Topic 8: Source terms

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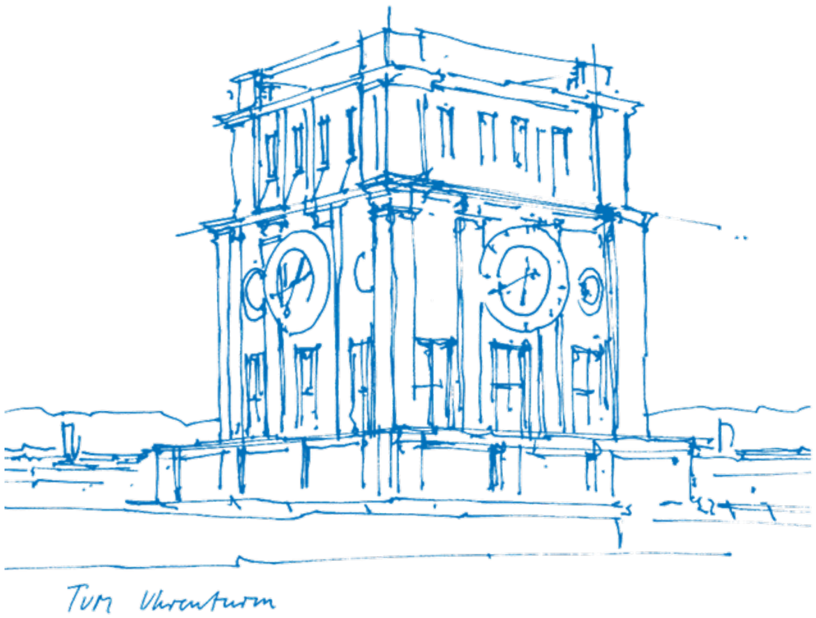

Burgers Equation

We consider the Burgers equation with source term:

$$\frac{\partial q}{\partial t} + \underbrace{\frac{1}{2}\frac{\partial q^2}{\partial x}}_{\textit{transport}} = \underbrace{\psi(q, x)}_{\textit{source}}$$

Operator $\mathcal{A}$ for the transport term:

$$\mathcal{A}q = -\frac{1}{2}\frac{\partial q^2}{\partial x}$$

Operator $\mathcal{B}$ for the source term:

$$\mathcal{B}q = \psi(q, x)$$

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We can rewrite the problem to the form:

$$\frac{\partial q}{\partial t} = (\mathcal{A} + \mathcal{B})q$$

The idea of splitting methods is to solve for $\mathcal{A}$ and $\mathcal{B}$ independently.

Godunov Splitting

- For classical **unsplit methods**, we look for a numerical scheme $\mathcal{C}_{\Delta t}$ which directly approximates $(\mathcal{A} + \mathcal{B})q$

$$\frac{\partial q}{\partial t} = \underbrace{(\mathcal{A} + \mathcal{B})q}_{\mathcal{C}q} \qquad \rightarrow \qquad Q(t + \Delta t) = \mathcal{C}_{\Delta t} Q(t)$$

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- For **Godunov splitting**, we first apply the operator $\mathcal{A}$ then $\mathcal{B}$

$$\frac{\partial q}{\partial t} = \mathcal{B}(\mathcal{A}q)$$

It is now enough to only consider the two subproblems:

$$\frac{\partial q^*}{\partial t} = \mathcal{A}q^* \qquad \text{and} \qquad \frac{\partial q^{**}}{\partial t} = \mathcal{B}q^{**}$$

for which we can find the numerical schemes $\mathcal{A}_{\Delta t}$ and $\mathcal{B}_{\Delta t}$

$$Q^*(t + \Delta t) = \mathcal{A}_{\Delta t} Q^*(t) \qquad \text{and} \qquad Q^{**}(t + \Delta t) = \mathcal{B}_{\Delta t} Q^{**}(t)$$

Godunov Splitting

To implement this scheme, we alternate $\mathcal{A}_{\Delta t}$ and $\mathcal{B}_{\Delta t}$. In each time step we apply:

- First step: $Q^*(t+\Delta t) = \mathcal{A}_{\Delta t} Q(t)$
- Second step: $Q(t+\Delta t) = \mathcal{B}_{\Delta t} Q^*(t+\Delta t)$

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No error if $\mathcal{A}$ and $\mathcal{B}$ commute! $\Leftrightarrow$ $\mathcal{B}(\mathcal{A}q) = \mathcal{A}(\mathcal{B}q)$

In the general case the error is of the order $\mathcal{O}(\Delta t)$

BUT...

