definition. — We define the **opposite ring** A^{op} to be the Λ-module A, endowed with the multiplication $a \cdot^{op} b := ba$. The **enveloping algebra** A^e is the Λ-algebra $A \otimes_\Lambda A^{op}$, where the multiplication is defined on simple tensors by

$$(a_1 \otimes b_1) \cdot^e (a_2 \otimes b_2) := a_1 a_2 \otimes b_2 b_1 \,.$$

Remark. — Notice that $A = A^{op}$ whenever A is commutative.

Definition. — Given two commutative unital rings B and R, a **B-R-bimodule** is an abelian group which is both a left B-module and a right R-module in such a way that the two structures are compatible, i.e. it holds

$$(bm)r = b(mr)\,, \quad \text{for every } b \in B, m \in M, r \in R\,.$$

Notice that an A-A-bimodule structure on M is the same as a left A^e-module, in which the scalar multiplication is given by $(a_1 \otimes a_2) \cdot m = a_1 m a_2$. For instance, if we consider M to be the n-fold tensor product

$$A^{\otimes n} := \underbrace{A \otimes_\Lambda \cdots \otimes_\Lambda A}_{n \text{ times}}\,,$$

then its A^e-structure is given by

$$(b \otimes c) \cdot (a_1 \otimes \cdots \otimes a_n) = ba_1 \otimes \cdots \otimes a_n c\,.$$

Definition. — The **bar resolution** of A is the sequence of A-A-bimodules

$$\mathrm{Bar}(A) : \ldots \longrightarrow A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A^{\otimes 2}\,,$$

where $d_n : A^{\otimes n+2} \to A^{\otimes n+1}$ is defined for every $n \geq 0$ by

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}\,. \tag{III.4.1}$$

III.5. Lemma. — *Let $d_0 : A \otimes A \to A$ be the multiplication $d_0(a \otimes b) = ab$. Then the* $\mathrm{Bar}(A) \to A \to 0$ *is an acyclic complex.*

Proof. For every $n \geq -1$, we define the Λ-linear map

$$s_n : A^{\otimes n+2} \longmapsto A^{\otimes n+3}\,, \quad s_n\,(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}\,.$$

To conclude, it is enough to check that $s_\bullet$ defines a contracting homotopy, i.e. $ds + sd$ is the identity at each level. For every simple tensor one computes

$$\begin{aligned}
d_{n+1} s_n\,(a_0 \otimes \cdots \otimes a_{n+1}) &= d_{n+1}\,(1 \otimes a_0 \otimes \cdots \otimes a_{n+1}) \\
&= a_0 \otimes \cdots \otimes a_{n+1} + \sum_{i=0}^{n} (-1)^{i+1} \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \\
&= a_0 \otimes \cdots \otimes a_{n+1} - s_{n-1} \left(\sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \right) \\
&= a_0 \otimes \cdots \otimes a_{n+1} - s_{n-1} d_n (a_0 \otimes \cdots \otimes a_{n+1})\,.
\end{aligned}$$

$\square$

Thus, the bar resolution is a chain complex in $C_\bullet({}_{A^e}\mathbf{Mod})$ which is quasi-isomorphic to $A[0]$.

Definition. — Let M be an A-A-bimodule. The **Hochschild chain complex with coefficients in** M is the complex

$$C_\bullet(A;M) := M \otimes_{A^e} \mathrm{Bar}(A)\,.$$

Dually, we define the **Hochschild cochains with coefficients in** M to be the complex

$$C^\bullet(A;M) := \mathrm{Hom}_{A^e}(\mathrm{Bar}(A), M)\,.$$

More explicitly, for every $n \geq 0$ we see there is an isomorphism of left A^e-modules

$$\begin{aligned} C_n(A;M) = M \otimes_{A^e} A^{\otimes(n+2)} &\xrightarrow{\sim} M \otimes_\Lambda A^{\otimes n}\,, \\ m \otimes_{A^e} (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) &\longmapsto a_{n+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_n\,, \end{aligned}$$

were we declare $A^{\otimes 0} = \Lambda$; thus the induced boundary δ map looks like

$$\begin{aligned} \delta_n\left(m \otimes a_1 \otimes \cdots \otimes a_n\right) = {} & m a_1 \otimes \cdots \otimes a_n \\ & + m \otimes d_{n-2}\left(a_1 \otimes \cdots \otimes a_n\right) \\ & + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}\,. \end{aligned}$$

Similarly, one can find that the n-cochains are

$$\mathrm{Hom}_{A^e}\left(A^{\otimes(n+2)}, M\right) \simeq \mathrm{Hom}_\Lambda\left(A^{\otimes n}, M\right)\,, \tag{III.5.1}$$

thus we get the coboundary maps

$$\begin{aligned} \partial_n f\left(a_1 \otimes \cdots \otimes a_n\right) = {} & a_1 f\left(a_2 \otimes \cdots \otimes a_n\right) \\ & + \sum_{i=1}^{n-1} (-1)^i f\left(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n\right) \\ & + (-1)^n f\left(a_1 \otimes \cdots \otimes a_{n-1}\right) a_n\,. \end{aligned}$$

By additivity of $M \otimes_{A^e} -$ and $\mathrm{Hom}_{A^e}(-, M)$, we see that the above sequences are indeed complexes, thus we can compute their (co)homology.

Definition. — The **Hochschild homology** $\mathrm{HH}_*(A;M)$ of A with coefficients in an A-A-bimodule M is the homology of the Hochschild chain complex, that is

$$\mathrm{HH}_n(A;M) := H_n\left(C_\bullet(A;M)\right) = H_n\left(M \otimes_\Lambda A^{\otimes \bullet}\right)\,,$$

and similarly the **Hochschild cohomology** $\mathrm{HH}^*(A;M)$ is

$$\mathrm{HH}^n(A;M) := H^n\left(C^\bullet(A;M)\right) = H^n\left(\mathrm{Hom}_\Lambda(A^{\otimes \bullet}, M)\right)\,.$$

In case $M = A$, we simply write $\mathrm{HH}_*(A)$, resp. $\mathrm{HH}^*(A)$.

