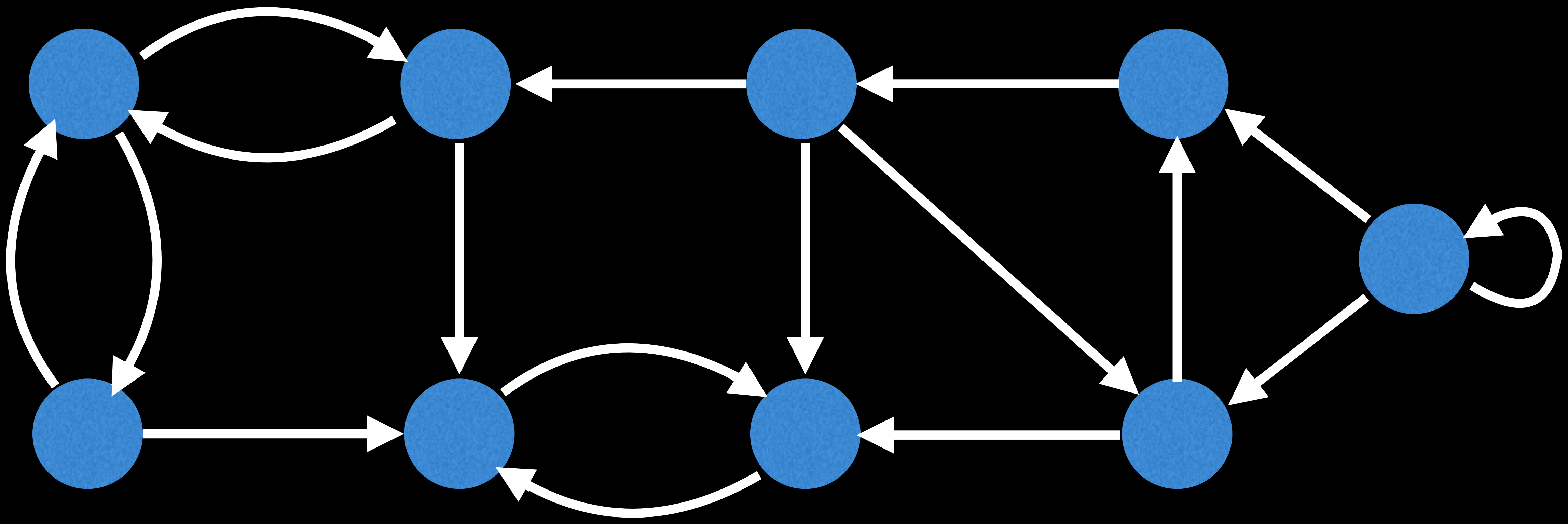
Tarjan’s Algorithm for Finding Strongly Connected Components

William Fiset

What are SCCs?

Strongly Connected Components (SCCs) can be thought of as **self-contained cycles** within a **directed graph** where every vertex in a given cycle can reach every other vertex in the same cycle.



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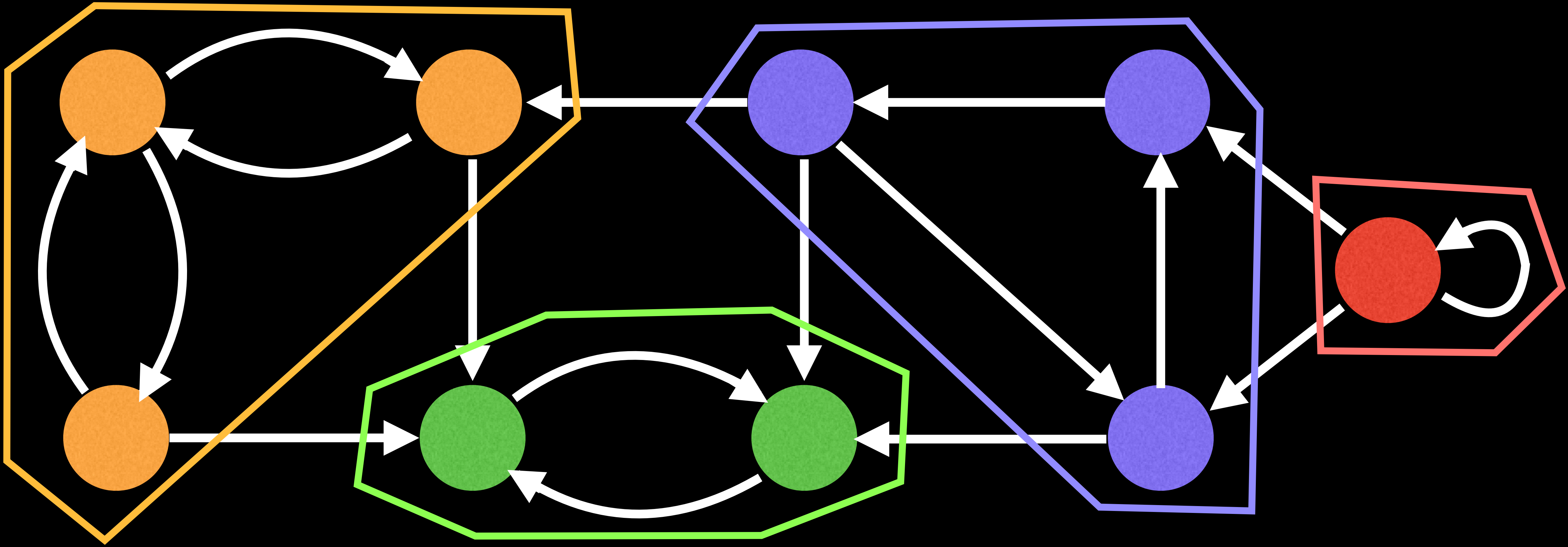
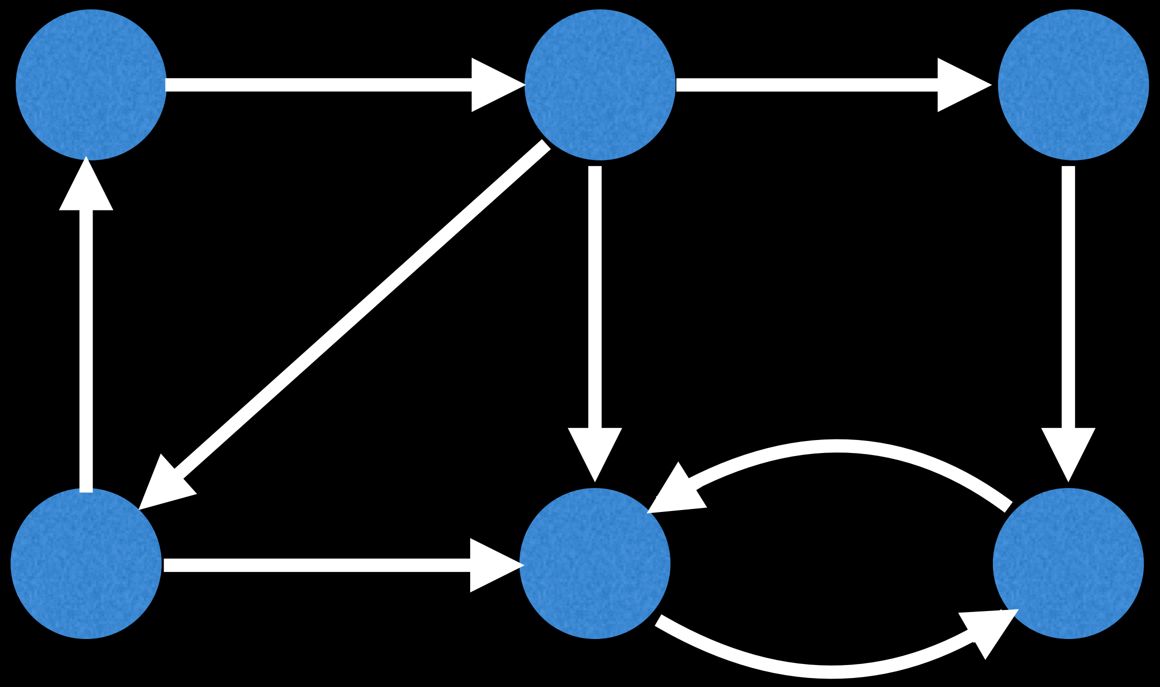


Low-Link Values

The **low-link** value of a node is the smallest [lowest] node id reachable from that node when doing a DFS (including itself).



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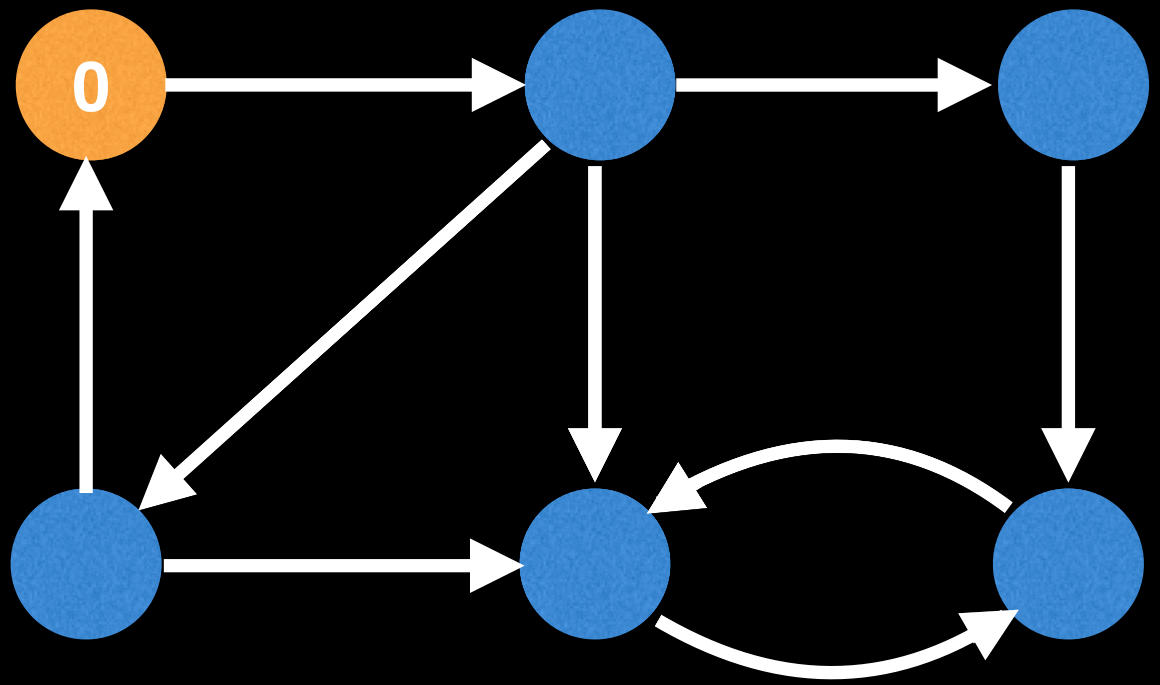

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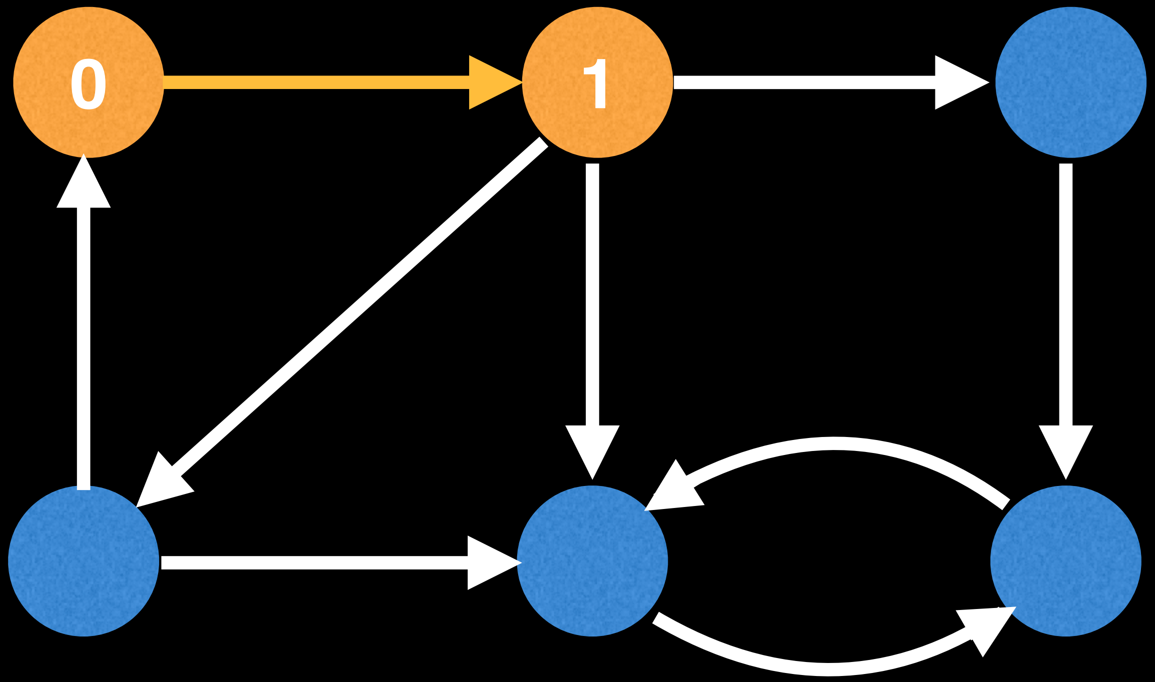

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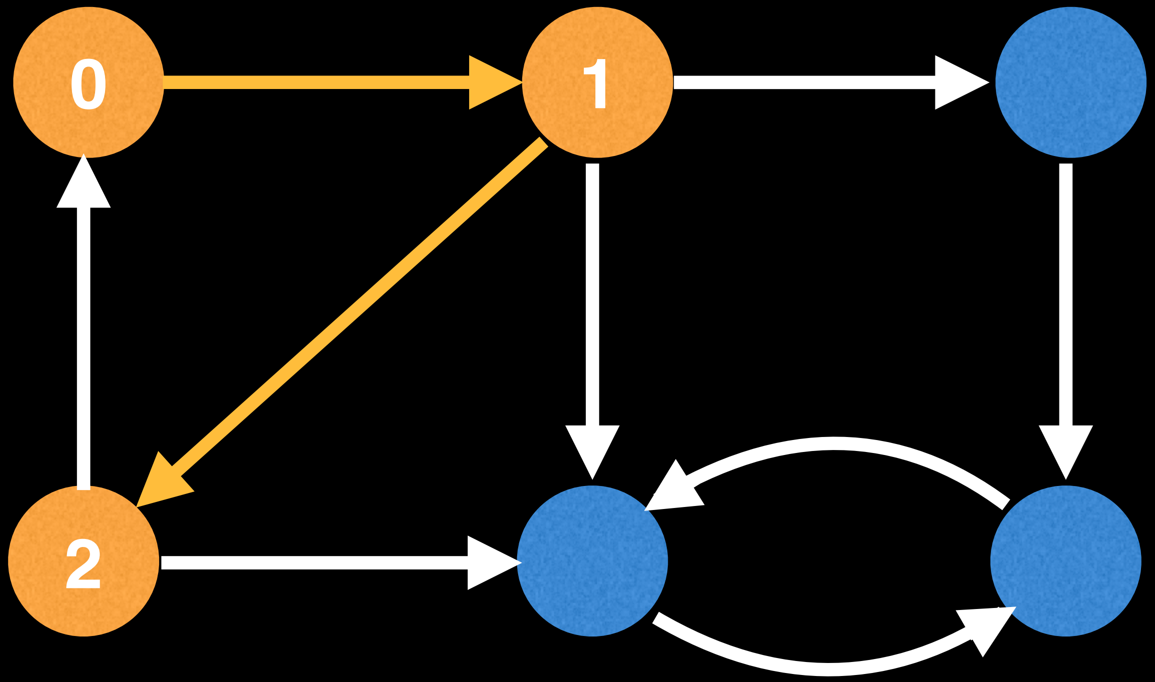

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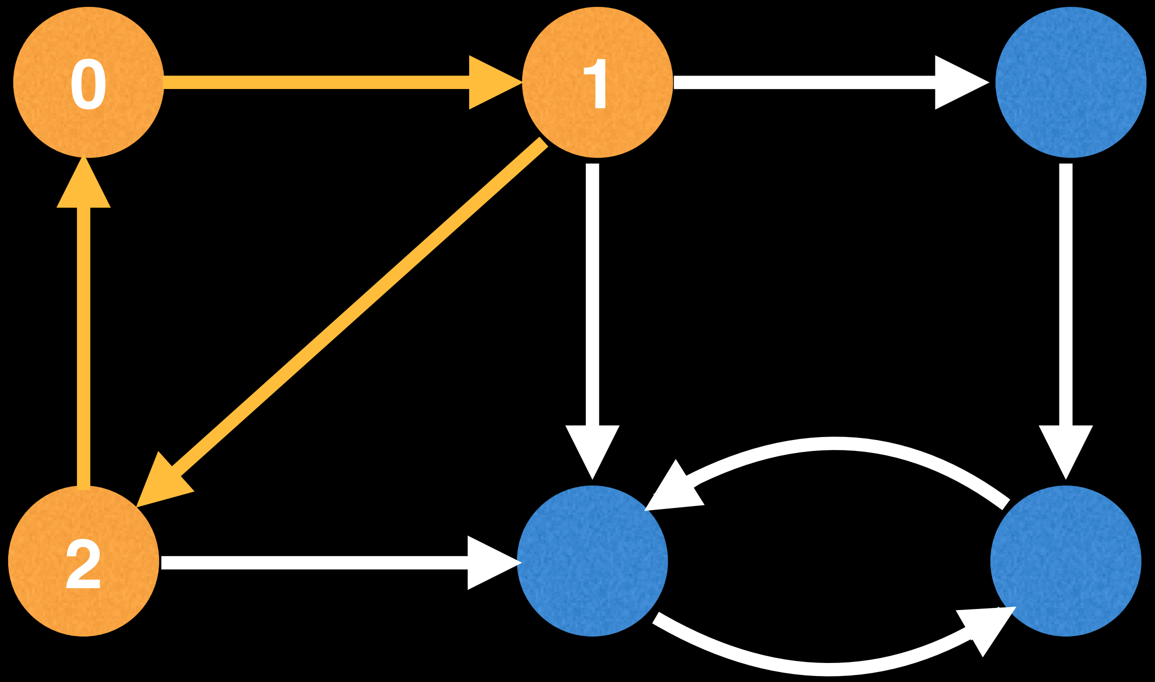

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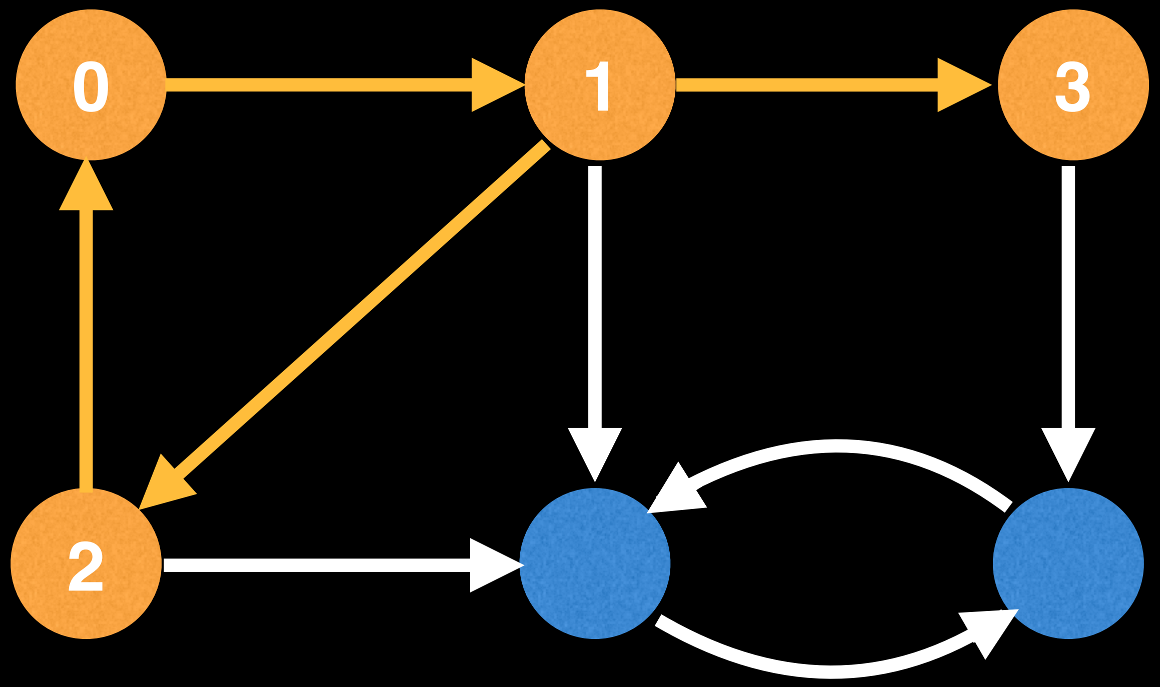

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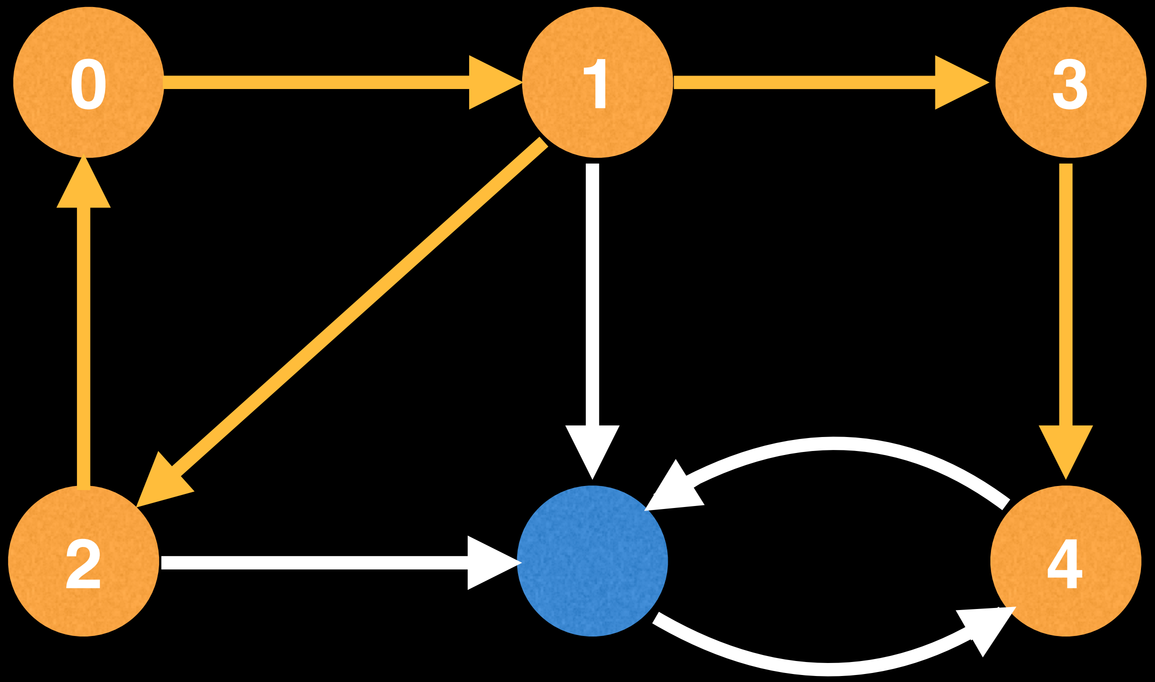

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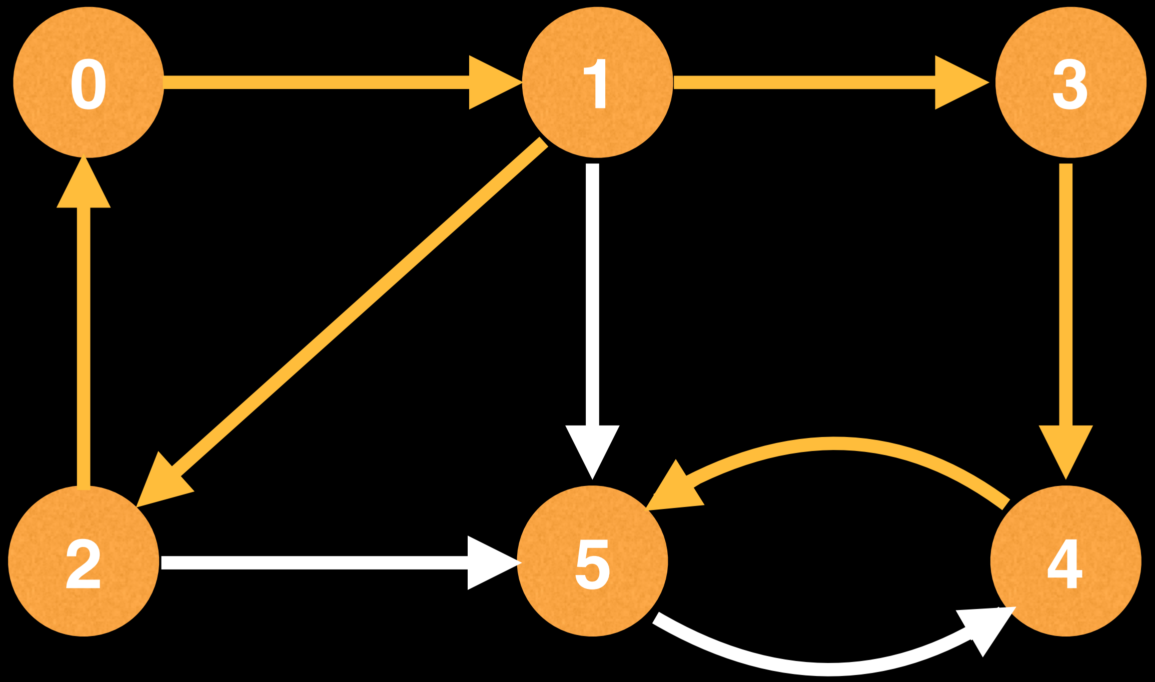

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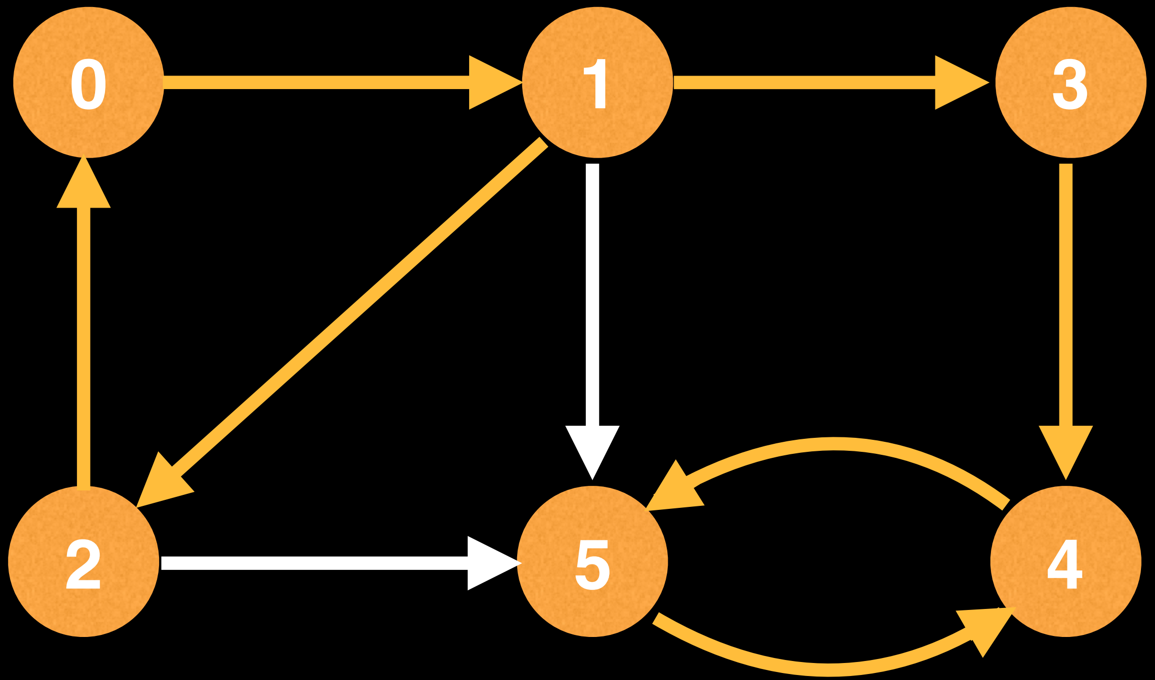

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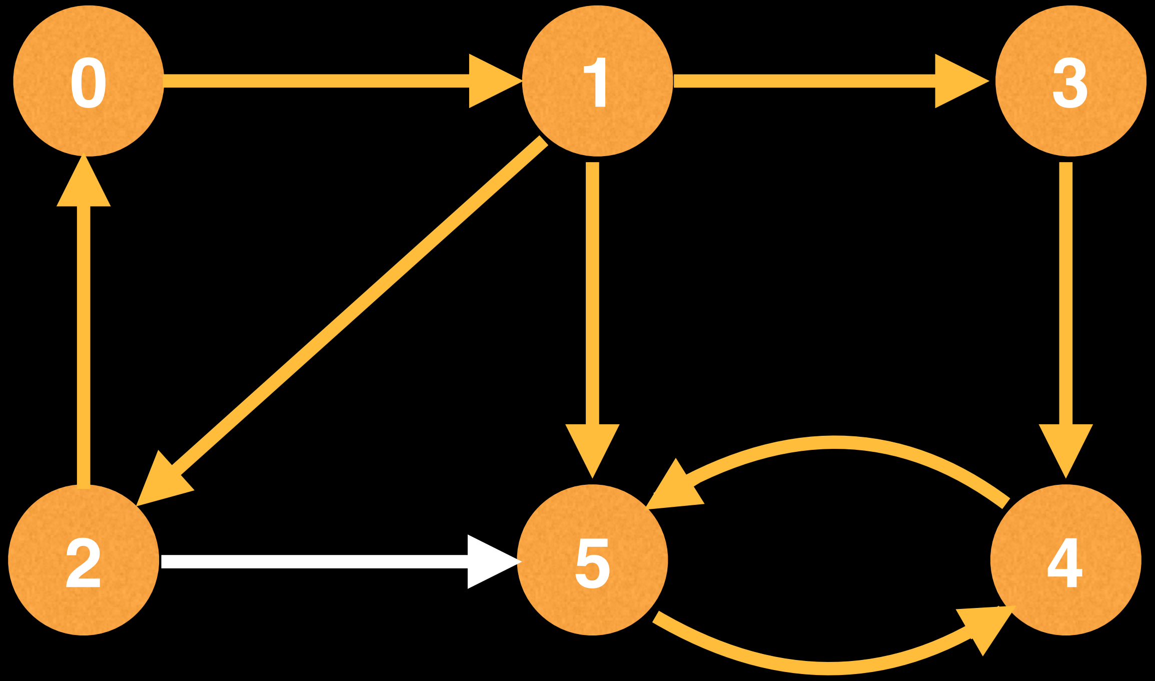