The accuracy of the method is decreased if the solvers $\mathcal{A}$ and $\mathcal{B}$ are of second order (or higher).

We need a fractional step method of higher order!

Strang Splitting

In **Strang splitting**, we apply $\mathcal{A}_{\frac{\Delta t}{2}}$ for half a time step before and after $\mathcal{B}_{\Delta t}$. Thus:

- First step: $Q^*(t + \frac{\Delta t}{2}) = \mathcal{A}_{\frac{\Delta t}{2}} Q(t)$
- Second step: $Q^{**}(t + \frac{\Delta t}{2}) = \mathcal{B}_{\Delta t} Q^*(t + \frac{\Delta t}{2})$
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In the general case the error is of the order $\mathcal{O}(\Delta t^2)$

In practice, Godunov and Strang splitting yield very similar results despite the theoretical difference.

An Example of a Stiff Source Term

$$\frac{\partial q}{\partial t} + \frac{1}{2}\frac{\partial q^2}{\partial x} = \frac{\mathbf{1}}{\boldsymbol{\tau}}\boldsymbol{q}(\boldsymbol{q} - \boldsymbol{\beta})(\mathbf{1} - \boldsymbol{q})$$

.

Stable equilibrium at $q = 0$ and $q = 1$

Unstable equilibrium at $q = \beta$

τ regulates the stiffness

Boundaries: 0 to the left, 1 to the right

Rarefaction wave as homogeneous solution

Wave travels with speed β to the right

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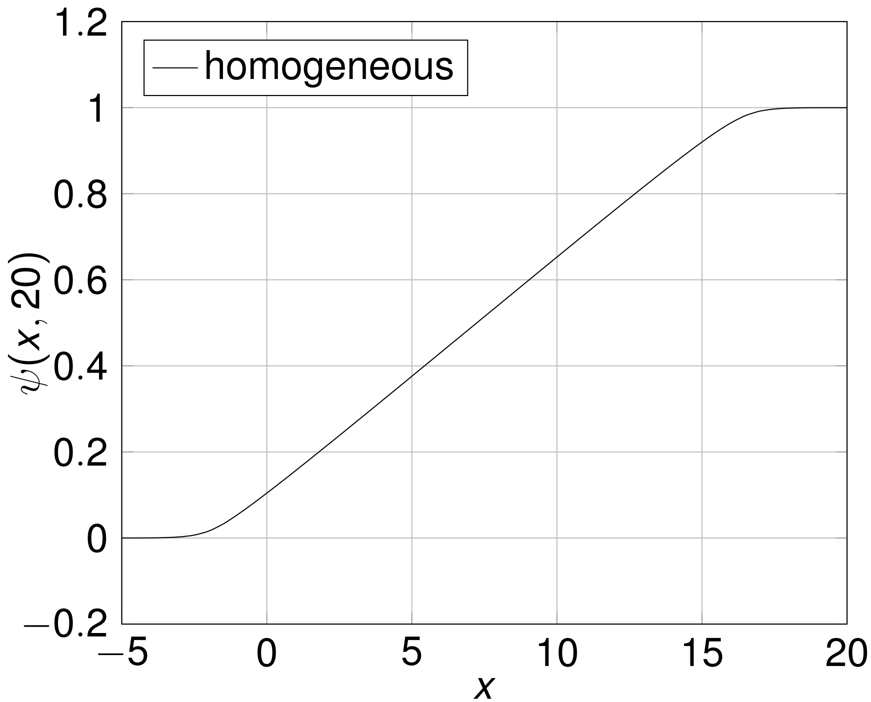