Example. — Let $A = \Lambda = k$ be a field. We compute $\mathrm{HH}_*(k)$ by following the definition: since $k \otimes k \simeq k$ via the multiplication, then we get the boundary maps

$$d_n(x_0 \otimes \cdots \otimes x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i x_0 x_1 \ldots x_{n+1} = (x_0 x_1 \ldots x_{n+1}) \sum_{i=0}^{n+1} (-1)^i$$

and hence the bar resolution turns out to be

$$\mathrm{Bar}(k): \qquad \ldots \longrightarrow k \xrightarrow{\sim} k \xrightarrow{0} k \xrightarrow{\sim} k \xrightarrow{0} k \longrightarrow 0\,.$$

Thus the only non trivial cohomology group is $\mathrm{HH}_0(k) = k$. Dually, if we want to compute the Hochschild cohomology we notice that $\mathrm{Hom}_k(k^{\otimes n}, k) \simeq k$ via the isomorphism $f \mapsto f(1 \otimes \cdots \otimes 1)$, and by composing with multiplications one gets

$$C^\bullet(k;k): \qquad 0 \longrightarrow k \xrightarrow{0} k \xrightarrow{\sim} k \xrightarrow{0} k \xrightarrow{\sim} k \xrightarrow{0} \ldots\,,$$

thus it holds $\mathrm{HH}^0(k) = k$ and $\mathrm{HH}^n(k) = 0$, for $n > 0$.

As the previous Example shows, computing the Hochschild cohomology of a Λ-algebra by using the bar resolution may be quite tedious and difficult; the next Proposition shows us a way to overcome this issue. We first state an easy

III.6. Lemma. — *If P_1 and P_2 are projective A-modules, then so is $P_1 \otimes_A P_2$.*

Proof. We have the natural isomorphism of functors

$$\mathrm{Hom}_A(P_1 \otimes_A P_2, -) \simeq \mathrm{Hom}_A(P_1, \mathrm{Hom}_A(P_2, -))\,,$$

from which we deduce that the left hand side is an exact functor. $\square$

III.7. Proposition. — *Assume the algebra A is projective as a Λ-module. For every A-A-bimodule M we have canonical isomorphisms*

$$\mathrm{HH}_*(A;M) \simeq \mathrm{Tor}_*^{A^e}(M, A)\,, \quad \mathrm{HH}^*(A;M) \simeq \mathrm{Ext}^*_{A^e}(A, M)\,.$$

Proof. Since by the Lemma III.6 the n-fold tensor product $A^{\otimes n}$ is a projective Λ-module, then $A^{\otimes n+2}$ is projective over A^e. This means that $\mathrm{Bar}(A) \to A \to 0$ is a *projective resolution* of A as an A^e-module, and hence computing Hochschild homology (resp. Hochschild cohomology), is the same as computing the left derived functor of $M \otimes_{A^e} -$, in other words $\mathrm{Tor}_*^{A^e}(M, -)$ (resp. is the right derived functor of $\mathrm{Hom}_{A^e}(-, M)$, i.e. $\mathrm{Ext}^*_{A^e}(-, M)$). $\square$

Example. — Let $\Lambda = k$ be a field and consider $A = k[t]$. We compute $\mathrm{HH}_*(k[t])$. Since $k[t]$ is commutative, $A^{op} = A$ and hence $A^e \simeq k[x,y]$, where $k[x,y]$ has the bimodule structure over $k[t]$ given by

$$p(t) \cdot f(x,y) := p(x) f(x,y)\,, \quad f(x,y) \cdot q(t) := f(x,y) q(y)\,,$$

and vice versa $k[t]$ is a $k[x,y]$-module via the structure map $f(x,y) \mapsto f(t,t)$. Thus, we may consider the following free resolution of $k[t]$, seen as a module over $k[x,y]$:

$$0 \longrightarrow k[x,y] \xrightarrow{(x-y)\cdot} k[x,y] \longrightarrow k[t] \longrightarrow 0\,.$$

After tensoring the truncated resolution with $k[t]$, the multiplication by $(x-y)$ becomes the zero map, thus from

$$0 \longrightarrow \underbrace{k[x,y] \otimes_{k[x,y]} k[t]}_{\simeq k[t]} \xrightarrow{0} \underbrace{k[x,y] \otimes_{k[x,y]} k[t]}_{\simeq k[t]} \longrightarrow 0\,,$$

we conclude that

$$\mathrm{HH}_n(k[t]) = \begin{cases} k[t]\,, & \text{if } n = 0, 1\,; \\ 0\,, & \text{if } n \geq 2\,. \end{cases}$$

Example. — Let $\Lambda = k$ be a field and $m \geq 2$. We now show how to compute the Hochschild cohomology of a *truncated polynomial algebra* $A = k[t]/(t^m)$. Define elements

$$u := t \otimes 1 - 1 \otimes t\,, \quad w := \sum_{i=0}^{m-1} t^{m-1-i} \otimes t^i\,,$$

then consider the 2-cyclic free resolution

$$\ldots \xrightarrow{u\cdot} A^e \xrightarrow{w\cdot} A^e \xrightarrow{u\cdot} A^e \xrightarrow{w\cdot} A^e \xrightarrow{u\cdot} A^e \longrightarrow A \longrightarrow 0\,.$$

One can show the above sequence is exact either by a straight computation, or by showing that the following left A-linear maps define a contracting homotopy: set $s_{-1}(1) = 1 \otimes 1$ and for $k \geq 0$ take

$$s_{2k}(1 \otimes x^j) = -\sum_{h=1}^{j} x^{j-h} \otimes x^{h-1}\,, \quad s_{2k+1}(1 \otimes x^j) = \delta_{j,m-1} \otimes 1\,,$$

where $\delta_{i,j}$ is the Kroeneker delta.