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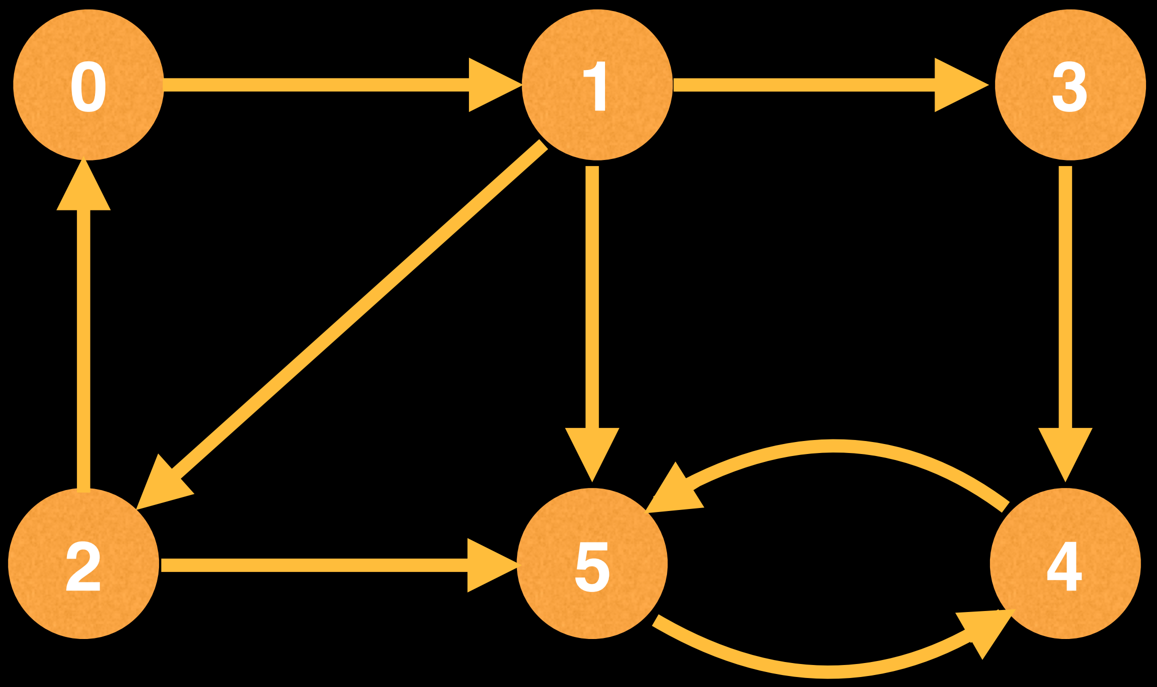

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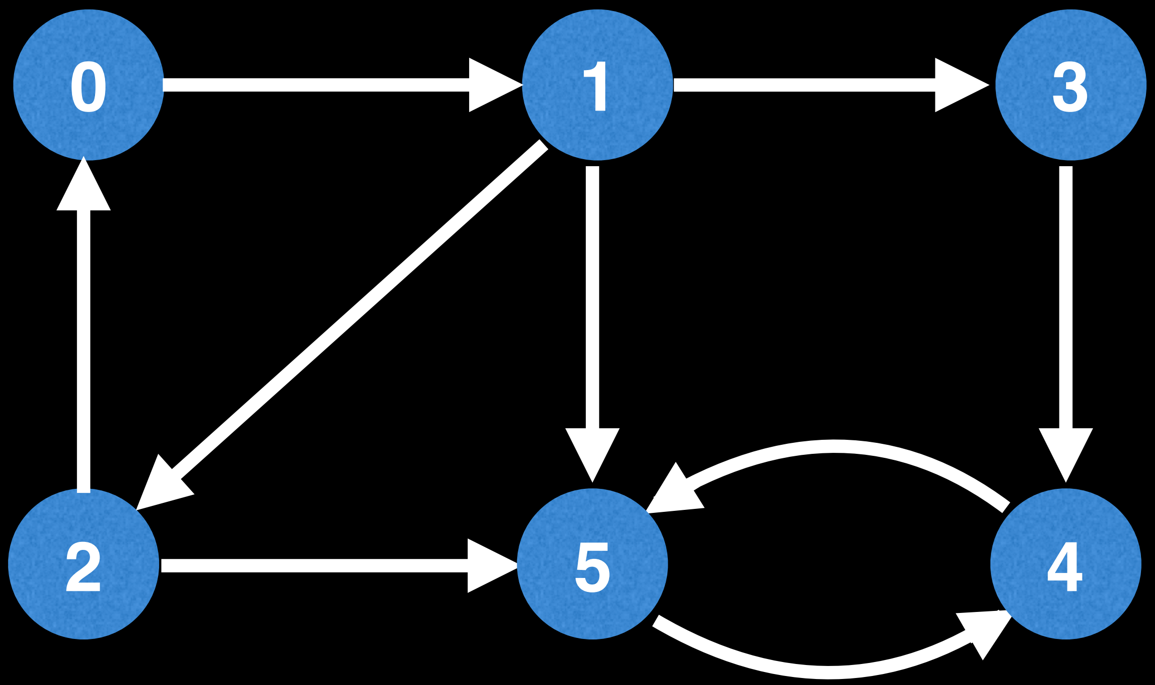

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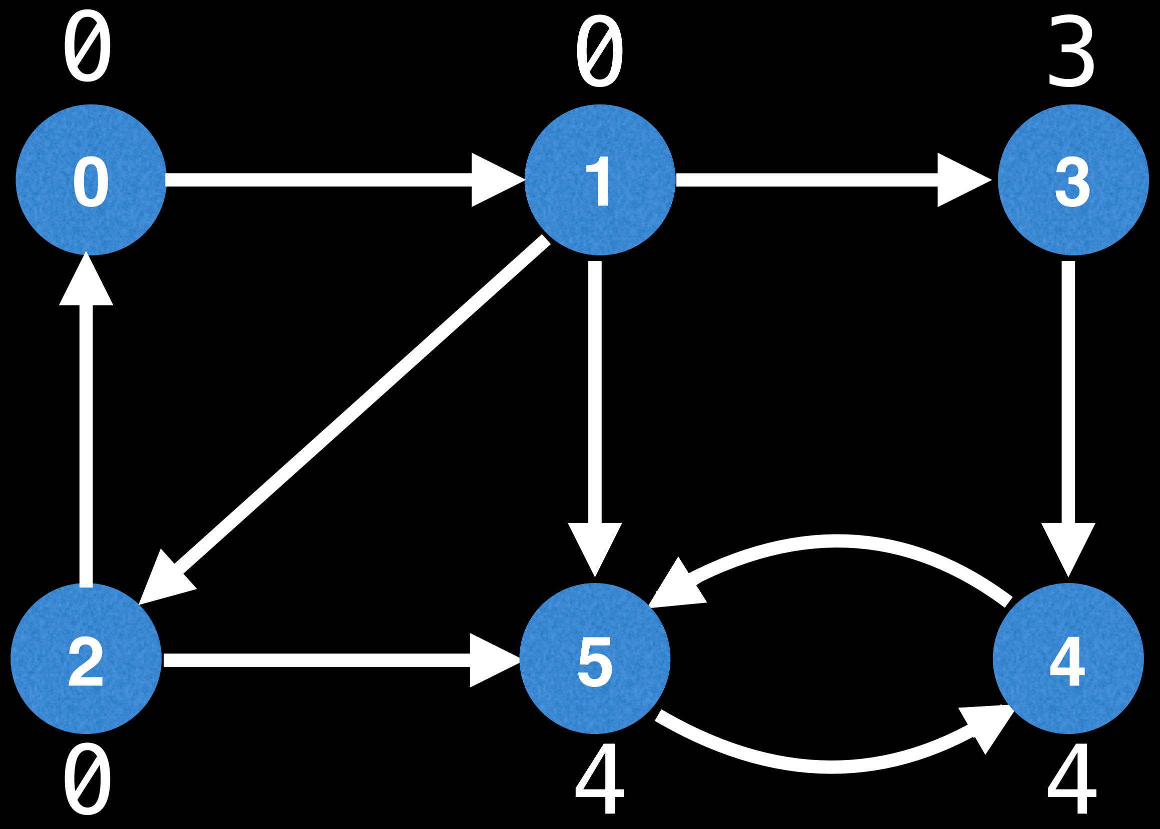

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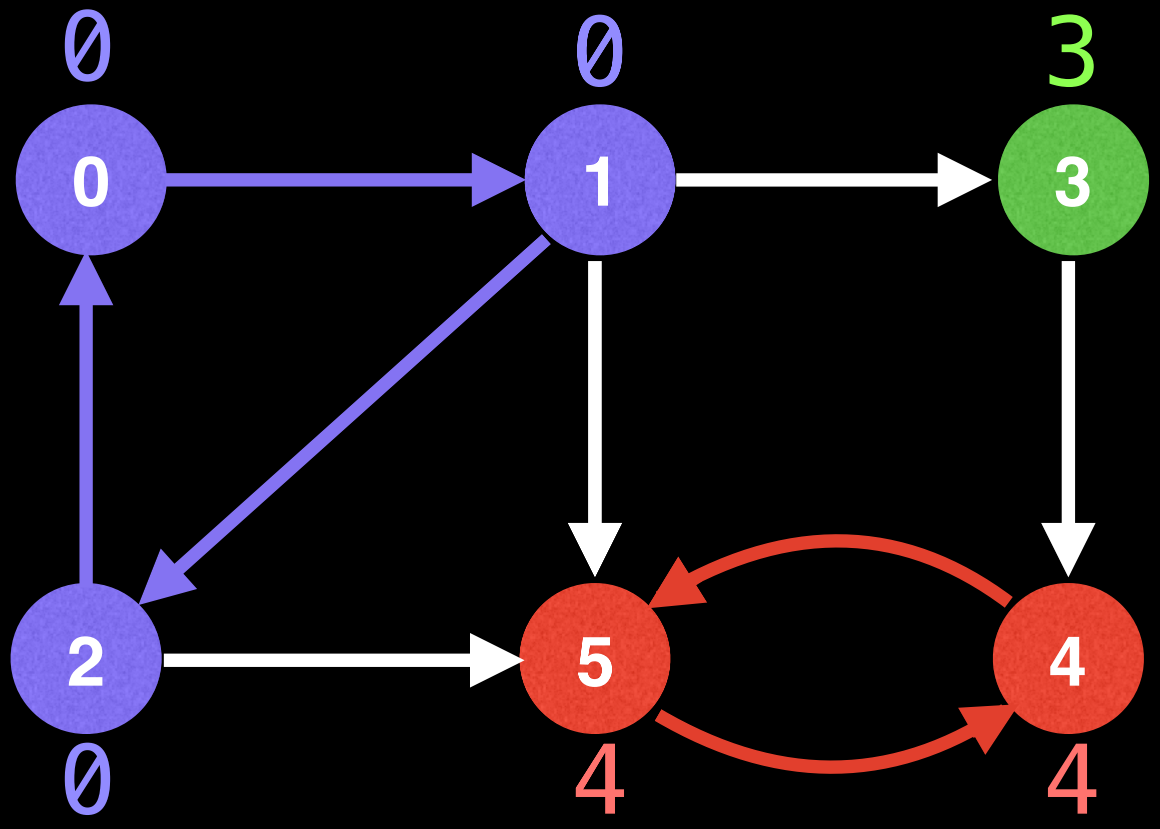

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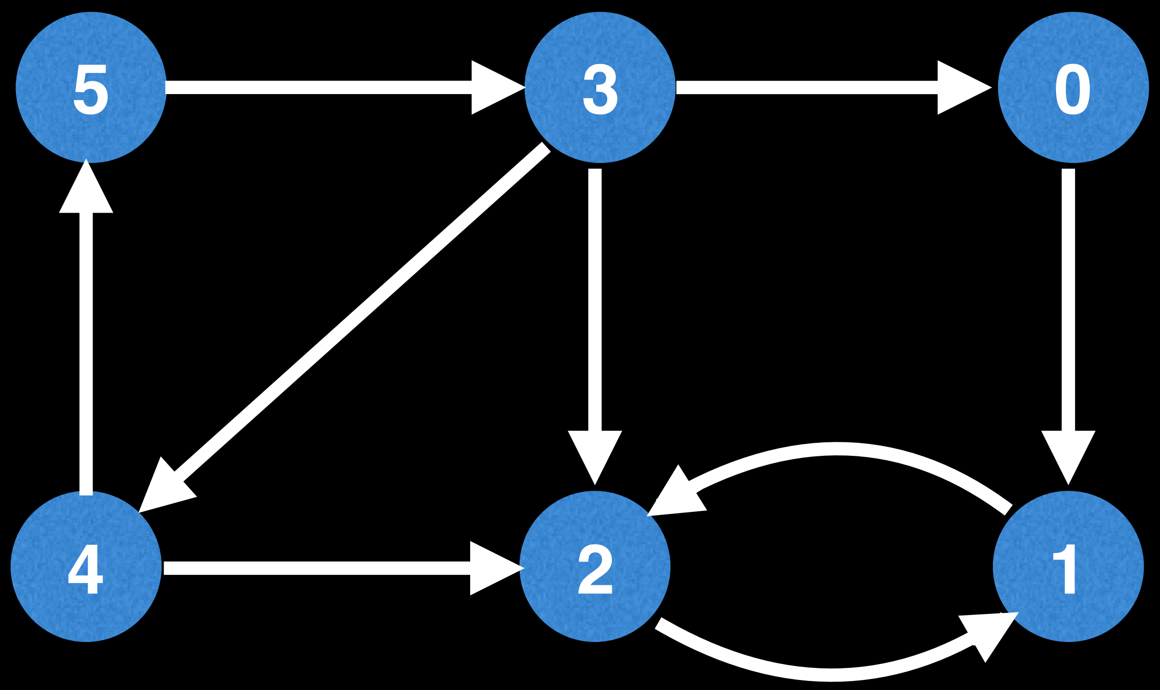

IMPORTANT: Depending on where the DFS starts and which edges are visited the low-link values could be wrong. In the context of Tarjan's SCC algorithm we maintain an invariant that prevents SCCs to interfere with each others' low-link values.

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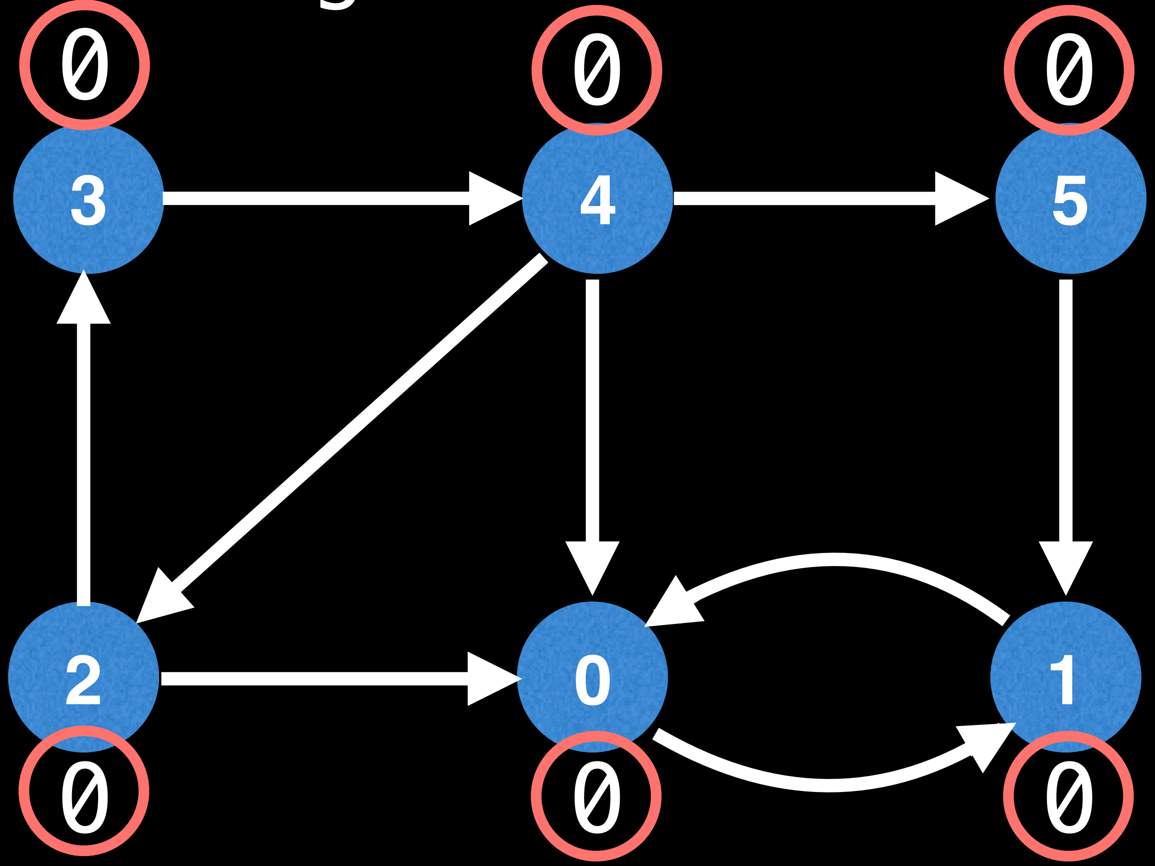

All low link values are the same but there are multiple SCCs!

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The Stack Invariant

To cope with the random traversal order of the DFS, Tarjan’s algorithm maintains a set (often as a stack) of valid nodes from which to update low—link values from.

Nodes are added to the stack [set] of valid nodes as they’re explored for the first time.

Nodes are removed from the stack [set] each time a complete SCC is found.

New low-link update condition

If u and v are nodes in a graph and we're currently exploring u then our new low-link update condition is that:

To update node u's low-link value to node v's low-link value there has to be a path of edges from u to v and **node v must be on the stack**.

Time Complexity

Another difference we’re going to make to finding all low-link values is that instead of finding low-link values after the fact we’re going to update them “on the fly” during the DFS so we can get a linear **O(V+E)** time complexity :)

Tarjan’s Algorithm Overview

Mark the id of each node as unvisited.

Start DFS. Upon visiting a node assign it an id and a low-link value. Also mark current nodes as visited and add them to a seen stack.

On DFS callback, if the previous node is on the stack then min the current node’s low-link value with the last node’s low-link value*.

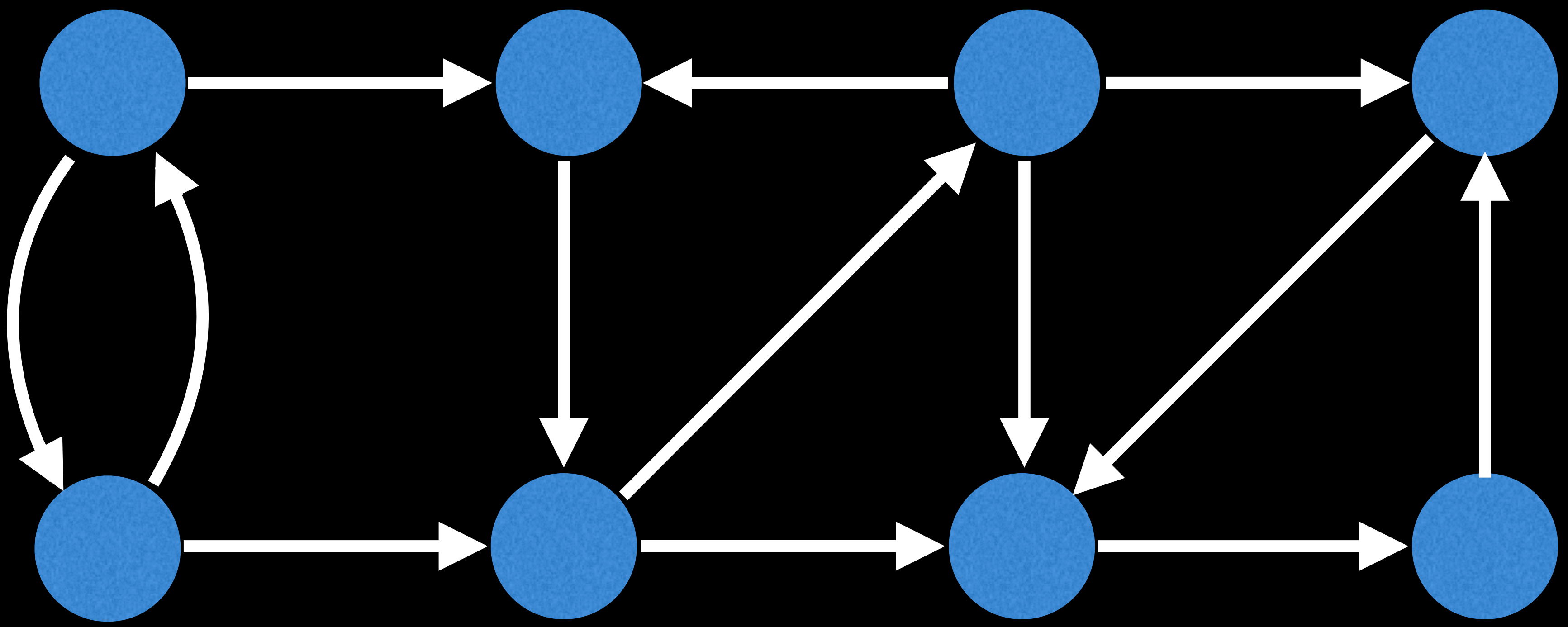
After visiting all neighbors, if the current node started a connected component** then pop nodes off stack until current node is reached.

*This allows low-link values to propagate throughout cycles.

As we will see, a node started a connected component if its **id equals its low link value

🔵 Unvisited 🟠 Visiting neighbours ⚪ Visited all neighbours

Stack



If a node’s colour is **grey** or **orange** then it is on the stack and we can update its low-link value.

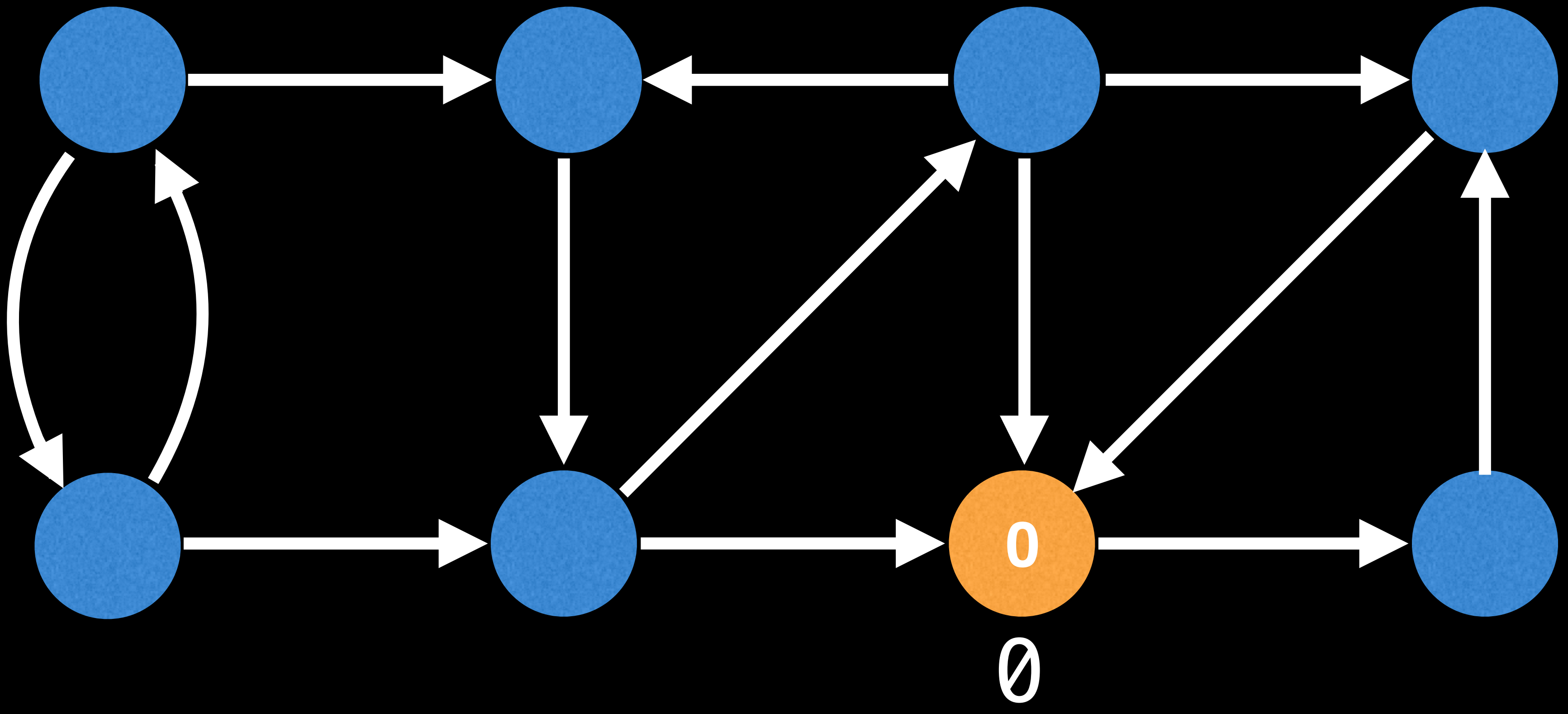
● Unvisited

● Visiting neighbours

● Visited all neighbours

Stack

0



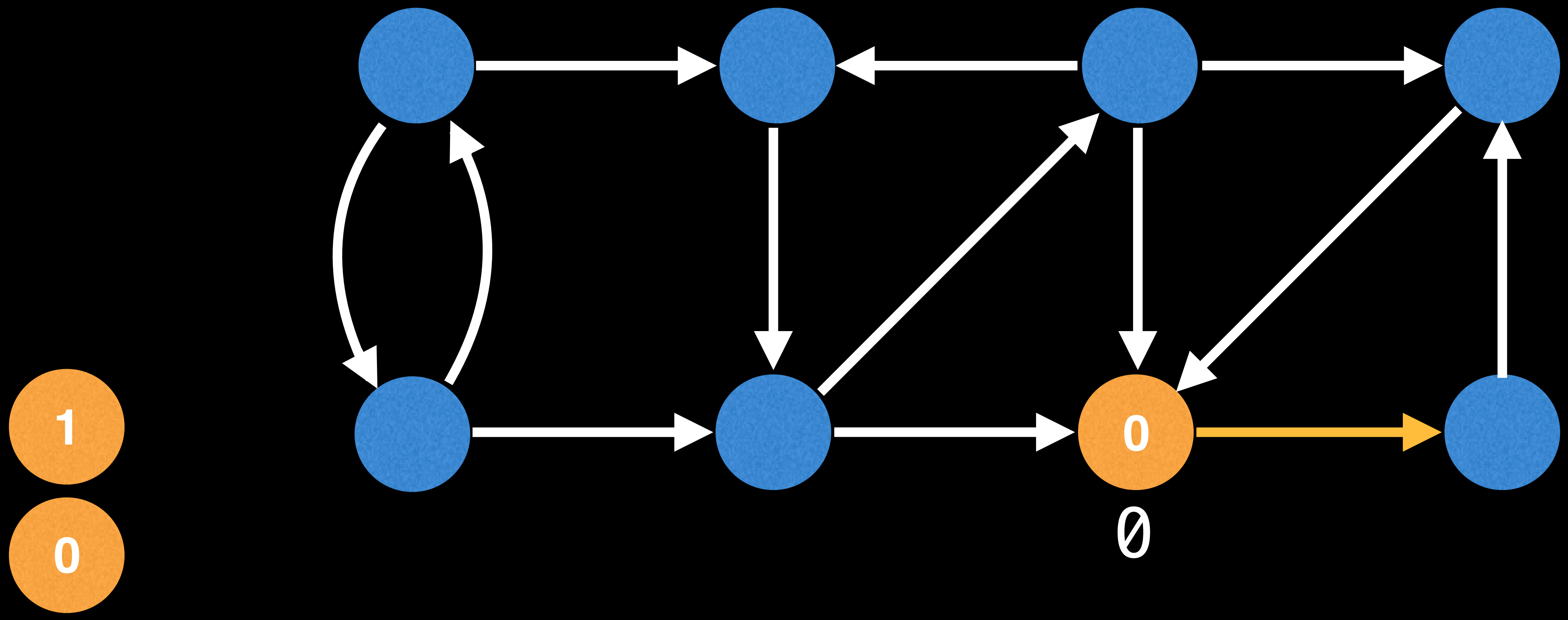
Start DFS anywhere.