Figure: Analytical solution after 20s

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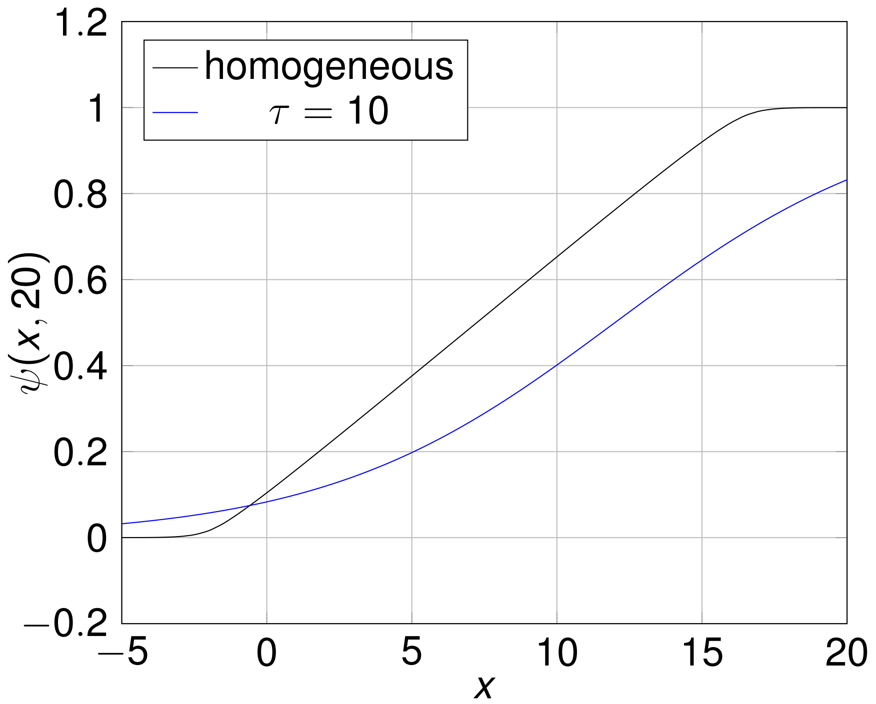

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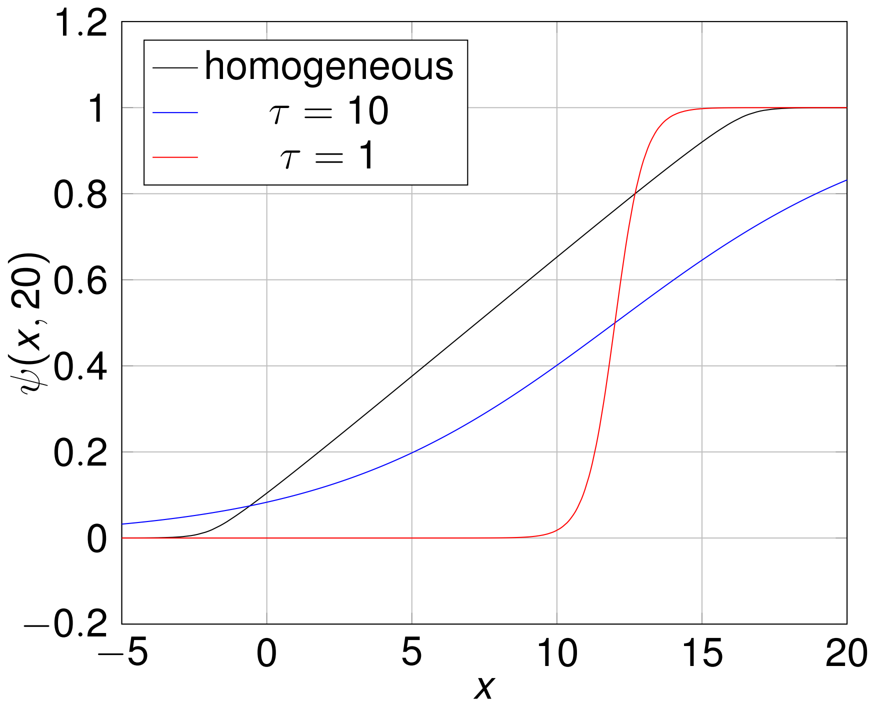

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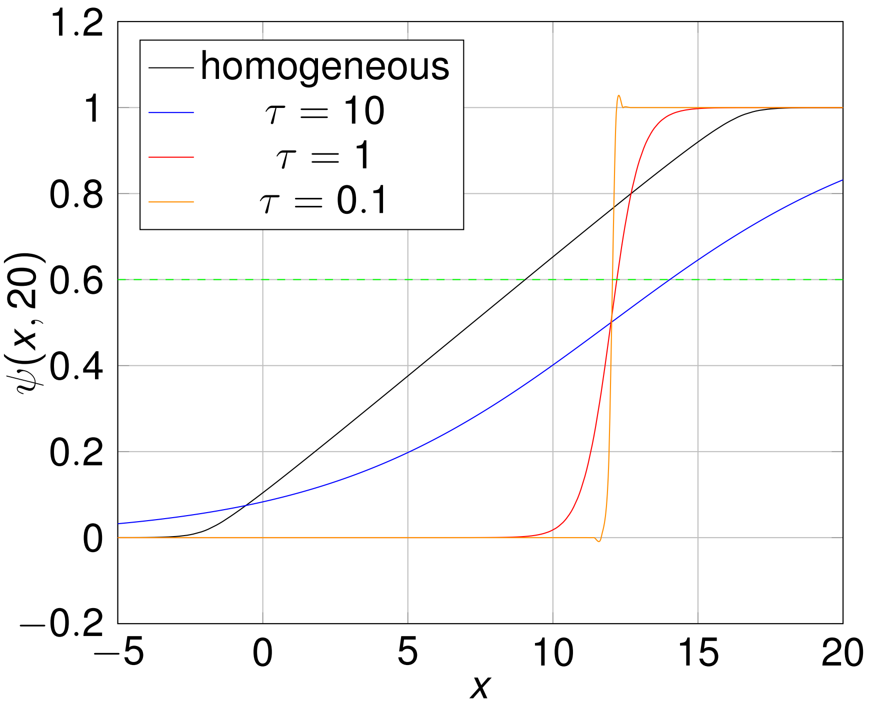