After applying $\mathrm{Hom}_{A^e}(-, A)$ and identifying $\mathrm{Hom}_{A^e}(A^{\otimes n}, A) \simeq A$, one gets the complex

$$\ldots \xleftarrow{0} A \xleftarrow{mx^{m-1}\cdot} A \xleftarrow{0} A \xleftarrow{mx^{m-1}\cdot} A \xleftarrow{0} A \longleftarrow 0\,,$$

thus one can compute the Hochschild cohomology, being careful to distinguish between the two cases:

- if $\mathrm{char}(k)$ divides m, then every coboundary map in the above complex is trivial and hence $\mathrm{HH}^n(A) \simeq A$ for every $n \geq 0$;
- if $\mathrm{char}(k)$ does *not* divide m, then we see that

$$\mathrm{HH}^n(A) = \begin{cases} A\,, & \text{if } n = 0\,; \\ (t)/(t^m)\,, & \text{if } n \text{ is odd}\,; \\ k[t]/(t^{m-1})\,, & \text{otherwise}\,. \end{cases}$$

Remark (**Interpretation of the Hochschild cohomology in low degree**)**.** — Given a bimodule M over A, in virtue of the isomorphism (III.5.1) we may identify Hochschild 0-cochains with elements of M:

$$M \simeq \mathrm{Hom}_{A^e}(A \otimes_\Lambda A, M)\,, \quad m \longmapsto (f : 1 \otimes 1 \mapsto m)\,,$$

thus $m \in M$ is a 0-cocycle if $ma - am = 0$, for every $a \in A$; we deduce that the zeroth Hochschild cohomology is the **submodule of invariants** of M:

$$\mathrm{HH}^0(A; M) = \{ m \in M \mid \forall_{a \in A}\, ma = am \} .$$

In particular, if $M = A$ we recover the **centre** of the algebra A, i.e. $\mathrm{HH}^0(A) = Z(A)$.

By using $\mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \simeq \mathrm{Hom}_k(A, M)$, a 1-cocycle is a k-linear map $f : A \to M$ such that

$$af(b) - f(ab) + f(a)b\,, \quad \text{for every } a, b \in A\,;$$

on other words, f satisfies the **Liebniz rule**

$$f(ab) = af(b) + f(a)b\,.$$

Thus, the Hochschild 1-cocycles $Z^1(C^\bullet(A; M))$ is the module of Λ-derivations from A into M, written $\mathrm{Der}_\Lambda(A, M)$. A derivation of the form $\partial_0 m$, for some $m \in M$, is called an **inner derivation** and by quotienting these derivations out, we obtain the module of **outer derivations**:

$$\mathrm{HH}^1(A; M) = \mathrm{OutDer}_\Lambda(A, M)\,.$$

In case $\Lambda = k$ is a field, outer derivations form a vector space which may be endowed with a Lie braket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$; then, we see that $\mathrm{HH}^1(A; M)$ has a natural **Lie algebra** structure.

As often happens with cohomology in degree 2, the module $\mathrm{HH}^2(A; M)$ classifies some particular A-A-bimodule extensions, called **Hochschild extensions**. One can show that $\mathrm{HH}^3(A; M)$ classifies other mathematical objects, namely c**rossed bimodules**. These spaces play an important role in the *deformation theory of algebras.*

Path algebras

Let k be a fixed field for the rest of this section.

Definition. — A **quiver** $Q = (Q_0, Q_1, s, t)$ is a *directed graph*, that is Q consists of a set Q_0 of **vertices**, a set Q_1 of **arrows** together with two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its **source** $s(\alpha)$ and its **target** $t(\alpha)$.

Definition. — A **non-trivial path** ρ in a quiver Q is a finite sequence $\alpha_1 \alpha_2 \dots \alpha_m$ of arrows such that $t(\alpha_i) = s(\alpha_{i+1})$ for every $1 \le i < m$. It can be represented as

$$\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} \bullet\,,$$

which helps visualizing that the path source is $s(\rho) = s(\alpha_1)$ and its target is $t(\rho) = t(\alpha_m)$. We call $m \ge 1$ its length. For every vertex $v \in Q_0$, there exists a **trivial path** e_v of length 0 which starts and ends at v.

As the above picture suggests, whenever the target of a path ρ coincides with the source of a second path σ, then we can join the two paths to form a new longer path $\rho\sigma$. This yields a well defined operation on paths, so that we can build an algebra.

Definition. — Given a quiver Q, the **path algebra** of Q is the associative k-algebra kQ whose basis (as a k-vector space) given by paths in Q, and product given by

$$(\alpha_1\alpha_2\dots\alpha_m)\cdot(\beta_1\beta_2\dots\beta_l) = \begin{cases} \alpha_1\alpha_2\dots\alpha_m\beta_1\beta_2\dots\beta_l\,, & \text{if } t(\alpha_m)=s(\beta_1)\,; \\ 0\,, & \text{otherwise}\,, \end{cases}$$

and for every $\alpha \in Q_1$ one sets $e_{s(\alpha)}\alpha = \alpha$ and $\alpha e_{t(\alpha)} = \alpha$.

We will be interested in *finite* quivers, that is quivers whose vertices and edges are finite sets.

Example. — Let Q be a quiver of m disjoint vertices

$$\bullet_1 \qquad \bullet_2 \qquad \cdots \qquad \bullet_m\,.$$

The path algebra kQ is the m-dimensional k-vector space with basis $e_1,\dots,e_m$ and since the following relations hold

$$e_i \cdot e_j = \delta_{ij}\,, \quad e_i^2 = e_i\,,$$

one deduces that this path algebra is the semisimple algebra $kQ \simeq k^m$.

Example. — Let Q be the quiver with one vertex and one loop

$$\begin{tikzcd} {}_1\bullet \arrow[loop above, "T"] \end{tikzcd}\,.$$

Then the loop T generates many paths $T^2, T^3, \dots$ unrelated to one another; they form a countable basis for the free algebra kQ, which can be identified with the polynomials $k[T]$.

Example. — If Q is a finite quiver such that there exists at most one path $v_i \to v_j$ for every $1 \le i,j \le m$, then by plugging a scalar into the (i,j)-th entry of an $m \times m$ matrix we can identify

$$kQ \simeq \{\, C \in M_{m\times m}(k) \mid C_{ij} = 0 \text{ if there is no path } v_i \to v_j \,\}\,.$$

For instance, the path algebra of the quiver

$$\bullet_1 \qquad \bullet_2 \qquad \cdots \qquad \bullet_m$$

is the subalgebra of upper triangular $m \times m$ matrices.

We now list some of the properties of these path algebras.