● Unvisited

● Visiting neighbours

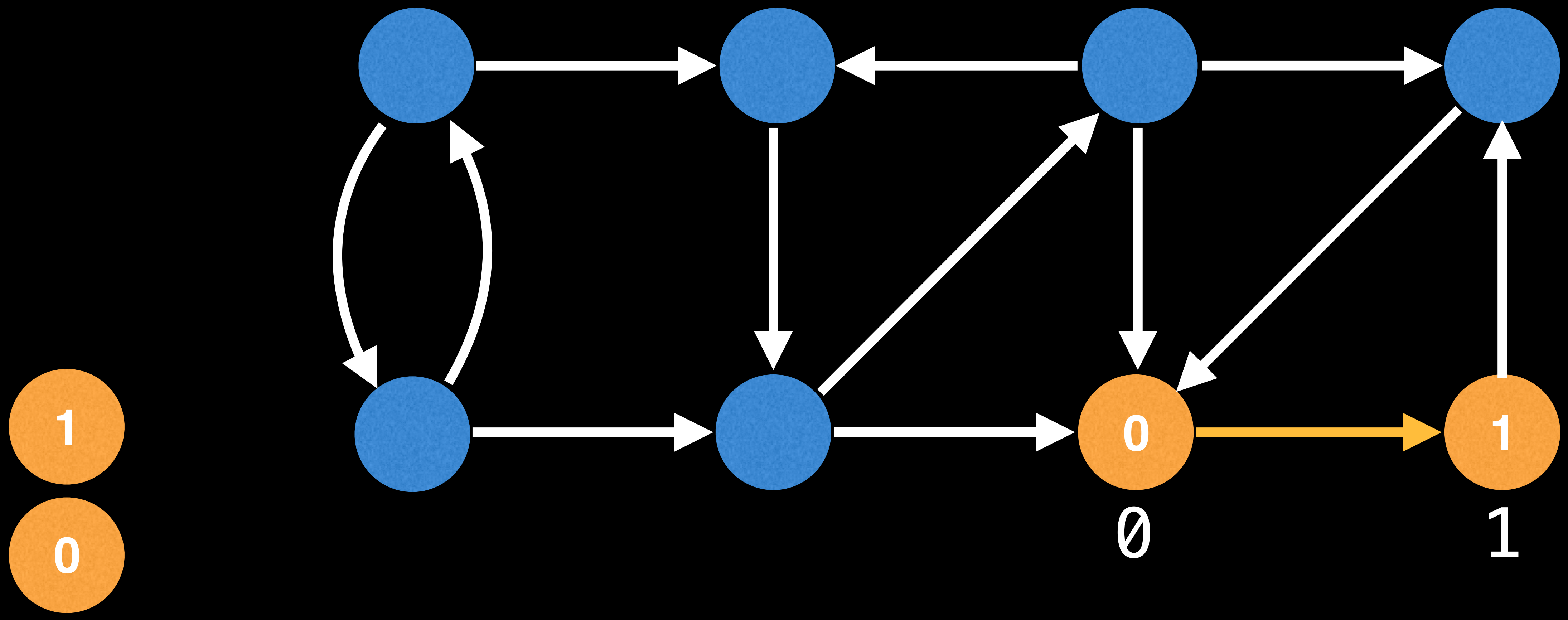
● Visited all neighbours

Stack



● Unvisited ● Visiting neighbours ● Visited all neighbours

Stack



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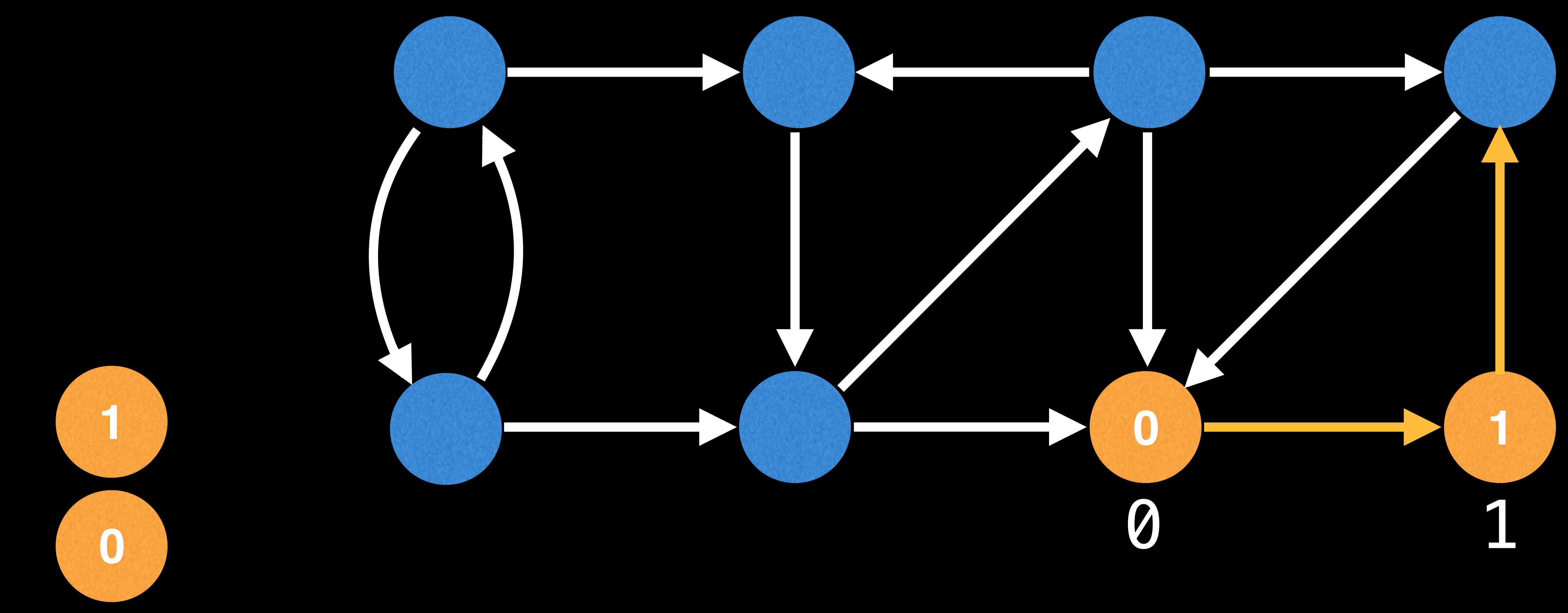
Stack



● Unvisited

● Visiting neighbours

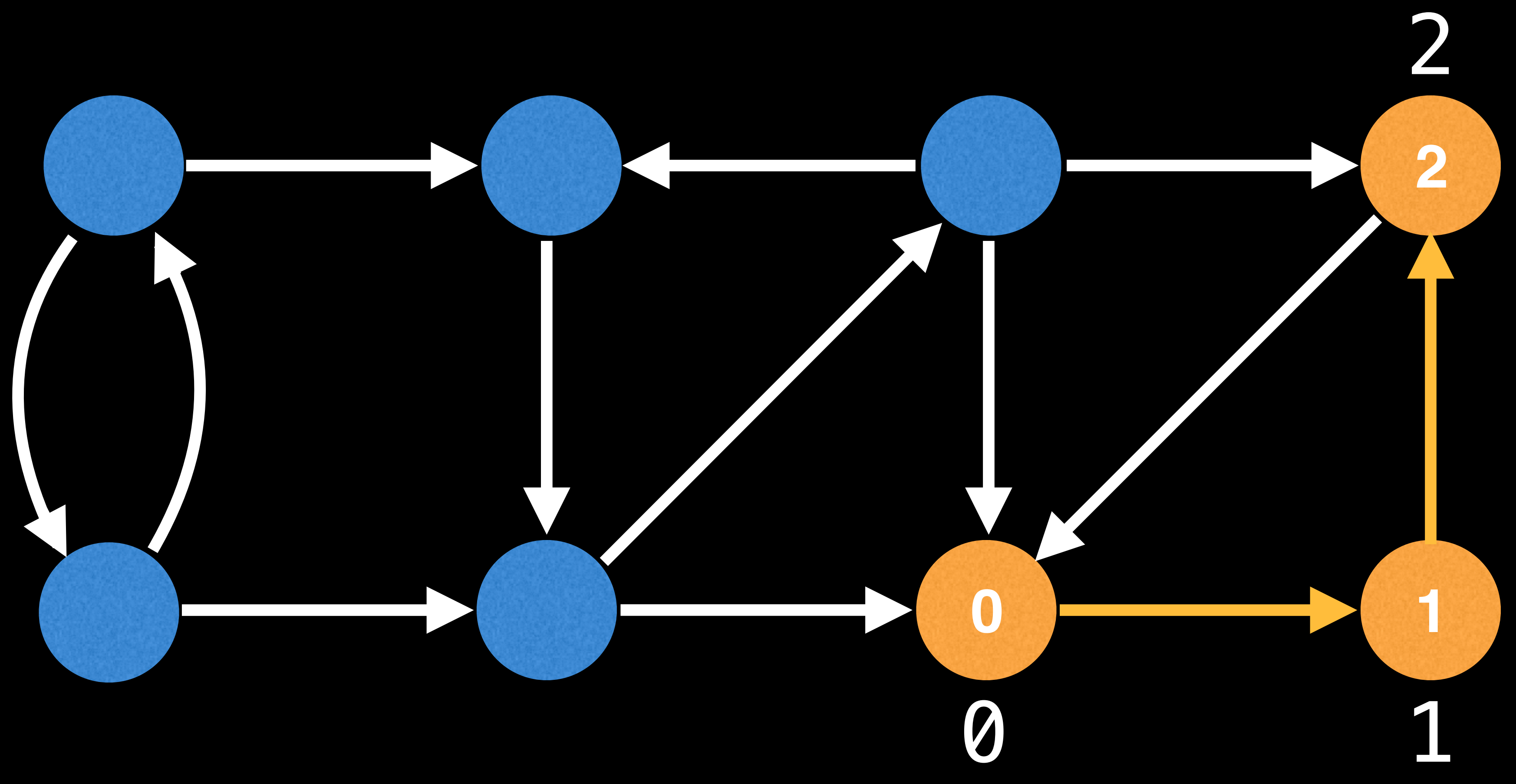
● Visited all neighbours

Stack

2

1

0



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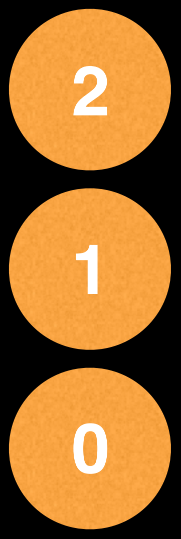
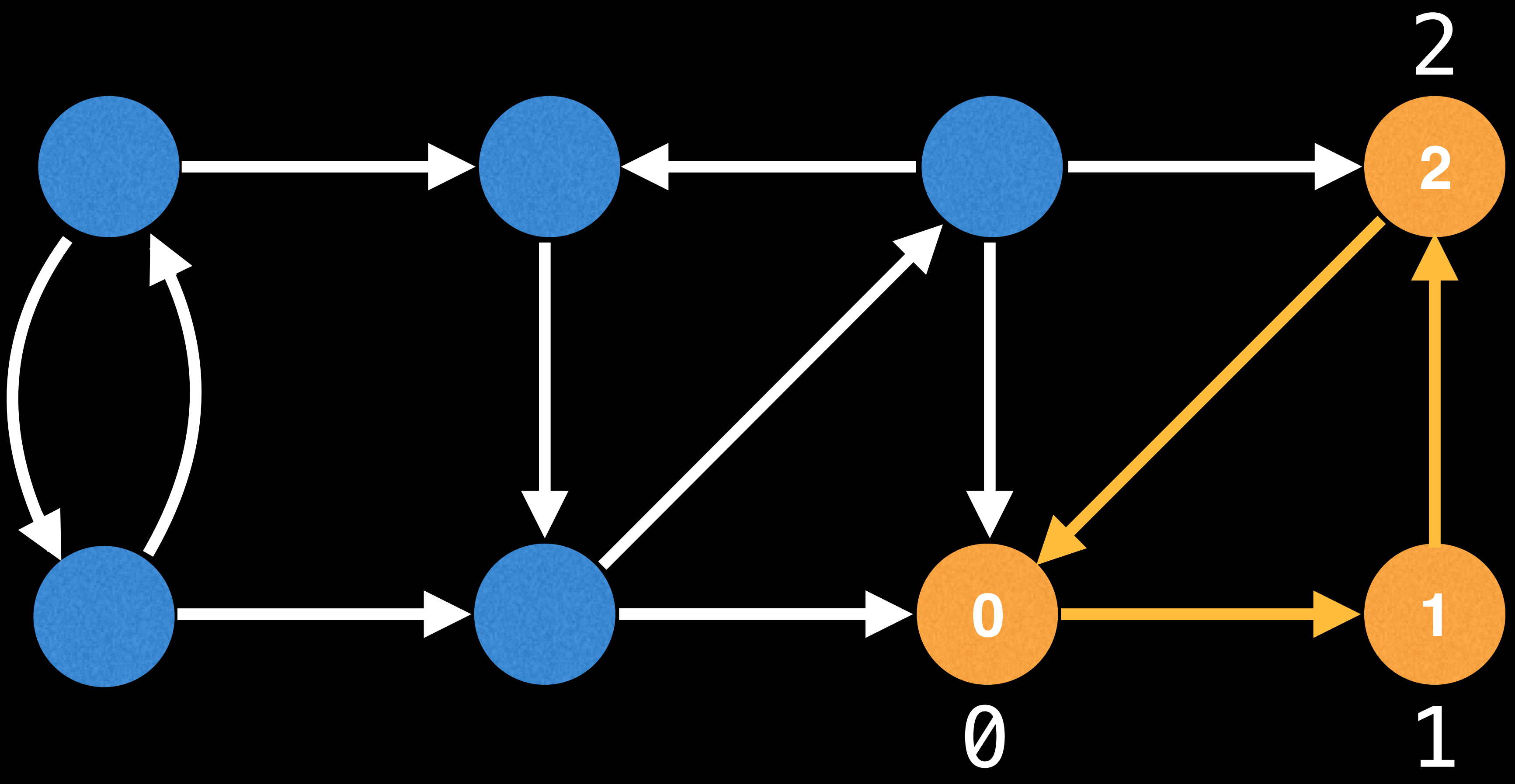
Stack





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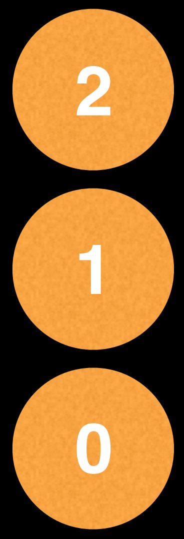
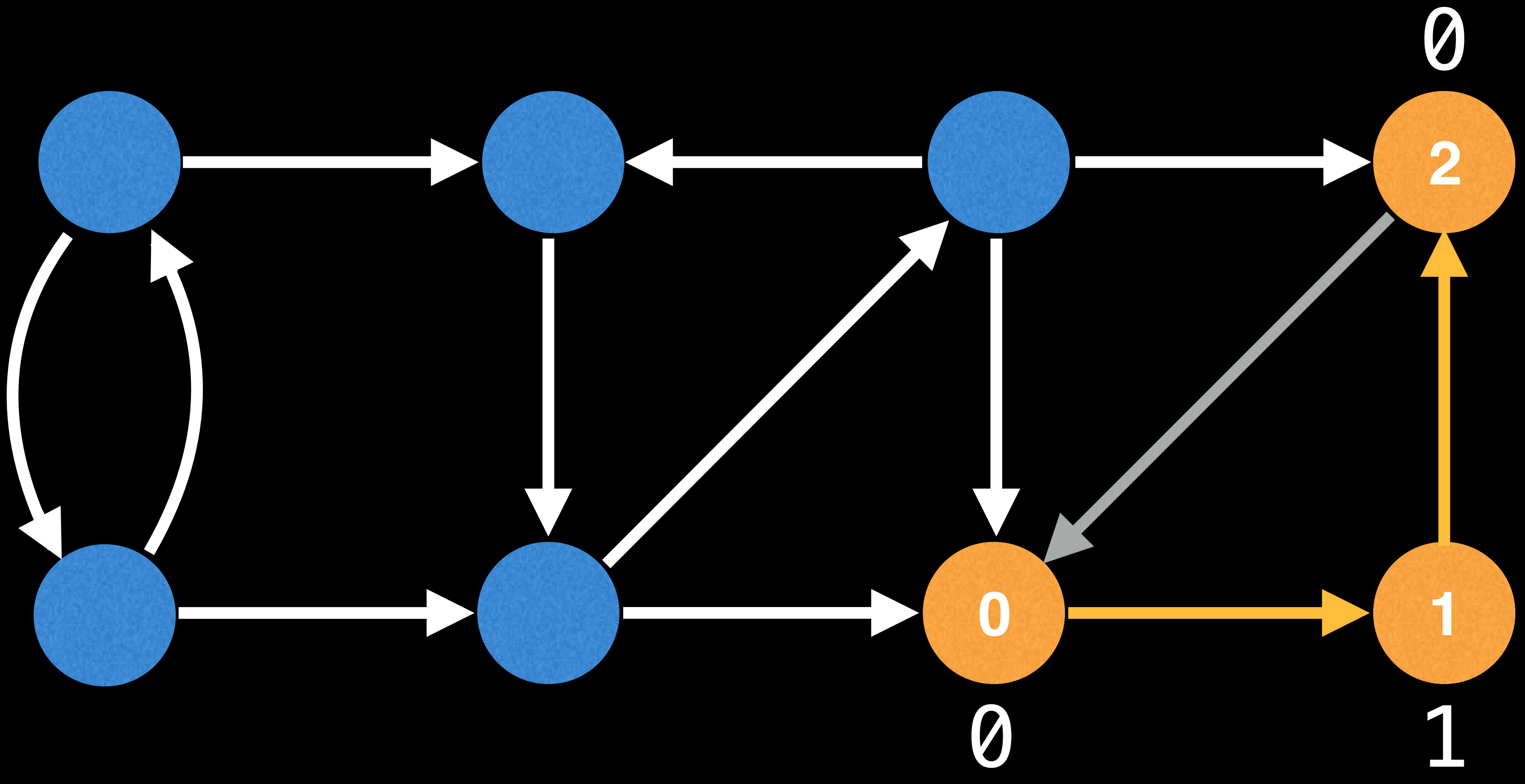
Stack





```
lowlink[2] = min(lowlink[2], lowlink[0])
           = 0
```

● Unvisited

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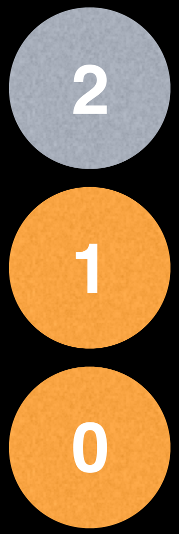
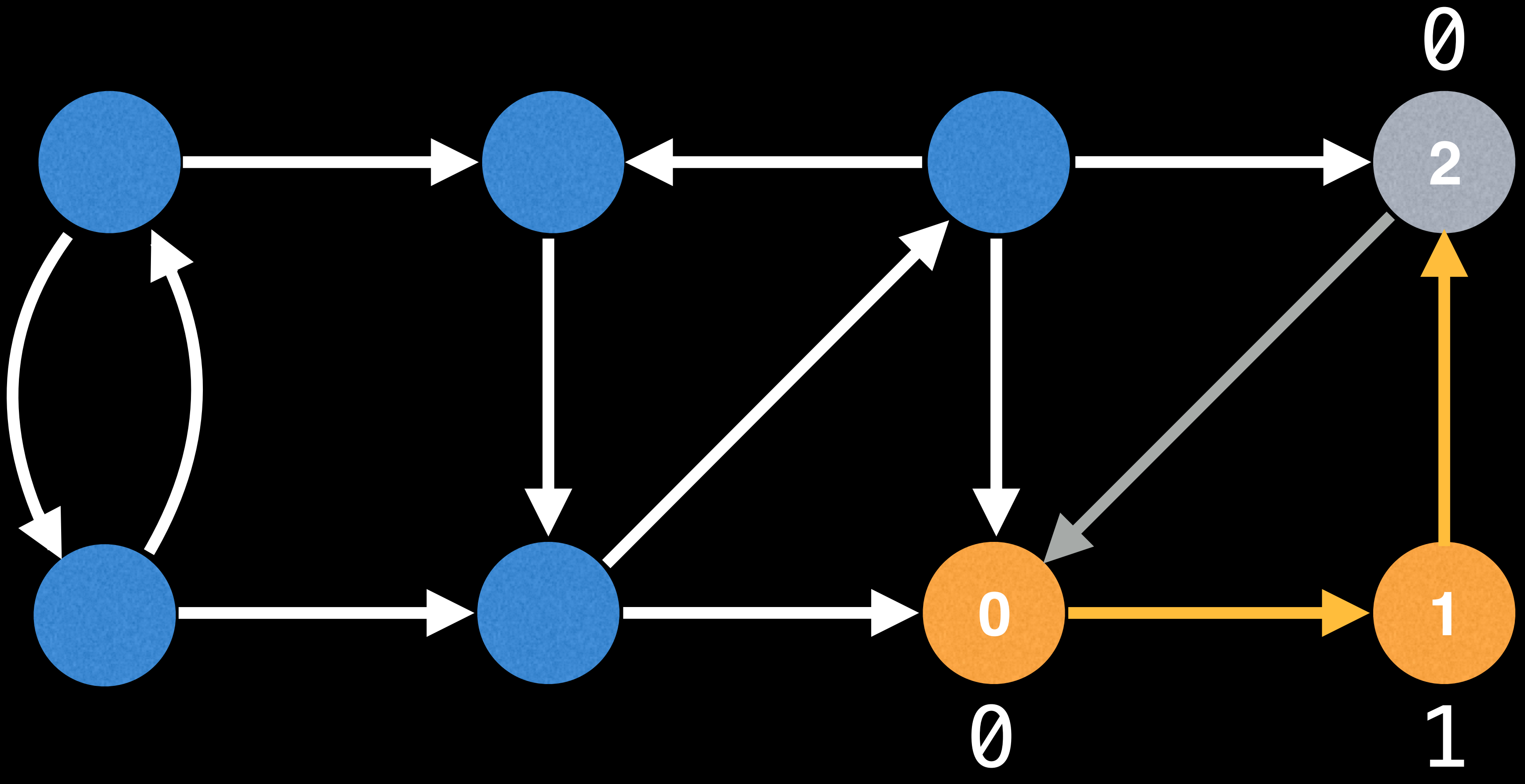
● Visited all neighbours

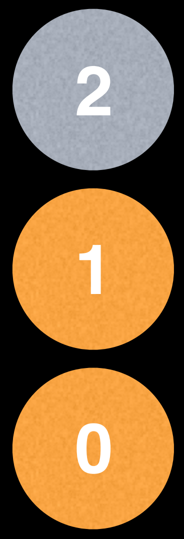
Stack

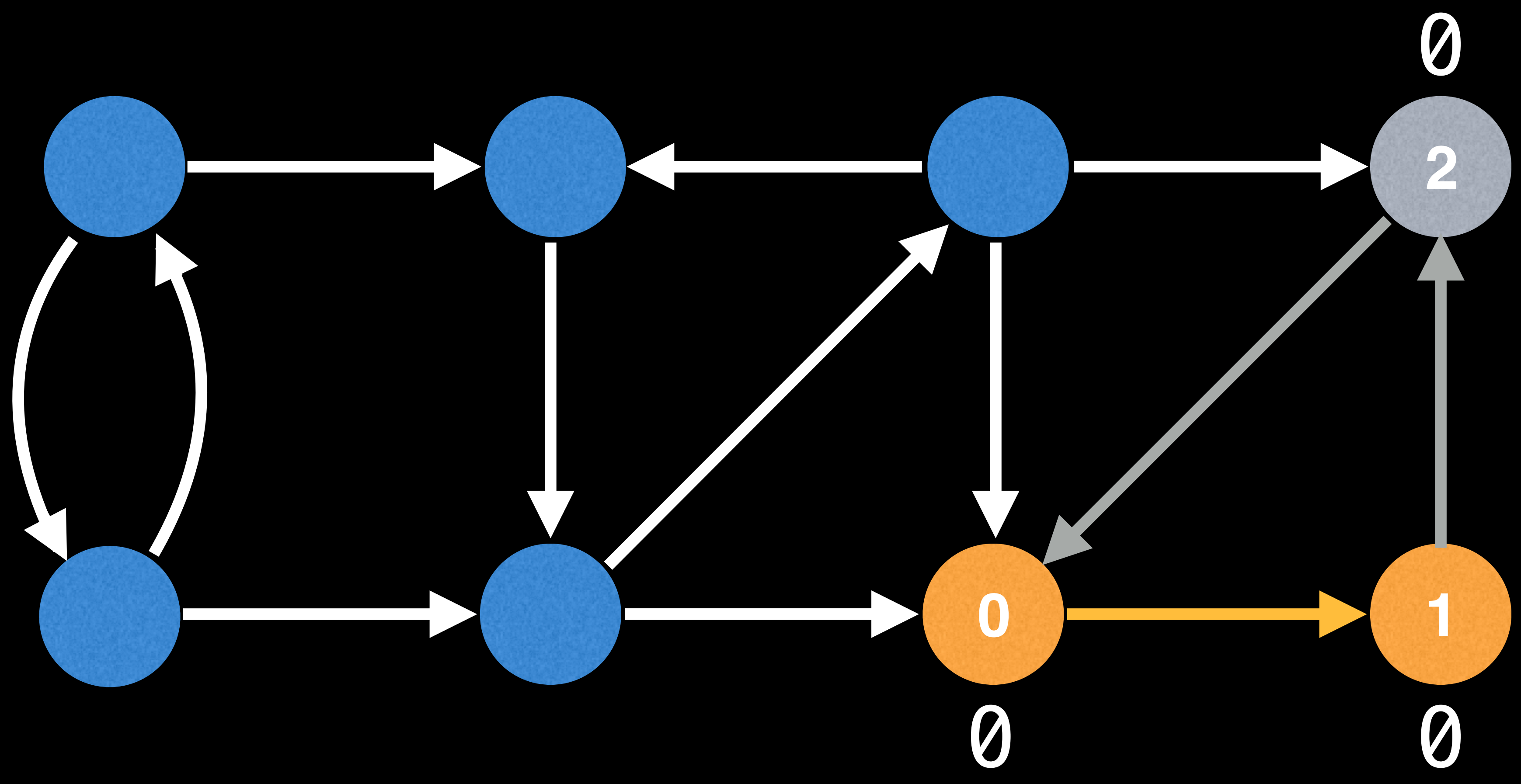




● Unvisited ● Visiting neighbours ● Visited all neighbours

Stack





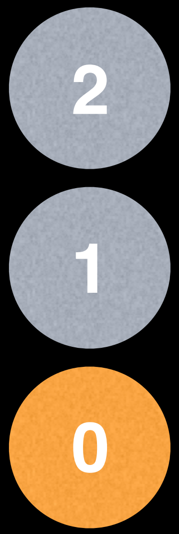
```
lowlink[1] = min(lowlink[1], lowlink[2])
           = 0
```

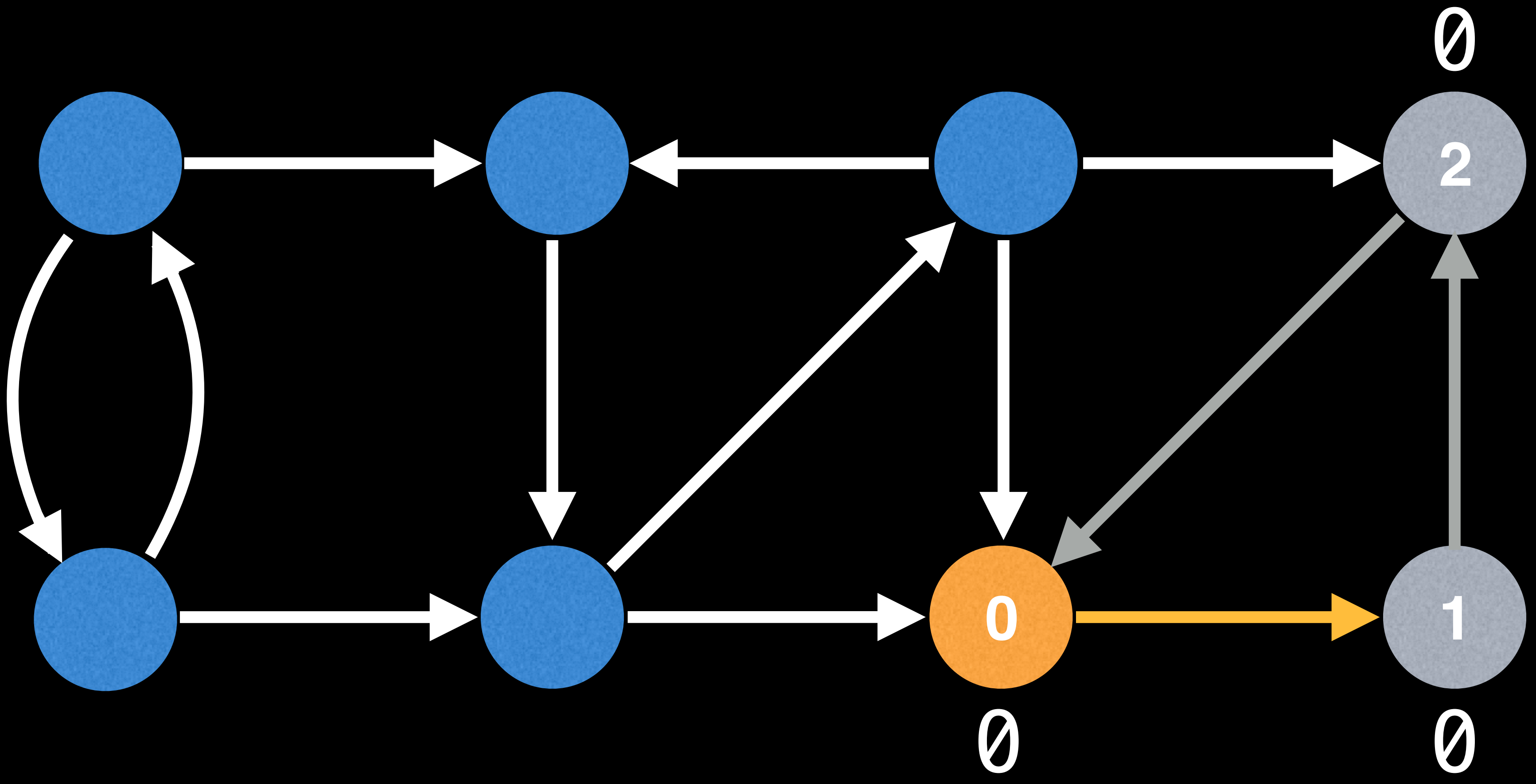
● Unvisited

● Visiting neighbours

● Visited all neighbours

Stack





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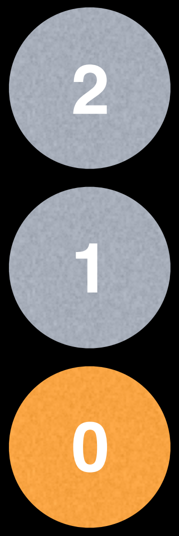
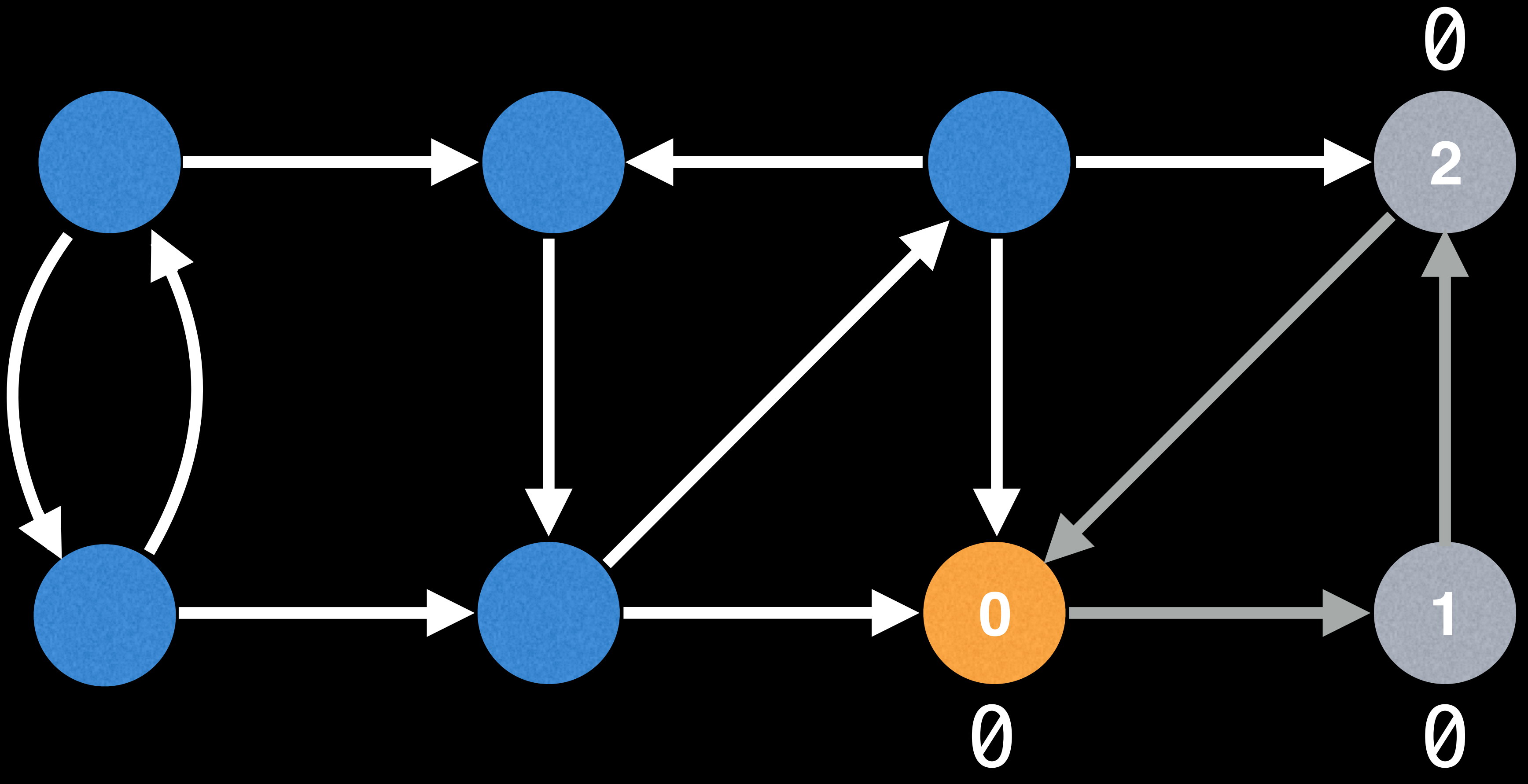




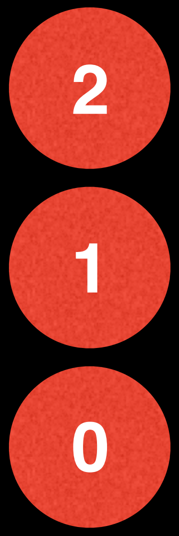
```
lowlink[0] = min(lowlink[0], lowlink[1])
           = 0
```

