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Dynamics of a stochastic tuberculosis model with constant recruitment and varying total population size

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HIGHLIGHTS

- A stochastic tuberculosis model with constant recruitment is studied.
- We establish sufficient conditions for the existence of ergodic stationary distribution.
- We establish sufficient conditions for the extinction of the disease.

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ABSTRACT

In this paper, we develop a mathematical model for a tuberculosis model with constant recruitment and varying total population size by incorporating stochastic perturbations. By constructing suitable stochastic Lyapunov functions, we establish sufficient conditions for the existence of an ergodic stationary distribution as well as extinction of the disease to the stochastic system.

1. Introduction

Tuberculosis (TB) is a bacterial disease caused by the infection of bacterium Mycobacterium tuberculosis. TB was so devasting that it became the motivating force in the development of the fields of bacteriology, modern epidemiology, and public health. Once thought under control using antibiotic therapies, TB made a dramatic come back in the late nineteen eighties and early nineteen nineties, large due to the emergence of antibiotic resistant stains and to co-infection with HIV [1]. The risk of developing active TB is greatest within the first several years following infection, however, sometimes disease can present many years after infection (see e.g. Refs. [2–5]). Globally, it was estimated that 2 billion people, one-third of the world total population, are infected with TB bacilli. This provides us a huge pool for future TB epidemic [6].

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Mathematical models which describe the dynamics of infectious diseases have been used to improve our understanding of the basic transmission dynamics of TB and to evaluate the effectiveness of various control strategies in the past few decades (see e.g. Refs. [7–16]). These models support the view that the acquisition of TB infection may not be as difficult as previously thought. The TB bacteria can spread in the air from a person with active TB disease to others when they are in close contact. The disease is most commonly transmitted from a person suffering from infectious (active) tuberculosis to other persons by infected droplets created when the person with active TB coughs or sneezes. When first infected with TB bacteria, a person shall goes through a latent, asymptomatic and non-infectious period during which the body's immune system fights the TB bacteria. By using a compartmental method, we can divide the total host population into three epidemiological class or subgroups: susceptible (S), exposed (E; infected but not infectious), and infected (I; assumed infectious) individuals. Motivated by these facts, we can formulate the following model for the transmission of TB:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Lambda - \beta cSI - \mu S + r_1 E + r_2 I, \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = \beta cSI - (\mu + k + r_1)E, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = kE - (\mu + d + r_2)I, \end{cases} \tag{1.1}$$

where the parameters have the following biological meanings: Λ is the constant recruitment rate, c is the per-capita contact rate, β represents the average number of susceptible individuals infected by one infectious individual per contact per unit of time, μ stands for per-capita natural death rate, k represents the rate at which an individual leaves the latent class by becoming infectious, d is the per-capita disease-induced death rate, r_1 and r_2 are the per-capita treatment rates of latent and infectious individuals, respectively. We assume that an individual can be infected only by contacting infectious individuals. All parameter values are assumed to be nonnegative and $\Lambda, \mu > 0$. In system (1.1), the basic reproduction number $R_0 = \frac{\beta\Lambda ck}{\mu(\mu+k+r_1)(\mu+d+r_2)}$ is the threshold which determines whether the epidemic occurs or not. If $R_0 \le 1$, then system (1.1) has only the disease-free equilibrium $E_0 = (S_0, 0, 0)$ and it is globally asymptotically stable in the invariant set D, where $D = \{(S, E, I) : S > 0, E > 0, I > 0, S + E + I \le \frac{\Lambda}{\mu}\}$. This means that the disease will disappear and the entire population will become susceptible, where $S_0 = \frac{\Lambda}{\mu}$. If $R_0 > 1$ and $r_1 + \mu > d$, then E_0 is unstable and system (1.1) has a unique positive endemic equilibrium $E^* = (S^*, E^*, I^*)$ which is globally asymptotically stable, where $S^* = \frac{(\mu+k+r_1)(\mu+d+r_2)}{\beta ck} = \frac{1}{R_0}$, $E^* = \frac{\mu+d+r_2}{k} I^* = \frac{(\mu+d+r_2)(\Lambda-\mu S^*)}{(\mu+d)(\mu+k)+r_2\mu}$ and $I^* = \frac{k(\Lambda-\mu S^*)}{(\mu+d)(\mu+k)+r_2\mu}$. It shows that the disease will prevail and persist in a population.

It is well known that in the real world, biological populations are always affected by the environmental noise which affects the environmental parameters and as a consequence affects the population dynamics significantly [17]. As we know, in several instances, environmental variations have an important effect on the development and propagation of an epidemic [18,19]. For human disease related epidemics, the nature of epidemic growth and spread is random due to the unpredictability in person-to-person contacts [20] and population is inevitably disturbed by a continuous spectrum of disturbances [21]. Therefore the variability and randomness of the environment is fed through the state of the epidemic [22]. Hence in epidemic dynamics, stochastic differential equation (SDE) models are more appropriate way of modeling epidemics in random environment. Thus many authors have introduced stochastic perturbations into the corresponding deterministic models to reveal the effect of environmental noise (see e.g. Refs. [17,23–28]). For example, Li et al. [17] investigated the basic features of a simple SI epidemic model of Feline immunodeficiency virus (FIV) within cat populations in presence of multiplicative noise terms to understand the effects of environmental driving forces on the disease dynamics. They proposed three threshold parameters, R_s^h, R_1 and R_2 to utilize in identifying the stochastic extinction and persistence and in the case of stochastic persistence, they proved that there is a stationary distribution. Lahrouz and Settati [25] considered a stochastic SIR epidemic model, they showed that the dynamics of the system are determined by a threshold value R_s. If $R_s < 1$, the disease-free equilibrium is globally asymptotically stable in probability, whereas if $R_s > 1$, the solution is positive recurrent and has a unique ergodic stationary distribution. Yang et al. [26] investigated the stochastic SIR and SEIR epidemic models with saturated incidence. They used the stochastic Lyapunov functions to prove under certain conditions, the solution admits the ergodic property as $R_s' > 1$, while exponential stability as $R_s' < 1$, where R_s' is defined by the model parameters and the intensity of the noise. Although there have been some papers concerned with the effect of stochastic parameter perturbation on SI model, SIR model, SEIR model and other epidemic models, we are aware of few literatures addressing this issue in stochastic tuberculosis model. Since there are difficulties in dealing with the stability by using the stochastic Lyapunov function methods.