III.8. Proposition. — *Let Q be a finite quiver with vertices $\{1,2,\dots,m\}$.*

(i) *The trivial paths e_i are* orthogonal idempotents*, that is $e_ie_j = \delta_{ij}e_i$.*

(ii) *The path algebra kQ is unital, with unit given by $1 = \sum_{i=1}^m e_i$.*

(iii) *The subalgebra $e_i \cdot kQ$ is generated by paths* starting *at i, the subalgebra $kQ \cdot e_j$ is the vector space of paths* ending *at j, while $e_i \cdot kQ \cdot e_j$ are paths starting at i and ending at j. One can also consider $kQ \cdot e_i \cdot kQ$ as the paths passing through the vertex i.*

(iv) *since* $kQ = \oplus_{i=1}^m e_i \cdot kQ$, *we deduce that each* $e_i \cdot kQ$ *is a projective right* kQ*-module.*

(v) *If* M *is a right* kQ*-module, then* $\mathrm{Hom}_{kQ}(e_i \cdot kQ, M) \simeq M \cdot e_i$.

(vi) *The* e_i *are* inequivalent, *i.e. if* $e_i \cdot kQ \simeq e_j \cdot kQ$, *then* $i = j$.

Proof. Only statement (vi) needs a proof. If there exists an isomorphism $f : e_i \cdot kQ \simeq e_j \cdot kQ$, then by (v) is can be seen as a path $f \in e_j \cdot kQ \cdot e_i$, whose inverse g is a path starting at e_i and ending at e_j. Thus their composition $fg = e_i$ should pass through j, which is possible only if $i = j$ because e_i is a trivial path. □

A_∞-algebras

Read the brief description in the article of braid groups and compare with something by Borislav Mladenov, which I found by chance.

STUDY THESE.

Definition. — An A_∞**-algebra** is a graded Λ-algebra $\mathcal{A} = \oplus_{i\in\mathbb{Z}} A_i$ together with graded Λ-linear maps

$$m_n : A^{\otimes n} \longrightarrow A\,, \quad n \geq 1\,,$$

where each m_n has degree $2 - n$ and satisfies the **Stasheff identities**

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0\,. \tag{III.8.1}$$

Sometimes we denote the A_∞-algebra by $(A, m_\bullet)$ to emphasize the notation chosen for these higher multiplication maps. If $m_1 = 0$, then A is called **minimal**.

We now give an interpretation to the *Stasheff identities* for small values of n. When $n = 1$, then we have to plug $s = t = 0$ in (III.8.1), so that m_1 is a degree 1 map which satisfies

$$m_1^2 = 0\,.$$

This means that m_1 is a differential on A, and hence (A, m_1) turns into a cochain complex. Thus, we will write $d := m_1$.

When $n = 2$, then in (III.8.1) we can either plug $s = 2$ and $s = t = 0$, or $s = 1$ and $t + s = 1$, so that the relation reads

$$dm_2 - m_2(1 \otimes d + d \otimes 1) = 0\,.$$

Since $m_2 : A \otimes A \to A$ is a degree zero map, we can think of it as a multiplication and thus write $m_2(a \otimes b) = a \cdot b$ for all homogeneous elements $a, b \in A$. Thus, the above relations can be rewritten as

$$d(a \cdot b) = da \cdot b + (-1)^{|a|} a \cdot db\,,$$

which means that $m_1 = d$ is a graded derivation with respect to the multiplication m_2.

Notice that the differential $m_1 = d$ induces a differential δ on the complex $\mathrm{Hom}_\Lambda(A^{\otimes 3}, A)$, which sends a map $f : A^{\otimes 3} \to A$ to the coboundary

$$\delta(f)(a\otimes b\otimes c) = d\big(f(a\otimes b\otimes c)\big) - f(da\otimes b\otimes c) + (-1)^{|a|-1} f(a\otimes db\otimes c) + (-1)^{|a|+|b|-1} f(a\otimes b\otimes dc)\,,$$

for any a, b, c homogeneous elements. Thus, the Stasheff identity for $n = 3$, given by

$$\begin{aligned} m_1 m_3 + m_2\,(m_2 \otimes 1) - m_2\,(1 \otimes m_2) \\ + m_3\,(m_1 \otimes 1 \otimes 1) + m_3\,(1 \otimes m_2 \otimes 1) + m_3\,(1 \otimes 1 \otimes m_1) = 0\,, \end{aligned}$$

can be thought of as

$$\delta(m_3) = m_2\,(1 \otimes m_2) - m_2\,(m_2 \otimes 1)\ ,$$

that is, m_2 is associative up to a coboundary in this Hom complex. It follows that the cohomology of (A, m_1) is a graded associative algebra with multiplication induced by m_2.

Witherspoon grives many examples.

Definition. — A **morphism** of A_∞-algebras, also called A_∞**-moprhism**, $f_\bullet : (A, m^A_\bullet) \to (B, m^B_\bullet)$ is a family of graded Λ-linear maps

$$f_n : A^{\otimes n} \longrightarrow B\,, \quad n \geq 1\,,$$

where each f_n has degree $1 - n$ and satisfies the relations

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(1^{\otimes r} \otimes m^A_s \otimes 1^{\otimes t}) = \sum_{i_1+\cdots+i_r=n} (-1)^u m^B_r(f_{i_1} \otimes \cdots \otimes f_{i_r})\,, \quad \text{(III.8.2)}$$

where $u = u(i_1, \ldots, i_r) := \sum_{k=1}^{r-1}(r-k)(i_k - 1)$.

The identity morphism $\mathbf{1}_\bullet : (A, m_\bullet) \to (A, m_\bullet)$ is given by the identity $\mathbf{1}_1 = \mathbf{1}_A$ and the zero map in higher degrees. If $f_\bullet : A \to B$ and $g : B \to C$ are two morphisms of A_∞-algebras, we define their composition $(gf)_\bullet$ to be given by the maps

$$(gf)_n := \sum_{i_1+\cdots+i_r=n} (-1)^u g_r\,(f_{i_1} \otimes \cdots \otimes f_{i_r})\ ,$$

where the integer u is defined as above.