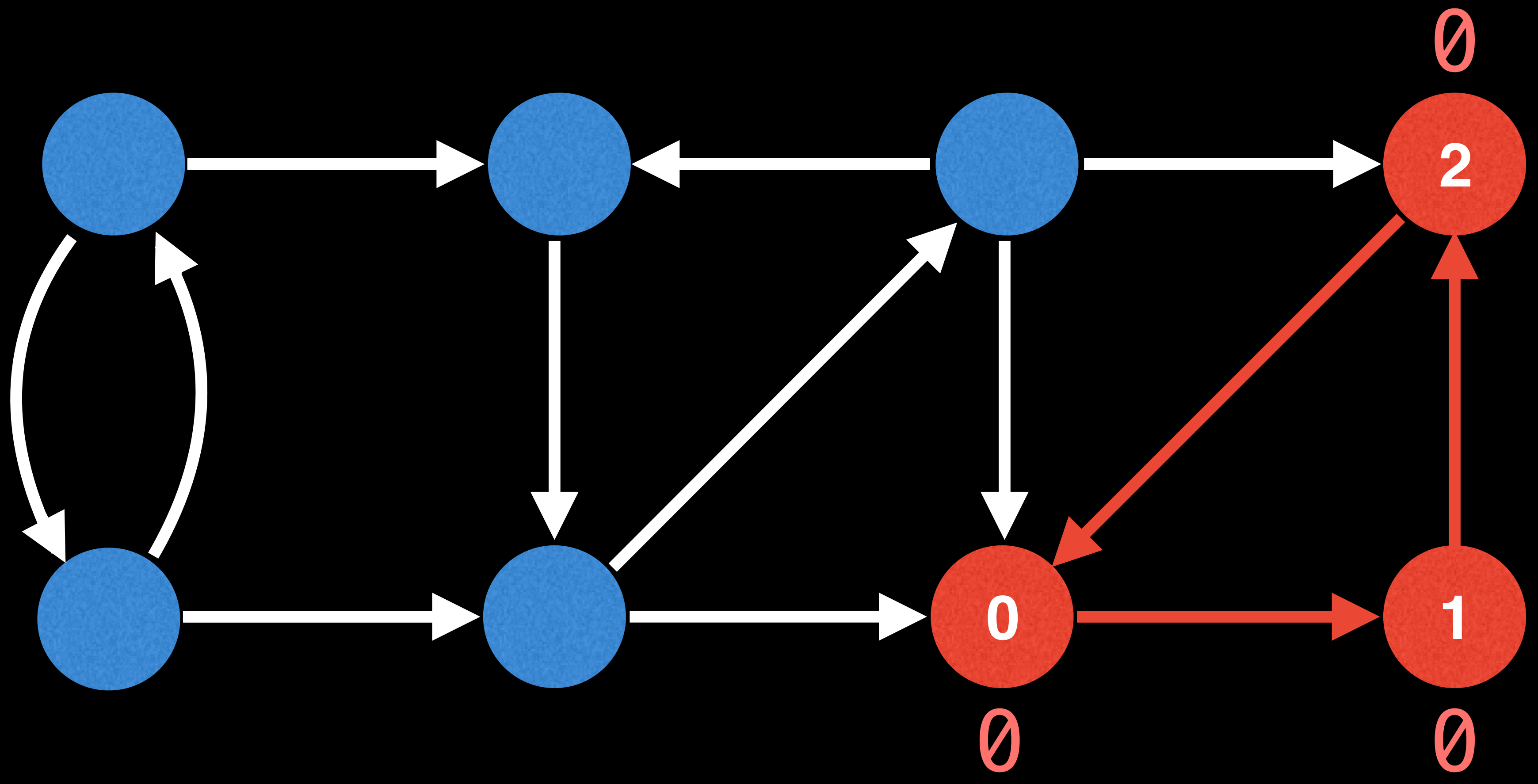
🔵 Unvisited

🟠 Visiting neighbours

⚪ Visited all neighbours

Stack





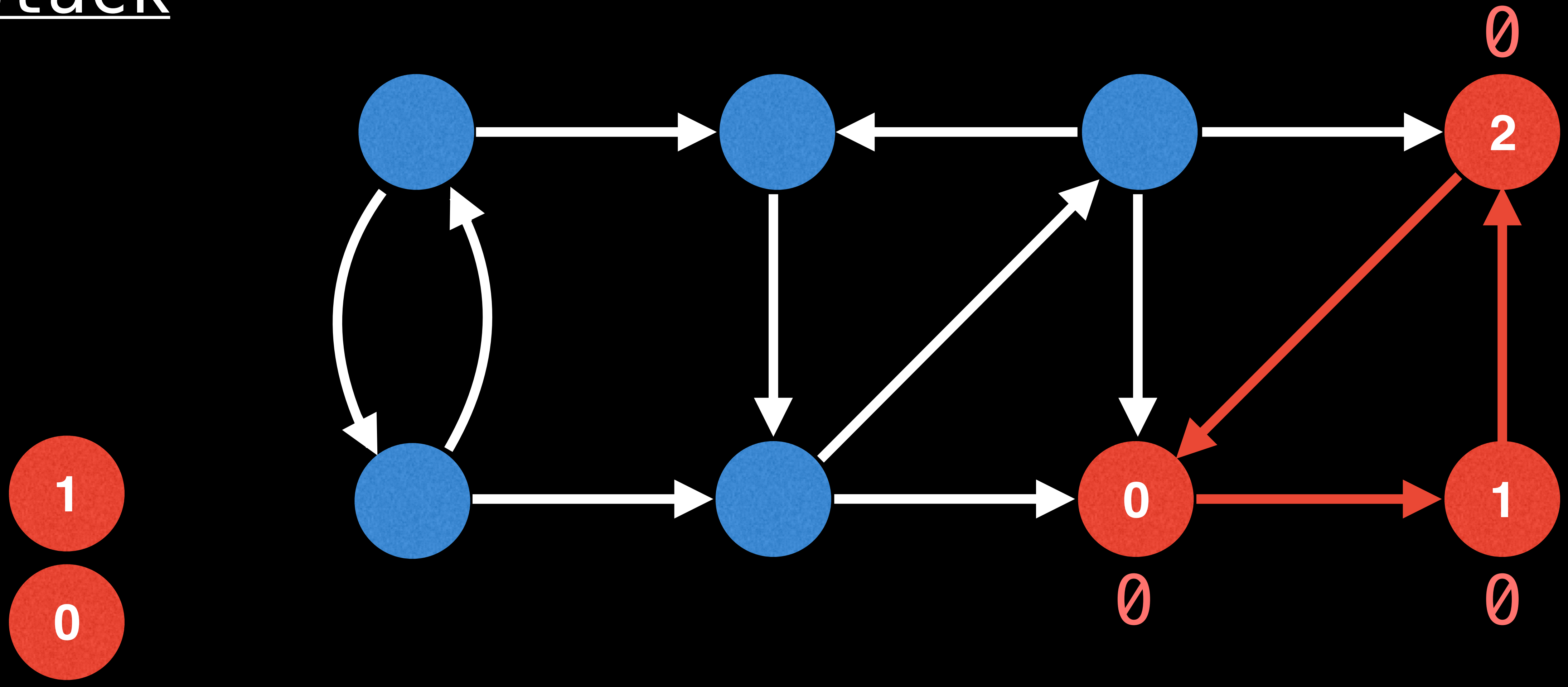
When a completed SCC is found (current node has visited all its neighbours and its lowlink value equals its id) pop off all associated nodes off the stack.

● Unvisited

● Visiting neighbours

● Visited all neighbours

Stack



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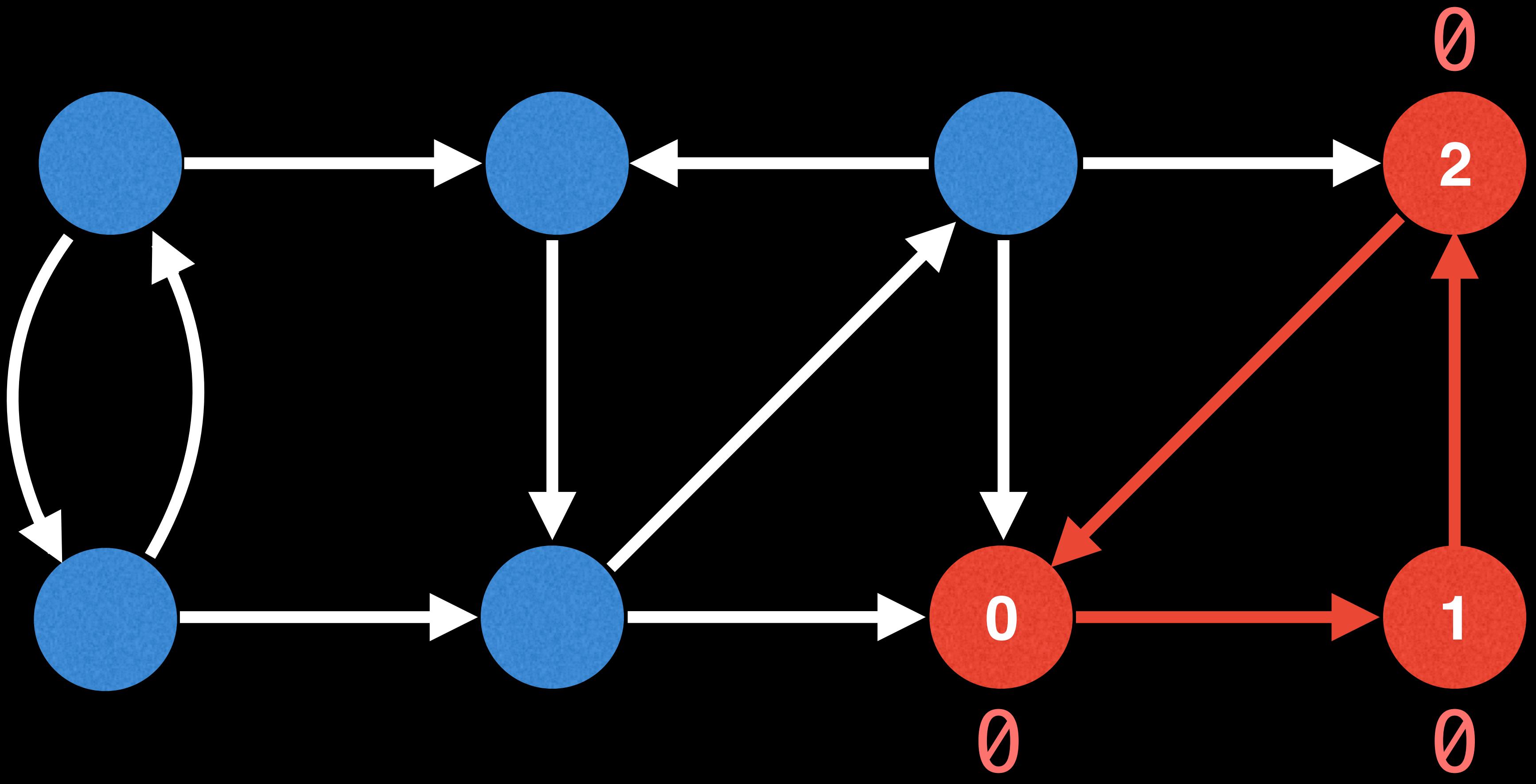

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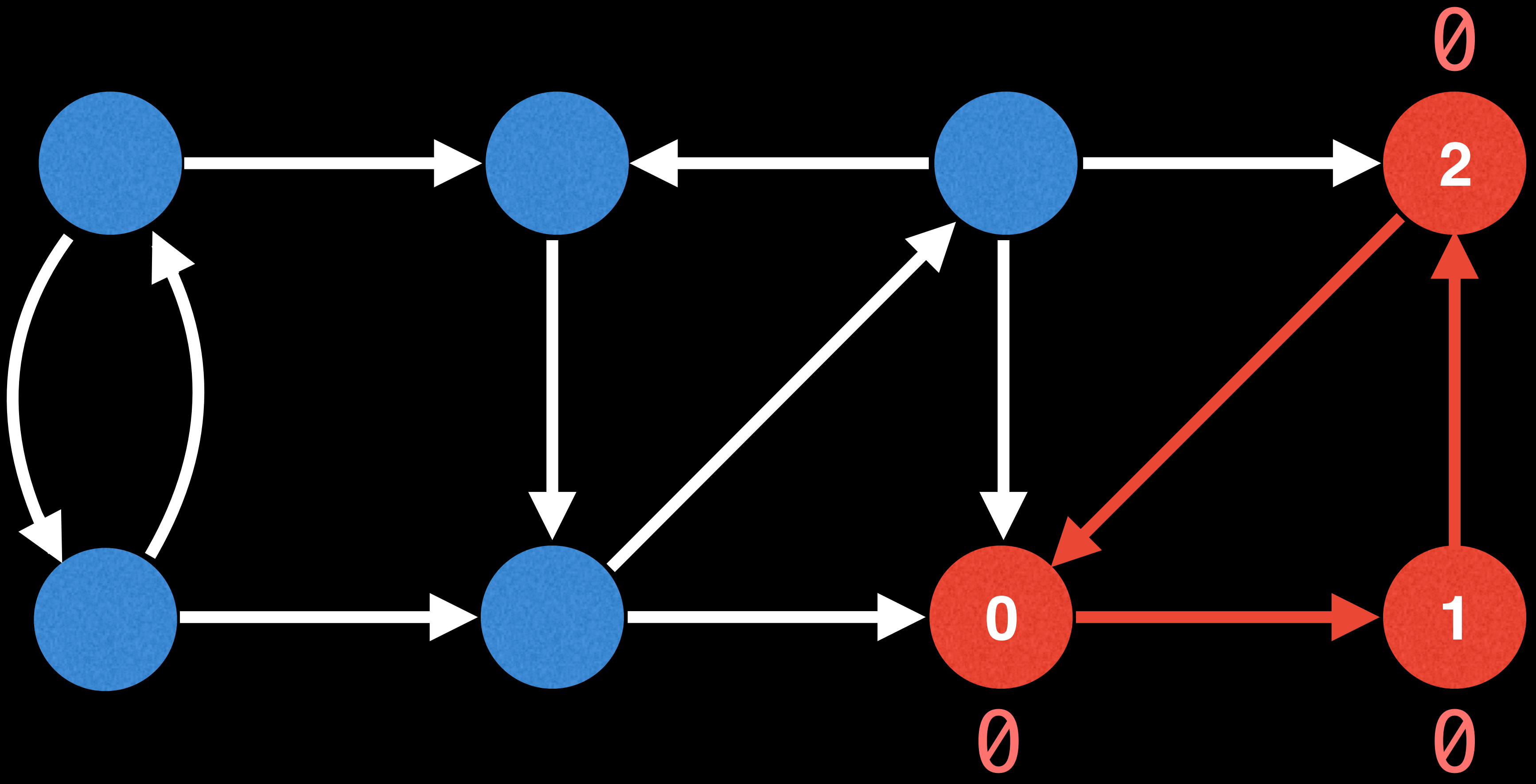

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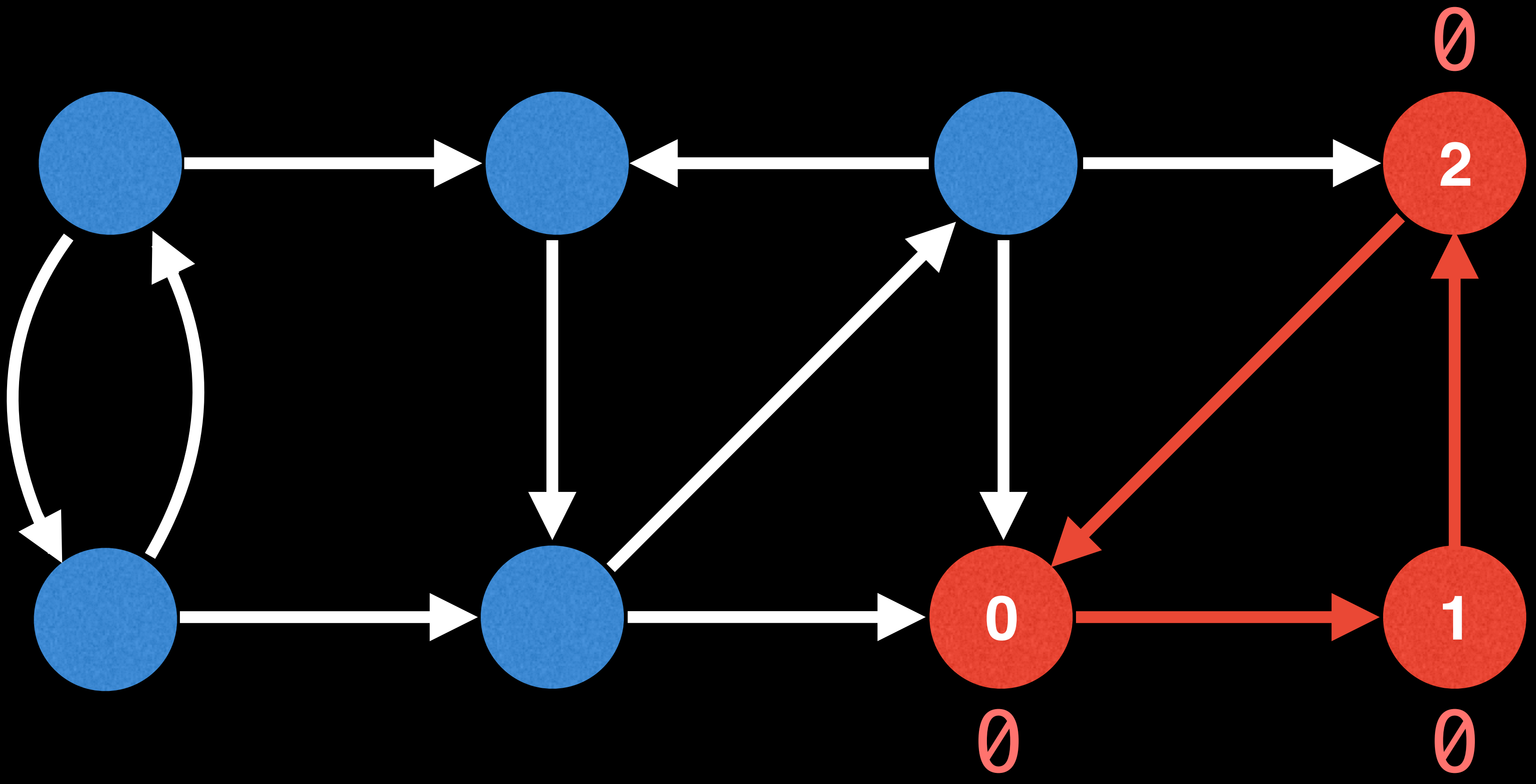


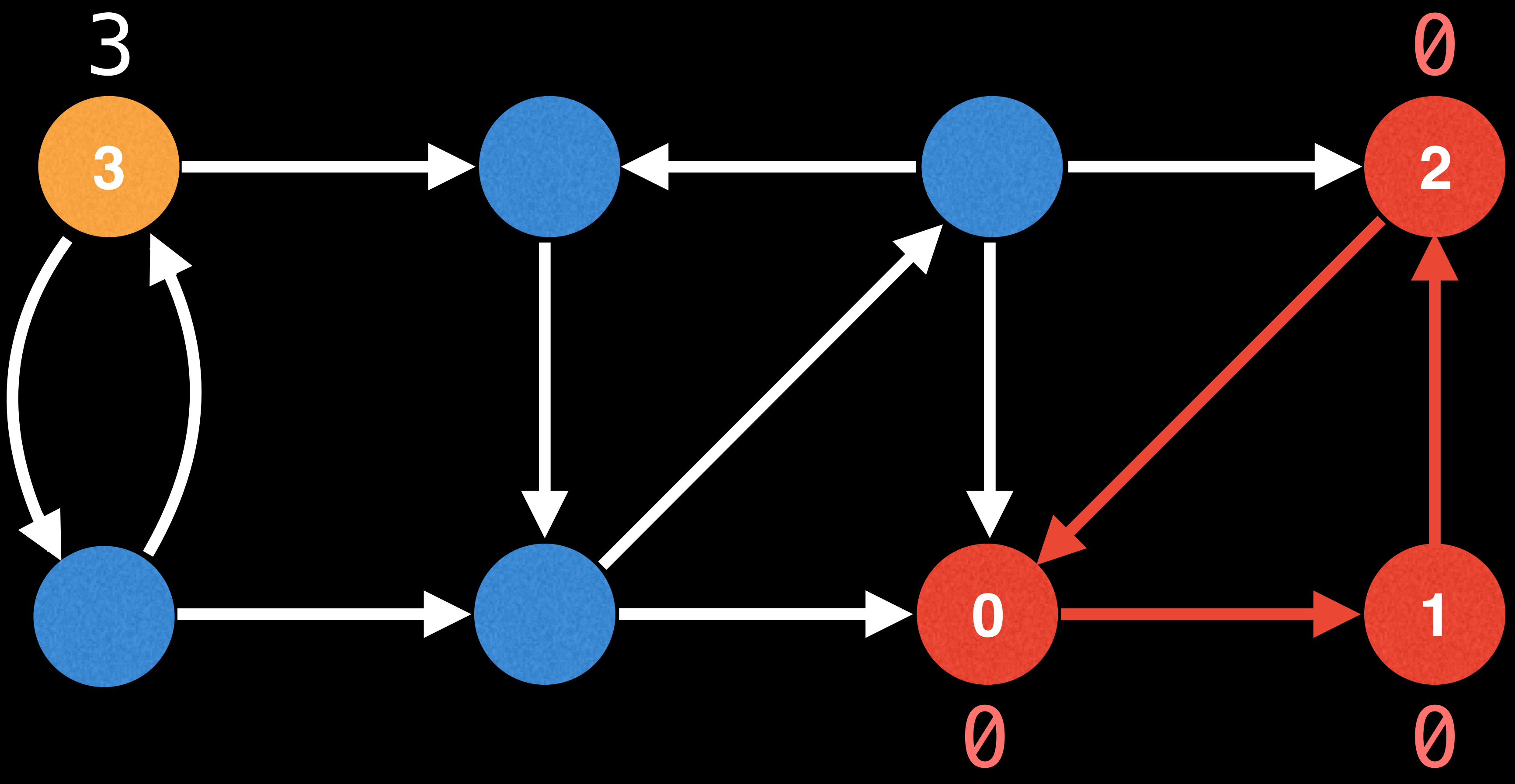
We're not done exploring the graph so pick another starting node at random.

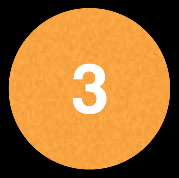
● Unvisited

● Visiting neighbours

● Visited all neighbours

Stack





🔵 Unvisited

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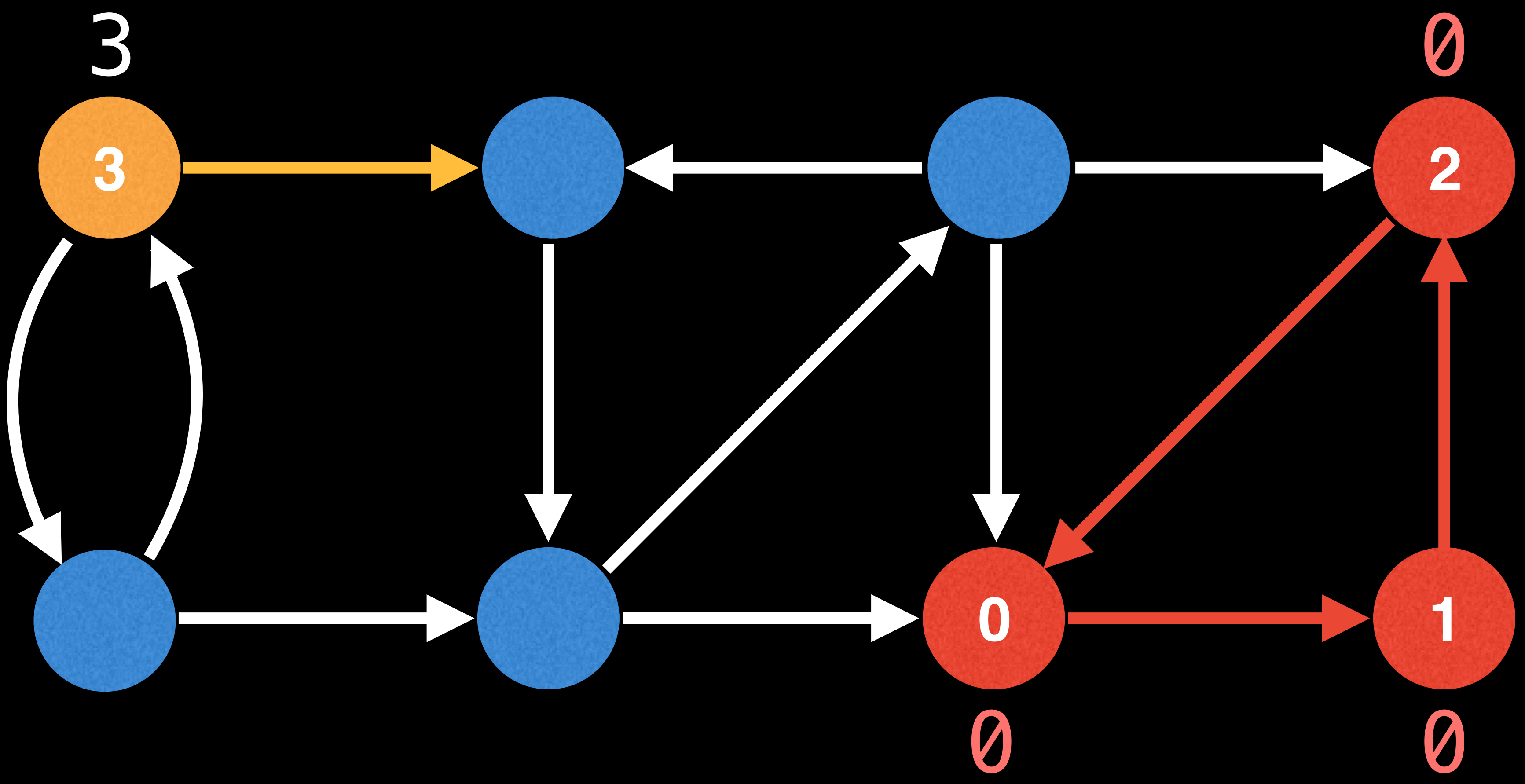


Stack

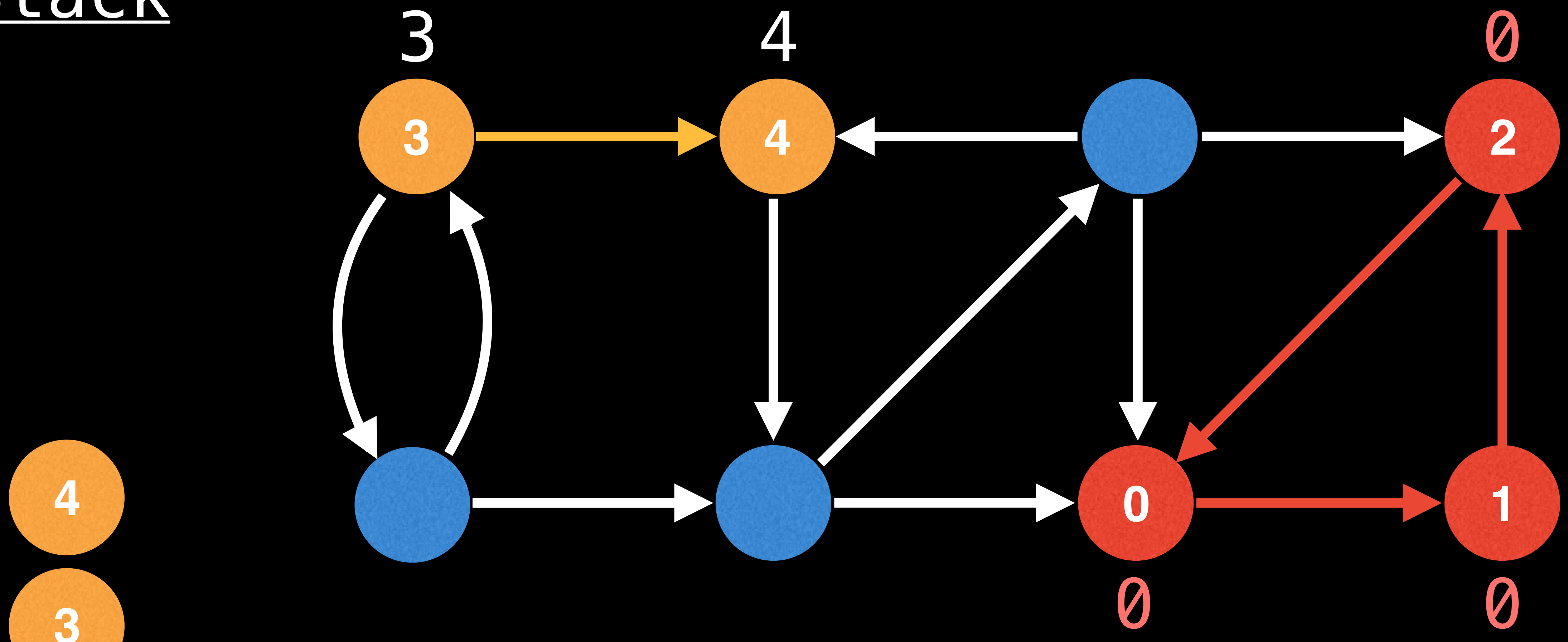
3

● Unvisited

● Visiting neighbours

● Visited all neighbours

Stack

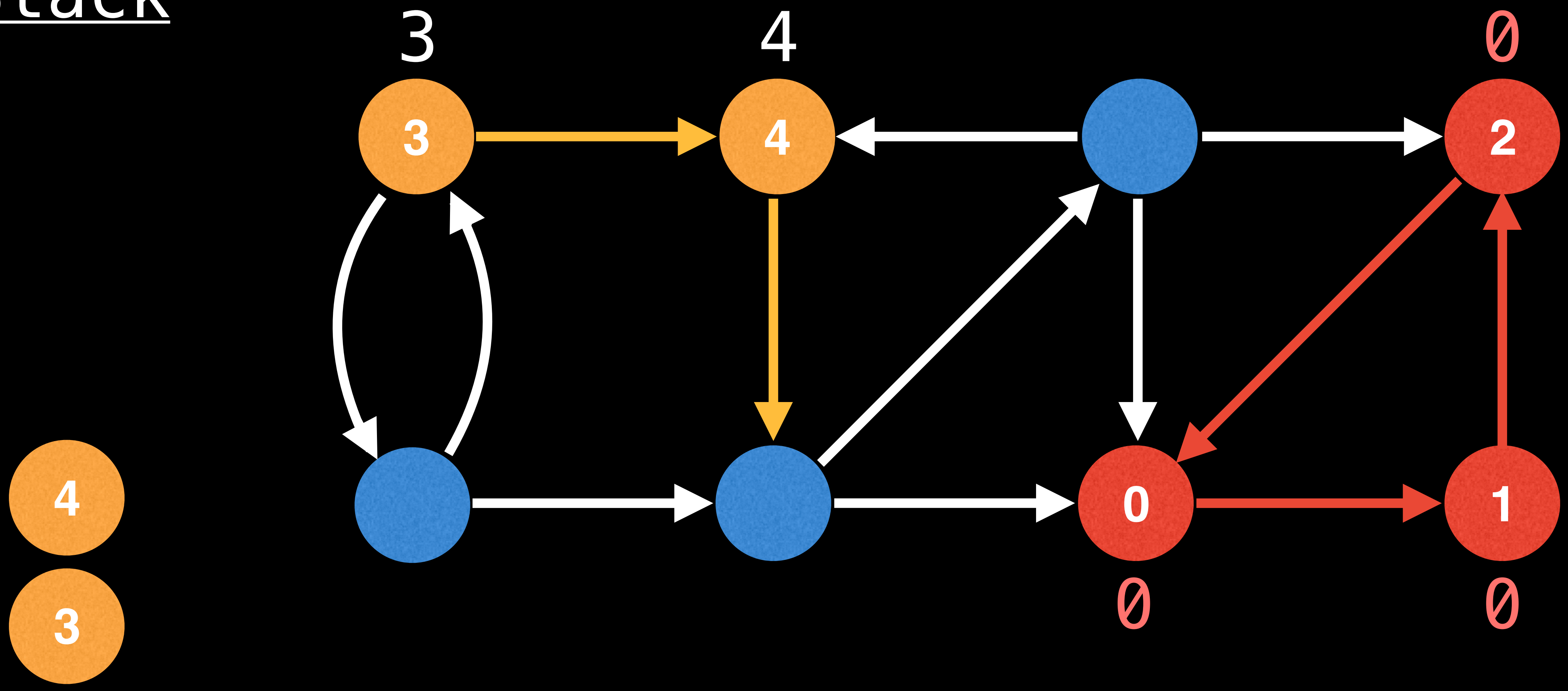


● Unvisited

● Visiting neighbours

● Visited all neighbours

Stack



● Unvisited

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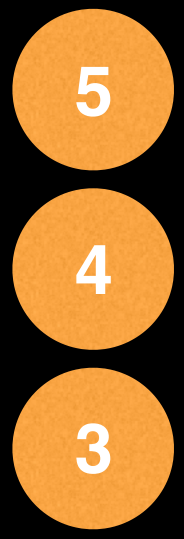
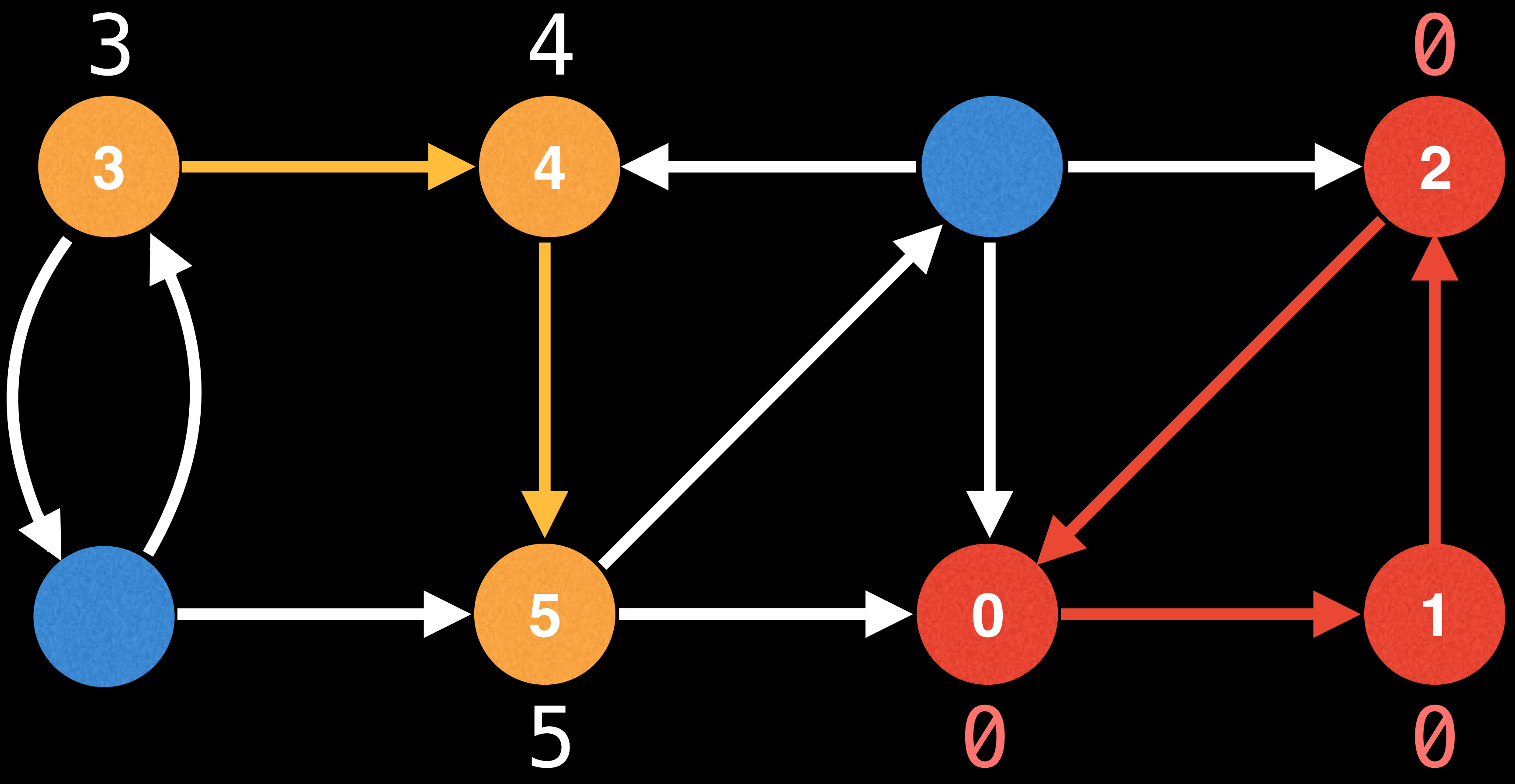
● Visited all neighbours

