Rate and state

Carsten Uphoff

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1 Test problem

$$0 = \tau - \sigma_n a \operatorname{arcsinh}\left(\frac{V}{2V_0}\exp\left(\frac{f_0 + b\log(V_0\psi/L)}{a}\right)\right) - \eta V \tag{1}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = 1 - \frac{V\psi}{L} \tag{2}$$

2 Manufactured solution

Let

$$V^*(t) = \frac{1}{\pi}\left(\arctan\left(\frac{t-t_e}{t_w}\right) + \frac{\pi}{2}\right) \tag{3}$$

$$\frac{\mathrm{d}V^*}{\mathrm{d}t} = \frac{1}{\pi t_w}\frac{1}{\left(\frac{t-t_e}{t_w}\right)^2 + 1} \tag{4}$$

Solve friction law for ψ:

$$\psi^*(t) = \frac{L}{V_0}\exp\left(\frac{a}{b}\log\left(\frac{2V_0}{V}\sinh\left(\frac{\tau-\eta V}{as_n}\right)\right) - \frac{f_0}{b}\right) \tag{5}$$

$$\begin{aligned}\frac{\mathrm{d}\psi^*}{\mathrm{d}t} = {} & \frac{aV\frac{L}{V_0}\exp\left(\frac{a}{b}\log\left(\frac{2V_0}{V}\sinh\left(\frac{\tau-\eta V}{as_n}\right)\right) - \frac{f_0}{b}\right)}{2bV_0\sinh\left(\frac{\tau-\eta V}{as_n}\right)} \\ & \times \left(\frac{2V_0}{Vas_n}\cosh\left(\frac{\tau-\eta V}{as_n}\right)\left(\frac{\mathrm{d}\tau}{\mathrm{d}t} - \eta\frac{\mathrm{d}V}{\mathrm{d}t}\right) - \frac{2V_0\frac{\mathrm{d}V}{\mathrm{d}t}}{V^2}\sinh\left(\frac{\tau-\eta V}{as_n}\right)\right)\end{aligned} \tag{6}$$

State ODE:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = 1 - \frac{V\psi}{L} - \left(1 - \frac{V^*\psi^*}{L}\right) + \frac{\mathrm{d}\psi*}{\mathrm{d}t} \tag{7}$$

3 Bounds

Assume $\sigma_n, \tau > 0$. Let $V = \tau/\eta$. Then

$$f(\tau/\eta) = -\sigma_n a \operatorname{arcsinh}\left(\frac{\tau/\eta}{2V_0}\exp\left(\frac{f_0 + b\log(V_0\psi/L)}{a}\right)\right) < 0 \tag{8}$$

Let $V = -\tau/\eta$.

$$f(-\tau/\eta) = 2\tau - \sigma_n a \operatorname{arcsinh}\left(\frac{-\tau/\eta}{2V_0}\exp\left(\frac{f_0 + b\log(V_0\psi/L)}{a}\right)\right) > 0 \tag{9}$$

(If $\tau < 0$ the situation reverses, if $\tau = 0$ then $V = 0$ is a zero of f.)

Therefore we may use $[-\tau/\eta, \tau/\eta]$ as bisection interval.

4 Steady state

Let

$$0 = \tau - \sigma_n a \operatorname{arcsinh}\left(\frac{V}{2V_0}\exp\left(\frac{f_0 + b\log(V_0\psi/L)}{a}\right)\right) \tag{10}$$

$$0 = 1 - \frac{V\psi}{L} \tag{11}$$

Steady state is

$$\psi_{ss}(V,\tau) = \frac{L}{V} \tag{12}$$

$$\tau_{ss}(V) = \sigma_n a \operatorname{arcsinh}\left(\frac{V}{2V_0}\exp\left(\frac{b\log(V_0/V) + f_0}{a}\right)\right) \tag{13}$$

$$\frac{\mathrm{d}\tau_{ss}}{\mathrm{d}V} = \sigma_n \frac{\exp\left(\frac{b\log(V_0/V)+f_0}{a}\right)(a-b)}{2V_0\sqrt{\exp\left(\frac{2b\log(V_0/V)+2f_0}{a}\right)V^2/(4V_0^2)+1}} \tag{14}$$

5 Critical sizes (Lapusta)

Take

$$0 = \tau - \sigma_n\left(f_0 + a\log\frac{V}{V_0} + b\log\frac{V_0\psi}{L}\right) \tag{15}$$

$$0 = 1 - \frac{V\psi}{L} \tag{16}$$

Steady state is

$$\psi_{ss}(V) = \frac{L}{V} \tag{17}$$

$$\tau_{ss}(V) = \sigma_n \left(f_0 + a \log \frac{V}{V_0} + b \log \frac{V_0}{V} \right) \tag{18}$$

$$\frac{\mathrm{d}\tau ss}{\mathrm{d}V} = \sigma_n \left(a \frac{1}{V} - b \frac{1}{V} \right) \tag{19}$$

Critical size is

$$h^* = \frac{\gamma \mu L}{\sigma_n (b - a)} \tag{20}$$

For the time-step we need

$$A^* = \sigma_n a \tag{21}$$

$$B^* - A^* = \sigma_n (a - b) \tag{22}$$

$$\chi = \frac{1}{4} \frac{B^* - A^*}{A^*} \left(\frac{h^*}{h} - 1 \right)^2 - \frac{h^*}{h} \tag{23}$$

$$\Delta t < \begin{cases} \frac{L}{V} \frac{A^*}{(B^* - A^*)(h^*/h - 1)} & \text{if} \quad \chi > 0 \\ \frac{L}{V}(1 - h/h^*) & \text{if} \quad \chi < 0 \end{cases} \tag{24}$$

6 Time-stepping elastodynamics

Let the semi-discrete form be given by

$$\ddot{u}_p = b_p(\boldsymbol{S}(t)) - \hat{K}_{pq} u_q \tag{25}$$

First order form is

$$\dot{u}_p = v_p$$
$$\dot{v}_p = b_p(\boldsymbol{S}(t)) - \hat{K}_{pq} u_q$$

Lets compactly write

$$\dot{y}_p = c_p(\boldsymbol{S}(t)) - G_{pq} y_p,$$

where

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} 0 \\ \boldsymbol{b} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & -I \\ \hat{K} & 0 \end{pmatrix}$$

The Runge-Kutta method is

$$y_p^{n+1} = y_p^n + h b_j k_{pj} \tag{26}$$

$$k_{pi} = c_p(\boldsymbol{S}(t_n + h c_i)) - G_{pq}(1_i y_p^n + h a_{ij} k_{pj}) \tag{27}$$

Collecting k on the LHS gives

$$k_{pi} + ha_{ij} G_{pq} k_{pj} = c_p(\boldsymbol{S}(t_n + hc_i)) - 1_i G_{pq} y_p^n$$

Obviously,

$$\underbrace{(\delta_{ij} + ha_{ij})}_{=:\alpha_{ij}} G_{pq} k_{pj} = c_p(\boldsymbol{S}(t_n + hc_i)) - 1_i G_{pq} y_p^n$$

Multiply with α_{si}^{-1}, which should exist (?), to obtain

$$G_{pq} k_{ps} = \alpha_{si}^{-1} c_p(\boldsymbol{S}(t_n + hc_i)) - (\alpha_{si}^{-1} 1_i) G_{pq} y_p^n$$