As before, we interpret low degree terms of an A_∞-morphism. When $n = 1$, equation (III.8.2) boils down to

$$f_1 m^A_1 = m^B_1 f_1\,,$$

which tells us that f_1 is a cochain map. Thus, as in the case of cochain complexes, we say that an A_∞-morphism $f_\bullet$ is a **quasi-isomorphism** if f_1 induces an isomorphism $H^*(A) \simeq H^*(B)$.

If $n = 2$, then (III.8.2) reads

$$f_1 m^A_2 = m^B_2(f_1 \otimes f_1) + \delta(f_2)\,,$$

where δ is the coboundary map of $\mathrm{Hom}_\Lambda(A^{\otimes 3}, A)$. That is, up to the coboundary $\delta(f_2)$, the map f_1 is an algebra homomorphism with respect to multiplication m_2.

Example. — If A is any DG-algebra, we may take m_1 to be its differential, m_2 its multiplication, and $m_n = 0$ for $n \geq 3$, to define an A_∞-algebra structure on A.

Example. — An associative algebra A itself may be viewed as a DG-algebra with zero differential, and thus as an A_∞-algebra in this way. If an A_∞-algebra A is concentrated in degree 0, i.e. $A_i = 0$ for all $i \neq 0$, then the maps m_n are necessarily zero maps for all $n \neq 2$ since $|m_n| = 2 - n$, so A is simply an associative algebra.

We would like to use the language of A_∞-morphisms in the case of DG-algebras. Let (A, ε) an augmented graded algebra and $\mathcal{B} = (B, d)$ a DG-algebra. As noticed in the previous examples, we may think of them as particular A_∞-algebras with non-trivial higher operations given by m_2^B, resp. m_2^A, which is the multiplication in $\mathcal{B}$, resp. in A, and $m_1^B = d$. Thus, an A_∞-morphism $f_\bullet : A \to \mathcal{B}$ is a sequence of $\Lambda - \Lambda$-bimodule morphisms

$$f_n \in \mathrm{Hom}_{\Lambda-\Lambda}\left((A^+)^{\otimes q}, \mathcal{B}[1-n]\right) , \quad n \geq 1 ,$$

which satisfy the Stasheff identities

$$\sum_{r+t=n-2} (-1)^{n-2+t} f_{n-1}(1^{\otimes r} \otimes m_2^A \otimes 1^{\otimes t}) = df_n + \sum_{i=1}^{n-1} (-1)^{i-1} m_2^B (f_i \otimes f_{n-i}) .$$

This means that, for homogeneous elements $a_1, \ldots, a_n \in A$, an A_∞-morphism satisfies the relations

$$\begin{aligned} df_n(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^u \Big(& f_n(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ & + f_i(a_1 \otimes \cdots \otimes a_i) f_{n-i}(a_{n+1} \otimes \cdots \otimes a_n) \Big) , \end{aligned}$$

where the sign is given by $u = -i + \sum_{k=1}^{i} |a_k|$.

The following construction shows how one can see A_∞-morphisms simply as a convenient way of encoding DG-morphisms from a certain large DG-algebra canonically associated to A, a kind of "*thickening* of A". Consider the graded Λ-bimodule $V := A^+[1]$ and form the tensor algebra

$$Th(A) := \Lambda \oplus \bigoplus_{r \geq 1} V^{\otimes r} ,$$

whose elements are linear combinations

Chapter IV.

Spherical objects

IV

Let k be a fixed field. All categories in this chapter are assumed to be k-linear. Recall that, in a k-linear category $\mathcal{A}$, for any two cochain complexes $C^\bullet, D^\bullet \in C^\bullet(\mathcal{A})$ one can define the complex $\mathrm{Hom}^*(C^\bullet, D^\bullet)$ of k-vector spaces, whose cohomology is given by

$$H^i \mathrm{Hom}^*(C^\bullet, D^\bullet) = \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(C^\bullet, D^\bullet[i]) \, .$$

I should see a precise definition

We already know that $\mathcal{A}$ is both tensored and cotensored over finite dimensional vector spaces. If $\mathcal{A}$ contains infinite products and arbitrary direct sums, then one can define objects $V \otimes_k A$ and $[V, C^\bullet]$ in $\mathcal{A}$, for any vector space V and any object $A \in \mathcal{A}$. One can check that in $C^\bullet(\mathcal{A})$ there are the following canonical monomorphism:

$$V \otimes \mathrm{Hom}^*(D^\bullet, C^\bullet) \longrightarrow \mathrm{Hom}^*(D^\bullet, V \otimes C^\bullet) \, , \tag{IV.0.1}$$

$$\mathrm{Hom}^*(D^\bullet, C^\bullet) \otimes V \longrightarrow \mathrm{Hom}^*([V, D^\bullet], C^\bullet) \, , \tag{IV.0.2}$$

$$\mathrm{Hom}^*(C^\bullet, [V, D^\bullet]) \otimes E^\bullet \longrightarrow [V, \mathrm{Hom}^*(C^\bullet, D^\bullet) \otimes E^\bullet] \, . \tag{IV.0.3}$$

I should learn these facts.

These maps are isomorphisms if V is finite dimensional, and they are quasi-isomorphisms if V has finite-dimensional cohomology.

From now on, fix an abelian category $\mathcal{A}$.

Definition. — A **Serre subcategory** of $\mathcal{A}$ is a non-empty full subcategory $\mathcal{S} \subset \mathcal{A}$ such that, given an exact sequence $A \to B \to C$, if both A and C are in $\mathcal{S}$, then also B is in $\mathcal{S}$.

Definition. — We define a full subcategory $\mathcal{B} \subset \mathcal{A}$ satisfying the following conditions:

(**C1**) $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$;

(**C2**) $\mathcal{B}$ contains infinite direct sums and products;

(**C3**) $\mathcal{B}$ has enough injectives and *any* direct sum of injectives is again injective;

(**C4**) given any epimorphism $f : A \to B$ with $B \in \mathcal{B}$, there exists an object A' in $\mathcal{B}$ and a morphism $g : A' \to A$ such that fg is again an epimorphism:

$$\begin{tikzcd} A \arrow[dr, "f"] & & \\ A' \arrow[u, dashed, "g"] \arrow[r] & B \arrow[r] & \mathbf{0} \, . \end{tikzcd}$$

Why this

Remark. — Since $\mathcal{B}$ is a Serre subcategory, the map g in property (**C4**) can be taken to be a monomorphism.