Stack





● Unvisited

● Visiting neighbours

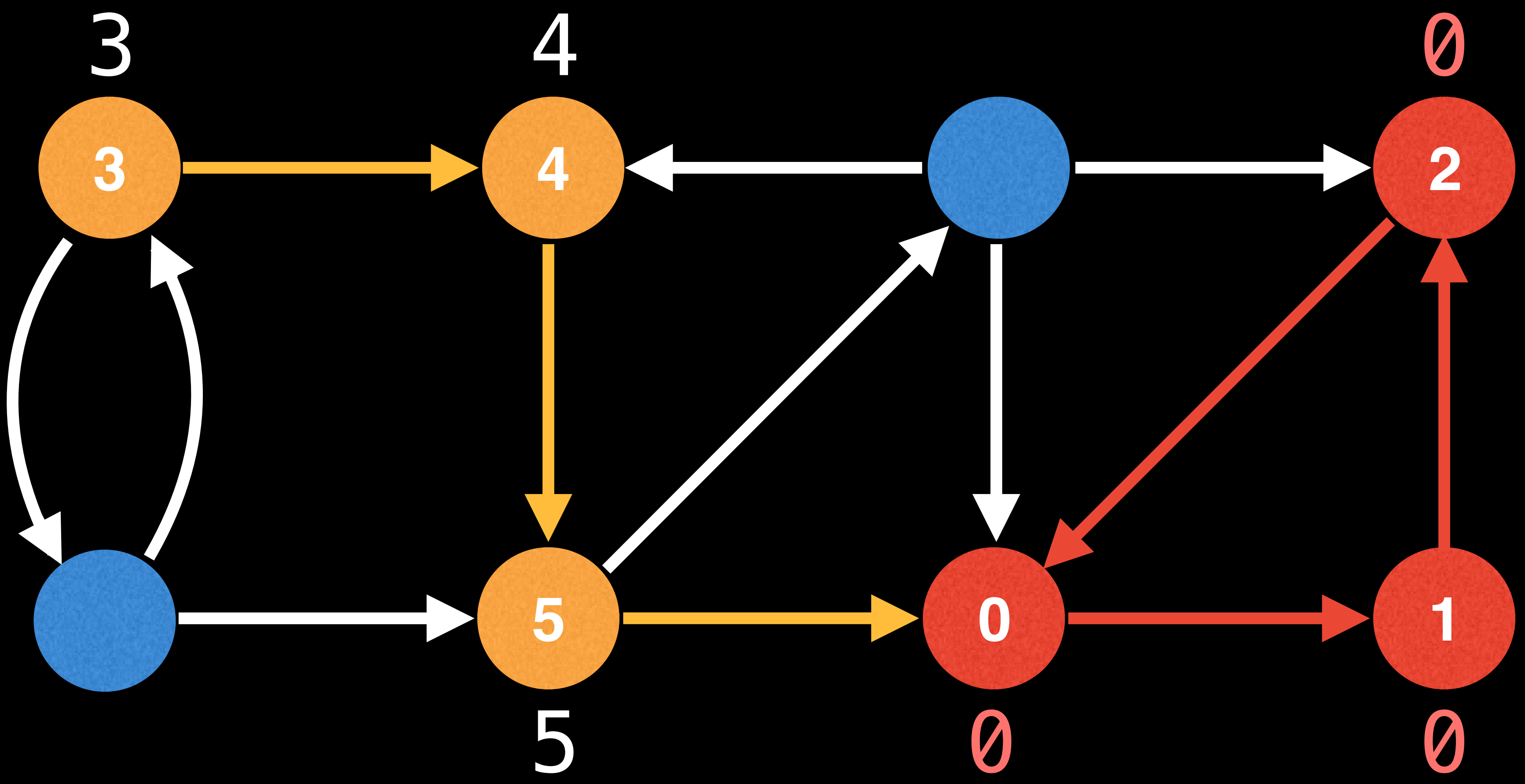
● Visited all neighbours

Stack

5

4

3



● Unvisited

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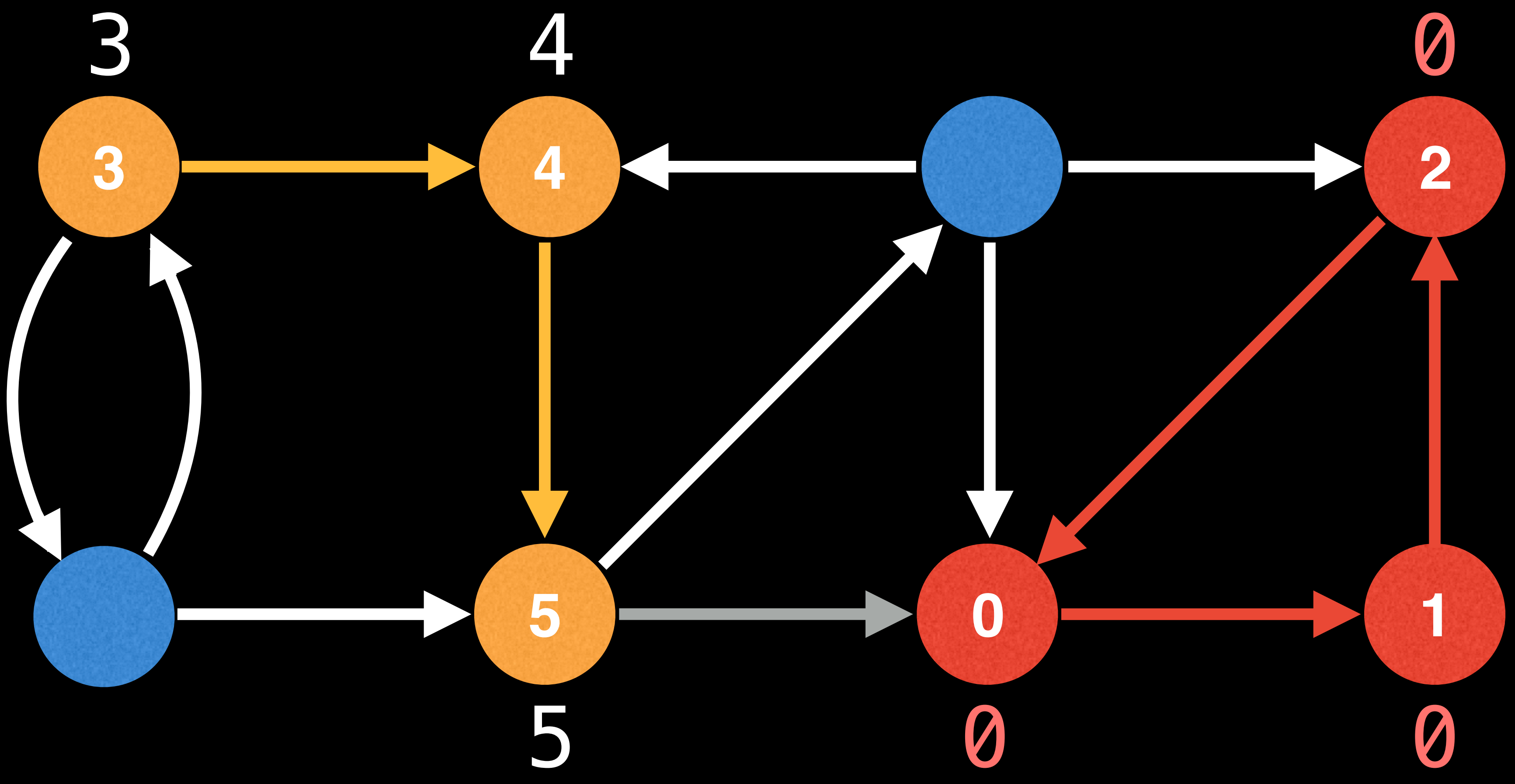
● Visited all neighbours

Stack

5

4

3



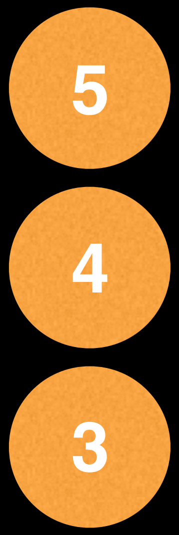
Node 0 is not on stack so don’t min with its low-link value.

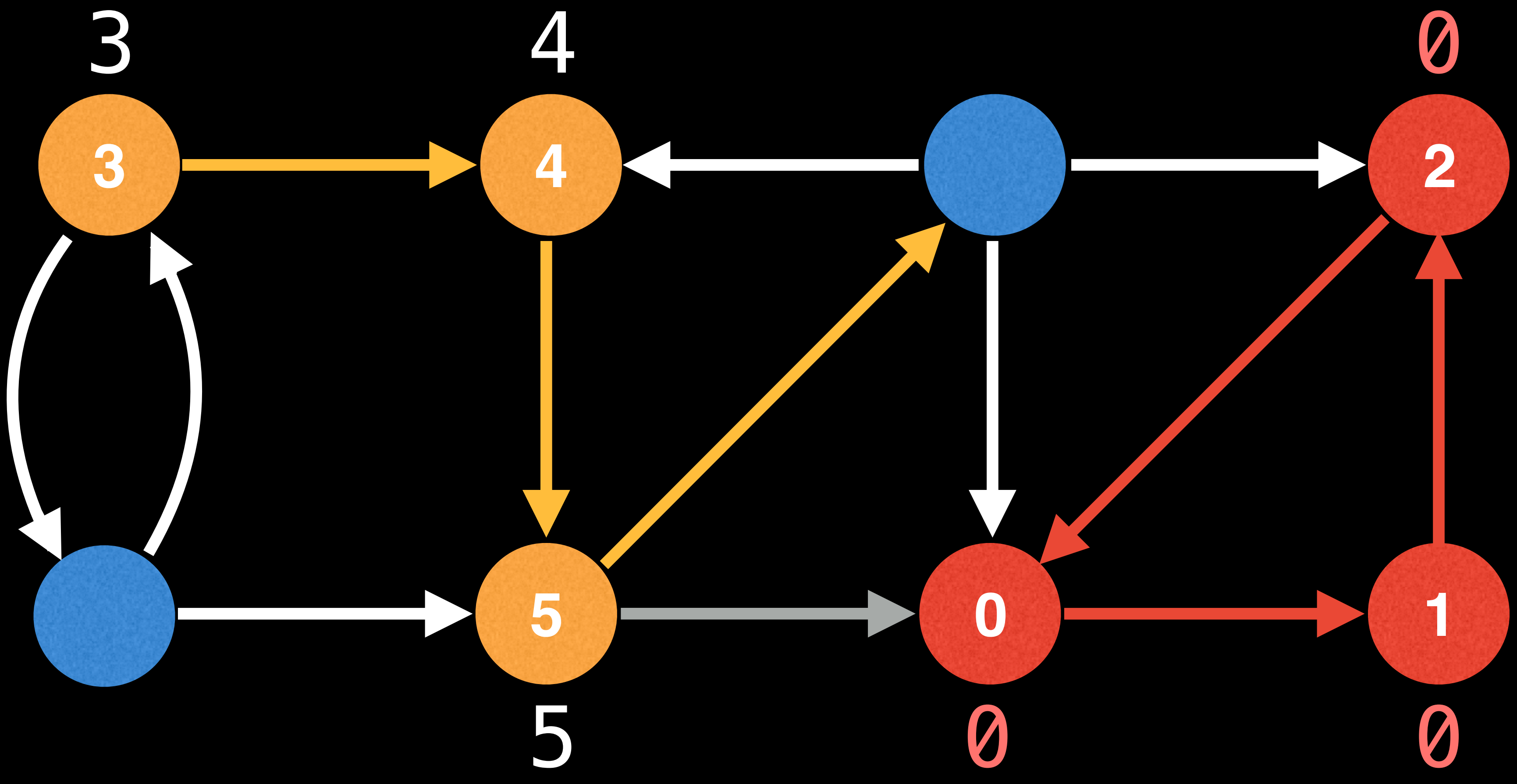
● Unvisited

● Visiting neighbours

● Visited all neighbours

Stack





● Unvisited

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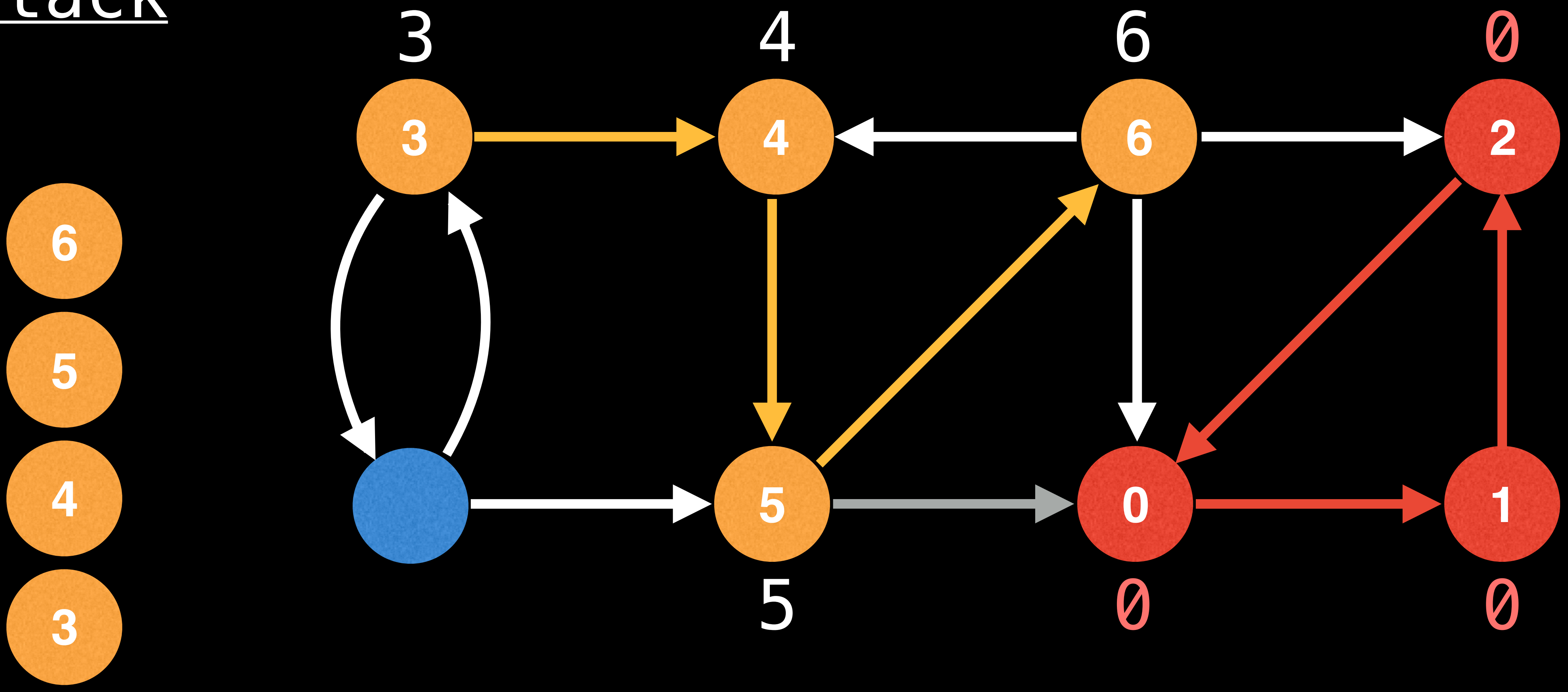
Stack



● Unvisited ● Visiting neighbours ● Visited all neighbours

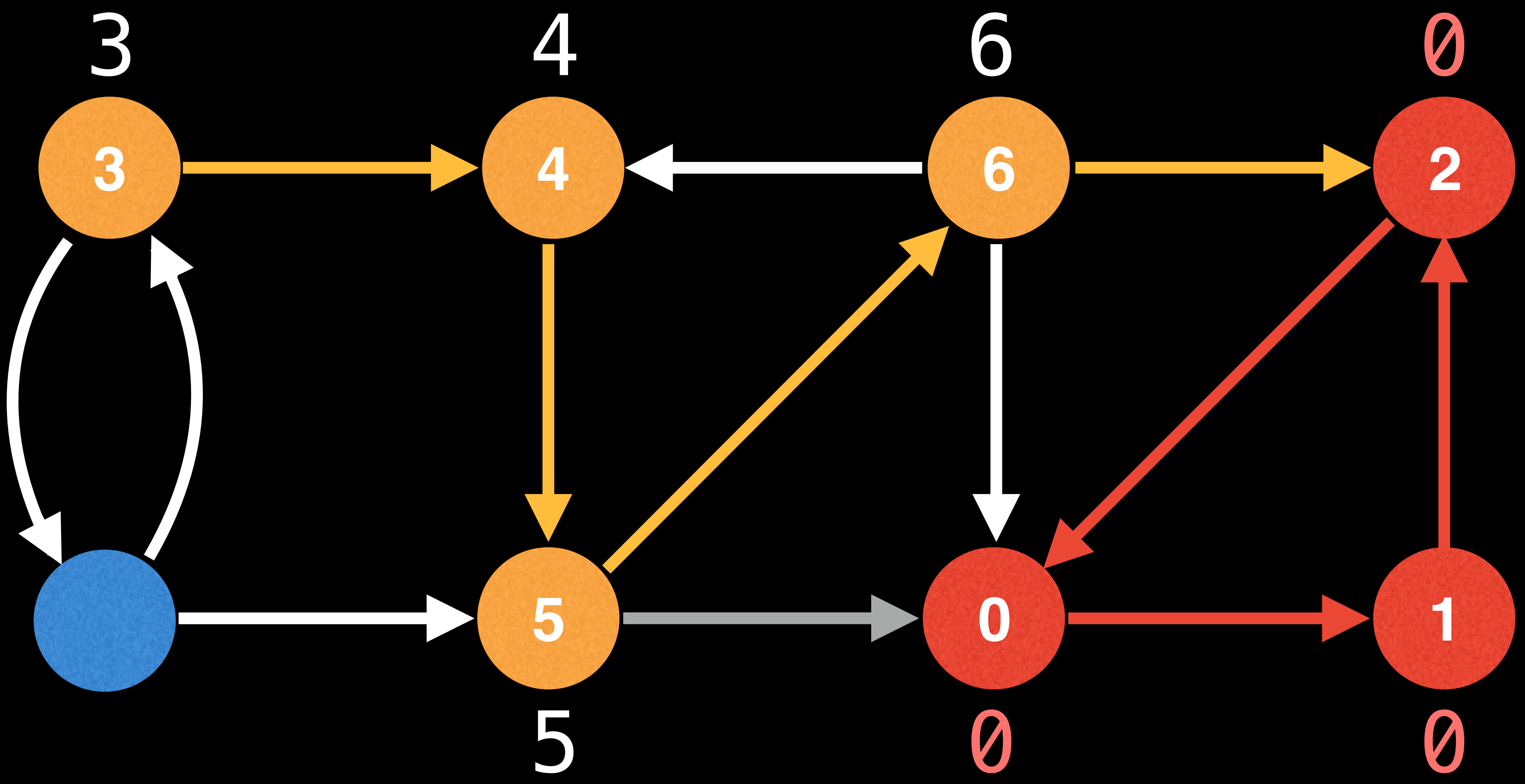
Stack

6

5

4

3



● Unvisited

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Stack

6

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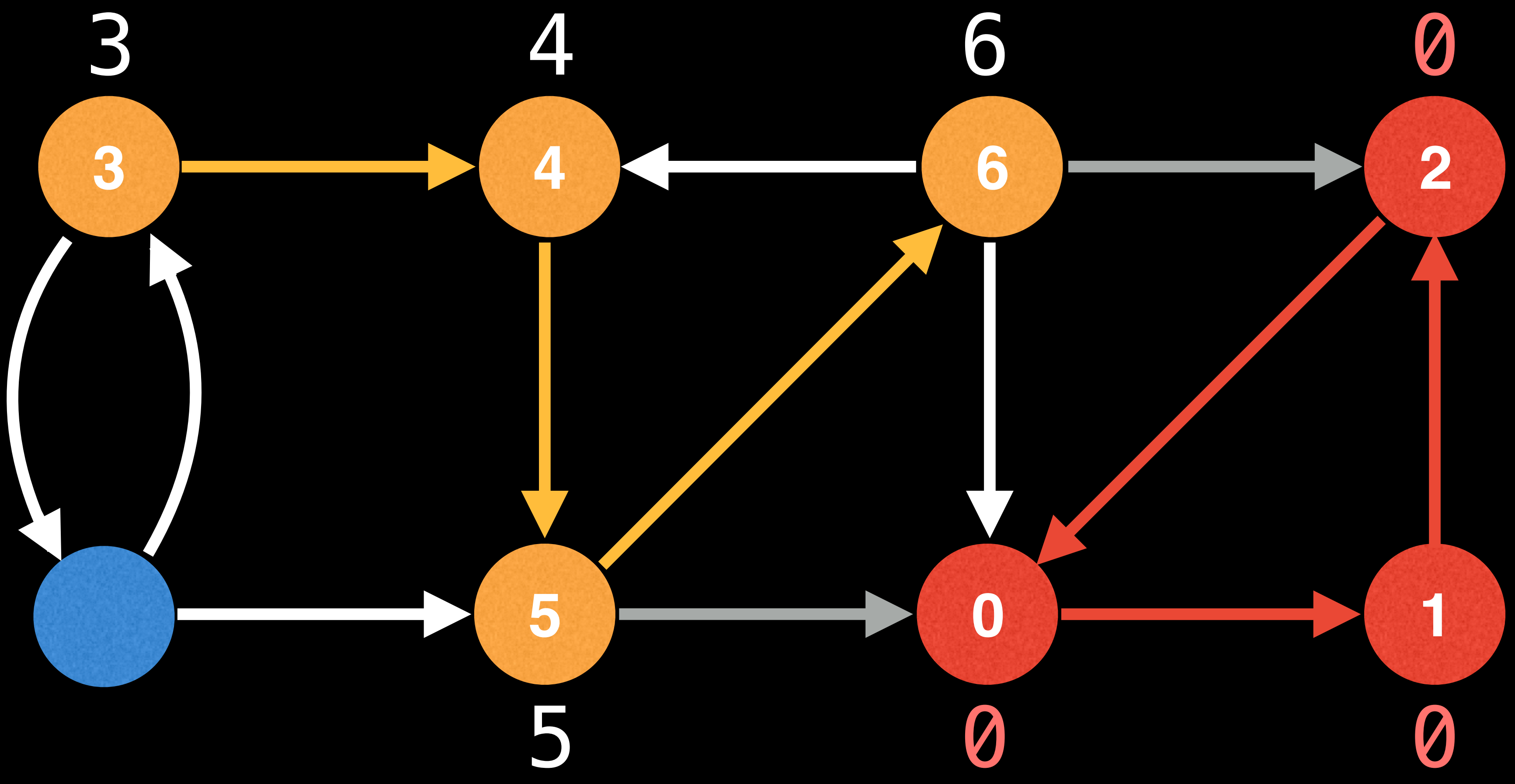
4

3



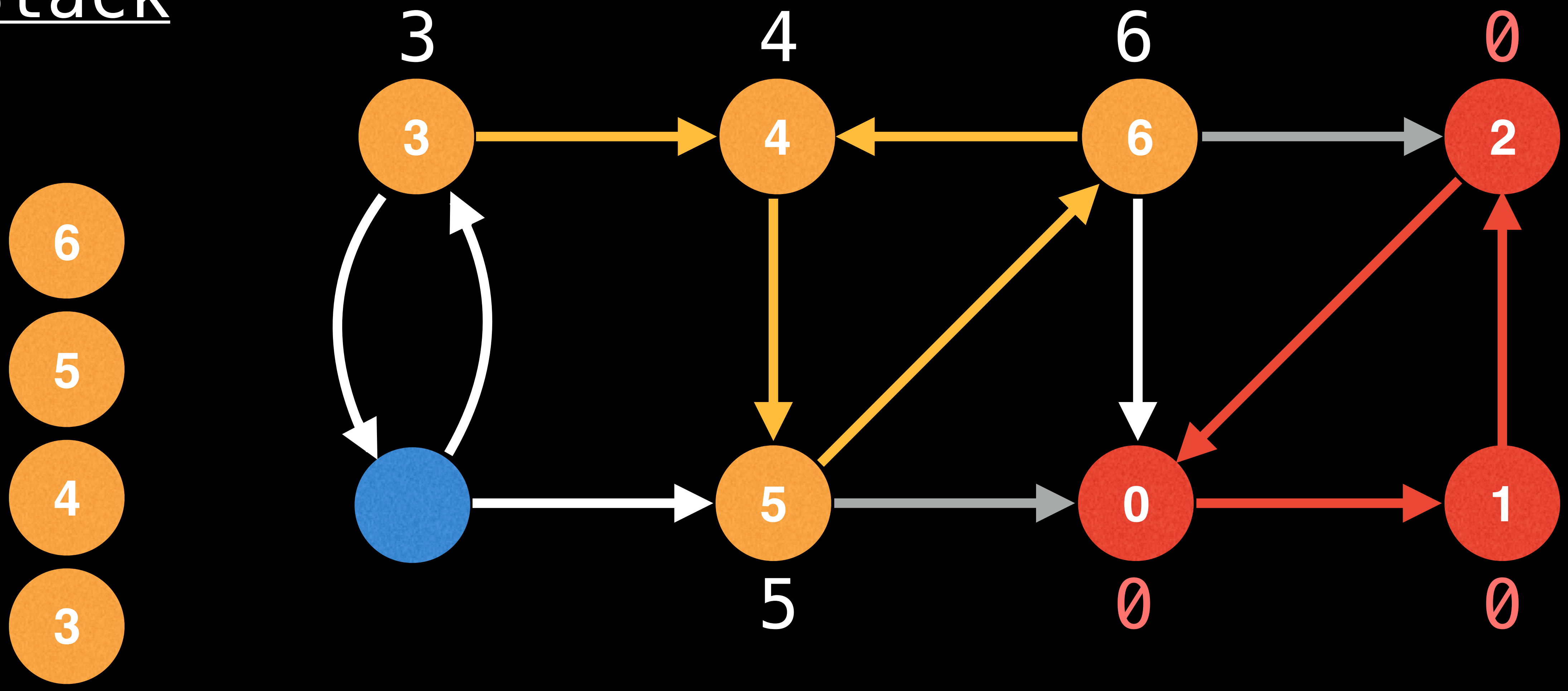
Node 2 is not on stack so don’t min with its low-link value.