Motivated by these facts, in this paper, we attempt to do some work in this field and we consider the existence of ergodic stationary distribution of the solution to stochastic tuberculosis model, our method to include stochastic perturbations is similar to that of Imhof and Walcher [29]. Here we assume that stochastic perturbations are of the white noise type which are directly proportional to S, E and I, influenced on the $\frac{\mathrm{d}S}{\mathrm{d}t}$, $\frac{\mathrm{d}E}{\mathrm{d}t}$ and $\frac{\mathrm{d}I}{\mathrm{d}t}$ in system (1.1). Then we can obtain the following stochastic version corresponding to deterministic model (1.1) (see Appendix for details on model construction):

$$\begin{cases} \mathrm{d}S = [\Lambda - \beta cSI - \mu S + r_1 E + r_2 I]\mathrm{d}t + \sigma_1 S\mathrm{d}B_1(t), \\ \mathrm{d}E = [\beta cSI - (\mu + k + r_1)E]\mathrm{d}t + \sigma_2 E\mathrm{d}B_2(t), \\ \mathrm{d}I = [kE - (\mu + d + r_2)I]\mathrm{d}t + \sigma_3 I\mathrm{d}B_3(t), \end{cases} \tag{1.2}$$

where $B_i(t)$ are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), $\sigma_i^2 \geq 0$ denote the intensity of the white noise, $i = 1, 2, 3$.

Throughout this paper, we let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0, i = 1, 2, 3\}$. The rest of this paper is organized as follows. In Section 2, we show that there is a unique global positive solution of system (1.2) with any initial value. In Section 3, we prove the existence and uniqueness of an ergodic stationary distribution of the solution to system (1.2) by constructing suitable stochastic Lyapunov functions and rectangular set. In Section 4, we establish sufficient conditions for extinction of the disease. Finally, we provide a brief discussion and summarize the main results obtained in this paper.

Here we present some basic theory in stochastic differential equations which are introduced in Ref. [30].

In general, consider the l-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \quad \text{for } t \geq t_0, \tag{1.3}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^l$. $B(t)$ stands for an n-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Denote by $C^{2,1}(\mathbb{R}^l \times [t_0, \infty]; \mathbb{R}_+)$ the family of all nonnegative functions $V(x, t)$ defined on $\mathbb{R}^l \times [t_0, \infty]$ such that they are continuously twice differentiable in x and once in t. The differential operator L of Eq. (1.3) is defined by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{l} f_i(x, t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{l}[g^T(x, t)g(x, t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^l \times [t_0, \infty]; \mathbb{R}_+)$, then

$$LV(x, t) = V_t(x, t) + V_x(x, t)f(x, t) + \frac{1}{2}trace[g^T(x, t)V_{xx}(x, t)g(x, t)],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_l})$, $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{l\times l}$. In view of Itô's formula, if $x(t) \in \mathbb{R}^l$, then

$$dV(x(t), t) = LV(x(t), t)dt + V_x(x(t), t)g(x(t), t)dB(t).$$

2. Existence and uniqueness of the positive solution

To study the dynamical behaviors of a population model, the first concern is whether the solution is global and positive. As we know, in order for a stochastic differential equation to have a unique global (i.e., no explosion at any finite time) solution, the coefficients of the system are generally required to satisfy the linear growth condition and local Lipschitz condition [30]. However, the coefficients of system (1.2) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of system (1.2) may explode at a finite time. In this section, motivated by the method in Ref. [31], we show that there is a unique global positive solution of system (1.2). We establish the following theorem.

Theorem 2.1. *For any initial value $(S(0), E(0), I(0)) \in \mathbb{R}^3_+$, there is a unique positive solution $(S(t), E(t), I(t))$ of system (1.2) on $t \geq 0$ and the solution will remain in $\mathbb{R}^3_+$ with probability one, that is to say, $(S(t), E(t), I(t)) \in \mathbb{R}^3_+$ for all $t \geq 0$ almost surely (a.s.).*

Proof. Since the coefficients of system (1.2) are locally Lipschitz continuous, then for any initial value $(S(0), E(0), I(0)) \in \mathbb{R}^3_+$ there is a unique local solution $(S(t), E(t), I(t))$ on $[0, \tau_e)$, where τ_e is the explosion time [30]. To show this solution is global, we only need to prove that $\tau_e = \infty$ a.s. First of all, we prove that $S(t)$, $E(t)$ and $I(t)$ do not explode to infinity in a finite time. Let $k_0 > 0$ be sufficiently large such that $S(0)$, $E(0)$ and $I(0)$ all lie within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : \min\{S(t), E(t), I(t)\} \leq \frac{1}{k} \text{ or } \max\{S(t), E(t), I(t)\} \geq k\right\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Obviously, τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$ a.s. and $(S(t), E(t), I(t)) \in \mathbb{R}^3_+$ a.s. For $t \geq 0$. That is to say, to complete the proof it needs to verify that $\tau_\infty = \infty$ a.s. If this assertion is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon.$$

Consequently there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \epsilon \quad \text{for all } k \geq k_1.$$

Define a C^2-function $V : \mathbb{R}^3_+ \to \mathbb{R}_+$ by

$$V(S,E,I) = \left(S - a - a\ln\frac{S}{a}\right) + (E - 1 - \ln E) + (I - 1 - \ln I),$$

where a is a positive constant to be determined later. The nonnegativity of this function can be seen from

$$u - 1 - \ln u \geq 0 \quad \text{for any } u > 0.$$

Let $k \geq k_0$ and $T > 0$ be arbitrary. Applying Itô's formula [30] to V yields

$$\mathrm{d}V(S,E,I) = LV\mathrm{d}t + \sigma_1(S-a)\mathrm{d}B_1(t) + \sigma_2(E-1)\mathrm{d}B_2(t) + \sigma_3(I-1)\mathrm{d}B_3(t),$$