IV.1. Definition. — Let $\mathfrak{K} \subset K^+(\mathcal{A})$ be the full subcategory whose objects are bounded below cochain complexes $I^\bullet$ with injective terms with bounded cohomology in $\mathcal{B}$, that is $H^i(I^\bullet) \in \mathcal{B}$ for all i and $H^i(I^\bullet) = 0$ for $i >> 0$.

Put these results in the derived cat part.

Our aim is to prove that $\mathfrak{K}$ is equivalent to the bounded derived category $\mathbf{D}^b(\mathcal{B})$. Now recall that $C^b_{\mathcal{B}}(\mathcal{A})$ is the category of bounded cochain complexes whose cohomology objects lie in $\mathcal{B}$.

IV.2. Proposition. — *Abelian categories have fibre products.*

Proof. Given $f : A \to M$ and $g : B \to M$ morphisms in an abelian category $\mathcal{A}$, the commutativity constraint of a square

$$\begin{tikzcd} T \arrow[r, "t_B"] \arrow[d, "t_A"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & M \end{tikzcd}$$

can be rewritten as $ft_A - gt_B = 0$. Thus, we can translate the fibre product universal property into a kernel property: we know that $\ker(f\pi_A - g\pi_B) \subset A \oplus B$ exists, and it satisfies

$$\begin{tikzcd} T \arrow[rrr, bend left, "t_B"] \arrow[dr, dashed] \arrow[ddr, bend right, "t_A"'] & & & \\ & \ker(f\pi_A - g\pi_B) \arrow[rr] \arrow[d] \arrow[r, hook, dashed] & A \oplus B \arrow[r, "\pi_B"'] \arrow[dl, "\pi_A"'] \arrow[dr] & B \arrow[d, "g"] \\ & A \arrow[rr, "f"] & & M\, , \end{tikzcd}$$

and hence $A \times_M B = \ker(f\pi_A - g\pi_B)$. $\square$

Definition. — Let $f : A \to M$ and $g : B \to M$ be morphisms in $\mathcal{A}$. If f is a monomorphism, then the fibre product will be denoted by $g^{-1}(A)$. If both f and g are monomorphisms, then their fibre product is called **intersection** $A \cap B$ and we define their **sum** $A + B := \operatorname{im}(f\pi_A - g\pi_B)$.

Move this part to the AbCat.

IV.3. Lemma. — *For any $C^\bullet \in C^b_{\mathcal{B}}(\mathcal{A})$, there is a complex $E^\bullet \in C^b(\mathcal{B})$ and a monomorphism $\iota^\bullet : E^\bullet \to C^\bullet$ which is a qis.*

Proof. Let N be the largest integer such that $C^N \neq \mathbf{0}$. We build the complex $E^\bullet$ inductively by setting $E^n = \mathbf{0}$, for every $n > N$, and by using property (**C4**) of $\mathcal{B}$ in the following way: for $n \geq N$, assume E^{n+1} has been built; then for (**C4**) there exists diagrams

$$\begin{tikzcd} (d^n_C)^{-1}(E^{n+1}) \arrow[dr, two heads, "d^n_C"] & \\ F^n \arrow[u, hook, dashed] \arrow[r, two heads, dashed] & E^{n+1} \cap \operatorname{im} d^n_C\, , \end{tikzcd} \qquad \begin{tikzcd} \ker d^n_C \arrow[dr, two heads] & \\ G^n \arrow[u, hook, dashed] \arrow[r, two heads, dashed] & H^n(C^\bullet)\, , \end{tikzcd}$$

with F^n and G^n in $\mathcal{B}$. For $n = N$, notice that $F^N, G^N \hookrightarrow C^N$, so inductively one gets F^n and G^n as subobjects of C^n. Thus, define $E^\bullet$ to be the complex with terms $E^n = F^n + G^n$ and differential inherited by $C^\bullet$, that is $d_E^n := d_C^n|_{E^n}$. We have a natural inclusion $\iota^\bullet : E^\bullet \to C^\bullet$, so the complex $E^\bullet$ is bounded, and by (**C1**) every E^n is an object in $\mathcal{B}$.

For every n, notice that the n-cocycles of $E^\bullet$ are by definition $\ker d_E^n = E^n \cap \ker d_C^n$, hence the map $E^n \cap \ker d_C^n \to H^n(C^n)$ is an epimorphism by the definition of G^n (see the commutative triangle on the right), with kernel given by $E^n \cap \operatorname{im} d_C^{n-1}$, which equals to

$$E^n \cap \operatorname{im} d_C^{n-1} = \operatorname{im}\left(F^{n-1} \twoheadrightarrow (E^n \cap \operatorname{im} d_C^{n-1})\right) = \operatorname{im} d_E^{n-1}$$

by construction. This shows that the inclusion $\iota^\bullet : E^\bullet \to C^\bullet$ induces $H^*(E^\bullet) \simeq H^*(C^\bullet)$. $\square$

Insert this in the part of DerCat

It follows by [GM02, Proposition III.2.10] that the functor induced by the inclusion on the derived cateogories $\mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b_{\mathcal{B}}(\mathcal{A})$ is an exact equivalence.