● Unvisited

● Visiting neighbours

● Visited all neighbours

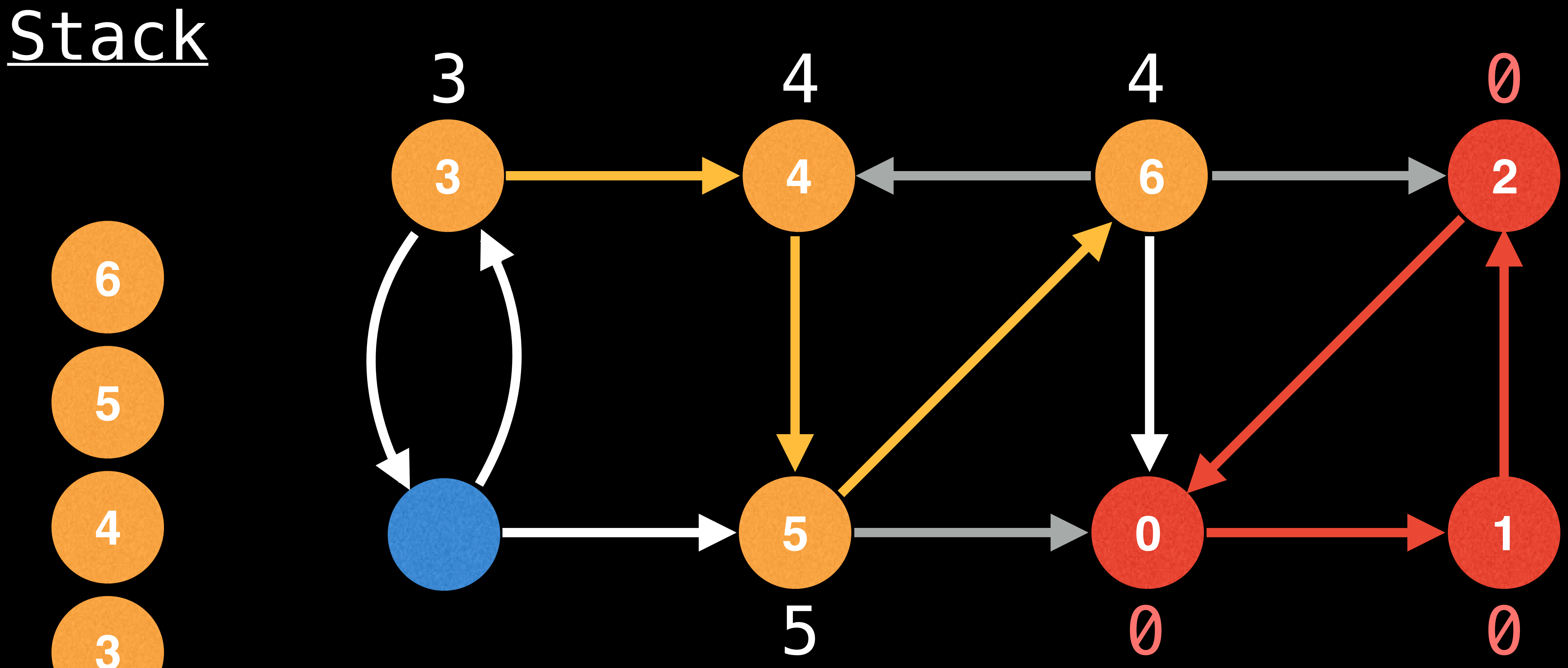
Stack



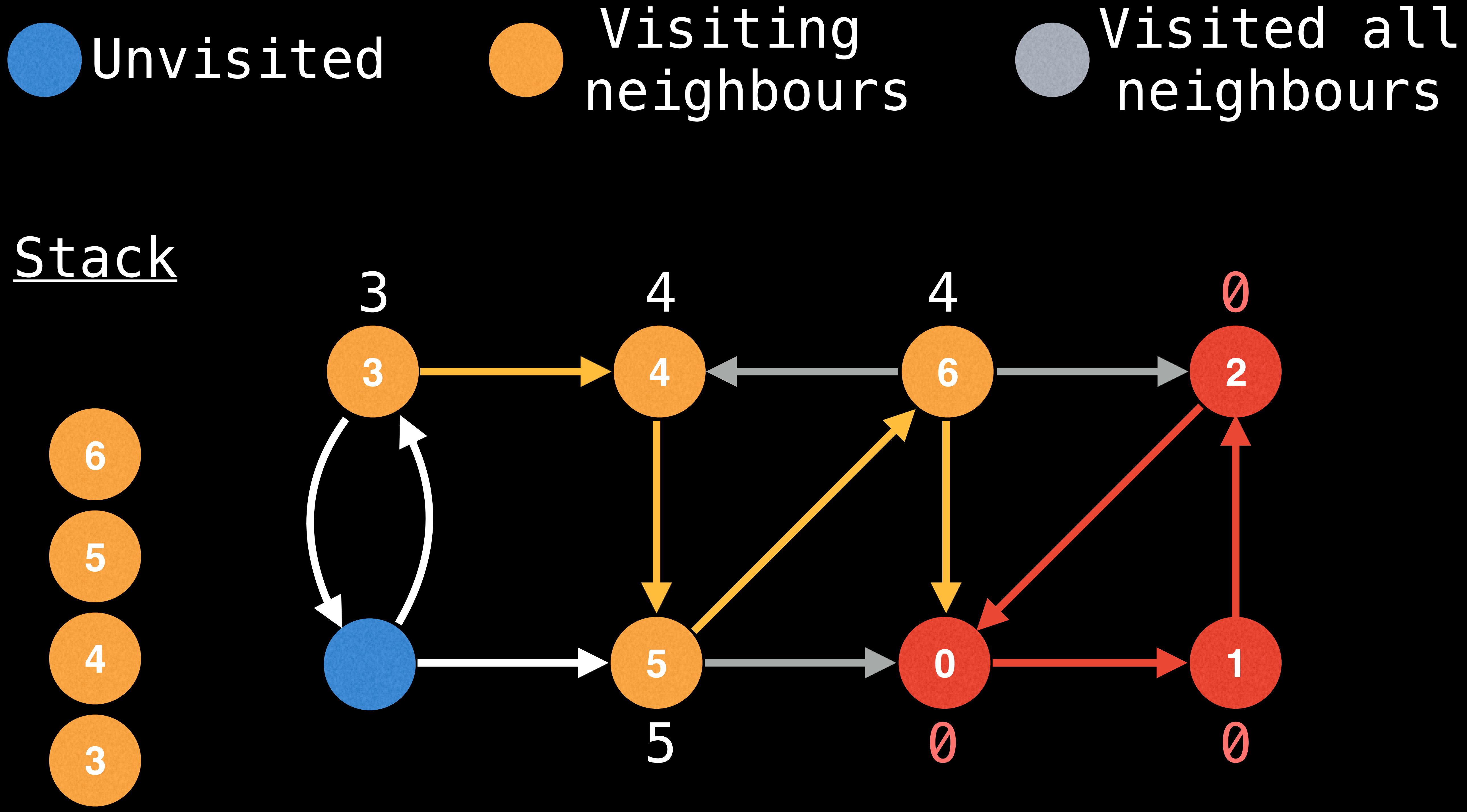
🔵 Unvisited

🟠 Visiting neighbours

⚪ Visited all neighbours

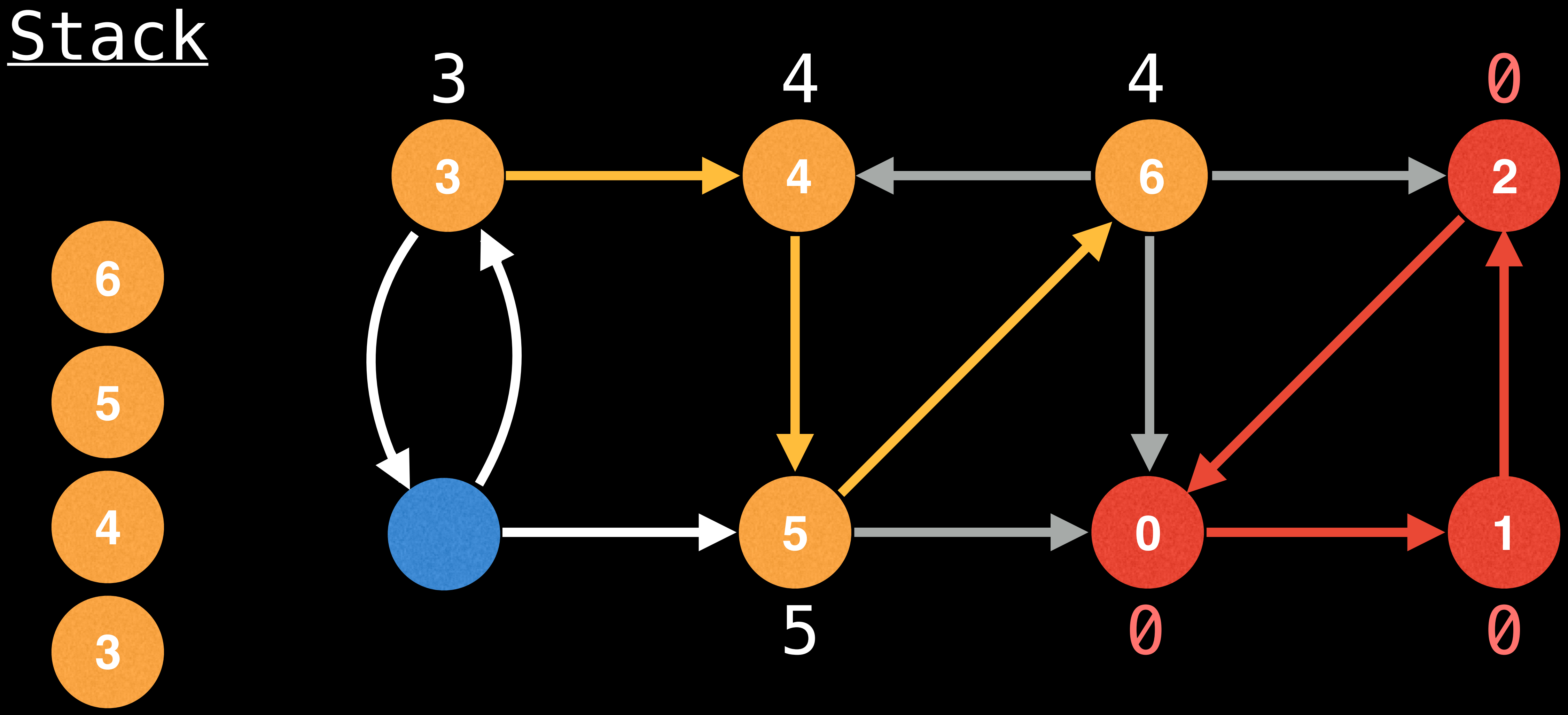


```
lowlink[6] = min(lowlink[6], lowlink[4])
           = 4
```



Unvisited
Visiting neighbours
Visited all neighbours
Stack
6
5
4
3
3
4
4
0
3
4
6
2
5
0
1
5
0
0

● Unvisited ● Visiting neighbours ● Visited all neighbours



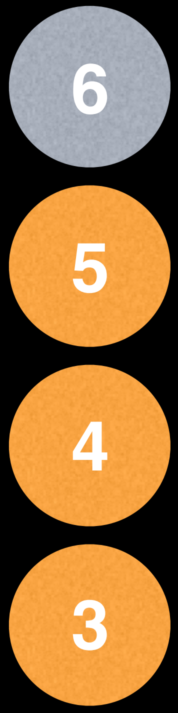
Node 0 is not on stack so don’t min with its low-link value.

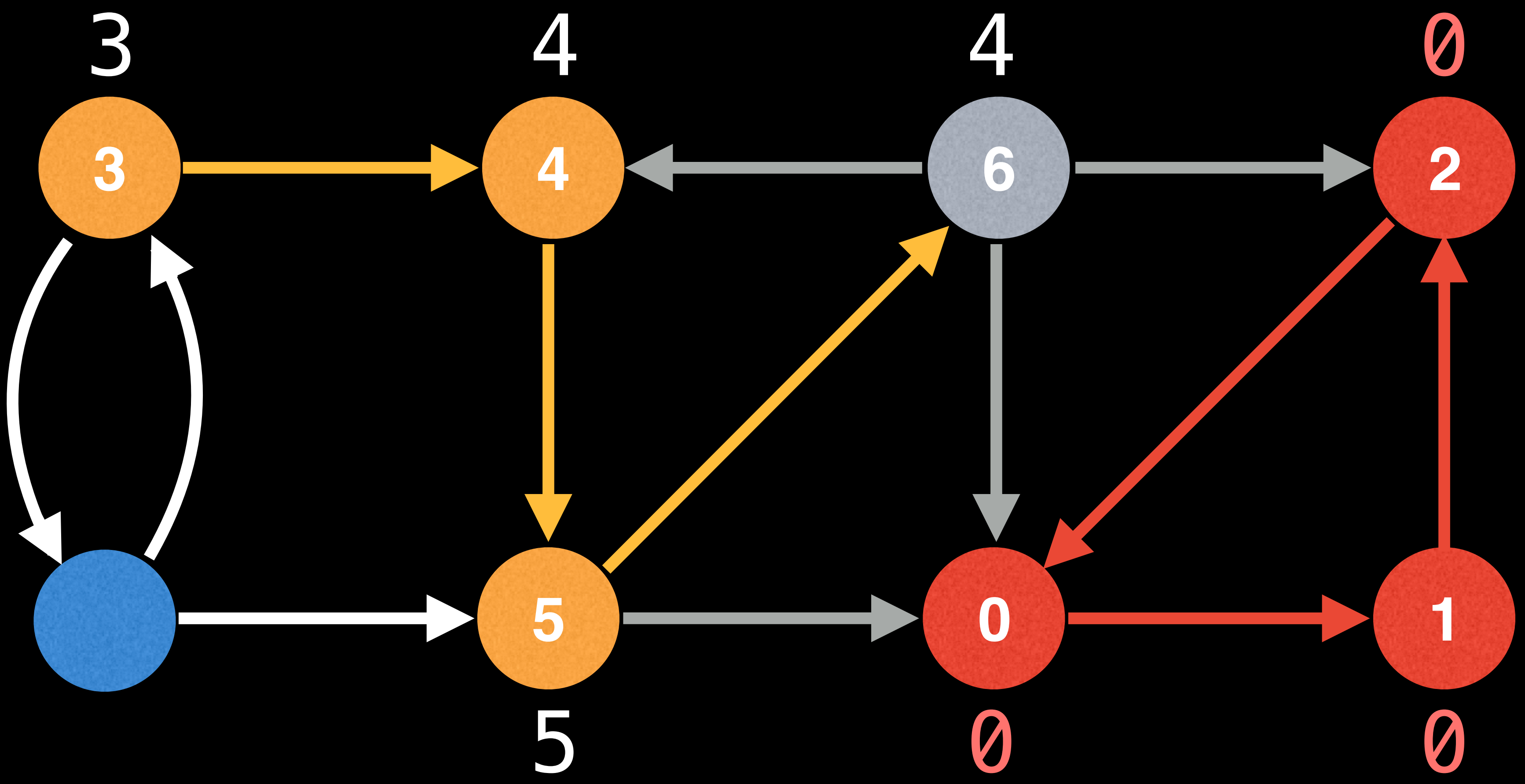
● Unvisited

● Visiting neighbours

● Visited all neighbours

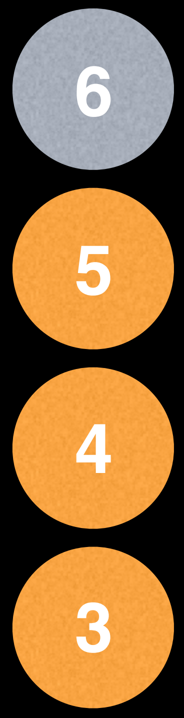
Stack

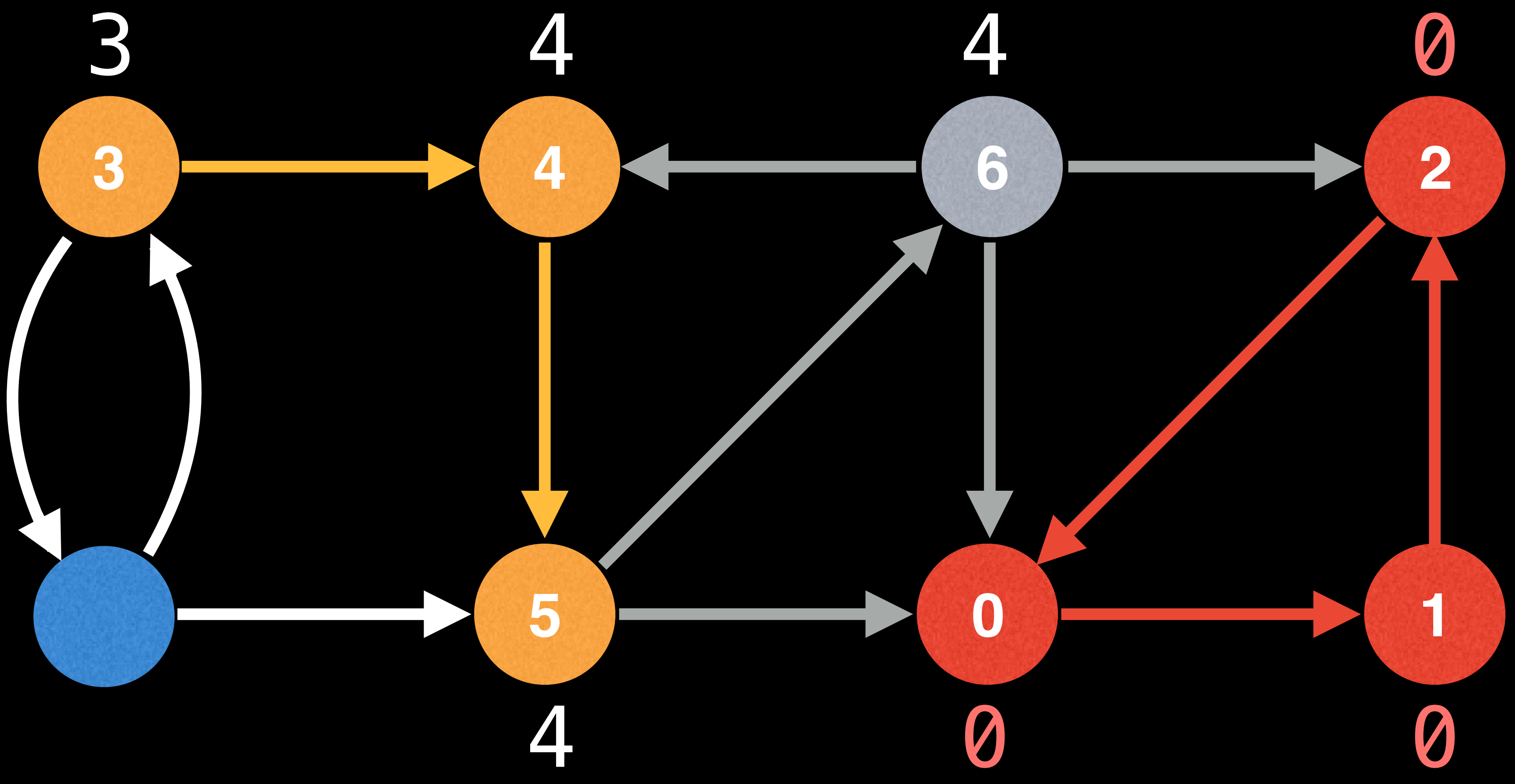




● Unvisited ● Visiting neighbours ● Visited all neighbours

Stack





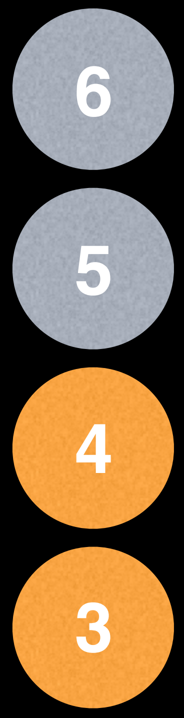
```
lowlink[5] = min(lowlink[5], lowlink[6])
           = 4
```

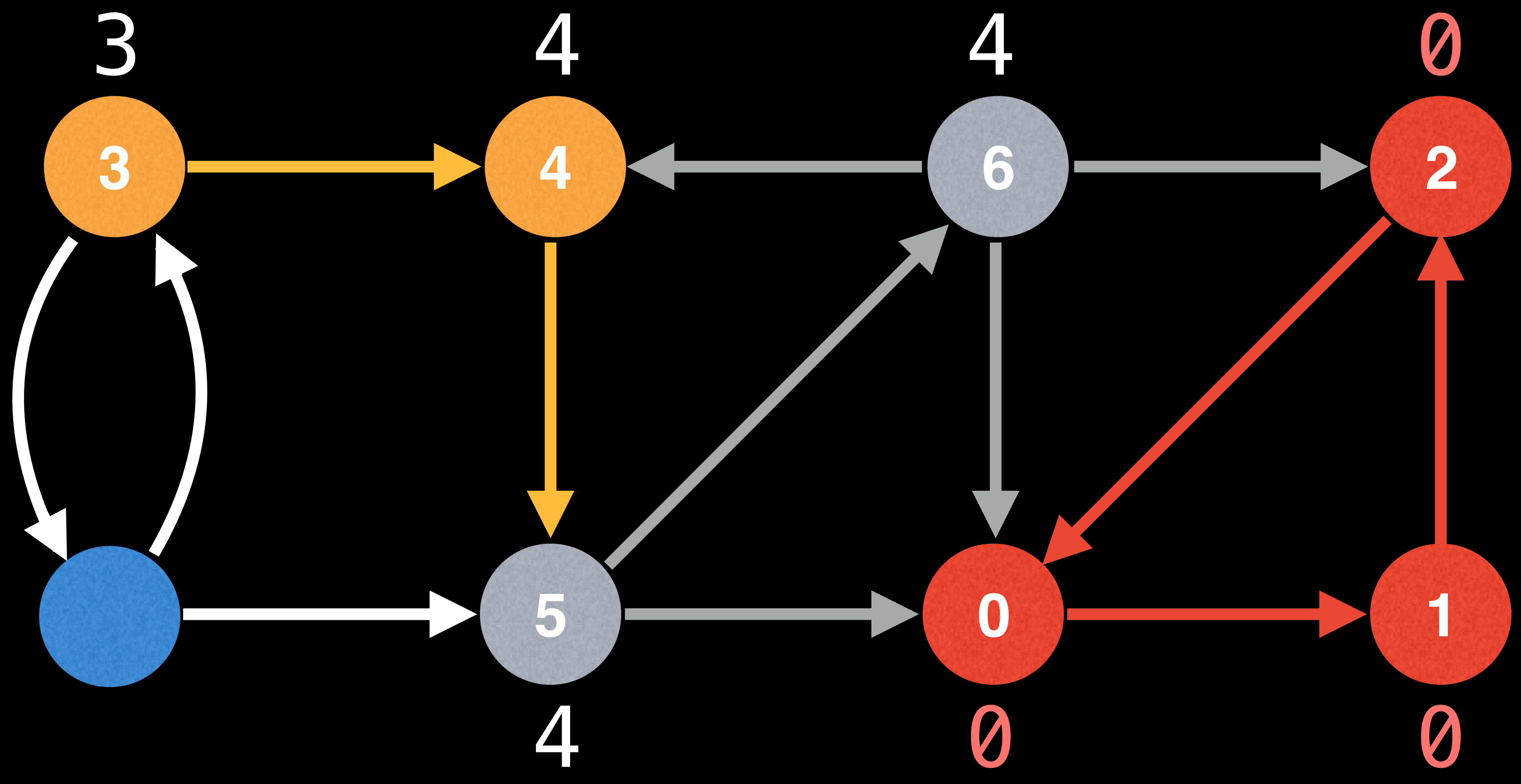
● Unvisited

● Visiting neighbours

● Visited all neighbours

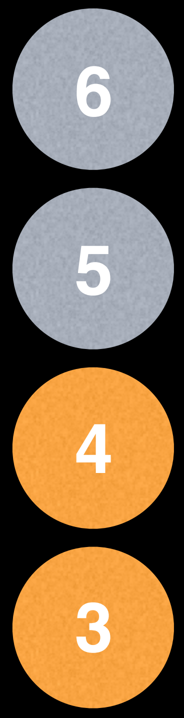
Stack

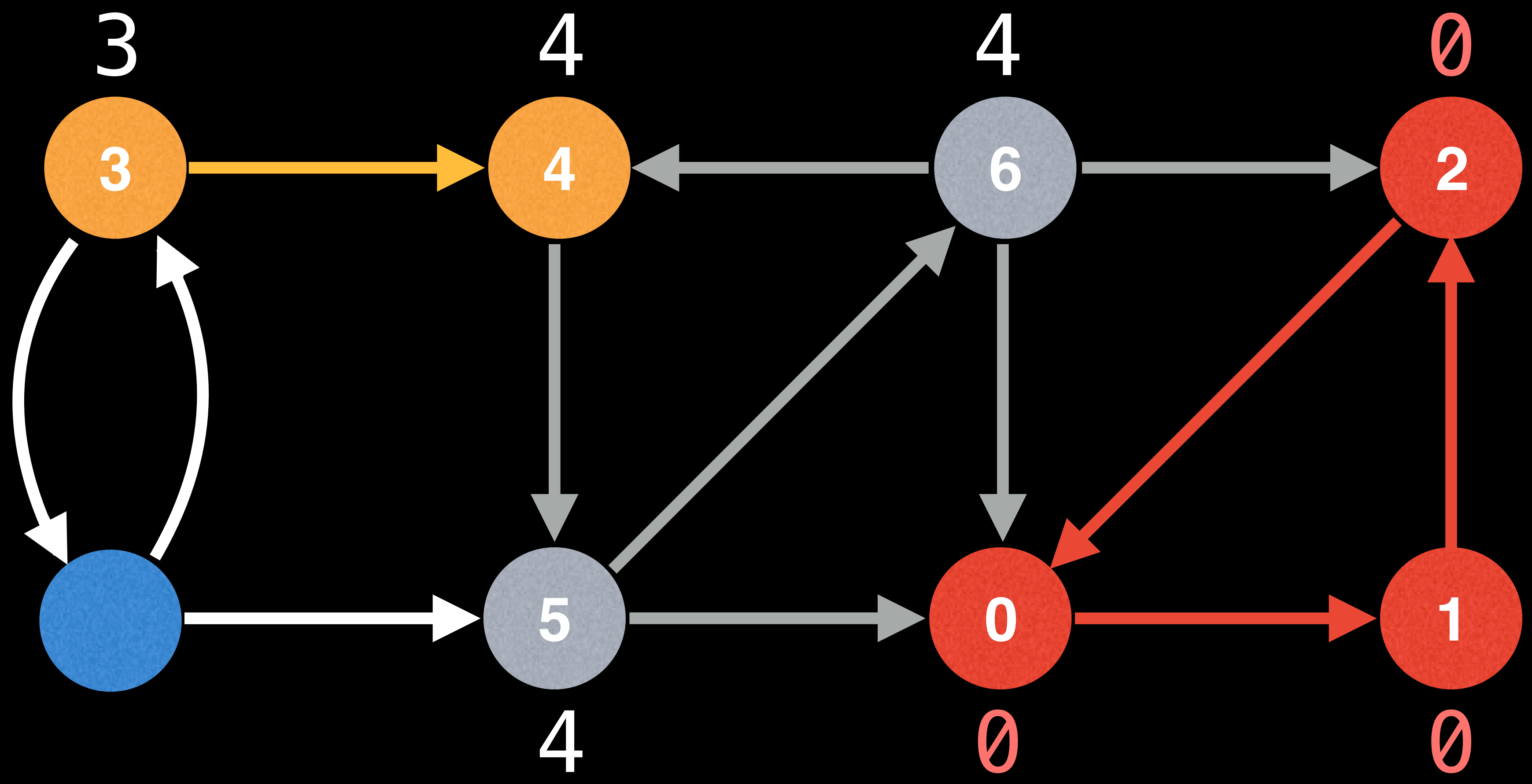




● Unvisited ● Visiting neighbours ● Visited all neighbours

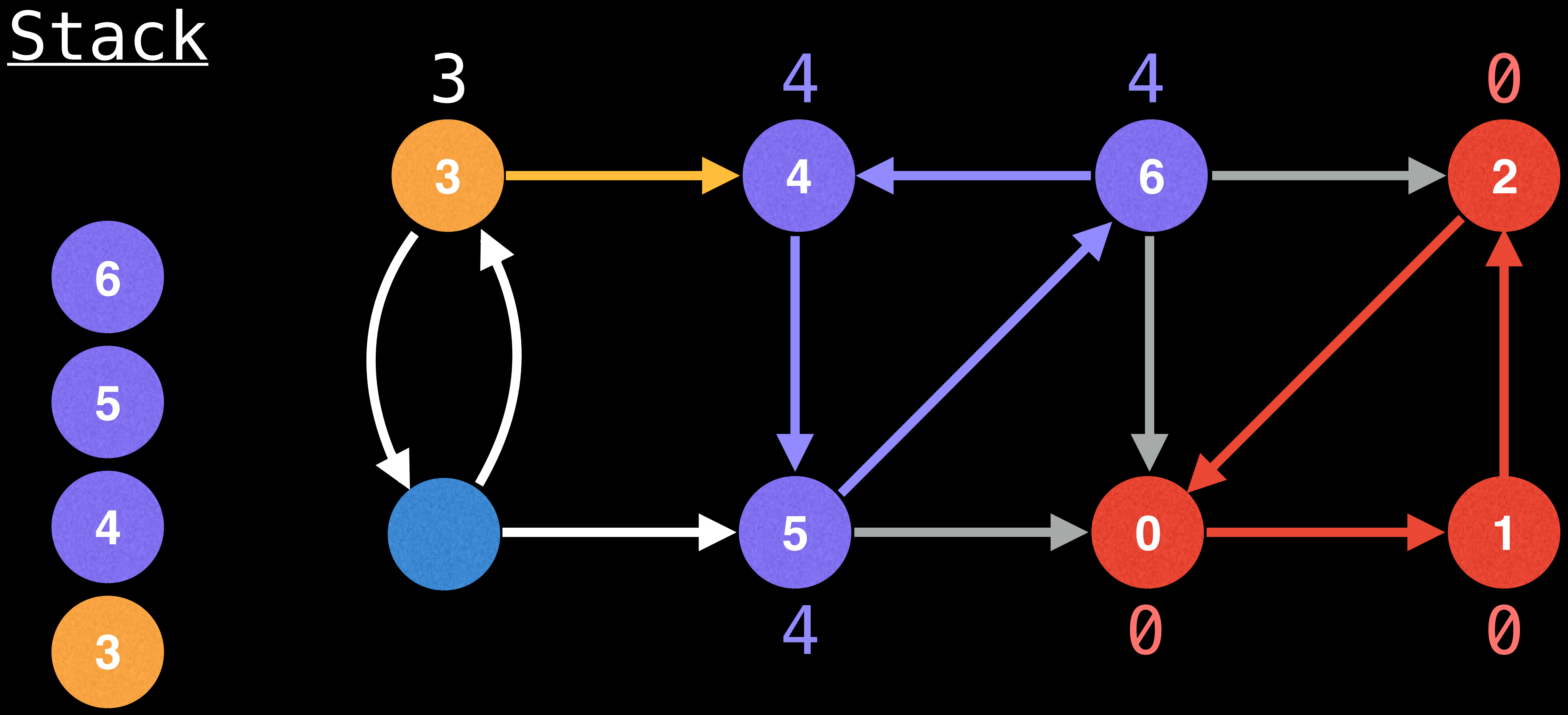
Stack





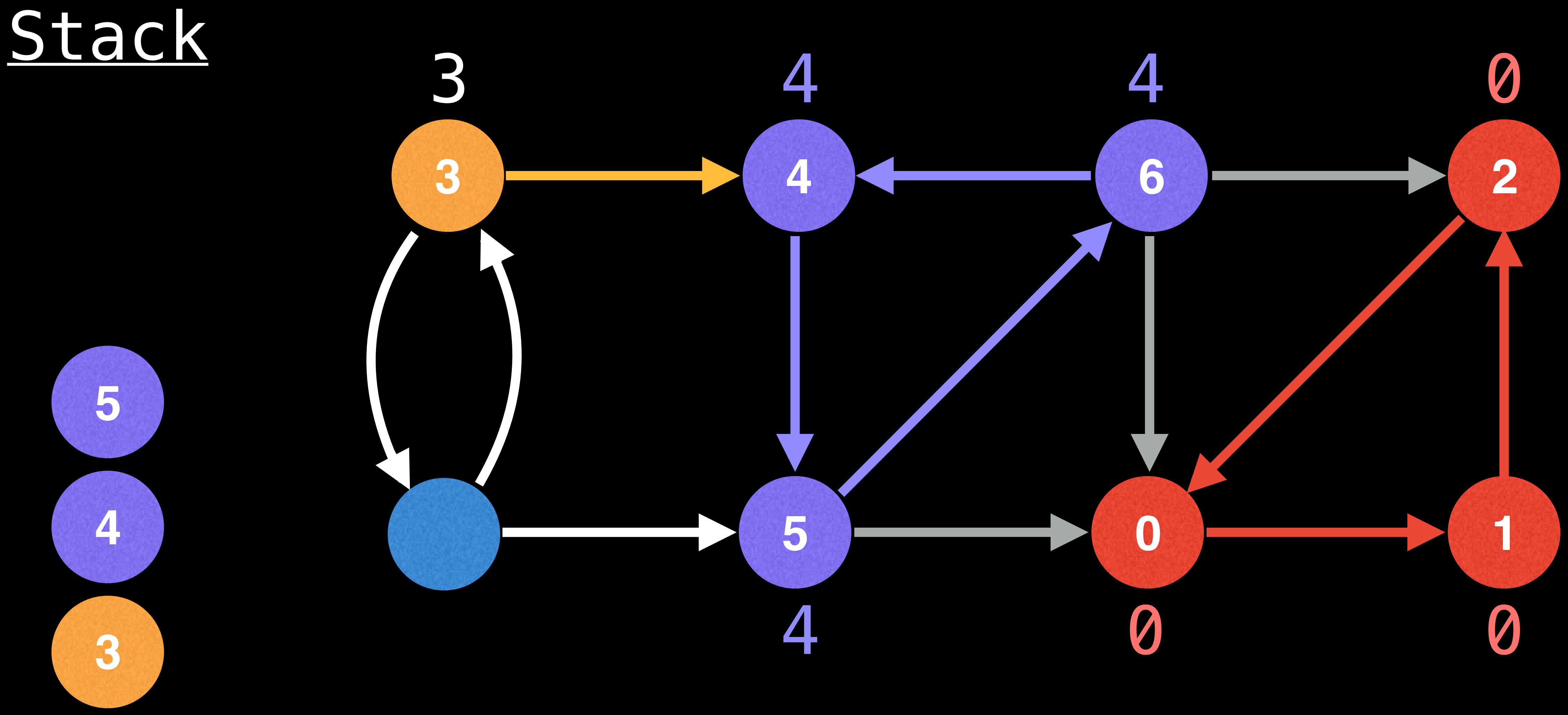
```
lowlink[4] = min(lowlink[4], lowlink[5])
           = 4
```

● Unvisited ● Visiting neighbours ● Visited all neighbours



When a completed SCC is found (current node has visited all its neighbours and its lowlink value equals its id) pop off all associated nodes off the stack.