where $LV : \mathbb{R}^3_+ \to \mathbb{R}_+$ is defined by

$$\begin{aligned}
LV(S,E,I) \;&=\; \Lambda - \beta cSI - \mu S + r_1E + r_2I - \frac{\Lambda a}{S} + \beta caI + \mu a - \frac{ar_1E}{S} \\
&\quad - \frac{ar_2I}{S} + \beta cSI - (\mu + k + r_1)E - \frac{\beta cSI}{E} + \mu + k + r_1 + \frac{1}{2}\sigma_2^2 \\
&\quad + kE - (\mu + d + r_2)I - \frac{kE}{I} + \mu + d + r_2 + \frac{1}{2}\sigma_3^2 \\
&= -\mu S - \mu E + [ac\beta - (\mu + d)]I - \frac{\Lambda a}{S} - \frac{ar_1E}{S} - \frac{ar_2I}{S} - \frac{\beta cSI}{E} \\
&\quad - \frac{kE}{I} + \mu a + \Lambda + 2\mu + k + d + r_1 + r_2 + \frac{1}{2}a\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 \\
&\leq [ac\beta - (\mu + d)]I + \mu a + \Lambda + 2\mu + k + d + r_1 + r_2 + \frac{1}{2}a\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2.
\end{aligned}$$

Choose $a = \frac{\mu+d}{c\beta}$ such that $ac\beta - (\mu + d) = 0$, then we get

$$\begin{aligned}
LV(S,E,I) \;&\leq\; \mu a + \Lambda + 2\mu + k + d + r_1 + r_2 + \frac{1}{2}a\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 \\
&:= K,
\end{aligned}$$

where K is a positive constant. The remainder of the proof is similar to Theorem 3.1 of Mao et al. [31] and hence is omitted. This ends the proof.

3. Positive recurrence and ergodic properties of system (1.2)

When considering epidemic dynamical systems, we are interested in when the disease will persist and prevail in a population. In the deterministic models, the problem can be solved by showing that the endemic equilibrium of the corresponding model is a global attractor or is globally asymptotically stable. But for system (1.2), there is no endemic equilibrium. In this section, based on the theory of Has'minskii [32], we show that there exists an ergodic stationary distribution, which reveals that the disease will persist.

Here we present some theory about the stationary distribution (see Has'minskii [32]).

Let $X(t)$ be a homogeneous Markov process in E_l (E_l represents l-dimensional Euclidean space), and be described by the following stochastic differential equation

$$\mathrm{d}X(t) = b(X)\mathrm{d}t + \sum_{r=1}^{k} g_r(X)\mathrm{d}B_r(t).$$

The diffusion matrix is defined as follows

$$A(x) = (a_{ij}(x)), \qquad a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

Lemma 3.1 ([32]). *The Markov process $X(t)$ has a unique ergodic stationary distribution $\mu(\cdot)$ if there exists a bounded domain $U \subset E_l$ with regular boundary Γ and*

A_1: *there is a positive number M such that $\sum_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j \geq M|\xi|^2$, $x \in U$, $\xi \in \mathbb{R}^l$.*

A_2: *there exists a nonnegative C^2-function V such that LV is negative for any $E_l \setminus U$. Then*

$$\mathbb{P}_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))\mathrm{d}t = \int_{E_l} f(x)\mu(\mathrm{d}x)\right\} = 1$$

for all $x \in E_l$, where $f(\cdot)$ is a function integrable with respect to the measure μ.

Define a parameter

$$R_0^s := \frac{\beta\Lambda ck}{\left(\mu+\frac{\sigma_1^2}{2}\right)\left(\mu+k+r_1+\frac{\sigma_2^2}{2}\right)\left(\mu+d+r_2+\frac{\sigma_3^2}{2}\right)}.$$

Theorem 3.1. *Assume that $R_0^s > 1$, then for any initial value $(S(0), E(0), I(0)) \in \mathbb{R}_+^3$, there exists a stationary distribution $\mu(\cdot)$ for system (1.2) and the ergodicity holds.*

Proof. The diffusion matrix of system (1.2) is given by

$$A = \begin{pmatrix} \sigma_1^2S^2 & 0 & 0 \\ 0 & \sigma_2^2E^2 & 0 \\ 0 & 0 & \sigma_3^2I^2 \end{pmatrix}.$$

Choosing $M = \min_{(S,E,I)\in\overline{D}_\sigma}\{\sigma_1^2S^2, \sigma_2^2E^2, \sigma_3^2I^2\}$, we obtain

$$\sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j = \sigma_1^2S^2\xi_1^2 + \sigma_2^2E^2\xi_2^2 + \sigma_3^2I^2\xi_3^2 \geq M|\xi|^2, \quad (S, E, I) \in \overline{D}_\sigma,\ \xi \in \mathbb{R}^3.$$

Then the condition A_1 in Lemma 3.1 is satisfied.

Define a C^2-function $\widetilde{V} : \mathbb{R}_+^3 \to \mathbb{R}$ as follows

$$\begin{aligned}\widetilde{V}(S, E, I) &= M\left(-\ln S - c_1\ln E - c_2\ln I + \frac{\beta c}{\mu+d+d_2}I\right) - \ln S - \ln I + \frac{1}{m+1}(S+E+I)^{m+1} \\ &:= MV_1 + V_2 + V_3 + V_4,\end{aligned}$$

where m is a positive constant satisfying

$$\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) > 0,$$

$$c_1 = \frac{\mu+\frac{\sigma_1^2}{2}}{\mu+k+r_1+\frac{\sigma_2^2}{2}}, \qquad c_2 = \frac{\mu+\frac{\sigma_1^2}{2}}{\mu+d+r_2+\frac{\sigma_3^2}{2}},$$

and $M > 0$ satisfies the following condition

$$-M\lambda + C \leq -2, \tag{3.1}$$

where

$$\lambda = 3\left[\frac{\beta\Lambda ck\left(\mu+\frac{\sigma_1^2}{2}\right)^2}{\left(\mu+k+r_1+\frac{\sigma_2^2}{2}\right)\left(\mu+d+r_2+\frac{\sigma_3^2}{2}\right)}\right]^{\frac{1}{3}} - 3\left(\mu+\frac{\sigma_1^2}{2}\right) = 3\left(\mu+\frac{\sigma_1^2}{2}\right)\left[(R_0^s)^{\frac{1}{3}} - 1\right] > 0,$$

$$B = \sup_{(S,E,I)\in\mathbb{R}_+^3}\left\{\Lambda(S+E+I)^m - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right](S+E+I)^{m+1}\right\} < \infty$$

and

$$C = \sup_{(S,E,I)\in\mathbb{R}_+^3}\left\{-\frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right](S^{m+1}+E^{m+1}+I^{m+1}) + \beta cI + B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2\right\}.$$