IV.4. Theorem. — *There is an exact equivalence of triangulated categories $\mathfrak{K} \simeq \mathbf{D}^b(\mathcal{B})$.*

Proof. Since $\mathcal{A}$ has enough injectives, then $\mathfrak{K}$ is equivalent to the full subcategory $\mathfrak{D} \subset \mathbf{D}^+(\mathcal{A})$ of bounded below complexes $C^\bullet$ with bounded cohomology, with each $H^n(C^\bullet) \in \mathcal{B}$, i.e. cohomology of $C^\bullet$ satisfies the property in Definition IV.1. We know that $\mathbf{D}^b_{\mathcal{B}}(\mathcal{A})$ is equivalent to $\mathfrak{D}$ via the functor induced by the inclusion, and together with Lemma IV.3 one gets the sequence of exact equivalences

$$\mathbf{D}^b(\mathcal{B}) \xrightarrow{\sim} \mathbf{D}^b_{\mathcal{B}}(\mathcal{A}) \xrightarrow{\sim} \mathfrak{D} \xleftarrow{\sim} \mathfrak{K},$$

from which we deduce that $\mathfrak{K}$ and $\mathbf{D}^b(\mathcal{B})$ are equivalent as triangulated categories. $\square$

Twist functors and spherical objects

Recall that, given any two complexes $C^\bullet, D^\bullet$ with terms in $\mathcal{A}$, we denote by

$$\operatorname{Hom}^*_{\mathbf{C}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{C}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet[n]),$$

and we do so for all other categories whose objects are complexes. Since $\mathcal{A}$ is a k-linear category, the set defined above is a k-vector space.

Definition. — We call an object $E^\bullet \in \mathfrak{K}$ **twistable** if it satisfies the following finiteness conditions:

(**S1**) it is bounded;

(**S2**) for any $F^\bullet$ in $\mathfrak{K}$, both $\operatorname{Hom}^*_{\mathfrak{K}}(E^\bullet, F^\bullet)$ and $\operatorname{Hom}^*_{\mathfrak{K}}(F^\bullet, E^\bullet)$ have finite dimension over k.

Definition. — Given a twistable complex $E^\bullet$, we define the **twist functor** around $E^\bullet$ to be the cone of the following evaluation map:

$$T_E : \mathfrak{K} \longrightarrow \mathfrak{K}, \quad T_E(F^\bullet) := \mathbf{C}\left(\operatorname{Hom}^*(E^\bullet, F^\bullet) \otimes E^\bullet \xrightarrow{ev} F^\bullet\right).$$

A remark on the well-definition of this functor may be needed: since $E^\bullet$ is bounded and $F^\bullet$ is bounded from below, then the complex $\mathrm{Hom}^*(E^\bullet, F^\bullet)$ is also bounded from below, thus $\mathrm{Hom}^*(E^\bullet, F^\bullet)$ is an object in $C^b(\mathbf{Vect}_k)$. As $E^\bullet$ has injective terms, by property (**C3**) we deduce that $\mathrm{Hom}^*(E^\bullet, F^\bullet) \otimes E^\bullet$ is again a bounded below complex made of injective objects. Notice that $\mathrm{Hom}^*(E^\bullet, F^\bullet)$ is quasi-isomorphic to the complex $\mathrm{Hom}^*_{\mathfrak{K}}(E^\bullet, F^\bullet)$, which is finite-dimensional by (**S2**), and since $- \otimes E^\bullet$ preserves quasi-isomorphisms it holds

$$H^*\left(\mathrm{Hom}^*(E^\bullet, F^\bullet) \otimes E^\bullet\right) \simeq H^*\left(\mathrm{Hom}^*_{\mathfrak{K}}(E^\bullet, F^\bullet) \otimes E^\bullet\right) \simeq \mathrm{Hom}^*_{\mathfrak{K}}(E^\bullet, F^\bullet) \otimes H^*(E^\bullet)\,,$$

where the last isomorphism is due to the additivity of H^*. It follows that the above cohomology is bounded, with objects in $\mathcal{B}$ by (**C2**), therefore $\mathrm{Hom}^*(E^\bullet, F^\bullet) \otimes E^\bullet \in \mathfrak{K}$. By inspecting the associated LECS, one sees that $T_E(F^\bullet$ lies in $\mathfrak{K}$, so T_E is a well-defined functor. Moreover, it is triangulated.

Check this!

IV.5. Proposition. — *A qis between twistable objects $E_1^\bullet \to E_2^\bullet$ gives rise to an isomorphism of twist functors $T_{E_1} \simeq T_{E_2}$.*

Proof. Given any $F^\bullet \in \mathfrak{K}$, since the complexes $\mathrm{Hom}^*(E_i^\bullet), F^\bullet) \otimes E_i^\bullet$ with $i, j \in \{1, 2\}$ are quasi-isomorphic, then by the diagram

$$\begin{tikzcd}
\mathrm{Hom}^*(E_1^\bullet, F^\bullet) \otimes E_1^\bullet \arrow[r, "ev"] & F^\bullet \arrow[r] & T_{E_1}(F^\bullet) \\
\mathrm{Hom}^*(E_2^\bullet, F^\bullet) \otimes E_1^\bullet \arrow[r] \arrow[u] \arrow[d] & F^\bullet \arrow[r] \arrow[u, equal] \arrow[d, equal] & C^\bullet \arrow[u, dashed] \arrow[d, dashed] \\
\mathrm{Hom}^*(E_2^\bullet, F^\bullet) \otimes E_2^\bullet \arrow[r, "ev"] & F^\bullet \arrow[r] & T_{E_2}(F^\bullet)
\end{tikzcd}$$

one deduces that $T_{E_1}(F^\bullet)$ and $T_{E_2}(F^\bullet)$ are quasi-isomorphic. Since $\mathfrak{K}$ is a category of complexes of injectives, quasi-isomorphisms are in fact isomorphisms. □

IV.6. Corollary. — *For any $j \in \mathbb{Z}$, the functor $T_{E[j]}$ is isomorphic to T_E.*

Proof. Given any $F^\bullet \in \mathfrak{K}$, the n-th term of the complex $\mathrm{Hom}^*(E^\bullet[j], F^\bullet) \otimes E^\bullet[j]$ is

$$\begin{aligned}
\Big(\mathrm{Hom}^*(E^\bullet[j], F^\bullet) \otimes E^\bullet[j]\Big)^n &= \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k(E^\bullet[j], F^\bullet) \otimes (E^\bullet[j])^{-k+n} \\
&\simeq \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k(E^\bullet, F^\bullet[-j]) \otimes E^{-k+j+n} \\
&= \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^{k-j}(E^\bullet, F^\bullet) \otimes E^{-(k-j)+n} \\
&= \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}^p(E^\bullet, F^\bullet) \otimes E^{-p+n} \\
&= \Big(\mathrm{Hom}^*(E^\bullet, F^\bullet) \otimes E^\bullet\Big)^n.
\end{aligned}$$