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Implementation

Code implemented in MATLAB

Godunov splitting of the 1D Burgers equation with stiff source term

2 solvers for the transport equation and 5 solvers for the source term

Choice of the solver:

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- For the transport equation $\mathcal{A}_{\Delta t}$: typical solvers for wave transport (**Lax-Wendroff** and **Godunov schemes**)
  The **Godunov scheme** is preferred because it gave better results than **Lax-Wendroff**.

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  The **Godunov scheme** is preferred because it gave better results than **Lax-Wendroff**.
- For the source term equation $\mathcal{B}_{\Delta t}$: depends a lot on the source $\psi(q, x)$
  - Direct solver - the easiest, not too much of interest
  - Explicit schemes - tend to be unstable (e.g. **explicit Euler**, **Runge-Kutta 2**)
  - Implicit schemes - usually stable, but expensive to compute (e.g. **implicit Euler**, **trapezoidal rule**, **TR-BDF2**)

Problem: Implicit schemes are not as reliable as expected for stiff source terms

Stability Issues

Results for different schemes after 20s simulation time

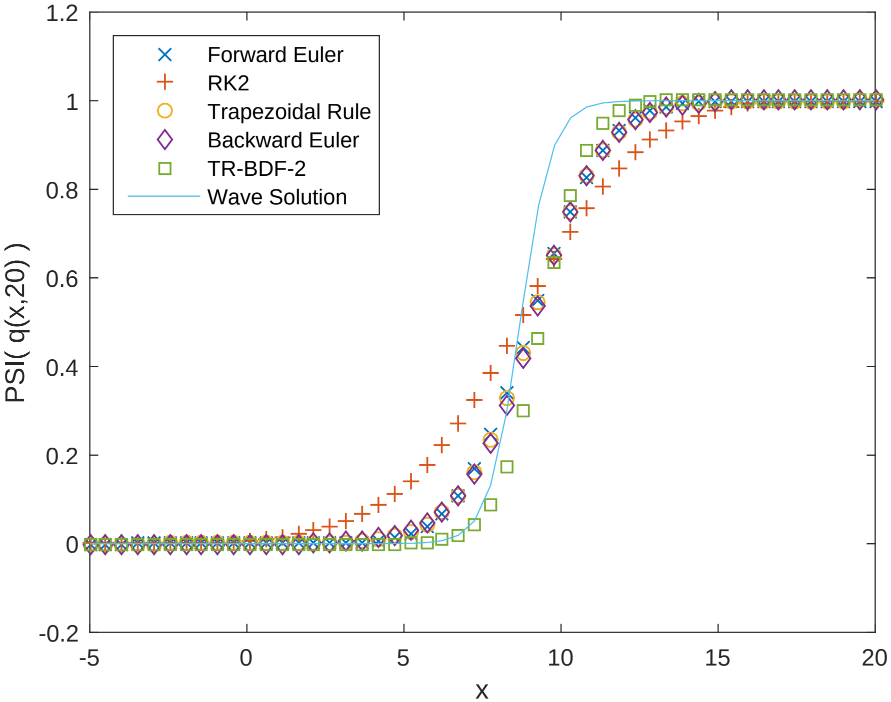

Figure: $\tau = 10^0$

- Everything is fine

Stability Issues

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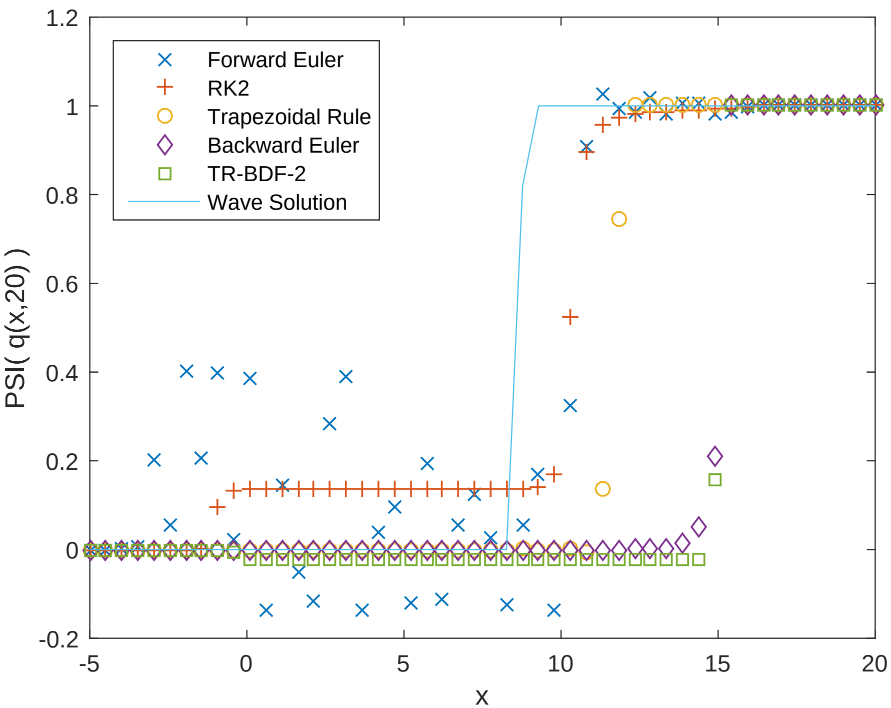

Figure: $\tau = 10^{-1}$

- Explicit methods fail (except RK2)
- Implicit methods work fine

Stability Issues

Results for different schemes after 20s simulation time

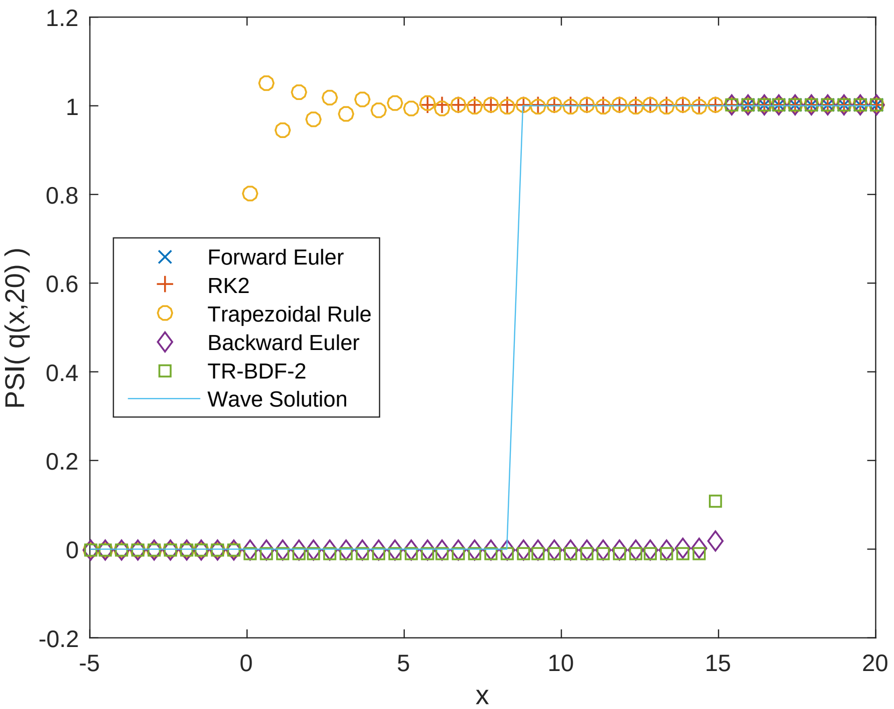

Figure: $\tau = 10^{-2}$

- All explicit methods fail
- Trapezoidal rule fails (A-stable)
- Backwards Euler and TR-BDF2 work fine (L-stable)