It is easy to check that

$$\liminf_{k\to\infty,(S,E,I)\in\mathbb{R}_+^3\setminus U_k} \widetilde{V}(S, E, I) = +\infty,$$

where $U_k = (\frac{1}{k}, k) \times (\frac{1}{k}, k) \times (\frac{1}{k}, k)$. Moreover, $\widetilde{V}(S, E, I)$ is a continuous function. Thereby $\widetilde{V}(S, E, I)$ must have a minimum point (S_0, E_0, I_0) in the interior of $\mathbb{R}^3_+$. Then we define a nonnegative C^2-function $V : \mathbb{R}^3_+ \to \mathbb{R}_+$ as follows

$$V(S, E, I) = \widetilde{V}(S, E, I) - \widetilde{V}(S_0, E_0, I_0).$$

The differential operator L acting on the function V_1 leads to

$$\begin{aligned}
LV_1 &= -\left(\frac{\Lambda}{S} + \frac{r_1 E}{S} + \frac{r_2 I}{S} + \frac{c_1\beta cSI}{E} + \frac{c_2 kE}{I}\right) + \frac{\beta ckE}{\mu + d + r_2} + \mu + c_1(\mu + k + r_1) \\
&\quad + c_2(\mu + d + r_2) + \frac{\sigma_1^2 + c_1\sigma_2^2 + c_2\sigma_3^2}{2} \\
&\le -\left(\frac{\Lambda}{S} + \frac{c_1\beta cSI}{E} + \frac{c_2 kE}{I}\right) + \frac{\beta ckE}{\mu + d + r_2} + \mu + c_1(\mu + k + r_1) + c_2(\mu + d + r_2) + \frac{\sigma_1^2 + c_1\sigma_2^2 + c_2\sigma_3^2}{2} \\
&\le -3(c_1c_2\Lambda\beta ck)^{\frac{1}{3}} + \mu + c_1(\mu + k + r_1) + c_2(\mu + d + r_2) + \frac{\sigma_1^2 + c_1\sigma_2^2 + c_2\sigma_3^2}{2} + \frac{\beta ckE}{\mu + d + r_2} \\
&= -3\left[\frac{\Lambda\beta ck\left(\mu + \frac{\sigma_1^2}{2}\right)^2}{\left(\mu + k + r_1 + \frac{\sigma_2^2}{2}\right)\left(\mu + d + r_2 + \frac{\sigma_3^2}{2}\right)}\right]^{\frac{1}{3}} + 3\left(\mu + \frac{\sigma_1^2}{2}\right) + \frac{\beta ckE}{\mu + d + r_2} \\
&:= -\lambda + \frac{\beta ckE}{\mu + d + r_2}.
\end{aligned}$$

Similarly

$$\begin{aligned}
LV_2 &= -\frac{\Lambda}{S} + \beta cI - \frac{r_1 E}{S} - \frac{r_2 I}{S} + \mu + \frac{1}{2}\sigma_1^2, \\
LV_3 &= -\frac{kE}{I} + \mu + d + r_2 + \frac{1}{2}\sigma_3^2
\end{aligned}$$

and

$$\begin{aligned}
LV_4 &= (S + E + I)^m[\Lambda - \mu S - \mu E - (\mu + d)I] + \frac{m}{2}(S + E + I)^{m-1}(\sigma_1^2 S^2 + \sigma_2^2 E^2 + \sigma_3^2 I^2) \\
&\le (S + E + I)^m[\Lambda - \mu(S + E + I)] + \frac{m}{2}(S + E + I)^{m+1}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2) \\
&= \Lambda(S + E + I)^m - \left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S + E + I)^{m+1} \\
&\le B - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S + E + I)^{m+1} \\
&\le B - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{m+1} + E^{m+1} + I^{m+1}),
\end{aligned}$$

where

$$B = \sup_{(S,E,I)\in\mathbb{R}^3_+}\left\{\Lambda(S + E + I)^m - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S + E + I)^{m+1}\right\} < \infty.$$

Consequently, we obtain

$$\begin{aligned}
LV &\le -M\lambda + \frac{M\beta ckE}{\mu + d + r_2} - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{m+1} + E^{m+1} + I^{m+1}) \\
&\quad - \frac{\Lambda}{S} - \frac{r_1 E}{S} - \frac{r_2 I}{S} - \frac{kE}{I} + \beta cI + B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 \\
&\le -M\lambda + \frac{M\beta ckE}{\mu + d + r_2} - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{m+1} + E^{m+1} + I^{m+1}) \\
&\quad - \frac{\Lambda}{S} - \frac{kE}{I} + \beta cI + B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2.
\end{aligned}$$

Define a bounded closed set

$$D_\epsilon = \left\{(S,E,I) \in \mathbb{R}^3_+ : \epsilon \le S \le \frac{1}{\epsilon}, \epsilon \le E \le \frac{1}{\epsilon}, \epsilon^2 \le I \le \frac{1}{\epsilon^2}\right\},$$

where $\epsilon > 0$ is a sufficiently small number. In the set $\mathbb{R}^3_+ \setminus D_\epsilon$, we can choose ϵ sufficiently small such that

$$-\frac{\Lambda}{\epsilon} + D \le -1, \tag{3.2}$$

$$-M\lambda + \frac{M\beta ck\epsilon}{\mu+d+r_2} + C \le -1, \tag{3.3}$$

$$-\frac{k}{\epsilon} + D \le -1, \tag{3.4}$$

$$-\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\frac{1}{\epsilon^{m+1}} + F \le -1, \tag{3.5}$$

$$-\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\frac{1}{\epsilon^{m+1}} + G \le -1, \tag{3.6}$$

$$-\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\frac{1}{\epsilon^{2m+2}} + H \le -1, \tag{3.7}$$

where D, F, G and H are positive constants which can be found from the following inequations (3.9), (3.13), (3.15) and (3.17). Note that for sufficiently small ϵ, condition (3.3) holds due to (3.1). For the sake of convenience, we divide $\mathbb{R}^3_+ \setminus D_\epsilon$ into six domains,

$$D_1 = \{(S,E,I) \in \mathbb{R}^3_+, 0 < S < \epsilon\}, \qquad D_2 = \{(S,E,I) \in \mathbb{R}^3_+, 0 < E < \epsilon\},$$

$$D_3 = \{(S,E,I) \in \mathbb{R}^3_+, E \ge \epsilon, 0 < I < \epsilon^2\}, \qquad D_4 = \left\{(S,E,I) \in \mathbb{R}^3_+, S > \frac{1}{\epsilon}\right\},$$

$$D_5 = \left\{(S,E,I) \in \mathbb{R}^3_+, E > \frac{1}{\epsilon}\right\}, \qquad D_6 = \left\{(S,E,I) \in \mathbb{R}^3_+, I > \frac{1}{\epsilon^2}\right\}.$$