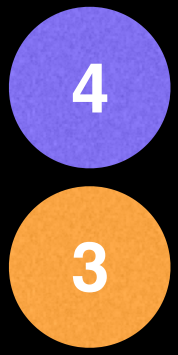
● Unvisited ● Visiting neighbours ● Visited all neighbours

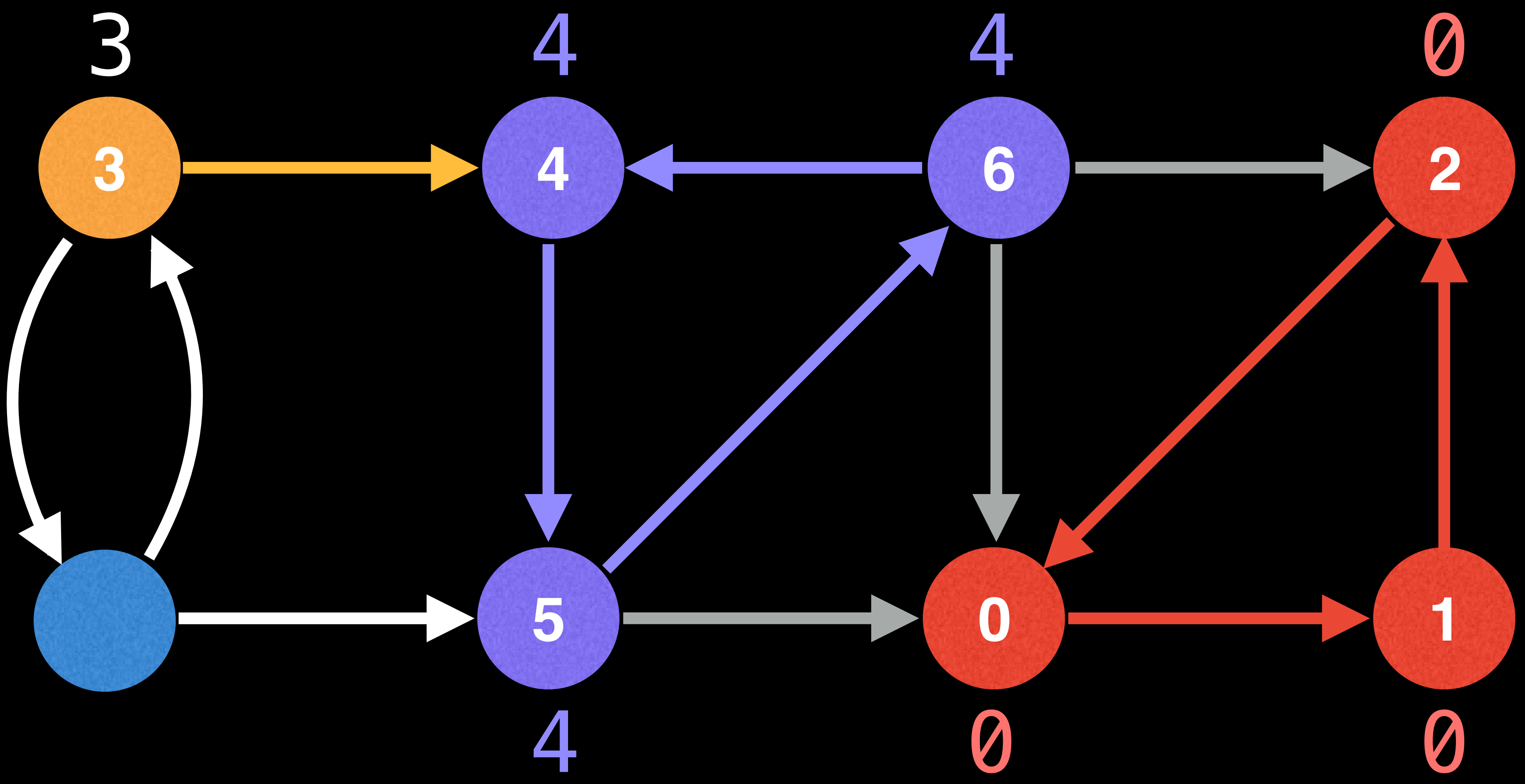


When a completed SCC is found (current node has visited all its neighbours and its lowlink value equals its id) pop off all associated nodes off the stack.

● Unvisited ● Visiting neighbours ● Visited all neighbours

Stack

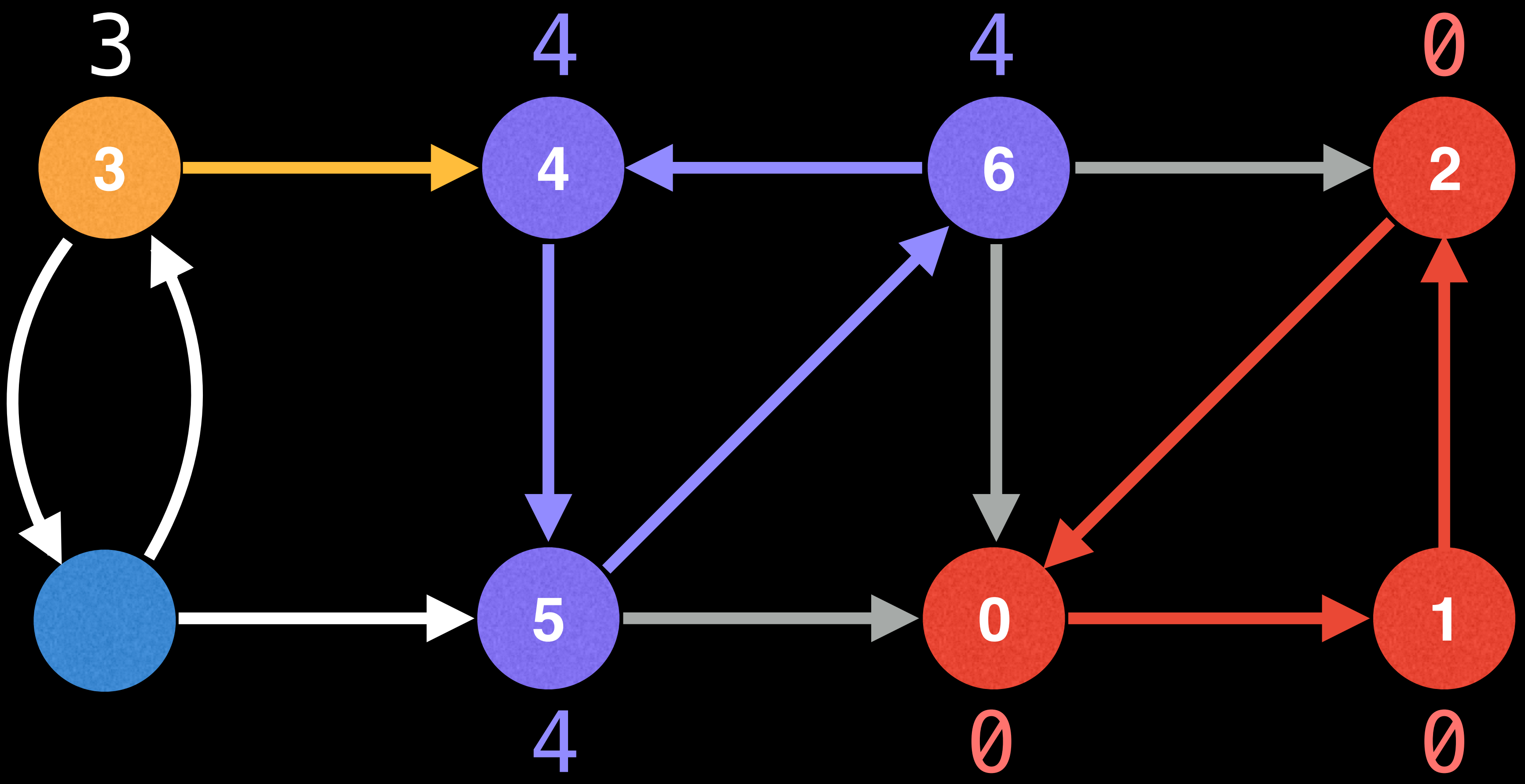


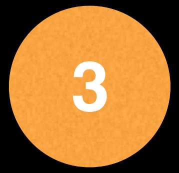


When a completed SCC is found (current node has visited all its neighbours and its lowlink value equals its id) pop off all associated nodes off the stack.

● Unvisited ● Visiting neighbours ● Visited all neighbours

Stack





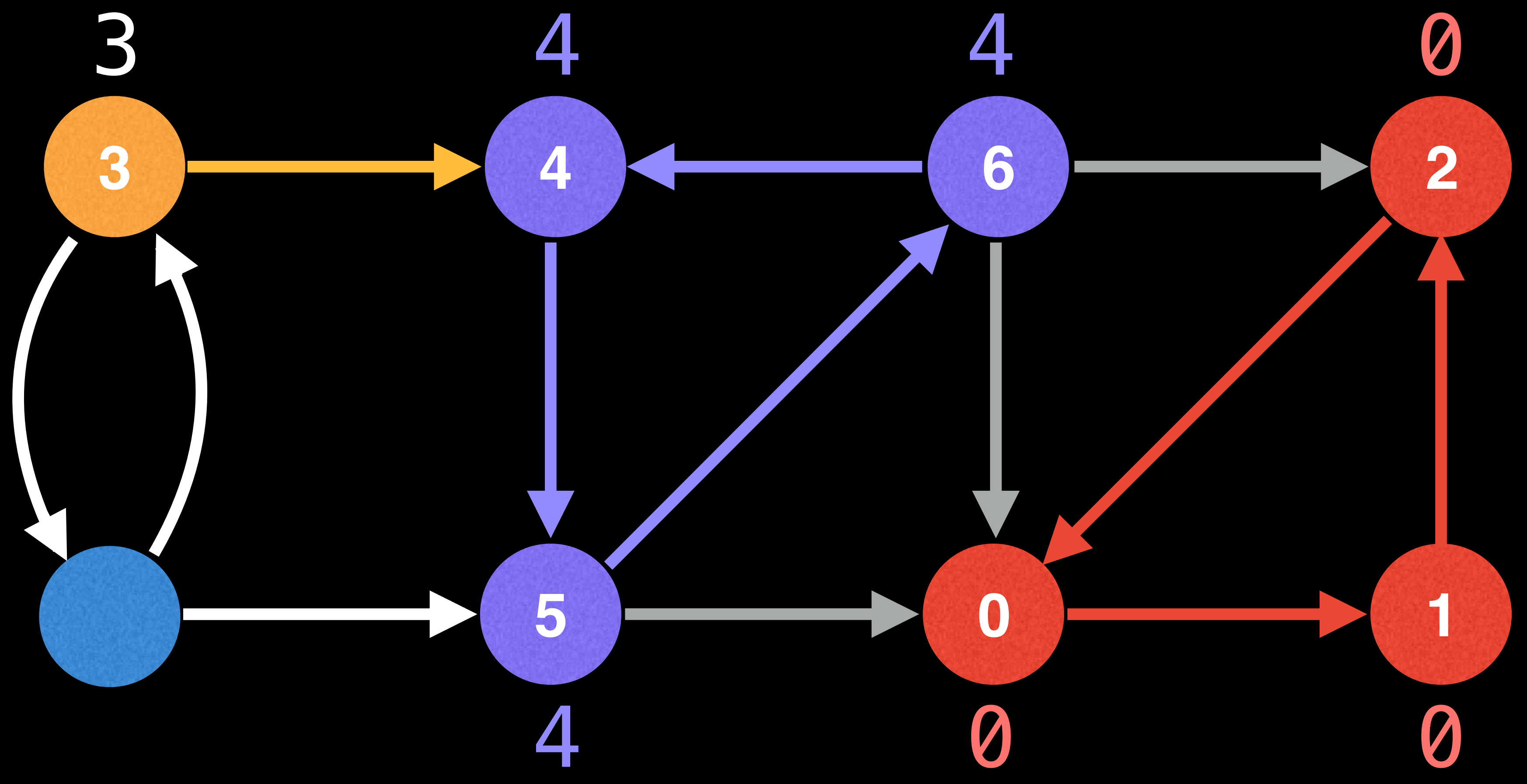
When a completed SCC is found (current node has visited all its neighbours and its lowlink value equals its id) pop off all associated nodes off the stack.

● Unvisited

● Visiting neighbours

● Visited all neighbours

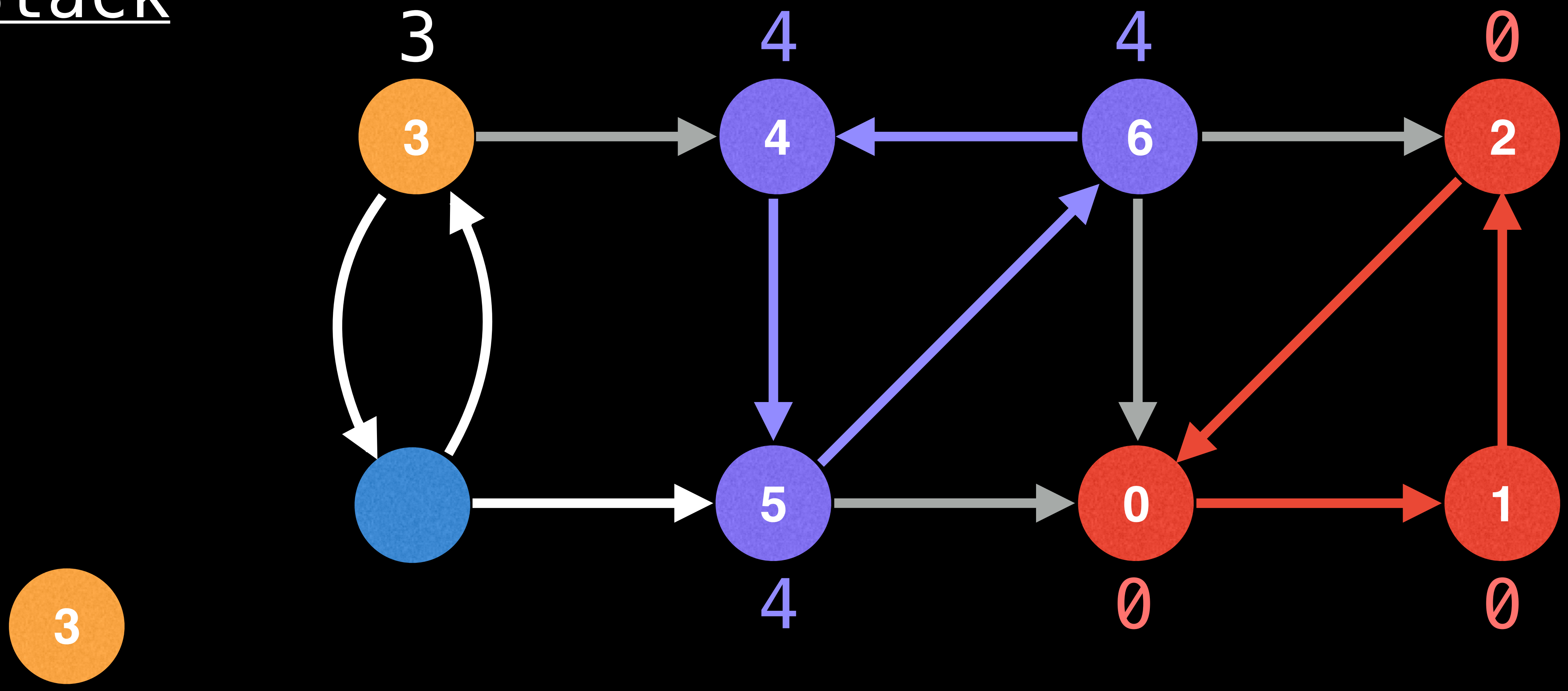
Stack



3

🔵 Unvisited 🟠 Visiting neighbours ⚪ Visited all neighbours

Stack



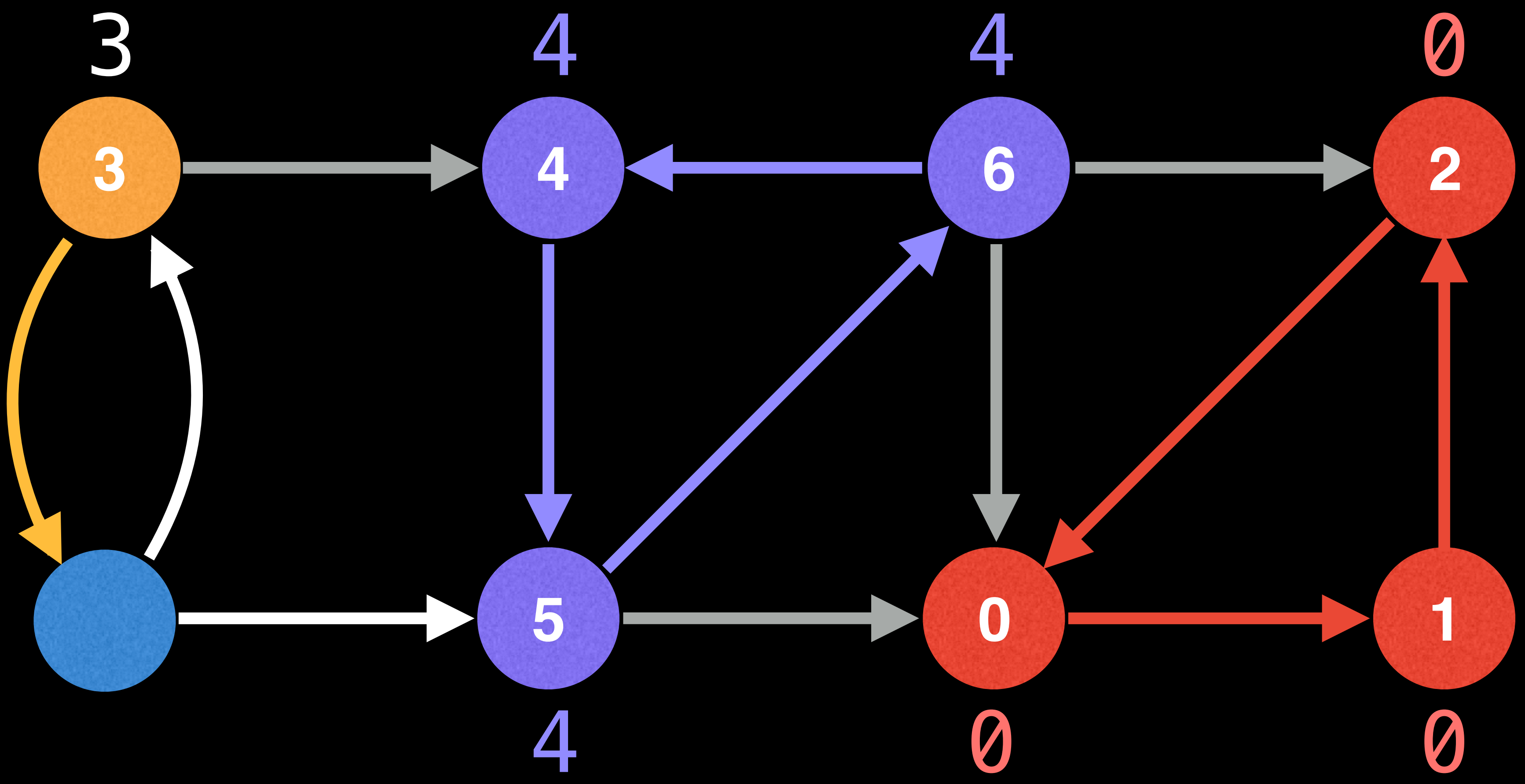
Node 4 is not on stack so don’t min with its low-link value.

● Unvisited

● Visiting neighbours

● Visited all neighbours

Stack



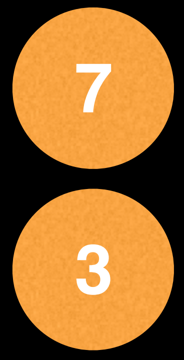
3

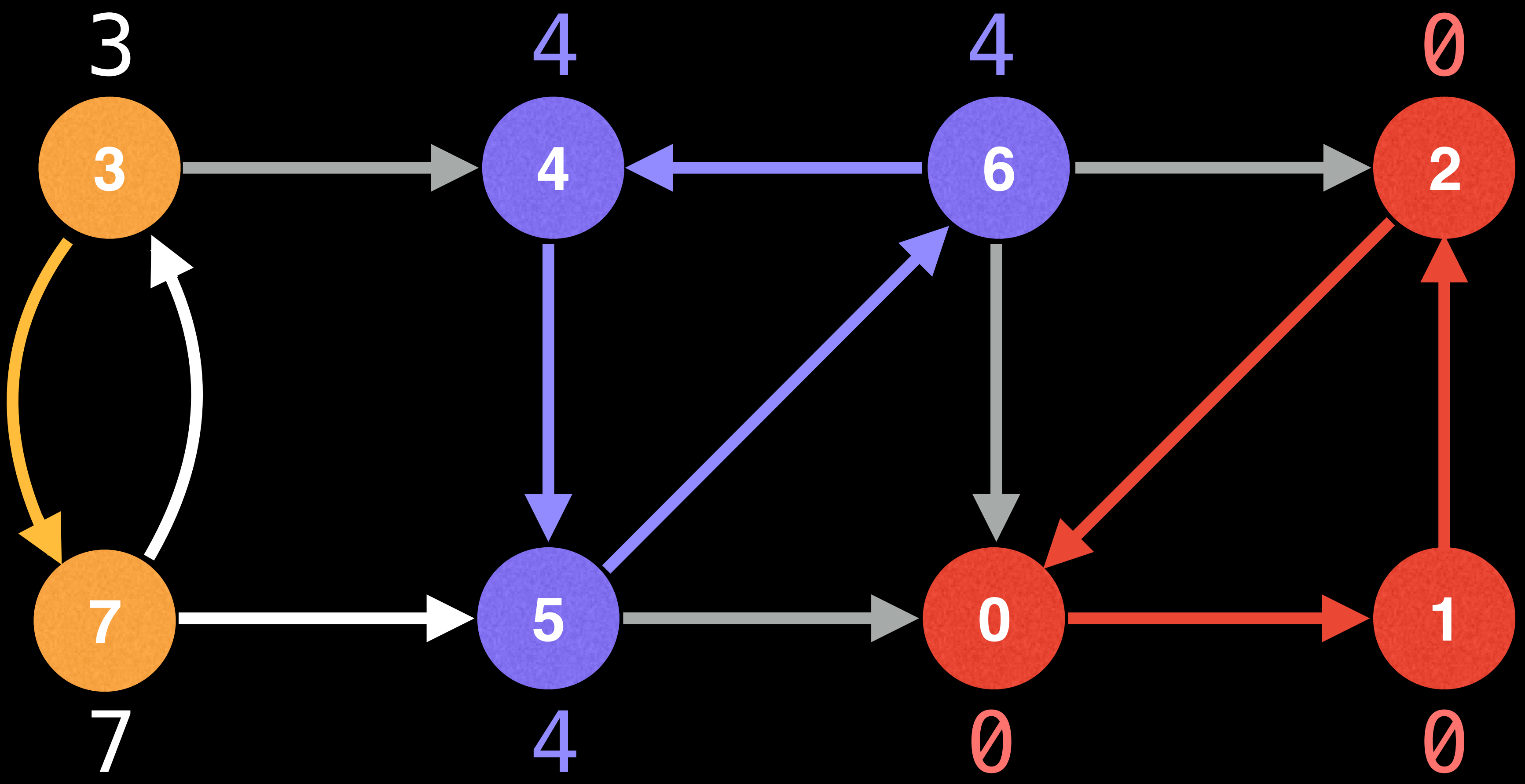
● Unvisited

● Visiting neighbours

● Visited all neighbours

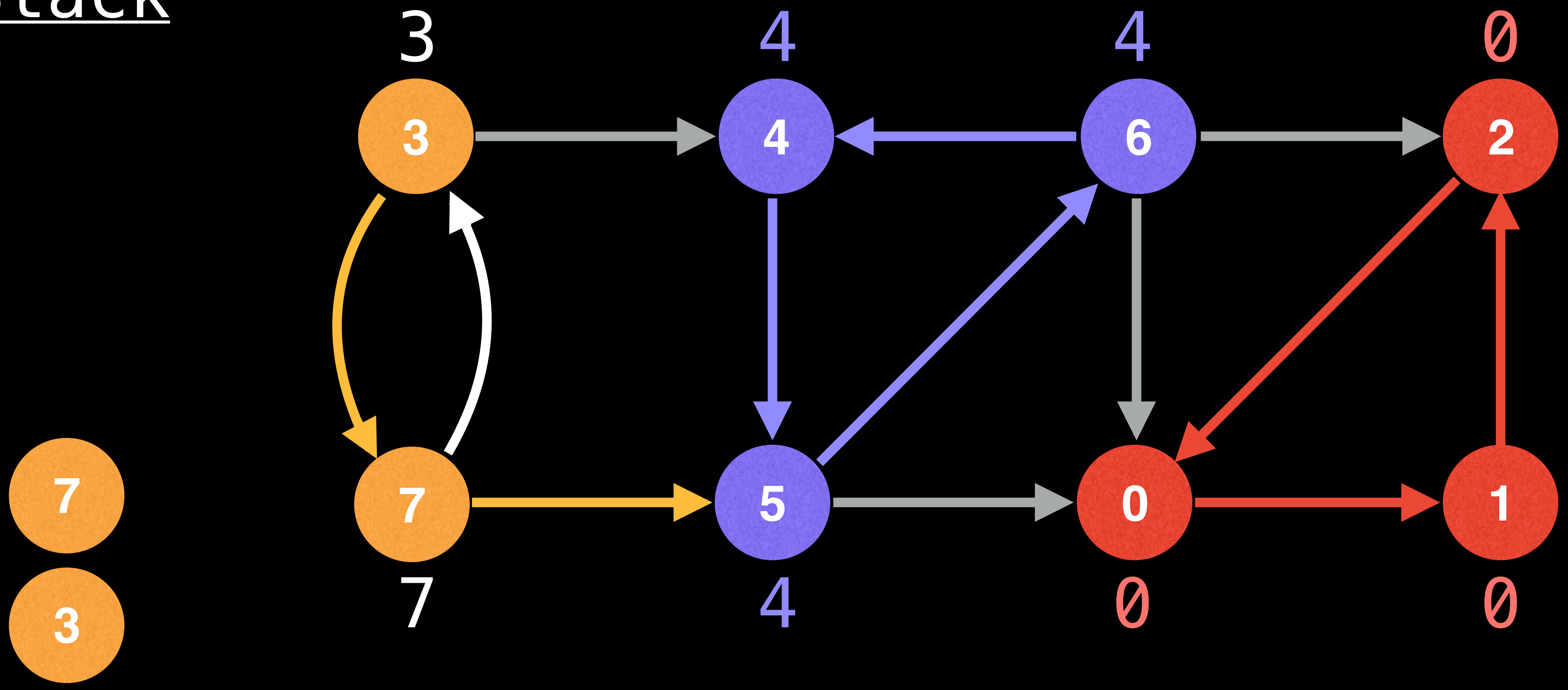
Stack





● Unvisited ● Visiting neighbours ● Visited all neighbours

Stack

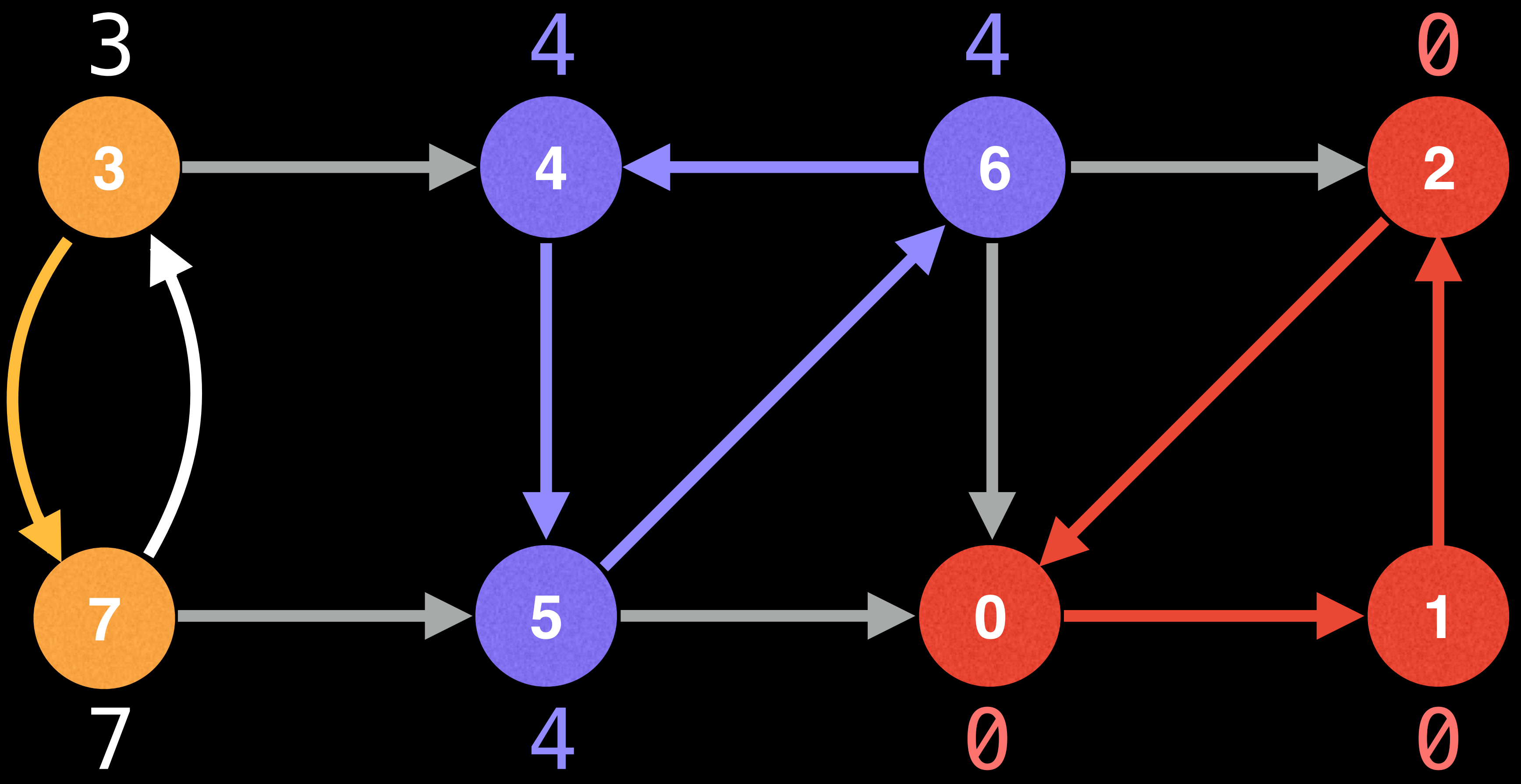


● Unvisited ● Visiting neighbours ● Visited all neighbours

Stack

7

3



Node 5 is not on stack so don’t min with its low-link value.