The sign convention is such that all these isomorphisms together define an isomorphism of complexes, thus the associated twist functors are isomorphic. □

Definition. — Given a twistable object $E^\bullet \in \mathfrak{K}$, we define the **dual twist functor** to be

$$T'_E : \mathfrak{K} \longrightarrow \mathfrak{K}\,, \quad T'_E(F^\bullet) := \mathbf{C}\left(F^\bullet \xrightarrow{ev'} [\mathrm{Hom}^*(F^\bullet, E^\bullet), E^\bullet]\right)$$

By a similar argument as for T_E, the dual twist functor actually takes values in $\mathfrak{K}$, and the name "*dual*" is justified by the following

IV.7. Lemma. — *For every $E^\bullet$ twistable, the functor T'_E is left adjoint to T_E.*

Proof. Given $F^\bullet, G^\bullet \in \mathfrak{K}$, we need to show the exists a natural isomorphism

$$\mathrm{Hom}^*_{\mathfrak{K}}(T'_E(F^\bullet), G^\bullet) \simeq \mathrm{Hom}^*_{\mathfrak{K}}(F^\bullet, T_E(G^\bullet))\,.$$

If we find a qis $\mathrm{Hom}^*(T'_E(F^\bullet), G^\bullet) \to \mathrm{Hom}^*(F^\bullet, T_E(G^\bullet))$, then the claim follows by taking H^0 on both sides.
By applying $\mathrm{Hom}^*(F^\bullet, -)$ to the distinguished triangle

$$\mathrm{Hom}^*(E^\bullet, G^\bullet) \otimes E^\bullet \xrightarrow{\ ev\ } G^\bullet \longrightarrow T_E(G^\bullet) \longrightarrow \left(\mathrm{Hom}^*(E^\bullet, G^\bullet) \otimes E^\bullet\right)[1]\,,$$

we notice that $\mathrm{Hom}^*(F^\bullet, T_E(G^\bullet)) \simeq \mathbf{C}(ev_*)$; dually, it holds

$$\mathrm{Hom}^*(T'_E(F^\bullet), G^\bullet) \simeq \mathbf{C}\left(\mathrm{Hom}^*\left([\mathrm{Hom}^*(F^\bullet, E^\bullet), E^\bullet], G^\bullet\right) \xrightarrow{(ev')^*} \mathrm{Hom}^*(F^\bullet, G^\bullet)\right)\,.$$

Thus, if we consider the composition morphism

$$\circ : \mathrm{Hom}^*(E^\bullet, G^\bullet) \otimes_k \mathrm{Hom}^*(F^\bullet, E^\bullet) \longrightarrow \mathrm{Hom}^*(F^\bullet, G^\bullet)\,, \quad \varphi^\bullet \otimes \psi^\bullet \longmapsto \varphi^\bullet \circ \psi^\bullet\,,$$

thanks to the natural quasi-isomorphisms (IV.0.1), we can build a zigzag of qis, which are natural both in $F^\bullet$ and in $G^\bullet$:

$$\mathrm{Hom}^*(E^\bullet, G^\bullet) \otimes E^\bullet \longleftarrow \mathbf{C}(ev_*) \longleftarrow \mathbf{C}(\circ) \longrightarrow \mathbf{C}\big((ev')^*\big) \longrightarrow \mathrm{Hom}^*(T'_E(F^\bullet), G^\bullet)\,,$$

hence the thesis. □

Definition. — An object $E^\bullet \in \mathfrak{K}$ is **n-spherical** for some $n > 0$ if it satisfies (**S1**), (**S2**) and the following two properties:

(**S3**) its endomorphism algebra is the cohomology of the n-sphere $\mathrm{Hom}^*_{\mathfrak{K}}(E^\bullet, E^\bullet) \simeq H^*_{\mathrm{sing}}(S^n; k)$, that is

$$\mathrm{Hom}^j_{\mathfrak{K}}(E^\bullet, E^\bullet) \simeq \begin{cases} k\,, & \text{if } j = 0, n\,; \\ 0\,, & \text{otherwise.} \end{cases}$$

(**S4**) **Poincaré duality**: for all $F^\bullet$ and all $j \in \mathbb{Z}$, the composition morphism

$$\mathrm{Hom}^j(F^\bullet, E^\bullet) \otimes_k \mathrm{Hom}^{n-j}(E^\bullet, F^\bullet) \longrightarrow \mathrm{Hom}^n(E^\bullet, E^\bullet) \simeq k$$

is a non degenerate pairing.

IV.8. Theorem. — *The spherical twist around an n-spherical object is an exact self-equivalence of $\mathfrak{K}$.*

Proof. Given $F^\bullet \in \mathfrak{K}$, then $T_E T'_E(F^\bullet)$ □

Applications of DG methods to homological algebra often hinge on constructing a chain of quasi-isomorphisms connecting two given DG-algebras. For instance, in the situation explained in the previous section, one can try to use the DG-algebra *end*(E) to study the twists T_{E_i} via the functor Ψ_E. What really matters for this purpose is only the quasi-isomorphism type of *end*(E). In general, quasi-isomorphism type is a rather subtle invariant. However, there are some cases where the cohomology already determines the quasi-isomorphism type.

Definition. — A DG-algebra $\mathcal{A}$ is called **formal** if it is quasi-isomorphic to its own cohomology algebra $H^*(\mathcal{A})$ thought as a DG-algebra with zero differentials.

Example. — Any algebra A seen as a DG-algebra concentrated in degree 0 is formal.

Definition. — A graded algebra A is called **intrinsically formal** if any two DG-algebras $\mathcal{B}, \mathcal{C}$ such that $H^*(\mathcal{B}) \simeq A \simeq H^*(\mathcal{C})$, then $\mathcal{B}$ and $\mathcal{C}$ are quasi-isomorphic.

Equivalently, one can say that A is intrinsically formal if any DG-algebra $\mathcal{B}$ whose cohomology is A is formal.

Explain why.

Example. — Any graded algebra A concentrated in degree zero is intrinsically formal.

The aim of this section is to find a characterization of intrinsic formality, which can be computed via Hochschild cohomology of the right A-modules obtianed by shifting A suitably.

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