Next we will show that $LV(S,E,I) \le -1$ on $\mathbb{R}^3_+ \setminus D_\epsilon$, which is equivalent to proving it on the above six domains.

Case 1. If $(S,E,I) \in D_1$, one can see that

$$\begin{aligned} LV \le& -\frac{\Lambda}{S} + \frac{M\beta ckE}{\mu+d+r_2} + \beta cI - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{m+1} + E^{m+1} + I^{m+1}) \\ &+ B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 \\ \le& -\frac{\Lambda}{S} + D \le -\frac{\Lambda}{\epsilon} + D, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} D =& \sup_{(S,E,I)\in\mathbb{R}^3_+}\left\{-\frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{m+1} + E^{m+1} + I^{m+1}) + \frac{M\beta ckE}{\mu+d+r_2}\right. \\ &\left. + \beta cI + B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2\right\}. \end{aligned} \tag{3.9}$$

According to (3.2), we have $LV \le -1$ for all $(S,E,I) \in D_1$.

Case 2. If $(S,E,I) \in D_2$, we get

$$\begin{aligned} LV \le& -M\lambda + \frac{M\beta ckE}{\mu+d+r_2} - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{m+1} + E^{m+1} + I^{m+1}) \\ &+ \beta cI + B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 \\ \le& -M\lambda + \frac{M\beta ckE}{\mu+d+r_2} + C \le -M\lambda + \frac{M\beta ck\epsilon}{\mu+d+r_2} + C, \end{aligned} \tag{3.10}$$

which together with (3.3) yields that for sufficiently small ϵ,

$$LV \leq -1 \quad \text{for any } (S, E, I) \in D_2.$$

Case 3. If $(S, E, I) \in D_3$, we derive

$$\begin{aligned} LV &\leq -\frac{kE}{I} + \frac{M\beta ckE}{\mu+d+r_2} - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{m+1} + E^{m+1} + I^{m+1}) \\ &\quad + \beta cI + B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 \\ &\leq -\frac{kE}{I} + D \leq -\frac{k}{\epsilon} + D. \end{aligned} \tag{3.11}$$

Combining with (3.4), one can get that for sufficiently small ϵ,

$$LV \leq -1 \quad \text{on } D_3.$$

Case 4. If $(S, E, I) \in D_4$, one can obtain

$$\begin{aligned} LV &\leq -\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]S^{m+1} - \frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]S^{m+1} \\ &\quad - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](E^{m+1} + I^{m+1}) + \frac{M\beta ckE}{\mu+d+r_2} + \beta cI + B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 \\ &\leq -\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]S^{m+1} + F \\ &\leq -\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\frac{1}{\epsilon^{m+1}} + F, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} F = \sup_{(S,E,I)\in\mathbb{R}_+^3} &\left\{-\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]S^{m+1} - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\right. \\ &\left.\times (E^{m+1} + I^{m+1}) + \frac{M\beta ckE}{\mu+d+r_2} + \beta cI + B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2\right\}. \end{aligned} \tag{3.13}$$

In view of (3.5), we can conclude that $LV \leq -1$ on D_4.

Case 5. If $(S, E, I) \in D_5$, one can see that

$$\begin{aligned} LV &\leq -\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]E^{m+1} - \frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]E^{m+1} \\ &\quad - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](S^{m+1} + I^{m+1}) + \frac{M\beta ckE}{\mu+d+r_2} + \beta cI + B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2 \\ &\leq -\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]E^{m+1} + G \\ &\leq -\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\frac{1}{\epsilon^{m+1}} + G, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} G = \sup_{(S,E,I)\in\mathbb{R}_+^3} &\left\{-\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]E^{m+1} - \frac{1}{2}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\right. \\ &\left.\times (S^{m+1} + I^{m+1}) + \frac{M\beta ckE}{\mu+d+r_2} + \beta cI + B + 2\mu + d + r_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_3^2\right\}. \end{aligned} \tag{3.15}$$

Therefore, in view of (3.6), we get $LV \leq -1$ in this domain.

Case 6. If $(S, E, I) \in D_6$, we obtain

$$LV \leq -\frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]I^{m+1} - \frac{1}{4}\left[\mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]I^{m+1}$$

$$-\frac{1}{2}\left[\mu-\frac{m}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right](S^{m+1}+E^{m+1})+\frac{M\beta ckE}{\mu+d+r_2}+\beta cI+B+2\mu+d+r_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_3^2$$
$$\le -\frac{1}{4}\left[\mu-\frac{m}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right]I^{m+1}+H$$
$$\le -\frac{1}{4}\left[\mu-\frac{m}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right]\frac{1}{\epsilon^{2m+2}}+H, \tag{3.16}$$

where

$$H=\sup_{(S,E,I)\in\mathbb{R}_+^3}\left\{-\frac{1}{4}\left[\mu-\frac{m}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right]I^{m+1}-\frac{1}{2}\left[\mu-\frac{m}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right]\right.$$
$$\left.\times(S^{m+1}+E^{m+1})+\frac{M\beta ckE}{\mu+d+r_2}+\beta cI+B+2\mu+d+r_2+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_3^2\right\}. \tag{3.17}$$

Together with (3.7), one can obtain that $LV\le -1$ in this domain.

Clearly, one can see from (3.8), (3.10)–(3.12), (3.14) and (3.16) that for a sufficiently small ϵ,

$$LV\le -1 \quad \text{for all } (S,E,I)\in\mathbb{R}_+^3\setminus D_\epsilon.$$

Hence A_2 in Lemma 3.1 is satisfied. According to Lemma 3.1, we can obtain that system (1.2) is ergodic and has a unique stationary distribution. This completes the proof.