● Unvisited

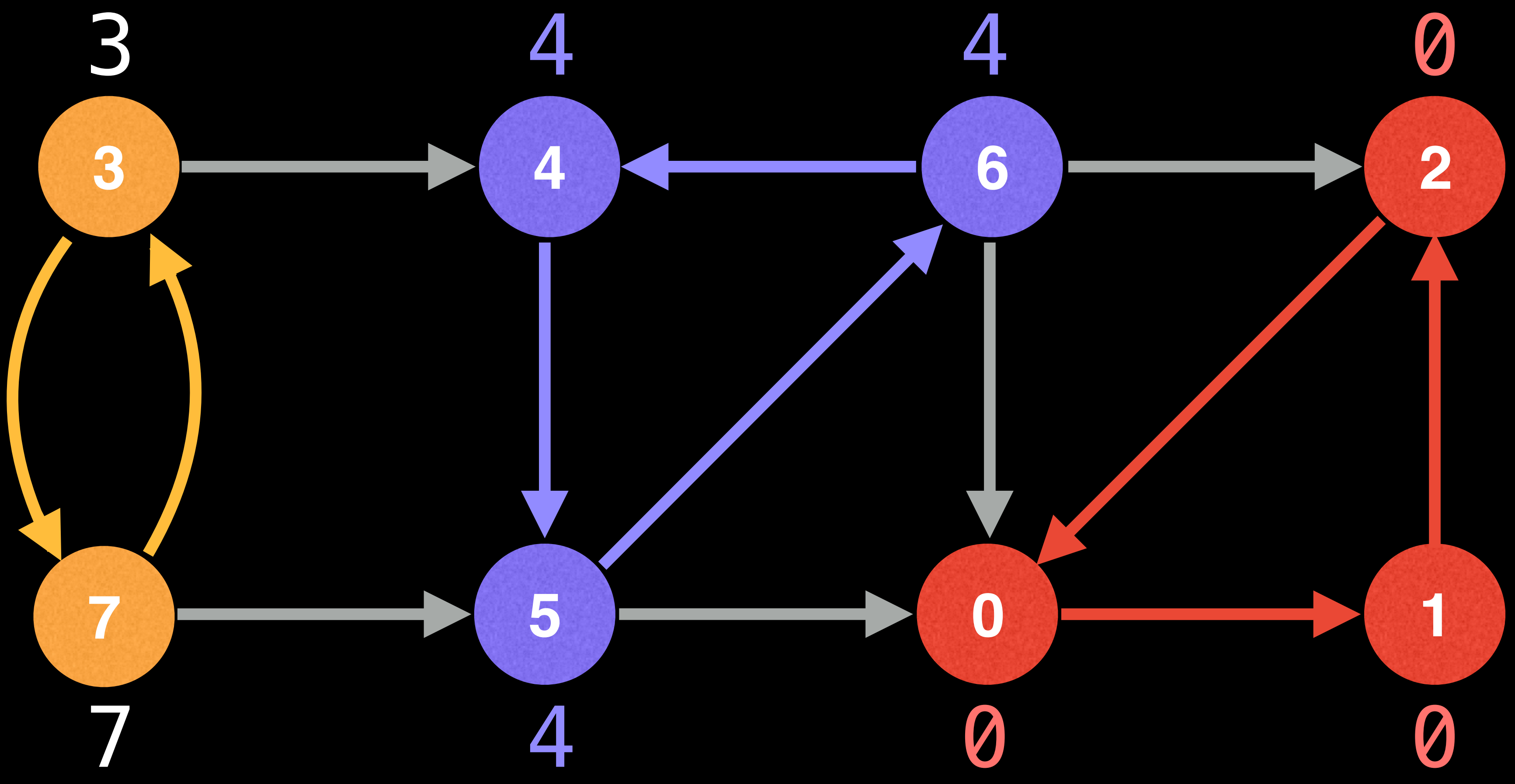
● Visiting neighbours

● Visited all neighbours

Stack

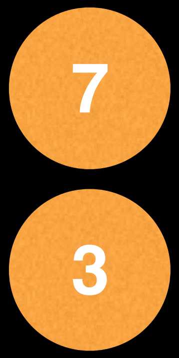
7

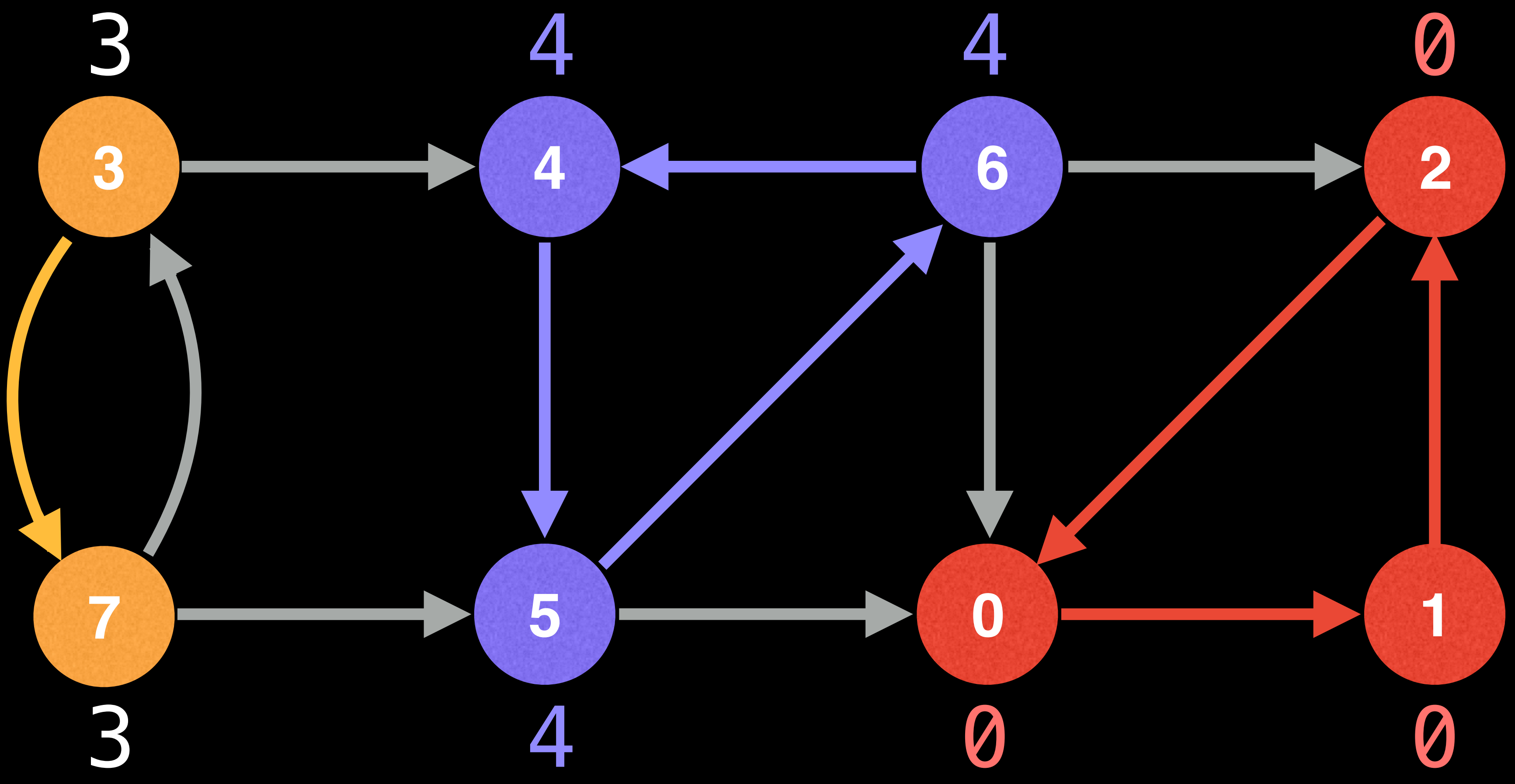
3



🔵 Unvisited 🟠 Visiting neighbours ⚪ Visited all neighbours

Stack

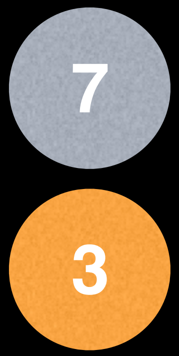


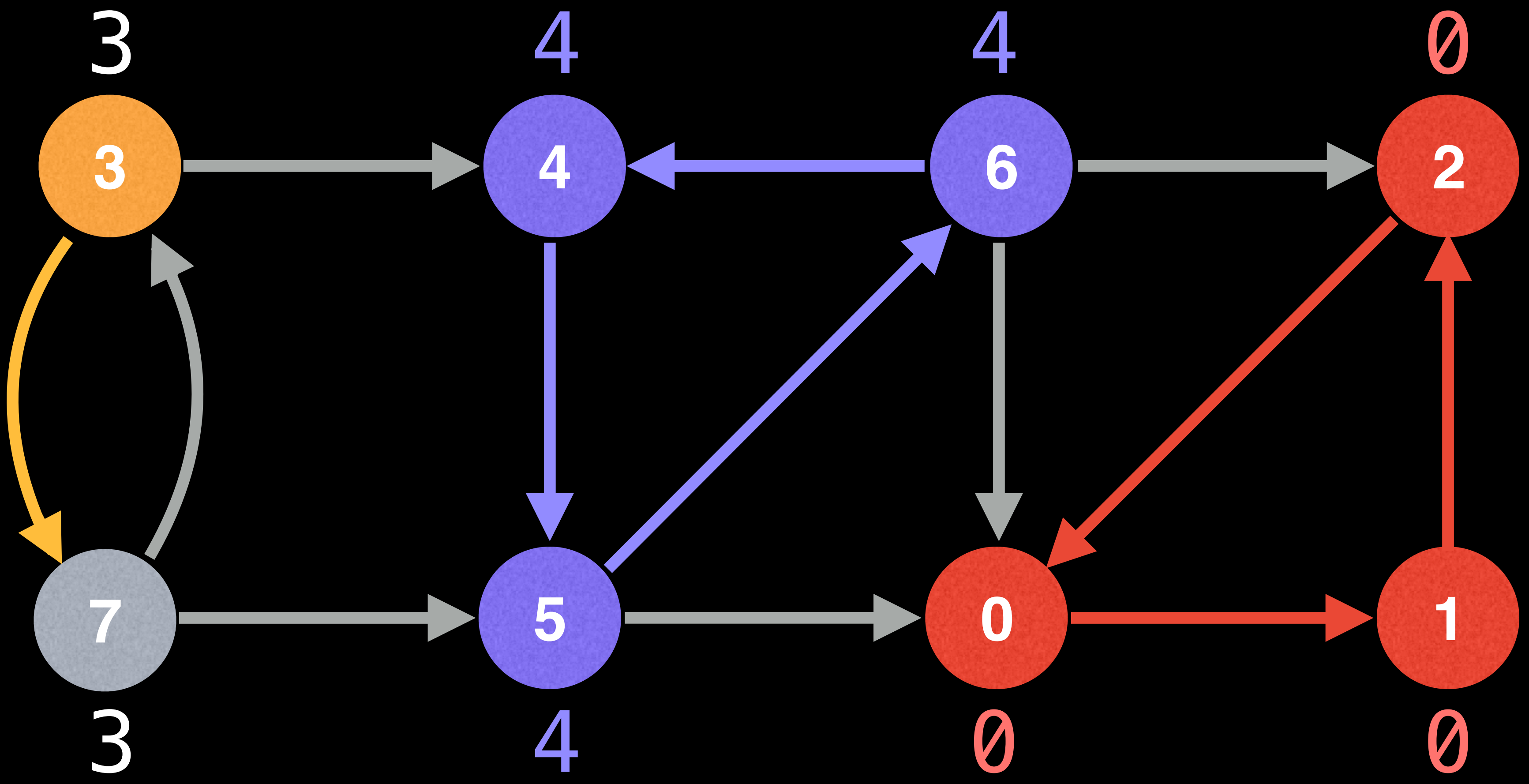


lowlink[6] = min(lowlink[6], lowlink[3])
= 3

● Unvisited ● Visiting neighbours ● Visited all neighbours

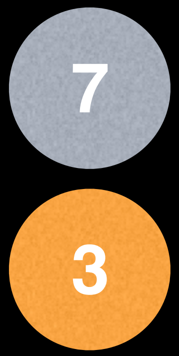
Stack

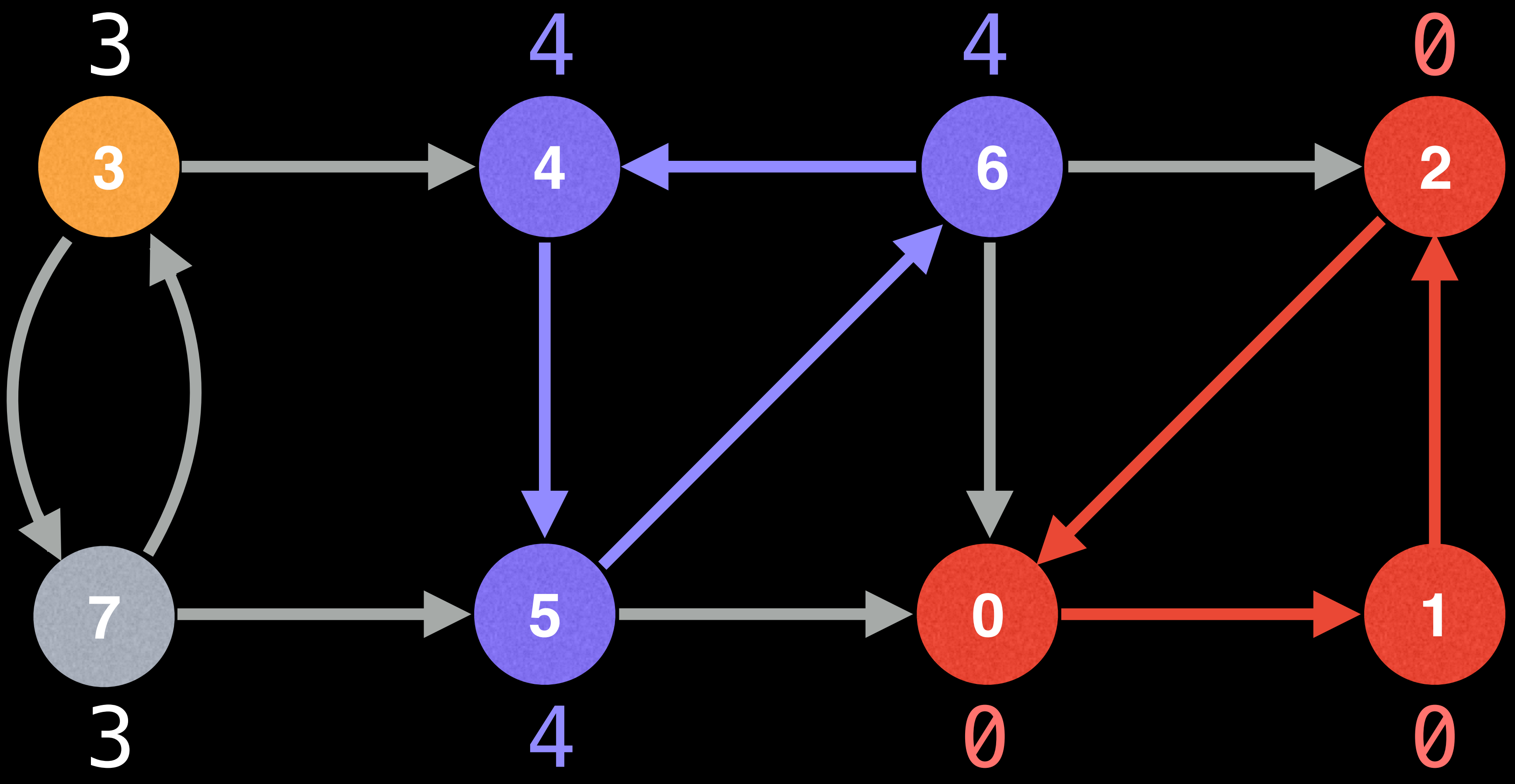




● Unvisited ● Visiting neighbours ● Visited all neighbours

Stack





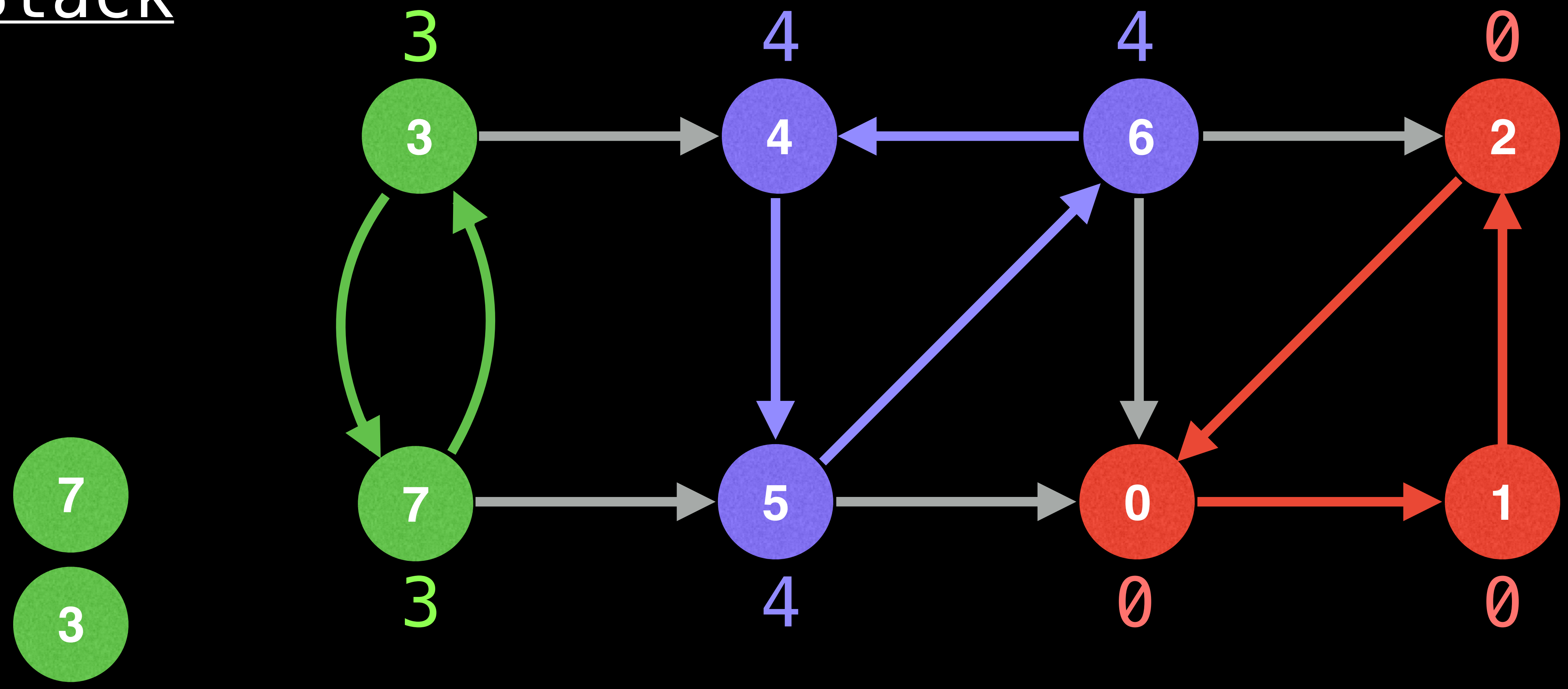
lowlink[3] = min(lowlink[3], lowlink[6])
= 3

● Unvisited

● Visiting neighbours

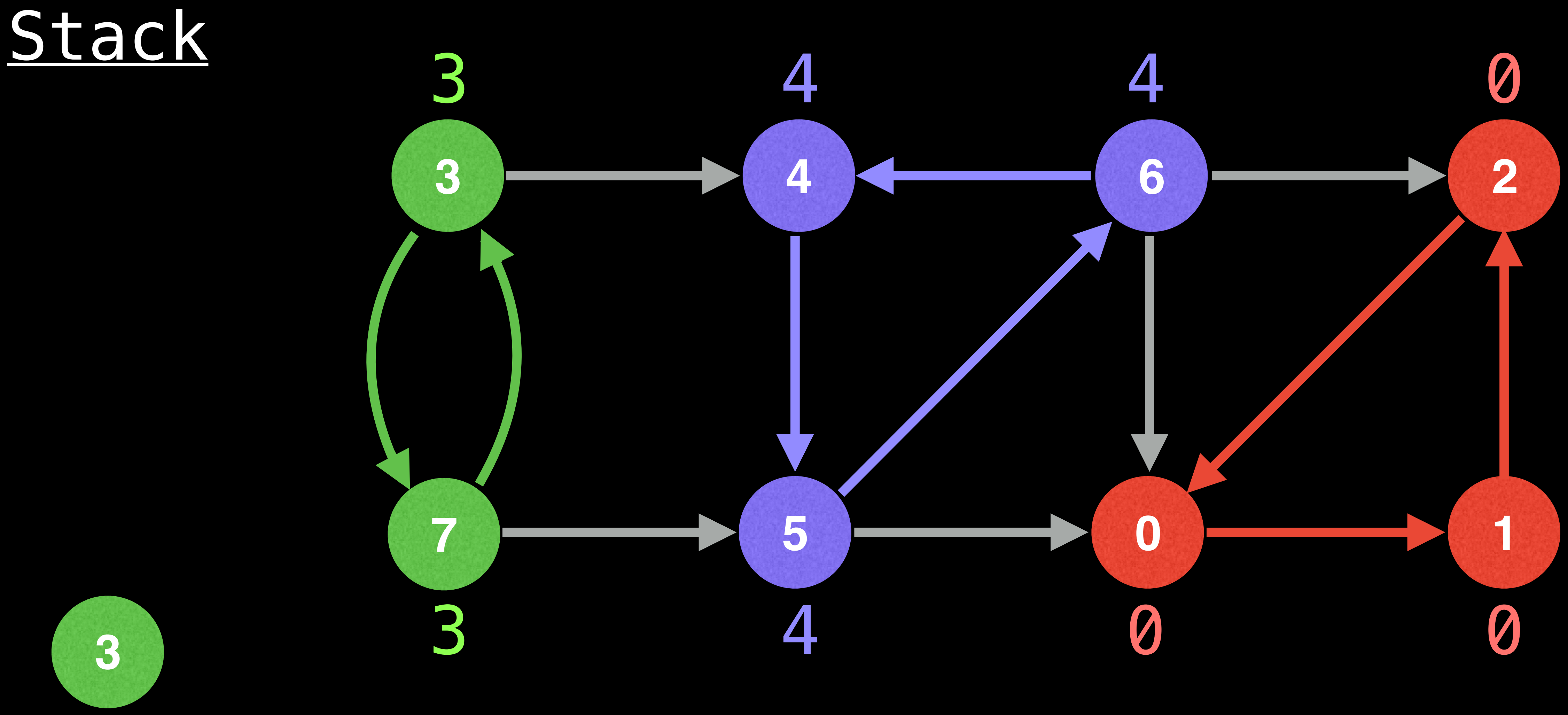
● Visited all neighbours

Stack



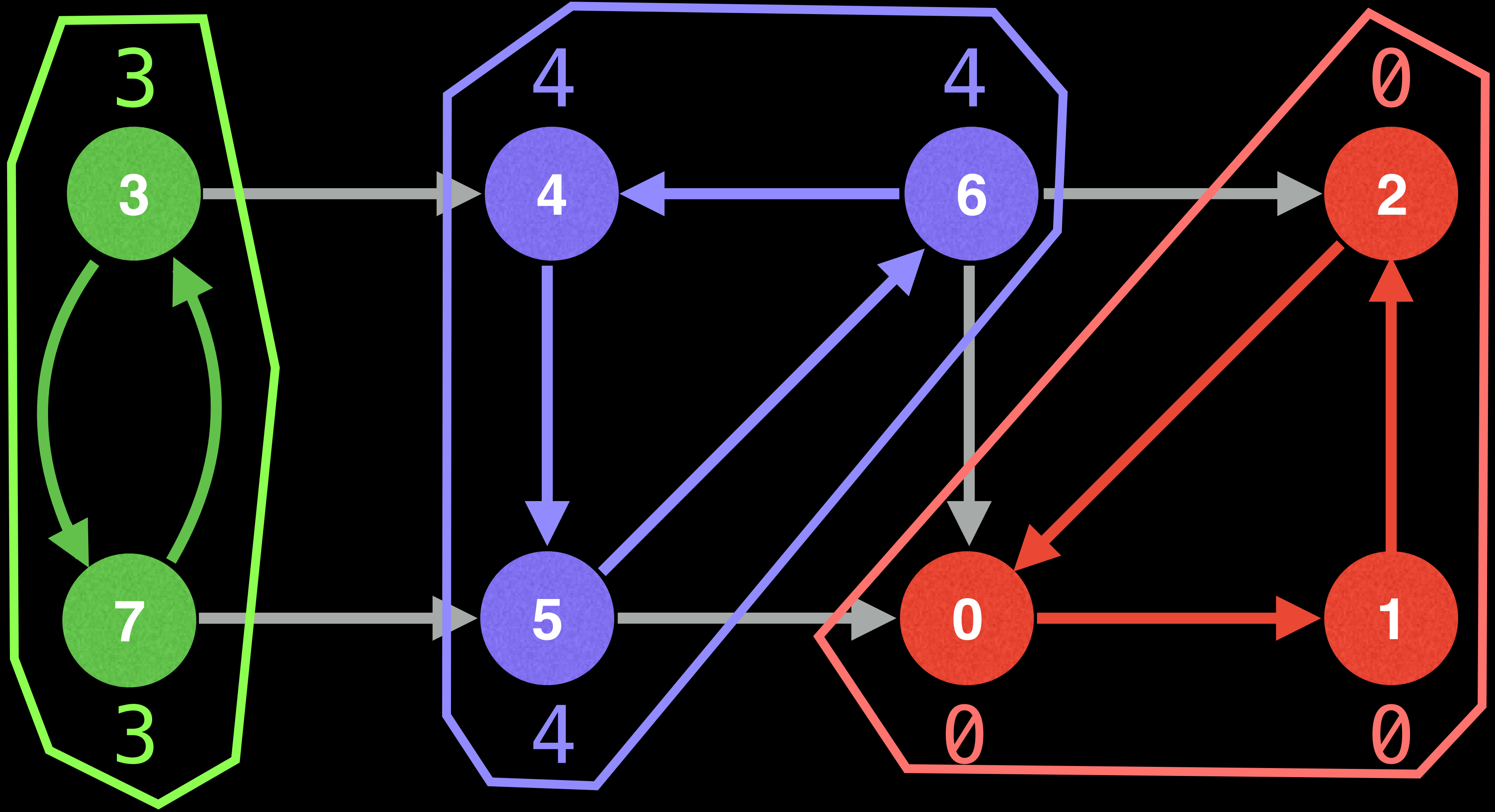
When a completed SCC is found (current node has visited all its neighbours and its lowlink value equals its id) pop off all associated nodes off the stack.

🔵 Unvisited 🟠 Visiting neighbours ⚪ Visited all neighbours



When a completed SCC is found (current node has visited all its neighbours and its lowlink value equals its id) pop off all associated nodes off the stack.

● Unvisited ● Visiting neighbours ● Visited all neighbours



```
UNVISITED = -1
n = number of nodes in graph
g = adjacency list with directed edges

id = 0       # Used to give each node an id
sccCount = 0 # Used to count number of SCCs found

# Index i in these arrays represents node i
ids = [0, 0, … 0, 0]               # Length n
low = [0, 0, … 0, 0]               # Length n
onStack = [false, false, …, false] # Length n
stack = an empty stack data structure

function findSccs():
  for(i = 0; i < n; i++): ids[i] = UNVISITED
  for(i = 0; i < n; i++):
    if(ids[i] == UNVISITED):
      dfs(i)
  return low
```

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```
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g = adjacency list with directed edges

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  for(i = 0; i < n; i++): ids[i] = UNVISITED
  for(i = 0; i < n; i++):
    if(ids[i] == UNVISITED):
      dfs(i)
  return low
```

```
function dfs(at):
  stack.push(at)
  onStack[at] = true
  ids[at] = low[at] = id++

  # Visit all neighbours & min low-link on callback
  for(to : g[at]):
    if(ids[to] == UNVISITED): dfs(to)
    if(onStack[to]): low[at] = min(low[at],low[to])

  # After having visited all the neighbours of ‘at’
  # if we're at the start of a SCC empty the seen
  # stack until we’re back to the start of the SCC.
  if(ids[at] == low[at]):
    for(node = stack.pop();;node = stack.pop()):
      onStack[node] = false
      low[node] = ids[at]
      if(node == at): break
    sccCount++
function dfs(at):
  stack.push(at)
  onStack[at] = true
  ids[at] = low[at] = id++

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      low[node] = ids[at]
      if(node == at): break
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      low[node] = ids[at]
      if(node == at): break
    sccCount++
function dfs(at):
  stack.push(at)
  onStack[at] = true
  ids[at] = low[at] = id++

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      onStack[node] = false
      low[node] = ids[at]
      if(node == at): break
    sccCount++
function dfs(at):
  stack.push(at)
  onStack[at] = true
  ids[at] = low[at] = id++

  # Visit all neighbours & min low-link on callback
  for(to : g[at]):
    if(ids[to] == UNVISITED): dfs(to)
    if(onStack[to]): low[at] = min(low[at],low[to])

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  # After having visited all the neighbours of ‘at’
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      onStack[node] = false
      low[node] = ids[at]
      if(node == at): break
    sccCount++
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  # Visit all neighbours & min low-link on callback
  for(to : g[at]):
    if(ids[to] == UNVISITED): dfs(to)
    if(onStack[to]): low[at] = min(low[at],low[to])

  # After having visited all the neighbours of ‘at’
  # if we're at the start of a SCC empty the seen
  # stack until we’re back to the start of the SCC.
  if(ids[at] == low[at]):
    for(node = stack.pop();;node = stack.pop()):
      onStack[node] = false
      low[node] = ids[at]
      if(node == at): break
    sccCount++
```