Wrong Wave Propagation Speed

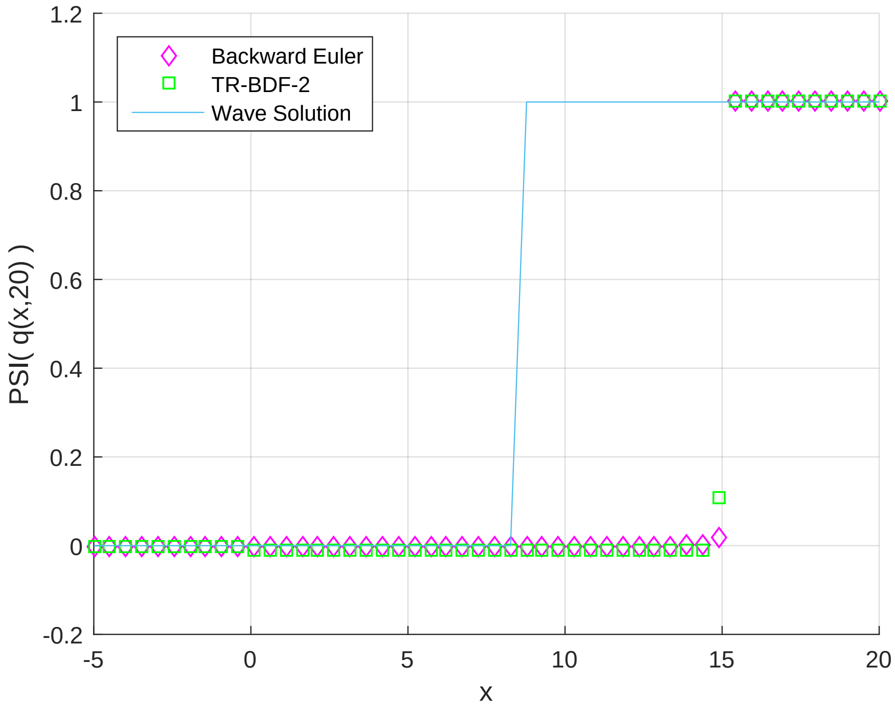

Figure: Implicit solvers at $\tau = 10^{-2}$

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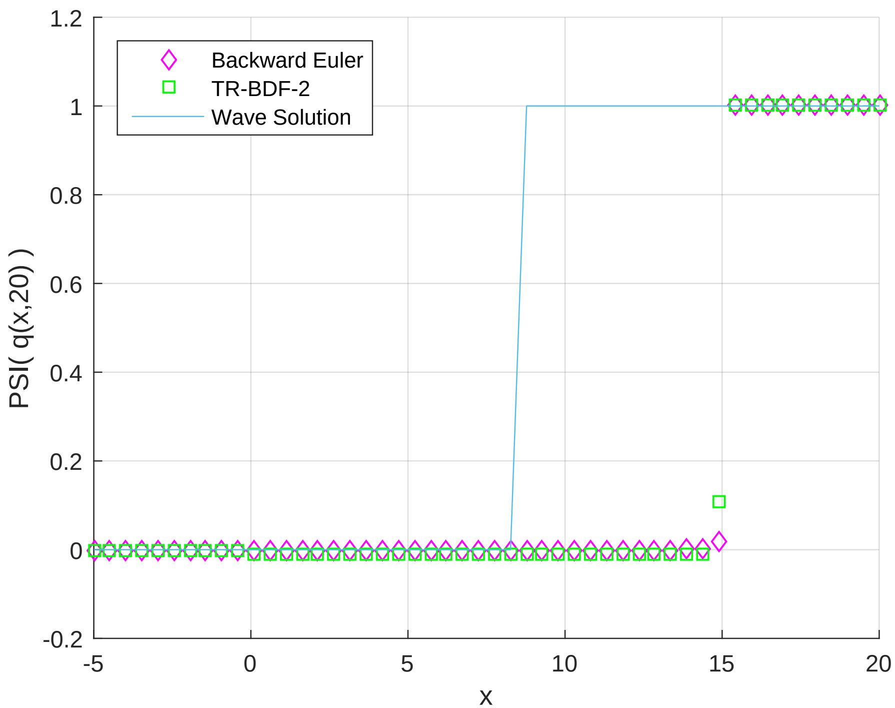

Figure: Implicit solvers at $\tau = 10^{-2}$

We expect the wave travel speed to be $\beta = 0.6$
We get a wave travel speed of $v = 1$
This is due to the Godunov-Strang splitting:

1. Godunov scheme forces first non-zero value to a value below β
2. The source term smears this value to 0

The wave travels at 1 grid point per timestep
As a result: unphysical wave propagation speed for stiff source terms