4. Extinction of the disease

In this section, we shall discuss the extinction of the disease. For convenience and simplicity in the following investigation, if f is an integrable function on $[0,\infty)$, define

$$\langle f\rangle=\frac{1}{t}\int_0^t f(s)\mathrm{d}s.$$

Lemma 4.1. *Let $(S(t),E(t),I(t))$ be the solution of system (1.2) with any initial value $(S(0),E(0),I(0))\in\mathbb{R}_+^3$, then*

$$\lim_{t\to\infty}\frac{S(t)}{t}=0,\qquad \lim_{t\to\infty}\frac{E(t)}{t}=0,\qquad \lim_{t\to\infty}\frac{I(t)}{t}=0 \quad a.s. \tag{4.1}$$

Furthermore, if $\mu>\frac{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2}{2}$, then

$$\lim_{t\to\infty}\frac{\int_0^t S(r)\mathrm{d}B_1(r)}{t}=0,\qquad \lim_{t\to\infty}\frac{\int_0^t E(r)\mathrm{d}B_2(r)}{t}=0,\qquad \lim_{t\to\infty}\frac{\int_0^t I(r)\mathrm{d}B_3(r)}{t}=0 \quad a.s. \tag{4.2}$$

Lemma 4.1 can be proved by using the same approaches as that in Lemmas 2.1 and 2.2 of Ref. [33], so we omit it here. Define

$$\widehat{R}_0=\frac{2k\beta c\Lambda(\mu+k+r_1)}{\mu\left[(\mu+k+r_1)^2\left(\mu+d+r_2+\frac{1}{2}\sigma_3^2\right)\wedge\frac{1}{2}k^2\sigma_2^2\right]}.$$

Theorem 4.1. *Let $(S(t),E(t),I(t))$ be the solution of system (1.2) with any initial value $(S(0),E(0),I(0))\in\mathbb{R}_+^3$. If $\mu>\frac{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2}{2}$ and $\widehat{R}_0<1$, then the solution $(S(t),E(t),I(t))$ of system (1.2) has the following property*

$$\lim_{t\to\infty}\langle S\rangle=\frac{\Lambda}{\mu}\quad a.s.$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\ln[kE(t)+(\mu+k+r_1)I(t)]$$
$$\le\frac{k\beta c\Lambda}{\mu(\mu+k+r_1)}-\frac{1}{2(\mu+k+r_1)^2}\left[(\mu+k+r_1)^2\left(\mu+d+r_2+\frac{1}{2}\sigma_3^2\right)\wedge\frac{1}{2}k^2\sigma_2^2\right]<0\quad a.s.,$$

Proof. Let $Q(t) = kE(t) + (\mu + k + r_1)I(t)$. Applying Itô's formula, we have

$$\begin{aligned}
\mathrm{d}\ln Q(t) &= \left\{\frac{k\beta cSI - (\mu+k+r_1)(\mu+d+r_2)I}{kE + (\mu+k+r_1)I} - \frac{k^2\sigma_2^2E^2 + (\mu+k+r_1)^2\sigma_3^2I^2}{2[kE + (\mu+k+r_1)I]^2}\right\}\mathrm{d}t \\
&\quad + \frac{k\sigma_2 E}{kE + (\mu+k+r_1)I}\mathrm{d}B_2(t) + \frac{(\mu+k+r_1)\sigma_3 I}{kE + (\mu+k+r_1)I}\mathrm{d}B_3(t) \\
&\le \frac{k\beta cS}{\mu+k+r_1}\mathrm{d}t - \frac{1}{[kE + (\mu+k+r_1)I]^2}\left\{\left[(\mu+k+r_1)^2(\mu+d+r_2)\right.\right. \\
&\quad \left.\left. + \frac{1}{2}(\mu+k+r_1)^2\sigma_3^2\right]I^2 + \frac{1}{2}k^2\sigma_2^2E^2\right\}\mathrm{d}t + \frac{k\sigma_2 E}{kE + (\mu+k+r_1)I}\mathrm{d}B_2(t) \\
&\quad + \frac{(\mu+k+r_1)\sigma_3 I}{kE + (\mu+k+r_1)I}\mathrm{d}B_3(t) \\
&\le \frac{k\beta cS}{\mu+k+r_1}\mathrm{d}t - \frac{1}{2(\mu+k+r_1)^2}\left[(\mu+k+r_1)^2\left(\mu+d+r_2+\frac{1}{2}\sigma_3^2\right) \wedge \frac{1}{2}k^2\sigma_2^2\right]\mathrm{d}t \\
&\quad + \frac{k\sigma_2 E}{kE + (\mu+k+r_1)I}\mathrm{d}B_2(t) + \frac{(\mu+k+r_1)\sigma_3 I}{kE + (\mu+k+r_1)I}\mathrm{d}B_3(t).
\end{aligned} \tag{4.3}$$

It follows from system (1.2) that

$$\begin{aligned}
\mathrm{d}(S(t)+E(t)+I(t)) &= [\Lambda - \mu S(t) - \mu E(t) - (\mu+d)I(t)]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t) \\
&\quad + \sigma_2 E(t)\mathrm{d}B_2(t) + \sigma_3 I(t)\mathrm{d}B_3(t) \\
&\le [\Lambda - \mu(S(t)+E(t)+I(t))]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t) + \sigma_2 E(t)\mathrm{d}B_2(t) + \sigma_3 I(t)\mathrm{d}B_3(t).
\end{aligned} \tag{4.4}$$

Integrating (4.4) from 0 to t and together with (4.2), we obtain

$$\limsup_{t\to\infty}\langle S(t)+E(t)+I(t)\rangle \le \frac{\Lambda}{\mu} \quad \text{a.s.} \tag{4.5}$$

