Rate and state

Carsten Uphoff

October 13, 2020

1 Test problem

$$0 = \tau - \sigma_n a \operatorname{arcsinh}\left(\frac{V}{2V_0}\exp\left(\frac{\psi}{a}\right)\right) - \eta V \tag{1}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{bV_0}{L}\left(\exp\left(\frac{f_0-\psi}{b}\right) - \frac{V}{V_0}\right) \tag{2}$$

2 Manufactured solution

Let

$$V^*(t) = \frac{1}{\pi}\left(\arctan\left(\frac{t-t_e}{t_w}\right) + \frac{\pi}{2}\right) \tag{3}$$

$$\frac{\mathrm{d}V^*}{\mathrm{d}t} = \frac{1}{\pi t_w}\frac{1}{\left(\frac{t-t_e}{t_w}\right)^2+1} \tag{4}$$

Solve friction law for ψ:

$$\psi^*(t) = a\log\left(\frac{2V_0}{V^*}\sinh\left(\frac{\tau^*-\eta V^*}{as_n}\right)\right) \tag{5}$$

$$\frac{\mathrm{d}\psi^*}{\mathrm{d}t} = \frac{a\left(\frac{2V_0}{Vas_n}\cosh\left(\frac{\tau^*-\eta V^*}{as_n}\right)\left(\frac{\mathrm{d}\tau^*}{\mathrm{d}t} - \eta\frac{\mathrm{d}V^*}{\mathrm{d}t}\right) - \frac{2V_0\frac{\mathrm{d}V^*}{\mathrm{d}t}}{(V^*)^2}\sinh\left(\frac{\tau^*-\eta V^*}{as_n}\right)\right)}{\frac{2V_0}{V^*}\sinh\left(\frac{\tau^*-\eta V^*}{as_n}\right)} \tag{6}$$

State ODE:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{bV_0}{L}\left(\exp\left(\frac{f_0-\psi}{b}\right) - \frac{V}{V_0}\right) - \frac{bV_0}{L}\left(\exp\left(\frac{f_0-\psi^*}{b}\right) - \frac{V^*}{V_0}\right) + \frac{\mathrm{d}\psi*}{\mathrm{d}t} \tag{7}$$

3 Bounds

Assume $\sigma_n, \tau > 0$. Let $V = \tau/\eta$. Then

$$f(\tau/\eta) = -\sigma_n a \operatorname{arcsinh}\left(\frac{\tau/\eta}{2V_0}\exp\left(\frac{\psi}{a}\right)\right) < 0 \tag{8}$$

Let $V = -\tau/\eta$.

$$f(-\tau/\eta) = 2\tau - \sigma_n a \operatorname{arcsinh}\left(\frac{-\tau/\eta}{2V_0}\exp\left(\frac{\psi}{a}\right)\right) > 0 \tag{9}$$

(If $\tau < 0$ the situation reverses, if $\tau = 0$ then $V = 0$ is a zero of f.)

Therefore we may use $[-\tau/\eta, \tau/\eta]$ as bisection interval.