Integrating (4.3) from 0 to t on both sides, together with (4.5), and note that $\widehat{R}_0 < 1$, one can see that

$$\limsup_{t\to\infty}\frac{\ln Q(t)}{t} \le \frac{k\beta c\Lambda}{\mu(\mu+k+r_1)} - \frac{1}{2(\mu+k+r_1)^2}\left[(\mu+k+r_1)^2\left(\mu+d+r_2+\frac{1}{2}\sigma_3^2\right) \wedge \frac{1}{2}k^2\sigma_2^2\right] < 0 \quad \text{a.s.},$$

which shows that

$$\lim_{t\to\infty} E(t) = 0, \qquad \lim_{t\to\infty} I(t) = 0 \quad \text{a.s.} \tag{4.6}$$

On the other hand, in view of (4.4), one can get that

$$\begin{aligned}
&\frac{S(t)-S(0)}{t} + \frac{E(t)-E(0)}{t} + \frac{I(t)-I(0)}{t} \\
&\quad = \Lambda - \mu\langle S\rangle - \mu\langle E\rangle - (\mu+d)\langle I\rangle + \frac{\sigma_1\int_0^t S(r)\mathrm{d}B_1(r)}{t} + \frac{\sigma_2\int_0^t E(r)\mathrm{d}B_2(r)}{t} + \frac{\sigma_3\int_0^t I(r)\mathrm{d}B_3(r)}{t},
\end{aligned}$$

which together with (4.1), (4.2) and (4.6), implies that

$$\lim_{t\to\infty}\langle S\rangle = \frac{\Lambda}{\mu} \quad \text{a.s.}$$

This completes the proof.

5. Concluding remarks

This paper is concerned with the dynamics of a stochastic tuberculosis model with constant recruitment and varying total population size. By constructing suitable stochastic Lyapunov functions, we establish sufficient conditions for the existence of an ergodic stationary distribution as well as extinction of the disease described by system (1.2).

There are some work deserving further consideration. In Theorem 4.1, the condition $\mu > \frac{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2}{2}$ is essential, what is the dynamics of system (1.2) in the case of $\mu < \frac{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2}{2}$? We try to research this problem. However, there are some technical obstacles that cannot be conquered at the present stage. On the other hand, one can propose some more realistic but complex models, such as considering the effects of impulsive perturbations on system (1.2). We leave these investigations for future work.

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Appendix. Construction of stochastic model (1.2)

Firstly, we consider a discrete time Markov chain. For a fixed time increment $\Delta t > 0$, we define a process $X^{(\Delta t)}(t) = (S^{(\Delta t)}(t), E^{(\Delta t)}(t), I^{(\Delta t)}(t))^T$ for $t = 0, \Delta t, 2\Delta t, \ldots$, where $S^{(\Delta t)}(t)$, $E^{(\Delta t)}(t)$ and $I^{(\Delta t)}(t)$ denote the susceptible, exposed (infected but not infectious), and infected individuals, respectively. Let the initial value $X^{(\Delta t)}(0) = \vartheta \in \mathbb{R}^3_+$ which is deterministic, three sequences of random variables are denoted by $\{R_i^{(\Delta t)}(k)\}_{k=1}^{\infty}$, $i = 1, 2, 3$. Assume that these variables are jointly independent and that within each sequence the variables are identically distributed such that

$$\mathbb{E}R_i^{(\Delta t)}(k) = 0, \qquad \mathbb{E}[R_i^{(\Delta t)}(k)]^2 = \sigma_i^2 \Delta t, \qquad \mathbb{E}[R_i^{(\Delta t)}(k)]^4 = o(\Delta t) \tag{$*1$}$$

for $i = 1, 2, 3$ and $k = 1, 2, \ldots$, where $\sigma_i \geq 0$ $(i = 1, 2, 3)$ are constants that reflect the sizes of stochastic influences.

The variables $R_i^{(\Delta t)}(k)$ $(i = 1, 2, 3)$ are assumed to capture the effects of random influences on the susceptible, exposed and infected individuals during the period $[k\Delta t, (k+1)\Delta t)$, respectively. We also suppose that $S^{(\Delta t)}$, $E^{(\Delta t)}$ and $I^{(\Delta t)}$ change within that time period according to the deterministic Eq. (1.1) and, furthermore, in view of the random amounts $R_1^{(\Delta t)}(k)S^{(\Delta t)}(k\Delta t)$, $R_2^{(\Delta t)}(k)E^{(\Delta t)}(k\Delta t)$, $R_3^{(\Delta t)}(k)I^{(\Delta t)}(k\Delta t)$, respectively. Specifically, for $k = 1, 2, \ldots$, we have

$$\begin{aligned} S^{(\Delta t)}((k+1)\Delta t) &= S^{(\Delta t)}(k\Delta t) + R_1^{(\Delta t)}(k)S^{(\Delta t)}(k\Delta t) + \{\Lambda - \beta c S^{(\Delta t)}(k\Delta t)I^{(\Delta t)}(k\Delta t) - \mu S^{(\Delta t)}(k\Delta t) \\ &\quad + r_1 E^{(\Delta t)}(k\Delta t) + r_2 I^{(\Delta t)}(k\Delta t)\}\Delta t, \\ E^{(\Delta t)}((k+1)\Delta t) &= E^{(\Delta t)}(k\Delta t) + R_2^{(\Delta t)}(k)E^{(\Delta t)}(k\Delta t) + \{\beta c S^{(\Delta t)}(k\Delta t)I^{(\Delta t)}(k\Delta t) - (\mu + k + r_1)E^{(\Delta t)}(k\Delta t)\}\Delta t \end{aligned}$$

and

$$I^{(\Delta t)}((k+1)\Delta t) = I^{(\Delta t)}(k\Delta t) + R_3^{(\Delta t)}(k)I^{(\Delta t)}(k\Delta t) + \{kE^{(\Delta t)}(k\Delta t) - (\mu + d + r_2)I^{(\Delta t)}(k\Delta t)\}\Delta t.$$

Next, we will prove that $X^{(\Delta t)}(t)$ converges to a diffusion process as $\Delta t \to 0$. We only need to determine the drift coefficient and diffusion coefficient of the diffusion process. Let $\mathbb{P}^{(\Delta t)}(x, \mathrm{d}z)$ represent the transition probabilities of the homogeneous Markov chain $\{X^{(\Delta t)}(k\Delta t)\}_{k=1}^{\infty}$, that is

$$\mathbb{P}^{(\Delta t)}(x, Z) = Prob\{X^{(\Delta t)}((k+1)\Delta t) \in Z : X^{(\Delta t)}(k\Delta t) = x\}$$

for all $x = (S, E, I)^T \in \mathbb{R}^3_+$ and all Borel sets $Z \subset \mathbb{R}^3_+$. Let

$$F^{(\Delta t)}(x) = (f_1^{(\Delta t)}(x), f_2^{(\Delta t)}(x), f_3^{(\Delta t)}(x))^T$$

and

$$G^{(\Delta t)}(x) = (g_{ij}^{(\Delta t)}(x))_{3\times 3}$$

represent the drift coefficient and diffusion coefficient, respectively. It follows from $(*1)$ that

$$\begin{aligned} f_1^{(\Delta t)}(x) &= \frac{1}{\Delta t}\int (z_1 - S)\mathbb{P}^{(\Delta t)}(x, \mathrm{d}z) = \Lambda - \beta cSI - \mu S + r_1 E + r_2 I + \frac{S}{\Delta t}\mathbb{E}R_1^{(\Delta t)}(0) \\ &= \Lambda - \beta cSI - \mu S + r_1 E + r_2 I, &(*2) \\ f_2^{(\Delta t)}(x) &= \frac{1}{\Delta t}\int (z_2 - E)\mathbb{P}^{(\Delta t)}(x, \mathrm{d}z) = \beta cSI - (\mu + k + r_1)E + \frac{E}{\Delta t}\mathbb{E}R_2^{(\Delta t)}(0) \\ &= \beta cSI - (\mu + k + r_1)E, &(*3) \\ f_3^{(\Delta t)}(x) &= \frac{1}{\Delta t}\int (z_3 - I)\mathbb{P}^{(\Delta t)}(x, \mathrm{d}z) = kE - (\mu + d + r_2)I + \frac{I}{\Delta t}\mathbb{E}R_3^{(\Delta t)}(0) \\ &= kE - (\mu + d + r_2)I, &(*4) \\ g_{11}^{(\Delta t)}(x) &= \frac{1}{\Delta t}\int (z_1 - S)^2\mathbb{P}^{(\Delta t)}(x, \mathrm{d}z) \\ &= \frac{1}{\Delta t}\mathbb{E}[(\Lambda - \beta cSI - \mu S + r_1 E + r_2 I)\Delta t + R_1^{(\Delta t)}(0)S]^2 \\ &= [\Lambda - \beta cSI - \mu S + r_1 E + r_2 I]^2\Delta t + \frac{S^2}{\Delta t}\mathbb{E}[R_1^{(\Delta t)}(0)]^2 \\ &\quad + 2[\Lambda - \beta cSI - \mu S + r_1 E + r_2 I]S\mathbb{E}[R_1^{(\Delta t)}(0)] \end{aligned}$$

$$
\begin{aligned}
&= [\Lambda - \beta cSI - \mu S + r_1E + r_2I]^2\Delta t + \sigma_1^2S^2, \\
g_{22}^{(\Delta t)}(x) &= \frac{1}{\Delta t}\int (z_2 - E)^2\mathbb{P}^{(\Delta t)}(x, \mathrm{d}z) \\
&= \frac{1}{\Delta t}\mathbb{E}[(\beta cSI - (\mu + k + r_1)E)\Delta t + R_2^{(\Delta t)}(0)E]^2 \\
&= [\beta cSI - (\mu + k + r_1)E]^2\Delta t + \frac{E^2}{\Delta t}\mathbb{E}[R_2^{(\Delta t)}(0)]^2 + 2[\beta cSI - (\mu + k + r_1)E]E\mathbb{E}[R_2^{(\Delta t)}(0)] \\
&= [\beta cSI - (\mu + k + r_1)E]^2\Delta t + \sigma_2^2E^2
\end{aligned}
$$

and

$$
\begin{aligned}
g_{33}^{(\Delta t)}(x) &= \frac{1}{\Delta t}\int (z_3 - I)^2\mathbb{P}^{(\Delta t)}(x, \mathrm{d}z) \\
&= \frac{1}{\Delta t}\mathbb{E}[(kE - (\mu + d + r_2)I)\Delta t + R_3^{(\Delta t)}(0)I]^2 \\
&= [kE - (\mu + d + r_2)I]^2\Delta t + \frac{I^2}{\Delta t}\mathbb{E}[R_3^{(\Delta t)}(0)]^2 + 2[kE - (\mu + d + r_2)I]I\mathbb{E}[R_3^{(\Delta t)}(0)] \\
&= [kE - (\mu + d + r_2)I]^2\Delta t + \sigma_3^2I^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lim_{\Delta t\to 0^+}\sup_{\|x\|\le K}|g_{11}^{(\Delta t)}(x) - \sigma_1^2S^2| &= \lim_{\Delta t\to 0^+}\sup_{\|x\|\le K}|g_{22}^{(\Delta t)}(x) - \sigma_2^2E^2| \\
&= \lim_{\Delta t\to 0^+}\sup_{\|x\|\le K}|g_{33}^{(\Delta t)}(x) - \sigma_3^2I^2| = 0 \qquad (*5)
\end{aligned}
$$

for $K \in (0, \infty)$. Analogously, one can also conclude that

$$
\lim_{\Delta t\to 0^+}\sup_{\|x\|\le K}|g_{ij}^{(\Delta t)}(x)| = 0 \qquad (*6)
$$

for $i, j = 1, 2, 3$ and $i \neq j$. Moreover, from $(*1)$, one can see that for all $K \in (0, \infty)$,

$$
\lim_{\Delta t\to 0^+}\sup_{\|x\|\le K}\frac{1}{\Delta t}\int \|z - x\|^3 = \mathbb{P}^{(\Delta t)}(x, \mathrm{d}z) = 0. \qquad (*7)
$$

According to Imhof and Walcher [29], the definition of $X^{(\Delta t)}(t)$ can be extended to all $t \geq 0$ by setting $X^{(\Delta t)}(t) = X^{(\Delta t)}(k\Delta t)$ for $t \in [k\Delta t, (k+1)\Delta t)$. By Theorem 7.1 and Lemma 8.2 in Ref. [34], and $(*2)$–$(*7)$, we can deduce that as $\Delta t \to 0, X^{(\Delta t)}(t)$ converges weakly to the solution of stochastic differential equation (1.2) with the initial value $X(0) = \vartheta$, provided its unique solution exists.

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